Assignment 4.

1. Ans:

$$R = \begin{bmatrix} 1 & 0 & 0 & -\frac{7}{8} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{11}{8} \end{bmatrix} \qquad P = \begin{bmatrix} -\frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}$$

2. Ans:

(a) For determining whether it is a subspace of F^3, ~~we~~ we need to check:

① The set contains 0:

Let $(x_1, x_2, x_3) = (0, 0, 0)$, then $x_1 + 2x_2 + 3x_3$

So 0 is in the set.

② The set is closed under addition:

Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

so satisfy $x_1 + 2x_2 + 3x_3 = 0$ and $y_1 + 2y_2 + 3y_3 = 0$

$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$x_1 + y_1 + 2(x_2 + y_2) + 3(x_3 + y_3) = x_1 + 2x_2 + 3x_3 + y_1$
$+ 2y_2 + 3y_3 = 0 + 0 = 0$

Therefore, closed under addition

③ The set is closed under scalar multiplication:

Let $x = (x_1, x_2, x_3)$ in the set and $\lambda \in F$

$\lambda \cdot x = (\lambda x_1, \lambda x_2, \lambda x_3)$

$\lambda x_1 + 2\lambda x_2 + 3\lambda x_3 = \lambda(x_1 + 2x_2 + 3x_3) = 0$

Therefore it is a subset of F^3

(b) This subset doesn't contain 0

Because when $(x_1, x_2, x_3) = (0,0,0)$, $x_1 + 2x_2 + 3x_3 = 0 \neq 4$

So it is not a subspace of F^3

(c) Let $x = (1,1,0)$, $y = (0,1,1)$

they are both in the set, because $x_1 x_2 x_3 = 0$

But $x+y = (1,2,1)$, then $x_1 x_2 x_3 = 2 \neq 0$

So it is not closed under addition

Therefore it is not subspace of F^3

(d) It contains $0 = (0,0,0)$ because $0 = 5 \times 0$

Then let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

so $x_1 = 5x_3$ and $y_1 = 5y_3$

$x+y = (x_1+y_1, x_2+y_2, x_3+y_3)$

So $x_1 + y_1 = 5x_3 + 5y_3 = 5(x_3+y_3)$, then

$x+y$ is in the set

Let $x = (x_1, x_2, x_3)$, then we can get $x_1 = I x_3$

$\lambda \in F$, $\lambda x = (\lambda x_1, \lambda x_2, \lambda x_3)$

$\lambda x_1 = \lambda (I x_3) = I(\lambda x_3)$

So it is in the set

Therefore it is the subspace of F^3

3. Ans:

The example set $U = \{(x_1, x_2): x_1, x_2 \in \mathbb{Z}\}$

Let $x, y \in U$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$x + y = (x_1 + y_1, x_2 + y_2)$

$x_1 + y_1$ and $x_2 + y_2$ are integers, because $x_1, x_2, y_1, y_2 \in \mathbb{Z}$

$-x = (-x_1, -x_2) \in U$, because $x_1, x_2 \in \mathbb{Z}$, then $-x_1, -x_2 \in \mathbb{Z}$

But U is not closed under scalar multiplication.

Let $x = (x_1, x_2) \in U$, $\lambda = \frac{1}{2}$

$\lambda x = (\frac{1}{2} x_1, \frac{1}{2} x_2) \notin U$, because $\frac{1}{2} x_1, \frac{1}{2} x_2$ are not integers.

4. Ans:

Since v_1, v_2, v_3 and v_4 spans V, we know there are $a_1, a_2, a_3, a_4 \in F$ such that

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = v \in V$$

We need find b_1, b_2, b_3, b_4 such that:

$$b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4 v_4 = v$$

$$b_1 v_1 + (b_2 - b_1) v_2 + (b_3 - b_2) v_3 + (b_4 - b_3) v_4 = v$$

So $b_1 = a_1$, $b_2 - b_1 = a_2$, $b_3 - b_2 = a_3$, $b_4 - b_3 = a_4$

$b_1 = a_1$, $b_2 = a_1 + a_2$, $b_3 = a_1 + a_2 + a_3$, $b_4 = a_1 + a_2 + a_3 + a_4$

~~Therefore~~ $v \in \text{span}(v_1, v_2, v_3, v_4)$

Therefore, the list $(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$ spans V.

5. (a) For linear dependent, there are nontrivial solutions for $a(1+i) + b(1-i) = 0$ where $a, b \in \mathbb{R}$. But We get $(a+b) + (a-b)i = 0$ only when $a = b = 0$. It is trivial solution. So the list $(1+i, 1-i)$ is linearly independent.

(b) Let $x, y \in \mathbb{C}$

$$x(1+i) + y(1-i) = 0$$

there are nontrivial solutions for this. for example $x = -1$, $y = i$. So the list $(1+i, 1-i)$ is linearly dependent over $\mathbb{C}$

Date . .

6, Ans:

① Exsitence: By definition $W_1 + W_2 = V$, we can know if $a \in V$ then there always exist $a_1 \in W_1$ and $a_2 \in W_2$ such that $a = a_1 + a_2$

② Uniqueness: Let $a \in V$. $a_1 \in W_1$, $a_2 \in W_2$, and suppose $a = a_1 + a_2$

Let $b_1 \in W_1$, $b_2 \in W_2$ and suppose $a = b_1 + b_2$

So $a_1 + a_2 = b_1 + b_2$

$a_1 - b_1 = b_2 - a_2$

Because W_1 and W_2 are subspaces, $a_1 - b_1 \in W_1$, $b_2 - a_2 \in W_2$.

Since by definition, $W_1 \cap W_2 = \{0\}$,

So $a_1 = b_1$ and $a_2 = b_2$. It is unique.

7. (a) Since $x_1 = 3x_2$, $x_3 = 7x_4$, $\dim(U) = 3$

First, we can find basis of three (x_2, x_4, x_5)

It can be $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$

Then the basis of U should be $(3,1,0,0,0)$, $(0,0,7,1,0)$ and $(0,0,0,0,1)$

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(b) $(1, 0, 0, 0, 0)$

$(3, 1, 0, 0, 0)$

$(0, 0, 7, 1, 0)$

$(0, 0, 1, 0, 0)$

$(0, 0, 0, 0, 1)$.

(c) ~~span((1,0,0,0,0))~~

$W = \text{span}((1, 0, 0, 0, 0), (0, 0, 1, 0, 0))$

8. Ans:

If X and Y are vector space over same set of scalars, then the set of all function from X to Y form a vector space.

Let $L, M \in LT[X, Y]$, scalar a.

We need to prove $(aL+M)(x) = aL(x) + M(x)$ in $LT[X,Y]$

For $x_1, x_2 \in X$, and scalar y

$(aL+M)(yx_1 + x_2)$

$= aLyx_1 + Myx_1 + aLx_2 + Mx_2$

$= y(aL+M)x_1 + (aL+M)x_2$

Therefore, $(aL+M) \in LT[X, Y]$, $LT[X,Y]$ is a vector space.