Assignment 7

1. Ans: $A = [P_1\ P_2\ P_3] = \begin{bmatrix} 1 & 4 & 3 \\ 2 & -2 & 4 \\ 3 & -6 & -2 \\ 4 & -7 & 1 \end{bmatrix}$

The projection matrix $P = A(A^H A)^{-1} A^H$

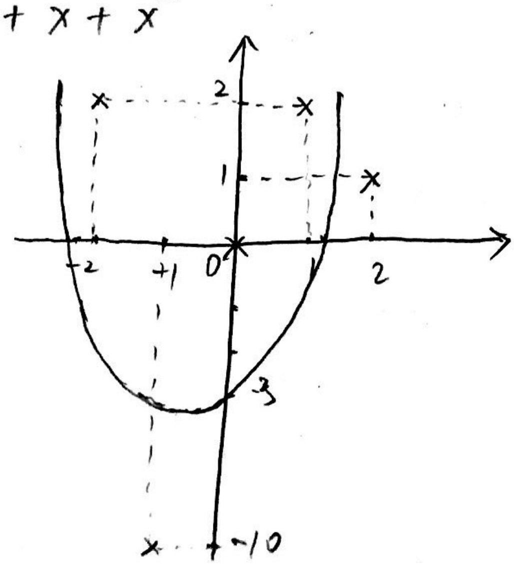
The best approximation $\hat{\underline{x}} = A\underline{s} = A(A^H A)^{-1} A^H \underline{x} \approx \begin{bmatrix} 0.8946 \\ 2.9665 \\ 4.1247 \\ 5.6996 \end{bmatrix}$

The projection of x onto the orthogonal complement of W should be error between x and $\hat{x}$: $x - \hat{x} = \begin{bmatrix} 0.1054 \\ -0.9665 \\ -1.1247 \\ 1.3004 \end{bmatrix}$

2. Ans: $A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ $\quad y = \begin{bmatrix} 2 \\ -10 \\ 0 \\ 2 \\ 1 \end{bmatrix}$

$\hat{z} = (A^H A)^{-1} A^H y = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

$\therefore\ y = -3 + x + x^2$



3. Ans: W is the subspace of polynomials with degree at most 2

Let $\underline{w}_1 = 1$, $\underline{w}_2 = x$, $\underline{w}_3 = x^2$

For calculating Gramian Matrix:

$\langle \underline{w}_1 | \underline{w}_1 \rangle = \int_0^1 dx = 1$, $\langle \underline{w}_1 | \underline{w}_2 \rangle = \int_0^1 x\,dx = \frac{1}{2}$, $\langle \underline{w}_1 | \underline{w}_3 \rangle = \frac{1}{3}$

$\langle \underline{w}_2 | \underline{w}_1 \rangle = \frac{1}{2}$, $\langle \underline{w}_2 | \underline{w}_3 \rangle = \frac{1}{3}$, $\langle \underline{w}_3 | \underline{w}_3 \rangle = \frac{1}{4}$, ...

$$\therefore G = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

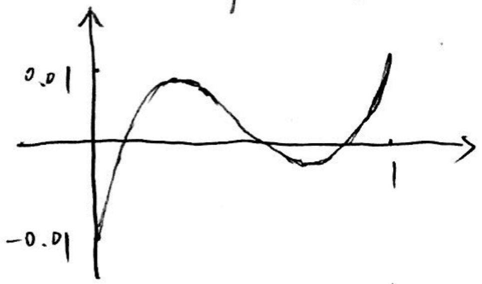
$\underline{v} = f(x) = e^x$

$\therefore \langle \underline{v} | \underline{w}_1 \rangle = e-1$, $\langle \underline{v} | \underline{w}_2 \rangle = \int_0^1 x e^x\,dx = 1$, $\langle \underline{v} | \underline{w}_3 \rangle = \int_0^1 x^2 e^x\,dx = e-2$

we get $\underline{t} = \begin{bmatrix} e-1 \\ 1 \\ e-2 \end{bmatrix}$ $\therefore \underline{s} = G^{-1}\underline{t} = \begin{bmatrix} 1.01 \\ 0.85 \\ 0.84 \end{bmatrix}$

$\therefore \hat{f}(x) = 1.01 + 0.85x + 0.84x^2$

error $= f(x) - \hat{f}(x)$



4. For showing that $I-P$ is a orthogonal projection matrix, we have to show it is idempotent and Hermitian: since P is an orthogonal projection matrix, $P^2=P$ and $P^H=P$

$$(I-P)^2 = (I-P)(I-P) = I - IP - PI + P^2 = I - P - P + P = I - P$$

$$(I-P)^H = I^H - P^H = I - P$$

range $(I-P)$ = null (P)

First, we show null$(P) \subseteq$ range$(I-P)$. We have a vector $\underline{v}$ such that $P\underline{v}=0$. Then $(I-P)\underline{v} = \underline{v} - P\underline{v} = \underline{v}$. Thus, any $\underline{v}$ in the nullspace of P is also in the range of $I-P$. Then, we show range$(I-P) \subseteq$ nullspace(P). If we have any $\underline{x} \in$ range$(I-P)$, then $\underline{x} = (I-P)\underline{v}$ for some $\underline{v}$.

$$\underline{x} = \underline{v} - P\underline{v} = -(P\underline{v} - \underline{v}) \qquad P\underline{x} = -P(P\underline{v} - \underline{v}) = -P^2\underline{v} + P\underline{v}$$

$$= -P\underline{v} + P\underline{v}$$

$$= 0$$

$\therefore x \in$ null(P) $\qquad \therefore$ range$(I-P)$ = null(P)

Similarly, we can prove null$(I-P)$ = range(P)

5. Ans:

(a) $\operatorname{cov}(x) \triangleq E[xx^T] + E[x]E[x]^T = \begin{bmatrix} 1 & 0.7 & 0.5 \\ 0.7 & 4 & 0.2 \\ 0.5 & 0.2 & 3 \end{bmatrix}$

Because x is zero-mean vector, $E[x]E[x]^T = 0$

$$E[xx^T] = \begin{bmatrix} E[x_1x_1^T] & E[x_2x_1^T] & E[x_3x_1^T] \\ E[x_1x_2^T] & E[x_2x_2^T] & E[x_3x_2^T] \\ E[x_1x_3^T] & E[x_2x_3^T] & E[x_3x_3^T] \end{bmatrix} = \begin{bmatrix} 1 & 0.7 & 0.5 \\ 0.7 & 4 & 0.2 \\ 0.5 & 0.2 & 3 \end{bmatrix}$$

$\hat{x}_1 = c_1x_2 + c_2x_3 \qquad$ Let $s = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$G \cdot \underline{s} = \underline{t}$$

$$\begin{bmatrix} E[x_2x_2^T] & E[x_3x_2^T] \\ E[x_2x_3^T] & E[x_3x_3^T] \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} E[x_1x_2^T] \\ E[x_1x_3^T] \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 0.2 \\ 0.2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 & 0.2 \\ 0.2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.167 \\ 0.156 \end{bmatrix}$$

(b) the minimum mean-square error is

$$\| x_1 - \hat{x}_1 \|^2 = E[x_1 x_1^T] - \underline{t}^H G^{-1} \underline{t} = 1 - \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix}^T \begin{bmatrix} 4 & 0.2 \\ 0.2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix}$$

$$\approx 1 - 0.195 = 0.805$$

(c) $E[x] = (1, 2, 3)^T$

Then $E[x] E[x]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \ 2 \ 3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

$$E[xx^T] = \begin{bmatrix} 1 & 0.7 & 0.5 \\ 0.7 & 4 & 0.2 \\ 0.5 & 0.2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 2.7 & 3.5 \\ 2.7 & 8 & 6.2 \\ 3.5 & 6.2 & 12 \end{bmatrix}$$

$\hat{x}_1 = c_1 x_2 + c_2 x_3$

$$\begin{bmatrix} 8 & 6.2 \\ 6.2 & 12 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2.7 \\ 3.5 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.186 \\ 0.196 \end{bmatrix}$$

the minimum mean-square error $= 2 - \begin{bmatrix} 2.7 \\ 3.5 \end{bmatrix}^T \begin{bmatrix} 8 & 6.2 \\ 6.2 & 12 \end{bmatrix}^{-1} \begin{bmatrix} 2.7 \\ 3.5 \end{bmatrix}$

≈ 0.813

6. ① to show $R(A) \subseteq R(U)$

assume $\underline{w} = V \Sigma^{-1} \underline{s}$

then for $\forall \underline{a} = A \underline{w} \in R(A)$

$\underline{a} = A \underline{w} = U \Sigma V^H V \Sigma^{-1} \underline{s} = U \underline{s}$

thus $R(A) \subseteq R(U)$

② to show $R(U) \subseteq R(A)$

assume $\underline{s} = \Sigma V^H \underline{w}$

then for $\forall \underline{b} = U \underline{s} \in R(U)$

$\underline{b} = U \underline{s} = U \Sigma V^H \underline{w} = A V \Sigma^{-1} \Sigma V^H \underline{w}$

$= A \underline{w}$

thus $R(U) \subseteq R(A)$

Therefore, $R(U) = R(A)$

$P_A = P_U = U(U^H U^{-1}) U^H$

$= U U^H$