Practice Midterm 2

October 23, 2019

"I have adhered to the Duke Community Standard in completing this assignment."

Signature:

Name:

The following list of mathematical statements may be useful.

Definition. A *subspace* of a vector space V over F is a subset W of V which itself is a vector space over F.

Theorem. A non-empty subset $W \subset V$ is a subspace of V if and only if, for every pair of vector $\underline{w}_1, \underline{w}_2 \in W$ and every scalar $s \in F$, the vector $s\underline{w}_1 + \underline{w}_2$ is also in W.

Definition. A *norm* on a vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ that satisfies

1. $\|\underline{v}\| \geq 0 \quad \forall \underline{v} \in V$; equality holds if and only if $\underline{v} = \underline{0}$ (positive definiteness)
2. $\|s\underline{v}\| = |s| \, \|\underline{v}\| \quad \forall \underline{v} \in V, s \in F$ (positive homogeneity)
3. $\|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\| \quad \forall \underline{v}, \underline{w} \in V.$ (triangle inequality)

Definition. Let V be a vector space over $F = \mathbb{R}$ or $F = \mathbb{C}$. An *inner product* on V is a function that assigns a scalar $\langle \underline{v} | \underline{w} \rangle \in F$ to each ordered pair of vectors $\underline{v}, \underline{w} \in V$ such that, for all $\underline{u}, \underline{v}, \underline{w} \in V$ and all $s \in F$,

1. $\langle \underline{u} + \underline{v} | \underline{w} \rangle = \langle \underline{u} | \underline{w} \rangle + \langle \underline{v} | \underline{w} \rangle$
2. $\langle s\underline{v} | \underline{w} \rangle = s \langle \underline{v} | \underline{w} \rangle$
3. $\langle \underline{v} | \underline{w} \rangle = \overline{\langle \underline{w} | \underline{v} \rangle}$, where the overbar denotes complex conjugation;
4. $\langle \underline{v} | \underline{v} \rangle \geq 0$ with equality iff $\underline{v} = \underline{0}$.

Theorem. Suppose $\mathcal{B} = \underline{v}_1, \ldots, \underline{v}_n$ is an ordered basis for the vector space V. Then, any inner product on V is completely determined by the values, $G_{ji} = \langle \underline{v}_i | \underline{v}_j \rangle$, it takes on pairs of vectors in $\mathcal{B}$.

Definition. For an inner product space V, two subsets $U, W \subseteq V$ are called *orthogonal* (i.e., $U \perp W$) if $\langle \underline{u} | \underline{w} \rangle = 0$ for all $\underline{u} \in U$ and $\underline{w} \in W$.

Definition. The *projection* of $\underline{v} \in V$ onto $\underline{w} \in V$ is given by $\underline{u} = (\langle \underline{v} | \underline{w} \rangle / \|\underline{w}\|^2)\underline{w}$ and satisfies $\underline{v} - \underline{u} \perp \underline{w}$.

Definition. Consider a complex $n \times n$ matrix A with elements a_{ij}. The *Hermitian transpose* $B = A^H$ of A is defined by the elements $b_{ij} = \overline{a_{ji}}$ where $\overline{a}$ denotes the complex conjugate of a.

Definition. A complex matrix A is *Hermitian* if $A^H = A$. It is *skew Hermitian* if $A^H = -A$.

Problems:

1. True or False:

   (a) **2.5 pt** – An orthogonal set of non-zero vectors must be linearly independent.

   (b) **2.5 pt** – A subset U of a vector space V is automatically a subspace of V.

   (c) **2.5 pt** – For inner product space V, if $\underline{w}$ is the projection of $\underline{v} \in V$ onto $\underline{u} \in V$ then $\underline{v} \perp \underline{w}$.

   (d) **2.5 pt** – A $n \times n$ matrix over a field F is invertible if and only its rows form a basis for F^n.

2. Assign one of the following terms to each sentence: basis, inner product, invertible, non-singular, nullity, nullspace, operator norm, orthogonal, range, rank, subspace.

   (a) **2.5 pt** – Two subspaces $U, W \subset V$ that satisfy $\langle \underline{u} | \underline{w} \rangle = 0$ for all $\underline{u} \in U$ and $\underline{w} \in W$.

   (b) **2.5 pt** – Let V be a vector space and $T : V \to W$ be a linear transformation that is injective.

   (c) **2.5 pt** – The largest scale factor by which a linear transform changes the length of a vector.

   (d) **2.5 pt** – The dimension of the column space of a matrix A.

3. Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let V_e be the subset of even functions satisfying $f(-x) = f(x)$; and let V_o be the subset of odd functions satisfying $f(-x) = -f(x)$.

   (a) **5 pt** – Show that V_e and V_o are subspaces of V.

   (b) **5 pt** – Prove that $V_e + V_o = V$.

   (c) **5 pt** – Prove that $V_e \cap V_o = \{0\}$.

   (d) **5 pt** – Let $f \in V$. Show that the decomposition $f = g + h$ where $g \in V_e$ and $h \in V_o$ is unique.

   (e) **2.5 pt** – Prove or disprove the claim: a polynomial in V can be expressed as the sum of an even polynomial and an odd polynomial.

   (f) **2.5 pt** – Prove or disprove the claim: a continuous function in V can be expressed as the sum of an even continuous function and an odd continuous function.

4. Let V be the vector space of all real polynomial functions of degree 2 or less , i.e., the space of all functions f of the form

   $$f(x) = c_0 + c_1 x + c_2 x^2 \quad \text{where } c_0, c_1, c_2 \in \mathbb{R}.$$

   Consider the elements $g_0(x) = 1$, $g_1(x) = 1 + x$, $g_2(x) = (1 + x)^2$.

   (a) **5 pt** – Prove that $\mathcal{B} = (g_0, g_1, g_2)$ is an ordered basis for V.

   (b) **5 pt** – If $f(x) = c_0 + c_1 x + c_2 x^2$, what are the coordinates of f in ordered basis $\mathcal{B}$?

   For the remainder of this problem, consider the linear transformation defined by

   $$T(b_0 g_0(x) + b_1 g_1(x) + b_2 g_2(x)) = (b_0 + 2b_1 - b_2) + b_2 x^2.$$

(c) **5 pt** – What is the rank and nullity of T? Substantiate your answer.

(d) **5 pt** – Find a matrix B such that $[Tf]_{\mathcal{B}} = B\,[f]_{\mathcal{B}}$ for any $f \in V$.

(e) **5 pt** – Let $\mathcal{A} = (1, x, x^2)$ be the standard ordered basis. Find a matrix A such that $[Tf]_{\mathcal{A}} = A\,[f]_{\mathcal{A}}$ for any $f \in V$.

5. The norm $\|\cdot\|_A$ is *norm equivalent* to the norm $\|\cdot\|_B$ if there exists an $M_{AB} < \infty$ such that

$$\frac{1}{M_{AB}}\|\underline{x}\|_B \le \|\underline{x}\|_A \le M_{AB}\|\underline{x}\|_B.$$

(a) **5 pt** – Show that norm equivalence is reflexive. In other words, show that "$\|\cdot\|_A$ norm equivalent to $\|\cdot\|_B$" implies "$\|\cdot\|_B$ norm equivalent to $\|\cdot\|_A$".

(b) **5 pt** – Show that norm equivalence is transitive. In other words, show that "$\|\cdot\|_A$ is norm equivalent to $\|\cdot\|_B$" and "$\|\cdot\|_B$ is norm equivalent to $\|\cdot\|_C$" implies "$\|\cdot\|_A$ is norm equivalent to $\|\cdot\|_C$".

Let $V = \mathbb{C}^n$ be the standard vector space over the complex numbers and define

$$\|x\|_p \triangleq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

to be the standard p-norm for $p \in [1, \infty)$.

(c) **5 pt** – Use simple bounds on $\|\cdot\|_p$ to show that any p-norm is norm equivalent to the ∞-norm for all $p \in [1, \infty)$

(d) **5 pt** – Now, show that an arbitrary norm $\|\cdot\|$ is equivalent to the 1-norm.
[Hint: You may assume that $\min_{\underline{x}:\|\underline{x}\|_1=1} \|\underline{x}\| = m > 0$ and that $\max_{i\in 1,\ldots,n} \|\underline{e}_i\| = M < \infty$.]

6. Consider the functions $f_i : [-1, 1] \mapsto \mathbb{R}$ given by $f_0(t) = 1$, $f_1(t) = t$, $f_2(t) = t^2$. Let $V = \operatorname{span}(f_0, f_1, f_2)$. Also, define the inner product

$$\langle f|h\rangle = \int_{-1}^{1} f(t)h(t)t^2 dt.$$

(a) **5 pt** – Since $\mathcal{B} = \{f_0, f_1, f_2\}$ is a basis for V, we know that any vector $f \in V$ can be expressed as $[f]_{\mathcal{B}} = [s_0\ s_1\ s_2]^T$ such that $f(t) = s_0 f_0(t) + s_1 f_1(t) + s_2 f_2(t)$. Find a matrix G such that

$$\langle f|h\rangle = [h]_{\mathcal{B}}^H\, G\, [f]_{\mathcal{B}}$$

for all $f, h \in V$.

(b) **5 pt** – Apply the Gram-Schmidt orthogonalization process to basis elements $\{f_0, f_1, f_2\}$ and derive an orthogonal basis for V. Call the resulting vectors $\mathcal{A} = \{h_0, h_1, h_2\}$.