Assignment 3

Due Wednesday 9/18/19

Reading Assignment:

- Required: Course Notes 2.1
- Supplemental: MMA 2.1

Problems:

1. (EF: 2.1.4) (5 pts) Let $\underline{x} = (x_1, \ldots, x_n), \underline{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and consider the function ρ given by
$$\rho\left(\underline{x}, \underline{y}\right) = \max\left\{|x_1 - y_1|, \ldots, |x_n - y_n|\right\}.$$
Show that ρ is a metric.

   Solution:

   We wish to prove that ρ is a metric. We must therefore show that it fulfills the three properties of a metric. First, since $|x - y| \geq 0$ for all $x, y \in \mathbb{R}$, it follows that
$$\rho\left(\underline{x}, \underline{y}\right) = \max\left\{|x_1 - y_1|, \ldots, |x_n - y_n|\right\} \geq 0.$$
Furthermore, $\rho\left(\underline{x}, \underline{y}\right) = 0$ if and only if $|x_1 - y_1| = \cdots = |x_n - y_n| = 0$. In particular, $\rho\left(\underline{x}, \underline{y}\right) = 0$ if and only if $x_i = y_i$ for $i = 1, \ldots, n$. Thus, $\rho\left(\underline{x}, \underline{y}\right) = 0$ if and only if $\underline{x} = \underline{y}$. The second property can be seen by noticing that
$$\begin{aligned} \rho\left(\underline{x}, \underline{y}\right) &= \max\left\{|x_1 - y_1|, \ldots, |x_n - y_n|\right\} \\ &= \max\left\{|y_1 - x_1|, \ldots, |y_n - x_n|\right\} = \rho\left(\underline{y}, \underline{x}\right). \end{aligned}$$
The triangle inequality is obtained by applying the triangle inequality componentwise. For any $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n$,
$$\begin{aligned} \rho\left(\underline{x}, \underline{z}\right) &= \max\left\{|x_1 - z_1|, \ldots, |x_n - z_n|\right\} \\ &\leq \max\left\{|x_1 - y_1| + |y_1 - z_1|, \ldots, |x_n - y_n| + |y_n - z_n|\right\} \\ &\leq \max\left\{|x_1 - y_1|, \ldots, |x_n - y_n|\right\} + \max\left\{|y_1 - z_1|, \ldots, |y_n - z_n|\right\} \\ &= \rho\left(\underline{x}, \underline{y}\right) + \rho\left(\underline{y}, \underline{z}\right). \end{aligned}$$
Hence, ρ is a metric.

2. (EF: 2.1.6) (5 pts) Suppose $a \in B_d(x, \epsilon)$ with $\epsilon > 0$. Find an explicit $\delta > 0$ such that the open ball $B_d(a, \delta)$ centered at a is contained in $B_d(x, \epsilon)$.

   Solution:

Let $a \in B_d(x, \epsilon)$ be given. Then, by definition, $d(x, a) < \epsilon$. Define $\delta = \epsilon - d(x, a)$ and note that $\delta > 0$. We claim that $B_d(a, \delta)$ is a open ball centered at a which is contained in $B_d(x, \epsilon)$. For any $y \in B_d(a, \delta)$, we have

$$\begin{aligned} d(x, y) &\leq d(x, a) + d(a, y) \\ &< d(x, a) + \delta \\ &= d(x, a) + (\epsilon - d(x, a)) \\ &= \epsilon. \end{aligned}$$

That is, the distance between x and y is less than ϵ, which implies that $y \in B_d(x, \epsilon)$. Since this is true for any $y \in B_d(a, \delta)$, we conclude that $B_d(a, \delta) \subseteq B_d(x, \epsilon)$, as desired.

3. (MMA: 2.1.20) (10 pts) Show that if $\{x_n\}$ is a sequence such that $d(x_n, x_{n+1}) \leq C r^n$ for $0 \leq r < 1$ and $C \geq 0$, then $\{x_n\}$ is a Cauchy sequence.

   Solution: From the definition of a Cauchy sequence, we need, for any $\epsilon > 0$, an N such that $d(x_n, x_m) < \epsilon$ for all $m, n > N$. One can assume $m > n$ (wolog) and apply the triangle inequality recursively to get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m) \\ &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} C\, r^i \\ &\leq \sum_{i=n}^{\infty} C\, r^i \\ &= \frac{C\, r^n}{1 - r} \end{aligned}$$

   Since this quantity is independent of m and strictly decreasing to zero as $n \to \infty$, there exists an N for any $\epsilon > 0$. This is also a key step in the proof of the contraction mapping theorem.

4. (MMA: 2.1.24) (5 pts) The fact that a sequence is Cauchy depends upon the metric employed. Consider the metric space $(C[a, b], d_\infty)$ of continuous functions mapping $[a, b] \to \mathbb{R}$ with

$$d_\infty(f, g) \triangleq \max_{t \in [a,b]} |f(t) - g(t)|.$$

   Let $f_n(t) \in C[-1, 1]$ be a sequence of functions defined by

$$f_n(t) = \begin{cases} 0 & t < -1/n, \\ nt/2 + 1/2 & -1/n \leq t \leq 1/n, \\ 1 & t > 1/n. \end{cases}$$

Show that

$$d_\infty(f_n, f_m) = \frac{1}{2} - \frac{n}{2m} \quad \text{for } m > n.$$

Is $f_n(t)$ a Cauchy sequence in this metric space? Hint: See Example 2.1.16 in MMA.

Solution: Consider the expression $f_m(t) - f_n(t)$ as a function of t for any $m > n$. It is easy to verify that this expression increasing in t for $0 \le t < \frac{1}{m}$ and decreasing for $\frac{1}{m} \le t \le 1$. From the symmetry about $t = 0$, we see that

$$|f_m(t) - f_n(t)| \le f_m\left(\frac{1}{m}\right) - f_n\left(\frac{1}{m}\right) = 1 - \left(\frac{n}{2m} + \frac{1}{2}\right) = \frac{1}{2} - \frac{n}{2m}.$$

In words, all points in a Cauchy sequence must eventually become arbitrarily close to each other. By choosing $m = 2n$, we see that this doesn't happen. Thus, we will prove the sequence is not Cauchy.

Formally, we can negate the definition of a Cauchy sequence to get an outline for the proof:

$$\neg\text{“}f_n \text{ is Cauchy”} \Leftrightarrow \exists\epsilon > 0, \forall N \in \mathbb{N}, \exists m, n > N, d_\infty(f_n, f_m) \ge \epsilon.$$

Following the implied path, we choose $\epsilon = \frac{1}{4}$ and observe that, for all $N \in \mathbb{N}$, we can choose $n = N + 1$ and $m = 2n$ to see that $d_\infty(f_n, f_m) \ge \epsilon$.

5. (TOP: 2.10.6) (5 pts) Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $\{f_n(x)\}$ converges for each $x \in [0, 1]$, but that the sequence $\{f_n\}$ does not converge uniformly. Recall that uniform convergence to f on $[a, b]$ implies that, for any $\epsilon > 0$, there exists an N such that $|f_n(t) - f(t)| < \epsilon$ for all $n > N$ and all $t \in [a, b]$.

   Solution: First, we note that the sequence converges pointwise. For $x \in [0, 1)$, the limit is

   $$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \lim_{n\to\infty} e^{n \ln x} = 0,$$

   since $\ln x < 0$. Also, we have $f(1) = \lim_{n\to\infty} f_n(1) = 1$. In words, a sequence of functions converges uniformly if the value of N in the definition of convergence can chosen independently of x. But, for this sequence, convergence becomes slower and slower as x approaches 1.

   Formally, we negate the definition of uniform convergence to get an outline for the proof:

   $$\neg\text{“}f_n \to f \text{ uniformly”} \Leftrightarrow \exists\epsilon > 0, \forall N \in \mathbb{N}, \exists n > N, \exists t \in X, d_Y(f_n(t), f(t)) \ge \epsilon.$$

   For this example, we can pick $\epsilon = 1/4$. Then, we let $t_n = (1/2)^{\frac{1}{n}}$ so that $f_n(t_n) = ((1/2)^{\frac{1}{n}})^n = 1/2$ and notice that $0 < t_n < 1$. Then, for all $n \in \mathbb{N}$, we have $d_Y(f_n(t_n) - f(t_n)) = |f_n(t_n) - f(t_n)| = |1/2 - 0| \ge 1/4$. Thus, the sequence of functions does not converge uniformly.

6. (EF: 2.1.7) (5 pts each) In this problem, we will numerically approximate the positive square-root of 2 using Newton's method to find the positive root of $g(x) = x^2 - 2$. Starting from some initial estimate $x_1 \in \mathbb{R}$, this gives

   $$x_{n+1} = f(x_n) \triangleq x_n - \frac{g(x_n)}{g'(x_n)}.$$

(a) For $A = [\sqrt{2}, 2]$, show that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $f(A) \subseteq A$ and is a contraction on A for some contraction coefficient $\gamma < 1$. Prove that $x_{n+1} = f(x_n)$ converges to $\sqrt{2}$ starting from $x_1 = 2$.

Solution: A little algebra shows that

$$f(x) - f(y) = \left(x - \frac{x^2-2}{2x}\right) - \left(y - \frac{y^2-2}{2y}\right) = (x-y)\left(\frac{1}{2} - \frac{1}{xy}\right).$$

Thus, $|f(x) - f(y)| \leq \frac{1}{2}|x-y|$ as long as $x, y \geq 1$. Since $f(x)$ is increasing for $x \geq \sqrt{2}$, $f(\sqrt{2}) = \sqrt{2}$, and $f(2) = 3/2$, it follows that $f(A) \subseteq A$. Using these results, we can apply the contraction mapping theorem to see that $x_n \to x^*$ where $x^* = \sqrt{2}$ is the unique solution of $x^* = f(x^*)$ in A.

(b) Determine some $\gamma < 1$ such that $|f(x) - f(y)| \leq \gamma |x-y|$ and use this value to find an n such that $|x_{n+1} - \sqrt{2}| \leq 10^{-3}$ (i.e., error is small after n iterations).

Solution: From above, we see that the best contraction coefficient on A is $\frac{1}{4}$. To bound the error as a function of n, we write

$$d(x^*, x_{n+1}) = d\left(f(x^*), f(x_n)\right) \leq 0.25\, d(x^*, x_n) \leq 0.25^n\, d(x^*, x_1).$$

Thus, the error is upper bounded by $0.25^n(2-\sqrt{2})$ and $n = 5$ suffices to meet the error bound.

(c) Write a program that uses this method and elementary computations (e.g., no `sqrt` or `log`) to compute the square root of an arbitrary real number $a \geq 1$ with error most 10^{-3}. Hint: Since the error is strictly decreasing faster than γ^n, it can be upper bounded by $\gamma/(1-\gamma)$ times the previous step size (i.e., use the other error bound).

Solution: Repeating the steps in (a) for $g(x) = x^2 - a$, one can: (i) choose $A = [\sqrt{a}, a]$ and show that $f(A) \subseteq A$ and (ii) show that Newton's method is a contraction on A for $\gamma = 0.5$. Choosing $x_1 = a$, we observe that $z_n \triangleq (x_{n-1}^2 - a)/(2x_{n-1})$ is non-negative if $x_{n-1}^2 \geq a \geq 1$. It follows that $x_{n+1} = x_n - z_n$ decreases down to $\sqrt{a}$. By substituting $x_n = \sqrt{a} + \delta$ into the expression for z_n, one can verify that $z_n \leq \delta$ as long as $\delta \geq 0$. Thus, the following code computes $\sqrt{a}$ within the specified error tolerance.

```
% Matlab Code
function x = cmsqrt(a)
  x = a;
  z = 0.001;
  while (z>=0.001)
    z = x/2 - a/(2*x);
    x = x - z;
  end
```

```
# Python Code
def cmsqrt(a) :
    x = a
    z = 0.001
    while (z>=0.001):
        z = x/2 - a/(2*x)
        x = x - z
    return x
```

Practice Problems (do not hand in):

1. (EF: 2.1.5) Let X be a metric space with metric d. Define $\bar{d}: X \times X \to \mathbb{R}$ by

$$\bar{d}(x,y) = \min\{d(x,y), 1\}.$$

Show that $\bar{d}$ is also a metric.

Solution:

We need to show that $\bar{d}$ satisfies the three properties of a metric. First, $d(x,y) \geq 0$ implies that $\min\{d(x,y),1\} \geq 0$ for all $x, y \in X$. Moreover, if $\bar{d}(x,y) = 0$ then $\min\{d(x,y),1\} = 0$ and $d(x,y) = 0$. This, in turn, implies that $x = y$ since d is a metric. We conclude that $\bar{d}(x,y) = 0$ if and only if $x = y$. The second property is obtained by the following string of equalities

$$\bar{d}(x,y) = \min\{d(x,y),1\} = \min\{d(y,x),1\} = \bar{d}(y,x),$$

which holds for all $x, y \in X$. The triangle inequality can be derived as follows. For $x, y, z \in X$, we have $d(x,y) + d(y,z) \geq d(x,z)$. If $d(x,y)$ or $d(y,z)$ is greater than or equal to one, then $\bar{d}(x,y) + \bar{d}(y,z) \geq 1 \geq \bar{d}(x,z)$. On the other hand, if $d(x,y)$ and $d(y,z)$ are less than one, then we have

$$\bar{d}(x,y) + \bar{d}(y,z) = d(x,y) + d(y,z) \geq d(x,y) \geq \bar{d}(x,z).$$

Thus, the triangle inequality holds. This shows that $\bar{d}$ is indeed a metric.

2. (EF: 2.2.2) Consider the metric space $(C[0,1], d_\infty)$ of continuous functions mapping $[0,1] \to \mathbb{R}$ with

$$d_\infty(f,g) = \max_{t \in [a,b]} |f(t) - g(t)|.$$

Prove that the sequence $f_n(x) = \sin(n\pi x)$ does not have a subsequence which converges.

Hint: Start by showing that

$$\max_{x \in [0,1]} |f_n(x) - f_m(x)|^2 \geq \int_0^1 (f_n(x) - f_m(x))^2 dx,$$

and then compute the integral for any integers $m \neq n$.

Solution: First, we point out that $(f_n(x) - f_m(x))^2 \leq \max_{x \in [0,1]} |f_n(x) - f_m(x)|^2$ for all $x \in [0,1]$. Integrating both sides implies the expression given in the hint. For integers $m, n \geq 0$, one can compute

$$\begin{aligned}\int_0^1 (f_n(x) - f_m(x))^2 dx &= \int_0^1 (\sin(n\pi x) - \sin(m\pi x))^2 dx \\ &= \int_0^1 \left(\sin^2(n\pi x) + \sin^2(m\pi x)\right) dx - 2\int_0^1 \sin(n\pi x)\sin(m\pi x) dx \\ &= 1 - \delta_{m,n}.\end{aligned}$$

This implies that $d(f_n, f_m) = \max_{x \in [0,1]} |f_n(x) - f_m(x)| \geq 1$ for all integers $m > n \geq 0$.

Now, assume that some subsequence $\{f_{n_i}\}$ converges to f. Then, for $\epsilon = 1/4$ there must exist some N where $d(f_{n_i}, f) < 1/4$ for all $i > N$. But, we also know that $1 = d(f_{n_i}, f_{n_j}) \leq d(f_{n_i}, f) + d(f, f_{n_j})$ for any f and all $i \neq j$. This gives a contradiction because $d(f_{n_i}, f) + d(f, f_{n_j}) < 1/2$ for all $i, j > N$. Therefore, the sequence has no subsequence which converges. Since this space is complete, one may also conclude that it is not totally bounded.

3. (EF: 2.1.8) Let $\mathbb{R}$ denote the standard metric space of real numbers and $\Psi\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping that satisfies $|\Psi(x) - \Psi(y)| \leq L|x-y|$ for all $x, y \in \mathbb{R}$. Let $X_{a,b} = C[a,b]$ be the metric space of continuous functions mapping $[a,b]$ to $\mathbb{R}$, equipped with the metric

$$d_\infty(f,g) = \max_{t\in[a,b]} |f(t) - g(t)|.$$

In this problem, you will show that the differential equation,

$$f'(t) = \Psi(f(t)), \quad t \in [a,b]$$

with boundary condition $f(a) = y_a$, has a unique solution $f \in X_{a,b}$.

(a) Show that, for some $c \in (a,b]$, the mapping $T : X_{a,c} \to X_{a,c}$, defined by

$$(Tf)(x) = y_a + \int_a^x \Psi(f(t))\mathrm{d}t,$$

is a well-defined contraction on $X_{a,c}$ with contraction coefficient $\gamma \leq 1/2$.

Solution: Since $\Psi(t)$ and $f(t)$ are continuous functions on $[a,b]$, it follows that $\Psi(f(t))$ is also continuous function on $[a,b]$. The mapping T is well-defined because the integral of a continuous functions always exists and is continuous. This implies that, for any $c \in (a,b]$ and all $f \in X_{a,c}$, we have $Tf \in X_{a,c}$.

Next, we observe that

$$\begin{aligned}
d_\infty(Tf,Tg) &= \max_{x\in[a,c]} |(Tf)(x) - (Tg)(x)| \\
&= \max_{x\in[a,c]} \left| y_a + \int_a^x \Psi(f(t))\mathrm{d}t - y_a - \int_a^x \Psi(g(t))\mathrm{d}t \right| \\
&= \max_{x\in[a,c]} \left| \int_a^x \left(\Psi(f(t)) - \Psi(g(t))\right) \mathrm{d}t \right| \\
&\leq \max_{x\in[a,c]} \int_a^x |\Psi(f(t)) - \Psi(g(t))| \,\mathrm{d}t \\
&\leq \max_{x\in[a,c]} \int_a^x L|f(t) - g(t)|\mathrm{d}t \\
&\leq L(c-a) \max_{x\in[a,c]} |f(t) - g(t)| \\
&\leq L(c-a) d_\infty(f,g).
\end{aligned}$$

For $c = a + \frac{1}{2L}$, it follows that T is a contraction with coefficient $\gamma = 1/2$.

(b) Use part (a) to show that the differential equation has a unique solution $f(t)$ for $t \in [a, c]$.

Solution: Consider the sequence $f_{n+1} = Tf_n$ starting from $f_1(x) = y_a$. Since $(X_{a,c}, d_\infty)$ is a complete metric space, the contraction mapping theorem shows that this sequence converges to the unique fixed point f^* satisfying $f^* = Tf^*$. The fixed-point equation implies f^* is differentiable and differentiating both sides shows that

$$f'(x) = \Psi(f(x)).$$

Since $f^*(a) = y_a$ also follows, we see that f is a solution to the differential equation if and only if it satisfies the fixed-point equation. Thus, the uniqueness of the solution is inherited from the uniqueness of the fixed point.

(c) How can one extend the uniqueness proof to the full range $[a, b]$?

Solution: Under the conditions given, it is easy to verify that

$$(T_c f)(x) = y_c + \int_c^x \Psi(f(t))\mathrm{d}t,$$

is a well-defined contraction on $X_{c,d}$ for all $c \in [a, b)$ and $d \leq \min\{b, c + \frac{1}{2L}\}$. We note that the value $y_c = f(c)$ is given by the unique solution computed in the previous part. By induction, one can iterate this idea to extend the proven range of uniqueness by $\frac{1}{2L}$ at each step. Thus, one can show uniqueness for the whole range by applying the contraction mapping theorem at most $(b - a)/(2L)$ times.

(d) Let $f(t)$ be the water height, as a function of time, in a bucket with a hole in the bottom. One can show that the water exits through the hole at a rate proportional to $-\sqrt{f(t)}$. Assuming this changes the water height at the same rate, it follows that $f(t)$ satisfies the differential equation $f'(t) = -\sqrt{f(t)}$. For $t \in [-1, 0]$, verify that $f(t) = t^2/4$ and $f(t) = 0$ are both solutions satisfying boundary condition $y_0 = 0$. How is this possible mathematically? To what physical situations do these two solutions apply?

Solution: Clearly $f(t) = 0$ has $f'(t) = 0$ and is a solution. For the second expression, we observe that

$$f'(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{4}t^2 = -\frac{1}{2}t = -\sqrt{\frac{1}{4}t^2}.$$

Mathematically, this does not violate our previous result because $\sqrt{x}$ is not Lipschitz continuous on any interval containing 0. Physically, when we see an empty bucket at time $t = 0$, we cannot tell from the height of the water whether it just became empty (i.e., $f(t) > 0$ for $t < 0$) or has been empty for some time (i.e., $f(t) = 0$ for $t < 0$).

4. (TOP: 2.10.5) Let $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$ with metric $d(x, x') = |x - x'|$. Show that

$$x_n + y_n \to x + y$$
$$x_n - y_n \to x - y$$
$$x_n y_n \to xy,$$

and provided that each $y_n \neq 0$ and $y \neq 0$,

$$x_n/y_n \to x/y.$$

[Hint: First show that $+, -, \cdot, /$ are continuous functions from $(\mathbb{R}^2, d_1)$ to $(\mathbb{R}, |\cdot|)$.]

Solution: Here is an outline of a proof. Consider the vector space $\mathbb{R}^2$ with metric

$$d_1\left((x,y),(x',y')\right) = |x-x'| + |y-y'|.$$

If $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$, then $(x_n, y_n) \to (x,y)$ in the metric space $(\mathbb{R}^2, d_1)$. Therefore, if a function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then $f\left((x_n, y_n)\right) \to f\left((x,y)\right)$. Using this, one only needs to establish the continuity of the functions implied by addition, subtraction, multiplication, and division.

For addition, we can write

$$|(x+y) - (x_0+y_0)| \leq |x-x_0| + |y-y_0| \leq d_1\left((x,y),(x_0,y_0)\right)$$

and choose $\delta = \epsilon$ to see $+$ is continuous for all $(x,y) \in \mathbb{R}^2$. The proof of subtraction is essentially the same. For multiplication, we can write

$$|(xy)-(x_0y_0)| = |xy - x_0y + x_0y - x_0y_0| \leq |y||x-x_0| + |x_0||y-y_0| \leq (|y_0|+\delta)|x-x_0| + |x_0||y-y_0|$$

and solve $(|y_0|+\delta)\delta + |x_0|\delta < \epsilon$ for δ to see that $\cdot$ is continuous for all $(x,y) \in \mathbb{R}^2$. For division, we can write

$$\left|\frac{x}{y} - \frac{x_0}{y_0}\right| = \left|\frac{x}{y} - \frac{x_0}{y} + \frac{x_0}{y} - \frac{x_0}{y_0}\right| \leq \frac{|x-x_0|}{|y|} + |x_0|\frac{|y-y_0|}{|y_0||y|} \leq \frac{|x-x_0|}{|y_0|-\delta} + |x_0|\frac{|y-y_0|}{|y_0|(|y_0|-\delta)}$$

and solve $\frac{\delta}{|y_0|-\delta} + \frac{|x_0|\delta}{|y_0|(|y_0|-\delta)} < \epsilon$ for δ to see that $/$ is continuous for all $(x,y) \in \mathbb{R}^2$ such that $y \neq 0$. In both cases, the LHS goes to zero as $\delta \to 0$. So, we can choose, for any $\epsilon > 0$, a $\delta > 0$ small enough to satisfy the inequality.

5. (TOP: 2.7.11) Let $f : A \to B$ and $g : C \to D$ be continuous functions with respect to the topologies generated by the metrics d_A, d_B, d_C, d_D. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)\left((a,c)\right) = \left(f(a), g(c)\right).$$

A simple product metric on $A \times C$ is the metric $d_{AC}\left((a,c),(a',c')\right) \triangleq d_A(a,a') + d_C(c,c')$. Show that $f \times g$ is continuous with respect to the product metrics d_{AC} and d_{BD}, where d_{BD} is defined similarly to d_{AC}.

Solution: Since f is continuous, there exists, for all $a \in A$, a $\delta_f > 0$ such that $d_B\left(f(a), f(a')\right) < \frac{\epsilon}{2}$ for all $a' \in B_{d_A}(a, \delta_f)$. Likewise, for all $c \in C$, there exists a $\delta_g > 0$ such that $d_D\left(g(c), g(c')\right) < \frac{\epsilon}{2}$ for all $c' \in B_{d_C}(c, \delta_g)$. But, if $d_{AC}\left((a,c),(a',c')\right) \leq \delta$ where $\delta = \min(\delta_f, \delta_g)$, then we have both $d_A(a,a') \leq \delta_f$ and $d_C(c,c') \leq \delta_g$. Therefore, for all $(a,c) \in A \times C$, we can write

$$d_{BD}\left(\left(f(a), g(c)\right), \left(f(a'), g(c')\right)\right) = d_B\left(f(a), f(a')\right) + d_D\left(g(c), g(c')\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $(a', c') \in B_{d_{AC}}\left((a,c), \delta\right)$. This implies that $f \times g$ is continuous w.r.t. to this product metric.