Assignment 5

Due Monday 10/18/19

Reading:

- Required: Course Notes 3.5-3.7,6.3.1
- Recommended: LADR Ch. 6

Problems:

1. (EF: 3.4.4) (5 pts each) Let $U = \mathbb{R}^3$ and $V = \mathbb{R}^4$ be vector spaces and $T \in L(U, V)$ be a linear transform mapping U to V. Suppose T maps $\underline{u}_1 = (1, 1, 0)^T$ to $\underline{v}_1 = (1, 1, 1, 1)^T$, $\underline{u}_2 = (0, 2, 1)^T$ to $\underline{v}_2 = (1, 0, 0, 1)^T$, and $\underline{u}_3 = (0, 1, 0)^T$ to $\underline{v}_3 = (1, 2, 3, 4)^T$.

   (a) Where will T map $\underline{u}_0 = (1, -1, -1)^T$?

   (b) Find a matrix representation $F \in \mathbb{R}^{4\times 3}$ of T such that $T\underline{u} = F\underline{u}$ for all $\underline{u} \in U$.

2. (LADR: 3.B.10) (5 pts) Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $Tv_1, \ldots, Tv_n$ spans the range of T.

3. (EF: 3.7.1) (5 pts each) Let $V = \mathbb{R}^4$ and $W = \mathbb{R}^3$ be vector spaces and $T : V \to W$ be a linear transformation. Since any linear transformation is defined completely by how it maps any set of basis vectors, we define T via

$$\begin{aligned} T\underline{e}_1 &= (1, 1, 0) \\ T\underline{e}_2 &= (0, 1, 1) \\ T\underline{e}_3 &= (1, 0, -1) \\ T\underline{e}_4 &= (2, 1, -1) \end{aligned}$$

   (a) Using the standard basis for V and W, express T as a 3 by 4 matrix.

   (b) Find a basis for the range of T.

   (c) Find a basis for the nullspace of T.

4. (EF: 3.4.1) (5 pts each) Let $(X, \|\cdot\|)$ be a normed vector space. Prove the following properties:

   (a) The induced distance $d(\underline{x}_1, \underline{x}_2) = \|\underline{x}_1 - \underline{x}_2\|$ is a metric.

   (b) $|\,\|\underline{x}_2\| - \|\underline{x}_1\|\,| \leq \|\underline{x}_2 - \underline{x}_1\|$ (Hint: try treating $\|\underline{x}_1\| \gtrless \|\underline{x}_2\|$ cases separately)

   (c) The norm $\|\cdot\| \colon X \to \mathbb{R}$ is a continuous function from the metric space (X, d), where $d(\underline{x}_1, \underline{x}_2) = \|\underline{x}_1 - \underline{x}_2\|$ is the induced metric, to the real numbers.

5. (EF: 5.3.1) (5 pts each) Let V and W be a normed vector spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$. Recall that the set, $L(V,W)$, of linear transforms from V to W forms a vector space when equipped with "vector addition" $(T+U)\underline{v} \triangleq T\underline{v} + U\underline{v}$ and "scalar multiplication" $(sT)\underline{v} \triangleq s(T\underline{v})$. For this setup, the *induced operator norm* is defined by

$$\|T\|_{V,W} \triangleq \sup_{\underline{v}\in V:\|\underline{v}\|_V=1} \|T\underline{v}\|_W.$$

   (a) Show that $\|T\|_{V,W}$ is a valid norm for the vector space of linear transforms, $L(V,W)$.

   (b) An operator norm $\|\cdot\|$ on $L(V,V)$ is called *submultiplicative* if, for all linear transforms $T\colon V\to V$ and $U\colon V\to V$, we have $\|TU\| \le \|T\|\|U\|$. Show that the induced operator norm $\|\cdot\|_{V,V}$ is submultiplicative.

6. (MMA: 4.2.11) (5 pts) Show that if $\|\cdot\|$ is a matrix norm satisfying the submultiplicative property and F is a matrix with $\|F\| < 1$, then $I-F$ is non-singular. Hint: If $I-F$ is singular, there is a vector $\mathbf{x}$ such that $(I-F)\mathbf{x} = \mathbf{0}$.

Practice Problems (do not hand in):

1. (EF: 3.5.1) Let V be a vector space over $\mathbb{R}$ that is equipped with a metric $d\colon V\times V\to\mathbb{R}$ satisfying: (i) $d(\underline{x},\underline{y}) = d(\underline{x}+\underline{z},\underline{y}+\underline{z})$ for all $\underline{x},\underline{y},\underline{z}\in V$ (shift invariance) and (ii) $d(s\underline{x},s\underline{y}) = |s|d(\underline{x},\underline{y})$ for all $s\in\mathbb{R}$ and $\underline{x},\underline{y}\in V$ (absolute scaling).

   (a) Show that $\|\underline{x}\| \triangleq d(\underline{x},\underline{0})$ defines a norm on V.

   (b) For a sequence $\underline{x}_n \in V$, prove that $\|\underline{x}_n - \underline{x}\|$ converges to 0 in the metric space of real numbers if and only if $\underline{x}_n$ converges to $\underline{x}$ in the metric space (X,d).

2. (TOP: 2.9.8) This problem outlines a proof that the Euclidean distance d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ and $c\in\mathbb{R}$, define

$$\mathbf{x}+\mathbf{y} = (x_1+y_1,\ldots,x_n+y_n) \qquad c\mathbf{x} = (cx_1,\ldots,cx_n),$$
$$\mathbf{x}\cdot\mathbf{y} = x_1y_1+\cdots+x_ny_n \qquad \|\mathbf{x}\| = (\mathbf{x}\cdot\mathbf{x})^{1/2}.$$

   (a) Show that $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{z})$.

   (b) Show that $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\|\mathbf{y}\|$. [Hint: If $\mathbf{x},\mathbf{y}\neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x}\pm b\mathbf{y}\| \ge 0$.]

   (c) Show that $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$. [Hint: Compute $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})$ and apply previous result.]

   (d) Verify that the Euclidean distance $d(\mathbf{x},\mathbf{y})$ is a metric.