ECE 586: Vector Space Methods
Chapter 1: Logic, Proofs, and Set Theory

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Introduction

- Statement (or proposition)
  - An assertion that is true or false, but not both
  - Ex. "There are no classes at Duke University today" ✓
  - Ex. "The real number $\sqrt{2}$ is irrational" ✓
  - Ex. "Wash your hands before dinner" ✗
- Combining statements
  - One can also form new statements from old ones using English expressions: and; or; not; if, then; if and only if
  - Ex. "Duke is located in Durham, NC or all real numbers are rational"
  - Note: symbols $P, Q, R, \ldots$ used to denote abstract statements

Basic Definitions

- Conjunction of P, Q (i.e., P AND Q)
  - Binary operation on logical propositions (denoted $P \wedge Q$) that:
    is true only if both statements are true, and is false otherwise
- Disjunction of P, Q (i.e., P OR Q)
  - Binary operation on logical propositions (denoted $P \vee Q$) that:
    is true if either statement is true, and false otherwise
- Negation OF P (i.e., NOT P)
  - Unary operation on a logical proposition (denoted $\neg P$) that:
    is true if the statement is false, and is true otherwise
- Truth Tables

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	$\neg P$
T	F
F	T

Conditional Statements (1)

- Conditional Connective $P \to Q$ (i.e., if P, then Q)
  - Binary operation on logical propositions that:
    is false if P is true and Q is false, and is true otherwise

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

  - P is called the antecedent and Q is called the consequent

- Meaning
  - When P is false, some people guess the truth value should be undefined. But, these values are universally accepted in logic
  - For motivation, one can think of $P \to Q$ as a promise that Q is true whenever P is true. When P is false, the promise is kept by default
  - Ex. suppose your friend promises "if it is sunny tomorrow, I will ride my bike". We will say this is true if they keep their promise. If it rains and they don't ride their bike, most people would agree that they have still kept their promise.

Conditional Statements (2)

- Biconditional $P \leftrightarrow Q$ (i.e., P if and only if Q)
  - Binary operation on logical propositions that is:
    true if P and Q have the same truth value, and false otherwise

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

  - Identical truth values as: $(P \rightarrow Q) \wedge (Q \rightarrow P)$
  - Ex. "John graduates this term if and only if he passes all his classes"
- Variations on the conditional connective $P \rightarrow Q$
  - The converse of $P \rightarrow Q$ is the statement $Q \rightarrow P$
  - The contrapositive of $P \rightarrow Q$ is the statement $\neg Q \rightarrow \neg P$

Compound Statements

- It is also useful to consider compound logical statements like

$$(P \to R) \wedge (Q \vee \neg R)$$

- There is also a mechanical way to compute their truth tables:

P	Q	R	$(P$	$\to$	$R)$	$\wedge$	$(Q$	$\vee$	$\neg R)$
T	T	T	T	T	T	T	T	T	F
T	T	F	T	F	F	F	T	T	T
T	F	T	T	T	T	F	F	F	F
T	F	F	T	F	F	F	F	T	T
F	T	T	F	T	T	T	T	T	F
F	T	F	F	T	F	T	T	T	T
F	F	T	F	T	T	F	F	F	F
F	F	F	F	T	F	T	F	T	T
			1	5	2	7	3	6	4

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- Ex. A tautology is a compound statement (e.g., $R(P, Q)$) that
  - is true for every valuation of its propositional variables
  - Ex. $R(P, Q) = P \vee \neg P \vee Q$ is a tautology
- Ex. A contradiction is a compound statement (e.g., $R(P, Q)$) that
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- An equivalence $P \Leftrightarrow Q$ (for compound statements P, Q)
  - states P is true if and only if Q is true (i.e., $P \leftrightarrow Q$ is a tautology)
  - Ex. $P \rightarrow Q \Leftrightarrow \neg P \vee Q$ because $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$ is a tautology

Quantifiers

- The logic we have discussed so far is called propositional logic
- Limitations of propositional logic
  - If "Socrates is a person" and "Every person is mortal"
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- First-order predicate logic
  - Let U be a collection of elements and $P(x)$ be a statement that can be applied to any $x \in U$
  - Ex. $P(x) =$ "x has 4 tires" for the collection U of all vehicles
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- Quantifiers
  - Quantifiers allow statements about collections of elements
  - Universal quantifier: $\forall x \in U, P(x) =$ "All vehicles have 4 tires"
  - Existential quantifier: $\exists x \in U, P(x) =$ "There is a vehicle with 4 tires"
  - Natural implication: $\forall x \in U, P(x) \Rightarrow \exists x \in U, P(x)$

Multiple Quantifiers

- Consider predicate $P(x, y)$ with free variables x, y
  - Ex. Let I be a collection of images and C be a collection of colors. Define predicate $P(x, y) =$ "x contains y" for $x \in I$ and $y \in C$
  - "$\forall x \in I, \forall y \in C, P(x, y)$" $=$ "All images in I contain all colors in C"
  - "$\forall x \in I, \exists y \in C, P(x, y)$" $=$ "All images in I contain some color in C"
  - 8 total possibilities to choose order and quantifiers for each variable
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- Negation
  - $\neg(\forall x \in I, \forall y \in C, P(x, y))) \Leftrightarrow \exists x \in I, \exists y \in C, \neg P(x, y)$
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- Natural Implications and Equivalences

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$$\begin{array}{ccccccc}
\forall x, \forall y, P(x, y) & \Rightarrow & \exists x, \forall y, P(x, y) & \Rightarrow & \forall y, \exists x, P(x, y) & \Rightarrow & \exists y, \exists x, P(x, y) \\
\Updownarrow & & & & & & \Updownarrow \\
\forall y, \forall x, P(x, y) & \Rightarrow & \exists y, \forall x, P(x, y) & \Rightarrow & \forall x, \exists y, P(x, y) & \Rightarrow & \exists x, \exists y, P(x, y)
\end{array}$$

Axiomatic Formulations

- "Ex falso quodlibet" is Latin for "from falsehood, anything"
  - Observe that $P \wedge \neg P \rightarrow Q$ is true regardless of Q
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- What do these words mean in logic?
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  - Consistent: implications of axioms do not contain a contradiction
  - Complete: all valid implications follow from the axioms
  - Decidable: algorithm can determine if any implication is valid or not
- First-Order Predicate Logic
  - Axiomatic formulation is consistent, complete, and semidecidable
  - Semidecidable: algorithm determines the truth of any postulated implication, if it terminates. But, termination is guaranteed only if postulate is true

Strategies for Proofs

- Background
  - Intuition identifies what might be true and why
  - Rigorous proofs verify and communicate that intuition
  - A proof is a sequence of verifiable steps from premises to the result
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- Types of Proofs for $P \to Q$
  - Direct: Assume P true and give steps that lead to Q
  - Contrapositive: Proof of equivalent statement $\neg Q \to \neg P$
  - Contradiction: Using $\neg(P \to Q) \Leftrightarrow P \wedge \neg Q$, one supposes that both P and $\neg Q$ are true and then gives steps leading to a contradiction
  - Induction: For predicate $P(n)$, prove $Q =$ "$\forall n \in \mathbb{N}, P(n)$" by establishing the premise $P =$ "$P(1) \wedge (\forall n \in \mathbb{N}, P(n) \to P(n+1))$"

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- Whiteboard Examples
  - Prove $\sqrt{2}$ is irrational by showing contrapositive via contradiction
  - For $P(n) =$ "$\sum_{i=1}^{n} i = \frac{n^2+n}{2}$", prove "$\forall n \in \mathbb{N}, P(n)$" via induction

Set Theory

- Foundation (along with logic) of all modern mathematics
  - Numbers, relations, functions, ... all defined using set theory
  - Not as easy as it seems because simple approaches include paradoxes (i.e., statements which are both true and false)
  - Axiomatic framework resolves paradoxes but not useful for engineers
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- Naive Set Theory
  - Set defined as "any collection of objects, mathematical or otherwise"
  - Ex. Consider "the set of all books published in 2007"
  - Objects in a set are called elements or members of the set
  - Logical statement: "a is a member of the set A" is written $a \in A$
  - Its negation: "a is not a member of the set A" is written $a \notin A$

Using Set Theory

- Defining Sets
  - One can present a set by listing elements: standard English vowels

$$A = \{a, e, i, o, u\}$$

  - Element order is irrelevant: $\{i, o, u, a, e\}$ is the same as A
  - Repeated elements have no effect: $\{a, e, i, o, u, e, o\}$ same as A
  - Singleton is a set containing exactly one element such as $\{a\}$
  - Standard sets: Integers $\mathbb{Z}$, Real numbers $\mathbb{R}$, and Complex numbers $\mathbb{C}$

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  - Standard sets: Integers $\mathbb{Z}$, Real numbers $\mathbb{R}$, and Complex numbers $\mathbb{C}$
- Building new sets from old
  - Set-builder notation: For logical predicate $P(x)$ defined on $x \in X$, "A is the set of elements in X such that $P(x)$ is true" is denoted by

    $$A = \{x \in X | P(x)\}$$

  - If no $x \in X$ satisfies $P(x)$, then result is an empty set, denoted by $\emptyset$
  - Ex. natural numbers $\mathbb{N}$ and positive prime integers P:

    $$\mathbb{N} = \{x \in \mathbb{Z} | x \geq 1\} = \{1, 2, 3, 4, \ldots\}$$
    $$P = \{x \in \mathbb{Z} | x \geq 1 \text{ and ``} x \text{ is prime''}\} = \{2, 3, 5, 7, 11, \ldots\}$$

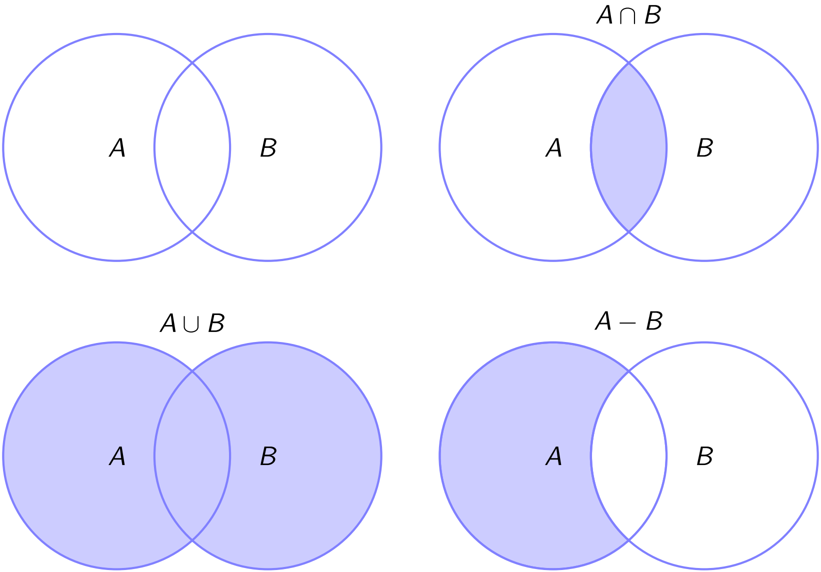
Set Properties

- Cardinality
  - For set A, the cardinality $|A|$ is the number of elements in A

$$|\{a, e, i, o, u\}| = 5$$

  - If there is a one-to-one correspondence between A and the natural numbers $\mathbb{N}$, then A is called countably infinite and $|A| = \infty$
  - If $|A| = \infty$ but not countably infinite, then A is uncountably infinite
  - Ex. Set of rational numbers $\mathbb{Q} \triangleq \{q \in \mathbb{R} \mid \exists z \in \mathbb{Z}, qz \in \mathbb{Z}\}$ is countably infinite but set of real numbers is uncountably infinite

Venn Diagrams*



Relationships and Operations Between Sets*

- Operations on sets A, B
  - Union of A and B ($A \cup B$): set of elements in either A or B
    $$x \in A \cup B \Leftrightarrow (x \in A) \vee (x \in B)$$
  - Intersection of A and B ($A \cap B$): set of elements in both A and B
    $$x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B)$$
  - Set difference $A - B$ (or $A \setminus B$): set of elements in A but not in B
    $$x \in A - B \Leftrightarrow (x \in A) \wedge (x \notin B)$$
  - Complement (A^c) for implied universal set U defined by $A^c = U - A$

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  - A equals B (denoted $A = B$) if both sets have the same elements
  $$\text{“}A = B\text{”} \Leftrightarrow \forall x\,((x \in A) \leftrightarrow (x \in B))$$
  - A is a subset of B (denoted $A \subseteq B$) if all elements in A are also in B
  $$\text{“}A \subseteq B\text{”} \Leftrightarrow \forall x\,((x \in A) \rightarrow (x \in B))$$
  - A is a proper subset of B (denoted $A \subset B$) if $A \subseteq B$ and $A \neq B$
  - Two sets are disjoint if $A \cap B = \emptyset$

De Morgan, Infinite Unions, and Tuples

- De Morgan's logical identity $\neg(P \vee Q) = \neg P \wedge \neg Q$ for sets:

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- Also holds for infinite unions and intersections:

$$\bigcup_{\alpha \in I} S_\alpha \triangleq \{x | x \in S_\alpha \text{ for some } \alpha \in I\}$$
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- Sets of tuples and vectors
  - For sets A, B, the Cartesian product $A \times B$ is the set of ordered pairs

$$\{(a, b) | a \in A, b \in B\}$$

  - For n-tuples from the same set, we have $A^n = A \times A \times \cdots \times A$
  - Ex. The set of all 3-tuples from with elements in $A = \{a, b\}$ is

$$A^3 = \{(a, a, a), (a, a, b), (a, b, a), (a, b, b), (b, a, a), (b, a, b), (b, b, a), (b, b, b)\}$$

Foundations of Set Theory

Example (Russell's Paradox)

Let $R = \{S | S \notin S\}$ be the set of all sets that do not contain themselves. This set exists in naive set theory (though it may empty) because it is described by the above sentence. The paradox arises from the fact that this definition leads to the logical contradiction: $R \in R \leftrightarrow R \notin R$.

- What does this mean?
  - This shows that naive set theory is not consistent because it allows constructions leading to contradictions
  - Issue avoided in axiomatic formulation by restricting constructions
  - Also implies that R cannot exist in any consistent set theory

Abstract Relations

Definition

A relation $\sim$ between elements of A is defined by the subset $E \subseteq A \times A$. Specifically, the relation $a \sim b$ holds if and only if $(x, y) \in E$.

- Properties: A relation on A is said to be:
  - Reflexive if $x \sim x$ holds for all $x \in A$
  - Symmetric if $x \sim y$ implies $y \sim x$ for all $x, y \in A$
  - Transitive if $x \sim y$ and $y \sim z$, then $x \sim z$ for all $x, y, z \in A$

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  - Transitive if $x \sim y$ and $y \sim z$, then $x \sim z$ for all $x, y, z \in A$
- Called an equivalence relation if reflexive, symmetric, and transitive
  - Ex. let A be a set of people and $P(x, y)$ be the statement: "x has the same birthday (month and day) as y"
  - Define relation $\sim$ such that $a \sim b$ holds iff $P(x, y)$ is true
  - Partitions A into disjoint equivalence classes: $[a] = \{x \in A | x \sim a\}$
  - In example, equivalence classes identify each day of the year
  - Set of equivalence classes called the quotient set: $A \backslash \sim = \{[a] | a \in A\}$

Functions

Definition

A function $f\colon X \to Y$ from X to Y is defined by a subset $F \subset X \times Y$ such that $A_x = \{y \in Y | (x, y) \in F\}$ has exactly one element for each $x \in X$. The value of f at $x \in X$, denoted $f(x)$, is the unique in A_x.

- Unpacking the definition
  - Function $f\colon X \to Y$ assigns one value $f(x) \in Y$ to each $x \in X$
  - Notation $f\colon X \to Y$ emphasizes the domain X and the codomain Y
  - Range of f is subset of Y achieved by f, $\{y \in Y | \exists x \in X, y = f(x)\}$
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  - Since term codomain is uncommon, people sometimes use the term range instead of codomain either intentionally or unintentionally
- In basic math, functions are described by graphs and formulas
  - This leads students to picture only "nice" functions
  - Ex. Cauchy published incorrect proof of false assertion: "sequence of continuous functions converging everywhere has a continuous limit"
  - Teachers now warn: the world is filled with "not so nice" functions

Properties of Functions

- Equality
  - Two functions are equal if they have the same domain, codomain, and value for all elements of the domain

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  - Two functions are equal if they have the same domain, codomain, and value for all elements of the domain
- A function f is called:
  - one-to-one or injective if, $\forall x, x' \in X$, if $f(x) = f(x')$ then $x = x'$
  - onto or surjective if its range $\{f(x) | x \in X\}$ equals Y
  - one-to-one correspondence or bijective if both one-to-one and onto

Properties of Functions

- Equality
  - Two functions are equal if they have the same domain, codomain, and value for all elements of the domain
- A function f is called:
  - one-to-one or injective if, $\forall x, x' \in X$, if $f(x) = f(x')$ then $x = x'$
  - onto or surjective if its range $\{f(x) | x \in X\}$ equals Y
  - one-to-one correspondence or bijective if both one-to-one and onto
- Inversion
  - A bijective function has a unique inverse function $f^{-1} \colon Y \to X$ satisfying: $\forall x \in X, f^{-1}(f(x)) = x$ and $\forall y \in Y, f(f^{-1}(y)) = y$
  - Any one-to-one function $f \colon X \to Y$ defines a bijective function $g \colon X \to R$ with $g(x) = f(x)$ by choosing range R to be codomain

Applying Functions to Sets

Definition

For $f\colon X \to Y$ and subset $A \subseteq X$, the image of A under f is

$$f(A) \triangleq \{y \in Y | \exists x \in A \text{ s.t. } f(x) = y\} = \{f(x) | x \in A\}.$$

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  - This implies that the range of f is simply $f(X)$
  - Allowing set-valued images means set-valued inverse always exists
  - For a one-to-one f, inverse image of singleton $\{f(x)\}$ is singleton $\{x\}$
  - In general, one can show: $f(f^{-1}(B)) \subseteq B$ and $f^{-1}(f(A)) \supseteq A$.

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  - Ex. For $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, let $A = [1, 2]$ and notice that $B = f(A) = [1, 4]$. But, $f^{-1}(B) = [-2, -1] \cup [1, 2] \supseteq A$

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  - Ex. For $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$, let $B = [0, 2]$ and notice that $A = f^{-1}(B) = [-1, 1]$. But, $f(A) = f([-1, 1]) = [1, 2] \subseteq B$