Spatial Transformations

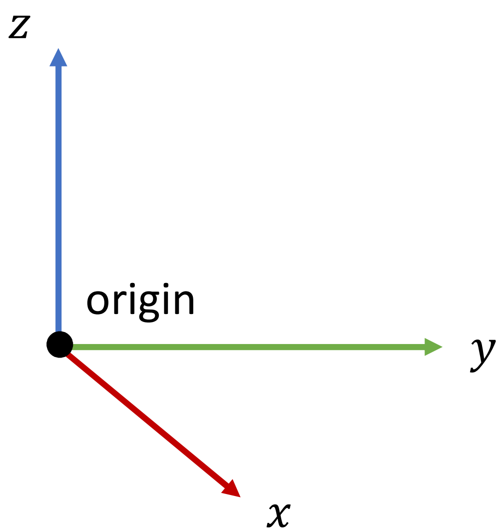
Jungdam Won

Computer Science & Engineering

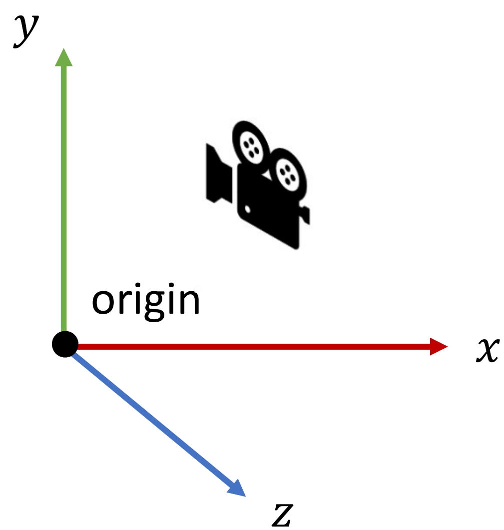
Seoul National Univ.

Many contents are adopted from the slides of the Computer Graphics course at SNU lectured by Jehee Lee

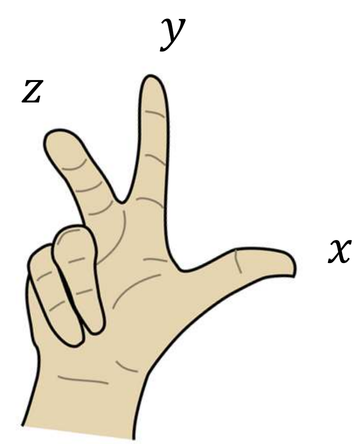
Graphics Coordinate System



mathematics



graphics



Right-handed coordinate

Spatial Transformations

- ***Linear*** transformation
- ***Rigid*** transformation
- ***Affine*** transformation
- ~~***Projective*** transformation~~

Linear Transformations

- A ***linear transformation*** T is a mapping between vector spaces
  - T maps vectors to vectors
  - Linear combination is invariant under T

$$T\left(\sum_{i=0}^{N} c_i\,\mathbf{v}_i\right) = c_0 T(\mathbf{v}_0) + c_1 T(\mathbf{v}_1) + \cdots + c_N T(\mathbf{v}_N)$$

- In 3D space, T can be represented by a 3x3 matrix

$$T(\mathbf{v}) = A_{3\times 3}\mathbf{v}_{3\times 1}$$

3D Matrix

capital

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{bmatrix}$$

vector

vector

$$\mathbf{I}\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{v}$$

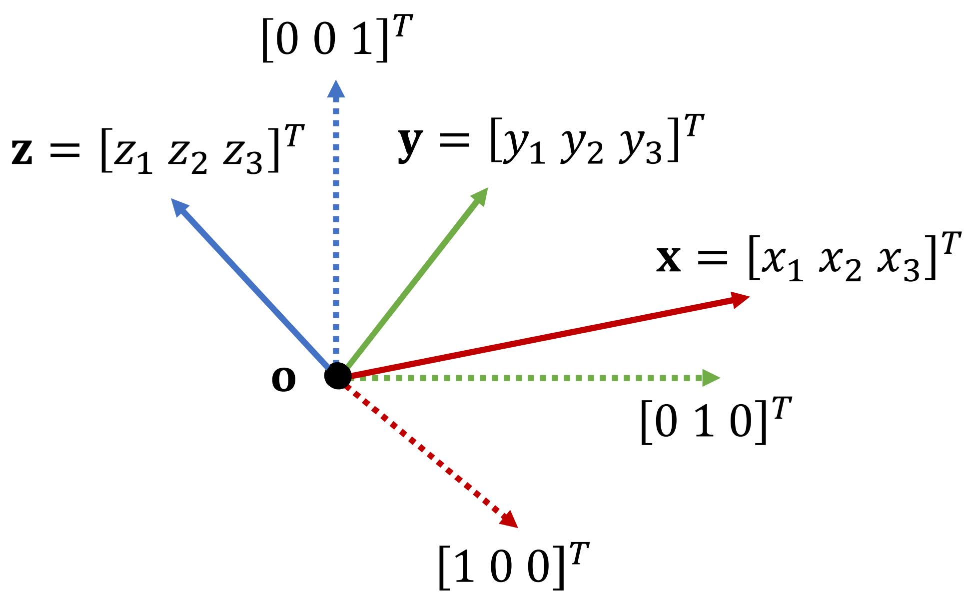
Identity matrix

Column Representation

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{bmatrix}$$

$$A\mathbf{v} = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1\,\boldsymbol{a}_1 + v_2\,\boldsymbol{a}_2 + v_3\,\boldsymbol{a}_3$$

Columns are Axes



$$A = [\mathbf{x} \quad \mathbf{y} \quad \mathbf{z}] = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

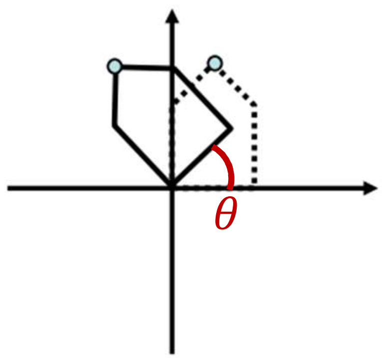
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

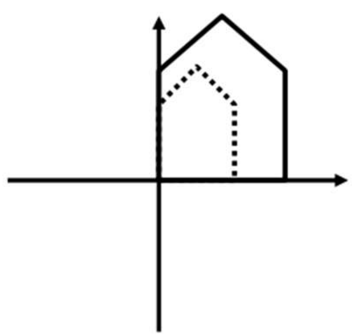
Examples of Linear Transformations

- 2D rotation



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

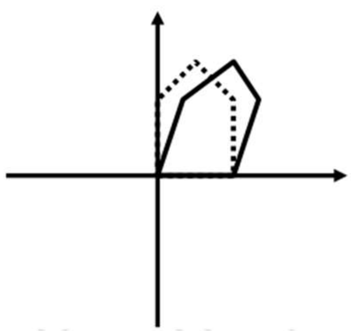
- 2D scaling



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

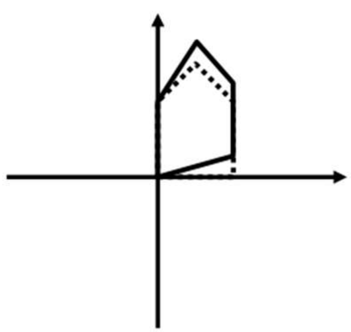
Examples of Linear Transformations

- 2D shear
  - Along x-axis



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + sy \\ y \end{bmatrix}$$

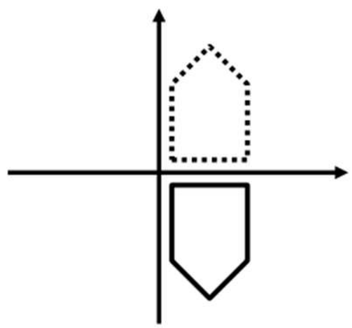
  - Along y-axis



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y + sx \end{bmatrix}$$

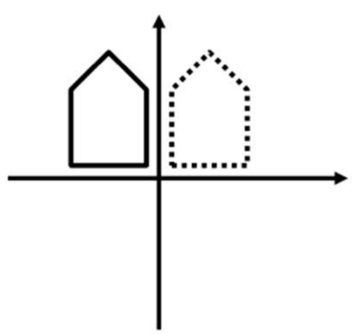
Examples of Linear Transformations

- 2D reflection
  - Along x-axis



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

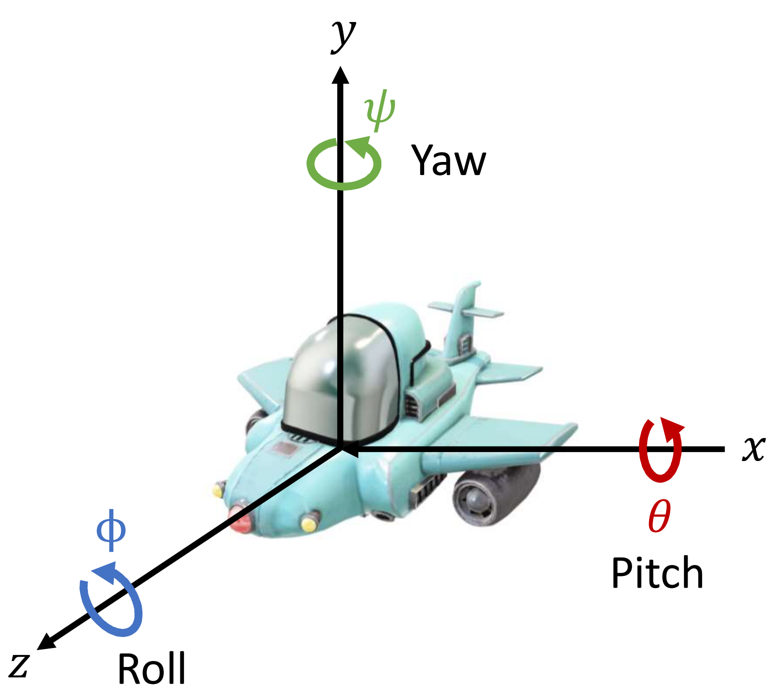
  - Along y-axis



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

Examples of Affine Transformations

- 3D rotation



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Properties of Linear Transformations

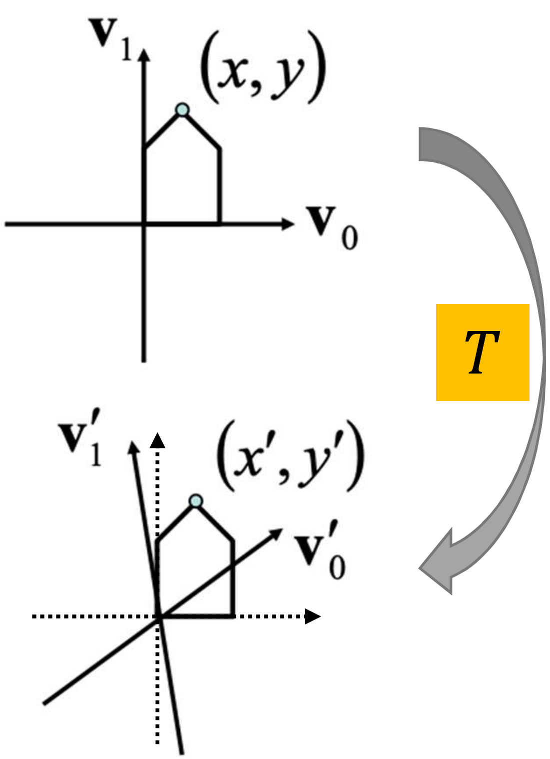
- Any linear transformation between 3D spaces can be represented by a 3x3 matrix

- Any linear transformation between 3D spaces can be represented as a combination of rotation, shear, and scaling

- Rotation can be represented as a combination of scaling and shear

Changing Bases

- Linear transformations as a change of bases



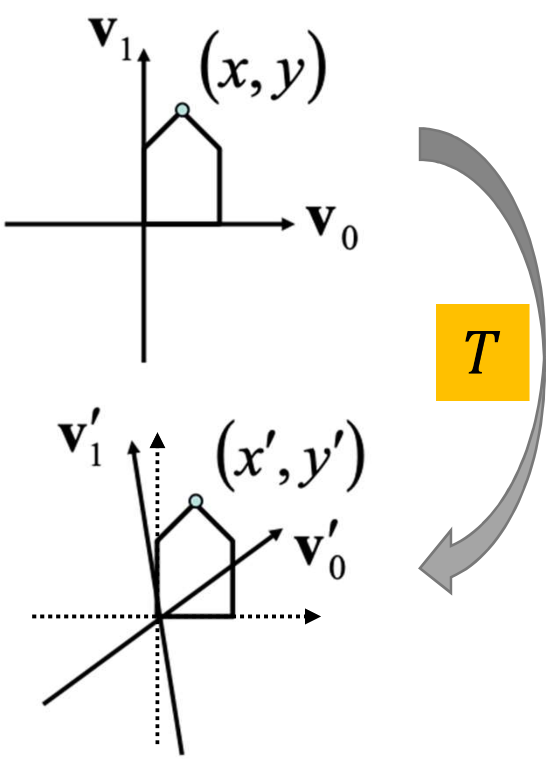
$$x\mathbf{v}_0 + y\mathbf{v}_1 = x'\mathbf{v}'_0 + y'\mathbf{v}'_1$$

$$[\mathbf{v}_0 \quad \mathbf{v}_1]\begin{bmatrix}x\\ y\end{bmatrix} = [\mathbf{v}'_0 \quad \mathbf{v}'_1]\begin{bmatrix}x'\\ y'\end{bmatrix}$$

$$[\mathbf{v}'_0 \quad \mathbf{v}'_1]^{-1}[\mathbf{v}_0 \quad \mathbf{v}_1]\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x'\\ y'\end{bmatrix}$$

Changing Bases

- Linear transformations as a change of bases



$$\mathbf{v}_0 = a_0\mathbf{v'}_0 + a_1\mathbf{v'}_1$$
$$\mathbf{v}_1 = b_0\mathbf{v'}_0 + b_1\mathbf{v'}_1$$

$$x\mathbf{v}_0 + y\mathbf{v}_1 = x'\mathbf{v'}_0 + y'\mathbf{v'}_1$$

$$\begin{bmatrix}\mathbf{v}_0 & \mathbf{v}_1\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}\mathbf{v'}_0 & \mathbf{v'}_1\end{bmatrix}\begin{bmatrix}x' \\ y'\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{v'}_0 & \mathbf{v'}_1\end{bmatrix}\begin{bmatrix}a_0 & b_0 \\ a_1 & b_1\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}\mathbf{v'}_0 & \mathbf{v'}_1\end{bmatrix}\begin{bmatrix}x' \\ y'\end{bmatrix}$$

$$\boxed{\begin{bmatrix}a_0 & b_0 \\ a_1 & b_1\end{bmatrix}}\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}x' \\ y'\end{bmatrix}$$

Affine Transformation

- An affine transformation T is a mapping between affine spaces
  - T maps vectors to vectors, and points to points
  - T is a linear transformation on vectors
  - Affine combination is invariant under T

$$T\left(\sum_{i=0}^{N} c_i\,\mathbf{p}_i\right) = c_0 T(\mathbf{p}_0) + c_1 T(\mathbf{p}_1) + \cdots + c_N T(\mathbf{p}_N)$$

- In 3D space, T can be represented by a 3x3 matrix together with a 3x1 translation vector

$$T(\mathbf{p}) = \mathrm{A}_{3\times 3}\mathbf{p}_{3\times 1} + \mathbf{t}_{3\times 1}$$

Properties of Affine Transformations

- Any affine transformation between 3D spaces can also be represented by a 4x4 matrix when using an extra coordinate
- Use an *extra* coordinate
  - **Point** : $[x\ y\ 1]^T$ in 2D, $[x\ y\ z\ 1]^T$ in 3D
  - **Vector** : $[x\ y\ 0]^T$ in 2D, $[x\ y\ z\ 0]^T$ in 3D

$$T(\mathbf{p}) = \begin{bmatrix} \mathrm{A}_{3\times3} & \mathbf{t}_{3\times1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{3\times1} \\ 1 \end{bmatrix}$$

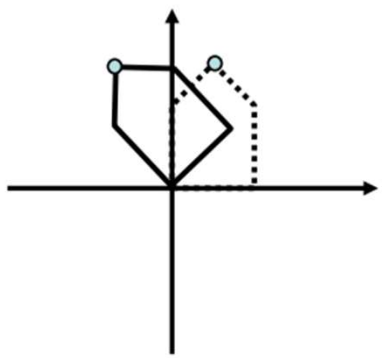
$$T(\mathbf{v}) = \begin{bmatrix} \mathrm{A}_{3\times3} & \mathbf{t}_{3\times1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_{3\times1} \\ 0 \end{bmatrix}$$

Properties of Affine Transformations

- An affine transformation maps ***lines*** to ***lines***
- An affine transformation maps ***parallel lines*** to ***parallel lines***
- An affine transformation preserves ***ratios of distance*** along a line
- An affine transformation does not preserve absolute distances and angles

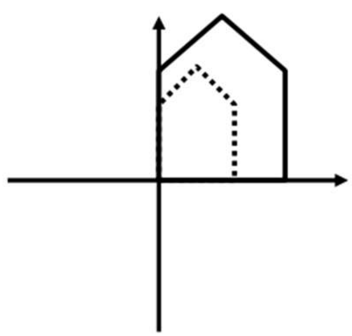
Examples of Affine Transformations

- 2D rotation



$$\begin{bmatrix} x' \\ y' \\ 0,1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0,1 \end{bmatrix}$$

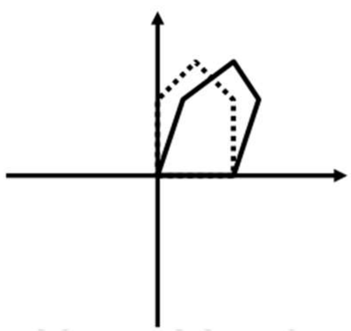
- 2D scaling



$$\begin{bmatrix} x' \\ y' \\ 0,1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0,1 \end{bmatrix}$$

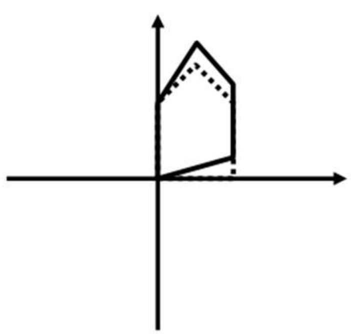
Examples of Affine Transformations

- 2D shear
  - Along x-axis



$$\begin{bmatrix} x' \\ y' \\ 0,1 \end{bmatrix} = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0/1 \end{bmatrix} = \begin{bmatrix} x + sy \\ y \\ 0,1 \end{bmatrix}$$

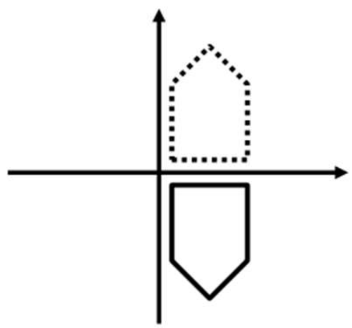
  - Along y-axis



$$\begin{bmatrix} x' \\ y' \\ 0,1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0,1 \end{bmatrix} = \begin{bmatrix} x \\ y + sx \\ 0,1 \end{bmatrix}$$

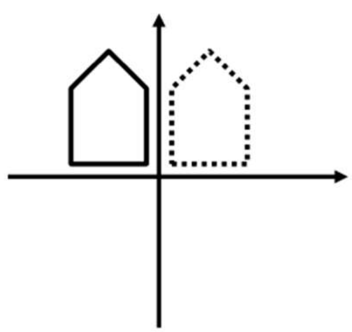
Examples of Affine Transformations

- 2D reflection
  - Along x-axis



$$\begin{bmatrix} x' \\ y' \\ 0,1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0,1 \end{bmatrix} = \begin{bmatrix} x \\ -y \\ 0,1 \end{bmatrix}$$

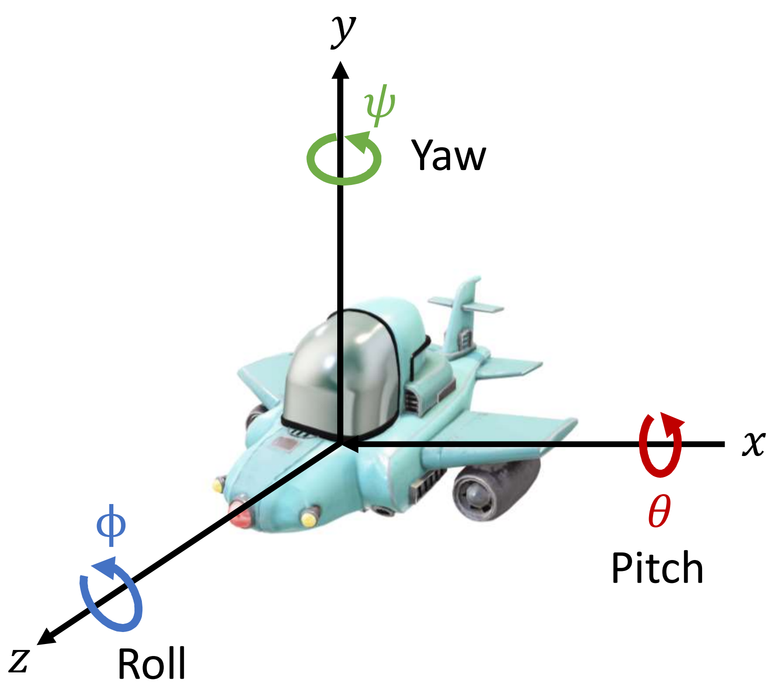
  - Along y-axis



$$\begin{bmatrix} x' \\ y' \\ 0,1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0,1 \end{bmatrix} = \begin{bmatrix} -x \\ y \\ 0,1 \end{bmatrix}$$

Examples of Affine Transformations

- 3D rotation



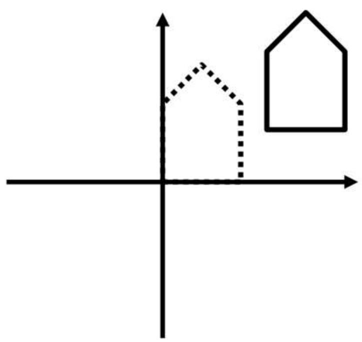
$$\begin{bmatrix} x' \\ y' \\ z' \\ 0,1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0,1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 0,1 \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & \sin\psi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\psi & 0 & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0,1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 0,1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0,1 \end{bmatrix}$$

Examples of Affine Transformations

- 2D translation



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{\mathrm{x}} \\ 0 & 1 & t_{\mathrm{y}} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

Points

$$\begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{\mathrm{x}} \\ 0 & 1 & t_{\mathrm{y}} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Vectors

Translation is simply ignored for vectors

Composite Transformations

- Composite 2D Translation

$$T = \mathbf{T}(\mathbf{t}_2) \cdot \mathbf{T}(\mathbf{t}_1)$$
$$= \mathbf{T}(\mathbf{t}_2 + \mathbf{t}_1)$$

$$\begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x2} + t_{x1} \\ 0 & 1 & t_{y2} + t_{y1} \\ 0 & 0 & 1 \end{bmatrix}$$

Composite Transformations

- Composite 2D Scaling

$$T = \mathbf{S}(\mathbf{s}_2) \cdot \mathbf{S}(\mathbf{s}_1)$$
$$= \mathbf{S}(\mathbf{s}_2 \odot \mathbf{s}_1)$$

$$\begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x2} \cdot s_{x1} & 0 & 0 \\ 0 & s_{y2} \cdot s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composite Transformations

- Composite 2D Rotation

$$T = \mathbf{R}(\theta_2) \cdot \mathbf{R}(\theta_1)$$

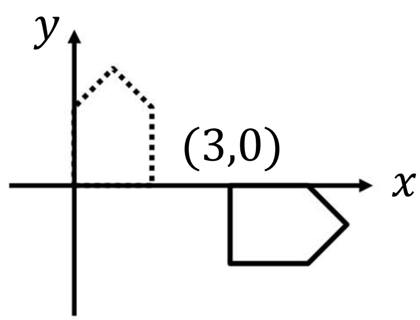
$$= \mathbf{R}(\theta_2 + \theta_1)$$

$$\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

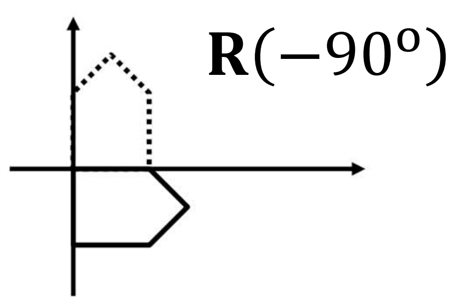
$$= \begin{bmatrix} \cos(\theta_2 + \theta_1) & -\sin(\theta_2 + \theta_1) & 0 \\ \sin(\theta_2 + \theta_1) & \cos(\theta_2 + \theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

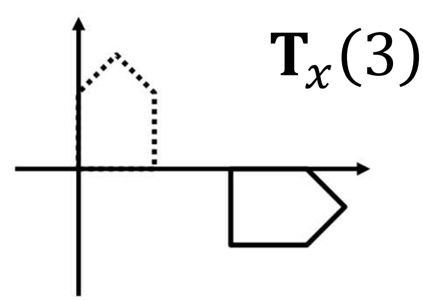
Composite Transformations

- Suppose we want,



- We have to compose two transformations

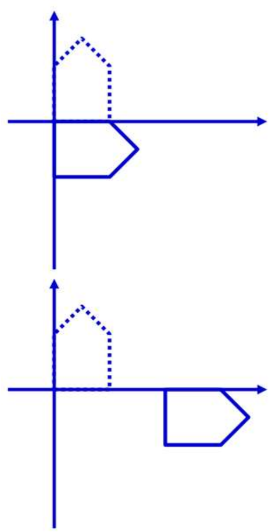




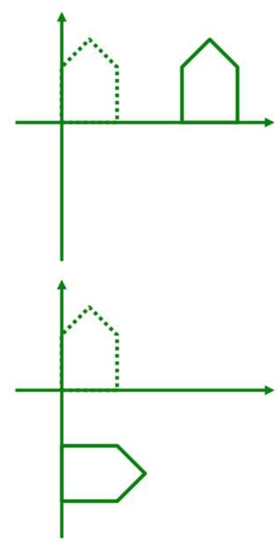
Composite Transformations

- Matrix multiplication is not commutative

$$\mathbf{T}_x(3) \cdot \mathbf{R}(-90^o) \neq \mathbf{R}(-90^o) \cdot \mathbf{T}_x(3)$$



Rotation followed by translation

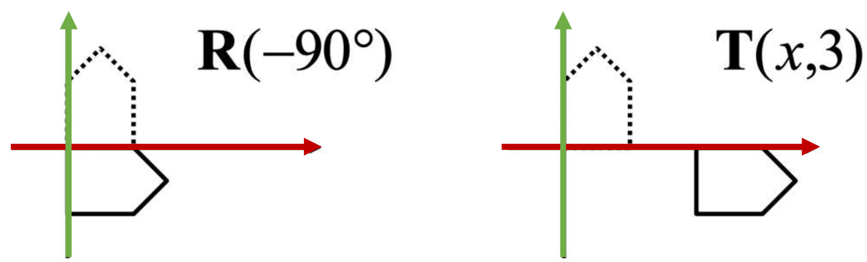


Translation followed by rotation

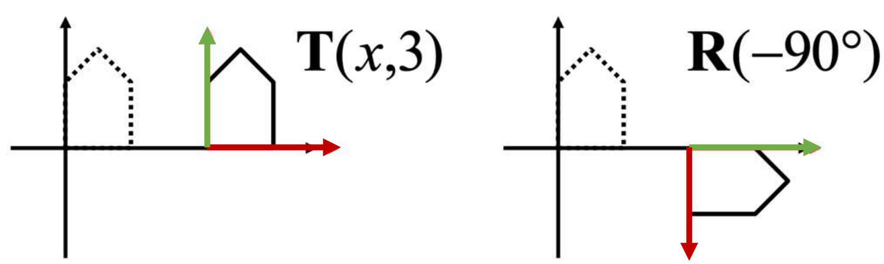
Composite Transformations

$$T\mathbf{p} = \mathbf{T}_x(3) \cdot \mathbf{R}(-90^o)\mathbf{p}$$ (Column major convention)

- There exist two interpretations
  - R-to-L : interpret operations w.r.t. fixed (world) coordinates

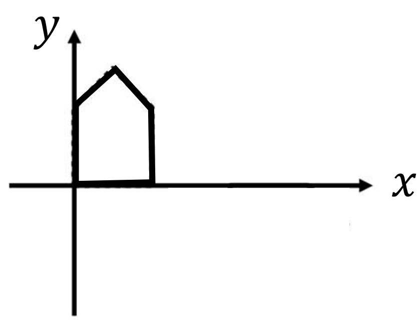


  - L-to-R : interpret operations w.r.t. moving (local) coordinates



Composite Transformations

- Given this configuration,



- What are the results of transformations below?

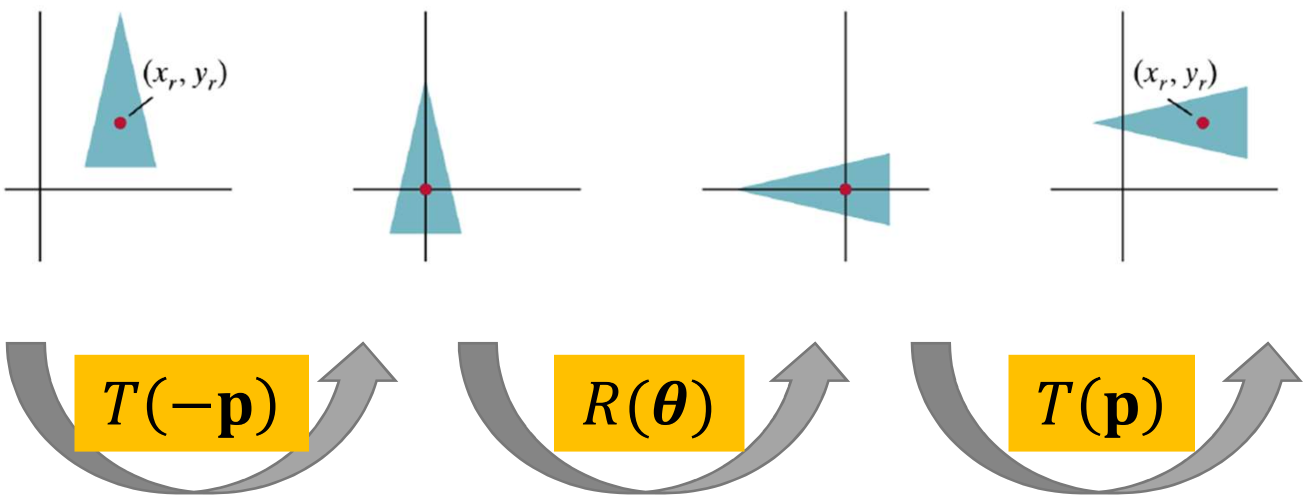
$$\mathbf{T}_x(3) \cdot \mathbf{R}(-90^{\mathrm{o}}) \qquad \mathbf{R}(-90^{\mathrm{o}}) \cdot \mathbf{T}_y(3)$$

Pivot-Point Rotation

- Rotation $\boldsymbol{\theta}$ w.r.t. a pivot point $\mathbf{p} = (\mathrm{x_r}, \mathrm{y_r})$

$$T(\mathbf{p}) \cdot R(\boldsymbol{\theta}) \cdot T(-\mathbf{p})$$

$$= \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

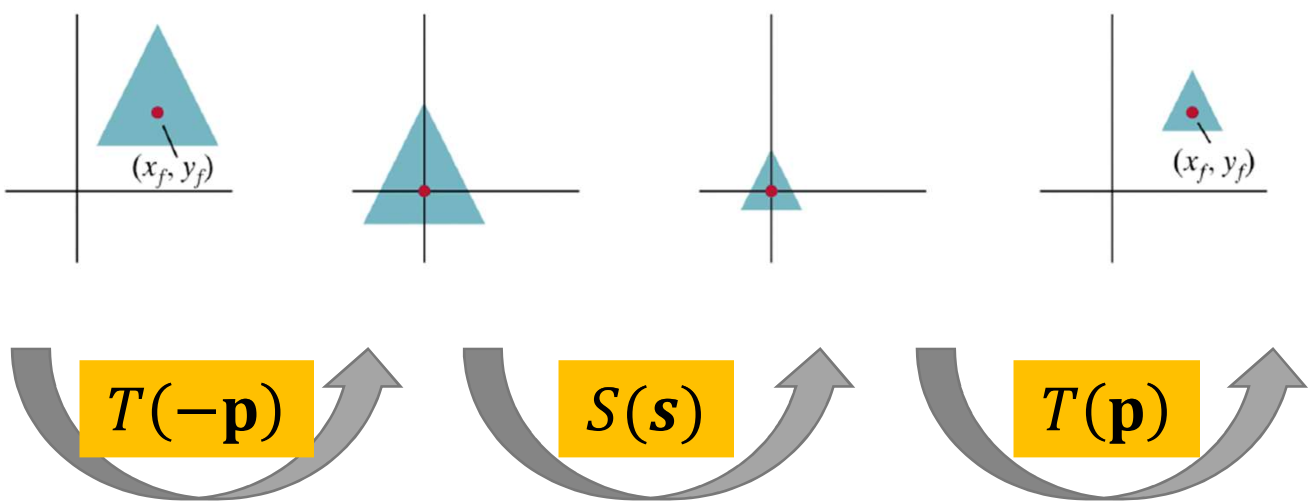


Fixed-Point Scaling

- Scaling by $\boldsymbol{s} = (\mathrm{s_x}, \mathrm{s_y})$ w.r.t. a pivot point $\mathbf{p} = (\mathrm{x_r}, \mathrm{y_r})$

$$T(\mathbf{p}) \cdot S(\boldsymbol{s}) \cdot T(-\mathbf{p})$$

$$= \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{s_x} & 0 & 0 \\ 0 & s_{\mathrm{y}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

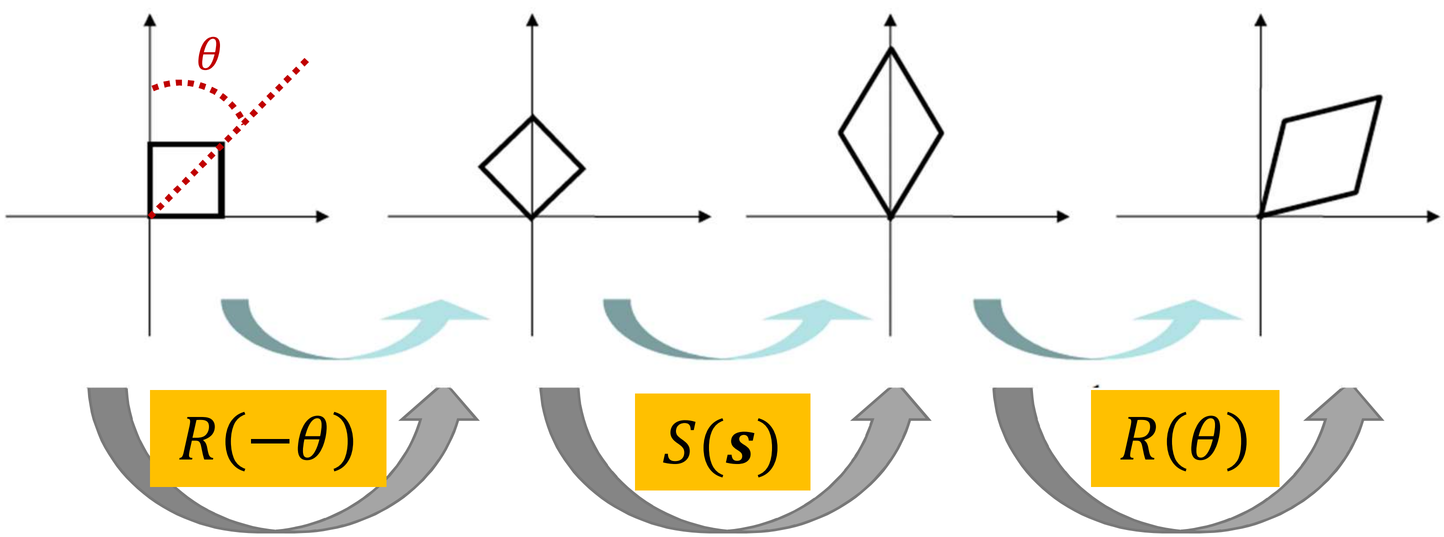


Scaling Direction

- Scaling by $\boldsymbol{s} = (\mathrm{s_x}, \mathrm{s_y})$ along an arbitrary axis

$$R(-\theta) \cdot S(\boldsymbol{s}) \cdot R(\theta)$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{s_x} & 0 & 0 \\ 0 & s_\mathrm{y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Review of Affine Frames

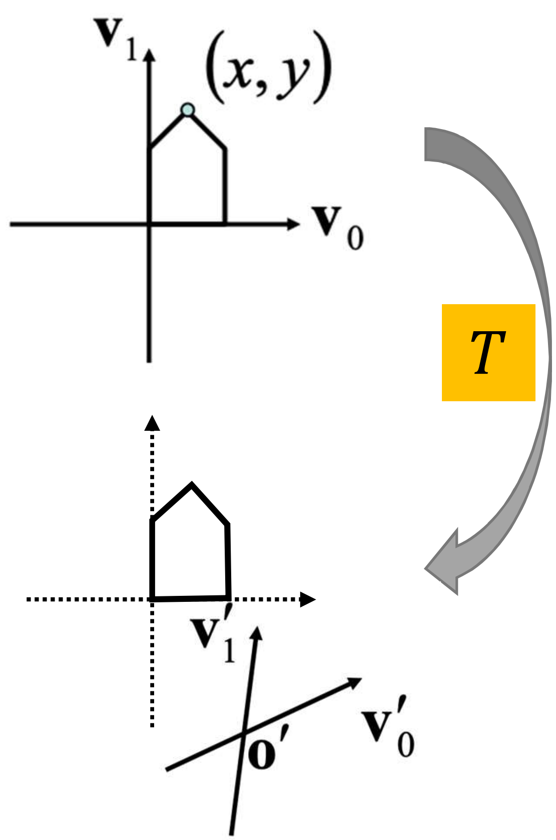
- A ***frame*** is defined as a set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_N\}$ and a point $\mathbf{o}$
  - Set of vectors are bases of the associated vector space
  - $\mathbf{o}$ is the origin of the frame
  - N is the dimension of the affine space

  - Any point $\mathbf{p}$ can be written as
$$\mathbf{p} = \mathbf{o} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_N\mathbf{v}_N$$

  - Any vector $\mathbf{v}$ can be written as
$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_N\mathbf{v}_N$$

Changing Frames

- Affine transformations as a change of frame



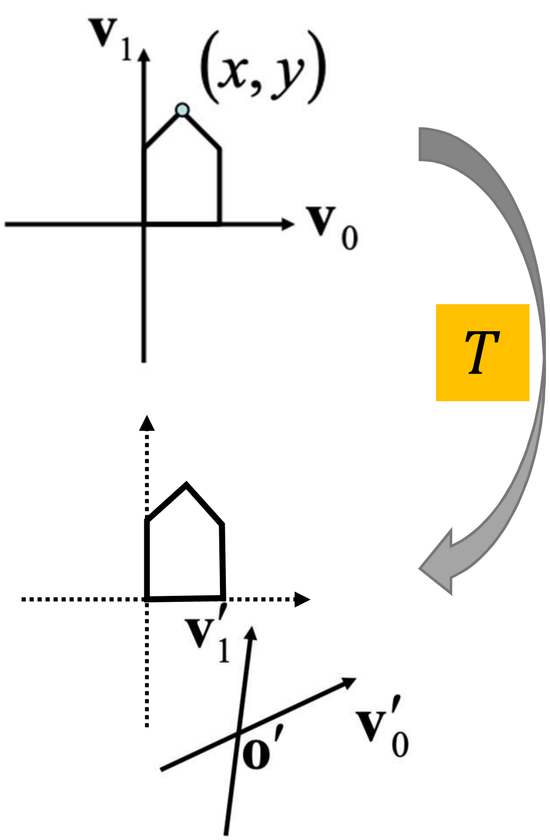
$$x\mathbf{v}_0 + y\mathbf{v}_1 + \mathbf{o} = x'\mathbf{v'}_0 + y'\mathbf{v'}_1 + \mathbf{o}'$$

$$[\mathbf{v}_0 \quad \mathbf{v}_1 \quad \mathbf{o}]\begin{bmatrix} x \\ y \\ 0{,}1 \end{bmatrix} = [\mathbf{v'}_0 \quad \mathbf{v'}_1 \quad \mathbf{o}']\begin{bmatrix} x' \\ y' \\ 0{,}1 \end{bmatrix}$$

$$[\mathbf{v'}_0 \quad \mathbf{v'}_1 \quad \mathbf{o}']^{-1}\,[\mathbf{v}_0 \quad \mathbf{v}_1 \quad \mathbf{o}]\begin{bmatrix} x \\ y \\ 0{,}1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 0{,}1 \end{bmatrix}$$

Changing Frames

- Affine transformations as a change of frame



$$\mathbf{v}_0 = a_0\mathbf{v}'_0 + a_1\mathbf{v}'_1$$
$$\mathbf{v}_1 = b_0\mathbf{v}'_0 + b_1\mathbf{v}'_1$$
$$\mathbf{o} = c_0\mathbf{v}'_0 + c_1\mathbf{v}'_1 + \mathbf{o}'$$

$$x\mathbf{v}_0 + y\mathbf{v}_1 + \mathbf{o} = x'\mathbf{v}'_0 + y'\mathbf{v}'_1 + \mathbf{o}'$$

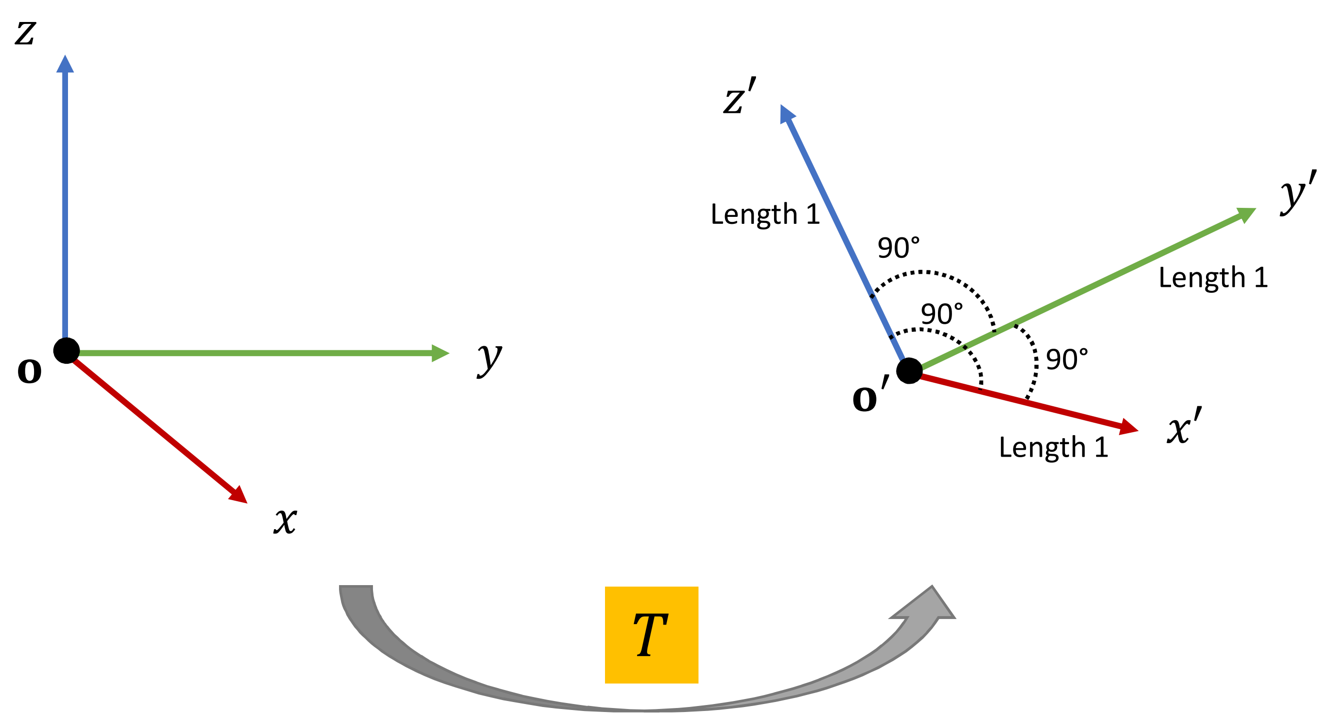
$$[\mathbf{v}_0 \quad \mathbf{v}_1 \quad \mathbf{o}]\begin{bmatrix} x \\ y \\ 0{,}1 \end{bmatrix} = [\mathbf{v}'_0 \quad \mathbf{v}'_1 \quad \mathbf{o}']\begin{bmatrix} x' \\ y' \\ 0{,}1 \end{bmatrix}$$

$$[\mathbf{v}'_0 \quad \mathbf{v}'_1 \quad \mathbf{o}']\begin{bmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 0{,}1 \end{bmatrix} = [\mathbf{v}'_0 \quad \mathbf{v}'_1 \quad \mathbf{o}]\begin{bmatrix} x' \\ y' \\ 0{,}1 \end{bmatrix}$$

$$\begin{bmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 0{,}1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 0{,}1 \end{bmatrix}$$

Rigid Transformation

- A rigid transformation T is a special case of affine transformation that consists of rotation and translation only



Rigid Transformation

- In 3D spaces, T can be represented as

$$T(\mathbf{p}) = \mathrm{R}_{3\times3}\mathbf{p}_{3\times1} + \mathbf{t}_{3\times1} = \begin{bmatrix} \mathrm{R}_{3\times3} & \mathbf{t}_{3\times1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{3\times1} \\ 1 \end{bmatrix}$$

$$\text{where } \mathrm{R}^T\mathrm{R} = \mathrm{I} \quad \text{and} \quad \det \mathrm{R} = 1$$

- Properties
  - T maps vectors to vectors, and points to points
  - T preserves distances between all points
  - T preserves cross product for all vectors (to avoid reflection)

Rigid Body Rotation

- $\mathrm{R}^T\mathrm{R} = \mathrm{I}$

$$\mathrm{R}^T\mathrm{R} = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{y}^T \\ \mathbf{z}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^T\mathbf{x} & \mathbf{x}^T\mathbf{y} & \mathbf{x}^T\mathbf{z} \\ \mathbf{y}^T\mathbf{x} & \mathbf{y}^T\mathbf{y} & \mathbf{y}^T\mathbf{z} \\ \mathbf{z}^T\mathbf{x} & \mathbf{z}^T\mathbf{y} & \mathbf{z}^T\mathbf{z} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_{3\times 3}$$

$$\mathrm{R}^T\mathrm{R} = \mathrm{I} \quad \Rightarrow \quad \mathrm{R}^{-1} = \mathrm{R}^T$$

Rigid Body Rotation

- R is normalized
  - The squares of the elements in any row or column sum to 1
- R is orthogonal

$$\mathrm{R}\mathrm{R}^T = \mathrm{R}^T\mathrm{R} = \mathrm{I}$$

  - The dot product of any pair of rows or any pair columns is 0
- The columns of R correspond to the vectors of the principle axes of the rotated coordinate frame

Rigid Body Rotation

- Rigid body transformations allow only rotation ($\mathrm{R}_{3\times 3}$) and translation ($\mathbf{t}_{3\times 1}$)

$$T(\mathbf{p}) = \mathrm{R}_{3\times 3}\mathbf{p}_{3\times 1} + \mathbf{t}_{3\times 1}$$

- Rotation matrices form SO(3)
  - Special orthogonal group

$$\mathrm{R}\mathrm{R}^T = \mathrm{R}^T\mathrm{R} = \mathrm{I}$$ (Distance preserving)

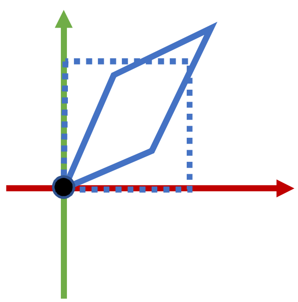
$$\det \mathrm{R} = 1$$ (No reflection)

Summary

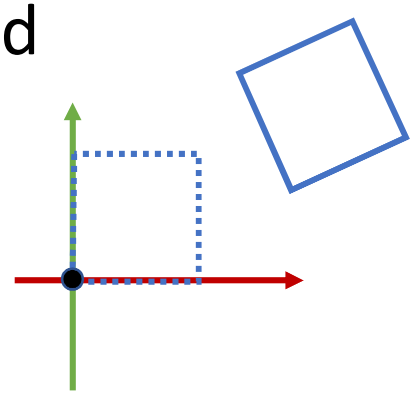
- ***Linear*** transformations
  - 3x3 matrix
  - Rotation + scaling + shear
- ***Rigid*** transformations
  - SO(3) for rotation
  - 3D vector for translation
- ***Affine*** transformation
  - (3x3 matrix + 3D vector) or 4x4 homogenous matrix
  - Linear transformation + translation
- ~~***Projective*** transformation~~
  - ~~4x4 matrix~~
  - ~~Affine transformation + perspective projection~~

Summary

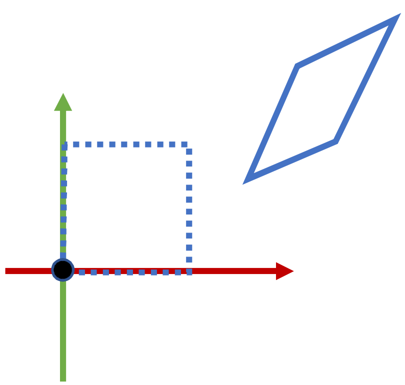
- Linear



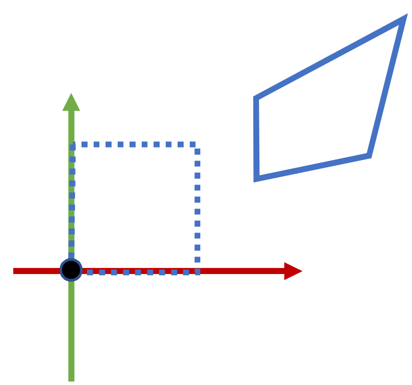
- Rigid



- Affine



- Projective



Questions

- What is the composition of linear/affine/rigid transformations?
- What is the linear (or affine) combination of linear (or affine) transformations?
- What is the linear (or affine) combination of rigid transformations ?