ECE 586: Vector Space Methods
Chapter 2: Metric Spaces and Topology

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Introduction

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  - Study of geometric properties preserved by continuous deformations

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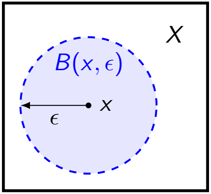
  - In engineering, a topology is typically defined using a metric

- Metric Spaces
  - A metric space (X, d) is a set X along with a well-defined metric d
  - A metric on a set X is a function $d\colon X \times X \to \mathbb{R}$ that satisfies:
    - $d(x, y) \geq 0 \quad \forall x, y \in X$; with equality if and only if $x = y$
    - $d(x, y) = d(y, x) \quad \forall x, y \in X$
    - $d(x, y) + d(y, z) \geq d(x, z) \quad \forall x, y, z \in X$.
  - $d(x, y)$ is called the distance between points x and y
  - Whiteboard Examples

Useful Abstractions



- Consider a metric space (X, d)
- "Set of points within distance ϵ from a point x"
  - The open ball of radius ϵ centered at x is

$$B_d(x, \epsilon) \triangleq \{y \in X | d(x, y) < \epsilon\}$$

  - $P =$ "For all $a \in B_d(x, \epsilon)$, there is $\delta > 0$ s.t. $B_d(a, \delta) \subset B_d(x, \epsilon)$"

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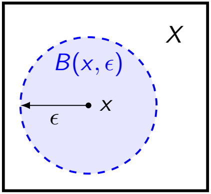

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- "Infinite list $x_1, x_2, x_3, \ldots$ of points in X"
  - A sequence $x_i \in X$ for $i \in \mathbb{N}$ equivalent to $x_i = f(i)$ for $f : \mathbb{N} \to X$
  - Ex. For $X = \mathbb{R}$ and $d(x, y) = |x - y|$, let $x_n = \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N}$

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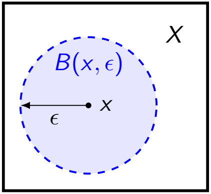

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- "A sequence of points approaches another point"
  - A sequence x_n converges to $x \in X$ (denoted $x_n \to x$) if, for any $\epsilon > 0$, there is natural number N such that $d(x, x_n) < \epsilon$ for all $n > N$

Convergence: Examples and Counterexamples

Definition

A sequence $x_1, x_2, \ldots$ in (X, d) is a Cauchy sequence if, for any $\epsilon > 0$, there is a natural number N (depending on ϵ) such that, for all $m, n > N$,

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- Theorem: Every convergent sequence is a Cauchy sequence
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  - One can show x_n is a Cauchy sequence and $|x_n - \sqrt{2}| \to 0$
- But, according to definition sequence does not converge!
  - Convergence requires limit lives in X but $\sqrt{2} \notin \mathbb{Q}$

Metric Topology

A topology is a collection of “open” sets satisfying certain properties

Definition

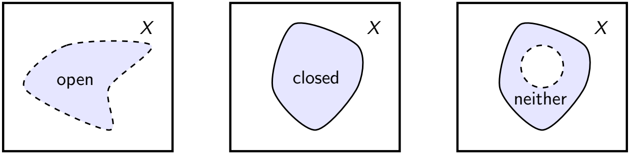
Let W be a subset of a metric space (X, d). The set W is called open if, for every $w \in W$, there is an $\epsilon > 0$ such that $B_d(w, \epsilon) \subseteq W$.

Definition

Subset W of (X, d) is closed if its complement $W^c = X - W$ is open.

Theorem

1. *$\emptyset$ and X are open*
2. *any union of open sets is open*
3. *any finite intersection of open sets is open*



Interior, Limit points, and Closure

For a metric space (X, d) and subset $W \subseteq X$:

Definition

A point $w \in W$ is in the interior of W (denoted W°) if there is a $\delta > 0$ such that, for all $x \in X$ with $d(x, w) < \delta$, it follows that $x \in W$.

Definition

A point $w \in W$ is a limit point of W if there is a sequence of distinct elements, $w_1, w_2, \ldots \in W$, that converges to w.

Definition

A point $x \in X$ is in the closure of W (denoted $\overline{W}$) if, for all $\delta > 0$, there is a $w \in W$ such that $d(x, w) < \delta$.

- Properties
  - The interior W° is open (see definition)
  - W is closed if and only if it contains all of its limit points
  - Closure $\overline{W}$ equals union of W and all its limit points (thus is closed)

Continuity

Let $f\colon X \to Y$ be a function between metric spaces (X, d_X) and (Y, d_Y):

Definition

The function f is continuous at $x_0 \in X$ if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in X$ satisfying $d_X(x_0, x) < \delta$,

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Theorem

If f is continuous at x_0, then $f(x_n) \to f(x_0)$ for all sequences $x_1, x_2, \ldots \in X$ such that $x_n \to x_0$. Conversely, if $f(x_n) \to f(x_0)$ for all sequences $x_1, x_2, \ldots \in X$ such that $x_n \to x_0$, then f is continuous at x_0.

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Definition

A function $f\colon X \to Y$ is called Lipschitz continuous on $A \subseteq X$ if there is a constant $L \in \mathbb{R}$ such that $d_Y(f(x), f(y)) \leq L d_X(x, y)$ for all $x, y \in A$.

Completeness

Definition

A metric space (X, d) is said to be complete if every Cauchy sequence in (X, d) converges to a limit $x \in X$.

Example

Consider the sequence $x_n \in \mathbb{Q}$ defined by $x_1 = 2$ and $x_{n+1} = \frac{1}{2}x_n + 1/x_n$. We have seen that this sequence satisfies $|x_n - \sqrt{2}| \to 0$ but $\sqrt{2}$ is not rational. Thus, the standard metric space of rationals is not complete.

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Key Point

The standard metric space of real numbers is a complete metric space. This can be shown using Cauchy sequences of rational numbers because $\mathbb{Q}$ is dense in $\mathbb{R}$. Note: proof not discussed but available on website.

Contraction Mapping Theorem

Definition

Let A be a subset of a metric space (X, d) and $f \colon X \to X$ be a function. Then, f is a contraction on A if $f(A) \subseteq A$ and there exists a constant $\gamma < 1$ such that $d\left(f(x), f(y)\right) \leq \gamma d(x, y)$ for all $x, y \in A$.

Example

Consider metric space $X = [0, 1]$ with absolute distance. Define $f \colon X \to X$ by $f(x) = 1 - \frac{1}{2}x$ and observe $|f(x) - f(y)| = \frac{1}{2}|x - y|$.

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Theorem (Contraction Mapping Theorem)

Let (X, d) be a complete metric space and f be contraction on a closed subset $A \subseteq X$. Then, f has a unique fixed point x^* in A such that $f(x^*) = x^*$ and $x_{n+1} = f(x_n)$ converges to x^* from any initial $x_1 \in A$. Moreover, x_n satisfies the error bounds:

$$d(x^*, x_n) \leq \gamma^{n-1} d(x^*, x_1) \text{ and } d(x^*, x_{n+1}) \leq d(x_n, x_{n+1})\gamma/(1 - \gamma).$$

Applications of the Contraction Mapping Theorem

The following important results in applied mathematics have relatively simple proofs based on the contraction mapping theorem.

- Picard's uniqueness theorem for differential equations
    - Differential equation $y'(t) = f(t, y(t))$ for $t \in [a, b]$ with $y(a) = y_0$
    - Assume $f(t, y)$ is Lipschitz continuous in y for $t \in [a, b]$
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- Implicit function theorem
  - Let $f \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable on open A
  - Let $g \colon \mathbb{R}^n \to \mathbb{R}^m$ be defined implicitly by $f(x, g(x)) = 0$
  - For $x_0 \in A$, assume $f(x_0, y_0) = 0$ and y-Jacobian invertible at (x_0, y_0)
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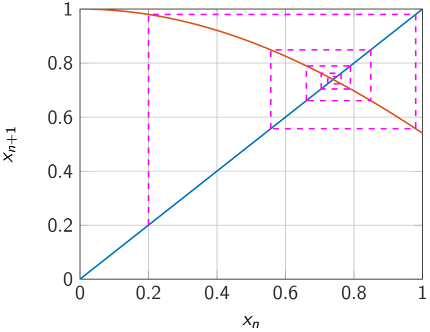
- Dynamic Programming for a Markov Decision Process (MDP)
  - State-action (s, a) defines probability $p(s'|s, a)$ and reward $R(s, a)$
  - Finite state + discounted reward $\Rightarrow$ stationary optimal policy

Contraction Mapping Example

Starting from $x_1 = 0.2$, define $x_{n+1} = \cos(x_n)$ and plot the points (x_n, x_{n+1}). Each point is connected to the slope-1 line to emphasize the path taken.



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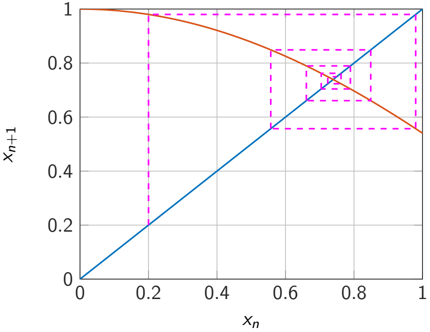

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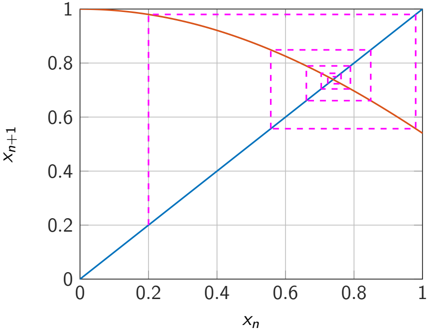

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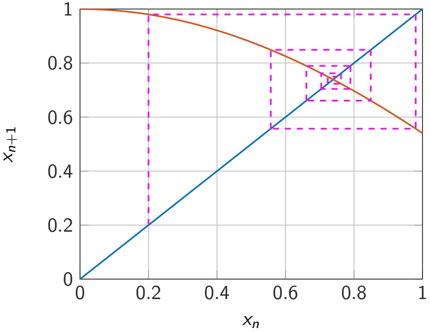

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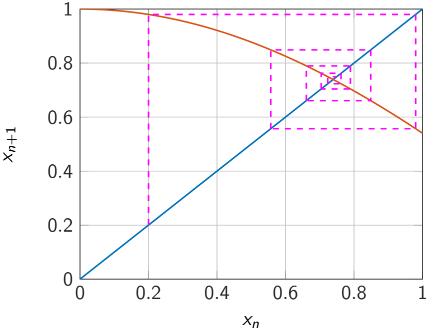

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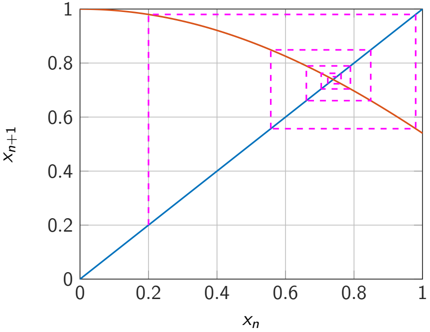

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- $f'(t) = -\sin(t)$ and $\sin([0,1]) = [0, \sin(1)]$ with $\sin(1) \approx 0.84$
- $|\cos(y) - \cos(x)| \leq 0.85\,|y - x|$ $\Rightarrow$ $f(x)$ is a contraction on $[0,1]$
- $x_{n+1} = \cos(x_n)$ converges to unique fixed point $x^* = \cos(x^*) \approx 0.739$

Compactness

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  - A subset of Euclidean $\mathbb{R}^n$ is compact iff it is closed and bounded
  - But, the standard metric space of real numbers is not compact because it is not totally bounded.

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Theorem

A closed subset A of a compact space X is itself a compact space.

Compactness and Sequences

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Theorem

A sequence in a compact metric space has a subsequence that converges.

Compactness and Sequences

Definition

Let $x_1, x_2, \ldots \in X$ be a sequence and $n_1, n_2, \ldots \in \mathbb{N}$ be a strictly increasing sequence. Then, $x_{n_1}, x_{n_2}, \ldots$ is called subsequence.

Theorem

A sequence in a compact metric space has a subsequence that converges.

Example

For the compact metric space $X = [-2, 2] \subset \mathbb{R}$ with absolute distance, let $x_n = (-1)^n + \frac{1}{n}$. Then, subsequence $x_2, x_4, x_6, \ldots$ converges to 1.

- Sketch proof on whiteboard in pictures

Properties of Real Numbers

- Let us consider extreme values for sets of real numbers
  - Extended Real Numbers: $\overline{\mathbb{R}} \triangleq \mathbb{R} \cup \{\infty, -\infty\}$
  - Compact metric space with metric $d_{\overline{\mathbb{R}}}(x, y) \triangleq |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$
  - "$x_n \to \infty$" equivalent to "$\forall M > 0$, $\exists N \in \mathbb{N}$, $\forall n > N$, $x_n > M$"

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Let X be a metric space and $f : X \to \mathbb{R}$ be a function from X to the real numbers. Let $M = \sup f(A)$ for some non-empty $A \subseteq X$. Then, there exists a sequence $x_1, x_2, \ldots \in A$ such that $\lim_n f(x_n) = M$.

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Theorem

Any bounded non-decreasing sequence of real numbers converges to its supremum.

Sequences of Functions

Let (X, d_X) and (Y, d_Y) be metric spaces and $f_n\colon X \to Y$ for $n \in \mathbb{N}$ be a sequence of functions mapping X to Y.

Definition

The sequence f_n converges pointwise to $f\colon X \to Y$ if, for all $x \in X$,

$$\lim_{n\to\infty} f_n(x) = f(x)$$

Definition

The sequence f_n converges uniformly to $f\colon X \to Y$ if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ \forall x \in X,\ d_Y\left(f_n(x), f(x)\right) < \epsilon.$$

Theorem

If each f_n is continuous and f_n converges uniformly to $f\colon X \to Y$, then f is continuous.

Two Important Results

Theorem

Let X be a metric space and $f : X \to \mathbb{R}$ be a continuous function from X to $\mathbb{R}$. If A is a compact subset of X, then there exists $x \in A$ such that $f(x) = \sup f(A)$ (i.e., f achieves a maximum on A).

Theorem

Let (X, d) be a compact metric space and $C_b(X)$ be the set of bounded continuous functions mapping X to $\mathbb{R}$. If we define the metric

$$d_\infty(f, g) = \max_{x \in X} |f(x) - g(x)|$$

on $C_b(X)$, then it becomes a complete metric space.