Practice Midterm 1 Solutions

September 17, 2019

Problems:

1. True or False:

   (a) **2.5 pt** – If a set is not open, then it is closed.
   Solution: False. The set $A = [-1, 1)$ is neither open nor closed.

   (b) **2.5 pt** – The conditional $P \to \neg P$ is a contradiction.
   Solution: False. This conditional is true if P is false.

   (c) **2.5 pt** – For sets $A, B \subseteq X$, we have $X - (A \cup B) = (X - A) \cap (X - B)$.
   Solution: True. Follows from De Morgan's law.

   (d) **2.5 pt** – It always holds that $\neg\big(\forall x,\ \exists y,\ P(x, y)\big) \Rightarrow \forall y,\ \exists x,\ \neg P(x, y)$.
   Solution: True, because $\neg\big(\forall x,\ \exists y,\ P(x, y)\big) \Leftrightarrow \exists x,\ \forall y,\ \neg P(x, y) \Rightarrow \forall y,\ \exists x,\ \neg P(x, y)$

2. Short answer questions:

   (a) **2.5 pt** – Given a conditional statement of the form $P \to Q$, what is the name of the related statement $\neg Q \to \neg P$?
   Solution: Contrapositive.

   (b) **2.5 pt** – If the statement $P \to Q$ is a tautology, what is the name of the meta-statement that characterizes P and Q?
   Solution: Implication, $P \Rightarrow Q$.

   (c) **2.5 pt** – Is the following statement a tautology: $(P \wedge Q) \to P$?
   Solution: This statement is a tautology,
   $$(P \wedge Q) \to P \Leftrightarrow \neg(P \wedge Q) \vee P \Leftrightarrow (\neg P \vee \neg Q) \vee P \Leftrightarrow \neg P \vee P.$$

   (d) **2.5 pt** – For metric spaces X, Y and $x \in X$, let $f : X \to Y$ be a function where $\lim_{n\to\infty} f(x_n) = f(x)$ for all sequences $x_n \in X$ satisfying $x_n \to x$. What property does f possess?
   Solution: It is continuous at x.

   (e) **2.5 pt** – Let $R(x, y)$ be an expression with free variables x and y. Let U and V be collections of possible values of x and y, respectively. Negate the statement: $\exists x \in U,\ \forall y \in V,\ R(x, y)$.
   Solution: This is accomplished as follows,
   $$\begin{aligned}\neg\left[(\exists x \in U)(\forall y \in V)R(x, y)\right] &\Leftrightarrow (\forall x \in U)\neg\left[(\forall y \in V)R(x, y)\right]\\ &\Leftrightarrow (\forall x \in U)(\exists y \in V)(\neg R(x, y)).\end{aligned}$$

(f) **2.5 pt** – Let (X, d) be the standard metric space of real numbers and $Y = [-10, -1] \cup [1, 10]$. Is (Y, d) a compact metric space?

Solution: Yes, $Y \subset X$ is closed and totally bounded so (Y, d) is complete and totally bounded.

(g) **2.5 pt** – Negate the statement, "Every horse that is black and flies is named midnight moon".

Solution: There exists a horse that is black and flies whose name is not midnight moon.

(h) **2.5 pt** – Suppose f and g are continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Is their composition $g \circ f$, defined pointwise by $(g \circ f)(x) = g(f(x))$, necessarily a continuous function?

Solution: Yes. By the continuity of f, we see that, for any $x_n \to x$, it follows that $y_n = f(x_n)$ satisfies $y_n \to y = f(x)$. By the continuity of g, we see also that $g(f(x_n)) = g(y_n) \to g(y) = g(f(x))$. Since $g(f(x_n)) \to g(f(x))$ for all $x_n \to x$, it follows that $g \circ f$ is continuous.

3. Let $f : X \to Y$, $A, B \subseteq X$, and $C, D \subseteq Y$. Recall also that $f(A) \triangleq \{f(x) | x \in A\}$ and $f^{-1}(C) \triangleq \{x \in X | f(x) \in C\}$.

In this problem, proofs should only use the rules of logic and the following statements:

$$
\begin{aligned}
f(x) \in C &\Leftrightarrow x \in f^{-1}(C) \\
x \in A &\Rightarrow f(x) \in f(A) \\
x \in A \cup B &\Leftrightarrow (x \in A) \vee (x \in B) \\
x \in A \cap B &\Leftrightarrow (x \in A) \wedge (x \in B) \\
A = B &\Leftrightarrow \forall x, (x \in A) \leftrightarrow (x \in B)
\end{aligned}
$$

(a) **5 pt** – Prove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ (i.e., f^{-1} preserves unions).

Solution: Applying the 1st and 3rd rules as needed, we find that

$$
\begin{aligned}
x \in f^{-1}(C \cup D) &\Leftrightarrow f(x) \in C \cup D \\
&\Leftrightarrow (f(x) \in C) \vee (f(x) \in D) \\
&\Leftrightarrow \big(x \in f^{-1}(C)\big) \vee \big(x \in f^{-1}(D)\big) \\
&\Leftrightarrow x \in f^{-1}(C) \cup f^{-1}(D).
\end{aligned}
$$

Applying the last rule, for all x, gives $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

(b) **5 pt** – Unfortunately, $f(x) \in f(A) \to x \in A$ is not true in general. Give a counterexample.

Solution: Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. If $x = -1$ and $A = \{1\}$, then this statement is false.

4. Your friend claims that the following recursive program computes the greatest common divisor of any two positive integers.

```
int gcd(int a,int b)
{
  if (a==b) { return a; }
```

```
  if (a>b) { return gcd(b,a-b); }
  else { return gcd(a,b-a); }
}
```

(a) **5 pt** – Use this program to write three statements that should be verified to prove that this program computes $\gcd(a, b)$ correctly for $a, b \in \mathbb{N}$, assuming it terminates.

Solution: The statements are: "if $a, b \in \mathbb{N}$ and $a = b$, then $\gcd(a, b) = a$", "if $a, b \in \mathbb{N}$ and $a > b$, then $\gcd(a, b) = \gcd(b, a - b)$ and $a - b \in \mathbb{N}$", and "if $a, b \in \mathbb{N}$ and $a < b$, then $\gcd(a, b) = \gcd(a, b - a)$ and $b - a \in \mathbb{N}$". To prove this recursion terminates, one would also need $\max(b, a - b) < \max(a, b)$ if $a > b$ and $\max(a, b - a) < \max(a, b)$ if $b > a$

(b) **5 pt** – Prove the statements listed in part (a). Write a proof that this program terminates and computes $\gcd(a, b)$ correctly as long as $a, b \in \mathbb{N}$.

Solution: The first statement holds because a divides a and all divisors of a must be $\leq a$. The second statement holds because: (i) $a - b > 0$ is a natural number because $a > b$, (ii) any divisor of a, b is also a divisor of $b - a$ and (iii) any divisor of $a, a - b$ is also a divisor of b. The third statement is the same as the second after swapping a, b.

The proof of correctness is by induction on $\max(a, b)$ If $\max(a, b) = 1$, then the algorithm correctly returns $1 = \gcd(1, 1)$. Now, we assume that the algorithm works for all (a, b) such that $\max(a, b) < n$. Due to symmetry, we can assume that $a > b$. If $a > b$, then $\max(a, b) = a$ and $\max(b, a - b) < a$. Thus, $\gcd(a, b) = \gcd(b, a - b)$ by the above argument and the program correctly computes $\gcd(b, a - b)$ by the induction hypothesis. Thus, the algorithm computes the correct answer.

(c) **5 pt** – Unlike Euclid's algorithm, this algorithm does not use division. While that seems like an improvement, can you think of a reason why Euclid's algorithm might be preferred?

Solution: Using this algorithm, computing $\gcd(a, 2)$ takes roughly $a/2$ iterations while Euclid's algorithm uses at most 2 iterations.

5. In a metric space (X, d), a set $A \subseteq X$ is called open if, for every $x_0 \in A$, there is a $\delta > 0$ such that $B(x_0, \delta) \subseteq A$. Use this definition to prove:

(a) **5 pt** – The intersection of any two open sets is open.

Solution: Let $A = A_1 \cap A_2$ where $A_1, A_2 \subseteq X$ are open. For all $x \in A$, the definition of A implies that $x \in A_1$ and $x \in A_2$. Sine A_1 and A_2 are open, it follows that there exist δ_1 and δ_2 such that $B(x, \delta_1) \subseteq A_1$ and $B(x, \delta_2) \subseteq A_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ and observe that $B(x, \delta) \subseteq B(x, \delta_1)$ and $B(x, \delta) \subseteq B(x, \delta_2)$ because $\delta \leq \delta_1$ and $\delta \leq \delta_2$. Thus, we have $B(x, \delta) \subseteq A$ and A is open.

(b) **5 pt** – The intersection of a finite number of open sets is open.

Solution: Let $A_1, A_2, \ldots$ be an arbitrary sequence of open sets in X. Let $C_n \triangleq \cap_{i \in \{1,2,\ldots,n\}} A_i$ and observe that $C_{n+1} = C_n \cap A_{n+1}$. Also, C_n is the finite intersection of n arbitrary open sets in X. Now, we will prove that "C_n is open" for all $n \in \mathbb{N}$. To start, we observe that the

$n = 1$ case is trivial and the $n = 2$ case follows from part (a). Next, we prove the inductive step by assuming C_n is open and showing that this implies C_{n+1} is open. If C_n is open, then $C_{n+1} = C_n \cap A_{n+1}$ is the intersection of 2 open sets. Thus, part (a) implies that C_{n+1} is open. By induction, the statement holds for all $n \in \mathbb{N}$.

(c) **5 pt** – The union of any two open sets is open.

Solution: Let $A = A_1 \cup A_2$ where $A_1, A_2 \subseteq X$ are open. For all $x \in A$, we find that $x \in A_1$ or $x \in A_2$. Without loss of generality, assume that $x \in A_1$. Then, there is a $\delta > 0$ such that $B(x, \delta) \subseteq A_1$. Since $A_1 \subseteq A$ (by definition of union), it follows that A is open.

(d) **5 pt** – If $\{A_t\}_{t \in T}$ is an arbitrary collection of open sets, then the union $U = \cup_{t \in T} A_t$ is open.

Solution: Since U is a union, if $x_0 \in U$, then there exists a $t_0 \in T$ such that $x_0 \in A_{t_0}$. By assumption, A_{t_0} is open and, thus there exists a $\delta > 0$ such that $B(x_0, \delta) \subseteq A_{t_0}$. Since U is a union, we see that $A_{t_0} \subseteq U$ and, thus $B(x_0, \delta) \subseteq U$. It is worth noting that one cannot use finite induction based on part (c) because T can be infinite.

6. Consider finding the real positive solution of the equation $x^3 - ax^2 - b = 0$ for some $a, b \in \mathbb{R}_{>0}$. Note: Descartes' rule of signs shows there is exactly one real positive root.

(a) **5 pt** – Show that any solution of this equation must also satisfy $x = f(x) \triangleq a + \frac{b}{x^2}$.

Solution: Adding $ax^2 + b$ to both sides and dividing by x^2 gives the desired equation. Since $x = 0$ cannot be a solution, this transformation preserves the set of solutions.

(b) **7.5 pt** – For $a = 4$ and $b = 32$, show that $f(x)$ is a contraction on $A = [4.5, 8]$ and compute a value for the contraction coefficient γ.

Solution: First, we note that the relevant metric space is $\mathbb{R}$ equipped with $d(x, y) = |x - y|$. Since $f(x)$ is decreasing for $x \geq 0$, $f(8) = 4.5$, and $f(4.5) = 4 + 128/81 \leq 4 + 128/64 = 6$, we see that $f([4.5, 8]) \subseteq [4.5, 6] \subseteq A$.

Taking the derivative gives $f'(x) = -\frac{64}{x^3}$ and integrating back shows that

$$|f(x) - f(y)| = \left| \int_x^y f'(s) ds \right| \leq |x - y| \frac{64}{x^3},$$

for $y \geq x$. To get a contraction coefficient less than 1, we need $x > 4$. Thus, f is a contraction on A with contraction coefficient $\gamma = 64/(4.5)^3 = (8/9)^3 < 1$.

The contraction property can also be shown via the decomposition

$$|f(x) - f(y)| = \frac{32}{xy} \left| \frac{1}{y} + \frac{1}{x} \right| |x - y| \leq \frac{64}{(4.5)^3} |x - y|.$$

However, bounding the numerator and denominator of $32(x+y)/(x^2y^2)$ separately (with worst-case values) does not work because $32(8 + 8)/(4.5)^4 \approx 1.24$.

(c) **5 pt** – For $a = 4$ and $b = 32$, prove that the sequence $x_{n+1} = f(x_n)$ converges starting from $x_1 \in A$? If so, what condition must its limit satisfy?

Solution: Since $\mathbb{R}$ (with absolute distance) is a complete metric space and f is a contraction on the closed subset A, we can apply the contraction mapping theorem. The contraction mapping theorem implies that $f(x)$ has unique fixed point x^* (in the set $[4.5, 8]$) and that the sequence x_n converges to this fixed point. This limit x must be the unique solution in A of the fixed-point equation $x^3 - 4x^2 - 32 = 0$.

(d) **2.5 pt** – For $a = 4$ and $b = 32$, what happens to the sequence $x_{n+1} = f(x_n)$ starting from $x_1 = 3$? Why?

Solution: One can compute $x_2 = f(x_1) = 4 + 32/3^2$ explicitly. Since $32/9 > 3$ and $32/9 < 4$, it also follows that $x_2 \in [7, 8]$. Since $x_2 \in A$, the sequence x_n still converges to the unique fixed point.

7. Let $\mathbb{R}$ be the field of real numbers equipped with absolute distance as a metric. Consider the set $X = B[a, b]$ of bounded functions mapping $[a, b]$ to $\mathbb{R}$ equipped with the metric

$$d_\infty(f, g) = \sup_{t \in [a,b]} |f(t) - g(t)|.$$

A sequence of functions $f_n(t)$ converges *pointwise* if $f_n(t) \to f(t)$ for all $t \in [a, b]$. *Uniform convergence* also requires that, for all $\epsilon > 0$, there is a single N such that $|f_n(t) - f(t)| < \epsilon$ for all $n > N$ and all $t \in [a, b]$.

(a) **5 pt** – Show that convergence $f_n \to f$ in the metric space (X, d_∞) implies uniform convergence.

Solution: We need to prove that, for all $\epsilon > 0$, there is a single N such that $|f_n(t) - f(t)| < \epsilon$ for all $n > N$ and all $t \in [a, b]$. Since $f_n \to f$ in X, we apply the definition of convergence in X (using the ϵ above) to see that there exists an N such that $d_\infty(f_n, f) < \epsilon$ for all $n > N$. Next, we note that

$$|f_n(t) - f(t)| \le \sup_{t \in [a,b]} |f_n(t) - f(t)| = d_\infty(f_n, f) < \epsilon$$

for all $n > N$ and all $t \in [a, b]$.

(b) **5 pt** – Let $W \subseteq X$ be the subset of continuous functions and let $g_n \in W$ be a sequence of functions that converges uniformly to g. Show that g is continuous.

[Hint: Apply the triangle inequality to $d_\infty(g(x), g(y))$ along the path $g(x), g_n(x), g_n(y), g(y)$.]

Solution: Following the hint, we get

$$d_\infty(g(x), g(y)) \le d_\infty(g(x), g_n(x)) + d_\infty(g_n(x), g_n(y)) + d_\infty(g_n(y), g(y)).$$

Since g_n converges uniformly to g, we see that, for any $\epsilon > 0$, there is an N such that $d_\infty(g(x), g_n(x)) < \epsilon/3$ and $d_\infty(g_n(y), g(y)) < \epsilon/3$ for all $n \ge N$. Notice that uniform convergence is required to use the same ϵ at x and y simultaneously. Since g_N is continuous, there is a δ (which depends on N, ϵ, x) such that $|g_N(x) - g_N(y)| < \epsilon/3$ for all $x, y \in [a, b]$ with $|x - y| < \delta$. This shows that, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|g(x) - g(y)| < \epsilon$ for all $x, y \in [a, b]$ with $|x - y| < \delta$. Therefore, $g(x)$ is continuous.

(c) **5 pt** – Let $x_n \to x$ be a sequence in $\mathbb{R}$ which converges. Use the above results to show that

$$\lim_{n\to\infty} \lim_{m\to\infty} g_n(x_m) = \lim_{m\to\infty} \lim_{n\to\infty} g_n(x_m).$$

Solution: The LHS expression is easily evaluated because the continuity of g_n implies that $\lim_{m\to\infty} g_n(x_m) = g_n(x)$ for all n. After that, the second limit requires only pointwise convergence to see that $\lim_{n\to\infty} g_n(x) = g(x)$.

The RHS expression can also be evaluated also by taking the limits separately. Since $g_n \to g$ is a sequence of continuous functions converging uniformly, the previous part implies that the limit function g is continuous. Thus, we can write

$$\lim_{m\to\infty} \lim_{n\to\infty} g_n(x_m) = \lim_{m\to\infty} g(x_m) = g(x),$$

where the first equality follows from the pointwise convergence of g_n to g (which follows from uniform convergence) and the second equality follows from the continuity of g.

To see how this can fail, try taking both these limits with $x_m = 1 - \frac{1}{m}$ and $g_n(x) = x^n$.