Full Waveform Inversion

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1 Seismic Imaging

In this article, we are in the particular setting of seismic imaging where we hope to detect mineral or oil resevoirs underground using seismic wave data. Mathematically, we assume the seismic waves propagate according to the acoustic wave equation [5]:

$$\frac{1}{\kappa(\boldsymbol{x})}\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) - \nabla\cdot\left(\frac{1}{\rho(\boldsymbol{x})}\nabla u(\boldsymbol{x},t)\right) = f(\boldsymbol{x},t) \quad \text{for } (\boldsymbol{x},t)\in\mathbb{R}^d\times(0,T]$$
$$u(\boldsymbol{x},0) = \frac{\partial u}{\partial t}(\boldsymbol{x},0) = 0 \quad \text{for } \boldsymbol{x}\in\mathbb{R}^d.$$

In practice, $d = 2$ or 3, but to reduce the computational requirement, we will consider solely the two-dimensional case. The solution of the equation, $u(\boldsymbol{x},t)$, is the pressure field, or the *incident wavefield*. We will denote the domain of interest as Ω; this is a finite, inhomogeneous medium that contains certain anomaly (see Figure 1).

The wavefield is governed by three parameters: the density ρ, the source function f, and the bulk modulus $\kappa = \rho v^2$ where v is known as the *P-wave velocity* in seismology. To keep things simple, we will assume the density is constant, and the source function is known, so that the only unknown parameter is the velocity v. Full waveform inversion then refers to the inverse problem of reconstructing the true v from the given seismic data.

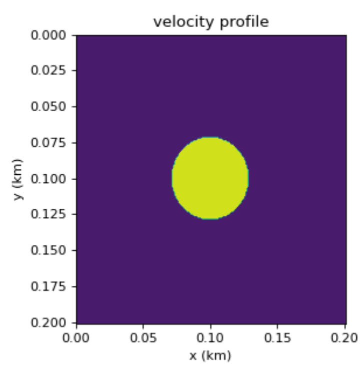

Figure 1: The velocity profile in this medium is homogeneous except for an anomaly at the center.

When waves travel in an inhomogeneous medium, they may be delayed, reflected, and refracted [3], and the wave data encodes information about the medium—this is what makes full waveform inversion possible. The setup to acquire data is known as the *acquisition geometry* [5] (see Figure 2). In practice, the sources are pulse signals generated by *e.g.* artificial explosions [3].

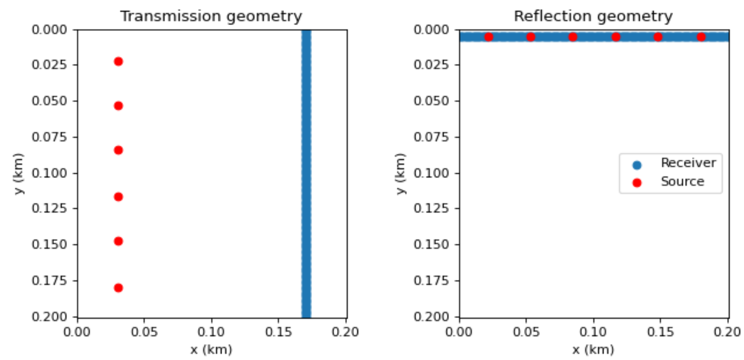

Figure 2: Two types of acquisiton geometries commonly used in seismic imaging.

2 Simulation of Wave Propagation

For our numerical study, we need to simulate wave propagation in an infinite domain to match reality. Clearly, this is impossible to be done numerically. So instead, we will simulate wave propagation and impose an *absorbing boundary condition* (ABC). In the words of Steven Johnson, ABC is a technique to simulate infinite domain in a finite computational cell. An ABC is not just forbidding waves from the reflecting at the computational boundary; a nowadays popular ABC seeks to devise a conceptual layer, in which the waves are attenuated intensely. This is known as the *perfectly matched layers* (PML). Consider the following diagram:

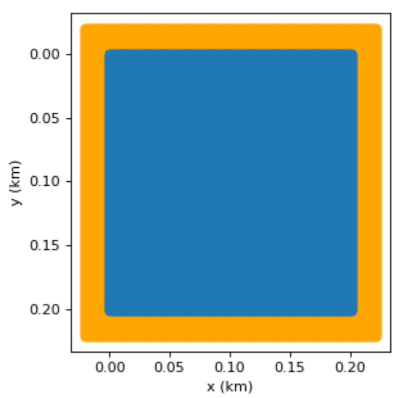

Figure 3: The blue region is Ω, and the orange region is the perfectly matched layer. We will denote the blue region plus orange region as Ω^+.

When wave travels to the interface between the blue and orange regions, some wave will be reflected [4]. Also, at the outer boundary of the orange layer, we still need to impose a boundary condition, *e.g.* Dirichlet, so some waves will be reflected. But if the PML is implemented properly, the reflected waves can be made so small so that they become negligible.

2.1 2D Wave Equation with PML

A good source of reference for understanding the PML is the note by Steven Johnson [4], and in what follows, I have consulted the note. Here is the idea of the PML: Consider the equation of interest

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f. \tag{1}$$

Suppose $u(x, y, t)$ solves this equation, then let $\hat{u}(k_1, k_2, \omega)$ be its Fourier transform in space-time[1], *i.e.*

$$u(x, y, t) = \int_{\mathbb{R}^2}\int_{\mathbb{R}} \hat{u}(k_1, k_2, \omega) e^{-i(k_1 x + k_2 y - \omega t)} d\omega dk_1 dk_2.$$

If we perform the change-of-variables

$$x \mapsto x + \frac{\varphi_1(x)}{i\omega}, \qquad y \mapsto y + \frac{\varphi_2(y)}{i\omega}, \tag{2}$$

then in effect, we have

$$\begin{aligned} u(x, y, t) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}} e^{-(k_1\varphi_1 + k_2\varphi_2)/\omega} \hat{u}(k_1, k_2, \omega) e^{-i(k_1 x + k_2 y - \omega t)}\, d\omega\, dk_1\, dk_2 \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}} e^{-\left(\frac{\varphi_1}{c_1} + \frac{\varphi_2}{c_2}\right)} \hat{u}(k_1, k_2, \omega) e^{-i(k_1 x + k_2 y - \omega t)}\, d\omega\, dk_1\, dk_2 \end{aligned} \tag{3}$$

where c_1, c_2 are the phase velocities in the $+x, +y$ directions, respectively[2]. By choosing appropriate[3] φ_1, φ_2, we can have the waves be attenuated when they come close to the computational boundary.

In practice, though, instead of performing the direct coordinate transformation, PML is better implemented by the auxiliary differential equations (ADE) approach. Basically, this means the original wave equation, plus certain extra terms, describes the same wave propagation as if the coordinates are changed as previously described:

1. Solve for u using

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \Delta u - \nabla \cdot (\boldsymbol{\sigma} \odot \boldsymbol{q}) - \frac{1}{v^2}(\sigma_1 + \sigma_2)\frac{\partial u}{\partial t} + \sigma_2 \frac{\partial q_1}{\partial x} + \sigma_1 \frac{\partial q_2}{\partial y} - \frac{1}{v^2}\sigma_1\sigma_2 u + f$$

2. Solve for $\boldsymbol{q}$ using

$$\frac{\partial \boldsymbol{q}}{\partial t} = \nabla u - \boldsymbol{\sigma} \odot \boldsymbol{q}.$$

2.2 Discretization

The system above admits the following semi-discretization:

$$\frac{1}{v^2}\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} = \Delta u^n - \nabla \cdot (\boldsymbol{\sigma} \odot \boldsymbol{q}^n) - \frac{1}{v^2}(\sigma_1 + \sigma_2)\frac{u^{n+1} - u^{n-1}}{2\Delta t} + \sigma_2 \frac{\partial q_1^n}{\partial x} + \sigma_1 \frac{\partial q_2^n}{\partial y} - \frac{1}{v^2}\sigma_1\sigma_2 u^n + f^n.$$

$$\frac{q_1^{n+1} - q_1^n}{\Delta t} = \frac{\partial}{\partial x}\left(\frac{u^n + u^{n+1}}{2}\right) - \left(\frac{q_1^n + q_1^{n+1}}{2}\right)\sigma_1$$

$$\frac{q_2^{n+1} - q_2^n}{\Delta t} = \frac{\partial}{\partial y}\left(\frac{u^n + u^{n+1}}{2}\right) - \left(\frac{q_2^n + q_2^{n+1}}{2}\right)\sigma_2.$$

[1] Apparently, according to the theory of special relativity, the signs for t and x, y needs to be the opposite, otherwise time is going backward.

[2] This explains why we had the convention of keeping an ω in the denominator.

[3] Clearly, the way to do this is to ensure that away from the boundary $\varphi_1 = \varphi_2 = 0$. How to choose φ_1, φ_2 near the boundary is a question that will be addressed later.

Rearranging and we have

$$
\begin{aligned}
u^{n+1} &= \Bigg[\left(\frac{\Delta t}{2}(\sigma_1+\sigma_2)-1\right) u^{n-1} + \left(2-(\Delta t)^2\sigma_1\sigma_2\right) u^n \\
&\qquad + (\Delta t)^2 v^2 \left(\Delta u^n - \nabla\cdot(\boldsymbol{\sigma}\odot\boldsymbol{q}^n) + \sigma_2 \frac{\partial q_1^n}{\partial x} + \sigma_1 \frac{\partial q_2^n}{\partial y} + f^n\right)\Bigg] \Big/ \left(\frac{\Delta t}{2}(\sigma_1+\sigma_2)+1\right) \\
q_1^{n+1} &= \left[\Delta t \frac{\partial}{\partial x}\left(\frac{u^n+u^{n+1}}{2}\right) + \left(1-\frac{\Delta t}{2}\sigma_1\right) q_1^n\right] \Big/ \left(1+\frac{\Delta t}{2}\sigma_1\right) \\
q_2^{n+1} &= \left[\Delta t \frac{\partial}{\partial y}\left(\frac{u^n+u^{n+1}}{2}\right) + \left(1-\frac{\Delta t}{2}\sigma_2\right) q_2^n\right] \Big/ \left(1+\frac{\Delta t}{2}\sigma_2\right).
\end{aligned}
\tag{4}
$$

2.3 Attenuation Coefficients

Now we explain how to select appropriate σ_1 and σ. Let us concentrate on the $+x$ direction for a second. In the formula (3), the attenuation is provided by the multiplicative factor $\exp(-\varphi_1/c_1)$. Let R be the value of the factor when the wave is at the computational boundary, *i.e.*

$$R = \exp\left(-\frac{\varphi_1(x_{\max})}{c_1}\right).$$

Typically, $10^{-5} \le R \le 10^{-3}$ [5]. Then by how we defined σ_1 in (**??**), we get

$$\int_{x_0}^{x_{\max}} \sigma_1(x)\,dx = \varphi_1(x_{\max}) - \varphi_1(x_0) = -c_1 \ln R$$

where x_0 is the point where we turn on the attentuation. Usually, σ_1 is taken to be a polynomial, *e.g.*

$$\sigma_1(x) = \sigma_{\max}(x-x_0)^m$$

where m is the order of the polynomial, and $\sigma_{\max}$ is the coefficient necessary to achieve the desired attenuation. To determine $\sigma_{\max}$, just know

$$\sigma_{\max}\int_{x_0}^{x_{\max}} (x-x_0)^m\,dx = -c_1 \ln R$$

implies

$$\sigma_{\max} = -\frac{m+1}{(x_{\max}-x_0)^{m+1}} c_1 \ln R.$$

2.4 Source Function

For seismic imaging, it is suggested that we use the Ricker wavelet as the source function [3]. The definition that I use is

$$h(t) = \left(1-2\pi^2 f_M^2 t^2\right) e^{-\pi^2 f_M^2 t^2}.$$

Here f_M is the frequency at which $\hat{h}$ attains its maximum. Sometimes if the total simulation time is short, h is shifted appropriately. The source functions are then pulse signals of the form

$$f_s(x,y,t) = h(t)\delta(x-x_s)\delta(y-y_s).$$

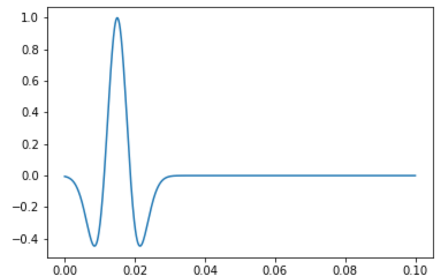

Figure 4: Ricker wavelet with $f_M = 60$ Hz.

2.5 Stability and Numerical Dispersion

In 2D, the CFL condition for the wave equation [5] is

$$\frac{\Delta t}{\max\{\Delta x, \Delta y\}} \sup_{(x,y)\in\Omega} v(x,y) \leq \frac{1}{\sqrt{2}}.$$

However, because of (2), it is necessary to pick a slightly smaller Δt.

In addition, to avoid grid dispersion, we need to ensure

$$\max\{\Delta x, \Delta y\} \leq \frac{\inf_{(x,y)\in\Omega} v(x,y)}{m f_M}$$

where m is the minimum number of grid points per wavelength, and f_M is as defined previously.

2.6 Numerical Experiment

Parameters for the experiments:

- Velocity profile: $c = 3$ km/s
- Computational domain: $[0, 0.2] \times [0, 0.2]$ km^2
- Source function: Ricker wavelet with 60 Hz at the center of the domain

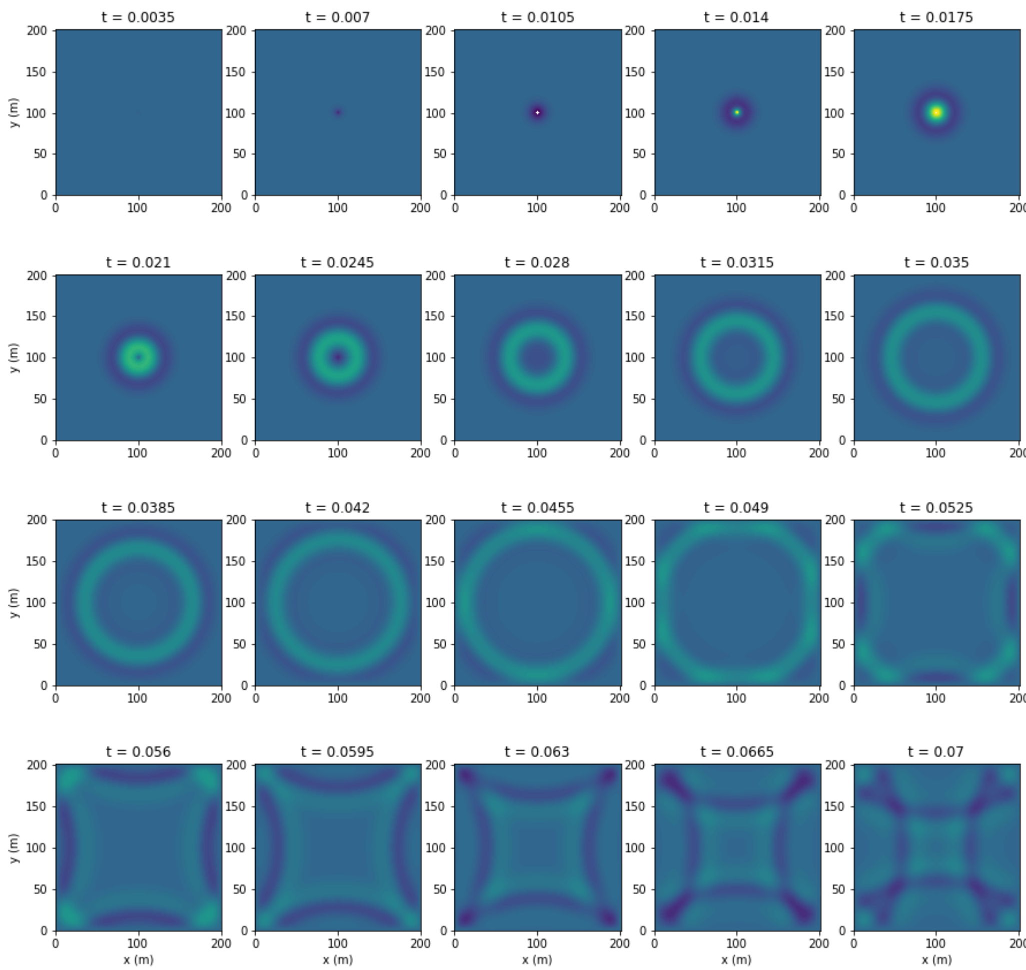

Figure 5: Snapshots of the solution of 2D wave equation with Dirichlet boundary condition. This is the most ordinary simulation of wave propagation; no PML is involved. The effect of imposing the Dirichlet boundary condition is the reflection from boundary.

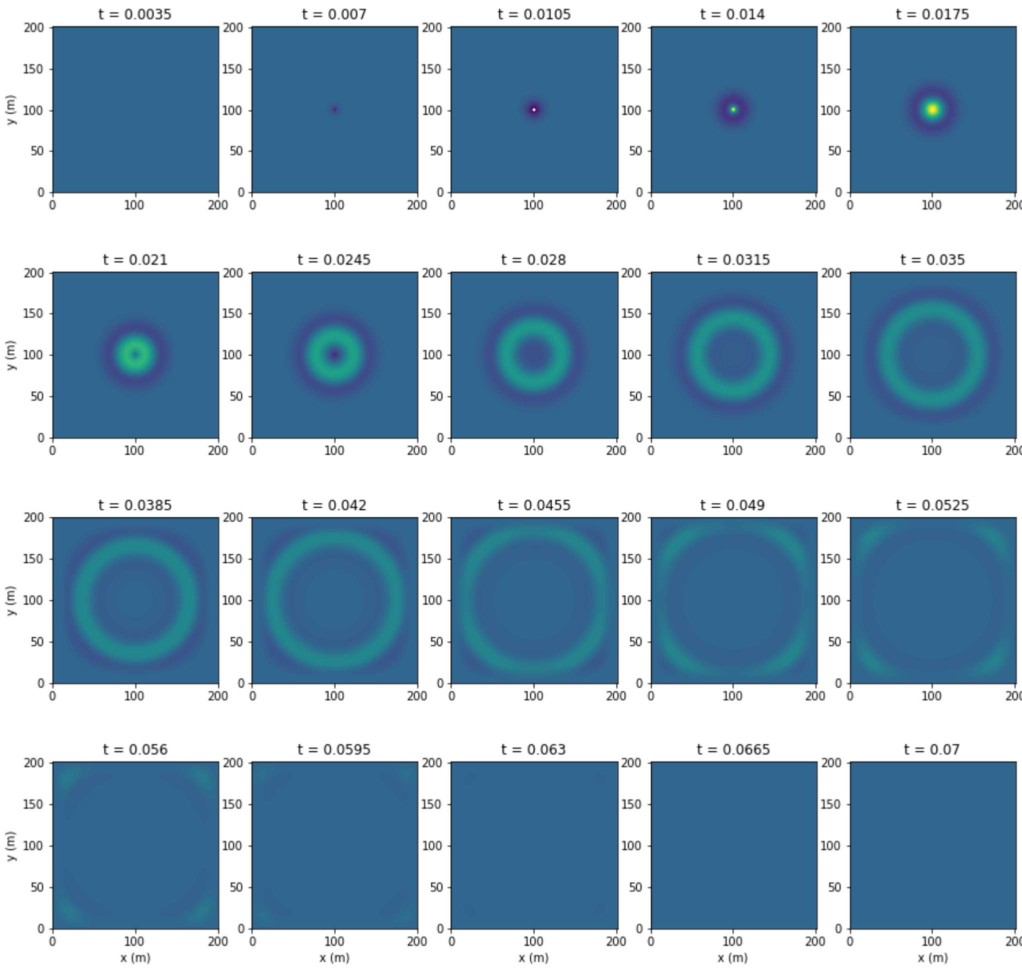

Figure 6: Solutions of the auxiliary differential equations derived for 2D wave equation with PML. The width of the PML is equal to 30 grid points, which means the area without PML is $[x_{\min} + 30 * \Delta x, x_{\max} - 30 * \Delta x] \times [y_{\min} + 30 * \Delta y, y_{\max} - 30 * \Delta y]$

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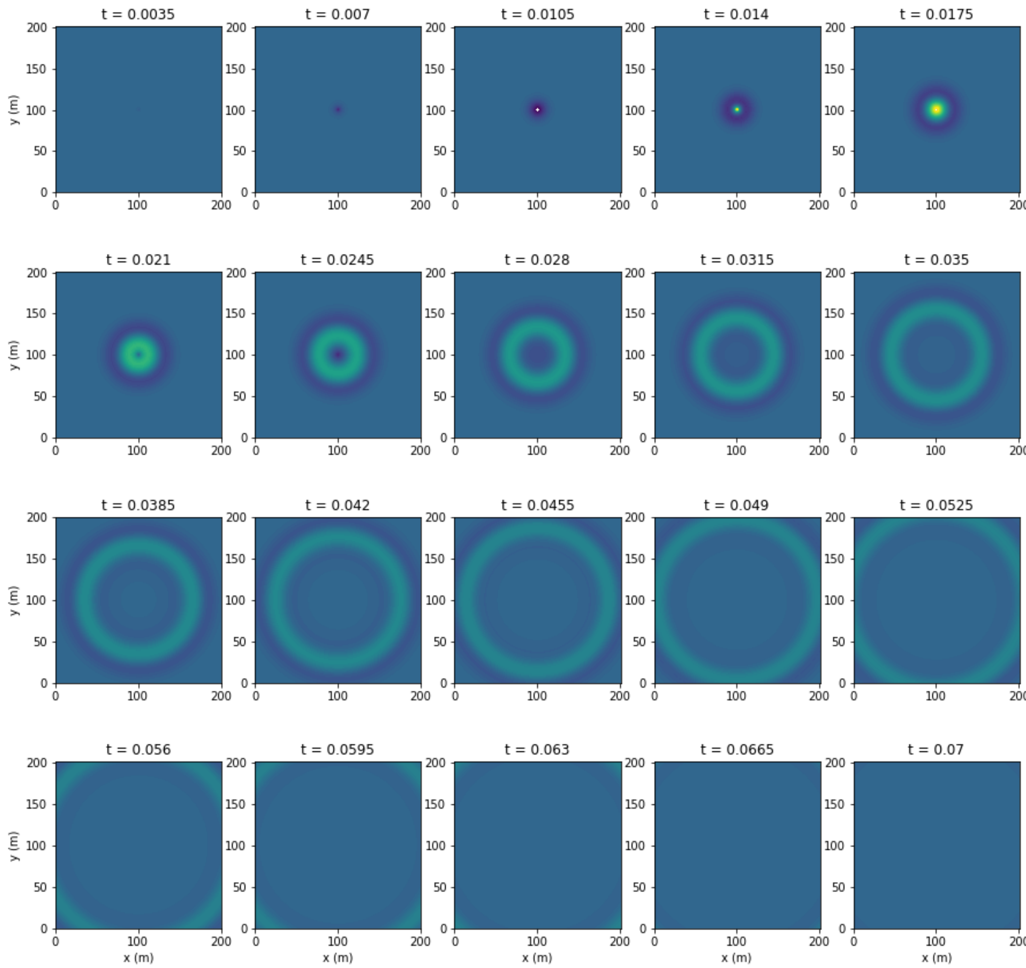

Figure 7: These are snapshots of waves propagating in actual free space. They are obtained by using a larger computational domain $[-0.2, 0.4] \times [-0.2, 0.4]$, and cropping out the part $[0, 0.2] \times [0, 0.2]$.

References

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