Assignment 4

Due Wednesday 10/9/19

Reading Assignment:

- Required: Course Notes 3.1-3.4
- Recommended: LADR Ch. 1, Ch. 2, Ch. 3ABC
- Supplemental: MMA 2.1-2.2

Problems:

1. (LA: 1.6.1) (5 pts) Let
$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}.$$
Find a row-reduced echelon matrix R which is row-equivalent to A and an invertible 3×3 matrix P such that $R = PA$.

2. (LADR: 1.C.1) (5 pts each) For each of the folowing subsets, determine whether it is a subspace of $\mathbf{F}^3$.
   (a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;
   (b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;
   (c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$;
   (d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$;

3. (LADR: 1.C.7) (5 pts) Give an example of a non-empty subset U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

4. (LADR: 2.A.1) (5 pts) Suppose the list v_1, v_2, v_3, v_4 spans V. Prove that the list
$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$
also spans V.

5. (LADR: 2.A.5) (5 pts each) Show that:
   (a) If we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $(1+i, 1-i)$ is linearly independent.
   (b) If we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $(1+i, 1-i)$ is linearly dependent.

6. (LA: 2.2.9) (5 pts) Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{\underline{0}\}$. Prove that for each vector α in V there are unique vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

   Note: If $S_1, S_2, \ldots, S_k$ are subsets of a vector space V, the set of all sums

   $$\alpha_1 + \alpha_2 + \cdots + \alpha_k$$

   or vectors α_i in S_i is called the sum of the subsets $S_1, S_2, \ldots, S_k$ and is denoted by

   $$S_1 + S_2 + \cdots + S_k$$

   of by

   $$\sum_{i=1}^{k} S_i.$$

7. (LADR: 2.B.3) (5 pts each)

   (a) Let U be the subspace of $\mathbf{R}^5$ defined by

   $$U = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2, x_3 = 7x_4\right\}.$$

   Find a basis for U.

   (b) Extend the basis from (a) to a basis for $\mathbf{R}^5$

   (c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

8. (MMA: 2.11.55) (5 pts) Let X and Y be vector spaces over the same set of scalars. Let $LT[X, Y]$ denote the set of all linear transformations (i.e., linear functions) from X to Y. Thus, the transformations L and M in $LT[X, Y]$ satisfy $L(ax + z) = aL(x) + L(z)$ for all scalars a and vectors $x, z \in X$. Define an addition operator between L and M as

   $$(L + M)(x) = L(x) + M(x),$$

   for all $x \in X$. Define scalar multiplication by

   $$(aL)(x) = a(L(x)).$$

   Show that $LT[X, Y]$ is a vector space.

Practice Problems (do not hand in):

1. (EF: 2.2.1) Let X, Y be metric spaces and $f : X \to Y$ be a continuous function. Prove that, if $A \subseteq X$ is a compact subset, then $f(A) \subseteq Y$ is a compact subset.

   [Hint: You may assume a continuous function is uniformly continuous on a compact set.]

2. (LA: 1.6.6) Suppose A is a 2×1 matrix and that B is a 1×2 matrix. Prove that $C = AB$ is not invertible.

3. (EF: 3.3.1) Let V be the vector space $\mathbb{C}^3$ over the *real* numbers. How many vectors are in any basis of V?

4. (LADR: 2.C.11) Suppose that U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

5. (LADR: 2.C.12) Suppose that U and W are both 5-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

6. (LA: 2.2.5) Let F be a field and let n be a positive integer ($n \geq 2$). Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A are subspaces of V?

   (a) all invertible A

   (b) all non-invertible A

   (c) all A such that $AB = BA$, where B is some fixed matrix in V

   (d) all A such that $A^2 = A$

7. (LADR: 2.A.15) Prove that $V = \mathbf{F}^\infty$ is infinite dimensional.