Dynamic Heterogeneous Panels

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1.1 Asymptotic property of LS estimator

Consider the dynamic heterogeneous panels data model:

$$y_{i,t} = \alpha_i + \phi_i y_{i,t-1} + \beta_i x_{i,t} + u_{i,t},\ \ for\ i = 1, \ldots N;\ t = 1, \ldots, T\,, \tag{1}$$

Assumption 1 *$x_{i,t}$ and $u_{i,t}$ are independently distributed for all t and s .*

Based on heterogenous dynamic panel data model (1), we can obtain fixed effect estimator as

$$\hat{\boldsymbol{\theta}}_{LS,i} = \begin{pmatrix} \hat{\phi}_i \\ \hat{\beta}_i \end{pmatrix} = \begin{pmatrix} \frac{\sum_{t=1}^{T} \tilde{y}_{i,t-1}^2}{T} & \frac{\sum_{t=1}^{T} \tilde{y}_{i,t-1}\tilde{x}_{i,t}}{T} \\ \frac{\sum_{t=1}^{T} \tilde{x}_{i,t}\tilde{y}_{i,t-1}}{T} & \frac{\sum_{t=1}^{T} \tilde{x}_{i,t}^2}{T} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\sum_{t=1}^{T} \tilde{y}_{i,t-1}\tilde{y}_{i,t}}{T} \\ \frac{\sum_{t=1}^{T} \tilde{x}_{i,t}\tilde{y}_{i,t}}{T} \end{pmatrix}, \tag{2}$$

where $\tilde{y}_{i,t} = y_{i,t} - \bar{y}_i$, $\tilde{y}_{i,t-1} = y_{i,t-1} - \bar{y}_{i,-1}\iota_T$ and $\tilde{x}_{i,t} = x_{i,t} - \bar{x}_i$ with $\bar{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{i,t}$, $\bar{y}_{i,-1} = \frac{1}{T}\sum_{t=1}^{T} y_{i,t-1}$, $\bar{x}_i = \frac{1}{T}\sum_{t=1}^{T} x_{i,t}$. Under equation (1), we have

$$\begin{pmatrix} \hat{\phi}_i - \phi_i \\ \hat{\beta}_i - \beta_i \end{pmatrix} = \begin{pmatrix} \frac{\sum_{t=1}^{T} \tilde{y}_{i,t-1}^2}{T} & \frac{\sum_{t=1}^{T} \tilde{y}_{i,t-1}\tilde{x}_{i,t}}{T} \\ \frac{\sum_{t=1}^{T} \tilde{x}_{i,t}\tilde{y}_{i,t-1}}{T} & \frac{\sum_{t=1}^{T} \tilde{x}_{i,t}^2}{T} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\sum_{t=1}^{T} \tilde{y}_{i,t-1}\tilde{u}_{i,t}}{T} \\ \frac{\sum_{t=1}^{T} \tilde{x}_{i,t}\tilde{u}_{i,t}}{T} \end{pmatrix}, \tag{3}$$

where $\tilde{u}_{i,t}$ is $u_{i,t} - \bar{u}_i$ with $\bar{u}_i = \frac{1}{T}\sum_{t=1}^{T} u_{i,t}$.

Now, we can investigate asymptotic bias by taking the probability limit as

$$A_{\phi i}^{(1)} = \operatorname*{plim}_{T\to\infty} \left(\frac{\sum_{t=1}^{T} \tilde{y}_{i,t-1}\tilde{u}_{i,t}}{T} \right). \tag{4}$$

Then $A_i^{(1)}$ can be taken expectations as

$$\begin{aligned} A_{\phi i}^{(1)} &= E\left(y_{i,t-1} - \bar{y}_{i,-1}\right)\left(u_{i,t} - \bar{u}_i\right) \\ &= E\left(y_{i,t-1}u_{i,t}\right) - E\left(y_{i,t-1}\bar{u}_i\right) - E\left(\bar{y}_{i,-1}u_{i,t}\right) + E\left(\bar{y}_{i,-1}\bar{u}_i\right), \end{aligned} \tag{5}$$

where $E\left(y_{i,t-1}u_{i,t}\right) = 0$.

And we assume $y_{i,t}$ has started from a long time period in the past, so we have

$$y_{i,t} = \frac{\alpha_i}{(1-\phi_i)} + \sum_{s=0}^{\infty} \beta_i \phi_i^s x_{i,t-s} + \sum_{s=0}^{\infty} \phi_i^s u_{i,t-s}, \tag{6}$$

Then, we have

$$\begin{aligned} A_{\phi i}^{(1)} = -E\left(\left(\sum_{s=0}^{\infty} \phi_i^s u_{i,t-s-1}\right)\left(\frac{1}{T}\sum_{t=1}^{T} u_{i,t}\right)\right) - E\left(\frac{u_{i,t}}{T}\sum_{t=1}^{T}\sum_{s=0}^{\infty} \phi_i^s u_{i,t-s-1}\right) + \\ \left(\frac{1}{T}\sum_{t=1}^{T}\sum_{s=0}^{\infty} \phi_i^s u_{i,t-s-1}\right)\left(\frac{1}{T}\sum_{t=1}^{T} u_{i,t}\right). \end{aligned} \tag{7}$$

Hence, from above equation, we have

$$
\begin{aligned}
A_{\phi i}^{(1)} = & -\frac{1}{T}E\left\{\left(u_{i,t-1}+u_{i,t-2}\phi_i^1+u_{i,t-3}\phi_i^2+\ldots\right)\left(u_{i,1}+\ldots+u_{i,t-1}+u_{i,t}+\ldots+u_{i,T}\right)\right\}- \\
& \frac{1}{T}E\left\{u_{i,t}\sum_{s=1}^{T}\left(u_{i,s-1}\phi_i^0+u_{i,s-2}\phi_i^1+\ldots+u_{i,s-t-1}\phi_i^t+\ldots+u_{i,s-T-1}\phi_i^T+\ldots\right)\right\}+ \\
& +\frac{1}{T}E\{\left(\sum_{s=1}^{T}u_{i,s-1}\phi_i^0+\sum_{s=1}^{T}u_{i,s-2}\phi_i^1+\ldots+\sum_{s=1}^{T}u_{i,s-t-1}\phi_i^t+\ldots+\sum_{s=1}^{T}u_{is-T-1}\phi_i^T+\ldots\right) \\
& \left(\frac{1}{T}\sum_{s=1}^{T}u_{i,s}\right)\} \\
& =-\frac{\sigma_u^2}{T}\frac{\left(1-\phi_i^{t-1}\right)}{1-\phi_i}-\frac{\sigma_u^2}{T}\frac{\left(1-\phi_i^{T-t}\right)}{\left(1-\phi_i\right)}+\frac{\sigma_u^2}{T}\left(\frac{1}{1-\phi_i}-\frac{1}{T}\frac{\left(1-\phi_i^T\right)}{\left(1-\phi_i\right)^2}\right) \\
& =-\frac{\sigma_u^2}{T(1-\phi_i)}\left(1-\phi_i^{t-1}-\phi_i^{T-t}+\frac{1}{T}\frac{\left(1-\phi_i^T\right)}{\left(1-\phi_i\right)}\right).
\end{aligned}
\tag{8}
$$

Therefore, we can see the bias of $\hat{\phi}_i$ is $O_p\left(T^{-1}\right)$.

To be more compact ,we can rewrite the model as,

$$\tilde{\boldsymbol{y}}_i=\tilde{\boldsymbol{W}}_i\boldsymbol{\theta}_i+\tilde{\boldsymbol{u}}_i, \tag{9}$$

where $\tilde{\boldsymbol{y}}_i=\left(\tilde{\boldsymbol{y}}_{i,1},\ldots,\tilde{\boldsymbol{y}}_{i,T}\right)'$ is $T\times 1$ vector, $\tilde{\boldsymbol{W}}_i=\left(\tilde{\boldsymbol{w}}_{i,1},\ldots,\tilde{\boldsymbol{w}}_{i,T}\right)'$ is $T\times 2$ matrix and $\tilde{\boldsymbol{u}}_i=\left(\tilde{\boldsymbol{u}}_{i,1},\ldots,\tilde{\boldsymbol{u}}_{i,T}\right)$ is $T\times 1$ vector with $\tilde{\boldsymbol{w}}_{i,t}=\left(y_{i,t-1}-\bar{y}_{i,-1},x_{i,t}-\bar{x}_i\right)'$, for $t=1,\ldots,T$ and $i=1,\ldots,N$.

Also, we define our interested parameter as

$$\left(\phi_i,\beta_i\right)'=\boldsymbol{\theta}_i=\boldsymbol{\theta}+\boldsymbol{\lambda}_i, \tag{10}$$

where $\boldsymbol{\lambda}_i\overset{i.i.d.}{\sim}\left(\mathbf{0},\boldsymbol{\Sigma}_\lambda\right)$. The lest square estimator, $\hat{\boldsymbol{\theta}}_{LS,i}$, can be expressed as

$$\hat{\boldsymbol{\theta}}_{LS,i}=\left(\frac{\tilde{\boldsymbol{W}}_i'\tilde{\boldsymbol{W}}_i}{T}\right)^{-1}\frac{\tilde{\boldsymbol{W}}_i'\tilde{\boldsymbol{y}}_i}{T}. \tag{11}$$

From above discussion and assumptions, we have following theorem

Theorem 1

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{IV,i}-\boldsymbol{\theta}_i\right)\xrightarrow{d}N\left(\mathbf{0},\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{LS,i}\boldsymbol{Q}^{-1}\right), \tag{12}$$

where $\boldsymbol{\Sigma}_{LS,i}=\operatorname{plim}_{T\to\infty}T^{-1}\tilde{\boldsymbol{W}}_i'\tilde{\boldsymbol{u}}_i\tilde{\boldsymbol{u}}_i'\tilde{\boldsymbol{W}}_i$ *and* $\boldsymbol{Q}_i=\operatorname{plim}_{T\to\infty}T^{-1}\tilde{\boldsymbol{W}}_i'\tilde{\boldsymbol{W}}_i$

1.1.1 Mean group LS estimator

Now, we define the mean group estimator of $\boldsymbol{\theta}$:

$$\hat{\boldsymbol{\theta}}_{LSMG} = \frac{1}{N}\sum_{i=1}^{N}\hat{\boldsymbol{\theta}}_{LSi}. \tag{13}$$

And we can show that the asymptotic property of $\hat{\boldsymbol{\theta}}_{LSMG}$, as

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{LSMG} &= N^{-1}\sum_{i=1}^{N}\left(\frac{\tilde{\boldsymbol{W}}_i^{'}\tilde{\boldsymbol{W}}_i}{T}\right)^{-1}\frac{\tilde{\boldsymbol{W}}_i^{'}\tilde{\boldsymbol{y}}_i}{T} \\ &= \bar{\boldsymbol{\theta}} + N^{-1}\sum_{i=1}^{N}\left(\frac{\tilde{\boldsymbol{W}}_i^{'}\tilde{\boldsymbol{W}}_i}{T}\right)^{-1}\frac{\tilde{\boldsymbol{W}}_i^{'}\tilde{\boldsymbol{u}}_i}{T},\end{aligned} \tag{14}$$

where $\bar{\boldsymbol{\theta}} = N^{-1}\sum_{i=1}^{N}\boldsymbol{\theta}_i$. For fixed N and large T, we have

$$\underset{T\to\infty}{\text{plim}}\,\hat{\boldsymbol{\theta}}_{LSMG} = \bar{\boldsymbol{\theta}} + N^{-1}\sum_{i=1}^{N}\underset{T\to\infty}{\text{plim}}\left(\frac{\tilde{\boldsymbol{W}}_i^{'}\tilde{\boldsymbol{W}}_i}{T}\right)^{-1}\underset{T\to\infty}{\text{plim}}\left(\frac{\tilde{\boldsymbol{W}}_i^{'}\tilde{\boldsymbol{u}}_i}{T}\right) \tag{15}$$

Then, from section 1.1, we know that $\text{plim}_{T\to\infty}\left(\frac{\tilde{\boldsymbol{W}}_i^{'}\tilde{\boldsymbol{u}}_i}{T}\right) = O_p(1)$. Thus, we can obtain

$$\underset{T\to\infty}{\text{plim}}\,\hat{\boldsymbol{\theta}}_{LSMG} = \bar{\boldsymbol{\theta}}. \tag{16}$$

When $N\to\infty$ and $T\to\infty$ and by the law of large numbers, we can see that

$$\underset{T\to\infty,N\to\infty}{\text{plim}}\,\hat{\boldsymbol{\theta}}_{LSMG} = \boldsymbol{\theta}. \tag{17}$$

And the variance estimator of $\hat{\boldsymbol{\theta}}_{LSMG}$ is given by

$$\hat{\boldsymbol{\Sigma}}_{LS,\lambda} = \frac{1}{N(N-1)}\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{LS,i} - \hat{\boldsymbol{\theta}}_{LSMG}\right)\left(\hat{\boldsymbol{\theta}}_{LS,i} - \hat{\boldsymbol{\theta}}_{LSMG}\right)^{'}. \tag{18}$$

We show that $\hat{\boldsymbol{\Sigma}}_{LS,\lambda}$ is consistent when $N\to\infty$ and $T\to\infty$.

$$\begin{aligned}&\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{LS,i} - \hat{\boldsymbol{\theta}}_{LSMG}\right)\left(\hat{\boldsymbol{\theta}}_{LS,i} - \hat{\boldsymbol{\theta}}_{LSMG}\right)^{'} = \sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{LS,i} - E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)\right)\left(\hat{\boldsymbol{\theta}}_{LS,i} - E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)\right)^{'} \\ &- \sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{LSMG} - E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)\right)\left(\hat{\boldsymbol{\theta}}_{LSMG} - E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)\right)^{'} \\ &+ \sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{LSMG} - E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)\right)\left(\hat{\boldsymbol{\theta}}_{LSMG} - E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)\right)^{'} \\ &- \sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{LS,i} - E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)\right)\left(\hat{\boldsymbol{\theta}}_{LS,i} - E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)\right)^{'}.\end{aligned} \tag{19}$$

Taking expectation on equation (19), we have

$$E\left(\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{LS,i}-\hat{\boldsymbol{\theta}}_{LSMG}\right)\left(\hat{\boldsymbol{\theta}}_{LS,i}-\hat{\boldsymbol{\theta}}_{LSMG}\right)^{'}\right)=\sum_{i=1}^{N}Var\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)+\sum_{i=1}^{N}\hat{\boldsymbol{\theta}}_{LSMG}\hat{\boldsymbol{\theta}}_{LSMG}^{'}+$$
$$\sum_{i=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)^{'}-\sum_{i=1}^{N}\hat{\boldsymbol{\theta}}_{LSMG}\hat{\boldsymbol{\theta}}_{LS,i}^{'}+\sum_{i=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)\hat{\boldsymbol{\theta}}_{LS,i}^{'}-\sum_{i=1}^{N}\hat{\boldsymbol{\theta}}_{LSMG}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)^{'}-$$
$$\sum_{i=1}^{N}\hat{\boldsymbol{\theta}}_{LS,i}\hat{\boldsymbol{\theta}}_{LSMG}^{'}+\sum_{i=1}^{N}\hat{\boldsymbol{\theta}}_{LS,i}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)^{'}-\sum_{i=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)^{'}=$$
$$\sum_{i=1}^{N}Var\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)-NE\left(\hat{\boldsymbol{\theta}}_{LSMG}\hat{\boldsymbol{\theta}}_{LSMG}^{'}\right)+\sum_{i=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)^{'} \tag{20}$$

and

$$E\left(\hat{\boldsymbol{\theta}}_{LSMG}\hat{\boldsymbol{\theta}}_{LSMG}^{'}\right)=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\hat{\boldsymbol{\theta}}_{LS,i}^{'}\right). \tag{21}$$

And, we also have

$$E\left(\hat{\boldsymbol{\theta}}_{LSMG}\hat{\boldsymbol{\theta}}_{LSMG}^{'}\right)=\frac{1}{N^2}\left(\sum_{i=1}^{N}Var\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)+\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\hat{\boldsymbol{\theta}}_{LS,i}^{'}\right)\right). \tag{22}$$

Then,

$$\begin{aligned}&E\left(\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{LS,i}-\hat{\boldsymbol{\theta}}_{LSMG}\right)\left(\hat{\boldsymbol{\theta}}_{LS,i}-\hat{\boldsymbol{\theta}}_{LSMG}\right)^{'}\right)=\\&\left(1-\frac{1}{N}\right)\sum_{i=1}^{N}Var\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)+\sum_{i=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\hat{\boldsymbol{\theta}}_{LS,i}^{'}\right)-\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\hat{\boldsymbol{\theta}}_{LS,i}^{'}\right).\end{aligned} \tag{23}$$

From above equation (23), we can observe the bias term as,

$$\aleph=\sum_{i=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\hat{\boldsymbol{\theta}}_{LS,i}^{'}\right)-\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(\hat{\boldsymbol{\theta}}_{LS,i}\hat{\boldsymbol{\theta}}_{LS,i}^{'}\right). \tag{24}$$

Taking expectation on equation (26), we have

$$E\left(\hat{\boldsymbol{\theta}}_{LS,i}\right)=\boldsymbol{\theta}+E\left(b_i\right), \tag{25}$$

From equation (24), we know that

$$\aleph=\sum_{i=1}^{N}E\left(b_i\right)E\left(b_i^{'}\right)-\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(b_i\right)E\left(b_j^{'}\right). \tag{26}$$

When $T\rightarrow\infty$, $\aleph=0$. Therefore, we have following theorem

Theorem 2 *When* $(T, N) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$,

1.
$$\sqrt{N}\left(\hat{\boldsymbol{\theta}}_{LSMG} - \boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0}, \boldsymbol{\Sigma}_{LS,\lambda}\right). \tag{27}$$

2.
$$\hat{\boldsymbol{\Sigma}}_{LS,\lambda} \xrightarrow{p} \boldsymbol{\Sigma}_{LS,\lambda} \tag{28}$$

where

$$\hat{\boldsymbol{\Sigma}}_{LS,\lambda} = \frac{1}{N(N-1)} \sum_{i=1}^{N} \left(\hat{\boldsymbol{\theta}}_{LS,i} - \hat{\boldsymbol{\theta}}_{LSMG}\right)\left(\hat{\boldsymbol{\theta}}_{LS,i} - \hat{\boldsymbol{\theta}}_{LSMG}\right)^{'}. \tag{29}$$

1.2 Asymptotic property of IV estimator

We use current and lagged values of $\boldsymbol{x}_i$ as instruments, as

$$\tilde{\boldsymbol{Z}}_{i,t} = \left(\tilde{\boldsymbol{x}}_i, \tilde{\boldsymbol{x}}_{i,-1}\right)^{'}, \tag{30}$$

where $\tilde{\boldsymbol{Z}}_i$ is $T \times 2$ vector.

Assumption 2 $\boldsymbol{A}_i = \mathrm{plim}_{T\to\infty} \tilde{\boldsymbol{A}}_{i,T}$ *has full column rank,* $\boldsymbol{B}_i = \mathrm{plim}_{T\to\infty} \tilde{\boldsymbol{B}}_{i,T}$ *and* $\boldsymbol{\Sigma}_i = \mathrm{plim}_{T\to\infty} T^{-1}\boldsymbol{Z}_i^{'}\boldsymbol{u}_i\boldsymbol{u}_i^{'}\boldsymbol{Z}$ *has positive definite, uniformly.*

Then, the IV estimator can be expressed as

$$\hat{\boldsymbol{\theta}}_{IV,i} = \left(\tilde{\boldsymbol{A}}_{i,T}^{'}\tilde{\boldsymbol{B}}_{i,T}^{-1}\tilde{\boldsymbol{A}}_{i,T}\right)\tilde{\boldsymbol{A}}_{i,T}^{'}\tilde{\boldsymbol{B}}_{i,T}^{-1}\tilde{\boldsymbol{g}}_{i,T}, \tag{31}$$

where

$$\tilde{\boldsymbol{A}}_{i,T} = \frac{1}{T}\tilde{\boldsymbol{Z}}_i^{'}\tilde{\boldsymbol{W}}_i,\ \tilde{\boldsymbol{B}}_{i,T} = \frac{1}{T}\tilde{\boldsymbol{Z}}_i^{'}\tilde{\boldsymbol{Z}}_i,\ \tilde{\boldsymbol{g}}_{i,T} = \frac{1}{T}\tilde{\boldsymbol{Z}}_i^{'}\tilde{\boldsymbol{y}}_i, \tag{32}$$

and $\tilde{\boldsymbol{W}}_i = \left(\tilde{w}_{i,1}^{'}, \ldots, \tilde{w}_{i,T}^{'}\right)^{'}$ is $T \times 2$ matrix

From above equation, we have

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right) = \left(\tilde{\boldsymbol{A}}_{i,T}^{'}\tilde{\boldsymbol{B}}_{i,T}^{-1}\tilde{\boldsymbol{A}}_{i,T}\right)\tilde{\boldsymbol{A}}_{i,T}^{'}\tilde{\boldsymbol{B}}_{i,T}^{-1}\left(T^{-1/2}\tilde{\boldsymbol{Z}}_i^{'}\tilde{\boldsymbol{u}}_i\right) \tag{33}$$

Then, the property of $T^{-1/2}\tilde{\boldsymbol{Z}}_i^{'}\tilde{\boldsymbol{u}}_i$ is given by following proposition.

proposition 1 *Under above assumptions, as* $(N, T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$, *for each* i, *we have*

$$\begin{gathered} N^{-1}\tilde{\boldsymbol{Z}}_i^{'}\tilde{\boldsymbol{u}}_i \xrightarrow{p} \mathbf{0}, \\ \textit{and} \\ T^{-1/2}\tilde{\boldsymbol{Z}}_i^{'}\tilde{\boldsymbol{u}}_i \xrightarrow{d} N\left(\mathbf{0}, \boldsymbol{\Sigma}_i\right). \end{gathered} \tag{34}$$

Thus, IV estimator, $\boldsymbol{\theta}_{IV,i}$ is $\sqrt{T}$ consistent to $\boldsymbol{\theta}_i$ and this estimator does not have Nickell's bias. Then, we have following theorem

Theorem 1

As $(N,T) \to \infty$ such that $N/T \to c$ with $0 < c < \infty$. for each i,

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right) \xrightarrow{d} N\left(\mathbf{0},\ \left(\boldsymbol{A}_i'\boldsymbol{B}_i^{-1}\boldsymbol{A}_i\right)^{-1}\boldsymbol{A}_i'\boldsymbol{B}_i^{-1}\boldsymbol{\Sigma}_i\boldsymbol{B}_i^{-1}\boldsymbol{A}_i\left(\boldsymbol{A}_i'\boldsymbol{B}_i^{-1}\boldsymbol{A}_i\right)\right). \quad (35)$$

1.2.1 Mean group IV estimator

Now, we define the mean group estimator of $\boldsymbol{\theta}$:

$$\hat{\boldsymbol{\theta}}_{IVMG} = \frac{1}{N}\sum_{i=1}^{N}\hat{\boldsymbol{\theta}}_{IVi}. \quad (36)$$

And we can show that the asymptotic property of $\hat{\boldsymbol{\theta}}_{IVMG}$, as

$$\hat{\boldsymbol{\theta}}_{IVMG} - \boldsymbol{\theta} = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{IVMG} - \boldsymbol{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{IVMG} - \boldsymbol{\theta}_i\right) + \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\lambda}_i. \quad (37)$$

$$\sqrt{N}\left(\hat{\boldsymbol{\theta}}_{IVMG} - \boldsymbol{\theta}\right) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{\lambda}_i + o_p(1). \quad (38)$$

And the variance estimator of $\hat{\boldsymbol{\theta}}_{IVMG}$ is given by

$$\hat{\boldsymbol{\Sigma}}_{IV,\lambda} = \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{IV,i} - \hat{\boldsymbol{\theta}}_{IVMG}\right)\left(\hat{\boldsymbol{\theta}}_{IV,i} - \hat{\boldsymbol{\theta}}_{IVMG}\right)'. \quad (39)$$

Follow Norkute et al. (2019), we can show that $\hat{\boldsymbol{\Sigma}}_{IV,\lambda}$ is consistent and it does not have small T bias. Firstly, we decompose (39) as

$$\begin{aligned}
&\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta} + \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{IVMG}\right)\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta} + \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{IVMG}\right)' = \\
&\sum_{i=1}^{N}\boldsymbol{\lambda}_i\boldsymbol{\lambda}_i' + \sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right)\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right)' + \sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right)\boldsymbol{\lambda}_i + \sum_{i=1}^{N}\boldsymbol{\lambda}_i\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right) - \\
&N\left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{IVMG}\right)'\left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{IVMG}\right).
\end{aligned} \quad (40)$$

Then we can show consistent of $\hat{\boldsymbol{\Sigma}}_{IV,\lambda}$ as

$$\begin{aligned}
&\hat{\boldsymbol{\Sigma}}_{IV,\lambda} - \boldsymbol{\Sigma}_{IV,\lambda} = \frac{1}{N-1}\sum_{i=1}^{N}\left(\boldsymbol{\lambda}_i\boldsymbol{\lambda}_i' - \boldsymbol{\Sigma}_{IV,\lambda}\right) + \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right)\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right)' \\
&+ \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right)\boldsymbol{\lambda}_i + \frac{1}{N-1}\sum_{i=1}^{N}\boldsymbol{\lambda}_i\left(\hat{\boldsymbol{\theta}}_{IV,i} - \boldsymbol{\theta}_i\right) - \\
&\frac{N}{N-1}\left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{IVMG}\right)'\left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{IVMG}\right) = o_p(1).(\textit{The proof is similar as })
\end{aligned} \quad (41)$$

Then, we can see that the asymptotic property of $\hat{\boldsymbol{\theta}}_{IVMG}$ as,

$$\sqrt{N}\left(\hat{\boldsymbol{\theta}}_{IVMG}-\boldsymbol{\theta}\right)\xrightarrow{d}N\left(\mathbf{0},\boldsymbol{\Sigma}_{IV,\lambda}\right). \tag{42}$$

2 Estimation method on dynamic heterogeneous panel data model with multifactor error structure

For convenient, we assume the number of regressor is 1 and we express the model as

$$y_{i,t}=\phi_i y_{i,t-1}+\sum_{\ell=1}^{k}\beta_{\ell i}x_{\ell i,t}+u_{i,t},\ \ for\ i=1,\dots N;\ t=1,\dots,T,\ \ell=1,\dots,k. \tag{43}$$

Consider the model (40), we drawn $x_{\ell i,t}$ as

$$x_{\ell i,t}=\boldsymbol{\gamma}_{xi}^{0\prime}\boldsymbol{f}_{xt}^{0}+\varepsilon_{xi,t} \tag{44}$$

and the idiosyncratic errors of the process for $y_{i,t}$ as

$$u_{i,t}=\boldsymbol{\gamma}_{yi}^{0\prime}\boldsymbol{f}_{yt}^{0}+\varepsilon_{yi,t}, \tag{45}$$

where $\boldsymbol{\gamma}_{yi}^{0}$ and $\boldsymbol{\gamma}_{xi}^{0}$ are $m_y\times 1$ and $m_x\times 1$ true factor loading respectively, $\boldsymbol{f}_{yt}^{0}$ and $\boldsymbol{f}_{xt}^{0}$ are $m_y\times 1$ and $m_x\times 1$ true vector of unobservable factors respectively.

2.1 Norkutes' (2019) IVMG estimator

We asymptotically eliminate the common factor in $\boldsymbol{x}_i$ by projecting matrix, $\boldsymbol{M}_{F_x^0}$.

$$\boldsymbol{M}_{F_x^0}=\boldsymbol{I}_T-\boldsymbol{F}_x^0\left(\boldsymbol{F}_x^{0\prime}\boldsymbol{F}_x^0\right)^{-1}\boldsymbol{F}_x^{0\prime};\boldsymbol{M}_{F_{x,-1}^0}=\boldsymbol{I}_T-\boldsymbol{F}_{x,-1}^0\left(\boldsymbol{F}_{x,-1}^{0\prime}\boldsymbol{F}_{x,-1}^0\right)^{-1}\boldsymbol{F}_{x,-1}^{0\prime} \tag{46}$$

And using the defactored covariates as instruments, as

$$\boldsymbol{Z}_{IVi}=\left(\boldsymbol{M}_{F_X^0}\boldsymbol{x}_i,\boldsymbol{M}_{F_{x,-1}^0}\boldsymbol{X}_{i,-1}\right) \tag{47}$$

The first step IV estimator can be expressed as

$$\begin{aligned}\hat{\boldsymbol{\varphi}}_{IVi}=&\left(\left(\frac{\boldsymbol{Z}_i^{\prime}\boldsymbol{M}_{F_X^0}\boldsymbol{W}_i}{T}\right)^{\prime}\left(\frac{\boldsymbol{Z}_i^{\prime}\boldsymbol{M}_{F_X^0}\boldsymbol{Z}_i}{T}\right)^{-1}\left(\frac{\boldsymbol{Z}_i^{\prime}\boldsymbol{M}_{F_X^0}\boldsymbol{W}_i}{T}\right)\right)^{-1}\\&\left(\left(\frac{\boldsymbol{Z}_i^{\prime}\boldsymbol{M}_{F_X^0}\boldsymbol{W}_i}{T}\right)^{\prime}\left(\frac{\boldsymbol{Z}_i^{\prime}\boldsymbol{M}_{F_X^0}\boldsymbol{Z}_i}{T}\right)^{-1}\left(\frac{\boldsymbol{Z}_i^{\prime}\boldsymbol{M}_{F_X^0}\boldsymbol{y}_i}{T}\right)\right).\end{aligned} \tag{48}$$

3 Monte Carlo simulation design

3.1 dynamic heterogeneous panels data model without error factor structure

The data generating process:

$$\begin{aligned} y_{i,t} &= \phi_i y_{i,t-1} + \sum_{\ell=1}^{k} \beta_{\ell i} x_{\ell i,t} + u_{i,t}, \ \ for\ i = 1, \ldots N;\ t = -49, \ldots, T\,, \\ x_{\ell i,t} &= \sum_{\ell=1}^{k} \phi_{\ell i} x_{\ell i,t-1} + v_{\ell i,t}, \end{aligned} \tag{49}$$

where $u_{i,t} \overset{i.i.d.}{\sim} \mathcal{N}(0,\,1)$, and $v_{\ell i,t} = \rho_{v,\ell} v_{\ell i,t-1} + \left(1-\rho_{v,\ell}^2\right)^{\frac{1}{2}} \varpi_{\ell i,t}, \varpi_{\ell i,t} \overset{i.i.d.}{\sim} U(0.5, 1.5)\,, \rho_{v,\ell} = 0.5$.

The slope coefficients are generated as

$$\phi_i = \phi + \eta_{\phi i},\ \ \beta_{1,i} = \beta_1 + \eta_{\beta_1 i}\, and\, \beta_{2,i} = \beta_2 + \eta_{\beta_2 i}. \tag{50}$$

Here we consider $\phi \in \{0.5\}$, $\beta_1 = 3$ and $\beta_2 = 1$. For the design of heterogenous slopes, $\eta_{\phi i} \overset{i.i.d.}{\sim} U\left(-c, c\right)$, and

$$\eta_{\beta_\ell i} = \left(1-\rho_\beta^2\right)^{1/2} \eta_{\phi i}. \tag{51}$$

Here, we set $c = 0.2$, $\rho_\beta = 0.4$ for $\ell = 1, 2$.

3.2 Dynamic heterogeneous panels data model with multi-factor error structure

This Monte Carlo simulation design same as Norkute et al. (2019). For convenience, we rewrite the data generating process as bellow

$$y_{i,t} = \alpha_i + \phi_i y_{i,t-1} + \sum_{\ell=1}^{k} \beta_{\ell i} x_{\ell i,t} + u_{i,t},\ \ for\ i = 1, \ldots N;\ t = -49, \ldots, T\,. \tag{52}$$

$$\tag{53}$$

We allow error factor structure in the model as

$$u_{i,t} = \sum_{s=1}^{m_y} \gamma_{si}^0 f_{s,t}^0 + \varepsilon_{i,t}, \tag{54}$$

where

$$f_{s,t}^0 = \rho_{s,t}^0 f_{s,t-1}^0 + \left(1-\rho_{fs}^2\right)^{1/2} \zeta_{s,t}, \tag{55}$$

with $\zeta_{s,t} \overset{i.i.d.}{\sim} N(0,1)$ for $s = 1, \ldots m_y$. We assume $k = 2$ and $m_y = 1 + k = 3$ and set $\rho_{s,t}^0 = 0.5$ for all s. The error term, $\varepsilon_{i,t}$, setting as

$$\varepsilon_{i,t} = \varsigma_\varepsilon \sigma_{it} \left(\epsilon_{it} - 1\right) / \sqrt{2}, \tag{56}$$

where $\epsilon_{it} \overset{i.i.d.}{\sim} \chi^2_1$, $\sigma^2_{it} = \eta_i \varphi_t$, $\eta_i \overset{i.i.d.}{\sim} \chi^2_2/2$, and $\varphi_t = t/T \; for \; t = 0, \ldots, T$. And we set

$$\varsigma_\varepsilon = \frac{\pi_\mu}{1-\pi_\mu} m_y. \tag{57}$$

we set $\pi_\mu \in \{3/4\}$.

The process of regressors is

$$x_{\ell it} = \mu_{\ell i} + + \sum_{\ell=1}^{k} \phi_{\ell i} x_{\ell i,t-1} + \sum_{s=1}^{m_x} \gamma^0_{\ell si} f^0_{s,t} + v_{\ell it}, \; for \; i = 1, \ldots N; \, t = -49, \ldots, T \, ; \, \ell = 1, 2. \tag{58}$$

We set number of factor, m_x, is 2. Therefore, $\boldsymbol{f}^0_{y,t} = \left(f^0_{1t}, f^0_{2t}, f^0_{3t}\right)'$ and $\boldsymbol{f}^0_{x,t} = \left(f^0_{1t}, f^0_{2t}\right)'$. We set

$$v_{\ell i,t} = \rho_{v,\ell} v_{\ell i,t-1} + \left(1 - \rho^2_{v,\ell}\right)^{\frac{1}{2}} \varpi_{\ell i,t}, for \; \ell = 1, 2, \tag{59}$$

where $\rho_{v,\ell} = 0.5$ for all ℓ. The individual effect is

$$\alpha^*_i \overset{i.i.d.}{\sim} N\left(0, (1-\rho_i)^2\right), \; \mu^*_{\ell i} = \rho_{\mu,\ell} \alpha^*_i + \left(1 - \rho^2_{\mu,\ell}\right)^{1/2} \omega_{\ell i}, \tag{60}$$

where $\omega \overset{i.i.d.}{\sim} N\left(0, (1-\rho_i)^2\right)$ and $\rho_{\mu,\ell} = 0.5$.

Now, we define the factor loading in $u_{i,t}$ are generated as $\gamma^{0*}_{si} \overset{i.i.d.}{\sim} N(0,1)$, for $s = 1, \ldots, m_y = 3$, and the factor loading in x_{1it} and x_{2it} are drawn as

$$\begin{aligned} \gamma^{0*}_{1si} &= \rho_{\gamma,1s} \gamma^{0*}_{3i} + \left(1 - \rho^2_{\gamma,1s}\right)^{1/2} \xi_{1si}; \; \xi_{1si} \overset{i.i.d.}{\sim} N(0,1); \\ \gamma^{0*}_{2si} &= \rho_{\gamma,2s} \gamma^{0*}_{si} + \left(1 - \rho^2_{\gamma,2s}\right)^{1/2} \xi_{2si}; \; \xi_{2si} \overset{i.i.d.}{\sim} N(0,1); \end{aligned} \tag{61}$$

for $s = 1, \ldots, m_x = 2$. We set $\rho_{\gamma,11} = \rho_{\gamma,12} \in \{0.5\}$ and $\rho_{\gamma,21} = \rho_{\gamma,22} = 0.5$. The factor loading are generated as

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^0 + \boldsymbol{\Gamma}^{0*}_i \tag{62}$$

where

$$\boldsymbol{\Gamma}^0_i = \begin{bmatrix} \gamma^0_{1i} & \gamma^0_{11i} & \gamma^0_{21i} \\ \gamma^0_{2i} & \gamma^0_{12i} & \gamma^0_{22i} \\ \gamma^0_{3i} & 0 & 0 \end{bmatrix} \tag{63}$$

and

$$\boldsymbol{\Gamma}^{0*}_i = \begin{bmatrix} \gamma^{0*}_{1i} & \gamma^{0*}_{11i} & \gamma^{0*}_{21i} \\ \gamma^{0*}_{2i} & \gamma^{0*}_{12i} & \gamma^{0*}_{22i} \\ \gamma^{0*}_{3i} & 0 & 0 \end{bmatrix}. \tag{64}$$

We set

$$\boldsymbol{\Gamma}^0 = \begin{bmatrix} 1/4 & 1/4 & -1 \\ 1/2 & -1 & 1/4 \\ 1/2 & 0 & 0 \end{bmatrix}. \tag{65}$$

And

$$\alpha_i = \alpha + \alpha_i^*,\ \mu_{\ell i} = \mu_\ell + \mu_{\ell i}^*, \tag{66}$$

where $\alpha = 1/2$, $\mu_1 = 1$, $\mu_2 = -1/2$.

The slope coefficients are generated as

$$\phi_i = \phi + \eta_{\phi i},\ \ \beta_{1,i} = \beta_1 + \eta_{\beta_1 i}\, and\, \beta_{2,i} = \beta_2 + \eta_{\beta_2 i}. \tag{67}$$

Here we consider $\phi \in \{0.5\}$, $\beta_1 = 3$ and $\beta_2 = 1$. For the design of heterogenous slopes, $\eta_{\phi i} \overset{i.i.d.}{\sim} U(-c, c)$, and

$$\eta_{\beta_\ell i} = \left[(2c)^2/12\right] \rho_\beta \xi_{\beta\ell i} + \left(1 - \rho_\beta^2\right)^{1/2} \eta_{\phi i}, \tag{68}$$

where

$$\xi_{\beta\ell i} = \frac{\bar{v^2_{\ell i}} - \bar{v^2_{\ell}}}{\left[N^{-1} \sum_{i=1}^N \left(\bar{v^2_{\ell i}} - \bar{v^2_{\ell}}\right)^2\right]^{1/2}}, \tag{69}$$

with $\bar{v^2_{elli}} = T^{-1} \sum_{t=1}^T v^2_{\ell it}$, $\bar{v^2_{\ell}} = N^{-1} \sum_{i=1}^N \bar{v^2_{\ell i}}$, for $\ell = 1, 2$. Here, we set $c = 0.2$, $\rho_\beta = 0.4$ for $\ell = 1, 2$. And

$$\varsigma_v^2 = \varsigma_\varepsilon^2 \left[SNR - \frac{\rho_v^2}{1 - \rho_v^2}\right] \left(\frac{\beta_1^2 + \beta_2^2}{1 - \rho_v^2}\right)^{-1}, \tag{70}$$

where $SNR = 4$. For the (T, N), we consider $T \in \{25, 50, 100, 200\}$ and $N \in \{25, 50, 100, 200\}$.

4 Monte Carlo simulation results

4.1 Dynamic Heterogeneous Panels without multifactor error structure

We consider ARDL(1,0) model.
$\phi \in \{0.5\}$.
$\beta_1 = 3$.
$\beta_2 = 1$.
$u_{i,t} \overset{i.i.d.}{\sim} \mathcal{N}(0,\ 1)$.
$\varpi_{\ell i,t} \overset{i.i.d.}{\sim} U(0.5, 1.5)$.
$\rho_{v,\ell} = 0.5$.
$c = 0.2$.
$\rho_\beta = 0.4$.
$T \in \{25, 50, 100, 200\}$.
$N \in \{25, 50, 100, 200\}$.

LSMG estimator is provided in sheet 1 of MC.xlsx file.
IVMG estimator is provided in sheet 2 of MC.xlsx file.

4.2 Dynamic Heterogeneous Panels with multifactor error structure

We consider ARDL(1,0) model.
$\phi \in \{0.5\}$.
$\beta_1 = 3$.
$\beta_2 = 1$.
$k = 2$.
$m_y = 1 + k = 3$.
$m_x = k = 2$.
$\zeta_{s,t} \overset{i.i.d.}{\sim} N(0, 1)$
$\pi_\mu \in \{3/4\}$.
$\rho^0_{s,t} = 0.5$.
$\rho_{v,\ell} = 0.5$.
$\rho_{\mu,\ell} = 0.5$.
$\gamma^{0*}_{si} \overset{i.i.d.}{\sim} N(0, 1)$.
$\xi_{1si} \overset{i.i.d.}{\sim} N(0, 1)$.
$\xi_{2si} \overset{i.i.d.}{\sim} N(0, 1)$.
$\rho_{\gamma,11} = \rho_{\gamma,12} \in \{0.5\}$.
$\rho_{\gamma,21} = \rho_{\gamma,22} = 0.5$.
$\mathbf{\Gamma}^0 = \begin{bmatrix} 1/4 & 1/4 & -1 \\ 1/2 & -1 & 1/4 \\ 1/2 & 0 & 0 \end{bmatrix}$.
$\alpha = 1/2$.
$\mu_1 = 1$.
$\mu_2 = -1/2$.
$c = 0.2$.
$\rho_\beta = 0.4$.
$SNR = 4$.
$T \in \{25, 50, 100, 200\}$.
$N \in \{25, 50, 100, 200\}$.

IVMG estimator is provided in sheet 3 of MC.xlsx file.

5 Short summary

5.1 Dynamic Heterogeneous Panels without multifactor error structure

1. The performance of IVMG estimator is better than LSMG estimator in bias and RMSE.

5.2 Dynamic Heterogeneous Panels with multifactor error structure

1. When N and T increase, the performance of IVMG estimator is good in bias and RMSE.
Related literature to dynamic Heterogeneous Panels with multifactor error structure: Chudik and Pesaran (2015) and Norkute et al. (2019).
Related literature to choosing number of instruments: Donald and Newey (2001), Swanson (2005), Carrasco (2012), Bai and Ng (2010) and Kang (2019).

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