Assignment 9

Due Monday 12/06/19

Reading:

- Required: Course Notes Chap 5,6.1-6.3,8.1-8.2

Problems:

1. (MMA: 6.2.8) (5 pts) Show that if λ^* is an eigenvalue of A, then $\lambda^* + r$ is an eigenvalue of $A + rI$, and that A and $A + rI$ have the same eigenvectors.

2. (MMA: 6.2.10) (5 pts) Suppose that A^{-1} exists; prove the following statements. If λ is an eigenvalue of A then $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^{-1}. The eigenvectors of A are the same as the eigenvectors of A^{-1}.

3. (MMA: 6.2.12) (5 pts) Show that the eigenvalues of a projection matrix P are either 1 or 0.

4. (EF: 3.4.2) (5 pts each) Let $(V, \|\cdot\|)$ be a complete normed vector space (i.e., a Banach space) and let A be a compact convex subset of V. For any $\underline{v} \in V$, consider the optimization problem

$$\inf_{\underline{u} \in A} \|\underline{u} - \underline{v}\|.$$

   (a) Show that any local minimum value is also global minimum value.

   (b) Show that the set of points equal to the minimum value is convex.

   (c) Does it have a unique minimum point $\underline{u}^* \in V$? Hint: Consider $V = \mathbb{R}^2$ with $\|\cdot\|_1$.

5. (EF: 4.6.2) (5 pts each) Let $V = \mathbb{R}^m$ be the standard inner-product space with subspaces $A, B \subseteq V$ whose intersection is $C = A \cap B$. Let the matrices P_A, P_B, P_C define orthogonal projections onto A, B, C. Starting from $\underline{v}_0 \in V$, the alternating projection algorithm generates the sequence

$$\underline{v}_{n+1} = \begin{cases} P_A \underline{v}_n & \text{if } n \text{ even} \\ P_B \underline{v}_n & \text{if } n \text{ odd.} \end{cases}$$

   (a) Suppose $m = 2$, $A = \text{span}\{\underline{a}\}$, and $B = \text{span}\{\underline{b}\}$, where $\underline{a} = (1/\sqrt{5}, 2/\sqrt{5})^T$ and $\underline{b} = (1, 0)^T$. Starting from $\underline{v}_0 = 50\underline{b} = (50, 0)^T$, compute $\underline{v}_n$ for $n = 1, 2, 3, 4$.

   (b) What do you observe about $\underline{v}_n$? Compare $\underline{v}_n$ to $\underline{v}_{n-1}$, $\underline{v}_{n-2}$, and $\underline{v}_{n-3}$. How are they related?

   (c) For all $n \geq 0$, prove that, for some $\alpha_{n+1} \in \mathbb{R}$, we have $\underline{v}_{n+1} = \alpha_{n+1}\underline{a}$ if n is even and $\underline{v}_{n+1} = \alpha_{n+1}\underline{b}$ if n is odd. Assuming that $\underline{v}_0 = \alpha_0 \underline{b}$, find a recursive formula for α_n. Hint: Try using the formulas $P_A = \underline{a}\,\underline{a}^T / \|\underline{a}\|^2$ and $P_B = \underline{b}\,\underline{b}^T / \|\underline{b}\|^2$.

(d) For the general case, assume that $\dim(C) = d$ and let the columns of $\tilde{C} = [\underline{c}_1, \ldots, \underline{c}_d]$ be an orthonormal basis for C. Similarly, assume that $\dim(A) = d + k$ and let the columns of $\tilde{A} = [\underline{a}_1, \ldots, \underline{a}_k]$ be orthonormal vectors such that the columns of $[\tilde{C}\ \tilde{A}]$ form an orthonormal basis for A. Likewise, assume that $\dim(B) = d + \ell$ and let $\tilde{B} = [\underline{b}_1, \ldots, \underline{b}_\ell]$ be orthonormal vectors such that the columns of $[\tilde{C}\ \tilde{B}]$ form an orthonormal basis for B. Based on this, we can assume $V = \mathrm{span}(A, B) = \mathbb{R}^{d+k+\ell}$. Using orthonormality, derive simplified expressions for P_A, P_B, P_C, and $P_A P_B$ in terms $\tilde{A}, \tilde{B}, \tilde{C}$. What object should we analyze to prove convergence in this case? What is the rate of convergence?

6. (EF: 5.3.4) (5 pts each) Let $V = \mathbb{R}^n$ be the standard inner product space and let $G \in \mathbb{R}^{n \times n}$ be a positive-definite matrix. Consider the optimization problem $\min_{\underline{v} \in V} \underline{v}^T G \underline{v} + \underline{c}^T \underline{v}$.

   (a) Use the eigenvalue decomposition of G to write $\underline{v}^T G \underline{v}$ as $\|A\underline{v}\|^2$ for some matrix A. Give an expression for A in terms of the eigenvalue decomposition of G.

   (b) Use the fact that $\underline{v}^T G \underline{v} = \|A\underline{v}\|^2$ to show that $\underline{v}^T G \underline{v} + \underline{c}^T \underline{v}$ is a convex function of $\underline{v}$.

   (c) For this optimization problem, there exists a translation vector $\underline{v}_0$ such that

   $$\underline{v}^T G \underline{v} + \underline{c}^T \underline{v} = (\underline{v} - \underline{v}_0)^T G (\underline{v} - \underline{v}_0) + d,$$

   where $d \in \mathbb{R}$. Use matrix algebra to find this translation vector and then use the translated form to identify the unique minimizer and the minimum value it achieves.

Practice Problems (do not hand in):

1. (MMA: 4.2.15) Show that $\|A\|_F^2 = \mathrm{tr}(A^H A)$.

2. (MMA: 3.15.30) Using the dual approximation theorem, solve the finite dimensional problem

$$\begin{aligned} &\text{minimize } \mathbf{x}^H Q \mathbf{x} \\ &\text{subject to } A\mathbf{x} = \mathbf{b}, \end{aligned}$$

   where $\mathbf{x} \in \mathbb{C}^n$, Q is a positive-definite symmetric matrix, and A is an $m \times n$ matrix with $m < n$.

3. (MMA: 4.11.46) Show that

$$B^{-1} = A^{-1} - B^{-1}(B - A)A^{-1}.$$