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Assignment 5

1. (a) $V_0 = (0, 1, 1, 0)^T$

(b) $F = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & -4 \\ -2 & 3 & -6 \\ -3 & 4 & -7 \end{bmatrix}$

2. Ans: Since $v_1, \cdots, v_n$ spans V, any $v \in V$ can be written as a linear combination of $v_1, \cdots, v_n$. There are $a_1, \cdots, a_n \in F$ such that $v = a_1 v_1 + \cdots + a_n v_n$

Since $T \in L(V, M)$, we can know that for $\forall w \in Range(T)$,

$\underline{w} = T\underline{v} = T \sum_{i=1}^{n} a_i \underline{v}_i = \sum_{i=1}^{n} a_i T \underline{v}_i$

Hence $range(T) \subset span(Tv_1, \cdots, Tv_n)$. On the other hand $Tv_1, \cdots, Tv_n$ are contained in $range(T)$. By the definition of span, therefore $Tv_1, \cdots, Tv_n$ spans range T.

3. (a) $T = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$

(b) do elementary column operation on T, we can get the matrix : $\begin{bmatrix} \cancel{0} & 0 & 1 & 1 \\ \cancel{0} & 1 & 0 & 1 \\ \cancel{0} & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \cancel{0} & \cancel{0} & 1 & 1 \\ \cancel{0} & \cancel{0} & 0 & 1 \\ \cancel{0} & \cancel{0} & -1 & 0 \end{bmatrix}$

So a basis for the range of T can be $(1, 0, -1)$ and $(1, 1, 0)$

(c) do elementary row operation on T, we can get a matrix

$$A = \begin{array}{cc} I & P \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $\tilde{A} = [I \ P]$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

Since $[I \ P] \begin{bmatrix} -P \\ I \end{bmatrix} = -IP + PI = 0$ and we want to find a matrix B such that $\tilde{A}B = 0$, so $B = \begin{bmatrix} -P \\ I \end{bmatrix}$

$$B = \begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

. Therefore the basis can be $(-1, 1, 1, 0)$ and $(-2, 1, 0, 1)$.

4. (a) A matric on a set X satisfies:

① $d(x, y) \geq 0 \quad \forall x, y \in X$; equals iff $x = y$

② $d(x, y) = d(y, x) \quad \forall x, y \in X$

③ $d(x, y) + d(y, z) \geq d(x, z) \quad \forall x, y, z \in X$

Since $(X, \|\cdot\|)$ is a normed vector space, accorrding its properties we know that $0 \leq \|x_1 - x_2\| < \infty$ and $\|x_1 - x_2\| = 0$ iff $x_1 = x_2$ and $d(x_1, x_2) + d(x_2, x_3) = \|x_1 - x_2\| + \|x_2 + x_3\| \geq \|x_1 - x_3\| = d(x_1, x_3)$

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Also, $d(\underline{x_1}, \underline{x_2}) = \|x_1 - x_2\| = \|\underline{x_2} - \underline{x_1}\| = d(x_2, x_1)$

The induced distance meet all the three properties, so it's a metric

b) when $\|\underline{x_1}\| < \|\underline{x_2}\|$

$\big|\|\underline{x_1} - \underline{x_2}\|\big| = \|\underline{x_2}\| - \|\underline{x_1}\| = \|\underline{x_2} + \underline{x_1} - \underline{x_1}\| - \|x_1\| \leq \|x_2 - x_1\| + \|x_1\| - \|x_1\|$

$= \|x_2 - x_1\|$

when $\|\underline{x_2}\| \leq \|\underline{x_1}\|$

$\big|\|\underline{x_1} - \underline{x_2}\|\big| = \|x_1\| - \|x_2\| = \|\underline{x_1} - \underline{x_2} + \underline{x_2}\| - \|\underline{x_2}\| \leq \|\underline{x_1} - \underline{x_2}\| + \|\underline{x_2}\| - \|\underline{x_2}\|$

$= \|\underline{x_1} - \underline{x_2}\|$

$= \|\underline{x_2} - \underline{x_1}\|$

Therefore $\big|\|\underline{x_2}\| - \|\underline{x_1}\|\big| \leq \|x_2 - x_1\|$

(c) Define $f = \|\cdot\| : X \to R \qquad f(x) = \|x\|$

For $\forall \varepsilon > 0$, if $d(\underline{x_1}, \underline{x_2}) = \|\underline{x_1} - \underline{x_2}\| < \varepsilon$

$\big|\|x_1\| - \|x_2\|\big| \leq \|\underline{x_1} - \underline{x_2}\| = d(x_1, x_2) < \varepsilon$ from question (b)

$\exists \delta > 0$, we assume $\delta = \varepsilon$

So $d(\|x_1\|, \|x_2\|) = d(f(x_1), f(x_2)) < \varepsilon = \delta$

Therefore, the norm $\|\cdot\| : X \to R$ is a continuous function.

J. (a) To prove this, we need to prove:

i) $\|T\|_{V,W} \geq 0$, equals iff $T=0$

ii) $\|sT\|_{V,W} = |s|\,\|T\|_{V,W}$

iii) $\|T+U\|_{V,W} \leq \|T\|_{V,W} + \|U\|_{V,W}$

$\Rightarrow$ i) when $T=0$, obviously $\|T\|=0$

when $T \neq 0$, $\underline{v}' \in V$ and $v' \geq 0$ such that $T\underline{v}' \neq 0$

$$\|T\|_{V,W} \triangleq \sup_{\underline{v}\in V:\|\underline{v}\|_V=1} \|T\underline{v}\|_W \geq \left\|T\frac{\underline{v}'}{\|v'\|}\right\| > 0$$

Therefore, $\|T\|_{V,W} \geq 0$, equals iff $T=0$

$\Rightarrow$ ii) $$\|sT\|_{V,W} = \sup_{\underline{v}\in V,\|v\|=1} \|sT\underline{v}\| = \sup_{\underline{v}\in V,\|v\|=1} |s|\,\|T\underline{v}\|$$

$$= |s|\,\|T\|_{V,W}$$

$\Rightarrow$ iii) $$\|T+U\|_{V,W} = \sup_{\underline{v}\in V,\|v\|=1} \|(T+U)\underline{v}\|_W$$

vector addition

$$= \sup_{\underline{v}\in V,\|v\|=1} \|T\underline{v} + U\underline{v}\|_W$$

vector properties

$$\leq \sup_{\underline{v}\in V} \|T\underline{v}\|_W + \|U\underline{v}\|_W$$

$$\leq \sup \|T\underline{v}\|_W + \sup \|U\underline{v}\|_W$$

$$= \|T\|_{V,W} + \|U\|_{V,W}$$

(b) From question (a), we get $\|T\|_{V,W} = \sup_{\underline{v} \in V : \|v\|=1} \|T\underline{v}\|_W \geq \|T\frac{\underline{v}}{\|\underline{v}\|}\|$

Then $\|T\underline{v}\|_W \leq \|T\|_{V,W} \|\underline{v}\|_V$

Replace W by V, then $\|T\underline{v}\|_V \leq \|T\|_{V,V} \|\underline{v}\|_V$

Since $U: V \to V$, then $U\underline{v} \in V$, we can replace $\underline{v}$ by $U\underline{v}$

Then take induced operator:

$$\sup_{v \in V, \|v\|=1} \|TU\underline{v}\|_V \leq \sup_{v \in V, \|v\|=1} \|T\|_{V,V} \|U\underline{v}\|_V$$

$$\leq \sup_{v \in V, \|v\|=1} \|T\|_{V,V} \|U\|_{V,V} \|\underline{v}\|_V = 1$$

$$= \sup_{v \in V, \|v\|=1} \|T\|_{V,V} \cdot \sup_{v \in V, \|v\|=1} \|U\|_{V,V}$$

Therefore, the induced operator norm $\|\cdot\|_{V,V}$ is submultiplicative

6. Ans: Since $\|\cdot\|$ is a matrix norm satisfying the submultiplicative

$\|TU\| \leq \|T\|\|U\|$

contropositive: if $I - F$ is singular, then $\|F\| \geq 1$

If $I - F$ is singular, there is a vector x such that $(I-F)x = 0$

Then $I\underline{x} = \underline{x} = F\underline{x}$

$$\|x\| = \|F x\| \leq \|F\|\|x\|$$

$$\|F\| \geq 1$$

Therefore, if $\|\cdot\|$ satisfys submultiplicative property and $\|F\| < 1$, then $I - F$ is non-singular