Practice Midterm 2 Solutions

October 23, 2019

Problems:

1. True or False:

   (a) **2.5 pt** – An orthogonal set of non-zero vectors must be linearly independent.
   Solution: True

   (b) **2.5 pt** – A subset U of a vector space V is automatically a subspace of V.
   Solution: False

   (c) **2.5 pt** – For inner product space V, if $\underline{w}$ is the projection of $\underline{v} \in V$ onto $\underline{u} \in V$ then $\underline{v} \perp \underline{w}$.
   Solution: False, in general, $(\underline{v} - \underline{w}) \perp \underline{w}$.

   (d) **2.5 pt** – A $n \times n$ matrix over a field F is invertible if and only its rows form a basis for F^n.
   Solution: True

2. Assign one of the following terms to each sentence: basis, inner product, invertible, non-singular, nullity, nullspace, operator norm, orthogonal, range, rank, subspace.

   (a) **2.5 pt** – Two subspaces $U, W \subset V$ that satisfy $\langle \underline{u} | \underline{w} \rangle = 0$ for all $\underline{u} \in U$ and $\underline{w} \in W$.
   Solution: orthogonal

   (b) **2.5 pt** – Let V be a vector space and $T : V \to W$ be a linear transformation that is injective.
   Solution: non-singular

   (c) **2.5 pt** – The largest scale factor by which a linear transform changes the length of a vector.
   Solution: operator norm

   (d) **2.5 pt** – The dimension of the column space of a matrix A.
   Solution: rank

3. Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let V_e be the subset of even functions satisfying $f(-x) = f(x)$; and let V_o be the subset of odd functions satisfying $f(-x) = -f(x)$.

   (a) **5 pt** – Show that V_e and V_o are subspaces of V.
   Solution: Let f and g be even functions. Then, for any $x \in \mathbb{R}$,

   $$(cf + g)(-x) = cf(-x) + g(-x) = cf(x) + g(x) = (cf + g)(x).$$

   Thus, V_e is a subspace of V. Similarly, suppose that f and g are odd functions. Then, for any $x \in \mathbb{R}$,

   $$(cf + g)(-x) = cf(-x) + g(-x) = -cf(x) - g(x) = -(cf + g)(x).$$

   Thus, V_o is also a subspace of V.

(b) **5 pt** – Prove that $V_e + V_o = V$.

Solution: Since V_e, V_o are subspaces of V, it follows that $V \subseteq V_e + V_o$. Thus, to prove $V = V_e + V_o$, we must show $V_e + V_o \subseteq V$. For any $f \in V$, let

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Using these definition, it is easy to verify that $f = f_e + f_o$, $f_e \in V_e$, and $f_o \in V_o$. This implies that $V \subseteq V_e + V_o$ and we conclude that $V_e + V_o = V$.

(c) **5 pt** – Prove that $V_e \cap V_o = \{0\}$.

Solution: Assume that $f \in V_e \cap V_o$. Then, for any $x \in \mathbb{R}$,

$$f(x) = f(-x) = -f(x).$$

This implies that $f(x) = 0$ for all $x \in \mathbb{R}$.

(d) **5 pt** – Let $f \in V$. Show that the decomposition $f = g + h$ where $g \in V_e$ and $h \in V_o$ is unique.

Solution: Let $f = f_e + f_o = g_e + g_o$ where $f_e, g_e \in V_e$ and $f_o, g_o \in V_o$. Then, $f_e - g_e = g_o - f_o$. Since V_e is a subspace, we have $f_e - g_e \in V_e$. Similarly, since V_o is a subspace, we have $g_o - f_o \in V_o$. This implies that

$$f_e - g_e = g_o - f_o \in V_e \cap V_o.$$

Or, equivalently, $f_e - g_e = g_o - f_o = 0$. Thus, $f_e = g_e$ and $f_o = g_o$. The decomposition is unique.

(e) **2.5 pt** – Prove or disprove the claim: a polynomial in V can be expressed as the sum of an even polynomial and an odd polynomial.

Solution: Let $p(x) = \sum_{i=1}^{n} p_i x^i$ be a polynomial in V. Then, $p_e(x) = (p(x) + p(-x))/2$ and $p_o(x) = (p(x) - p(-x))/2$ are also polynomials in V. As such, we conclude that a polynomial in V can be expressed as the sum of an even polynomial and an odd polynomial.

(f) **2.5 pt** – Prove or disprove the claim: a continuous function in V can be expressed as the sum of an even continuous function and an odd continuous function.

Solution: If $f(x)$ is continuous then $f(-x)$ is continous and so is $-f(x)$. To see that, let $x_1, x_2, \ldots$ be a sequence that converges to x. Then $-x_1, -x_2, \ldots$ is a sequence that converges to $-x$. Since $f(\cdot)$ is continuous, then $f(-x_1), f(-x_2), \ldots$ converges to $f(-x)$ and also $-f(x_1), -f(x_2), \ldots$ conveges to $-f(x)$. It follows that f_e and f_o as defined above are continuous functions. Thus, a continuous function in V can be expressed as the sum of an even continuous function and an odd continuous function.

4. Let V be the vector space of all real polynomial functions of degree 2 or less , i.e., the space of all functions f of the form

$$f(x) = c_0 + c_1 x + c_2 x^2 \quad \text{where } c_0, c_1, c_2 \in \mathbb{R}.$$

Consider the elements $g_0(x) = 1$, $g_1(x) = 1 + x$, $g_2(x) = (1 + x)^2$.

(a) **5 pt** – Prove that $\mathcal{B} = (g_0, g_1, g_2)$ is an ordered basis for V.

Solution: Consider the linear combination

$$\begin{aligned} b_0 g_0(x) + b_1 g_1(x) + b_2 g_2(x) &= b_0 + b_1 + b_1 x + b_2 + 2b_2 x + b_2 x^2 \\ &= (b_0 + b_1 + b_2) + (b_1 + 2b_2)x + b_2 x^2. \end{aligned}$$

Setting this equation to zero, we get

$$\begin{aligned} b_0 + b_1 + b_1 &= 0 \\ b_1 + 2b_2 &= 0 \\ b_2 &= 0. \end{aligned}$$

The unique solution to this system of linear equations is $b_0 = b_1 = b_2 = 0$. That is, the vectors g_0, g_1, g_2 are linearly independent. Since V has dimension three, as shown by the standard basis $\mathcal{A} = (1, x, x^2)$, we conclude that $\mathcal{B}$ is a basis for V.

(b) **5 pt** – If $f(x) = c_0 + c_1 x + c_2 x^2$, what are the coordinates of f in ordered basis $\mathcal{B}$?

Solution: Suppose $[f]_{\mathcal{B}} = (b_0, b_1, b_2)$, then we have the system of linear equations

$$\begin{aligned} b_0 + b_1 + b_2 &= c_0 \\ b_1 + 2b_2 &= c_1 \\ b_2 &= c_2. \end{aligned}$$

Or, equivalently,

$$Q\,[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}.$$

This leads to the solution

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = Q^{-1} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}.$$

Thus, $[f]_{\mathcal{B}} = (c_0 - c_1 + c_2, c_1 - 2c_2, c_2)$.

For the remainder of this problem, consider the linear transformation defined by

$$T(b_0 g_0(x) + b_1 g_1(x) + b_2 g_2(x)) = (b_0 + 2b_1 - b_2) + b_2 x^2.$$

(c) **5 pt** – What is the rank and nullity of T? Substantiate your answer.

Solution: The range of T is spanned by $\{1, x^2\}$, a linearly independent set of dimension two. Thus, the rank of T is two. Since V has dimension three, we deduce that the nullity of T is one.

(d) **5 pt** – Find a matrix B such that $[Tf]_{\mathcal{B}} = B\,[f]_{\mathcal{B}}$ for any $f \in V$.

Solution: First, we note that

$$\begin{aligned} Tg_0 &= g_0 \\ Tg_1 &= 2g_0 \\ Tg_2 &= x^2 - 1 = -2(1+x) + (1+x)^2 = -2g_1 + g_2. \end{aligned}$$

Collecting these results, we immediately get

$$[Tf]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} [f]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}.$$

(e) **5 pt** – Let $\mathcal{A} = (1, x, x^2)$ be the standard ordered basis. Find a matrix A such that $[Tf]_{\mathcal{A}} = A\,[f]_{\mathcal{A}}$ for any $f \in V$.

Solution: First, we note that

$$\begin{aligned} A\,[f]_{\mathcal{A}} = [Tf]_{\mathcal{A}} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} [Tf]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} B\,[f]_{\mathcal{B}} \\ &= QBQ^{-1}\,[f]_{\mathcal{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} [f]_{\mathcal{A}}. \end{aligned}$$

Hence, we gather that

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. The norm $\|\cdot\|_A$ is *norm equivalent* to the norm $\|\cdot\|_B$ if there exists an $M_{AB} < \infty$ such that

$$\frac{1}{M_{AB}} \|\underline{x}\|_B \le \|\underline{x}\|_A \le M_{AB} \|\underline{x}\|_B.$$

(a) **5 pt** – Show that norm equivalence is reflexive. In other words, show that "$\|\cdot\|_A$ norm equivalent to $\|\cdot\|_B$" implies "$\|\cdot\|_B$ norm equivalent to $\|\cdot\|_A$".

Solution: Solving for norm B in each inequality gives

$$\frac{1}{M_{AB}} \|\underline{x}\|_A \le \|\underline{x}\|_B \le M_{AB} \|\underline{x}\|_A.$$

Therefore, norm equivalence is reflexive.

(b) **5 pt** – Show that norm equivalence is transitive. In other words, show that "$\|\cdot\|_A$ is norm equivalent to $\|\cdot\|_B$" and "$\|\cdot\|_B$ is norm equivalent to $\|\cdot\|_C$" implies "$\|\cdot\|_A$ is norm equivalent to $\|\cdot\|_C$".

Solution: We can upper bound norm A with $\|\underline{x}\|_A \leq M_{AB}\|\underline{x}\|_B \leq M_{BC}M_{AB}\|\underline{x}\|_C$. We can lower bound norm B with $\|\underline{x}\|_A \geq \frac{1}{M_{AB}}\|\underline{x}\|_B \leq \frac{1}{M_{BC}M_{AB}}\|\underline{x}\|_C$. Therefore, the equivalence between norm A and norm C follows by choosing $M_{AC} = M_{AB}M_{BC}$.

Let $V = \mathbb{C}^n$ be the standard vector space over the complex numbers and define

$$\|x\|_p \triangleq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

to be the standard p-norm for $p \in [1, \infty)$.

(c) **5 pt** – Use simple bounds on $\|\cdot\|_p$ to show that any p-norm is norm equivalent to the ∞-norm for all $p \in [1, \infty)$

Solution: We can lower bound the p-norm by the largest single term and upper bound the sum by n times the largest term to get

$$\max_{i=1,\ldots,n} |x_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \leq n^{1/p} \max_{i=1,\ldots,n} |x_i|.$$

By the transitive property, all p-norms are therefore equivalent.

(d) **5 pt** – Now, show that an arbitrary norm $\|\cdot\|$ is equivalent to the 1-norm.

[Hint: You may assume that $\min_{\underline{x}:\|\underline{x}\|_1=1} \|\underline{x}\| = m > 0$ and that $\max_{i\in 1,\ldots,n} \|\underline{e}_i\| = M < \infty$.]

Solution: The required upper bound is given by

$$\begin{aligned}
\|\underline{x}\| &= \left\| \sum_{i=1}^{n} x_i \underline{e}_i \right\| \\
&\leq \sum_{i=1}^{n} |x_i| \|\underline{e}_i\| \\
&\leq \left(\sum_{i=1}^{n} |x_i| \right) \max_{j\in 1,\ldots,n} \|\underline{e}_j\| \\
&\leq M\|\underline{x}\|_1.
\end{aligned}$$

The necessary lower bound is given by

$$\begin{aligned}
\|\underline{x}\| &= \|\underline{x}\|_1 \left\| \frac{\underline{x}}{\|\underline{x}\|_1} \right\| \\
&\geq \|\underline{x}\|_1 \min_{\underline{x}:\|\underline{x}\|_1=1} \|\underline{x}\| \\
&= m\|\underline{x}\|_1.
\end{aligned}$$

(e) **5 pt** – Let $(V, \|\cdot\|)$ be a normed vector space. The first hint in part (d) is based on the continuity of the norm. Show that $\|\cdot\|$ is a continuous function from V (in the induced metric) to $\mathbb{R}$.

[Hint: One method starts by showing $\|\underline{x} - \underline{y}\| \geq \|\underline{x}\| - \|\underline{y}\|$.]

Solution: To prove continuity, we need to show that for any $\epsilon > 0$ there is a $\delta > 0$ such that $|\|\underline{x}\| - \|\underline{y}\|| < \epsilon$ for all $\|\underline{x} - \underline{y}\| < \delta$. Applying the triangle inequality to $\|\underline{x} - \underline{y} + \underline{y}\|$ gives $\|\underline{x}\| = \|\underline{x} - \underline{y} + \underline{y}\| \leq \|\underline{x} - \underline{y}\| + \|\underline{y}\|$. This gives (by swapping $\underline{x}, \underline{y}$ if $\|y\| \geq \|x\|$) $|\|\underline{x}\| - \|\underline{y}\|| < \|\underline{x} - \underline{y}\|$. Choosing $\delta = \epsilon$ in the definition of continuity suffices.

6. Consider the functions $f_i : [-1, 1] \mapsto \mathbb{R}$ given by $f_0(t) = 1$, $f_1(t) = t$, $f_2(t) = t^2$. Let $V = \mathrm{span}(f_0, f_1, f_2)$. Also, define the inner product

$$\langle f|h\rangle = \int_{-1}^{1} f(t)h(t)t^2 dt.$$

(a) **5 pt** – Since $\mathcal{B} = \{f_0, f_1, f_2\}$ is a basis for V, we know that any vector $f \in V$ can be expressed as $[f]_{\mathcal{B}} = [s_0\ s_1\ s_2]^T$ such that $f(t) = s_0 f_0(t) + s_1 f_1(t) + s_2 f_2(t)$. Find a matrix G such that

$$\langle f|h\rangle = [h]_{\mathcal{B}}^H \, G \, [f]_{\mathcal{B}}$$

for all $f, h \in V$.

Solution: This matrix can be formed using

$$G_{ij} = \langle f_{j-1}|f_{i-1}\rangle = \int_{-1}^{1} t^{j-1} t^{i-1} t^2 dt = \left. \frac{t^{i+j+1}}{i+j+1} \right|_{-1}^{1},$$

which yields

$$G = \begin{bmatrix} \frac{2}{3} & 0 & \frac{2}{5} \\ 0 & \frac{2}{5} & 0 \\ \frac{2}{5} & 0 & \frac{2}{7} \end{bmatrix}.$$

(b) **5 pt** – Apply the Gram-Schmidt orthogonalization process to basis elements $\{f_0, f_1, f_2\}$ and derive an orthogonal basis for V. Call the resulting vectors $\mathcal{A} = \{h_0, h_1, h_2\}$.

Solution:

$$h_0(t) = f_0(t) = 1$$
$$h_1(t) = f_1(t) - \frac{\langle f_1|h_0\rangle}{\|h_0\|^2} h_0(t) = f_1(t) = t$$
$$h_2(t) = f_2(t) - \frac{\langle f_2|h_1\rangle}{\|h_1\|^2} h_1(t) - \frac{\langle f_2|h_0\rangle}{\|h_0\|^2} h_0(t) = t^2 - \frac{3}{5}.$$