Practice Midterm 1

September 17, 2019

"I have adhered to the Duke Community Standard in completing this assignment."

Signature:

Name:

The following list of mathematical statements may be useful.

Definition. A *metric* on a set X is a function $d\colon X \times X \to \mathbb{R}$ that satisfies

1. $d(x,y) \geq 0 \quad \forall x,y \in X$ where equality holds if and only if $x = y$ (non-negativity)
2. $d(x,y) = d(y,x) \quad \forall x,y \in X$ (symmetry)
3. $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x,y,z \in X.$ (triangle inequality)

Definition. For a metric space (X,d), a subset $A \subset X$ is *open* if, for every $x_0 \in A$, there is a $\delta > 0$ such that $B(x_0,\delta) \subseteq A$, where $B(x_0,\delta) \triangleq \{x \in X \mid d(x,x_0) < \delta\}$ is the *open ball* with radius ϵ centered at x_0.

Definition. Let (X,d_X) and (Y,d_Y) be metric spaces. A function $f : X \to Y$ is *continuous* if, for all $x_0 \in X$ and any $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in X$ and $d_X(x,x_0) < \delta$, then $d_Y(f(x),f(x_0)) < \epsilon$.

Definition. For a metric space (X,d), the sequence $x_1, x_2, \ldots \in X$ *converges* to $x \in X$ if, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, we have $d(x_n,x) < \epsilon$.

Theorem. Let $f : X \to Y$ be a function between metric spaces. The function f is *continuous* at $x \in X$ if and only if, for every convergent sequence $x_n \to x$, the sequence $f(x_n)$ converges to $f(x)$.

Definition. Let A be a subset of a metric space (X,d) and $f : X \to X$ be a function. The function f is a *contraction* on A if $f(A) \subseteq A$ and $d\left(f(x),f(y)\right) \leq \gamma d(x,y)$ for all $x,y \in A$ and some $\gamma \in [0,1)$.

Theorem. Let (X,d) be a complete metric space and f be a contraction on a closed subset $A \subseteq X$. Then, f has a unique fixed point x^* in A such that $f(x^*) = x^*$ and the sequence $x_{n+1} = f(x_n)$ converges to x^* starting from any point $x_1 \in A$. Moreover, x_n satisfies the error bounds $d(x^*,x_n) \leq \gamma^{n-1} d(x_1,x_2)$ and $d(x^*,x_{n+1}) \leq d(x_n,x_{n+1})\gamma/(1-\gamma)$.

Euclid's Algorithm. For $a_1, a_2 \in \mathbb{N}$ with $a_1 > a_2$, define $a_{n+1} = a_{n-1} \bmod a_n$ and $n^* = \min\{n \in \mathbb{N} \mid a_n = 0\}$. Then, $\gcd(a_1,a_2) = a_{n^*-1}$.

Problems:

1. True or False:

   (a) **2.5 pt** – If a set is not open, then it is closed.

   (b) **2.5 pt** – The conditional $P \to \neg P$ is a contradiction.

   (c) **2.5 pt** – For sets $A, B \subseteq X$, we have $X - (A \cup B) = (X - A) \cap (X - B)$.

   (d) **2.5 pt** – It always holds that $\neg\big(\forall x,\ \exists y,\ P(x, y)\big) \Rightarrow \forall y,\ \exists x,\ \neg P(x, y)$.

2. Short answer questions:

   (a) **2.5 pt** – Given a conditional statement of the form $P \to Q$, what is the name of the related statement $\neg Q \to \neg P$?

   (b) **2.5 pt** – If the statement $P \to Q$ is a tautology, what is the name of the meta-statement that characterizes P and Q?

   (c) **2.5 pt** – Is the following statement a tautology: $(P \wedge Q) \to P$?

   (d) **2.5 pt** – For metric spaces X, Y and $x \in X$, let $f : X \to Y$ be a function where $\lim_{n\to\infty} f(x_n) = f(x)$ for all sequences $x_n \in X$ satisfying $x_n \to x$. What property does f possess?

   (e) **2.5 pt** – Let $R(x, y)$ be an expression with free variables x and y. Let U and V be collections of possible values of x and y, respectively. Negate the statement: $\exists x \in U,\ \forall y \in V,\ R(x, y)$.

   (f) **2.5 pt** – Let (X, d) be the standard metric space of real numbers and $Y = [-10, -1] \cup [1, 10]$. Is (Y, d) a compact metric space?

   (g) **2.5 pt** – Negate the statement, "Every horse that is black and flies is named midnight moon".

   (h) **2.5 pt** – Suppose f and g are continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Is their composition $g \circ f$, defined pointwise by $(g \circ f)(x) = g(f(x))$, necessarily a continuous function?

3. Let $f : X \to Y$, $A, B \subseteq X$, and $C, D \subseteq Y$. Recall also that $f(A) \triangleq \{f(x) | x \in A\}$ and $f^{-1}(C) \triangleq \{x \in X | f(x) \in C\}$.

   In this problem, proofs should only use the rules of logic and the following statements:

$$
\begin{aligned}
f(x) \in C &\Leftrightarrow x \in f^{-1}(C) \\
x \in A &\Rightarrow f(x) \in f(A) \\
x \in A \cup B &\Leftrightarrow (x \in A) \vee (x \in B) \\
x \in A \cap B &\Leftrightarrow (x \in A) \wedge (x \in B) \\
A = B &\Leftrightarrow \forall x, (x \in A) \leftrightarrow (x \in B)
\end{aligned}
$$

   (a) **5 pt** – Prove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ (i.e., f^{-1} preserves unions).

   (b) **5 pt** – Unfortunately, $f(x) \in f(A) \to x \in A$ is not true in general. Give a counterexample.

4. Your friend claims that the following recursive program computes the greatest common divisor of any two positive integers.

```
int gcd(int a,int b)
{
  if (a==b) { return a; }
  if (a>b) { return gcd(b,a-b); }
  else { return gcd(a,b-a); }
}
```

   (a) **5 pt** – Use this program to write three statements that should be verified to prove that this program computes $\gcd(a, b)$ correctly for $a, b \in \mathbb{N}$, assuming it terminates.

   (b) **5 pt** – Prove the statements listed in part (a). Write a proof that this program terminates and computes $\gcd(a, b)$ correctly as long as $a, b \in \mathbb{N}$.

   (c) **5 pt** – Unlike Euclid's algorithm, this algorithm does not use division. While that seems like an improvement, can you think of a reason why Euclid's algorithm might be preferred?

5. In a metric space (X, d), a set $A \subseteq X$ is called open if, for every $x_0 \in A$, there is a $\delta > 0$ such that $B(x_0, \delta) \subseteq A$. Use this definition to prove:

   (a) **5 pt** – The intersection of any two open sets is open.

   (b) **5 pt** – The intersection of a finite number of open sets is open.

   (c) **5 pt** – The union of any two open sets is open.

   (d) **5 pt** – If $\{A_t\}_{t\in T}$ is an arbitrary collection of open sets, then the union $U = \cup_{t\in T} A_t$ is open.

6. Consider finding the real positive solution of the equation $x^3 - ax^2 - b = 0$ for some $a, b \in \mathbb{R}_{>0}$. Note: Descartes' rule of signs shows there is exactly one real positive root.

   (a) **5 pt** – Show that any solution of this equation must also satisfy $x = f(x) \triangleq a + \frac{b}{x^2}$.

   (b) **7.5 pt** – For $a = 4$ and $b = 32$, show that $f(x)$ is a contraction on $A = [4.5, 8]$ and compute a value for the contraction coefficient γ.

   (c) **5 pt** – For $a = 4$ and $b = 32$, prove that the sequence $x_{n+1} = f(x_n)$ converges starting from $x_1 \in A$? If so, what condition must its limit satisfy?

   (d) **2.5 pt** – For $a = 4$ and $b = 32$, what happens to the sequence $x_{n+1} = f(x_n)$ starting from $x_1 = 3$? Why?

7. Let $\mathbb{R}$ be the field of real numbers equipped with absolute distance as a metric. Consider the set $X = B[a,b]$ of bounded functions mapping $[a,b]$ to $\mathbb{R}$ equipped with the metric

$$d_\infty(f,g) = \sup_{t\in[a,b]} |f(t) - g(t)|.$$

A sequence of functions $f_n(t)$ converges *pointwise* if $f_n(t) \to f(t)$ for all $t \in [a,b]$. *Uniform convergence* also requires that, for all $\epsilon > 0$, there is a single N such that $|f_n(t) - f(t)| < \epsilon$ for all $n > N$ and all $t \in [a,b]$.

(a) **5 pt** – Show that convergence $f_n \to f$ in the metric space (X, d_∞) implies uniform convergence.

(b) **5 pt** – Let $W \subseteq X$ be the subset of continuous functions and let $g_n \in W$ be a sequence of functions that converges uniformly to g. Show that g is continuous.

[Hint: Apply the triangle inequality to $d_\infty(g(x), g(y))$ along the path $g(x), g_n(x), g_n(y), g(y)$.]

(c) **5 pt** – Let $x_n \to x$ be a sequence in $\mathbb{R}$ which converges. Use the above results to show that

$$\lim_{n\to\infty}\lim_{m\to\infty} g_n(x_m) = \lim_{m\to\infty}\lim_{n\to\infty} g_n(x_m).$$