Assignment 2

Due Wednesday 9/11/19

Reading:

- Required: Course Notes 1.4-1.5
- Recommended: PAF 3.1-5.3

Problems:

1. (PAF: 1.5.2) (1 pt each) Suppose that the possible values of x and y are all cars. Let $L(x, y)$ = "x is as fast as y," let $M(x, y)$ = "x is as expensive as y" and $N(x, y)$ = "x is as old as y." Translate the following statements into words.

   (a) $(\exists x)(\forall y)L(x, y)$.
   Solution: There exists a car that is as fast as any other car.

   (b) $(\forall x)(\exists y)M(x, y)$.
   Solution: For every car, there is a car that is as expensive as it.

   (c) $(\exists y)(\forall x)[L(x, y) \vee N(x, y)]$.
   Solution: There is a car such that every car is as fast or as old as it.

   (d) $(\forall y)(\exists x)[\neg M(x, y) \to L(x, y)]$.
   Solution: For every car, there exists a car such that if the latter car is not as expensive as the former then the latter car is as fast as the former car.

2. (PAF: 1.5.4) (1 pt each) Suppose that the possible values of p and q are all fruit. Let $A(p, q)$ = "p tastes better than q," let $B(p, q)$ = "p is riper than q" and $C(p, q)$ = "p is the same species as q." Translate the following into symbols.

   (a) There is a fruit such that all fruit taste better than it.
   Solution: $(\exists q)(\forall p)A(p, q)$.

   (b) For every fruit, there is a fruit that is riper than it.
   Solution: $(\forall q)(\exists p)B(p, q)$.

   (c) There is a fruit such that all fruit taste better than it and are not riper than it.
   Solution: $(\exists q)(\forall p)[A(p, q) \wedge \neg B(p, q)]$.

   (d) For every fruit, there is a fruit of the same species that does not taste better than it.
   Solution: $(\forall q)(\exists p)[C(p, q) \wedge \neg A(p, q)]$.

3. (PAF: 1.5.6) (1 pt each) Write a negation of each statement. Do not write the word "not" applied to any of the objects being quantified (for example, do not write "Not all boys are good" for the first part).

(a) All boys are good.

Solution: There exists a boy that is bad.

(b) There are bats that weigh 50 lbs. or more.

Solution: All bats weigh less than 50 lbs.

(c) The equation $x^2 - 2x > 0$ holds for all real numbers x.

Solution: There exists a real number x such that $x^2 - 2x \leq 0$.

(d) Every parent has to change diapers.

Solution: There is a parent that does not have to change diapers.

(e) Every flying saucer is aiming to conquer some galaxy.

Solution: There exists a flying saucer that is not aiming to conquer any galaxy.

(f) There is an integer n such that n^2 is a perfect number.

Solution: The square n^2 of every integer n is not a perfect number.

(g) There is a house in Kansas such that every one who enters the house goes blind.

Solution: All houses in Kansas are such that one person who enters the house does not go blind.

(h) Every house has a door that is white.

Solution: There exists a house that does not have a door that is white.

(i) At least one person in New York City owns every book published in 1990.

Solution: Every person in New York City does not own at least one book published in 1990.

4. (PAF: 1.5.8) (5 pts) Negate the following statement: For every real number $\epsilon > 0$ there exists a positive integer k such that for all positive integers n, it is the case that $|a_n - k^2| < \epsilon$.

Solution: There exists a real number $\epsilon > 0$ such that for any positive integer k there exists a positive integer n such that $|a_n - k^2| \geq \epsilon$.

5. (PAF: 2.3.2*)(5 pts each) Prove the following by giving direct proofs for their contrapositives.

(a) Let n be an integer. If n^2 is even, then n is even.

Solution: The contrapositive of this statement is: If n is odd, then n^2 is odd. To prove this directly, we note that "n is odd" $\Leftrightarrow$ $\exists k \in \mathbb{N}, n = 2k + 1$. Thus, if n is odd, then $n = 2k + 1$ and

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2j + 1,$$

where $j = 2k^2 + 2k \in \mathbb{N}$. Thus, n^2 is odd.

(b) Let x and y be real numbers. If xy is irrational, then x is irrational or y is irrational.

Solution: The contrapositive of this statement is: If x and y are rational, then xy is rational. To prove this, we note that "x is rational" $\Leftrightarrow$ $\exists a, b \in \mathbb{Z}, x = a/b$. Thus, if x and y are rational, then $\exists a, b, c, d \in \mathbb{Z}, (x = a/b) \wedge (y = c/d)$. This implies that $xy = ac/(bd)$. Since $ac, bd \in \mathbb{Z}$, it follows that xy is rational.

Of course, we have simply verified the well-known fact that the rationals are closed under multiplications.

6. (EF: 1.4.1) (5 pts each) Let X, Y be sets and $f : X \to Y$ be a function.

   (a) Show that the relation $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$ is an equivalence relation.

   Solution: One can verify that $\sim$ inherits the reflexive, symmetric, and transitive properties from the same properties of equality $=$ on the real numbers.

   (b) Describe the set of equivalence classes $X/\sim$ for $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$.

   Solution: The equivalence class $[x]$ of any $x \in \mathbb{R}$ is given by $[x] = \{x + 2\pi k | k \in \mathbb{Z}\} \cup \{\pi - x + 2\pi k | k \in \mathbb{Z}\}$. In general, there is a natural bijection $g : f(X) \to X/\sim$ between range $f(X) = [-1, 1]$ of f and the set of equivalence classes. In this case, we have $g(y) = [\sin^{-1}(y)]$ for $y \in [-1, 1]$.

   (c) Consider any $\widetilde{f} : X/\sim \to Y$ satisfying $\widetilde{f}([x]) = f(x)$ for $x \in X$. Is $\widetilde{f}$ unique? one-to-one?

   Solution: The expression $\widetilde{f}([x])$ is really the composition of the projection $x \to [x]$ and the function $\widetilde{f}$. The function $\widetilde{f}$ is unique because, for each $x \in X$, we must assign the value $f(x)$ to the function $\widetilde{f}$ evaluated at $[x]$. Fortunately, we find that $\widetilde{f}$ is well-defined because the image of the set $[x]$ under f is the singleton set $\{f(x)\}$.

   The function $\sin(x)$ is not one-to-one on $\mathbb{R}$. But, the resulting $\widetilde{f}$ is one-to-one because each element in the range of f defines a distinct equivalence class. In fact, this provides a canonical method of constructing invertible functions from non-invertible functions.

Practice Problems (do not hand in):

1. (PAF: 1.5.3) Suppose that the possible values of x are all cows. Let $P(x) =$ "x is brown," let $Q(x) =$ "x is four years old" and $R(x) =$ "x has white spots." Translate the following statements into symbols.

   (a) There is a brown cow.

   Solution: $(\exists x)P(x)$.

   (b) All cows are four years old.

   Solution: $(\forall x)Q(x)$.

   (c) There is a brown cow with white spots.

   Solution: $(\exists x)[P(x) \wedge R(x)]$.

   (d) All four year old cows have white spots.

   Solution: $(\forall x)[Q(x) \to R(x)]$.

   (e) There exists a cow such that if it is four years old, then it has no white spots.

   Solution: $(\exists x)[Q(x) \to \neg R(x)]$.

(f) All cows are brown if and only if they are not four years old.

Solution: $(\forall x)[P(x) \leftrightarrow \neg Q(x)]$.

(g) There are no brown cows.

Solution: $(\forall x)[\neg P(x)]$.

2. (TOP: 1.2.1) The image of a function applied to a set-valued argument is defined by $f(A) \triangleq \{f(x)|x \in A\}$ and $f^{-1}(B) \triangleq \{x \in X|f(x) \in B\}$. Let $f : X \to Y$, $A \subseteq X$, and $B \subseteq Y$.

   (a) Show that $f^{-1}(f(A)) \supseteq A$ and that equality holds if f is injective.

   Solution:

   – Consider $f^{-1}(f(x))$ for any element $x \in A$.
   – It is easy to see that $B = f(x)$ is a single element in the range of f because a function is (by def.) single valued.
   – From the definition of f^{-1}, it follows that $x \in f^{-1}(B)$.
   – It follows that $x \in A \implies x \in f^{-1}(f(A))$ and that this implies $f^{-1}(f(A)) \supseteq A$.
   – If f is injective, then $f(a) = f(b) \implies a = b$; So $x = f^{-1}(f(x))$ for all $x \in A$ and $f^{-1}(f(A)) = A$.

   (b) Show that $f(f^{-1}(B)) \subseteq B$ and that equality holds if f is surjective.

   Solution:

   – Consider $f(f^{-1}(x))$ for any element $x \in B$.
   – Notice that, if x is in the range of f, then $B = f^{-1}(x)$ is not empty and $f(B) = x$.
   – If x is not in the range of f, then $B = \emptyset$ and $f(B) = \emptyset$.
   – It follows that $f(f^{-1}(B)) = \{x \in B|x \text{ in the range of } f\} \subseteq B$.
   – If f is surjective, then its range is Y; So, B is in the range of f and $f(f^{-1}(B)) = B$.

3. (PAF: 2.2.4) Let n be an integer. Find a direct proof that shows: if n is even then n^2 is even, and if n is odd then n^2 is odd.

   Solution: First, we note that "n is even" $\Leftrightarrow \exists k \in \mathbb{Z}, n = 2k$. Applying this definition shows that $n = 2k$ and $n^2 = 4k^2 = 2j$, where $j = 2k^2$. Thus, n^2 is even.

   Similarly, we note that "n is odd" $\Leftrightarrow$ $\exists k \in \mathbb{N}, n = 2k + 1$. Thus, if n is odd, then $n = 2k + 1$ and

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2j + 1,$$

   where $j = 2k^2 + 2k \in \mathbb{N}$. Thus, n^2 is odd.

4. (PAF: 2.2.5) Let n and m be integers. Assume that n and m are divisible by 3.

   (a) Find a direct proof that shows $n + m$ is divisible by 3

   Solution: If n and m are divisible by 3, then there exist $j, k \in \mathbb{Z}$ such that $n = 3j$ and $m = 3k$. This implies that $n + m = 3j + 3k = 3(j + k)$ by the distributive property of addition over multiplication. Since $j + k \in \mathbb{Z}$, we see that $n + m$ is divisible by 3.

(b) Find a direct proof that shows nm is divisible by 3

Solution: If n and m are divisible by 3, then there exist $j, k \in \mathbb{Z}$ such that $n = 3j$ and $m = 3k$. This implies that $nm = (3j)(3k) = 3(3jk)$. Since $3jk \in \mathbb{Z}$, we see that nm is divisible by 3.

5. (PAF: 2.2.6) Let a, b, c, m and n be integers. Find a direct proof to show that if $a|b$ and $a|c$, then $a|(bm + cn)$.

Solution: First, we note that $a|b$ denotes "a divides b" or $\exists k \in \mathbb{Z}, b = ak$. If $a|b$ and $a|c$, then $b = aj$ and $c = ak$ for $j, k \in \mathbb{Z}$. Thus, we can write

$$bm + cn = ajm + akn = a(jm + kn).$$

Since $jm + kn \in \mathbb{Z}$, it follows that a divides $a(jm + kn) = bm + cn$.