No.

Date Assignment 3

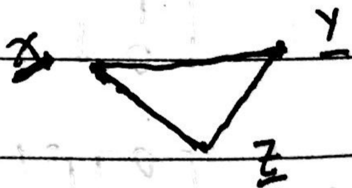
1. Ans:

The function ρ needs satisfying three condition to be a matric:

① Since $\rho(\underline{x}, \underline{y}) = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$, we get absolute value. So $\rho(\underline{x}, \underline{y}) \geq 0$. $\forall \underline{x}, \underline{y} \in R^n$.

②
$$\rho(\underline{x}, \underline{y}) = \max\{|x_1 - y_1|, \cdots |x_n - y_n|\}$$
$$= \max\{|y_1 - x_1|, \cdots |y_n - x_n|\}$$
$$< \rho(\underline{y}, \underline{x}), \quad \forall \underline{x}, \underline{y} \in R^n$$

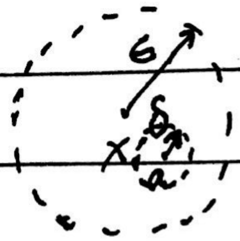
③ Let $\underline{z} = (z_1, \cdots, z_n) \in R^n$



$$\rho(\underline{x}, \underline{y}) + \rho(\underline{y}, \underline{z})$$
$$= \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\} + \max\{|y_1 - z_1|, \cdots, |y_n - z_n|\}$$
$$= \max\{d(x_1, y_1) + d(y_1, z_1) + \cdots + d(x_n, y_n) + d(y_n, z_n)\}$$
$$\geq \max\{d(x_1, z_1) + \cdots + d(x_n, z_n)\}$$
$$= \rho(\underline{x}, \underline{z}\} \quad \text{for } \forall \underline{x}, \underline{y}, \underline{z} \in R^n$$

Therefore, the function ρ is a matric.

2. Ans:



To satisfy the open ball $B_d(a, \delta)$ centered at a is contained in $B_d(x, \epsilon)$, the δ should be $(0, \epsilon - d(a, x))$.

3. Ans:

To show that $\{x_n\}$ is a Cauchy sequence, it should satisfy $\forall \varepsilon > 0, \exists N \in \mathbb{N} \, [\forall n, m \geq N, d(x_m, x_n) < \varepsilon, x_m, x_n \in X]$.

Assume $m > n$,

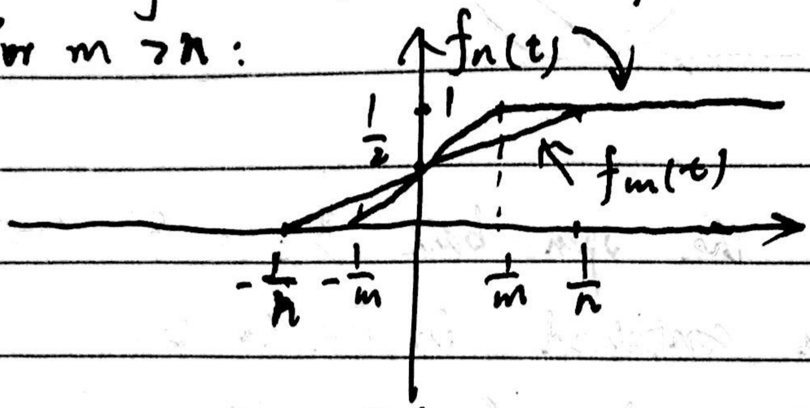
$$
\begin{aligned}
d(x_m, x_n) &\le d(x_m, x_{n+1}) + d(x_n, x_{n+1}) \\
&\le d(x_m, x_{n+2}) + d(x_{n+1}, x_{n+2}) + Cr^n \\
&\le d(x_m, x_{n+3}) + d(x_{n+2}, x_{n+3}) + Cr^n + Cr^{n+1} \\
&\quad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&\le d(x_m, x_{m-1}) + Cr^n + Cr^{n+1} + \cdots Cr^{m-1} \\
&\le Cr^n + Cr^{n+1} + \cdots + Cr^m \\
&= \frac{Cr^n - Cr^{m+1}}{1-r} < \varepsilon
\end{aligned}
$$

thus $\{x_n\}$ is a Cauchy sequence.

4. Ans:

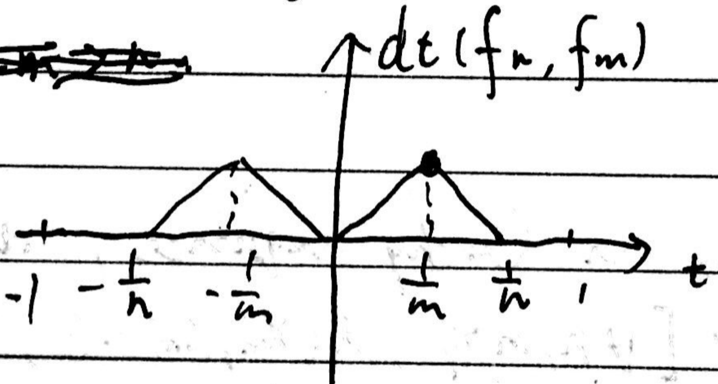
According to function definition of $f_n(t) \in C[-1,1]$,

For $m > n$:



Then we can get

~~Assume m > n~~



Therefore, $d_\infty(f_n, f_m) = f_m(\frac{1}{m}) - f_n(\frac{1}{m})$

$$= \frac{1}{2} - \frac{n}{2m} \quad \text{for } m > n$$

$f_n(t)$ is not a Cauchy sequence in this metrix space. The counterexample is $m = 2n$. $d_\infty(f_n, f_m) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. It does not satisfy $\forall \varepsilon > 0, \exists N \in \mathbb{N} [\forall n, m > N, d(f_m, f_n) < \varepsilon, f_m, f_n \in C[-1,1]]$

J. Ans:

If $x=0$, $f_n(x)=0$ for $\forall n$

if $x=1$, $f_n(x)=1$ for $\forall n$

If $0<x<1$, $f_n(x)=x^n \to 0$ as $n\to\infty$

So the sequence $\{f_n(x)\}$ coverges for $x\in[0,1]$.

To prove sequence $\{f_n(x)\}$ does not converge uniformly, we can prove its contrapositive : $\exists\, \varepsilon>0$, $\forall N\in\mathbb{N}$, $\exists\, n>N$, $\exists\, t\in X$, $d(f_n(t), f(t))\geq\varepsilon$

Assume $\varepsilon=\frac{1}{4}$, $t_n=(\frac{1}{2})^{\frac{1}{n}}$

so $f_n(t_n)=((\frac{1}{2})^{\frac{1}{n}})^n=\frac{1}{2}$

$d_Y(f_n(t_n), f(t))=\frac{1}{2}-0=\frac{1}{2}\geq\frac{1}{4}$

Therefore the sequence $\{f_n\}$ coverges for $x\in[0,1]$, but doesn't coverge uniformly.

6. Ans:

(a) $f(x_n) = x_n - \frac{x_n^2-2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$

For $\forall a \in A = [\sqrt{2}, 2]$, $f(a) = \frac{a}{2} + \frac{1}{a}$

$f(a)$ increase in $[\sqrt{2}, 2]$, then $\sqrt{2} \le f(a) \le \frac{3}{2}$

So $f(A) \subseteq A$

For contraction, assume $x, y \in A$

$$d(f(x), f(y)) = f(x) - f(y)$$
$$= \frac{x-y}{2} + \frac{1}{x} - \frac{1}{y}$$
$$= \frac{x-y}{2} + \frac{y-x}{xy}$$
$$= \left(\frac{1}{2} - \frac{1}{xy}\right) d(x, y)$$

$\frac{1}{2} - \frac{1}{xy} < \frac{1}{2}$, so we can find a $\gamma < 1$ such that $d(f(x), f(y)) \le \gamma d(x, y)$ for $\forall x, y \in A$

Proof: $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \ge 2\sqrt{\frac{x_n}{2} \cdot \frac{1}{x_n}} = \sqrt{2}$

$$x_{n+1} - \sqrt{2} = \frac{1}{2}\left(\frac{x_n}{2} + \frac{1}{x_n} - 2\sqrt{2}\right)$$
$$= \frac{1}{2x_n}(x_n^2 + 2 - 2\sqrt{2}x_n)$$
$$= \frac{1}{2x_n}(x_n - \sqrt{2})^2$$
$$\le \frac{1}{2\sqrt{2}}(2-\sqrt{2})(x_n - \sqrt{2})$$
$$< \frac{1}{4}(x_n - \sqrt{2})$$

$x_n \to \sqrt{2}$ as $n \to \infty$

(b) $d(f(x^*), f(x_{n}))\le r\,d(x^*, x_n)$

$d(x^*, x_{n+1}) \le r\,d(x^*, x_n)$

$|x_{n+1}-\sqrt{2}| \le r\,|x_n-\sqrt{2}|$

let $r=\frac{1}{4}<\frac{1}{2}$

$|x_{n+1}-\sqrt{2}| \le \frac{1}{4}|x_n-\sqrt{2}|$

$\le \frac{1}{4^n}|x_1-\sqrt{2}|$

when $n\ge 5$, $\frac{1}{4^n}<10^{-3}$

thus, $r=\frac{1}{4}$, $n\ge 5$

(c) $x_{n+1} = x_n - \frac{x_n^2-a}{2x_n}$

~~c(2 x_{n+1} ;~~

~~b(2 x_n~~

~~ε/f a-b~~

```
int main() {
    x = x_{n+1} ;
    y = x_n ;
    ε = x - y ;
    for (n = 0 ; ; n++) {
        if (ε < 10^-3) {
            return n ;
        }
    }
}
```