Dynamic Heterogeneous Panels

Contents

1 Asymptotic bias of LS estimator

1.1 Brief the source of bias

Consider the dynamic heterogeneous panels data model:

$$y_{i,t} = \phi_i y_{i,t-1} + \beta_{1i} x_{i,t} + u_{i,t},\ for\ i = 1, \ldots N;\ t = 1, \ldots, T\,, \tag{1}$$

Form above model, the model can rewritten as

$$\Delta y_{i,t} = -\left(1 - \phi_i\right)\left(y_{i,t} - \pi_i x_{i,t}\right) + u_{i,t}, \tag{2}$$

where $\pi_i = \frac{\beta_i}{1-\phi_i}$. And we defined $\theta_i = (1 - \phi_i)$. Then we suppose

$$\begin{aligned} \theta_i &= \theta + \eta_{i1}, \\ \pi_i &= \pi + \eta_{i2}. \end{aligned} \tag{3}$$

Therefore, we know

$$\beta_i = \pi_i \theta_i = \left(\pi + \eta_{i2}\right)\left(\theta + \eta_{i1}\right) = \pi\theta + \pi\eta_{i,1} + \theta\eta_{i2} + \eta_{i1}\eta_{i2} \tag{4}$$

And we defined $\eta_{i3} = \pi\eta_{i,1} + \theta\eta_{i2} + \eta_{i1}\eta_{i2}$. Then,we know $\beta_i = \pi\theta + \eta_{i3}$. Therefore, from equation (1), we have

$$\begin{aligned} y_{i,t} &= \phi_i y_{i,t-1} + \left(\pi\theta + \pi\eta_{i,1} + \theta\eta_{i2} + \eta_{i1}\eta_{i2}\right) x_{i,t} + u_{i,t} \\ &= \left(1 - \theta_i\right) y_{i,t-1} + \beta x_{i,t} + \eta_{i,3} x_{i,t} + u_{i,t} \\ &= (1 - \theta) y_{i,t-1} - \eta_{i1} y_{i,t-1} + \beta x_{i,t} + \eta_{i3} x_{i,t} + u_{i,t} \\ &= \phi y_{i,t-1} + \beta x_{i,t} + \left(u_{i,t} - \eta_{i1} y_{i,t-1} + \eta_{i3} x_{i,t}\right) \\ &= \phi y_{i,t-1} + \beta x_{i,t} + v_{i,t}, \end{aligned} \tag{5}$$

where $v_{i,t} = (u_{i,t} - \eta_{i1} y_{i,t-1} + \eta_{i3} x_{i,t})$. Then,we can see that $y_{i,t-1}$ and $x_{i,t}$ are correlated with $v_{i,t}$.

1.2 Asymptotic bias of LS estimator

To be continue.

2 Estimation method on dynamic heterogeneous panel data model

For convenient, we assume the number of regressor is 1 and we express the model as

$$y_{i,t} = \phi_i y_{i,t-1} + \sum_{\ell=1}^{k} \beta_{\ell i} x_{\ell i,t} + u_{i,t},\ for\ i = 1, \ldots N;\ t = 1, \ldots, T,\ \ell = 1, \ldots, k. \tag{6}$$

We stack the T observations for each i yield

$$\boldsymbol{y}_i = \boldsymbol{y}_{i,-1}\phi_i + \sum_{\ell=1}^{k} \boldsymbol{x}_{\ell i}\boldsymbol{\beta}_{\ell i} + \boldsymbol{u}_i, \tag{7}$$

where $\boldsymbol{y}_i = (y_{i,1}, \dots, y_{i,T})'$, $\boldsymbol{y}_{i,-1} = (y_{i,0}, \dots, y_{i,T-1})'$, $\boldsymbol{x}_{\ell i} = (x_{\ell i,1}, \dots, x_{\ell i,T})'$ and $\boldsymbol{u}_i = (u_{i,1}, \dots, u_{i,T})$. To be more compressive, the model can be expressed as

$$\boldsymbol{y}_i = \boldsymbol{W}_i \boldsymbol{\varphi}_i + \boldsymbol{u}_i, \tag{8}$$

where $\boldsymbol{W}_i = \left(\boldsymbol{y}_{i,-1}, \boldsymbol{X}_i\right)$ and $\boldsymbol{\varphi}_i = \left(\phi_i, \boldsymbol{\beta}_i'\right)'$

2.1 LSMG estimator

The LS (least square) estimator is defined as

$$\hat{\boldsymbol{\varphi}}_{LSi} = \left(\frac{\boldsymbol{W}_i' \boldsymbol{W}_i}{T}\right)^{-1} \left(\frac{\boldsymbol{W}_i' \boldsymbol{y}_i}{T}\right) \tag{9}$$

Follow Pesaran and Smith (1995), we define the LSMG (least square mean group) estimator as

$$\hat{\boldsymbol{\varphi}}_{LSMG} = \frac{1}{N} \sum_{i=1}^{N} \hat{\boldsymbol{\varphi}}_{LSi}. \tag{10}$$

2.2 IVMG estimator

We use current and lagged values of $\boldsymbol{x}_i$ as instruments, as

$$\boldsymbol{Z}_i = \left(\boldsymbol{X}_i, \boldsymbol{X}_{i,-1}\right), \tag{11}$$

where $\boldsymbol{Z}_i$ is $T \times k$ matrix.

The IV (instrument variable) estimator is defined as

$$\hat{\boldsymbol{\varphi}}_{IVi} = \left(\left(\frac{\boldsymbol{Z}_i' \boldsymbol{W}_i}{T}\right)' \left(\frac{\boldsymbol{Z}_i' \boldsymbol{Z}_i}{T}\right)^{-1} \left(\frac{\boldsymbol{Z}_i' \boldsymbol{W}_i}{T}\right)\right)^{-1} \left(\left(\frac{\boldsymbol{Z}_i' \boldsymbol{W}_i}{T}\right)' \left(\frac{\boldsymbol{Z}_i' \boldsymbol{Z}_i}{T}\right)^{-1} \left(\frac{\boldsymbol{Z}_i' \boldsymbol{y}_i}{T}\right)\right) \tag{12}$$

We also define the IVMG (instrument variable mean group) estimator as

$$\hat{\boldsymbol{\varphi}}_{IVMG} = \frac{1}{N} \sum_{i=1}^{N} \hat{\boldsymbol{\varphi}}_{IVi}. \tag{13}$$

3 Estimation method on dynamic heterogeneous panel data model with multifactor error structure

Consider the model (6), we drawn $x_{\ell i,t}$ as

$$x_{\ell i,t} = \boldsymbol{\gamma}_{xi}^{0\prime} \boldsymbol{f}_{xt}^{0} + \varepsilon_{xi,t} \tag{14}$$

and the idiosyncratic errors of the process for $y_{i,t}$ as

$$u_{i,t} = \boldsymbol{\gamma}_{yi}^{0\prime} \boldsymbol{f}_{yt}^{0} + \varepsilon_{yi,t}, \tag{15}$$

where $\boldsymbol{\gamma}_{yi}^{0}$ and $\boldsymbol{\gamma}_{xi}^{0}$ are $m_y \times 1$ and $m_x \times 1$ true factor loading respectively, $\boldsymbol{f}_{yt}^{0}$ and $\boldsymbol{f}_{xt}^{0}$ are $m_y \times 1$ and $m_x \times 1$ true vector of unobservable factors respectively.

3.1 Norkutes' (2019) IVMG estimator

We asymptotically eliminate the common factor in $\boldsymbol{x}_i$ by projecting matrix, $\boldsymbol{M}_{F_x^0}$.

$$\boldsymbol{M}_{F_x^0} = \boldsymbol{I}_T - \boldsymbol{F}_x^0 \left(\boldsymbol{F}_x^{0\prime} \boldsymbol{F}_x^0 \right)^{-1} \boldsymbol{F}_x^{0\prime}; \boldsymbol{M}_{F_{x,-1}^0} = \boldsymbol{I}_T - \boldsymbol{F}_{x,-1}^0 \left(\boldsymbol{F}_{x,-1}^{0\prime} \boldsymbol{F}_{x,-1}^0 \right)^{-1} \boldsymbol{F}_{x,-1}^{0\prime} \tag{16}$$

And using the defactored covariates as instruments, as

$$\boldsymbol{Z}_{IVi} = \left(\boldsymbol{M}_{F_X^0} \boldsymbol{x}_i, \boldsymbol{M}_{F_{x,-1}^0} \boldsymbol{X}_{i,-1} \right) \tag{17}$$

The first step IV estimator can be expressed as

$$\begin{aligned} \hat{\boldsymbol{\varphi}}_{IVi} = & \left(\left(\frac{\boldsymbol{Z}_i^{\prime} \boldsymbol{M}_{F_X^0} \boldsymbol{W}_i}{T} \right)^{\prime} \left(\frac{\boldsymbol{Z}_i^{\prime} \boldsymbol{M}_{F_X^0} \boldsymbol{Z}_i}{T} \right)^{-1} \left(\frac{\boldsymbol{Z}_i^{\prime} \boldsymbol{M}_{F_X^0} \boldsymbol{W}_i}{T} \right) \right)^{-1} \\ & \left(\left(\frac{\boldsymbol{Z}_i^{\prime} \boldsymbol{M}_{F_X^0} \boldsymbol{W}_i}{T} \right)^{\prime} \left(\frac{\boldsymbol{Z}_i^{\prime} \boldsymbol{M}_{F_X^0} \boldsymbol{Z}_i}{T} \right)^{-1} \left(\frac{\boldsymbol{Z}_i^{\prime} \boldsymbol{M}_{F_X^0} \boldsymbol{y}_i}{T} \right) \right). \end{aligned} \tag{18}$$

4 Monte Carlo simulation design

4.1 dynamic heterogeneous panels data model without error factor structure

The data generating process:

$$\begin{aligned} y_{i,t} &= \phi_i y_{i,t-1} + \sum_{\ell=1}^{k} \beta_{\ell i} x_{\ell i,t} + u_{i,t}, \;\; for \;\; i = 1, \ldots N; \; t = -49, \ldots, T, \\ x_{\ell i,t} &= \sum_{\ell=1}^{k} \phi_{\ell i} x_{\ell i,t-1} + v_{\ell i,t}, \end{aligned} \tag{19}$$

where $u_{i,t} \overset{i.i.d.}{\sim} \mathcal{N}(0,\, 1)$, and $v_{\ell i,t} = \rho_{v,\ell} v_{\ell i,t-1} + \left(1 - \rho_{v,\ell}^2\right)^{\frac{1}{2}} \varpi_{\ell i,t}, \varpi_{\ell i,t} \overset{i.i.d.}{\sim} U(0.5, 1.5)$, $\rho_{v,\ell} = 0.5$.

The slope coefficients are generated as

$$\phi_i = \phi + \eta_{\phi i}, \;\; \beta_{1,i} = \beta_1 + \eta_{\beta_1 i} \, and \, \beta_{2,i} = \beta_2 + \eta_{\beta_2 i}. \tag{20}$$

Here we consider $\phi \in \{0.5\}$, $\beta_1 = 3$ and $\beta_2 = 1$. For the design of heterogenous slopes, $\eta_{\phi i} \overset{i.i.d.}{\sim} U\left(-c, c\right)$, and

$$\eta_{\beta_\ell i} = \left(1 - \rho_\beta^2\right)^{1/2} \eta_{\phi i}. \tag{21}$$

Here, we set $c = 0.2$, $\rho_\beta = 0.4$ for $\ell = 1, 2$.

4.2 Dynamic heterogeneous panels data model with multi-factor error structure

This Monte Carlo simulation design same as Norkute et al. (2019). For convenience, we rewrite the data generating process as bellow

$$y_{i,t} = \alpha_i + \phi_i y_{i,t-1} + \sum_{\ell=1}^{k} \beta_{\ell i} x_{\ell i,t} + u_{i,t}, \;\; for\ i = 1, \dots N;\ t = -49, \dots, T\,. \tag{22}$$

$$\tag{23}$$

We allow error factor structure in the model as

$$u_{i,t} = \sum_{s=1}^{m_y} \gamma_{si}^0 f_{s,t}^0 + \varepsilon_{i,t}, \tag{24}$$

where

$$f_{s,t}^0 = \rho_{s,t}^0 f_{s,t-1}^0 + \left(1 - \rho_{fs}^2\right)^{1/2} \zeta_{s,t}, \tag{25}$$

with $\zeta_{s,t} \overset{i.i.d.}{\sim} N(0,1)$ for $s = 1, \dots m_y$. We assume $k = 2$ and $m_y = 1 + k = 3$ and set $\rho_{s,t}^0 = 0.5$ for all s. The error term, $\varepsilon_{i,t}$, setting as

$$\varepsilon_{i,t} = \varsigma_\varepsilon \sigma_{it} \left(\epsilon_{it} - 1\right) / \sqrt{2}, \tag{26}$$

where $\epsilon_{it} \overset{i.i.d.}{\sim} \chi_1^2$, $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \overset{i.i.d.}{\sim} \chi_2^2/2$, and $\varphi_t = t/T\ for\ t = 0, \dots, T$. And we set

$$\varsigma_\varepsilon = \frac{\pi_\mu}{1 - \pi_\mu} m_y. \tag{27}$$

we set $\pi_\mu \in \{3/4\}$.

The process of regressors is

$$x_{\ell it} = \mu_{\ell i} + + \sum_{\ell=1}^{k} \phi_{\ell i} x_{\ell i,t-1} + \sum_{s=1}^{m_x} \gamma_{\ell si}^0 f_{s,t}^0 + v_{\ell it}, \;\; for\ i = 1, \dots N;\ t = -49, \dots, T\,;\ \ell = 1, 2. \tag{28}$$

We set number of factor, m_x, is 2. Therefore, $\boldsymbol{f}_{y,t}^0 = \left(f_{1t}^0, f_{2t}^0, f_{3t}^0\right)'$ and $\boldsymbol{f}_{x,t}^0 = \left(f_{1t}^0, f_{2t}^0\right)'$. We set

$$v_{\ell i,t} = \rho_{v,\ell} v_{\ell i,t-1} + \left(1 - \rho_{v,\ell}^2\right)^{\frac{1}{2}} \varpi_{\ell i,t}, for\ \ell = 1, 2, \tag{29}$$

where $\rho_{v,\ell} = 0.5$ for all ℓ. The individual effect is

$$\alpha_i^* \overset{i.i.d.}{\sim} N\left(0, (1-\rho_i)^2\right), \;\; \mu_{\ell i}^* = \rho_{\mu,\ell} \alpha_i^* + \left(1 - \rho_{\mu,\ell}^2\right)^{1/2} \omega_{\ell i}, \tag{30}$$

where $\omega \overset{i.i.d.}{\sim} N\left(0, (1-\rho_i)^2\right)$ and $\rho_{\mu,\ell} = 0.5$.

Now, we define the factor loading in $u_{i,t}$ are generated as $\gamma_{si}^{0*} \overset{i.i.d.}{\sim} N(0,1)$, for $s = 1, \ldots, m_y = 3$, and the factor loading in x_{1it} and x_{2it} are drawn as

$$\begin{aligned} \gamma_{1si}^{0*} &= \rho_{\gamma,1s}\gamma_{3i}^{0*} + \left(1-\rho_{\gamma,1s}^{2}\right)^{1/2}\xi_{1si};\ \xi_{1si} \overset{i.i.d.}{\sim} N(0,1); \\ \gamma_{2si}^{0*} &= \rho_{\gamma,2s}\gamma_{si}^{0*} + \left(1-\rho_{\gamma,2s}^{2}\right)^{1/2}\xi_{2si};\ \xi_{2si} \overset{i.i.d.}{\sim} N(0,1); \end{aligned} \tag{31}$$

for $s = 1, \ldots, m_x = 2$. We set $\rho_{\gamma,11} = \rho_{\gamma,12} \in \{0.5\}$ and $\rho_{\gamma,21} = \rho_{\gamma,22} = 0.5$. The factor loading are generated as

$$\mathbf{\Gamma} = \mathbf{\Gamma}^{0} + \mathbf{\Gamma}_{i}^{0*} \tag{32}$$

where

$$\mathbf{\Gamma}_{i}^{0} = \begin{bmatrix} \gamma_{1i}^{0} & \gamma_{11i}^{0} & \gamma_{21i}^{0} \\ \gamma_{2i}^{0} & \gamma_{12i}^{0} & \gamma_{22i}^{0} \\ \gamma_{3i}^{0} & 0 & 0 \end{bmatrix} \tag{33}$$

and

$$\mathbf{\Gamma}_{i}^{0*} = \begin{bmatrix} \gamma_{1i}^{0*} & \gamma_{11i}^{0*} & \gamma_{21i}^{0*} \\ \gamma_{2i}^{0*} & \gamma_{12i}^{0*} & \gamma_{22i}^{0*} \\ \gamma_{3i}^{0*} & 0 & 0 \end{bmatrix}. \tag{34}$$

We set

$$\mathbf{\Gamma}^{0} = \begin{bmatrix} 1/4 & 1/4 & -1 \\ 1/2 & -1 & 1/4 \\ 1/2 & 0 & 0 \end{bmatrix}. \tag{35}$$

And

$$\alpha_i = \alpha + \alpha_i^*,\ \mu_{\ell i} = \mu_\ell + \mu_{\ell i}^*, \tag{36}$$

where $\alpha = 1/2$, $\mu_1 = 1$, $\mu_2 = -1/2$.

The slope coefficients are generated as

$$\phi_i = \phi + \eta_{\phi i},\ \ \beta_{1,i} = \beta_1 + \eta_{\beta_1 i}\, and\, \beta_{2,i} = \beta_2 + \eta_{\beta_2 i}. \tag{37}$$

Here we consider $\phi \in \{0.5\}$, $\beta_1 = 3$ and $\beta_2 = 1$. For the design of heterogenous slopes, $\eta_{\phi i} \overset{i.i.d.}{\sim} U(-c, c)$, and

$$\eta_{\beta_\ell i} = \left[(2c)^2/12\right]\rho_\beta \xi_{\beta\ell i} + \left(1-\rho_\beta^2\right)^{1/2}\eta_{\phi i}, \tag{38}$$

where

$$\xi_{\beta\ell i} = \frac{\bar{v_{\ell i}^2} - \bar{v_{\ell}^2}}{\left[N^{-1}\sum_{i=1}^{N}\left(\bar{v_{\ell i}^2} - \bar{v_{\ell}^2}\right)^2\right]^{1/2}}, \tag{39}$$

with $\bar{v_{elli}^2} = T^{-1}\sum_{t=1}^{T} v_{\ell it}^2$, $\bar{v_\ell^2} = N^{-1}\sum_{i=1}^{N}\bar{v_{\ell i}^2}$, for $\ell = 1, 2$. Here, we set $c = 0.2$, $\rho_\beta = 0.4$ for $\ell = 1, 2$. And

$$\varsigma_v^2 = \varsigma_\varepsilon^2\left[SNR - \frac{\rho_v^2}{1-\rho_v^2}\right]\left(\frac{\beta_1^2+\beta_2^2}{1-\rho_v^2}\right)^{-1}, \tag{40}$$

where $SNR = 4$. For the (T, N), we consider $T \in \{25, 50, 100, 200\}$ and $N \in \{25, 50, 100, 200\}$.

5 Monte Carlo simulation results

5.1 Dynamic Heterogeneous Panels without multifactor error structure

We consider ARDL(1,0) model.
$\phi \in \{0.5\}$.
$\beta_1 = 3$.
$\beta_2 = 1$.
$u_{i,t} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$.
$\varpi_{\ell i,t} \overset{i.i.d.}{\sim} U(0.5, 1.5)$.
$\rho_{v,\ell} = 0.5$.
$c = 0.2$.
$\rho_\beta = 0.4$.
$T \in \{25, 50, 100, 200\}$.
$N \in \{25, 50, 100, 200\}$.

LSMG estimator is provided in sheet 1 of MC.xlsx file.
IVMG estimator is provided in sheet 2 of MC.xlsx file.
Short summary: The performance of IVMG estimator is better than LSMG estimator in bias and RMSE.

5.2 Dynamic Heterogeneous Panels with multifactor error structure

We consider ARDL(1,0) model.
$\phi \in \{0.5\}$.
$\beta_1 = 3$.
$\beta_2 = 1$.
$k = 2$.
$m_y = 1 + k = 3$.
$m_x = k = 2$.
$\zeta_{s,t} \overset{i.i.d.}{\sim} N(0, 1)$
$\pi_\mu \in \{3/4\}$.
$\rho^0_{s,t} = 0.5$.
$\rho_{v,\ell} = 0.5$.
$\rho_{\mu,\ell} = 0.5$.
$\gamma^{0*}_{si} \overset{i.i.d.}{\sim} N(0, 1)$.
$\xi_{1si} \overset{i.i.d.}{\sim} N(0, 1)$.
$\xi_{2si} \overset{i.i.d.}{\sim} N(0, 1)$.
$\rho_{\gamma,11} = \rho_{\gamma,12} \in \{0.5\}$.
$\rho_{\gamma,21} = \rho_{\gamma,22} = 0.5$.
$$\mathbf{\Gamma}^0 = \begin{bmatrix} 1/4 & 1/4 & -1 \\ 1/2 & -1 & 1/4 \\ 1/2 & 0 & 0 \end{bmatrix}.$$
$\alpha = 1/2$.

$\mu_1 = 1.$
$\mu_2 = -1/2.$
$c = 0.2.$
$\rho_\beta = 0.4.$
$SNR = 4.$
$T \in \{25, 50, 100, 200\}.$
$N \in \{25, 50, 100, 200\}.$

IVMG estimator is provided in sheet 3 of MC.xlsx file.

Short summary: When N and T increase, the performance of IVMG estimator is good in bias and RMSE.

References

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Pesaran, M. and R. Smith (1995). Estimating long-run relationships from dynamic heterogeneous panels. *Journal of Econometrics 68*, 79 – 113.