Assignment 4

Due Wednesday 10/9/19

Reading Assignment:

- Required: Course Notes 3.1-3.4
- Recommended: LADR Ch. 1, Ch. 2, Ch. 3ABC
- Supplemental: MMA 2.1-2.2

Problems:

1. (LA: 1.6.1) (5 pts) Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}.$$

Find a row-reduced echelon matrix R which is row-equivalent to A and an invertible 3×3 matrix P such that $R = PA$.

Solution: Using elementary row operations on an augmented matrix, we get

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 5 & 0 & 1 & 0 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & -4 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 2 & 4 & 5 & 1 & 1 & 0 \\ 0 & 0 & 8 & 11 & 1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|ccc} 1 & 0 & -3 & -5 & 0 & -1 & 0 \\ 0 & 1 & 2 & 5/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 11/8 & 1/8 & 1/4 & 1/8 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -7/8 & 3/8 & -1/4 & 3/8 \\ 0 & 1 & 0 & -1/4 & 1/4 & 0 & -1/4 \\ 0 & 0 & 1 & 11/8 & 1/8 & 1/4 & 1/8 \end{array}\right].$$

where $\sim$ is used here to denote the equivalence relation on matrices that holds if the two matrices have the same row space. Thus, we have

$$R = \begin{bmatrix} 1 & 0 & 0 & -7/8 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 11/8 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 3/8 & -1/4 & 3/8 \\ 1/4 & 0 & -1/4 \\ 1/8 & 1/4 & 1/8 \end{bmatrix}.$$

From $PA = R$, the right-most column of R is the solution to linear system below because

$$P \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = P \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7/8 \\ -1/4 \\ 11/8 \end{bmatrix}$$

2. (LADR: 1.C.1) (5 pts each) For each of the folowing subsets, determine whether it is a subspace of $\mathbf{F}^3$.

   (a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;
   Solution: Closed under addition.

$$\begin{aligned} x, y &\in V \\ x &= \left(x_1, x_2, \frac{x_1 + 2x_2}{-3}\right) \\ y &= \left(y_1, y_2, \frac{y_1 + 2y_2}{-3}\right) \\ x + y &= \left(x_1 + y_1, x_2 + y_2, \frac{(x_1 + y_1) + 2(x_2 + y_2)}{-3}\right) \\ x + y &\in V \end{aligned}$$

   Closed under scalar multiplication.

$$\begin{aligned} x &\in V \\ x &= \left(x_1, x_2, \frac{x_1 + 2x_2}{-3}\right) \\ ax &= \left(ax_1, ax_2, \frac{ax_1 + 2ax_2}{-3}\right) \\ ax &=\in V \end{aligned}$$

   Contains additive identity. $(0, 0, 0)$ is a solution to the defining equation. $0 \in V$.
   $\therefore$ **YES**, it is a subspace of $\mathbb{F}^3$.

   (b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;
   Solution: The additive identity $(0, 0, 0)$ is not a solution to the defining equation. $0 \notin V$.
   $\therefore$ **NO**, it is not a subspace of $\mathbb{F}^3$.

   (c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$;
   Solution:

$$\begin{aligned} x, y &\in V \\ x &= (x_1, x_2, 0) : x_1, x_2 \neq 0 \\ y &= (0, y_2, y_3) : y_2, y_3 \neq 0 \\ x + y &= (x_1, x_2 + y_2, y_3) \\ x + y &\notin V. \end{aligned}$$

   Subset is not closed under addition.
   $\therefore$ **NO**, it is not a subspace of $\mathbb{F}^3$.

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$;

Solution: Closed under addition.

$$\begin{aligned} x, y &\in V \\ x &= (x_1, x_2, x_1/5) \\ y &= (y_1, y_2, y_1/5) \\ x + y &= (x_1 + y_1, x_2 + y_2, (x_1 + y_1)/5) \\ x + y &\in V \end{aligned}$$

Closed under scalar multiplication.

$$\begin{aligned} x &\in V \\ x &= (x_1, x_2, x_1/5) \\ ax &= (ax_1, ax_2, a(x_1/5)) \\ ax &= \in V \end{aligned}$$

Contains additive identity. $(0, 0, 0)$ is a solution to the defining equation. $0 \in V$.

$\therefore$ **YES**, it is a subspace of $\mathbb{F}^3$.

3. (LADR: 1.C.7) (5 pts) Give an example of a non-empty subset U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

   Solution: Define the set of integers $\mathbf{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Consider the following subset of $\mathbf{R}^2$:

   $$U = \{(x_1, x_2) \in \mathbf{R}^2 : x_1, x_2 \in \mathbf{Z}\}$$

   This subset is closed under addition (the sum of any two integers is an integer) and under taking additive inverses (if $u = (x_1, x_2) \in U, (-x_1, -x_2) \in U$). However the subset is not closed under scalar multiplication, since the original vector space had the reals as the set of scalars.

4. (LADR: 2.A.1) (5 pts) Suppose the list v_1, v_2, v_3, v_4 spans V. Prove that the list

   $$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

   also spans V.

   Solution: We need to show that v_1, v_2, v_3, v_4 can be expressed as a linear combination of $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$. Let u be a vector in $\operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$, then

   $$\begin{aligned} u &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 \qquad (\exists b_i \in \mathbf{R}) \\ &= b_1 v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4. \end{aligned}$$

   That is to say, $u \in \operatorname{span}(v_1, v_2, v_3, v_4)$, by the definition of *span*.

5. (LADR: 2.A.5) (5 pts each) Show that:

   (a) If we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $(1+i, 1-i)$ is linearly independent.
   Solution: Let $a, b \in \mathbf{R}$, we need to check whether there are nontrivial solutions of the linear combination $a(1+i) + b(1-i) = 0$. Distributing and rearraging we have

   $$0 = a(1+i) + b(1-i) = (a+b) + (a-b)i,$$

   which can only be true if $a = b = 0$. That is, the only solution is trivial. Hence, the list $(1+i, 1-i)$ is linearly independent over $\mathbf{R}$.

   (b) If we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $(1+i, 1-i)$ is linearly dependent.
   Solution: Now, let $w, z \in \mathbf{C}$, we (again) need to check whether there are nontrivial solutions of the linear combination $w(1+i) + z(1-i) = 0$. Since w, z are complex we have more than one solution. One of them is $w = 1-i,\ z = -1-i$, another is $w = -1,\ z = i$. Thus, the list $(1+i, 1-i)$ is linearly dependent over $\mathbf{C}$.

6. (LA: 2.2.9) (5 pts) Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{\underline{0}\}$. Prove that for each vector α in V there are unique vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

   Note: If $S_1, S_2, \dots, S_k$ are subsets of a vector space V, the set of all sums

   $$\alpha_1 + \alpha_2 + \cdots + \alpha_k$$

   or vectors α_i in S_i is called the sum of the subsets $S_1, S_2, \dots, S_k$ and is denoted by

   $$S_1 + S_2 + \cdots + S_k$$

   of by

   $$\sum_{i=1}^{k} S_i.$$

   Solution: The existence part comes from the definition of $W_1 + W_2 = V$. In particular, if $\alpha \in V$ then there exist $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$. We wish to prove the uniqueness of this decomposition. Let α be any vector in V. Suppose that $\alpha_1 + \alpha_2 = \alpha$ with $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$. Furthermore, suppose that $\beta_1 + \beta_2 = \alpha$ with $\beta_1 \in W_1$ and $\beta_2 \in W_2$. It follows that $\alpha = \alpha_1 + \alpha_2 = \beta_1 + \beta_2$. This, in turn, implies that $\alpha_1 - \beta_1 = \beta_2 - \alpha_2$. Since W_1 is a subspace, we have $\alpha_1 - \beta_1 \in W_1$. Similarly, $\beta_2 - \alpha_2 \in W_2$. Combining these two facts, we get

   $$\alpha_1 - \beta_1 = \beta_2 - \alpha_2 \in W_1 \cap W_2 = \{0\}.$$

   Hence, $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. That is, the decomposition of α in $W_1 + W_2$ is unique.

7. (LADR: 2.B.3) (5 pts each)

(a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2, x_3 = 7x_4\right\}.$$

Find a basis for U.

Solution: The subspace definition is equivalent to

$$U = \{(3x_2, x_2, 7x_4, x_4, x_5) : x_2, x_4, x_5 \in \mathbf{R}^5\}$$

In this form it becomes easy to see that $\dim U = 3$, as any particular vector will be fully defined by specifying three numbers (x_2, x_4, x_5). Thus our basis will be a list of three vectors in $\mathbf{R}^5$. A space of three dimensions can be spanned by a variety of bases, but the simplest is perhaps the standard basis $((1,0,0),(0,1,0),(0,0,1))$. Of course, we have to identify the coordinates of this basis with the three numbers that we have to specify. We take the first coordinate to be x_2, the second to be x_4, and the third to be x_5. Thus the standard basis on three dimensions is equivalent to the coordinate choices

i. $x_2 = 1, x_4 = 0, x_5 = 0$
ii. $x_2 = 0, x_4 = 1, x_5 = 0$
iii. $x_2 = 0, x_4 = 0, x_5 = 1$

Finally, we have to turn these coordinate choices into vectors in $\mathbf{R}^5$. The remaining two coordinates are constrained by the definition of the subspace to $x_1 = 3x_2$ and $x_3 = 7x_4$. So, the final list of basis vectors becomes:

i. $(3,1,0,0,0)$
ii. $(0,0,7,1,0)$
iii. $(0,0,0,0,1)$

Of course, choosing something other than the standard basis will result in a different (but still valid) basis.

(b) Extend the basis from (a) to a basis for $\mathbf{R}^5$

Solution:

i. $(1,0,0,0,0)$
ii. $(3,1,0,0,0)$
iii. $(0,0,7,1,0)$
iv. $(0,0,1,0,0)$
v. $(0,0,0,0,1)$

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

Solution: $W = \operatorname{span}((1,0,0,0,0),(0,0,1,0,0))$

8. (MMA: 2.11.55) (5 pts) Let X and Y be vector spaces over the same set of scalars. Let $LT[X,Y]$ denote the set of all linear transformations (i.e., linear functions) from X to Y.

Thus, the transformations L and M in $LT[X,Y]$ satisfy $L(ax+z) = aL(x) + L(z)$ for all scalars a and vectors $x, z \in X$. Define an addition operator between L and M as

$$(L+M)(x) = L(x) + M(x),$$

for all $x \in X$. Define scalar multiplication by

$$(aL)(x) = a(L(x)).$$

Show that $LT[X,Y]$ is a vector space.

Solution: Recall that, if X and Y are vector spaces over the same set of scalars, then the set of all functions from X to Y also forms a vector space. To show that $LT[X,Y]$ is a vector space, we need only show that it fulfills the subspace condition. Let $L, M \in LT[X,Y]$ and a be a scalar. Then, we need to show that

$$(aL+M)(x) = aL(x) + M(x)$$

is in $LT[X,Y]$. For $x_1, x_2 \in X$ and scalar s, we have

$$\begin{aligned}
(aL+M)(sx_1+x_2) &= aL(sx_1+x_2) + M(sx_1+x_2) \\
&= asL(x_1) + aL(x_2) + sM(x_1) + M(x_2) \\
&= saL(x_1) + sM(x_1) + aL(x_2) + M(x_2) \\
&= s(aL+M)(x_1) + (aL+M)(x_2).
\end{aligned}$$

Thus, we conclude that $(aL+M) \in LT[X,Y]$ as desired.

Practice Problems (do not hand in):

1. (EF: 2.2.1) Let X, Y be metric spaces and $f : X \to Y$ be a continuous function. Prove that, if $A \subseteq X$ is a compact subset, then $f(A) \subseteq Y$ is a compact subset.

   [Hint: You may assume a continuous function is uniformly continuous on a compact set.]

   Solution: First, we show that the metric space $(f(A), d_Y)$ is complete. Let $y_1, y_2, \ldots$ be any Cauchy sequence in $f(A) \subseteq Y$ and $x_1, x_2, \ldots$ be defined by choosing $x_n \in f^{-1}(\{y_n\}) \cap A$ arbitrarily. Since A is compact, the sequence $x_1, x_2, \ldots$ has a subsequence which converges to some $x \in A$. Moreover, the continuity of f implies that $f(x_{n_1}), f(x_{n_2}), \ldots$ converges to $f(x) \in f(A)$. Therefore, $f(A)$ is complete.

   Next, we prove (by contradiction) that a continuous function on a compact set is uniformly continuous. Suppose f is not uniformly continuous on A. Then, there is an $\epsilon > 0$ such that, for all $\delta > 0$, there exist $x, x_0 \in A$ where $d_X(x, x_0) < \delta$ and $d_Y(f(x_0), f(x)) \geq \epsilon$. So, for each $n \in \mathbb{N}$, we can choose $\delta = \frac{1}{n}$ and find $w_n, z_n \in A$ (i.e., x, x_0 from above) such that $d_X(w_n, z_n) < \frac{1}{n}$. There must be an $\epsilon > 0$ such that $d_Y(f(w_n), f(z_n)) \geq \epsilon$ for all $n \in \mathbb{N}$. But A is compact, so we can choose a subsequence where $w_n \to w$ and $z_n \to z$. This causes a contradiction because $d_X(w_n, z_n) < \frac{1}{n}$ implies that $w = z$ and $d_Y(f(w_n), f(z_n)) \to 0$. Therefore, f is uniformly continuous on A and there exists a function $\delta(\epsilon)$ such that $d_Y(f(x_0), f(x)) < \epsilon$ for all $d_X(x, x_0) \leq \delta(\epsilon)$ and $x, x_0 \in A$.

Now, we show that $f(A)$ is totally bounded. Since A is totally bounded, we can, for any $\epsilon > 0$, cover the set A with $n < \infty$ d-open balls centered at $x_1, \ldots, x_n \in A$ of radius $\delta(\epsilon) > 0$. Since f is uniformly continuous on A, we have, for any $\epsilon > 0$, a $\delta(\epsilon) > 0$ such that $f(B_{d_X}(x, \delta(\epsilon))) \subseteq B_{d_Y}(f(x), \epsilon)$ for all $x \in A$. Since the image of the covering of A under f must cover $f(A)$, we find that the set of d-open balls of radius ϵ centered at $f(x_1), \ldots, f(x_n)$ also covers $f(A)$. This approach works for any $\epsilon > 0$, so we find that $f(A)$ is totally bounded.

2. (LA: 1.6.6) Suppose A is a 2×1 matrix and that B is a 1×2 matrix. Prove that $C = AB$ is not invertible.

   Solution: An $n \times n$ matrix C is invertible iff $C\underline{x} = 0$ implies $\underline{x} = 0$. But, any $m \times n$ matrix B with $m < n$ must have a vector $\underline{x}$ such that $B\underline{x} = \underline{0}$. For example, if $B = (b_1, b_2)$, then $\underline{x} = (b_2, -b_1)^T$ shows that $B\underline{x} = \underline{0}$.

3. (EF: 3.3.1) Let V be the vector space $\mathbb{C}^3$ over the *real* numbers. How many vectors are in any basis of V?

   Solution: There are 6 vectors in any basis of V. For example, one could use $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(j, 0, 0)$, $(0, j, 0)$, and $(0, 0, j)$.

4. (LADR: 2.C.11) Suppose that U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

   Solution: We have that $\dim U = 3$, $\dim W = 5$, $\dim(U + W) = \dim \mathbb{R}^8 = 8$. Invoking Theorem 2.43 in LADR, we see that

$$\begin{aligned}
\dim(U + W) &= \dim U + \dim W - \dim(U \cap W) \\
8 &= 3 + 5 + \dim(U \cap W) \\
0 &= \dim(U \cap W).
\end{aligned}$$

   Thus $U \cap W = \{0\}$, and by Theorem 1.45, the set $U + W$ must be a direct sum. Therefore, $U \oplus W = U + W = \mathbb{R}^8$, which proves the result.

5. (LADR: 2.C.12) Suppose that U and W are both 5-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

   Solution: Since U and W are both subspaces of $\mathbf{R}^9$, $\dim(U + W) \leq 9$. Using Theorem 2.43 in LADR, we see that

$$\begin{aligned}
9 &\geq \dim(U + W) = \dim U + \dim W - \dim(U \cap W) \\
9 &\geq 5 + 5 - \dim(U \cap W) \\
9 &\geq 10 - \dim(U \cap W) \\
\dim(U \cap W) &\geq 1
\end{aligned}$$

   $\therefore U \cap W \neq \{0\}$.

6. (LA: 2.2.5) Let F be a field and let n be a positive integer ($n \geq 2$). Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A are subspaces of V?

   (a) all invertible A

   (b) all non-invertible A

   (c) all A such that $AB = BA$, where B is some fixed matrix in V

   (d) all A such that $A^2 = A$

   Solution: Only (c) is a subspace.

7. (LADR: 2.A.15) Prove that $V = \mathbf{F}^\infty$ is infinite dimensional.

   Solution: Suppose V is finite dimensional and let $v_1, \ldots, v_n$ be a basis for V. Then, any vector $v \in V$ can be written uniquely as $v = \sum_{j=1}^{n} a_j v_j$. Let e_j be the j-th standard basis vector (which is all 0 except for a 1 in ths j-th position). Then, there exists scalars $a_{j,k}$ such that $e_j = \sum_{k=1}^{n} a_{j,k} v_k$ for $j = 1, \ldots, n+1$. Since $e_1 \ldots, e_{n+1}$ are clearly linearly independent, if $b_1, \ldots, b_{n+1}$ are not all zero, then

$$\sum_{j=1}^{n+1} b_j e_j = \sum_{j=1}^{n+1} \sum_{k=1}^{n} b_j a_{j,k} v_k = \sum_{k=1}^{n} \left(\sum_{j=1}^{n+1} b_j a_{j,k} \right) v_k \neq 0.$$

   But, $a_{j,k}$ defines an $n+1$ by n matrix and, hence, there is a non-zero vector $b_1, \ldots, b_{n+1}$ such that $\sum_{j=1}^{n+1} b_j a_{j,k} = 0$ for $k = 1, \ldots, n$. This gives a contradiction and shows that V cannot be finite dimensional.