Assignment 6

Due Friday 10/25/19

Reading:

- Required: Course Notes 3.6-3.7,5.1-5.2,6.3.2
- Required: LADL 6.A-6.B

Problems:

1. (LA: 8.1.2) (5 pts) Let V be a vector space over F. Show that the sum of two inner products on V is an inner product on V. Is the difference of two inner products an inner product? Show that a positive multiple of an inner product is an inner product.

2. (LA: 8.1.9) (5 pts) Let V be a real or complex vector space with an inner product. Show that the quadratic form determined by the inner product satisfies the *parallelogram law*
$$\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2.$$

3. (EF: 3.6.3) (5 pts each) Assume $\underline{v}_1 = (1,0,1)$, $\underline{v}_2 = (0,1,1)$, and $\underline{v} = (1,3,3)$ forms a basis for the standard inner product space $V = \mathbb{R}^3$.
   (a) Apply the Gram-Schmidt process to get an orthonormal basis $\mathcal{B} = (\underline{u}_1, \underline{u}_2, \underline{u}_3)$ for V. Note: One can either normalize after each projection or normalize all vectors at the end.
   (b) For the vector $\underline{v} = (1,1,2)$, compute the coordinate vector $[\underline{v}]_{\mathcal{B}}$.

4. (EF: 3.6.2) (5 pts each) Let V be the vector space of real polynomials on $[-1,1]$ with inner product
$$\langle f|h\rangle = \int_{-1}^{1} f(t)h(t)dt.$$
   Since polynomials have finite degree by definition, the ordered list $\mathcal{B} = (1, x, x^2, \ldots)$ forms a Hamel basis for V.
   (a) Let $f, h \in V$ have the unique representations $f(t) = \sum_{j=0}^{\infty} f_j t^j$ and $h(t) = \sum_{i=0}^{\infty} h_i t^i$. Find an expression for g_{ij} such that
$$\langle f|h\rangle = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} h_i g_{ij} f_j.$$
   (b) Apply the Gram-Schmidt process to the vector sequence $\underline{v}_i = t^{i-1}$ for $i \in \{1,2,3,4\}$. Note: This gives the first four unnormalized Legendre polynomials.
   (c) Project $f(t) = \max\{0, t\}$ onto the four orthogonal Gram-Schmidt vectors from part (b).

(d) Explain why the following inequality is true.

$$\left(\int_{-1}^{1} f(t)h(t)dt\right)^2 \leq \left(\int_{-1}^{1} |f(t)|^2 dt\right)\left(\int_{-1}^{1} |g(t)|^2 dt\right)$$

5. (LA: 8.2.13) (5 pts) Let S be a subset of an inner product space V. Show that $(S^{\perp})^{\perp}$ contains the subspace spanned by S. When V is finite-dimensional, show that $(S^{\perp})^{\perp}$ equals the subspace spanned by S.

6. (EF: 3.7.4) (5 pts each) Let $V = \mathbb{R}^n$ be the standard inner-product space and let $f\colon V \to \mathbb{R}$ be a differentiable real functional. The function $\nabla f(\underline{v}) \triangleq \left(\frac{\partial f}{\partial x_1}(\underline{v}), \ldots, \frac{\partial f}{\partial x_n}(\underline{v})\right)$ maps V to V and is called the gradient of f at $\underline{v} \in V$. The gradient defines a 1st-order approximation $f(\underline{v}+\underline{u}) \approx f(\underline{v}) + \langle \underline{u} | \nabla f(\underline{v})\rangle$ that is accurate for small $\underline{u}$ because

$$\lim_{\underline{u}\to\underline{0}} \frac{|f(\underline{v}+\underline{u}) - f(\underline{v}) - \langle \underline{u}|\nabla f(\underline{v})\rangle|}{\|\underline{u}\|} = 0,$$

where the norm is induced by the inner product.

(a) Find the unit-norm vector $\underline{u}$ that minimizes the value of the linear approximation $f(\underline{v}) + \langle \underline{u}|\nabla f(\underline{v})\rangle$? What does this imply about the direction of the vectors $\underline{u}$ and $\nabla f(\underline{v})$?

(b) Under what condition is there a step-size $\delta > 0$ such that $f(\underline{v}+\delta\underline{u}) < f(\underline{v})$? Prove it.

(c) Consider an algorithm for minimizing f that constructs the sequence

$$\underline{v}_{i+1} = \underline{v}_i - \delta_i \nabla f(\underline{v}_i),$$

where $\delta_i = \arg\min_{\delta\geq 0} f(\underline{v}_i - \delta\nabla f(\underline{v}_i))$ is the step size that minimizes $f(\underline{v}_{i+1})$. Show that $f(\underline{v}_{i+1}) \leq f(\underline{v}_i)$ with equality iff $\nabla f(\underline{v}_i) = \underline{0}$.

(d) Assume $f(\underline{v}) \geq M$ for all $\underline{v} \in V$ and that ∇f satisfies $\|\nabla f(\underline{u}) - \nabla f(\underline{v})\| \leq L\|\underline{u} - \underline{v}\|$ for all $\underline{u}, \underline{v} \in V$ (i.e., f has a Lipschitz gradient). For the update in (c) with $\delta_i = \frac{1}{L}$, show that $\nabla f(\underline{v}_i) \to \underline{0}$. Hint: First show $f(\underline{v}_{i+1}) \leq f(\underline{v}_i) + \nabla f(\underline{v}_i)\cdot(\underline{v}_{i+1} - \underline{v}_i) + \frac{L}{2}\|\underline{v}_{i+1} - \underline{v}_i\|^2$.

Practice Problems (do not hand in):

1. (MMA: 4.6.34) Show that if the linear operator $A : X \to Y$ has an inverse, then the inverse is linear.

2. (MMA: 4.6.35) Show that if A has both a left and a right inverse, they must be the same.

3. (LA: 8.2.10) Let V be the vector space of all $n \times n$ matrices over C, with the inner product $(A|B) = \mathrm{tr}(AB^H)$. Find the orthogonal complement of the subspace of diagonal matrices.

4. (LA: 8.2.9) Let V be the vector space of real polynomials with degree at most 3. Equip V with the inner product

$$\langle f|g\rangle = \int_0^1 f(t)g(t)dt.$$

(a) Find the orthogonal complement of the subspace of constant polynomials.

(b) Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$.

5. (LADR: 6.A.6) Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in \mathbf{F}$.

6. (MMA: 2.3.37) Let p be in the range $0 < p < 1$, and consider the space $L_p[0,1]$ of all functions with

$$\|x\| = \left[\int_0^1 |x(t)|^p dt\right]^{1/p} < \infty.$$

Show that $\|x\|$ is not a norm on $L_p[0,1]$. Hint: for a real number α such that $0 \leq \alpha \leq 1$, note that $\alpha \leq \alpha^p \leq 1$.