Assignment 3

Due Wednesday 9/18/19

Reading Assignment:

- Required: Course Notes 2.1
- Supplemental: MMA 2.1

Problems:

1. (EF: 2.1.4) (5 pts) Let $\underline{x} = (x_1, \ldots, x_n), \underline{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and consider the function ρ given by
$$\rho\left(\underline{x}, \underline{y}\right) = \max\left\{|x_1 - y_1|, \ldots, |x_n - y_n|\right\}.$$
Show that ρ is a metric.

2. (EF: 2.1.6) (5 pts) Suppose $a \in B_d(x, \epsilon)$ with $\epsilon > 0$. Find an explicit $\delta > 0$ such that the open ball $B_d(a, \delta)$ centered at a is contained in $B_d(x, \epsilon)$.

3. (MMA: 2.1.20) (10 pts) Show that if $\{x_n\}$ is a sequence such that $d\left(x_n, x_{n+1}\right) \leq Cr^n$ for $0 \leq r < 1$ and $C \geq 0$, then $\{x_n\}$ is a Cauchy sequence.

4. (MMA: 2.1.24) (5 pts) The fact that a sequence is Cauchy depends upon the metric employed. Consider the metric space $(C[a, b], d_\infty)$ of continuous functions mapping $[a, b] \to \mathbb{R}$ with
$$d_\infty(f, g) \triangleq \max_{t \in [a,b]} |f(t) - g(t)|.$$
Let $f_n(t) \in C[-1, 1]$ be a sequence of functions defined by
$$f_n(t) = \begin{cases} 0 & t < -1/n, \\ nt/2 + 1/2 & -1/n \leq t \leq 1/n, \\ 1 & t > 1/n. \end{cases}$$
Show that
$$d_\infty(f_n, f_m) = \frac{1}{2} - \frac{n}{2m} \quad \text{for } m > n.$$
Is $f_n(t)$ a Cauchy sequence in this metric space? Hint: See Example 2.1.16 in MMA.

5. (TOP: 2.10.6) (5 pts) Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $\{f_n(x)\}$ converges for each $x \in [0, 1]$, but that the sequence $\{f_n\}$ does not converge uniformly. Recall that uniform convergence to f on $[a, b]$ implies that, for any $\epsilon > 0$, there exists an N such that $|f_n(t) - f(t)| < \epsilon$ for all $n > N$ and all $t \in [a, b]$.

6. (EF: 2.1.7) (5 pts each) In this problem, we will numerically approximate the positive square-root of 2 using Newton's method to find the positive root of $g(x) = x^2 - 2$. Starting from some initial estimate $x_1 \in \mathbb{R}$, this gives

$$x_{n+1} = f(x_n) \triangleq x_n - \frac{g(x_n)}{g'(x_n)}.$$

   (a) For $A = [\sqrt{2}, 2]$, show that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $f(A) \subseteq A$ and is a contraction on A for some contraction coefficient $\gamma < 1$. Prove that $x_{n+1} = f(x_n)$ converges to $\sqrt{2}$ starting from $x_1 = 2$.

   (b) Determine some $\gamma < 1$ such that $|f(x) - f(y)| \leq \gamma |x - y|$ and use this value to find an n such that $|x_{n+1} - \sqrt{2}| \leq 10^{-3}$ (i.e., error is small after n iterations).

   (c) Write a program that uses this method and elementary computations (e.g., no `sqrt` or `log`) to compute the square root of an arbitrary real number $a \geq 1$ with error most 10^{-3}. Hint: Since the error is strictly decreasing faster than γ^n, it can be upper bounded by $\gamma/(1-\gamma)$ times the previous step size (i.e., use the other error bound).

Practice Problems (do not hand in):

1. (EF: 2.1.5) Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by

$$\bar{d}(x, y) = \min \{d(x, y), 1\}.$$

   Show that $\bar{d}$ is also a metric.

2. (EF: 2.2.2) Consider the metric space $(C[0,1], d_\infty)$ of continuous functions mapping $[0,1] \to \mathbb{R}$ with

$$d_\infty(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|.$$

   Prove that the sequence $f_n(x) = \sin(n\pi x)$ does not have a subsequence which converges.

   Hint: Start by showing that

$$\max_{x \in [0,1]} |f_n(x) - f_m(x)|^2 \geq \int_0^1 (f_n(x) - f_m(x))^2 dx,$$

   and then compute the integral for any integers $m \neq n$.

3. (EF: 2.1.8) Let $\mathbb{R}$ denote the standard metric space of real numbers and $\Psi\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping that satisfies $|\Psi(x) - \Psi(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$. Let $X_{a,b} = C[a, b]$ be the metric space of continuous functions mapping $[a, b]$ to $\mathbb{R}$, equipped with the metric

$$d_\infty(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|.$$

   In this problem, you will show that the differential equation,

$$f'(t) = \Psi(f(t)), \quad t \in [a, b]$$

   with boundary condition $f(a) = y_a$, has a unique solution $f \in X_{a,b}$.

(a) Show that, for some $c \in (a, b]$, the mapping $T : X_{a,c} \to X_{a,c}$, defined by

$$(Tf)(x) = y_a + \int_a^x \Psi(f(t))\mathrm{d}t,$$

is a well-defined contraction on $X_{a,c}$ with contraction coefficient $\gamma \leq 1/2$.

(b) How can one extend the uniqueness proof to the full range $[a, b]$?

(c) Let $f(t)$ be the water height, as a function of time, in a bucket with a hole in the bottom. One can show that the water exits through the hole at a rate proportional to $-\sqrt{f(t)}$. Assuming this changes the water height at the same rate, it follows that $f(t)$ satisfies the differential equation $f'(t) = -\sqrt{f(t)}$. For $t \in [-1, 0]$, verify that $f(t) = t^2/4$ and $f(t) = 0$ are both solutions satisfying boundary condition $y_0 = 0$. How is this possible mathematically? To what physical situations do these two solutions apply?

4. (TOP: 2.10.5) Let $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$ with metric $d(x, x') = |x - x'|$. Show that

$$\begin{aligned} x_n + y_n &\to x + y \\ x_n - y_n &\to x - y \\ x_n y_n &\to xy, \end{aligned}$$

and provided that each $y_n \neq 0$ and $y \neq 0$,

$$x_n/y_n \to x/y.$$

[Hint: First show that $+, -, \cdot, /$ are continuous functions from $(\mathbb{R}^2, d_1)$ to $(\mathbb{R}, |\cdot|)$.]

5. (TOP: 2.7.11) Let $f : A \to B$ and $g : C \to D$ be continuous functions with respect to the topologies generated by the metrics d_A, d_B, d_C, d_D. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)((a, c)) = (f(a), g(c)).$$

A simple product metric on $A \times C$ is the metric $d_{AC}((a, c), (a', c')) \triangleq d_A(a, a') + d_C(c, c')$. Show that $f \times g$ is continuous with respect to the product metrics d_{AC} and d_{BD}, where d_{BD} is defined similarly to d_{AC}.