Assigment 6.

Q1. Ans: $\underline{u}, \underline{v}, \underline{w} \in V$ and $s \in F$

Let $\langle \underline{u} | \underline{w} \rangle_a$ and $\langle \underline{u} | \underline{w} \rangle_b$ be two inner products, we will show that $\langle \underline{u} | \underline{w} \rangle = \langle \underline{u} | \underline{w} \rangle_a + \langle \underline{u} | \underline{w} \rangle_b$ satisfy properties of an inner product

(a) $\langle \underline{u}+\underline{v} | \underline{w} \rangle = \langle \underline{u}+\underline{v} | \underline{w} \rangle_a + \langle \underline{u}+\underline{v} | \underline{w} \rangle_b = \langle \underline{u} | \underline{w} \rangle_a + \langle \underline{v} | \underline{w} \rangle_a + \langle \underline{u} | \underline{w} \rangle_b + \langle \underline{v} | \underline{w} \rangle_b$

$= \langle \underline{u} | \underline{w} \rangle_a + \langle \underline{u} | \underline{w} \rangle_b + \langle \underline{v} | \underline{w} \rangle_a + \langle \underline{v} | \underline{w} \rangle_b$

$= \langle \underline{u} | \underline{w} \rangle + \langle \underline{v} | \underline{w} \rangle$

(b) $\langle s\underline{u} | \underline{w} \rangle : \langle s\underline{u} | \underline{w} \rangle_a + \langle s\underline{u} | \underline{w} \rangle_b = s\langle \underline{u} | \underline{w} \rangle_a + s\langle \underline{u} | \underline{w} \rangle_b$

$= s(\langle \underline{u} | \underline{w} \rangle_a + \langle \underline{u} | \underline{w} \rangle_b)$

$= s\langle \underline{u} | \underline{w} \rangle$

(c) $\langle \underline{u} | \underline{w} \rangle = \langle \underline{u} | \underline{w} \rangle_a + \langle \underline{u} | \underline{w} \rangle_b = \overline{\langle \underline{w} | \underline{u} \rangle_a} + \overline{\langle \underline{w} | \underline{u} \rangle_b} = \overline{\langle \underline{w} | \underline{u} \rangle_a + \langle \underline{w} | \underline{u} \rangle_b}$

$= \overline{\langle \underline{w} | \underline{u} \rangle}$

(d) If $u \neq 0$, $\langle \underline{u} | \underline{u} \rangle_a > 0$ and $\langle \underline{u} | \underline{u} \rangle_b > 0$ therefore $\langle \underline{u} | \underline{u} \rangle = \langle \underline{u} | \underline{u} \rangle_a + \langle \underline{u} | \underline{u} \rangle_b > 0$

If $u = 0$, $\langle \underline{u} | \underline{u} \rangle_a = 0$ and $\langle \underline{u} | \underline{u} \rangle_b = 0$ therefore $\langle \underline{u} | \underline{u} \rangle = \langle \underline{u} | \underline{u} \rangle_a + \langle \underline{u} | \underline{u} \rangle_b = 0$

The difference of inner product is NOT an inner product. Counterexample: assume that $\langle \underline{u} | \underline{v} \rangle_a = \langle \underline{u} | \underline{v} \rangle_b$ and $\langle \underline{u} | \underline{v} \rangle = \langle \underline{u} | \underline{v} \rangle_a - \langle \underline{u} | \underline{v} \rangle_b$, let $u \neq 0$, then $\langle \underline{u} | \underline{u} \rangle = \langle \underline{u} | \underline{u} \rangle_a - \langle \underline{u} | \underline{u} \rangle_b = 0$ which contradicts with (d)

The positive multiple of an inner product IS an inner product. Let $\langle \underline{u} | \underline{w} \rangle_i$ be inner products, $\langle \underline{u} | \underline{w} \rangle = \langle \underline{u} | \underline{v} \rangle_1 + \langle \underline{u} | \underline{v} \rangle_2 + \dots + \langle \underline{u} | \underline{v} \rangle_n = \sum_{i=1}^{n} \langle \underline{u} | \underline{v} \rangle_i$

(a) $\langle \underline{u}+\underline{v} | \underline{w} \rangle = \sum_{i=1}^{n} \langle \underline{u}+\underline{v} | \underline{w} \rangle_i = \sum_{i=1}^{n} (\langle \underline{u} | \underline{w} \rangle_i + \langle \underline{v} | \underline{w} \rangle_i) = \langle \underline{u} | \underline{w} \rangle + \langle \underline{v} | \underline{w} \rangle$

(b) $\langle s\underline{u} | \underline{v} \rangle = \sum_{i=1}^{n} \langle s\underline{u} | \underline{v} \rangle_i = \sum_{i=1}^{n} s\langle \underline{u} | \underline{v} \rangle_i = s\sum_{i=1}^{n} \langle \underline{u} | \underline{v} \rangle_i = s(u|\underline{v})$

(c) $\langle \underline{u} | \underline{v} \rangle = \sum_{i=1}^{n} \langle \underline{u} | \underline{v} \rangle_i = \sum_{i=1}^{n} \overline{\langle \underline{v} | \underline{u} \rangle_i} = \overline{\langle \underline{v} | \underline{u} \rangle}$

(d) If $\underline{u} = 0$, $\langle \underline{u} | \underline{u} \rangle = \sum_{i=1}^{n} \langle \underline{u} | \underline{u} \rangle_i = 0$

If $\underline{u} \neq 0$, $\langle \underline{u} | \underline{u} \rangle = \sum_{i=1}^{n} \langle \underline{u} | \underline{u} \rangle_i > 0$

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2 Ans:

$$\|\alpha+\beta\|^2+\|\alpha-\beta\|^2 = \langle\alpha+\beta|\alpha+\beta\rangle + \langle\alpha-\beta|\alpha-\beta\rangle$$

$$= (\langle\alpha|\alpha\rangle + \langle\alpha|\beta\rangle + \langle\beta|\alpha\rangle + \langle\beta|\beta\rangle) + (\langle\alpha|\alpha\rangle - \langle\alpha|\beta\rangle - \langle\beta|\alpha\rangle + \langle\beta|\beta\rangle)$$

Since V is a real or complex vector space with an inner product

$$= 2\langle\alpha|\alpha\rangle + 2\langle\beta|\beta\rangle$$

$$= 2\|\alpha\|^2 + 2\|\beta\|^2$$

$$= RHS$$

3. (a) Apply the Gram-Schmidt:

$\underline{u}_1 = (1,0,1)$ $\quad \frac{u_1}{\|u_1\|} = (\frac{\sqrt2}{2}, 0, \frac{\sqrt2}{2})$

$$\underline{u}_2 = (0,1,1) - \frac{\langle(1,0,1)|(0,1,1)\rangle}{2}(1,0,1)$$

$$= (0,1,1) - \frac{1}{2}(1,0,1)$$

$$= (-\frac{1}{2}, 1, \frac{1}{2}) \qquad \frac{u_2}{\|u_2\|} = (-\frac{\sqrt6}{6}, \frac{\sqrt6}{3}, \frac{\sqrt6}{6})$$

$$u_3 = (1,3,3) - \frac{\langle(1,3,3)|(1,0,1)\rangle}{2}(1,0,1) - \frac{\langle(1,3,3)|(-\frac12,1,\frac12)\rangle}{\frac{3}{2}}(-\frac12,1,\frac12)$$

$$= (\frac13, \frac13, -\frac13) \qquad \frac{u_3}{\|u_3\|} = (\frac{\sqrt3}{3}, \frac{\sqrt3}{3}, -\frac{\sqrt3}{3})$$

(b) $\underline{V} = \alpha_1\underline{u}_1 + \alpha_2\underline{u}_2 + \alpha_3\underline{u}_3$

$$\Rightarrow \begin{cases} \frac{\sqrt2}{2}\alpha_1 - \frac{\sqrt6}{6}\alpha_2 + \frac{\sqrt3}{3}\alpha_3 = 1 \\ \frac{\sqrt6}{3}\alpha_2 + \frac{\sqrt3}{3}\alpha_3 = 1 \\ \frac{\sqrt2}{2}\alpha_1 + \frac{\sqrt6}{6}\alpha_2 - \frac{\sqrt3}{3}\alpha_3 = 2 \end{cases}$$

The coordinate vector $[\underline{v}]_B = (\frac{3}{2}\sqrt2, \frac{\sqrt6}{2}, 0)$

4. Ans:

(a) Since $\langle f|h\rangle = \int_{-1}^{1} f(t)h(t)\,dt$ and $f(t) = \sum_{j=0}^{\infty} f_j t^j$ and $h(t) = \sum_{i=0}^{\infty} h_i t^i$

$$\langle f|h\rangle = \int_{-1}^{1}(\textstyle\sum_{j=0}^{\infty} f_j t^j)(\sum_{i=0}^{\infty} h_i t^i)\,dt$$

Rearrange so that we can get $\langle f|h\rangle = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} h_i \int_{-1}^{1} t^i t^j\,dt\, f_j$

Therefore we can know $g_{ij} = \int_{-1}^{1} t^i t^j\,dt = \langle t^i|t^j\rangle = \left[\frac{t^{i+j+1}}{i+j}\right]_{-1}^{1}$

$$= \begin{cases} 0, & i+j = \text{even} \\ \frac{2}{i+j+1}, & i+1 = \text{odd} \end{cases}$$

(b) $\underline{v}_1 = 1$, $\underline{v}_2 = t$, $\underline{v}_3 = t^2$, $\underline{v}_4 = t^3$

Apply the Gram - Schmidt process to $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$

$$\underline{w}_1 = \underline{v}_1 = 1$$

$$\underline{w}_2 = \underline{v}_2 - \frac{\langle \underline{v}_2 | \underline{w}_1 \rangle}{\|\underline{w}_1\|^2} \underline{w}_1$$

$$= t - 0$$

$$= t$$

$$\underline{w}_3 = \underline{v}_3 - \left(\frac{\langle \underline{v}_3 | \underline{w}_1 \rangle}{\|w_1\|^2} w_1 + \frac{\langle \underline{v}_3 | \underline{w}_2 \rangle}{\|\underline{w}_2\|^2} w_2 \right)$$

$$= t^2 - \left(\frac{\frac{2}{3}}{2} + 0 \right)$$

$$= t^2 - \frac{1}{3}$$

$$\underline{w}_4 = v_4 - \left(\frac{\langle v_4 | \underline{w}_1 \rangle}{\|w_1\|^2} w_1 + \frac{\langle v_4 | \underline{w}_2 \rangle}{\|w_2\|^2} w_2 + \frac{\langle \underline{v}_4 | \underline{w}_3 \rangle}{\|\underline{w}_3\|^2} w_3 \right)$$

$$= t^3 - \left(0 + \frac{\frac{2}{5}}{\frac{2}{3}} t + 0 \right)$$

$$= t^3 - \frac{3}{5} t$$

(c) $$S_1 = \frac{\langle f(t) | w_1 \rangle}{\|w_1\|^3} \underline{w}_1 = \frac{\int_0^1 t \cdot 1}{2} \underline{w}_1 = \frac{\frac{1}{2}}{2} \underline{w}_1 = \frac{1}{4}$$

$$S_4 = \frac{\langle f(t) | w_4 \rangle}{\|w_4\|^2} \underline{w}_4 = 0$$

$$S_2 = \frac{\langle f(t) | w_2 \rangle}{\|w_2\|^2} \underline{w}_2 = \frac{\frac{1}{3}}{\frac{2}{3}} \underline{w}_2 = \frac{1}{2} t$$

$$S_3 = \frac{\langle f(t) | w_3 \rangle}{\|w_3\|^2} \underline{w}_3 = \frac{\frac{1}{12}}{\frac{8}{45}} \underline{w}_3 = \frac{15}{32} t^2 - \frac{5}{32}$$

(d) To prove $\left(\int_{-1}^1 f(t) h(t)\, dt \right)^2 \le \left(\int_{-1}^1 |f(t)|^2 dt \right) \left(\int_{-1}^1 |h(t)|^2 dt \right)$

We can prove : $|\langle \underline{u} | \underline{v} \rangle|^2 \le \langle \underline{u} | \underline{u} \rangle \langle \underline{v} | \underline{v} \rangle = \|\underline{u}\|^2 \|\underline{v}\|^2$

If $\underline{v} = 0$, then both sides of inequality equal to 0. Thus we can assume that $\underline{v} \neq 0$

Orthogonal decomposition : $\underline{u} = \frac{\langle \underline{u} | \underline{v} \rangle}{\|\underline{v}\|^2} \underline{v} + \underline{w}$ and by the Pythagorean theorem

$$\|u\|^2 = \left\| \frac{\langle \underline{u} | \underline{v} \rangle}{\|\underline{v}\|^2} \right\|^2 + \|w\|^2 = \frac{|\langle \underline{u} | \underline{v} \rangle|^2}{\|\underline{v}\|^2} + \|w\|^2 \ge \frac{|\langle \underline{u} | \underline{v} \rangle|^2}{\|\underline{v}\|^2}$$

Therefore $|\langle u | v \rangle|^2 \le \|u\|^2 \|v\|^2$

Note that we get equality if and only if $w = 0$ which means u and v are linearly dependent.

J. Ans:

① Prove $(S^{\perp})^{\perp} \supset \text{Span}(S)$

For $\forall \alpha \in \text{span}(S)$, $\forall \beta \in S^{\perp}$,

$$\beta \in S^{\perp},$$

$$\Rightarrow \langle \beta | \gamma \rangle = 0, \quad \forall \gamma \in S,$$

$$\Rightarrow \langle \beta | \alpha \rangle = 0, \quad \forall \alpha \in \text{Span}(S)$$

$$\Rightarrow \alpha \in (S^{\perp})^{\perp}$$

It follows that $(S^{\perp})^{\perp} \supset \text{Span}(S)$

② prove that if V is finite dimensional, then $(S^{\perp})^{\perp} = \text{span}(S)$

Suppose $\dim(V) = n$ and $\dim(\text{span}(S)) = k$, where $k \leq n$.

We know that $\dim((\text{span}(S))^{\perp}) = n - k$

$$\Rightarrow \dim\left(((\text{span}(S))^{\perp})^{\perp}\right) = n - (n-k) = k$$

It shows that $\dim(((\text{span}(S))^{\perp})^{\perp}) = \dim(\text{span}(S))$

$\because$ we know that $(S^{\perp})^{\perp} \supset \text{span}(S)$ from ①

then we can conclude that $(S^{\perp})^{\perp} = \text{span}(S)$

6. a) $f(\underline{v}+\underline{u}) \approx f(\underline{v}) + \langle \underline{u} | \nabla f(\underline{v}) \rangle$

We need to minimize $\langle \underline{u} | \nabla f(\underline{v}) \rangle$

$|\langle \underline{u} | \nabla f(\underline{v}) \rangle| \le \|\underline{u}\| \|\nabla f(\underline{v})\|$ with equality iff $\underline{u}=0$, $\nabla f(\underline{v})=0$ or $\underline{u} = s \nabla f(\underline{v})$

$\because \underline{u} \ne 0$ when $\nabla f(\underline{v}) = 0$, $\langle \underline{u} | \nabla f(\underline{v}) \rangle = 0$

In order to get negative value of $\langle \underline{u} | \nabla f(\underline{v}) \rangle$

we set $s = -1$ to get unit-norm $\underline{u} = \dfrac{-\nabla f(\underline{v})}{\|\nabla f(\underline{v})\|}$

b) $\delta > 0$, to show $f(\underline{v} + \delta \underline{u}) < f(\underline{v})$

$$\lim_{h \to 0} \frac{|f(\underline{v}+\delta\underline{u}) - f(\underline{v}) - \langle \delta \underline{u} | \nabla f(\underline{v}) \rangle|}{\|\delta \underline{u}\|} = 0$$

the limit implies that $\forall \varepsilon > 0$, $\exists \delta_0$, $\forall \delta \in (0, \delta_0)$

$$\frac{f(\underline{v}+\delta\underline{u}) - f(\underline{v}) - \langle \delta \underline{u} | \nabla f(\underline{v}) \rangle}{\|\delta \underline{u}\|} - 0 < \varepsilon$$

$$\begin{aligned} f(\underline{v}+\delta\underline{u}) &< f(\underline{v}) + \langle \delta \underline{u} | \nabla f(\underline{v}) \rangle + \varepsilon \|\delta \underline{u}\| \\ &< f(\underline{v}) - \frac{\delta \langle \nabla f(\underline{v}) | \nabla f(\underline{v}) \rangle}{\|\nabla f(\underline{v})\|} + \varepsilon \delta \|\nabla f(\underline{v})\| \\ &< f(\underline{v}) - \delta \|\nabla f(\underline{v})\| + \varepsilon \delta \|\nabla f(\underline{v})\| \\ &< f(\underline{v}) \underbrace{- \delta \|\nabla f(\underline{v})\| (1-\varepsilon)}_{<0} \end{aligned}$$

$\therefore$ when $u = -\dfrac{\nabla f(\underline{v})}{\|\nabla f(\underline{v})\|}$ $\quad 0 < \varepsilon < 1 \quad \nabla f(\underline{v}) \ne 0$

we can get $f(\underline{v}+\delta\underline{u}) < f(\underline{v})$

c) $\because \delta_i = \arg\min_{\delta \ge 0} f(\underline{v}_i - \delta \nabla f(\underline{v}_i))$

$\therefore f(\underline{v}_i - \delta \nabla f(\underline{v}_i)) \le f(\underline{v}_i)$

$\therefore f(\underline{v}_{i+1}) = f(\underline{v}_i - \delta \nabla f(\underline{v}_i)) \le f(\underline{v}_i)$

when $\nabla f(v_i) = 0$ $\quad \underline{v}_{i+1} = v_i$

$\therefore f(v_{i+1}) = f(v_i)$

d) First show that $f(\underline{V}_{i+1}) \le f(\underline{V}_i) + \nabla f(\underline{V}_i)(\underline{V}_{i+1} - \underline{V}_i) + \frac{L}{2}\|\underline{V}_{i+1} - \underline{V}_i\|^2$

$$f(x_1) - f(x_2) = \int_{x_2}^{x_1} f'(x)\,dx = f(x)\Big|_{x_1}^{x_2} = f(x_1) - f(x_2)$$

$$\phi(t) = f(\underline{V}_{i+1} + t(\underline{V}_{i+1} - \underline{V}_i))$$

$$\int_0^1 \phi'(t)\,dt = \phi(t)\Big|_0^1 = f(V_{i+1}) - f(V_i)$$

$$\int_0^1 \phi'(0)\,dt = \nabla f(\underline{V}_i)\Big|_0^1 = \nabla f(\underline{V}_i)(\underline{V}_{i+1} - \underline{V}_i)$$

$$\therefore f(\underline{V}_{i+1}) - f(\underline{V}_i) - \nabla f(\underline{V}_i)(\underline{V}_{i+1} - \underline{V}_i)$$

$$= \int_0^1 \phi'(t)\,dt - \int_0^1 \phi'(0)\,dt$$

$$= \int_0^1 (\phi'(t)\,dt - \phi'(0))\,dt$$

$$= \int_0^1 \left(\nabla f(\underline{V}_i + t(\underline{V}_{i+1} - \underline{V}_i))(\underline{V}_{i+1} - \underline{V}_i) - \nabla f(\underline{V}_i)(\underline{V}_{i+1} - \underline{V}_i)\right) dt$$

$$= \int_0^1 [\nabla f(\underline{V}_i + t(\underline{V}_{i+1} - \underline{V}_i)) - \nabla f(\underline{V}_i)][\underline{V}_{i+1} - \underline{V}_i]\,dt$$

$$\le \int_0^1 \|\nabla f(\underline{V}_i + t(\underline{V}_{i+1} - V_i)) - \nabla f(\underline{V}_i)\|\,\|\underline{V}_{i+1} - \underline{V}_i\|\,dt$$

$$\le \int_0^1 |t|\,\|\underline{V}_{i+1} - \underline{V}_i\|^2\,dt = \frac{L}{2}\|V_{i+1} - V_i\|^2$$

$$f(\underline{V}_1) - f(\underline{V}_t) = \sum_{i=1}^{t} f(\underline{V}_i) - f(\underline{V}_{i+1})$$

$$= f(\underline{V}_i) - f(\underline{V}_{i+1})$$

$$\ge \sum_{i=1}^{t} \frac{1}{2L}\|\nabla f(\underline{V}_i)\|^2 + \frac{1}{2L}\|\nabla f(\underline{V}_{i+1})\|^2$$

$$\nabla f(\underline{V}_{i+1}) = f(\underline{V}_i)$$

$$\nabla f(\underline{V}_i) \Rightarrow 0$$