Assignment 5

Due Friday 10/18/19

Reading:

- Required: Course Notes 3.5-3.7,6.3.1
- Recommended: LADR Ch. 6

Problems:

1. (EF: 3.4.4) (5 pts each) Let $U = \mathbb{R}^3$ and $V = \mathbb{R}^4$ be vector spaces and $T \in L(U, V)$ be a linear transform mapping U to V. Suppose T maps $\underline{u}_1 = (1,1,0)^T$ to $\underline{v}_1 = (1,1,1,1)^T$, $\underline{u}_2 = (0,2,1)^T$ to $\underline{v}_2 = (1,0,0,1)^T$, and $\underline{u}_3 = (0,1,0)^T$ to $\underline{v}_3 = (1,2,3,4)^T$.

   (a) Where will T map $\underline{u}_0 = (1,-1,-1)^T$?

   Solution: Since $\underline{u}_0 = (1,-1,-1)^T = (1,1,0)^T - (0,2,1)^T$, it follows that $T\underline{u}_0 = \underline{v}_1 - \underline{v}_2 = (0,1,1,0)^T$.

   (b) Find a matrix representation $F \in \mathbb{R}^{4\times 3}$ of T such that $T\underline{u} = F\underline{u}$ for all $\underline{u} \in U$.

   Solution: Since $\underline{e}_1 = (1,0,0)^T = \underline{u}_1 - \underline{u}_3$, the first column of F is given by $T\underline{e}_1 = \underline{v}_1 - \underline{v}_3 = (0,-1,-2,-3)^T$. Similarly, $\underline{e}_3 = (0,0,1)^T = \underline{u}_2 - 2\underline{u}_3$ implies that the third column of F is given by $T\underline{e}_3 = \underline{v}_2 - 2\underline{v}_3 = (-2,-3,-5,-8)^T$ Along with $\underline{e}_2 = \underline{u}_3$, this implies

$$F = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 2 & -3 \\ -2 & 3 & -5 \\ -3 & 4 & -8 \end{bmatrix}.$$

2. (LADR: 3.B.10) (5 pts) Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that the list $Tv_1, \ldots, Tv_n$ spans the range of T.

   Solution: By definition, for any vector $\underline{w}$ in the range of T, there is a vector $\underline{v} \in V$ satisfying $T\underline{v} = \underline{w}$. Since $v_1, \ldots, v_n$ spans V, for any $\underline{v} \in V$, there are also scalars $s_1, \ldots, s_n$ such that

$$\underline{v} = \sum_{i=1}^{n} s_i \underline{v}_i.$$

   This implies that

$$\underline{w} = T\underline{v} = \sum_{i=1}^{n} s_i T\underline{v}_i.$$

   Thus, $Tv_1, \ldots, Tv_n$ spans the range of T.

3. (EF: 3.7.1) (5 pts each) Let $V = \mathbb{R}^4$ and $W = \mathbb{R}^3$ be vector spaces and $T : V \to W$ be a linear transformation. Since any linear transformation is defined completely by how it maps any set of basis vectors, we define T via

$$\begin{aligned} T\underline{e}_1 &= (1,1,0) \\ T\underline{e}_2 &= (0,1,1) \\ T\underline{e}_3 &= (1,0,-1) \\ T\underline{e}_4 &= (2,1,-1) \end{aligned}$$

(a) Using the standard basis for V and W, express T as a 3 by 4 matrix.

Solution: The matrix representation of T is given by

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}.$$

(b) Find a basis for the range of T.

Solution: The range of T is equivalent to the column space A. Since elementary column operations do not affect the range of A, we first put it into column reduced echelon form

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}.$$

Then, it is clear that the first two columns form a basis for the range of T.

(c) Find a basis for the nullspace of T.

Solution: The nullspace of T is equivalent to the solution set of $A\underline{s} = \underline{0}$. Since elementary row operations do not affect the nullspace of A, we first put it into row reduced echelon form

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

After dropping the row of zeros (i.e., a meaningless equation), we notice that the first two rows can be written as $\widetilde{A} = [IP]$ with

$$P = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}.$$

Therefore, $\widetilde{B} = [-P^H \ I]$ satisfies $A\widetilde{B}^H$ equals a zero matrix and the columns of

$$\widetilde{B}^H = \begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the nullspace of A (and T).

4. (EF: 3.4.1) (5 pts each) Let $(X, \|\cdot\|)$ be a normed vector space. Prove the following properties:

   (a) The induced distance $d(\underline{x}_1, \underline{x}_2) = \|\underline{x}_1 - \underline{x}_2\|$ is a metric.
   Solution: The induced distance is non-negative because $\|\underline{x}\| \geq 0$ with equality iff $\underline{x} = \underline{0}$. It is symmetric because the vector addition is commutative and the norm is scale invariant (e.g., choose $s = -1$), so we have $\|\underline{x}_1 - \underline{x}_2\| = \|\underline{x}_2 - \underline{x}_1\|$. It satisfies the triangle inequality because the triangle inequality property of the norm implies that

$$\|\underline{x}_1 - \underline{x}_3\| = \|\underline{x}_1 - \underline{x}_2 + \underline{x}_2 - \underline{x}_3\| \leq \|\underline{x}_1 - \underline{x}_2\| + \|\underline{x}_2 - \underline{x}_3\|.$$

   (b) $|\,\|\underline{x}_2\| - \|\underline{x}_1\|\,| \leq \|\underline{x}_2 - \underline{x}_1\|$ (Hint: try treating $\|\underline{x}_1\| \gtrless \|\underline{x}_2\|$ cases separately)
   Solution: When $\|\underline{x}_2\| \geq \|\underline{x}_1\|$, this follows from

$$\|\underline{x}_2\| = \|\underline{x}_2 - \underline{x}_1 + \underline{x}_1\| \leq \|\underline{x}_2 - \underline{x}_1\| + \|\underline{x}_1\|.$$

   When $\|\underline{x}_1\| \geq \|\underline{x}_2\|$, this follows from

$$\|\underline{x}_1\| = \|\underline{x}_1 - \underline{x}_2 + \underline{x}_2\| \leq \|\underline{x}_1 - \underline{x}_2\| + \|\underline{x}_2\|.$$

   (c) The norm $\|\cdot\|\colon X \to \mathbb{R}$ is a continuous function from the metric space (X, d), where $d(\underline{x}_1, \underline{x}_2) = \|\underline{x}_1 - \underline{x}_2\|$ is the induced metric, to the real numbers.
   Solution: The previous result shows that

$$|\,\|\underline{x}_2\| - \|\underline{x}_1\|\,| \leq \|\underline{x}_2 - \underline{x}_1\| = d(\underline{x}_1, \underline{x}_2).$$

   Therefore, to prove continuity, we let $\delta = \epsilon$ and observe that $d(\underline{x}_1, \underline{x}_2) < \epsilon$ implies $|\,\|\underline{x}_2\| - \|\underline{x}_1\|\,| < \epsilon$.

5. (EF: 5.3.1) (5 pts each) Let V and W be a normed vector spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$. Recall that the set, $L(V, W)$, of linear transforms from V to W forms a vector space when equipped with "vector addition" $(T + U)\underline{v} \triangleq T\underline{v} + U\underline{v}$ and "scalar multiplication" $(sT)\underline{v} \triangleq s(T\underline{v})$. For this setup, the *induced operator norm* is defined by

$$\|T\|_{V,W} \triangleq \sup_{\underline{v} \in V : \|\underline{v}\|_V = 1} \|T\underline{v}\|_W.$$

   (a) Show that $\|T\|_{V,W}$ is a valid norm for the vector space of linear transforms, $L(V, W)$.
   Solution: To do this, we must check the three conditions. First, if T is the zero-transform, then $\|T\|_{V,W} = 0$. If T does not equal the zero-transform, then there is some $\underline{v} \in V$ such that $T\underline{v} \neq \underline{0}$ and

$$\sup_{\underline{v} \in V : \|\underline{v}\|_V = 1} \|T\underline{v}\|_W \geq \left\| T \frac{\underline{v}}{\|\underline{v}\|} \right\| > 0$$

because $\|\underline{v}\| < \infty$ and $T\underline{v} \neq \underline{0}$ implies $\|T\underline{v}\|_W > 0$. The second property is inherited easily from the vector norm because

$$\|sT\|_{V,W} = \sup_{\underline{v}\in V:\|\underline{v}\|_V=1} \|sT\underline{v}\|_W = \sup_{\underline{v}\in V:\|\underline{v}\|_V=1} |s|\|T\underline{v}\|_W = |s|\|T\|_{V,W}.$$

The third property follows from

$$\begin{aligned}\|T+U\|_{V,W} &= \sup_{\underline{v}\in V:\|\underline{v}\|_V=1} \|(T+U)\underline{v}\|_W \\ &= \sup_{\underline{v}\in V:\|\underline{v}\|_V=1} \|T\underline{v}+U\underline{v}\|_W \\ &\leq \sup_{\underline{v}\in V:\|\underline{v}\|_V=1} \big(\|T\underline{v}\|_W + \|U\underline{v}\|_W\big) \\ &\leq \|T\|_{V,W} + \|U\|_{V,W},\end{aligned}$$

where the first inequality follows from the triangle inequality of the vector norm and the second inequality follows from optimizing both terms over a single $\underline{v}$ rather than separately.

(b) An operator norm $\|\cdot\|$ on $L(V,V)$ is called *submultiplicative* if, for all linear transforms $T\colon V\to V$ and $U\colon V\to V$, we have $\|TU\| \leq \|T\|\|U\|$. Show that the induced operator norm $\|\cdot\|_{V,V}$ is submultiplicative.

Solution: First, for any $T\in L(V,W)$, one can show that $\|T\underline{v}\|_W \leq \|T\|_{V,W}\|\underline{v}\|_V$ using

$$\|T\|_{V,W} = \sup_{\underline{v}\in V:\|v\|_V=1} \|T\underline{v}\|_W \geq \left\| T\frac{\underline{v}}{\|\underline{v}\|_V}\right\|_W, \forall \underline{v}\in V.$$

Rearranging terms shows that $\|T\underline{v}\|_W \leq \|T\|_{V,W}\|\underline{v}\|_V$. Next, we observe that

$$\|TU\underline{v}\|_V \leq \|T\|\|U\underline{v}\|_V \leq \|T\|\|U\|\|\underline{v}\|_V.$$

The proof is completed by taking the supremum over all $\underline{v}$ satisfying $\|\underline{v}\|=1$.

6. (MMA: 4.2.11) (5 pts) Show that if $\|\cdot\|$ is a matrix norm satisfying the submultiplicative property and F is a matrix with $\|F\|<1$, then $I-F$ is non-singular. Hint: If $I-F$ is singular, there is a vector $\mathbf{x}$ such that $(I-F)\mathbf{x}=\mathbf{0}$.

Solution: If one assumes the matrix norm is an induced operator norm given by some vector norm, then the following simple proof applies.

We prove the contrapositive: if $I-F$ singular, then $\|F\|\geq 1$. If $I-F$ is singular, then there exists a non-zero vector $\mathbf{x}$ such that $(I-F)\mathbf{x}=\mathbf{0}$. For this non-zero vector, we have $\mathbf{x} = I\mathbf{x} = F\mathbf{x}$. So, by the induced operator norm property, we have $\|\mathbf{x}\| = \|F\mathbf{x}\| \leq \|F\|\|\mathbf{x}\|$. Since $\|\mathbf{x}\|\neq 0$, we conclude that $\|F\|\geq 1$. This shows that, if $\|\cdot\|$ is an induced operator norm and F is a matrix with $\|F\|<1$, then $I-F$ is non-singular.

Since there is no mention of vector norms, a stricter reading requires the following proof.

If $\|F\| < 1$, then the Neumann expansion implies $(I-F)^{-1}$ exists and is given by

$$(I-F)^{-1} = \sum_{i=0}^{\infty} F^i.$$

Now, suppose that $I-F$ is singular. Then, there is a non-zero vector $\mathbf{x}$ satisfying $(I-F)\mathbf{x} = \mathbf{0}$ and $(I-F)^{-1}(I-F)\mathbf{x} = \mathbf{0}$. But, $(I-F)^{-1}(I-F)\mathbf{x} = \mathbf{x}$ by definition. This contradiction implies that $I-F$ is non-singular.

Practice Problems (do not hand in):

1. (EF: 3.5.1) Let V be a vector space over $\mathbb{R}$ that is equipped with a metric $d\colon V \times V \to \mathbb{R}$ satisfying: (i) $d(\underline{x}, \underline{y}) = d(\underline{x}+\underline{z}, \underline{y}+\underline{z})$ for all $\underline{x}, \underline{y}, \underline{z} \in V$ (shift invariance) and (ii) $d(s\underline{x}, s\underline{y}) = |s| d(\underline{x}, \underline{y})$ for all $s \in \mathbb{R}$ and $\underline{x}, \underline{y} \in V$ (absolute scaling).

   (a) Show that $\|\underline{x}\| \triangleq d(\underline{x}, \underline{0})$ defines a norm on V.

   Solution: First, we observe that $\|\underline{x}\| = d(\underline{x}, \underline{0}) \geq 0$ with equality iff $\underline{x} = \underline{0}$ because $d(\underline{x}, \underline{y}) \geq 0$ with equality iff $\underline{x} = \underline{y}$. Next, we observe that

   $$\|s\underline{x}\| = d(s\underline{x}, s\underline{0}) = |s| d(\underline{x}, \underline{0}) = |s| \|x\|.$$

   Finally, we note that $\|\underline{x}+\underline{y}\| = d(\underline{x}+\underline{y}, \underline{0}) \leq d(\underline{x}, \underline{0}) + d(\underline{y}, \underline{0}) = \|\underline{x}\| + \|\underline{y}\|$ for all $\underline{x}, \underline{y} \in V$. Thus, $\|\cdot\|$ satisfies all axioms required by a norm.

   (b) For a sequence $\underline{x}_n \in V$, prove that $\|\underline{x}_n - \underline{x}\|$ converges to 0 in the metric space of real numbers if and only if $\underline{x}_n$ converges to $\underline{x}$ in the metric space (X, d).

   Solution: If $\|\underline{x}_n - \underline{x}\|$ converges to 0 in the metric space of real numbers, then

   $$d(\underline{x}_n, \underline{x}) = \|\underline{x}_n - \underline{x}\| \to 0$$

   and, for any $\epsilon > 0$, there is an N such that, for all $n > N$, $d(\underline{x}_n, \underline{x}) < \epsilon$. Thus, $\underline{x}_n$ converges to $\underline{x}$ in (X, d).

   If $\underline{x}_n$ converges to $\underline{x}$ in (X, d), then for any $\epsilon > 0$, there is an N such that, for all $n > N$, $d(\underline{x}_n, \underline{x}) < \epsilon$. Thus, $\|\underline{x}_n - \underline{x}\| = d(\underline{x}_n, \underline{x})$ converges to 0 in the metric space of real numbers.

2. (TOP: 2.9.8) This problem outlines a proof that the Euclidean distance d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

   $$\mathbf{x}+\mathbf{y} = (x_1+y_1, \ldots, x_n+y_n) \qquad c\mathbf{x} = (cx_1, \ldots, cx_n),$$
   $$\mathbf{x}\cdot\mathbf{y} = x_1 y_1 + \cdots + x_n y_n \qquad \|\mathbf{x}\| = (\mathbf{x}\cdot\mathbf{x})^{1/2}.$$

   (a) Show that $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) = (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{z})$.

Solution: We show this inequality by first applying the definitions above, and then using simple properties of the real numbers,

$$\begin{aligned}
\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) &= x_1(y_1 + z_1) + \cdots + x_n(y_n + z_n) \\
&= x_1y_1 + x_1z_1 + \cdots + x_ny_n + x_nz_n \\
&= x_1y_1 + \cdots + x_ny_n + x_1z_1 + \cdots + x_nz_n \\
&= (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z}).
\end{aligned}$$

(b) Show that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$. [Hint: If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$.]

Solution: If $\|\mathbf{x}\| = 0$ or $\|\mathbf{y}\| = 0$, then the result is obvious. Assume that $\|\mathbf{x}\| \neq 0$ and $\|\mathbf{y}\| \neq 0$. Using the hint, we have

$$\begin{aligned}
0 &\leq \|a\mathbf{x} + b\mathbf{y}\|^2 \\
&= (a\mathbf{x} + b\mathbf{y}) \cdot (a\mathbf{x} + b\mathbf{y}) \\
&= a^2\|\mathbf{x}\|^2 + 2ab(\mathbf{x} \cdot \mathbf{y}) + b^2\|\mathbf{y}\|^2.
\end{aligned}$$

Thus, for $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, we get

$$\mathbf{x} \cdot \mathbf{y} \geq \frac{a^2\|\mathbf{x}\|^2 + b^2\|\mathbf{y}\|^2}{-2ab} = -\|\mathbf{x}\|\|\mathbf{y}\|.$$

Similarly, we have

$$0 \leq \|a\mathbf{x} - b\mathbf{y}\|^2 = a^2\|\mathbf{x}\|^2 - 2ab(\mathbf{x} \cdot \mathbf{y}) + b^2\|\mathbf{y}\|^2.$$

For $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, we obtain

$$\mathbf{x} \cdot \mathbf{y} \leq \frac{a^2\|\mathbf{x}\|^2 + b^2\|\mathbf{y}\|^2}{2ab} = \|\mathbf{x}\|\|\mathbf{y}\|.$$

Putting these two results together, we get $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.

(c) Show that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. [Hint: Compute $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$ and apply previous result.]

Solution: We apply the hint and get

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\
&= \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\
&\leq \|\mathbf{x}\|^2 + |\mathbf{x} \cdot \mathbf{y}| + |\mathbf{y} \cdot \mathbf{x}| + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.
\end{aligned}$$

Taking the square root of each side, we conclude that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

(d) Verify that the Euclidean distance $d(\mathbf{x}, \mathbf{y})$ is a metric.

Solution: The distance between two vectors $\mathbf{x}$ and $\mathbf{y}$ is given by $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$. By the properties of the norm, we have that $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \geq 0$ with equality if and only if $\mathbf{x} = \mathbf{y}$. Furthermore, $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\| = d(\mathbf{y}, \mathbf{x})$. Finally, for vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, we have

$$\begin{aligned} d(\mathbf{x}, \mathbf{z}) &= \|\mathbf{x} - \mathbf{z}\| = \|\mathbf{x} - \mathbf{y} + \mathbf{y} - \mathbf{z}\| \\ &\leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}). \end{aligned}$$

The triangle inequality for the norm necessarily implies the triangle inequality for the induced metric.