Rigid Body Dynamics

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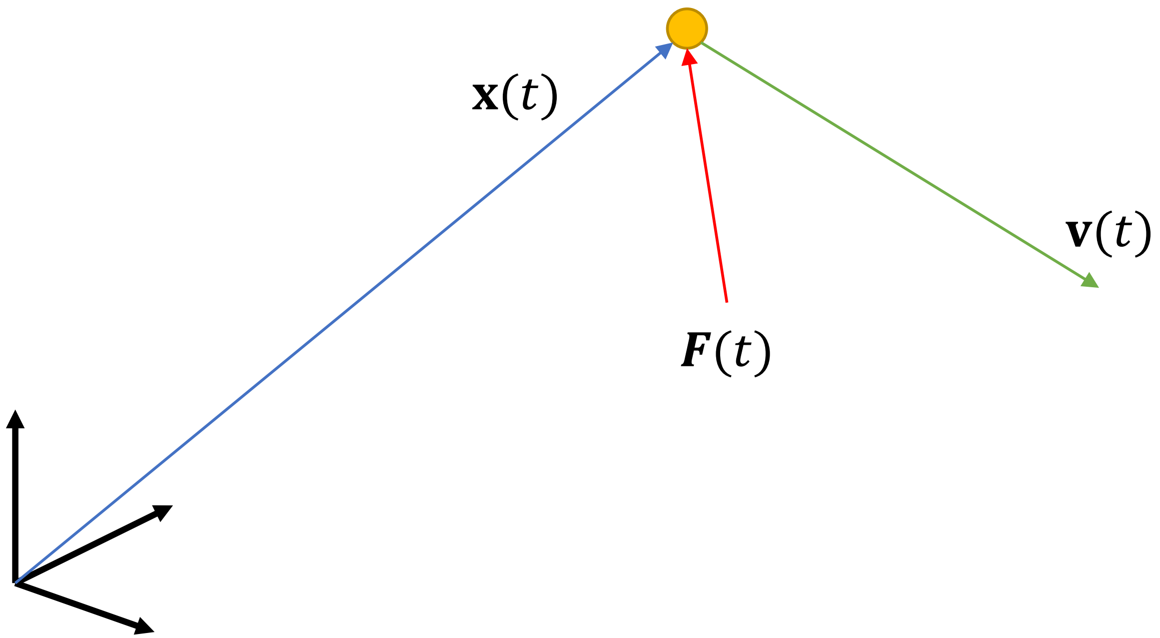
This material was created based on the slides and lecture notes of
Physically Based Modeling (SIGGRAPH 2001 course) by Andrew Witkin

Particle Motion



Particle State

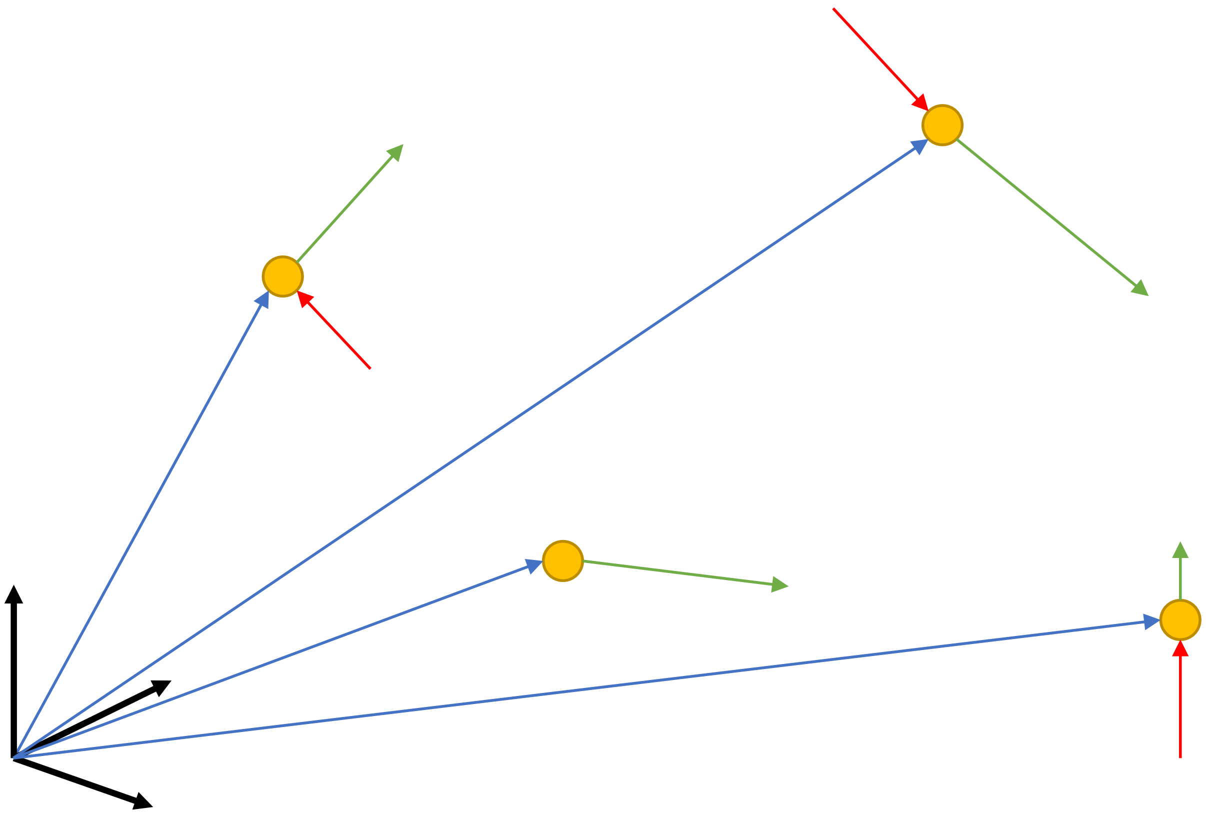
$$\mathbf{X}(t) = \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{pmatrix}$$

$$\mathbf{X} = \overbrace{\boxed{}\boxed{}\boxed{}}^{\mathbf{x}(t)}\overbrace{\boxed{}\boxed{}\boxed{}}^{\mathbf{v}(t)}$$

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ F(t)/m \end{pmatrix}$$

$$\frac{d}{dt}\mathbf{X} = \overbrace{\boxed{}\boxed{}\boxed{}}^{\mathbf{v}(t)}\overbrace{\boxed{}\boxed{}\boxed{}}^{F(t)/m}$$

Multiple Particles



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State Derivative of Multi. Particles

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix}\mathbf{x}_1(t)\\ \mathbf{v}_1(t)\\ \vdots\\ \mathbf{x}_n(t)\\ \mathbf{v}_n(t)\end{pmatrix} = \begin{pmatrix}\mathbf{v}_1(t)\\ F_1(t)/m_1\\ \vdots\\ \mathbf{v}_n(t)\\ F_n(t)/m_n\end{pmatrix}$$

$$\frac{d}{dt}\mathbf{X} = \begin{array}{|c|c|c|c|c|c|c|}\hline \quad & \quad & \quad & \cdots\ 6n \text{ elements}\ \cdots & \quad & \quad & \quad \\ \hline\end{array}$$

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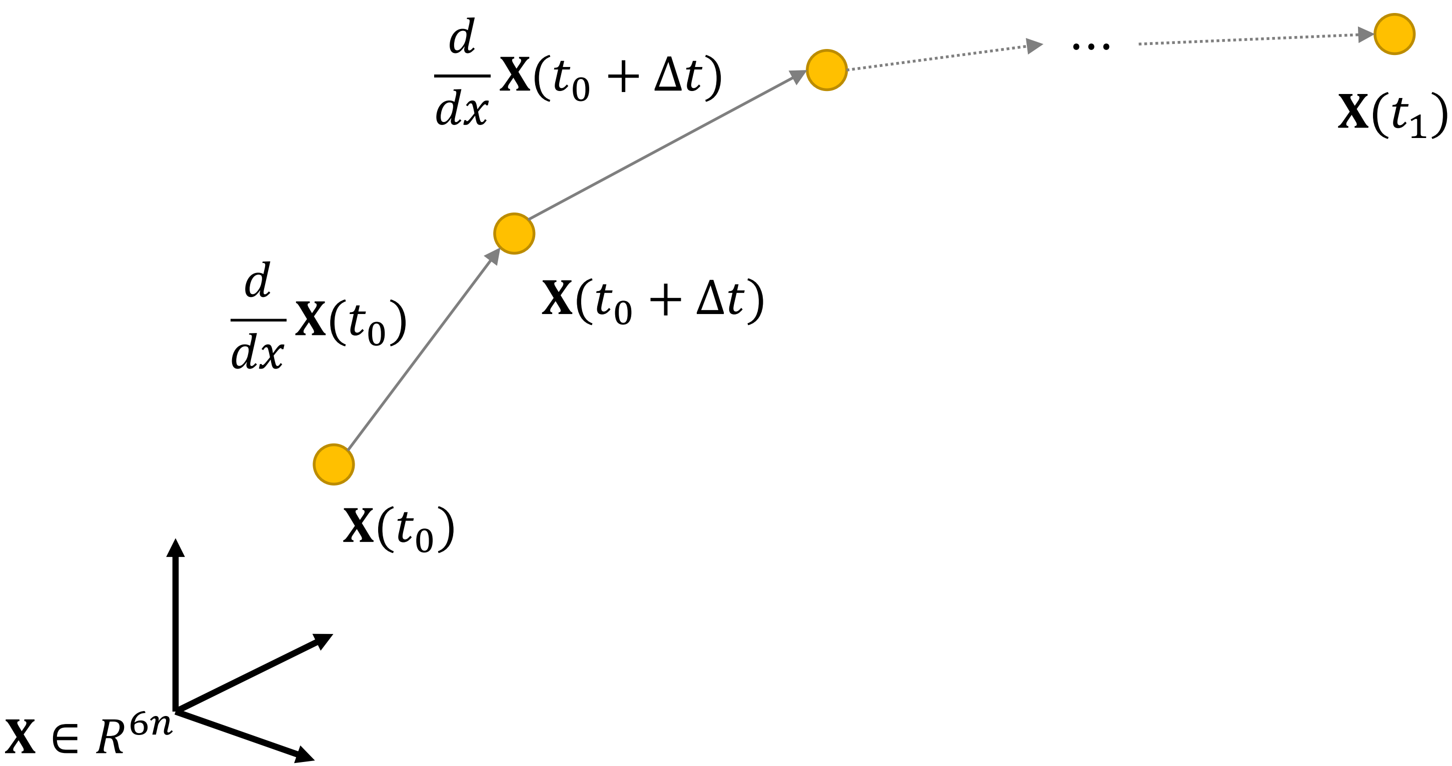
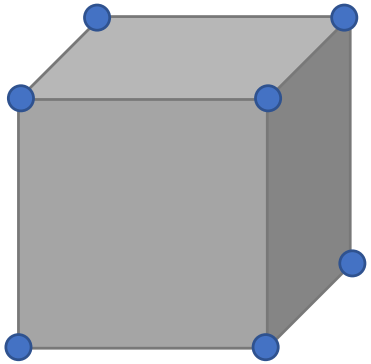
ODE Solution in State Space



Rigid Body Simulation



- Ideally, $\mathbf{X}(t)$ should represent the exact number of degrees-of-freedom of what we want to simulate
- In soft bodies, all the particles can move independently, so their positions and velocities should be included in the state vector $\mathbf{X}(t)$
- In rigid bodies, all the particles are firmly tied together, so we can use the state vector $\mathbf{X}(t)$ that is more compact

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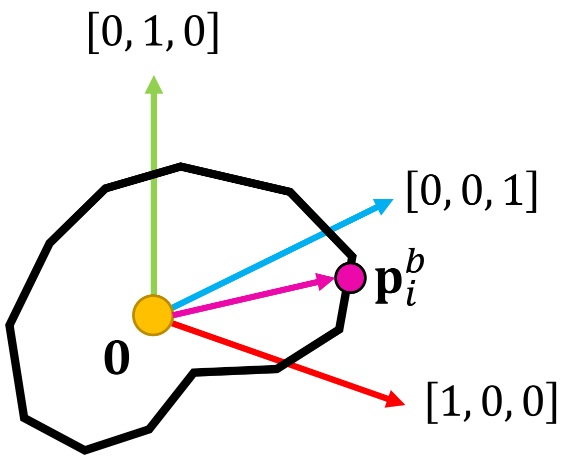
Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{bmatrix} \mathbf{x}(t) \\ ? \\ \mathbf{v}(t) \\ ? \end{bmatrix} = \begin{bmatrix} \mathbf{v}(t) \\ ? \\ F(t)/M \\ ? \end{bmatrix}$$

- The degree-of-freedom of a rigid body never change regardless of its shape, which is 3DoF in 2D and 6DoF in 3D
- We will drive the ***rigid body equation of motion*** by assuming a rigid body is composed of many particles

Motions of Attached Particles



body space

- We will assume that its center-of-mass corresponds to the origin in ***body space***
- $\mathbf{p}_i^b$ is the position of i-th particle in ***body space***, which is attached to a rigid body

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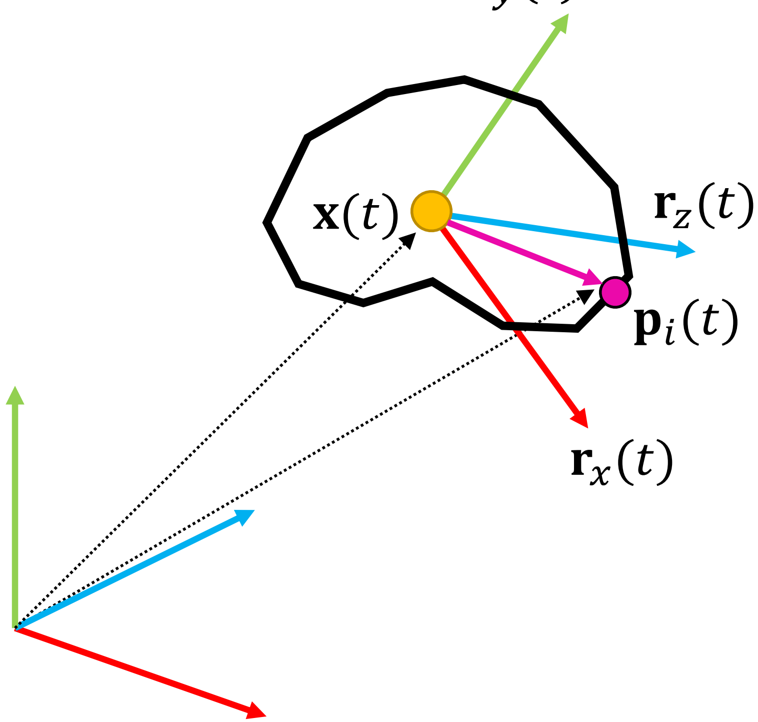
Motions of Attached Particles



world space

By representing orientation as a rotation matrix $R(t)$, we can represent the motion of $\mathbf{p}_i$ as:

$$\mathbf{p}_i(t) = R(t)\mathbf{p}_i^b + \mathbf{x}(t)$$

$$= \begin{bmatrix} | & | & | \\ \boldsymbol{r}_x(t) & \boldsymbol{r}_y(t) & \boldsymbol{r}_z(t) \\ | & | & | \end{bmatrix} \mathbf{p}_i^b + \mathbf{x}(t)$$

where $\mathbf{x}(\mathrm{t})$ is the position of center-of-mass in ***world space***

Then, the velocity of $\mathbf{p}_i(t)$ is:

$$\frac{d}{dt}\mathbf{p}_i(t) = \frac{d}{dt}R(t)\mathbf{p}_i^b + \frac{d}{dt}\mathbf{x}(t)$$

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Motions of Attached Particles

$$\frac{d}{dt}\mathbf{p}_i(t) = \frac{d}{dt}R(t)\mathbf{p}_i^b + \frac{d}{dt}\mathbf{x}(t)$$

$$= \underbrace{\left(\frac{d}{dt}R(t)\right)}\mathbf{p}_i^b + \mathbf{v}(t)$$

The time derivative of $R(t)$ means how axes (columns of rotation matrix) change instaneously

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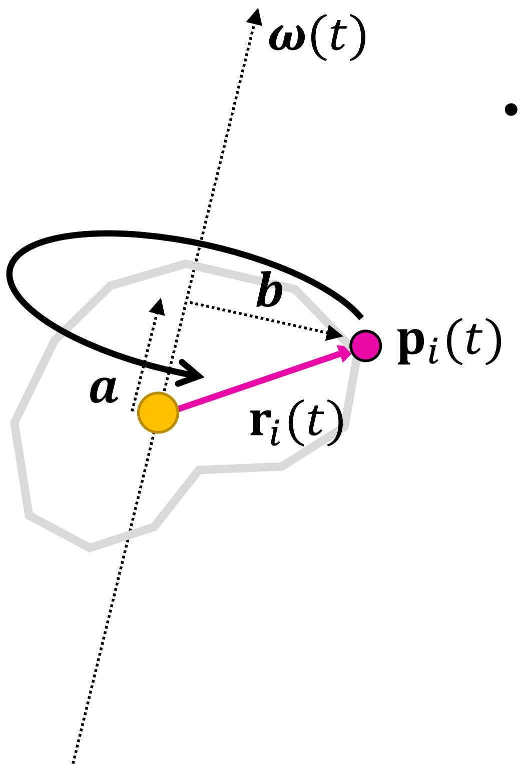
Angular Velocity



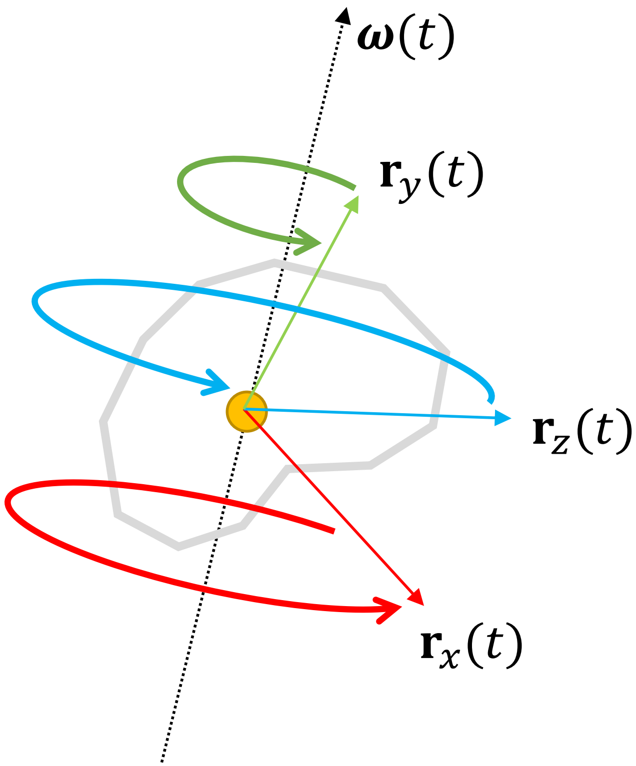
- $\boldsymbol{r}_i(t)$ is defined as a vector from the center-of-mass to a point on the body, and $\boldsymbol{\omega}(t)$ is an angular velocity where its axis $\boldsymbol{\omega}(t)/|\boldsymbol{\omega}(t)|$ and magnitude $|\boldsymbol{\omega}(t)|$ represent the axis of spinning and revolutions per time (rad/s), respectively
- Since the tip of the vector $\boldsymbol{r}_i(t)$ is instantaneously moving along this circle, the instantaneous change of $\boldsymbol{r}_i(t)$ is perpendicular to both $\boldsymbol{b}$ and $\boldsymbol{\omega}(t)$, and the instantaneous velocity of $\boldsymbol{r}_i(t)$ has magnitude $|\mathbf{b}||\boldsymbol{\omega}(t)|$

$$\dot{\boldsymbol{r}}_i(t) = \boldsymbol{\omega}(t)\times\mathbf{b} = \boldsymbol{\omega}(t)\times(\mathbf{a}+\mathbf{b})$$

$$= \boldsymbol{\omega}(t)\times\mathbf{r}_i(t) = [\boldsymbol{\omega}(t)]_\times\mathbf{r}_i(t)$$

$$\boldsymbol{a}\times\boldsymbol{b} = [\boldsymbol{a}]_\times\boldsymbol{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} -a_z b_y + a_y b_z \\ a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{bmatrix}$$

Angular Velocity



- The change of the axes (columns) representing the orientation can be written as

$$\dot{R}(t)$$

$$= [\boldsymbol{\omega}(t)]_\times \begin{bmatrix} | & | & | \\ \boldsymbol{r}_x(t) & \boldsymbol{r}_y(t) & \boldsymbol{r}_z(t) \\ | & | & | \end{bmatrix}$$

$$= [\boldsymbol{\omega}(t)]_\times R(t)$$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{bmatrix} \mathbf{x}(t) \\ R(t) \\ \mathbf{v}(t) \\ \boldsymbol{\omega}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{v}(t) \\ [\boldsymbol{\omega}(t)]_\times R(t) \\ F(t)/M \\ ? \end{bmatrix}$$

Motions of Attached Particles

$$\begin{aligned}
\frac{d}{dt}\mathbf{p}_i(t) &= \frac{d}{dt}R(\mathrm{t})\mathbf{p}_i^b + \frac{d}{dt}\mathbf{x}(\mathrm{t}) \\
&= \left(\frac{d}{dt}R(\mathrm{t})\right)\mathbf{p}_i^b + \mathbf{v}(\mathrm{t}) \\
&= [\boldsymbol{\omega}(t)]_\times R(t)\mathbf{p}_i^b + \mathbf{v}(\mathrm{t}) \\
&= [\boldsymbol{\omega}(t)]_\times \left(R(t)\mathbf{p}_i^b + \mathbf{x}(t) - \mathbf{x}(t)\right) + \mathbf{v}(\mathrm{t}) \\
&= \underbrace{[\boldsymbol{\omega}(t)]_\times \left(\mathbf{p}_i(t) - \mathbf{x}(t)\right)}_{\text{Angular component}} + \underbrace{\mathbf{v}(\mathrm{t})}_{\text{Linear component}}
\end{aligned}$$

Center-of-Mass

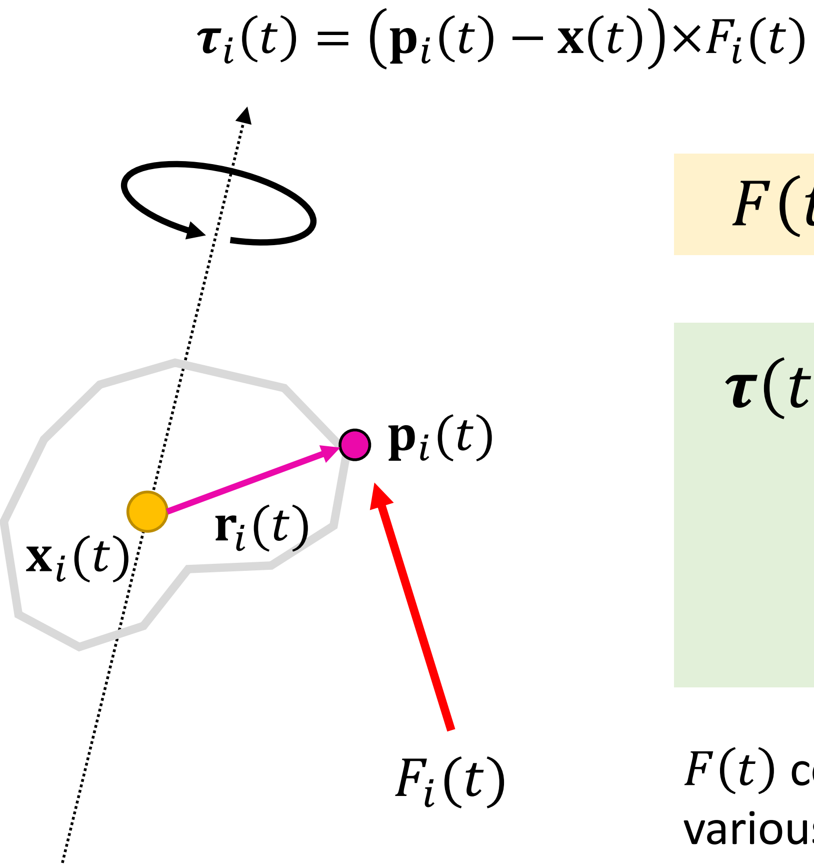
- We have assumed so far that $\mathbf{x}(\mathrm{t})$ is the center-of-mass of our rigid body, is it really true?

$$CoM(t) \equiv \frac{\Sigma m_i \mathbf{p}_i(t)}{\Sigma m_i} = \frac{\Sigma m_i \mathbf{p}_i(t)}{M}$$

$$\frac{\Sigma m_i \mathbf{p}_i(t)}{M} = \frac{\Sigma m_i (R(t)\mathbf{p}_i^b + \mathbf{x}(t))}{M} = \cancel{\frac{R(t)\Sigma m_i \mathbf{p}_i^b}{M}} + \frac{\Sigma m_i \mathbf{x}(t)}{M} = \mathbf{x}(t)\frac{\Sigma m_i}{M} = \mathbf{x}(t)$$

$$\Sigma m_i (\mathbf{p}_i(t) - \mathbf{x}(t)) = \mathbf{0}$$

Force and Torque



$$F(t) = \sum F_i(t)$$

$$\begin{aligned}\boldsymbol{\tau}(t) &= \sum \boldsymbol{\tau}_i(t) \\ &= \sum \Big(\left(\mathbf{p}_i(t) - \mathbf{x}(t)\right) \times F_i(t)\Big) \\ &= \sum \left(\mathbf{r}_i(t) \times F_i(t)\right)\end{aligned}$$

$F(t)$ conveys no information about where the various forces acted on the body; however, $\boldsymbol{\tau}(t)$ does tell us something about the distribution of the forces $F_i(t)$ over the body

Linear Momentum

- The linear momentum p of a particle with mass m and velocity $\mathbf{v}$ is defined as

$$p = m\mathbf{v}$$

- The total linear momentum $P(t)$ of a rigid body is the sum of the products of the mass and velocity of each particle:

$$\begin{aligned}
P(t) &= \sum m_i \dot{\mathbf{p}}_i(t) \\
&= \sum \Big(m_i \boldsymbol{\omega}(t) \times \big(\mathbf{p}_i(t) - \mathbf{x}(t)\big) + m_i \mathbf{v}(\mathrm{t}) \Big) \\
&= \boldsymbol{\omega}(t) \times \sum \Big(m_i \big(\mathbf{p}_i(t) - \mathbf{x}(t)\big) \Big) + \sum m_i \mathbf{v}(\mathrm{t}) \\
&= \sum m_i \mathbf{v}(\mathrm{t}) \\
&= M\mathbf{v}(\mathrm{t})
\end{aligned}$$

Linear Momentum

- The total linear momentum of our rigid body is the same as if the body was simply a particle with mass M and velocity $\mathbf{v}(t)$

$$P(t) = M\mathbf{v}(\mathrm{t})$$

- And, this gives us a relationship below

$$\dot{P}(t) = M\dot{\mathbf{v}}(\mathrm{t}) = F(t)$$

- Physical meaning of linear momentum is:
  - If you have a body floating through space with no force acting on it, the body's linear momentum is constant

Angular Momentum

- Angular momentum has a physical meaning similar to linear momentum
  - If you have a body floating through space with no torque acting on it, the body's angular momentum is constant

$$L(t) = \mathrm{I}(t)\boldsymbol{\omega}(\mathrm{t}) \qquad P(t) = M\mathbf{v}(\mathrm{t})$$

$$\dot{L}(t) = \boldsymbol{\tau}(\mathrm{t}) \qquad \dot{P}(t) = F(t)$$

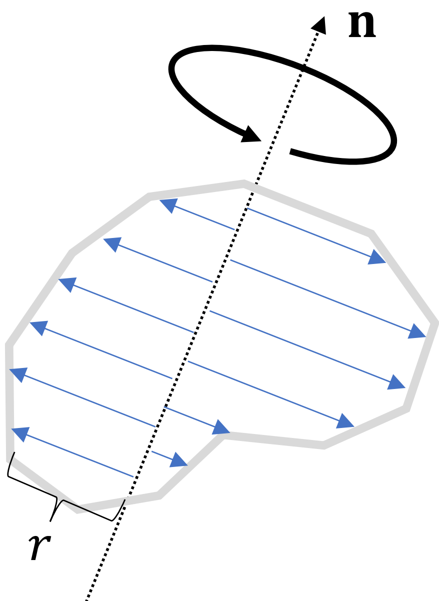
- Note that a body's angular velocity might not be constant even if its angular momentum is constant, this is what confuses many people when compared to linear momentum

Angular Momentum

https://en.wikipedia.org/wiki/Moment_of_inertia

Inertia Tensor

- For the same object, different axes of rotation will have different moments of inertia about those axes. In general, the moments of inertia are not equal unless the object is symmetric about all axes. The inertia tensor is a convenient way to summarize all moments of inertia of an object with one quantity



$$I = mr^2$$: a moment of inertia of a point when rotating around $\mathbf{n}$

$$I = \sum m_i r_i^2$$: a moment of inertia of a body when rotating around $\mathbf{n}$

Inertia Tensor

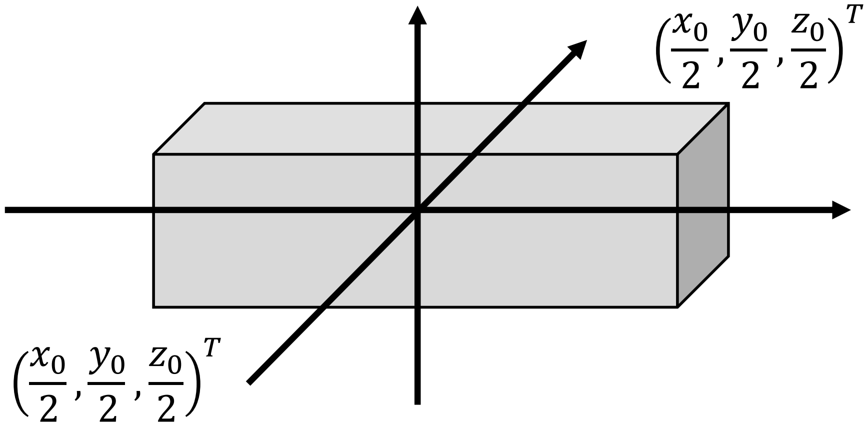
- For the same object, different axes of rotation will have different moments of inertia about those axes. In general, the moments of inertia are not equal unless the object is symmetric about all axes. The inertia tensor is a convenient way to summarize all moments of inertia of an object with one quantity

$$I(t) = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} \sum m_i(r_{iy}^2 + r_{iz}^2) & -\sum m_i r_{ix} r_{iy} & -\sum m_i r_{ix} r_{iz} \\ -\sum m_i r_{iy} r_{ix} & \sum m_i(r_{ix}^2 + r_{iz}^2) & -\sum m_i r_{iy} r_{iz} \\ -\sum m_i r_{iz} r_{ix} & -\sum m_i r_{iz} r_{iy} & m_i(r_{ix}^2 + r_{iy}^2) \end{bmatrix}$$

$$\text{where } \boldsymbol{r}_i = \left[r_{ix}, r_{iy}, r_{iz}\right]^T = \boldsymbol{r}_i(t) = \boldsymbol{p}_i(t) - \mathbf{x}(t)$$

Inertia Tensor of a Block (Cube)



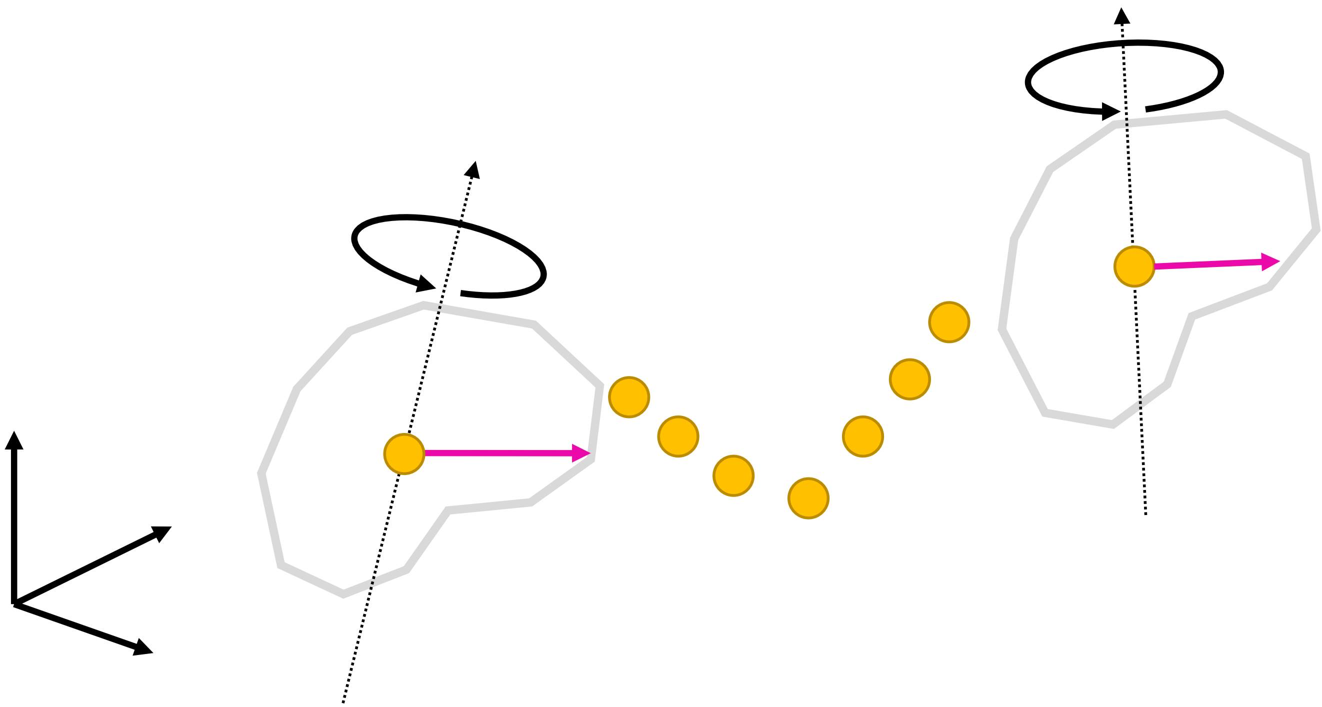
$$I_{body} = \frac{M}{12}\begin{bmatrix} y_0^2 + z_0^2 & 0 & 0 \\ 0 & x_0^2 + z_0^2 & 0 \\ 0 & 0 & x_0^2 + y_0^2 \end{bmatrix}$$

$$I_{xx} = \int_{\frac{-x_0}{2}}^{\frac{x_0}{2}} \int_{\frac{-y_0}{2}}^{\frac{y_0}{2}} \int_{\frac{-z_0}{2}}^{\frac{z_0}{2}} \rho(x, y, z)(y^2 + z^2)\, dx\, dy\, dz$$
$$\vdots$$
$$= \frac{M}{12}(y_0^2 + z_0^2)$$

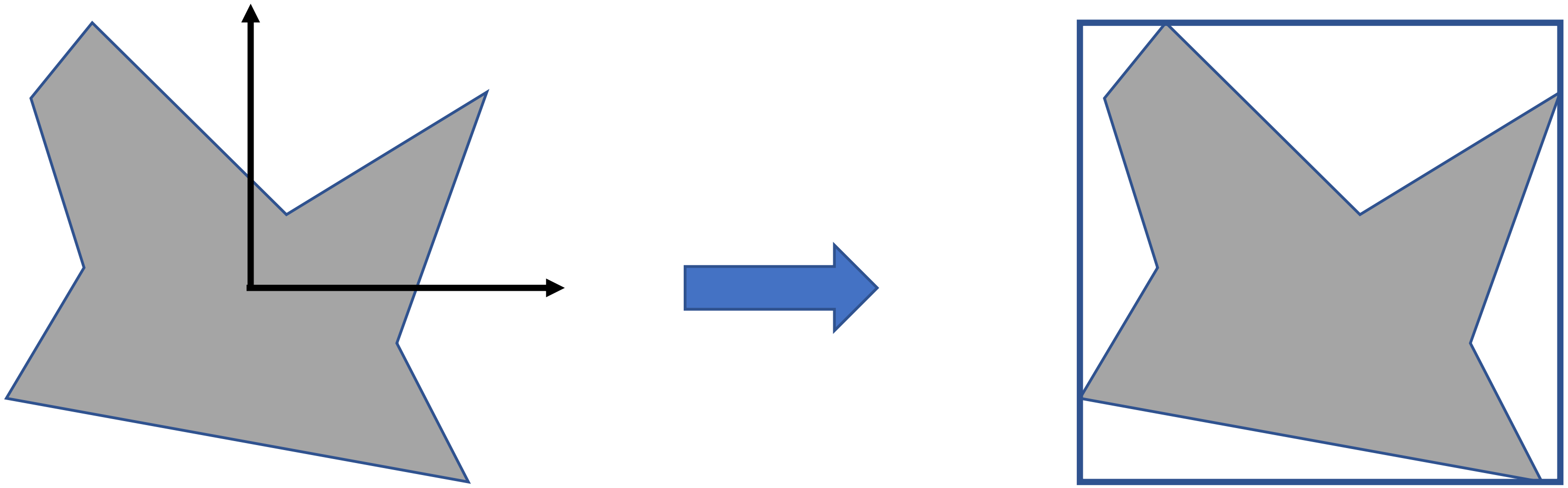
$$I_{xy} = \int_{\frac{-x_0}{2}}^{\frac{x_0}{2}} \int_{\frac{-y_0}{2}}^{\frac{y_0}{2}} \int_{\frac{-z_0}{2}}^{\frac{z_0}{2}} \rho(x, y, z)(xy)\, dx\, dy\, dz$$
$$\vdots$$
$$= 0$$

Inertia Tensor

- Inertia tensors vary in world space when object is moving unless the object shape and its movent are both symmetric
- This is trivial because the mass distribution changes over time when it measured in world space

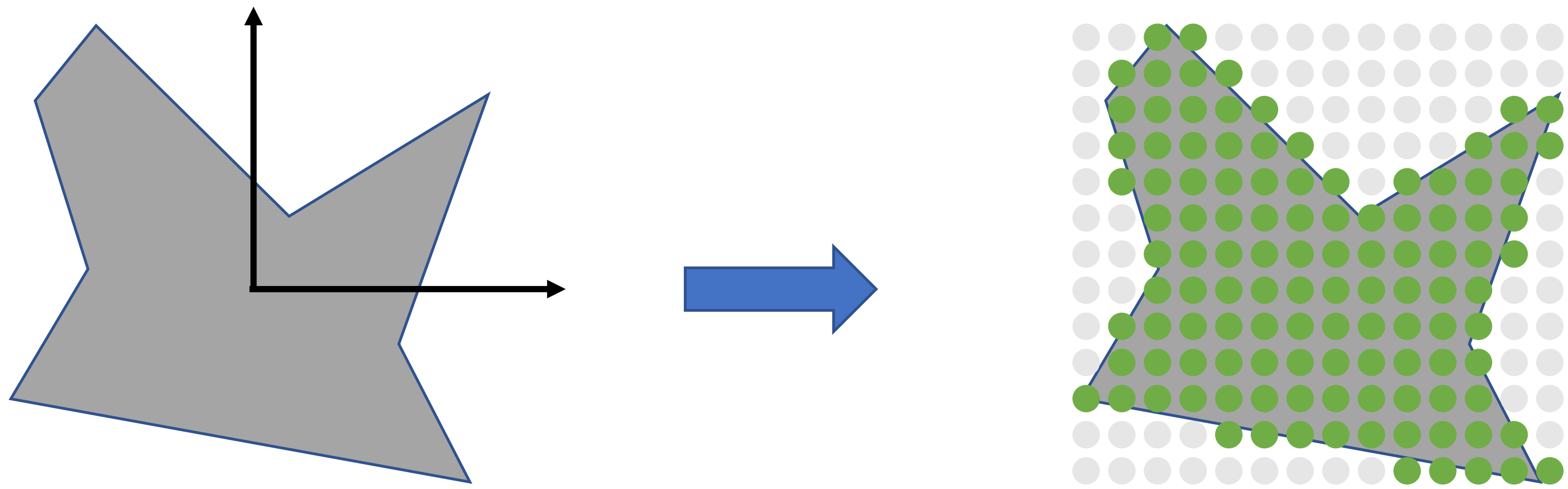


Approximating I_{body}: Bounding Boxes



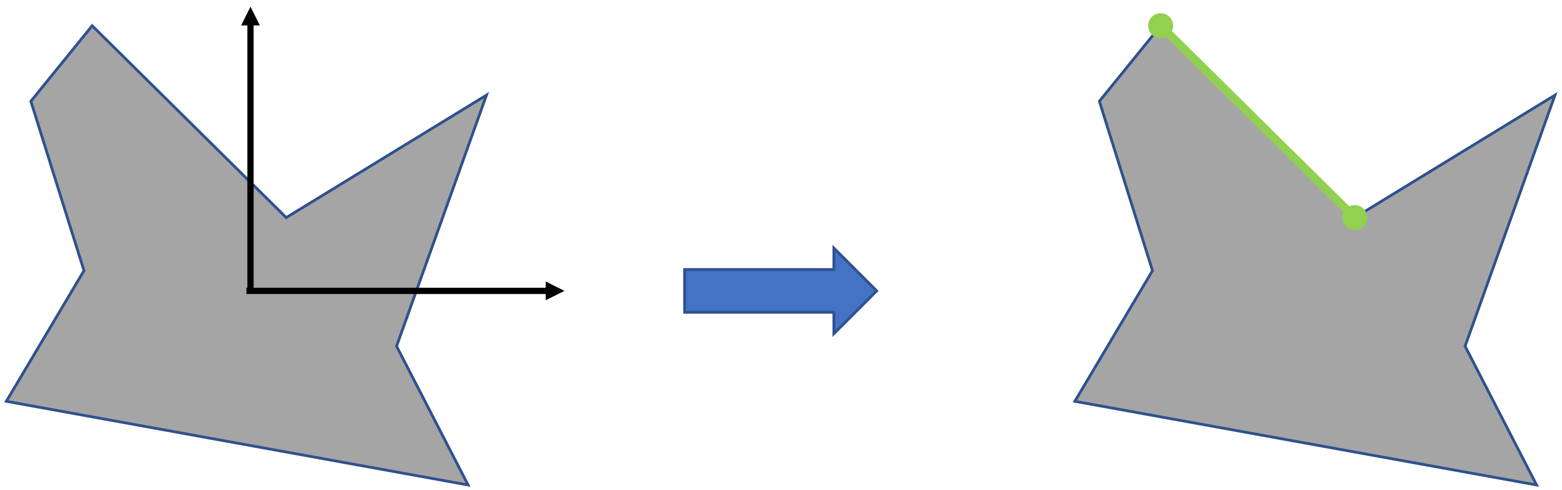
- Pros
  - Simple
- Cons
  - Inaccurate when bounding box is not a good fit

Approximating I_{body}: Bounding Boxes



- Pros
  - Simple, fairly accurate, no B-rep needed
- Cons
  - Expensive, require volume test

Computing I_{body}: Green's Theorem



- Pros
  - Simple, exact, no volumes needed
- Cons
  - Requires boundary representation

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{bmatrix}\mathbf{x}(t)\\ R(t)\\ P(t)\\ L(t)\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}\mathbf{x}(t)\\ R(t)\\ M\mathbf{v}(t)\\ I(t)\boldsymbol{\omega}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{v}(t)\\ [\boldsymbol{\omega}(t)]_\times R(t)\\ F(t)\\ \boldsymbol{\tau}(t)\end{bmatrix}$$

$P(t) = M\mathbf{v}(t)$: Linear Momentum

$L(t) = I(t)\boldsymbol{\omega}(t)$: Angular Momentum

Rigid Body Equation of Motion

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{x}(t) \\ R(t) \\ P(t) \\ L(t) \end{bmatrix} \qquad \frac{d}{dt}\mathbf{X}(t) = \begin{bmatrix} \mathbf{v}(t) \\ [\boldsymbol{\omega}(t)]_\times R(t) \\ F(t) \\ \boldsymbol{\tau}(t) \end{bmatrix}$$

These are given at time t

These are to compute at time t

$$\mathbf{v}(t) = P(t)/M$$

$$\boxed{\boldsymbol{\omega}(t) = \left(I(t)\right)^{-1} L(t) \qquad \text{Computational bottleneck!}}$$

$$F(t) = \text{given}$$

$$\boldsymbol{\tau}(t) = \text{given}$$

$P(t) = M\mathbf{v}(t)$: Linear Momentum

$L(t) = I(t)\boldsymbol{\omega}(t)$: Angular Momentum

Inertia Tensor

- Fortunately, by using body-space coordinates we can cheaply compute the inertia tensor $I(t)$ for any orientation by using the rotation matrix $R(t)$ and the inertia tensor I_{body} precomputed in body-space coordinates

$$I(t) = R(t) I_{body} R(t)^T$$

Inertia Tensor (Proof)

$$I(t) = R(t)I_{body}R(t)^T$$

$$\begin{aligned}
I(t) &= \sum m_i\big((\mathbf{r}_i^T\mathbf{r}_i)\mathrm{I} - \mathbf{r}_i\mathbf{r}_i^T\big) \\
&= \sum m_i\left(\left(R(t)\mathbf{p}_i^b\right)^T\left(R(t)\mathbf{p}_i^b\right)\mathrm{I} - \left(R(t)\mathbf{p}_i^b\right)\left(R(t)\mathbf{p}_i^b\right)^T\right) \\
&= \sum m_i\left(\left(\mathbf{p}_i^b\right)^T\mathbf{p}_i^b\mathrm{I} - R(t)\mathbf{p}_i^b\left(\mathbf{p}_i^b\right)^T R(t)^T\right) \\
&= \sum m_i\left(R(t)\left(\mathbf{p}_i^b\right)^T\mathbf{p}_i^b R(t)^T\mathrm{I} - R(t)\mathbf{p}_i^b\left(\mathbf{p}_i^b\right)^T R(t)^T\right) \\
&= R(t)\left(\sum m_i\left(\left(\mathbf{p}_i^b\right)^T\mathbf{p}_i^b\mathrm{I} - \mathbf{p}_i^b\left(\mathbf{p}_i^b\right)^T\right)\right)R(t)^T \\
&= R(t)I_{body}R(t)^T
\end{aligned}$$

$R^T R = I$

Rigid transf. does not change its norm

Rearrangement

$$\mathbf{r}_i = \mathbf{p}_i(t) - \mathbf{x}(t) = R(t)\mathbf{p}_i^b$$

A Uniform Force Field

- How does a uniform force field (e.g. gravity) affect the movement of a rigid body?
- The ***net force*** is

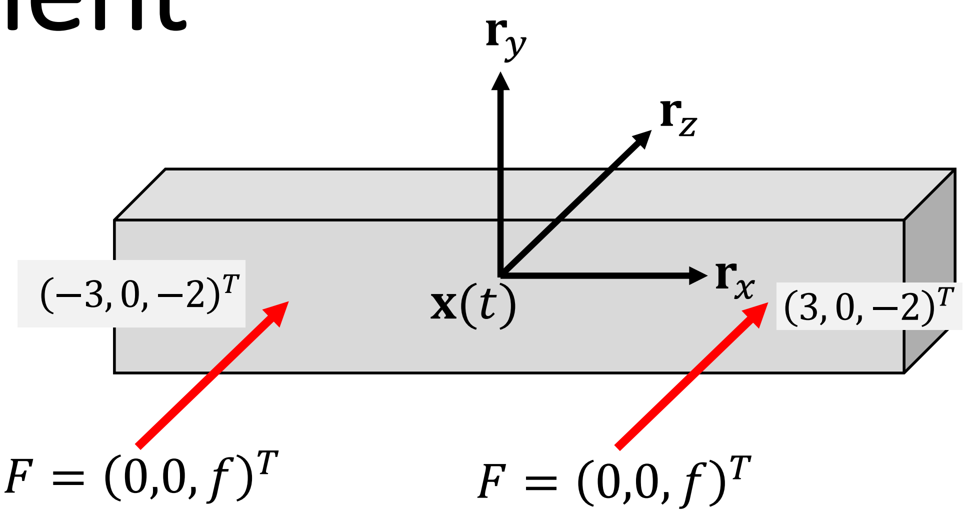
$$F_{\mathrm{g}} = \sum m_i \mathbf{g} = M\mathbf{g}$$

- The ***net torque*** is

$$\boldsymbol{\tau}_{\mathrm{g}} = \sum \big(\mathbf{p}_i(t) - \mathbf{x}(t)\big) \times (m_i \mathbf{g}) = \sum m_i \big(\mathbf{p}_i(t) - \mathbf{x}(t)\big) \times \mathbf{g} = \mathbf{0}$$

- A uniform force field does not affect rotational movement!

An Example: Rotational-free Movement



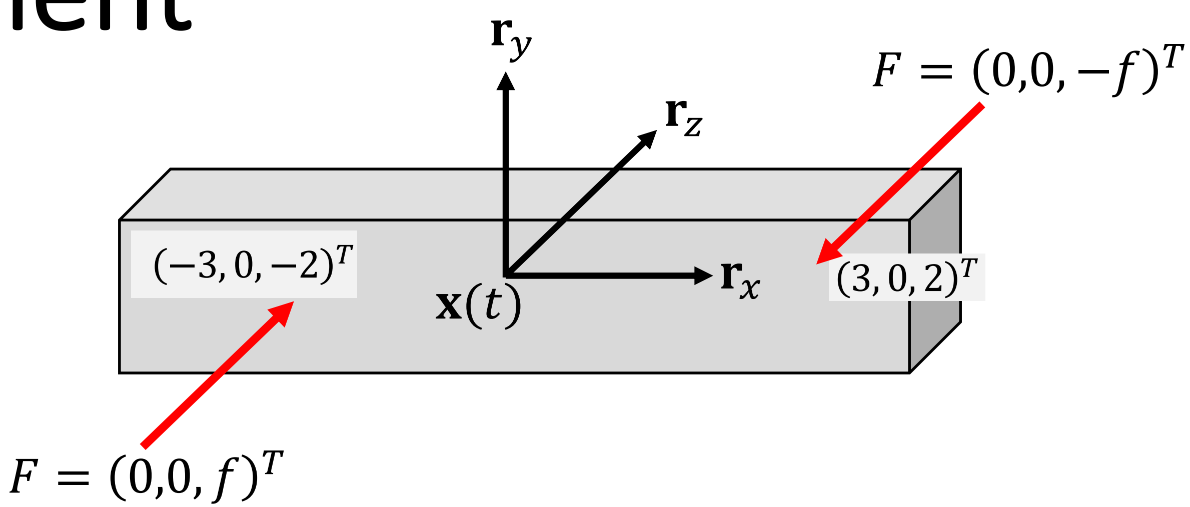
- Translational movement

$$F_{net} = \begin{pmatrix} 0 \\ 0 \\ 2f \end{pmatrix}$$

- Rotational movement

$$\boldsymbol{\tau}_{net} = \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} = \mathbf{0}$$

An Example: Translational-free Movement



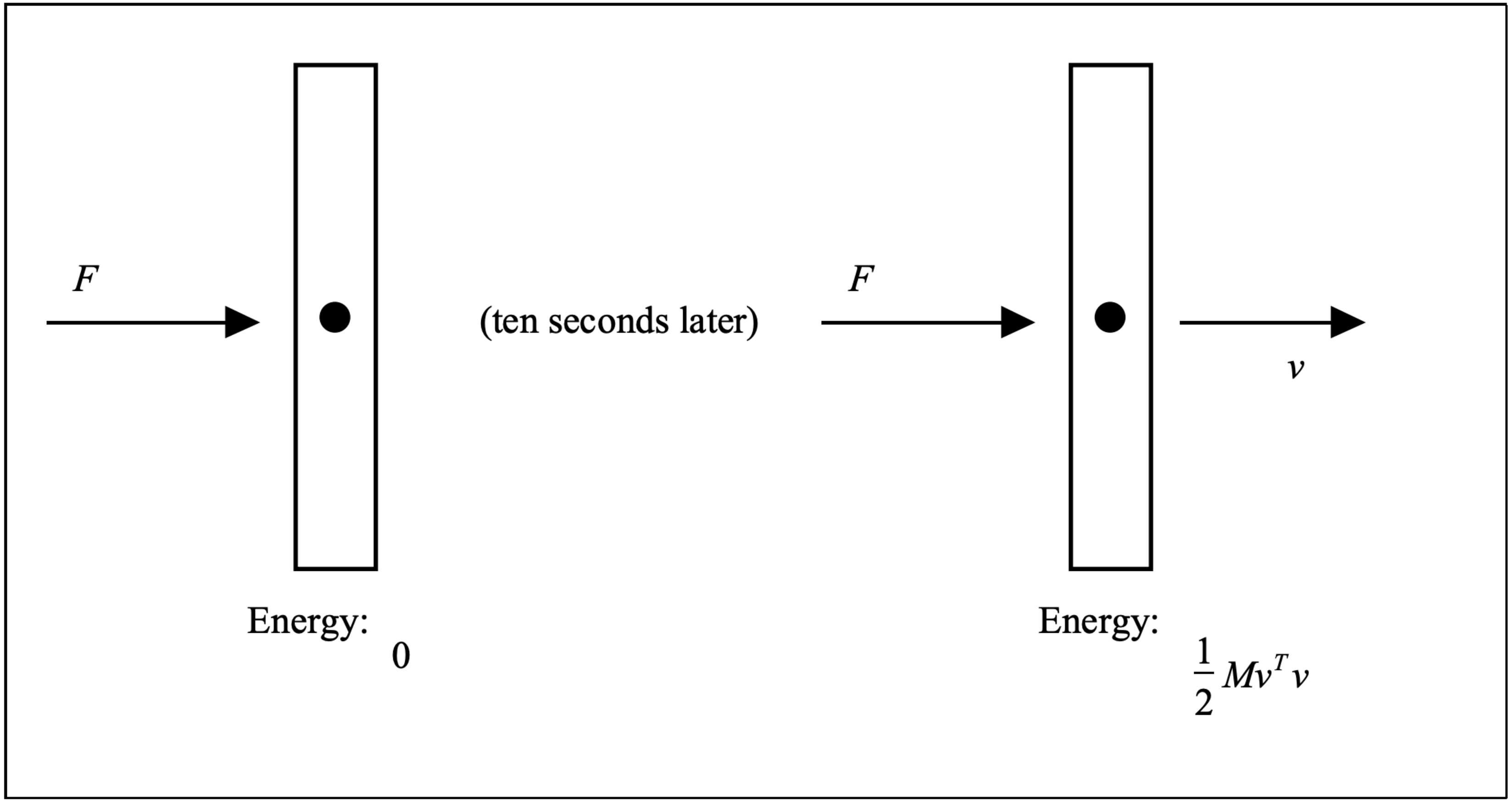
- Translational movement

$$F_{net} = \mathbf{0}$$

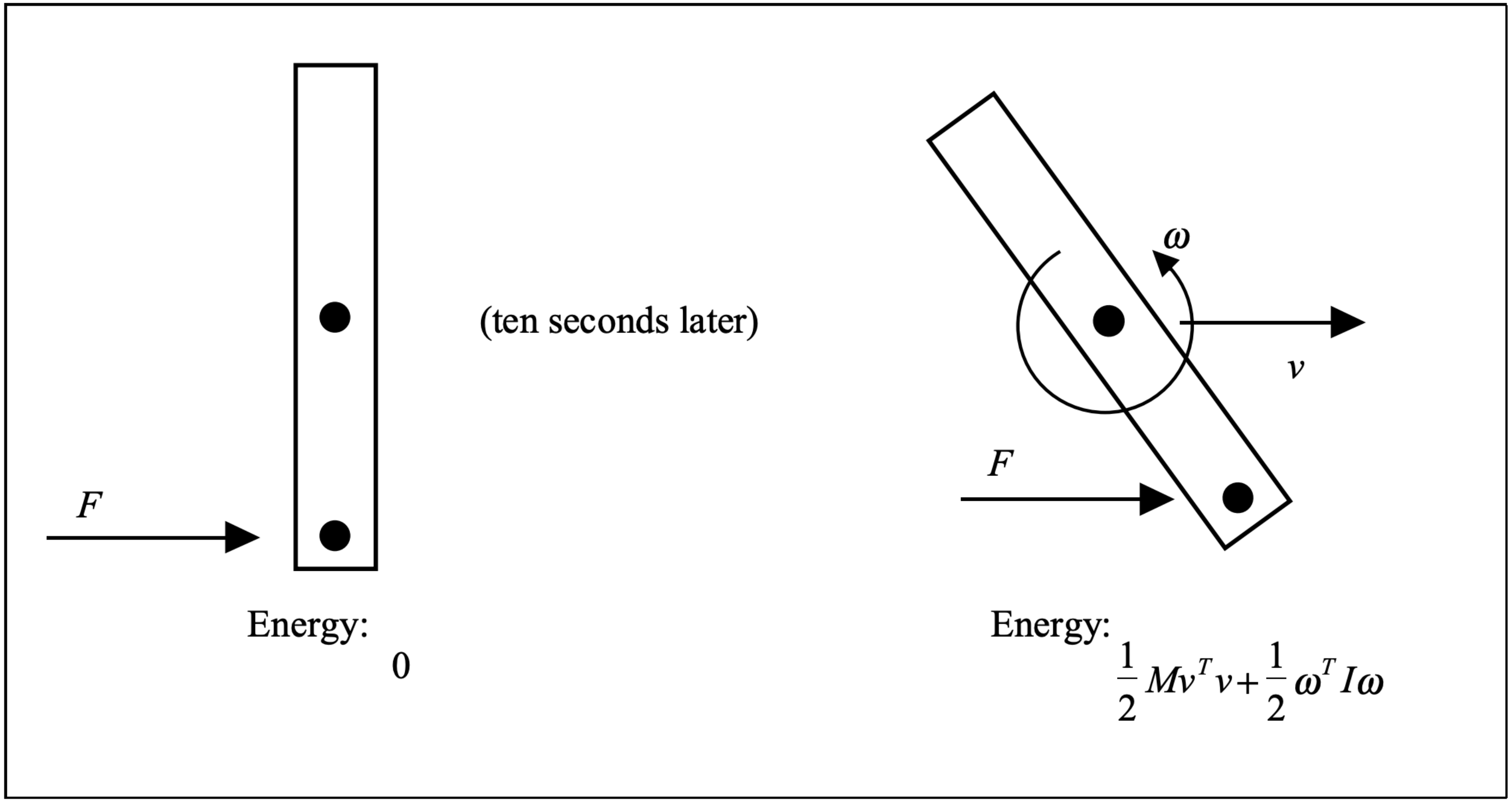
- Rotational movement

$$\boldsymbol{\tau}_{net} = \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -4 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 6f \\ 0 \end{pmatrix}$$

Force vs. Torque Puzzle



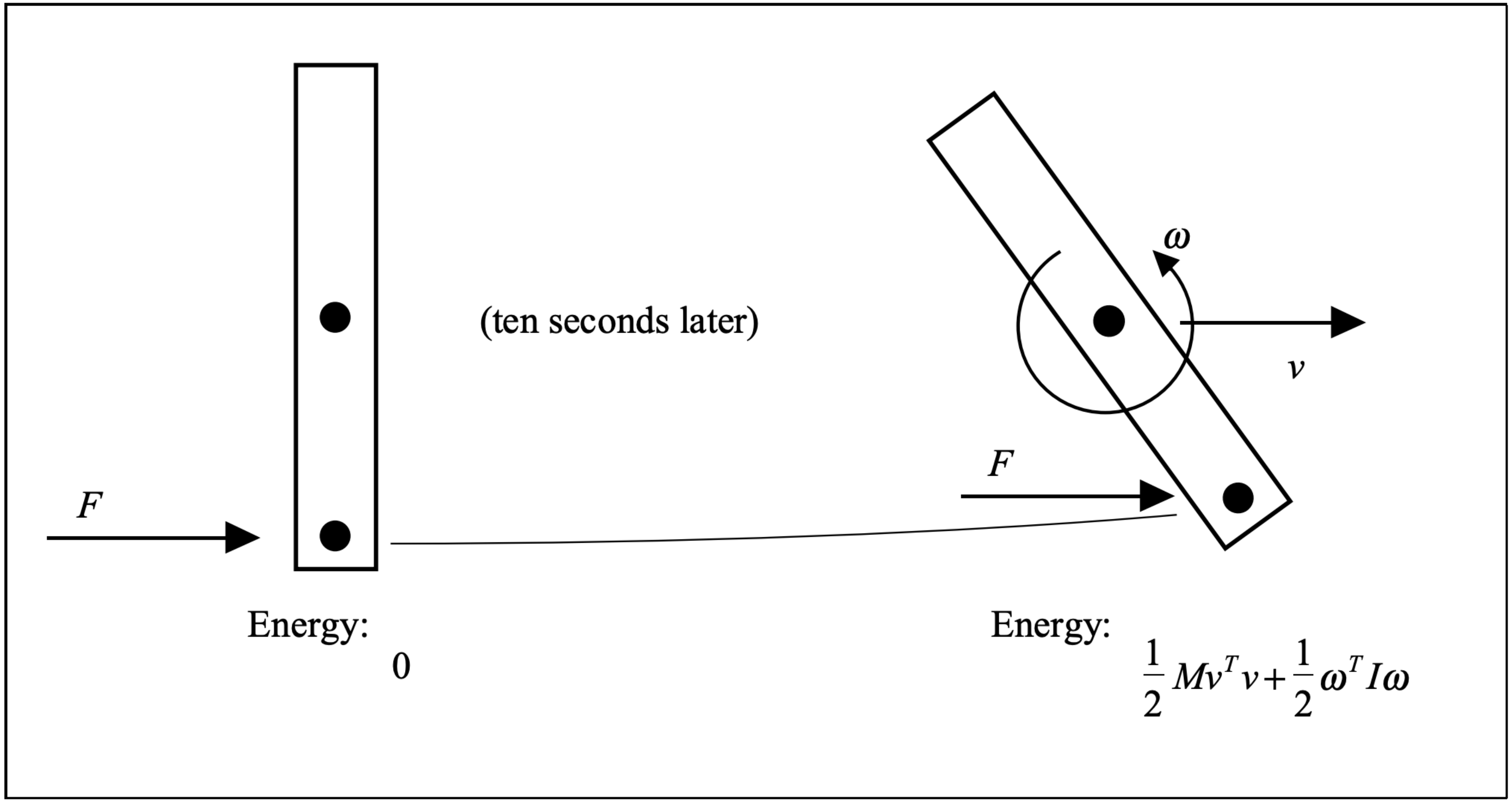
Force vs. Torque Puzzle



If identical forces pushed the block in both cases for the same duration, how can the energy of the block be different?

(Hint: Energy, or work, is the integral of force over distance)

Force vs. Torque Puzzle



Answer: The path in the second case is longer than the path in the first case, which means that we performed more work in the second case

Summary

- The idea used in particle dynamics were extended to rigid bodies, for which, new concepts like angular velocity, linear/angular momentums, inertia tensors, and etc have introduced
- In the end, the equation of motion for rigid body is defined, where we used linear/angular momentums instead of linear/angular velocities
- Now, we have an ability to implement and run rigid body simulation in free space