Hw 8

1. The orthogonal complement using the standard inner-product of the range of A : $R(A)^{\perp} = N(A^H)$

$$A^H = \begin{bmatrix} 1 & 3 & 0 & 5 \\ 2 & -1 & 1 & 7 \end{bmatrix}$$

do echelon row-reduction : $A^H = \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{7} & \frac{26}{7} \\ 0 & 1 & -\frac{1}{7} & \frac{3}{7} \end{array}\right] = [I \mid P]$

Let $\underline{v} \in N(A^H)$

$A^H \underline{v} = 0$ $\quad \because [I \mid P]\begin{bmatrix} -P \\ \hline I \end{bmatrix} = 0$ $\quad \therefore \underline{v} = \begin{bmatrix} -P \\ I \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & -\frac{26}{7} \\ \frac{1}{7} & -\frac{3}{7} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ the orthogonal complement of the range of A is span$\left(\begin{bmatrix} -\frac{3}{7} \\ \frac{1}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{26}{7} \\ -\frac{3}{7} \\ 0 \\ 1 \end{bmatrix}\right)$

2. For any $\underline{v} \in N(A)$ and $\underline{u} \in V$

we can get $A\underline{v} = 0$

$\therefore \langle A\underline{v} \mid \underline{u} \rangle = \underline{u}^H A \underline{v} = 0$

$R(A^H) = A^H \underline{u}$

$\langle \underline{v} \mid A^H \underline{u} \rangle = \underline{u}^H A \underline{v} = 0$

$\therefore R(A^H) = N(A)^{\perp}$

$\because A$ is n by n

$\therefore \dim(R(A^H)) = \dim(R(A))$

$\because \dim(N(A)) + \dim(N(A)^{\perp}) = n$

$\dim(N(A)) + \dim(R(A)) = n$

$\therefore \dim(N(A)^{\perp}) = \dim(R(A)) = \dim(R(A^H))$

$\therefore$ proved

3. 1) $AA^+ = U\Sigma \underbrace{\underbrace{V^H V}_{I} \Sigma^{-1}}_{I} U^H = UU^H$

① $\because (AA^+)^2 = (UU^H)(UU^H) = U(U^HU)U^H = UU^H = AA^+$

$(AA^+)^H = (UU^H)^H = UU^H = AA^+$

$\therefore AA^+$ is a projection operator

② Range (AA^+) = Range (UU^H)

$\therefore P_{AA^+} = P_{UU^H} = UU^H[(UU^H)^H UU^H]^{-1}(UU^H)^H = UU^H(UU^H)^{-1}(UU^H)^H = UU^H \rightarrow \underline{R(A)}$

2) ① $A^+A = V\Sigma^{-1}U^HU\Sigma V^H = VV^H$ $\because (A^+A)^2 = (VV^H)(VV^H) = V(V^HV)V^H = VV^H = A^+A$

$(A^+A)^H = (VV^H)^H = VV^H = A^+A$ $\therefore A^+A$ is a projection operator

② Range (A^+A) = Range (VV^H) $\therefore P_{A^+A} = P_{VV^H} = VV^H[(VV^H)^H VV^H]^{-1}(VV^H)^H = VV^H \rightarrow \underline{R(A^H)}$

4. (a) $U = \begin{bmatrix} -0.1898 & -0.1991 & 0.7929 & 0.5437 \\ -0.5053 & -0.1428 & -0.5823 & 0.6207 \\ -0.8207 & -0.0864 & 0.1543 & -0.5432 \\ 0.1872 & -0.9657 & -0.0912 & -0.1553 \end{bmatrix}$

$\Sigma = \begin{bmatrix} 17.3585 & 0 & 0 \\ 0 & 1.9193 & 0 \\ 0 & 0 & 0.000\ldots \\ 0 & 0 & 0 \end{bmatrix}$

$V^H = \begin{bmatrix} -0.2002 & -0.3562 & -0.9127 \\ -0.8906 & 0.4543 & 0.018 \\ -0.4082 & -0.8165 & 0.4082 \end{bmatrix}$

rank $(A) = 2$

(b) $U\Sigma V = [U_1\ U_2]\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} - V_1^H - \\ - V_2^H - \end{bmatrix}$ where $\Sigma_2 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$

$A^+ = V_1\Sigma_1^{-1}U_1^H$ $\quad A = U_1\Sigma_1V_1^H = U\Sigma V^H$

$P_A = AA^+ = U_1U_1^H$

$A^+ = \begin{bmatrix} 0.0946 & 0.0721 & -0.0495 & 0.4439 \\ -0.0432 & -0.0234 & 0.0036 & -0.2324 \\ 0.008 & 0.0252 & 0.0423 & -0.0189 \end{bmatrix}$

$$P_A = \begin{bmatrix} 0.0757 & 0.1243 & 0.1386 & 0.1567 \\ 0.1243 & 0.2757 & 0.4024 & 0.0433 \\ 0.1386 & 0.4024 & 0.681 & -0.2371 \\ 0.1567 & 0.0433 & -0.2371 & 0.9676 \end{bmatrix}$$

(c) projection onto $R(A)$ is $\hat{\underline{v}} = P_A \underline{v} = [1.4703 \quad 2.1297 \quad 2.7892 \quad 3.9027]^T$

projection onto $R(A)^{\perp}$ is $(I - P_A)\underline{v} = \underline{v} - \hat{\underline{v}} = [-0.4703 \quad -0.1297 \quad 0.2108 \quad 0.0973]^T$

5. (a) $P_C(v) = \underset{\underline{u} \in C}{\arg\min} \|\underline{u} - \underline{v}\|$

$P_{C+\underline{w}}(v) = \underset{\underline{u} \in C+\underline{w}}{\arg\min} \|\underline{u} - \underline{v}\|$

suppose we have $u' = \underline{u} - \underline{w}$, $u' \in C$

$P_{C+\underline{w}=A}(v) = \underset{u' \in C}{\arg\min} \|u' + \underline{w} - \underline{v}\| + \underline{w}$

$P_A(v) = \underset{u' \in C}{\arg\min} \|u' - (\underline{v} - \underline{w})\| + \underline{w}$

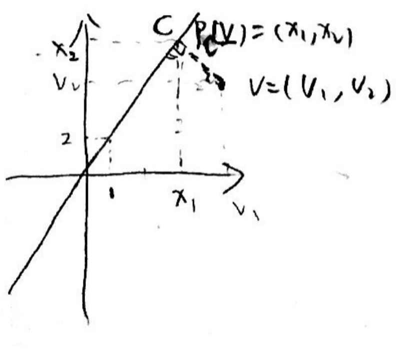
$= P_C(\underline{v} - \underline{w}) + \underline{w}$

(b) suppose $c_1, c_2 \in C$

$\lambda c_1 + (1-\lambda) c_2 = \lambda s_1 (1,2) + (1-\lambda) s_2 (1,2) = (\lambda s_1 + (1-\lambda) s_2,\ 2\lambda s_1 + 2(1-\lambda) s_2)$

$= [\lambda s_1 + (1-\lambda) s_2](1,2) \in C$

$\therefore$ C is convex



let $v = (v_1, v_2)$. Assume $P_A(\underline{v}) = (x_1, x_2)$ where $x_2 = 2x_1$

$v_1 - x_1 = 2(x_2 - v_2)$

$v_1 - x_1 = 2(2x_1 - v_2)$

$v_1 - x_1 = 4x_1 - 2v_2$

$x_1 = \frac{1}{5}(v_1 + 2v_2)$

$\therefore P_C(\underline{v}) = \frac{1}{5}(v_1 + 2v_2)(1,2)$

(c) Suppose $A_1, A_2 \in A$, $C_1, C_2 \in C$

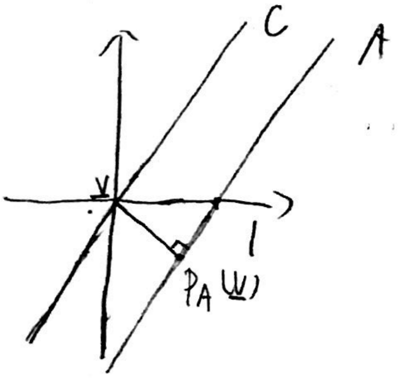
$$\lambda A_1 + (1-\lambda)A_2 = \lambda(C_1 + (1,0)) + (1-\lambda)(C_2 + (1,0))$$

$$= \lambda(S_1(1,2) + (1,0)) + (1-\lambda)(S_2(1,2) + (1,0))$$

$$= \lambda S_1(1,2) + \cancel{\lambda(1,0)} + (1-\lambda)S_2(1,2) + (1,0) - \cancel{\lambda(1,0)}$$

$\because$ we have proved $\lambda S_1(1,2) + (1-\lambda)S_2(1,2) \in C$

And $C + (1,0) \in A$ $\therefore$ A is convex.

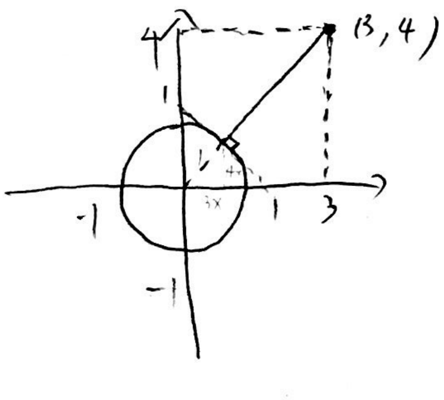


$$P_A(\underline{v}) = P_C(\underline{v} - (1,0)) + (1,0)$$

$$= \frac{1}{5}(v_1 - 1 + 2v_2)(1,2) + (1,0)$$

$$P_A((0,0)) = -\frac{1}{5}(1,2) + (1,0) = \left(-\frac{4}{5}, -\frac{2}{5}\right)$$

(d)



$$P_C((3,4)) = \left(\frac{3}{5}, \frac{4}{5}\right)$$