HW 9

1. Since λ^* is an eigenvalue of A,

assume $\underline{v}$ is the eigenvector $\therefore A\underline{v} = \lambda^* \underline{v}$

$(\lambda^* + r)\underline{v} = \lambda^* \underline{v} + r\underline{v}$

$= A\underline{v} + r \cdot I\underline{v}$

$= (A + rI)\underline{v}$

$\therefore \lambda^* + r$ is an eigenvalue of $A + rI$, and $\underline{v}$ is the eigenvector of both A and $A+rI$

$\therefore$ A and $A + rI$ have the same eigenvectors

---

2. Assume $\underline{v}$ is the eigenvector of A

$\therefore A\underline{v} = \lambda \underline{v}$

$A^{-1} \cdot A\underline{v} = A^{-1}\lambda \underline{v}$

$\underline{v} = A^{-1}\lambda \underline{v} = \lambda A^{-1}\underline{v}$

If $\lambda = 0$, then $\underline{v} = 0$, but eigenvector cannot be 0

$\therefore \lambda \neq 0$

$\therefore A^{-1}\underline{v} = \frac{1}{\lambda}\underline{v}$

$\therefore \frac{1}{\lambda}$ is an eigenvalue of A^{-1}

Also, $\underline{v}$ is the eigenvector of A^{-1} and A

$\therefore$ The eigenvectors of A are the same as the eigenvectors of A^{-1}

---

3. Assume that λ is the eigenvalue of P

Then $\lambda^2 - \lambda$ is an eigenvalue of $P^2 - P$

$\because P$ is a projection matrix

$\therefore P^2 = P$

$\therefore P^2 - P = 0$ and the eigenvalue of $P^2 - P$ is also 0

$\therefore \lambda^2 - \lambda = 0$

$\therefore \lambda = 0$ or $\lambda = 1$

The eigenvalues of P are either 0 or 1

4. (a) proving by contradiction

Assume that x_0 is a local minimum value, Then we assume that there is a point x^* such that $f(x^*) < f(x_0)$

Since $f(x)$ is a convex function, for any $t \in (0,1)$

$$f(tx^* + (1-t)x_0) \leq tf(x^*) + (1-t)f(x_0)$$

When $t \to 0$, $(tx^* + (1-t)x_0) \to x_0$ And $f(tx^* + (1-t)x_0) < f(x_0)$

so $f(x_0)$ can't be a local minimum value.

So there is a contradiction. Therefore, x_0 is a global minimum value

(b) assume that the set of points equals to minimum value is $S = \{u \mid f(u) = f(u_0)\}$, where u_0 is the point that minimize the value

Consider $\underline{u}_1, \underline{u}_2 \in S$, $\lambda \in (0,1)$

$$u_s = \lambda \underline{u}_1 + (1-\lambda)\underline{u}_2$$

$\because \underline{u}_1, \underline{u}_2 \in A$, A is convex

$\therefore f(\lambda \underline{u}_1 + (1-\lambda)\underline{u}_2) \leq \lambda f(\underline{u}_1) + (1-\lambda) f(\underline{u}_2)$

$$f(\underline{u}_s) \leq f(u_0)$$

$\because f(\underline{u}_s) \geq f(\underline{u}_0)$ ($f(\underline{u}_0)$ is the minimum value)

$\therefore f(\underline{u}_s) = f(\underline{u}_0)$

$\therefore u_s \in S$

$\therefore \lambda \underline{u}_1 + (1-\lambda)\underline{u}_2 \in S$

$\therefore S$ is convex

(c) There is no such a point

Consider $V = R^2$ with $\|\cdot\|_1$

assume $V = \{\underline{v} \in R^2 \mid \|v\| \leq 1\}$. $f((\frac{1}{2}, \frac{1}{2})) = |\frac{1}{2}-1| + |\frac{1}{2}-1| = 1$

$f((0,1)) = 1$ $\therefore (0,1)$ and $(\frac{1}{2}, \frac{1}{2})$ are also u^*

$\therefore u^*$ is not unique.

5. (a) $V_1 = P_A \underline{V_0} = \frac{\langle V_0 | a \rangle}{\|a\|^2} \underline{a} = \frac{50}{\sqrt{5}} \underline{a} = (10, 20)^T$

$V_2 = P_B \underline{V_1} = \frac{\langle V_1 | b \rangle}{\|b\|^2} \underline{b} = 10 \underline{b} = (10, 0)^T$

$V_3 = P_A \underline{V_2} = \frac{\langle V_2 | a \rangle}{\|a\|^2} \underline{a} = \frac{10}{\sqrt{5}} \underline{a} = (2, 4)^T$

$V_4 = \frac{\langle V_3 | b \rangle}{\|b\|^2} \underline{b} = 2\underline{b} = (2, 0)^T$

(b) when n is odd, it is a scalar multiple of $\underline{a}$

when n is even, it is a scalar multiple of $\underline{b}$

And the scalar is multiplied by $\frac{\sqrt{5}}{5}$ when n increasing 1.

(c) when n is even, $n+1$ is odd, $\underline{V_n} = \alpha_n \underline{b}$

$\underline{V_{n+1}} = P_A \underline{V_n} = \frac{\underline{a}\,\underline{a}^T}{\|\underline{a}\|^2} \alpha_n \underline{b} = \alpha_n \frac{\underline{a}^T b}{\|\underline{a}\|^2} \underline{a} = \alpha_{n+1} \underline{a}$

when n is odd, $n+1$ is even, $\underline{V_n} = \alpha_n \underline{a}$

$V_{n+1} = P_B \underline{V_n} = \alpha_n \frac{\underline{b}\underline{b}^T}{\|\underline{b}\|^2} \underline{a} = \alpha_n \frac{\underline{b}^T \underline{a}}{\|\underline{b}\|^2} \underline{b} = \alpha_{n+1} \underline{b}$

$\because \alpha_{n+1} = \alpha_{n+1}$

$\therefore a^T b = b^T a = 4$

$\|a\| = \|b\| = 5$

$\therefore \alpha_{n+1} = \frac{4}{5} \alpha_n$

(d) $P_C = \tilde{C} \tilde{C}^T$ $\quad P_A = [\tilde{C}\tilde{A}][\tilde{C}\tilde{A}]^T = \tilde{C}\tilde{C}^T + \tilde{A}\tilde{A}^T = P_C + \tilde{A}\tilde{A}^T$

$P_B = [\tilde{C}\tilde{B}][\tilde{C}\tilde{B}]^T = \tilde{C}\tilde{C}^T + \tilde{B}\tilde{B}^T = P_C + \tilde{B}\tilde{B}^T$

$P_A P_B = (P_C + \tilde{A}\tilde{A}^T)(P_C + \tilde{B}\tilde{B}^T) = P_C + \tilde{A}\tilde{A}^T\tilde{B}\tilde{B}^T$

For n odd, $\underline{V_n} = P_C \underline{V_0} + \tilde{A}\underline{s}$

$V_{n+1} = P_B \underline{V_n} = P_C \underline{V_0} + \tilde{B}(\tilde{B}^T\tilde{A}\underline{s}) = P_C \underline{V_0} + \tilde{B}\underline{t}$

for $\underline{t} = \tilde{B}^T\tilde{A}\underline{s}$, $\gamma \triangleq \max_{\underline{s} \neq 0} \frac{\|\tilde{B}^T\tilde{A}\underline{s}\|}{\|\underline{s}\|} = \max_{\underline{s} \in R^n} \|\tilde{B}^T\tilde{A}\underline{s}\| = \|\tilde{B}^T\tilde{A}\|$

Similarly, when n is even, the decay rate equals to $\tilde{A}^T\tilde{B}$

Thus, γ equals the largest singular value of $\tilde{B}^T\tilde{A}$ and $\|\underline{V_n} - P_C \underline{V_0}\| \le \gamma^{n-1} \|\underline{V_1} - P_C \underline{V_0}\|$

6. (a) Since G is positive definite, we can write $G = U^T \Lambda U$ where U is orthogonal and Λ is a positive diagnal matrix

$$\underline{v}^T G \underline{v} = \underline{v}^T U^T \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} U \underline{v} = (\underline{v}^T U^T \Lambda^{\frac{1}{2}})(\Lambda^{\frac{1}{2}} U \underline{v}) = \|\Lambda^{\frac{1}{2}} U \underline{v}\|^2$$

(b) We can use the triangle inequality

$$(\alpha \underline{u} + (1-\alpha)\underline{v})^T G (\alpha \underline{u} + (1-\alpha)\underline{v}) + \underline{c}^T(\alpha \underline{u} + (1-\alpha)\underline{v})$$

$$= \|A(\alpha \underline{u} + (1-\alpha)\underline{v})\|^2 + \alpha \underline{c}^T \underline{u} + (1-\alpha)\underline{c}^T \underline{v}$$

$$\leq \alpha \|A\underline{u}\|^2 + (1-\alpha)\|A\underline{v}\|^2 + \alpha \underline{c}^T \underline{u} + (1-\alpha)\underline{c}^T \underline{v}$$

$$= \alpha(\underline{u}^T G \underline{u} + \underline{c}^T \underline{u}) + (1-\alpha)(\underline{v}^T G \underline{v} + \underline{c}^T \underline{v})$$

(c) $$\underline{v}^T G \underline{v} + \underline{c}^T \underline{v} = (\underline{v} - v_0)^T G (\underline{v} - \underline{v_0}) + d$$

$$\underline{v}^T G \underline{v} + \underline{c}^T \underline{v} = \underline{v}^T G \underline{v} - v_0^T G v_0 + d$$

$$d = \underline{v_0}^T G \underline{v_0} \qquad v_0 = G^{-1} \underline{c}$$

G is invertible $\because$ it is positive definite

the minimum value $\underline{d} = v_0^T G v_0$ where $\underline{v_0} = G^{-1} \underline{c}$