Assignment 6

Due Friday 10/25/19

Reading:

- Required: Course Notes 3.6-3.7,5.1-5.2,6.3.2
- Required: LADL 6.A-6.B

Problems:

1. (LA: 8.1.2) (5 pts) Let V be a vector space over F. Show that the sum of two inner products on V is an inner product on V. Is the difference of two inner products an inner product? Show that a positive multiple of an inner product is an inner product.

   Solution: Let $\langle\cdot|\cdot\rangle_a$ and $\langle\cdot|\cdot\rangle_b$ be two inner products on V. Furthermore, for $\underline{v}, \underline{u} \in V$, define $\langle\underline{u}|\underline{v}\rangle = \langle\underline{u}|\underline{v}\rangle_a + \langle\underline{u}|\underline{v}\rangle_b$. Then, the following properties hold.

   (a) For any $\underline{u}, \underline{v}, \underline{w} \in V$,

   $$\begin{aligned}\langle\underline{u}+\underline{v}|\underline{w}\rangle &= \langle\underline{u}+\underline{v}|\underline{w}\rangle_a + \langle\underline{u}+\underline{v}|\underline{w}\rangle_b \\ &= \langle\underline{u}|\underline{w}\rangle_a + \langle\underline{v}|\underline{w}\rangle_a + \langle\underline{u}|\underline{w}\rangle_b + \langle\underline{v}|\underline{w}\rangle_b \\ &= \langle\underline{u}|\underline{w}\rangle + \langle\underline{v}|\underline{w}\rangle.\end{aligned}$$

   (b) For any $\underline{v}, \underline{w} \in V$ and $s \in F$,

   $$\langle s\underline{v}|\underline{w}\rangle = \langle s\underline{v}|\underline{w}\rangle_a + \langle s\underline{v}|\underline{w}\rangle_b = s\langle\underline{v}|\underline{w}\rangle_a + s\langle\underline{v}|\underline{w}\rangle_b = s\langle\underline{v}|\underline{w}\rangle.$$

   (c) For any $\underline{v}, \underline{w} \in V$,

   $$\langle\underline{v}|\underline{w}\rangle = \langle\underline{v}|\underline{w}\rangle_a + \langle\underline{v}|\underline{w}\rangle_b = \overline{\langle\underline{w}|\underline{v}\rangle_a} + \overline{\langle\underline{w}|\underline{v}\rangle_b} = \overline{\langle\underline{w}|\underline{v}\rangle}.$$

   (d) If $\underline{v} \neq \underline{0}$ then $\langle\underline{v}|\underline{v}\rangle_a > 0$ and $\langle\underline{v}|\underline{v}\rangle_b > 0$, which implies that

   $$\langle\underline{v}|\underline{v}\rangle = \langle\underline{v}|\underline{v}\rangle_a + \langle\underline{v}|\underline{v}\rangle_b > 0.$$

   That is, the sum of two inner products on V is itself an inner product on V.

   The difference of two inner products is not necessarily an inner product. Suppose that $\langle\cdot|\cdot\rangle_a$ is an inner product on V. Then $\langle\cdot|\cdot\rangle_a - \langle\cdot|\cdot\rangle_a = 0$ is not an inner product, whereas $2\langle\cdot|\cdot\rangle_a - \langle\cdot|\cdot\rangle_a = \langle\cdot|\cdot\rangle_a$ is obviously an inner product.

   A positive multiple of an inner product is also an inner product. Let $\langle\cdot|\cdot\rangle$ be an inner product on V and let c be a positive number. Then, for all $\underline{u}, \underline{v}, \underline{w} \in V$ and $s \in F$, we have

   (a) $c\langle\underline{u}+\underline{v}|\underline{w}\rangle = c\langle\underline{u}|\underline{w}\rangle + c\langle\underline{v}|\underline{w}\rangle$

(b) $c\langle s\underline{v}|\underline{w}\rangle = cs\langle \underline{v}|\underline{w}\rangle = sc\langle \underline{v}|\underline{w}\rangle$

(c) $c\langle \underline{v}|\underline{w}\rangle = c\overline{\langle \underline{v}|\underline{w}\rangle} = \overline{c\langle \underline{v}|\underline{w}\rangle}$

(d) $c\langle \underline{v}|\underline{v}\rangle > 0$ if $\underline{v} \neq \underline{0}$.

2. (LA: 8.1.9) (5 pts) Let V be a real or complex vector space with an inner product. Show that the quadratic form determined by the inner product satisfies the *parallelogram law*

$$\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2.$$

Solution: The parallelogram law can be shown as follows,

$$\begin{aligned}\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 &= \langle \alpha+\beta|\alpha+\beta\rangle - \langle \alpha-\beta|\alpha-\beta\rangle \\ &= \langle\alpha|\alpha\rangle + \langle\alpha|\beta\rangle + \langle\beta|\alpha\rangle + \langle\beta|\beta\rangle + \langle\alpha|\beta\rangle - \langle\alpha|\beta\rangle - \langle\beta|\alpha\rangle + \langle\beta|\beta\rangle \\ &= 2\langle\alpha|\alpha\rangle + 2\langle\beta|\beta\rangle = 2\|\alpha\|^2 + 2\|\beta\|^2.\end{aligned}$$

3. (EF: 3.6.3) (5 pts each) Assume $\underline{v}_1 = (1,0,1)$, $\underline{v}_2 = (0,1,1)$, and $\underline{v} = (1,3,3)$ forms a basis for the standard inner product space $V = \mathbb{R}^3$.

(a) Apply the Gram-Schmidt process to get an orthonormal basis $\mathcal{B} = (\underline{u}_1, \underline{u}_2, \underline{u}_3)$ for V. Note: One can either normalize after each projection or normalize all vectors at the end.
Solution:

$$\begin{aligned}\underline{u}_1 &= \frac{1}{\sqrt{2}}(1,0,1) \\ \underline{u}_2 &= \frac{1}{\sqrt{6}}(-1,2,1) \\ \underline{u}_3 &= \frac{1}{\sqrt{3}}(1,1,-1).\end{aligned}$$

(b) For the vector $\underline{v} = (1,1,2)$, compute the coordinate vector $[\underline{v}]_{\mathcal{B}}$.
Solution: This gives $[\underline{v}]_{\mathcal{B}} = (3/\sqrt{2}, 3/\sqrt{6}, 0)$ because

$$\underline{v} = \frac{3}{\sqrt{2}}\underline{u}_1 + \frac{3}{\sqrt{6}}\underline{u}_2 = (3/2, 0, 3/2) + (-1/2, 1, 1/2) = (1,1,2).$$

4. (EF: 3.6.2) (5 pts each) Let V be the vector space of real polynomials on $[-1,1]$ with inner product

$$\langle f|h\rangle = \int_{-1}^{1} f(t)h(t)dt.$$

Since polynomials have finite degree by definition, the ordered list $\mathcal{B} = (1, x, x^2, \ldots)$ forms a Hamel basis for V.

(a) Let $f, h \in V$ have the unique representations $f(t) = \sum_{j=0}^{\infty} f_j t^j$ and $h(t) = \sum_{i=0}^{\infty} h_i t^i$. Find an expression for g_{ij} such that

$$\langle f|h\rangle = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} h_i g_{ij} f_j.$$

Solution: Using bilinearity, we see that

$$g_{ij} = \langle t^j|t^i\rangle = \int_{-1}^{1} t^i t^j \, dt = \left.\frac{t^{i+j+1}}{i+j+1}\right|_{-1}^{1} = \begin{cases} \frac{2}{i+j+1} & \text{if } i+j \text{ is even} \\ 0 & \text{if } i+j \text{ is odd.} \end{cases}$$

(b) Apply the Gram-Schmidt process to the vector sequence $\underline{v}_i = t^{i-1}$ for $i \in \{1,2,3,4\}$. Note: This gives the first four unnormalized Legendre polynomials.
Solution:

$$\begin{aligned}
\underline{u}_1 &= 1 \\
\underline{u}_2 &= t - \frac{\langle t|1\rangle}{\|1\|^2}1 = t \\
\underline{u}_3 &= t^2 - \frac{\langle t^2|1\rangle}{\|1\|^2}1 - \frac{\langle t^2|t\rangle}{\|t\|^2}t = t^2 - \frac{2/3}{2} = t^2 - \frac{1}{3} \\
\underline{u}_4 &= t^3 - \frac{\langle t^3|1\rangle}{\|1\|^2}1 - \frac{\langle t^3|t\rangle}{\|t\|^2}t - \frac{\langle t^3|t^2\rangle}{\|t^2\|^2}t = t^3 - \frac{2/5}{2/3}t = t^3 - \frac{3}{5}t.
\end{aligned}$$

(c) Project $f(t) = \max\{0, t\}$ onto the four orthogonal Gram-Schmidt vectors from part (b).
Solution: Let $\underline{w}_i$ be the projection of $f(t)$ onto $\underline{u}_i$. Then,

$$\begin{aligned}
\underline{w}_1 &= \left(\int_{-1}^{1} f(t)1 \, dt\right) \frac{1}{\|\underline{u}_1\|^2}\underline{u}_1 = \frac{1}{4}1 \\
\underline{w}_2 &= \left(\int_{-1}^{1} f(t)t \, dt\right) \frac{1}{\|\underline{u}_2\|^2}\underline{u}_2 = \frac{1/3}{2/3}t = \frac{1}{2}t \\
\underline{w}_3 &= \left(\int_{-1}^{1} f(t)\left(t^2 - \frac{1}{3}\right) dt\right) \frac{1}{\|\underline{u}_3\|^2}\underline{u}_3 = \frac{1/12}{8/45}\left(t^2 - \frac{1}{3}\right) \\
\underline{w}_4 &= \left(\int_{-1}^{1} f(t)\left(t^3 - \frac{3}{5}t\right) dt\right) \frac{1}{\|\underline{u}_4\|^2}\underline{u}_4 = 0\left(t^3 - \frac{3}{5}t\right).
\end{aligned}$$

(d) Explain why the following inequality is true.

$$\left(\int_{-1}^{1} f(t)h(t)dt\right)^2 \le \left(\int_{-1}^{1} |f(t)|^2 dt\right)\left(\int_{-1}^{1} |h(t)|^2 dt\right)$$

Solution: This is given by squaring the Cauchy-Schwarz inequality for this inner product space.

5. (LA: 8.2.13) (5 pts) Let S be a subset of an inner product space V. Show that $(S^\perp)^\perp$ contains the subspace spanned by S. When V is finite-dimensional, show that $(S^\perp)^\perp$ equals the subspace spanned by S.

   Solution: The set $S^\perp$ is defined by

$$S^\perp = \{\beta \in V | \langle \alpha|\beta\rangle = 0 \text{ for all } \alpha \in S\}.$$

   Similarly, we have

$$(S^\perp)^\perp = \{\gamma \in V | \langle \beta|\gamma\rangle = 0 \text{ for all } \beta \in S^\perp\}.$$

   Since $\langle \beta|\gamma\rangle = \overline{\langle \gamma|\beta\rangle} = 0$, for all $\gamma \in S$ and $\beta \in S^\perp$ it follows that $\gamma \in (S^\perp)^\perp$ and $S \subseteq (S^\perp)^\perp$. Moreover, if $\gamma \in \text{span}(S)$, then there are $s_1, \ldots, s_n$ and $\alpha_1, \ldots, \alpha_n \in S$ such that

$$\gamma = \sum_{i=1}^{n} s_i \alpha_i.$$

   Since the orthogonal complement is always a subspace and we have $\alpha_1, \ldots, \alpha_n \in (S^\perp)^\perp$, it follows that $\gamma \in (S^\perp)^\perp$. Therefore, $\text{span}(S) \subseteq (S^\perp)^\perp$.

   If V is finite-dimensional, then we can form an orthogonal basis $\mathcal{B}_S$ for $\text{span}(S)$ and extend this basis to an orthogonal basis for V. Since the set of added vectors form a basis for the subspace of all vectors orthogonal to $\text{span}(S)$, they form a basis for $S^\perp$ and we call them $\mathcal{B}_{S^\perp}$. Starting with $\mathcal{B}_{S^\perp}$, we can also work in reverse and extend this basis to an orthogonal basis for V. Of course, the added vectors must be a basis for $(S^\perp)^\perp$ (by orthogonality) and also a basis for $\text{span}(S)$ (by comparison with $\mathcal{B}_S$). Therefore, we find that $\text{span}(S) = (S^\perp)^\perp$.

6. (EF: 3.7.4) (5 pts each) Let $V = \mathbb{R}^n$ be the standard inner-product space and let $f\colon V \to \mathbb{R}$ be a differentiable real functional. The function $\nabla f(\underline{v}) \triangleq \left(\frac{\partial f}{\partial x_1}(\underline{v}), \ldots, \frac{\partial f}{\partial x_n}(\underline{v})\right)$ maps V to V and is called the gradient of f at $\underline{v} \in V$. The gradient defines a 1st-order approximation $f(\underline{v}+\underline{u}) \approx f(\underline{v}) + \langle \underline{u}|\nabla f(\underline{v})\rangle$ that is accurate for small $\underline{u}$ because

$$\lim_{\underline{u}\to\underline{0}} \frac{|f(\underline{v}+\underline{u}) - f(\underline{v}) - \langle \underline{u}|\nabla f(\underline{v})\rangle|}{\|\underline{u}\|} = 0,$$

   where the norm is induced by the inner product.

   (a) Find the unit-norm vector $\underline{u}$ that minimizes the value of the linear approximation $f(\underline{v}) + \langle \underline{u}|\nabla f(\underline{v})\rangle$? What does this imply about the direction of the vectors $\underline{u}$ and $\nabla f(\underline{v})$?

   Solution: The value of the linear approximation is minimized by minimizing $\langle \underline{u}|\nabla f(\underline{v})\rangle$. The Cauchy-Schwarz inequality shows that $|\langle \underline{u}|\nabla f(\underline{v})\rangle| \leq \|\underline{u}\|\|\nabla f(\underline{v})\|$, with equality iff $\underline{u} = \alpha \nabla f(\underline{v})$ for some scalar α or $\nabla f(\underline{v}) = \underline{0}$. The value of the linear approximation is minimized by choosing $\underline{u} = -\nabla f(\underline{v})/\|\nabla f(\underline{v})\|$ because the vector $\nabla f(\underline{v})$ points in the direction that provides the maximum increase in f per unit length and moving the opposite direction gives the maximum decrease.

(b) Under what condition is there a step-size $\delta > 0$ such that $f(\underline{v} + \delta \underline{u}) < f(\underline{v})$? Prove it.

Solution: If the gradient $\nabla f(\underline{v})$ exists and is non-zero, then such a δ exists. In contrast, if $\nabla f(\underline{v}) = \underline{0}$, then such a δ is not guaranteed. In the first case, the choice of $\underline{u} = -\nabla f(\underline{v})/\|\nabla f(\underline{v})\|$ can be combined with the definition of the derivative to see that, for any $\epsilon > 0$, there is a $\delta_0 > 0$ such that

$$f(\underline{v} + \delta \underline{u}) \leq f(\underline{v}) - \delta \frac{\langle \nabla f(\underline{v}) | \nabla f(\underline{v}) \rangle}{\|\nabla f(\underline{v})\|} + \epsilon \delta \|\nabla f(\underline{v})\| = f(\underline{v}) - \delta(1-\epsilon)\|\nabla f(\underline{v})\|$$

for all $\delta \in (0, \delta_0)$. Thus, one can choose $\epsilon = \frac{1}{2}$ to obtain a δ that will decrease the function value.

(c) Consider an algorithm for minimizing f that constructs the sequence

$$\underline{v}_{i+1} = \underline{v}_i - \delta_i \nabla f(\underline{v}_i),$$

where $\delta_i = \arg\min_{\delta \geq 0} f(\underline{v}_i - \delta \nabla f(\underline{v}_i))$ is the step size that minimizes $f(\underline{v}_{i+1})$. Show that $f(\underline{v}_{i+1}) \leq f(\underline{v}_i)$ with equality iff $\nabla f(\underline{v}_i) = \underline{0}$.

Solution: This algorithm is called *gradient descent* with line search. Using the results of (a) and (b), we see that, if $\nabla f(\underline{v}_i) \neq \underline{0}$, then there exists a step-size $\delta_i > 0$ that decreases the function value. It follows that the sequence $\underline{v}_i$ satisfies $f(\underline{v}_{i+1}) \leq f(\underline{v}_i)$ with equality iff $\nabla f(\underline{v}_i) = \underline{0}$.

(d) Assume $f(\underline{v}) \geq M$ for all $\underline{v} \in V$ and that ∇f satisfies $\|\nabla f(\underline{u}) - \nabla f(\underline{v})\| \leq L\|\underline{u} - \underline{v}\|$ for all $\underline{u}, \underline{v} \in V$ (i.e., f has a Lipschitz gradient). For the update in (c) with $\delta_i = \frac{1}{L}$, show that $\nabla f(\underline{v}_i) \to \underline{0}$. Hint: First show $f(\underline{v}_{i+1}) \leq f(\underline{v}_i) + \nabla f(\underline{v}_i) \cdot (\underline{v}_{i+1} - \underline{v}_i) + \frac{L}{2}\|\underline{v}_{i+1} - \underline{v}_i\|^2$.

Solution: The hint follows from defining $\phi(t) = f(\underline{v}_i + t(\underline{v}_{i+1} - \underline{v}_i))$ and writing

$$\begin{aligned}
f(\underline{v}_{i+1}) - f(\underline{v}_i) - \nabla f(\underline{v}_i) \cdot (\underline{v}_{i+1} - \underline{v}_i) &= \int_0^1 \big(\phi'(t) - \phi'(0)\big)\, \mathrm{d}t \\
&= \int_0^1 (\nabla f(\underline{v}_i + t(\underline{v}_{i+1} - \underline{v}_i)) - \nabla f(\underline{v}_i)) \cdot (\underline{v}_{i+1} - \underline{v}_i)\, \mathrm{d}t \\
&\leq \int_0^1 \|\nabla f(\underline{v}_i + t(\underline{v}_{i+1} - \underline{v}_i)) - \nabla f(\underline{v}_i)\| \cdot \|\underline{v}_{i+1} - \underline{v}_i\|\, \mathrm{d}t \\
&\leq \int_0^1 Lt\|\underline{v}_{i+1} - \underline{v}_i\| \cdot \|\underline{v}_{i+1} - \underline{v}_i\|\, \mathrm{d}t \\
&= \frac{L}{2}\|\underline{v}_{i+1} - \underline{v}_i\|^2.
\end{aligned}$$

Combining $\nabla f(\underline{v}_i) \cdot (\underline{v}_{i+1} - \underline{v}_i) = -\delta_i \|\nabla f(\underline{v}_i)\|^2$ and $\|\underline{v}_{i+1} - \underline{v}_i\|^2 = \delta_i^2 \|\nabla f(\underline{v}_i)\|^2$ with the hint, we find that

$$f(\underline{v}_{i+1}) - f(\underline{v}_i) \leq -\frac{1}{L}\|\nabla f(\underline{v}_i)\|^2 + \frac{L}{2L^2}\|\nabla f(\underline{v}_i)\|^2 = -\frac{1}{2L}\|\nabla f(\underline{v}_i)\|^2.$$

Multiplying by -1 and summing both sides from $i = 1$ to $i = T$ shows that

$$f(\underline{v}_1) - f(\underline{v}_T) = \sum_{i=1}^{T} f(\underline{v}_i) - f(\underline{v}_{i+1}) \geq \sum_{i=1}^{T} \frac{1}{2L} \|\nabla f(\underline{v}_i)\|^2.$$

Since $f(\underline{v}_i)$ is decreasing and lower bounded, the LHS must converge to a limit as $T \to \infty$. Since the RHS is upper bounded by the LHS, it follows that $\|\nabla f(\underline{v}_i)\| \to 0$.

Practice Problems (do not hand in):

1. (MMA: 4.6.34) Show that if the linear operator $A : X \to Y$ has an inverse, then the inverse is linear.

   Solution: Suppose that the operator A is invertible and linear. Let $\mathbf{y}_1, \mathbf{y}_2 \in Y$ and $s \in F$. Since A is invertible, there exists $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that

   $$\begin{aligned} \mathbf{y}_1 &= A\mathbf{x}_1 \\ \mathbf{y}_2 &= A\mathbf{x}_2. \end{aligned}$$

   Furthermore, by the linearity of A we have

   $$A(s\mathbf{x}_1 + \mathbf{x}_2) = sA\mathbf{x}_1 + A\mathbf{x}_2 = s\mathbf{y}_1 + \mathbf{y}_2.$$

   It follows that $A^{-1}(s\mathbf{y}_1 + \mathbf{y}_2) = s\mathbf{x}_1 + \mathbf{x}_2$. Using this fact, consider the equation

   $$\begin{aligned} A^{-1}(s\mathbf{y}_1 + \mathbf{y}_2) &= A^{-1}(A(s\mathbf{x}_1 + \mathbf{x}_2)) = s\mathbf{x}_1 + \mathbf{x}_2 \\ &= sA^{-1}A\mathbf{x}_1 + A^{-1}A\mathbf{x}_2 = sA^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2. \end{aligned}$$

   The second and third equalities follow from the fact that $A^{-1}A\mathbf{x} = \mathbf{x}$ for any $\mathbf{x} \in X$. The last equality comes from applying the definiton of $\mathbf{x}_1$ and $\mathbf{x}_2$. Since $\mathbf{y}_1$, $\mathbf{y}_2$ and s are arbitrary, we conclude that A^{-1} is linear.

2. (MMA: 4.6.35) Show that if A has both a left and a right inverse, they must be the same.

   Solution: Suppose that L is the left inverse of A, and R is its right inverse. In other words, $I = LA = AR$. Then, we have

   $$L = LI = L(AR) = (LA)R = IR = R.$$

   That is, if A has both a left and a right inverse then they must be the same.

3. (LA: 8.2.10) Let V be the vector space of all $n \times n$ matrices over C, with the inner product $(A|B) = \mathrm{tr}(AB^H)$. Find the orthogonal complement of the subspace of diagonal matrices.

   Solution: First, we observe that

   $$[AB^H]_{i,k} = \sum_j A_{i,j} \overline{B}_{k,j}$$

and

$$\operatorname{tr}(AB^H) = \sum_{i,j} A_{i,j}\overline{B}_{i,j}.$$

If B is diagonal, then we find that

$$(A|B) = \sum_i A_{i,i}\overline{B}_{i,i}.$$

For any A with a non-zero diagonal, this quantity can be made positive by choosing $B_{i,i} = A_{i,i}$. Therefore, the only matrices that satisfy $(A|B) = 0$ for all digaonal B are the matrices with zeros on the diagonal.

4. (LA: 8.2.9) Let V be the vector space of real polynomials with degree at most 3. Equip V with the inner product

$$\langle f|g\rangle = \int_0^1 f(t)g(t)dt.$$

(a) Find the orthogonal complement of the subspace of constant polynomials.

Solution: If $W = \operatorname{span}\{1\}$, then $W^\perp$ is the set of all polynomials of degree at most 3 such that $\langle f|1\rangle = 0$. That is,

$$\int_0^1 \left(f_3x^3 + f_2x^2 + f_1x + f_0\right) dx = \frac{f_3}{4} + \frac{f_2}{3} + \frac{f_1}{2} + f_0 = 0.$$

We can rewrite this set as

$$W^\perp = \left\{ f_3x^3 + f_2x^2 + f_1 - \frac{f_3}{4} - \frac{f_2}{3} - \frac{f_1}{2} \right\}.$$

(b) Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$.

Solution:

$$\begin{aligned}
p_1(x) &= 1 \\
p_2(x) &= x - \frac{\int_0^1 x dx}{\int_0^1 1dx}1 = x - \frac{1}{2} \\
p_3(x) &= x^2 - \frac{\int_0^1 x^2 dx}{\int_0^1 1dx}1 - \frac{\int_0^1 \left(x^3 - x^2/2\right) dx}{\int_0^1 \left(x^2 - x + 1/4\right) dx}\left(x - \frac{1}{2}\right) \\
&= x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) = x^2 - x + \frac{1}{6} \\
p_4(x) &= x^3 - \frac{\int_0^1 x^3 dx}{\int_0^1 1dx}1 - \frac{\int_0^1 \left(x^4 - x^3/2\right) dx}{\int_0^1 \left(x^2 - x + 1/4\right) dx}\left(x - \frac{1}{2}\right) \\
&\quad - \frac{\int_0^1 \left(x^5 - x^4 + x^3/6\right) dx}{\int_0^1 \left(x^4 - 2x^3 + 4x^2/3 - x/3 + 1/36\right) dx}\left(x^2 - x + \frac{1}{6}\right) \\
&= x^3 - \frac{1}{4} - \frac{9}{10}\left(x - \frac{1}{2}\right) - \frac{3}{2}\left(x^2 - x + \frac{1}{6}\right) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}
\end{aligned}$$

5. (LADR: 6.A.6) Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0$ if and only if

$$\|u\| \leq \|u + av\|$$

for all $a \in \mathbf{F}$.

Solution: To see this, we first expand

$$\|u + av\|^2 = \langle u + av, u + av\rangle = \|u\|^2 + 2\mathrm{Re}(a\langle v, u\rangle) + |a|^2\|v\|^2.$$

First, we prove the "if". If $\langle u, v\rangle = 0$, then this implies that

$$\|u + av\|^2 = \|u\|^2 + |a|^2\|v\|^2 \geq \|u\|^2$$

for all a. To show the converse, we prove the contrapositive of the converse: if $\langle v, u\rangle \neq 0$, then there exists an a such that $\|u + av\|^2 < \|u\|^2$. To see this, we can choose $a = -\overline{\langle v, u\rangle}/\|v\|^2$ in the expansion so that

$$\|u + av\|^2 - \|u\|^2 = 2\mathrm{Re}(a\langle v, u\rangle) + |a|^2\|v\|^2 = -2\frac{|\langle v, u\rangle|^2}{\|v\|^2} + \frac{|\langle v, u\rangle|^2}{\|v\|^4}\|v\|^2 = -\frac{|\langle v, u\rangle|^2}{\|v\|^2} < 0.$$

6. (MMA: 2.3.37) Let p be in the range $0 < p < 1$, and consider the space $L_p[0, 1]$ of all functions with

$$\|x\| = \left[\int_0^1 |x(t)|^p dt\right]^{1/p} < \infty.$$

Show that $\|x\|$ is not a norm on $L_p[0, 1]$. Hint: for a real number α such that $0 \leq \alpha \leq 1$, note that $\alpha \leq \alpha^p \leq 1$.

Solution: To show that this is not a norm on $L_p[0, 1]$, we find a counterexample to the triangle inequality. Consider the following two functions, which lie in $L_p[0, 1]$,

$$x(t) = \begin{cases} 1 & t \in [0, 1/2] \\ 0 & t \in (1/2, 1] \end{cases},$$

$$y(t) = \begin{cases} 0 & t \in [0, 1/2] \\ 1 & t \in (1/2, 1] \end{cases}.$$

Then, $\|x\|$ is given by

$$\|x\| = \left[\int_0^1 |x(t)|^p dt\right]^{1/p} = \left[\int_0^{1/2} 1^p dt\right]^{1/p} = \left(\frac{1}{2}\right)^{1/p}.$$

Similarly, $\|y\|$ is equal to

$$\|y\| = \left[\int_{1/2}^1 1^p dt\right]^{1/p} = \left(\frac{1}{2}\right)^{1/p}.$$

It is easy to see that $\|x+y\| = 1$. Note that, for $0 < p < 1$, we have $(1/2)^{1/p} < 1/2$. Putting these results together, we obtain

$$\|x+y\| = 1 = \frac{1}{2} + \frac{1}{2} > \left(\frac{1}{2}\right)^{1/p} + \left(\frac{1}{2}\right)^{1/p} = \|x\| + \|y\|.$$

This shows that $\|x\|$ is not a norm, as it does not fulfill the triangle inequality.