Hierarchical Modeling & Kinematics

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Many contents are adopted from the slides of the Computer Graphics course at SNU lectured by Jehee Lee

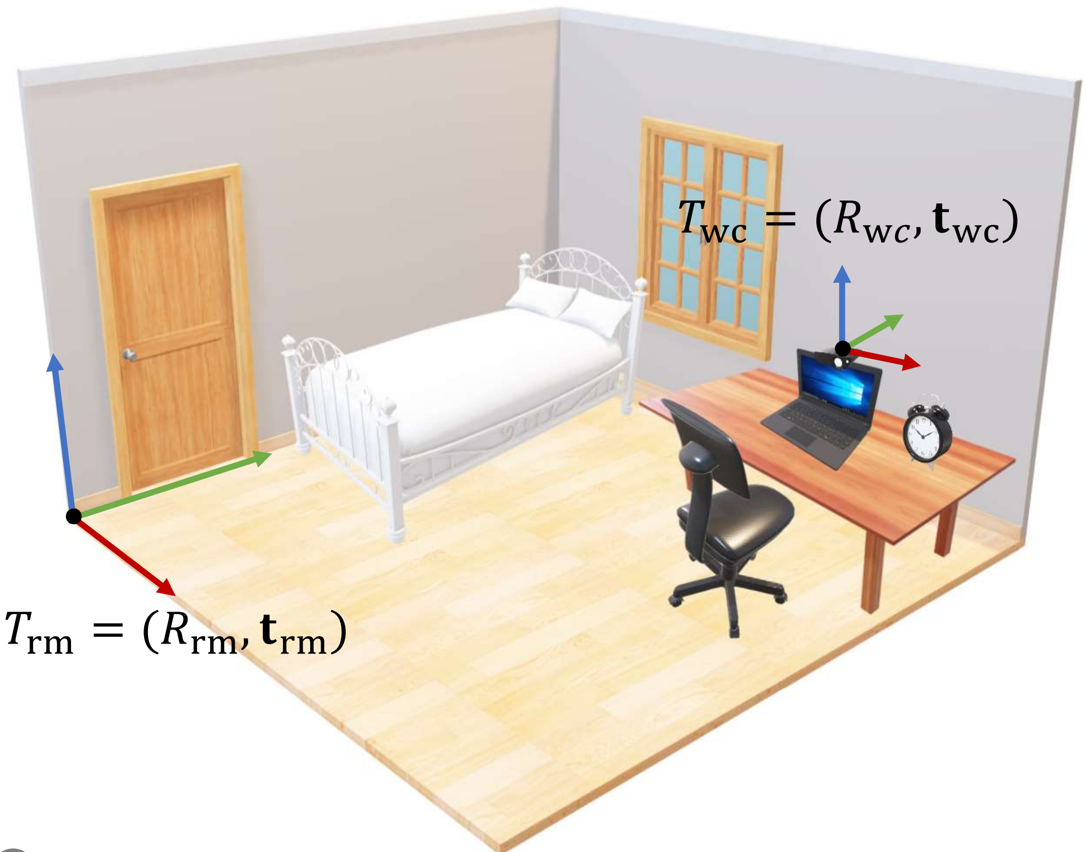
Hierarchical Modeling

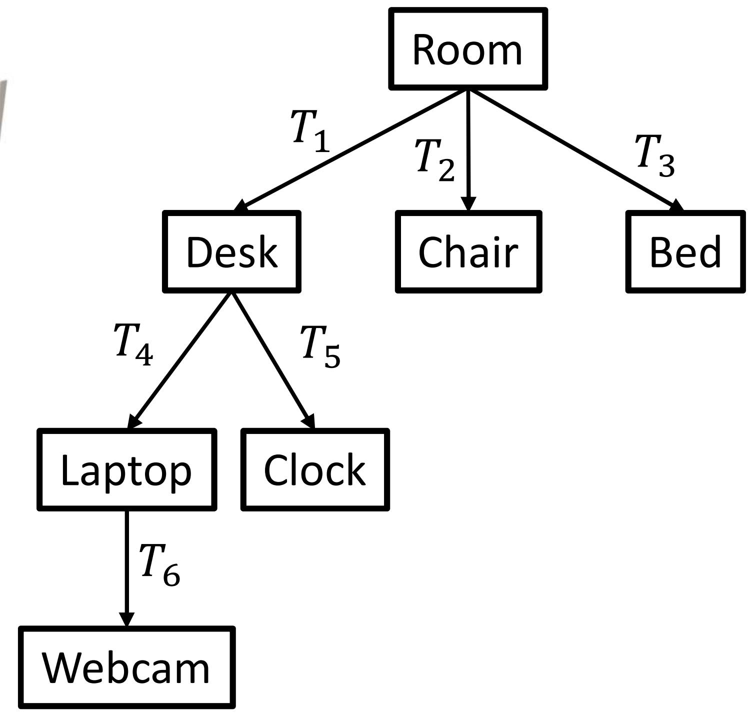
- Motivation
  - Many objects are composed of multiple subcomponents. For example, a car is approximately made up of 30,000 parts, and what if you want to model a New York city?

  - Modeling all the components in the same coordinate system would not be an ideal choice as the complexity of the model increases. Furthermore, it may also hinder collaboration among many designers

Hierarchical Modeling

- Tree structure is typically used to represent the hierarchy





$$T_{\mathrm{wc}} = T_{\mathrm{rm}} \cdot T_1 \cdot T_4 \cdot T_6$$

Static vs. Dynamic Models

- Static Hierarchical Model
  - Relative transf. between subcomponents do not change
- Dynamic Hierarchical Model
  - Relative transf. between subcomponents may change on the fly (i.e., the model is animatable)
  - The motions between connected components are often described separately by introducing ***joints***

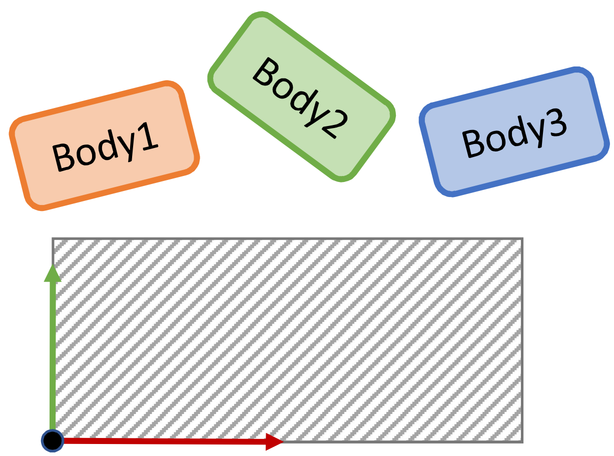
Kinematics

- ***Kinematics*** is the study on motion of systems of bodies (i.e., dynamic hierarchical models) without considering the forces acting it
- It was developed in classical mechanics to describe motion of complex objects
- Tree structure of **joints** and **bodies** (links) is used
  - The root link can be chosen arbitrarily

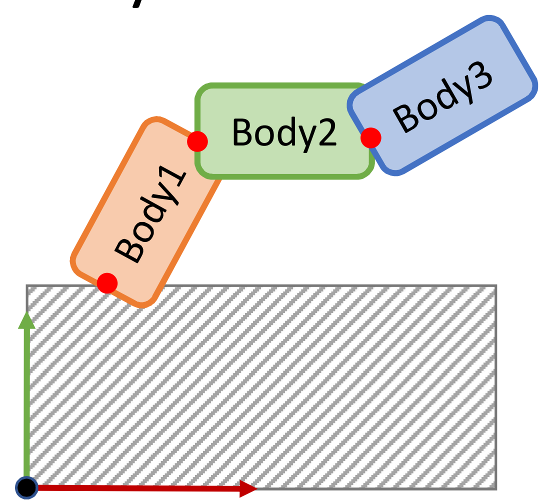
What Is a Joint?

- A joint is an area (**connection**) where two or more bodies meet to allow movement

- Mathematically, joints are the constraints that reduce ***degree-of-freedom*** of the system



(3+3+3) DoFs



(1+1+1) DoFs

Types of Mathematical Joints

- A fixed (weld) joint does not allow any movement between connected links (0-DoF)

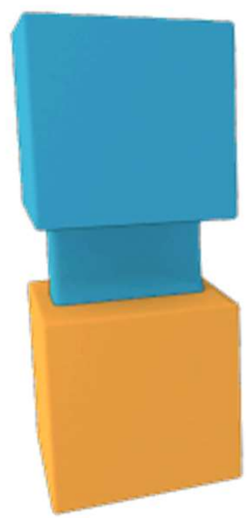


Image source: https://light-and-shadows.com/documentation/interact/

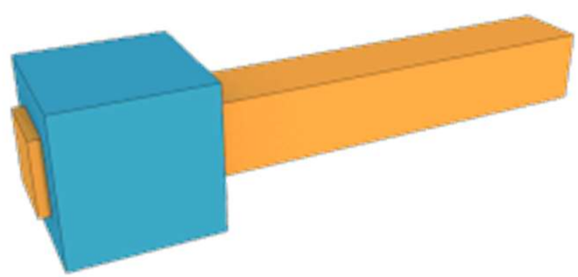
Types of Mathematical Joints

- A revolute (hinge) joint allows rotation about a fixed axis (1-DoF)

Image source: https://light-and-shadows.com/documentation/interact/

Types of Mathematical Joints

- A prismatic (translational) joint allows translation about a fixed axis (1-DoF)



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Types of Mathematical Joints

- A universal joint allows rotation about two fixed axes (2-DoFs)

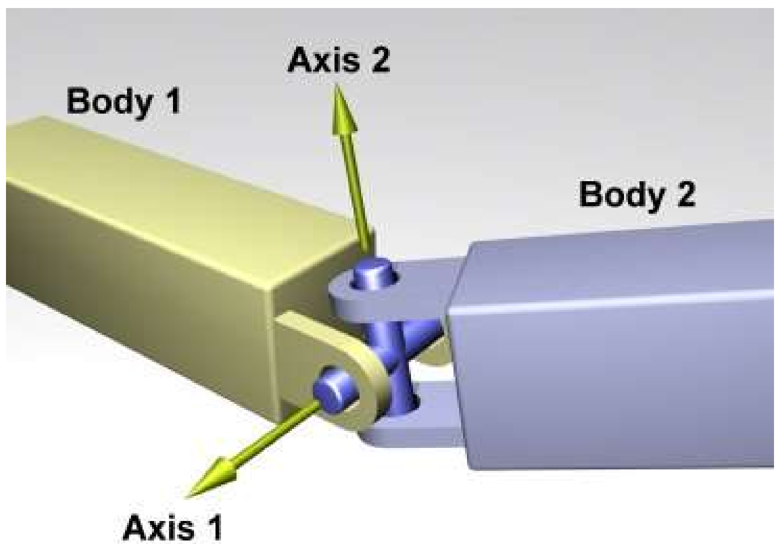


Image source: https://www.ode.org/

Types of Mathematical Joints

- Ball-and-socket joint allows rotation about an arbitrary axis (3-DoFs)

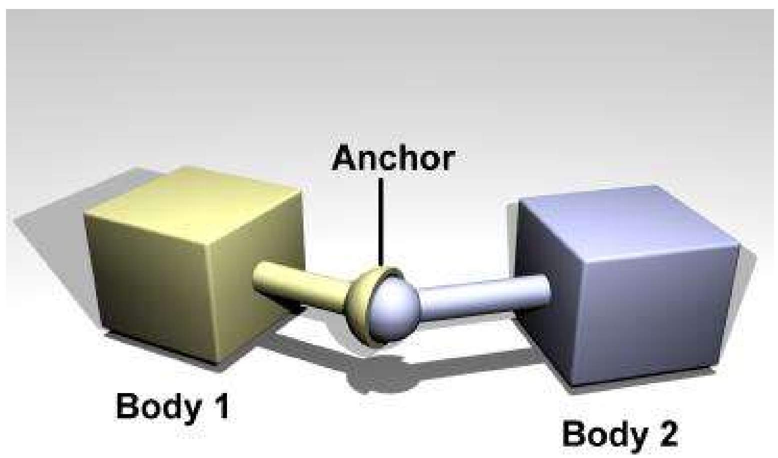


Image source: https://www.ode.org/

Types of Mathematical Joints

- A free joint can freely rotate and translate in the space (6-DoFs)

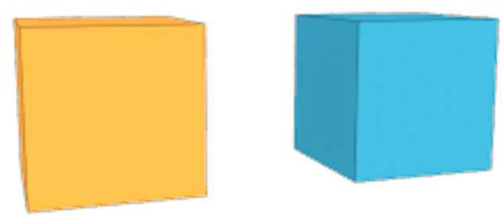
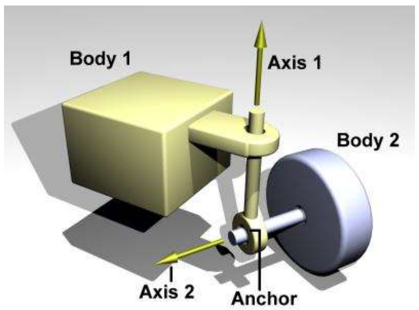
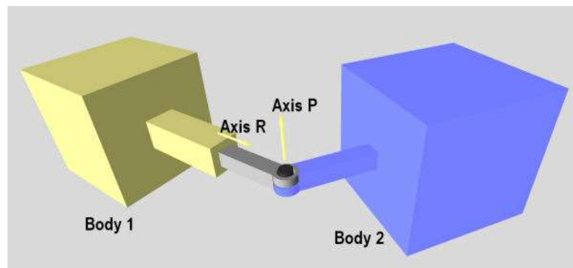


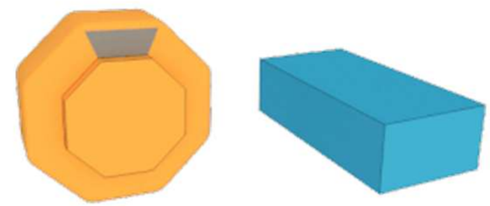
Image source: https://light-and-shadows.com/documentation/interact/

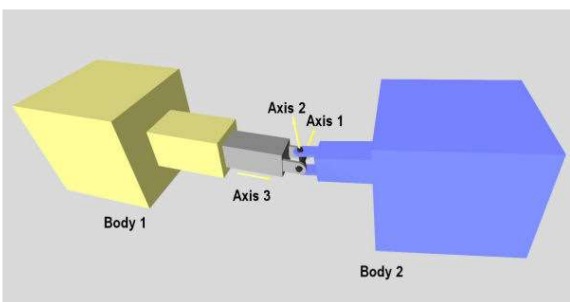
Types of Mathematical Joints

- And many more…









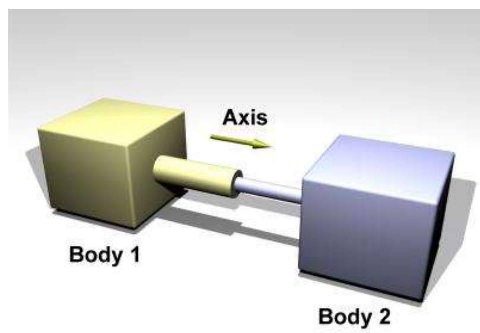
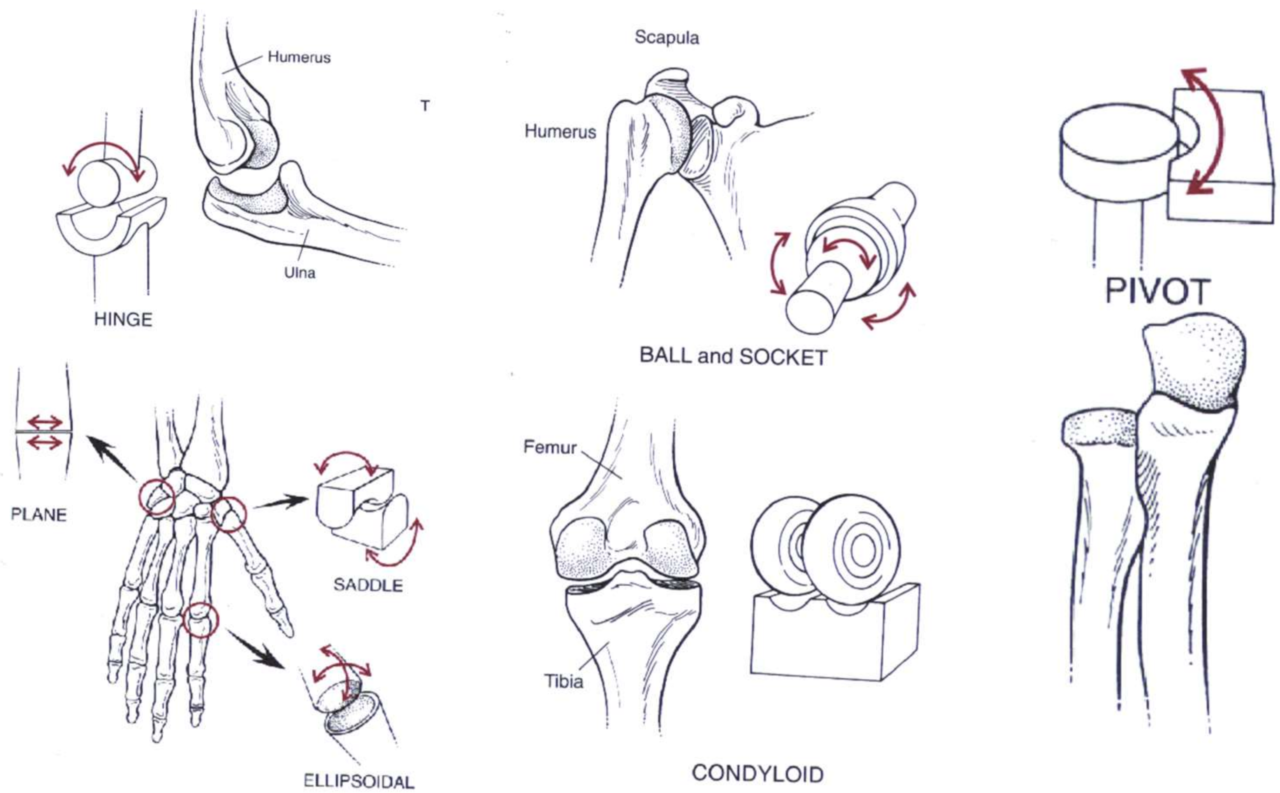


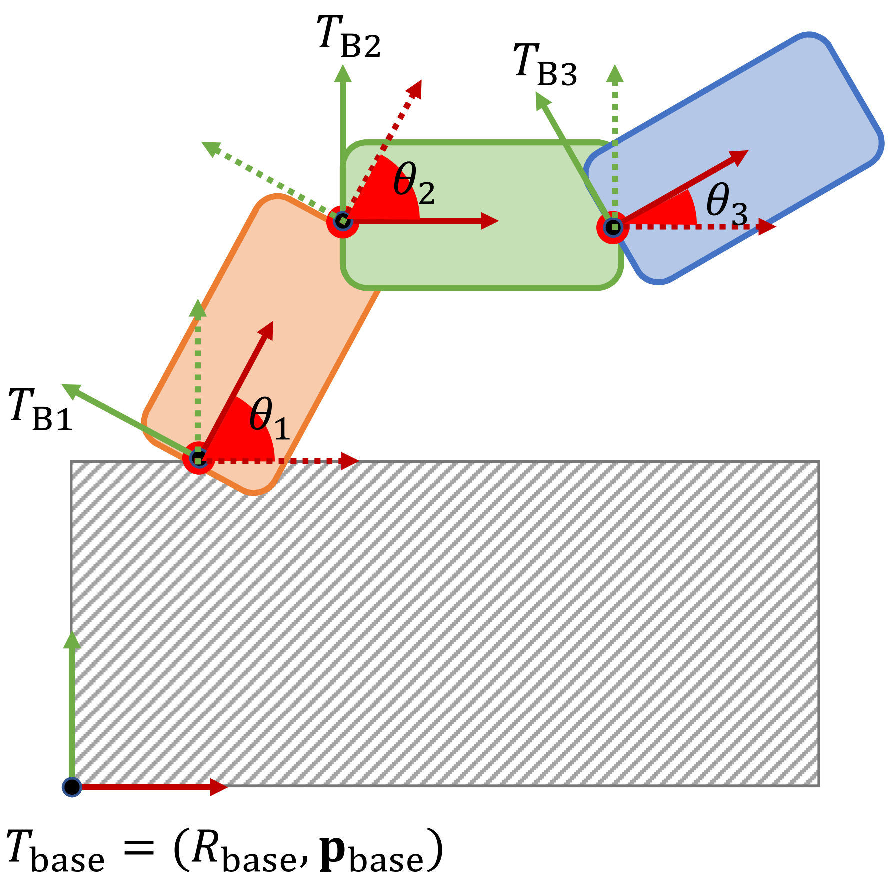
Image sources:
(a) https://www.ode.org/
(b) https://light-and-shadows.com/documentation/interact/

Biological Joints

- Human/Animal joints are much more complicated!



Local vs. Global Configuration



- Local (Joint) Configuration
  - Values that minimally describe the system (i.e., minimal coordinate system)
  - Joint angles $(\theta_1, \theta_2, \theta_3)$
- Global Configuration
  - Values that maximally describe the system (i.e., maximal coordinate system)
  - Rotations and translations (T_{B1}, T_{B2}, T_{B3}) of each body w.r.t. T_{base}

Forward and Inverse Kinematics

- Forward Kinematics (FK)
  - The process to compute global configuration given joint (local) configuration

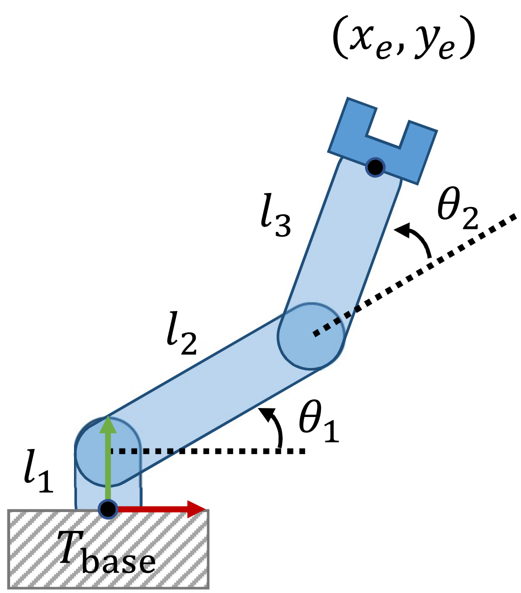
$$\underbrace{(R_i, \mathbf{p}_i)}_{\text{Unknown}} = F_i\underbrace{(\theta_1, \theta_2, \dots, \theta_N)}_{\text{Known}}$$

- Inverse Kinematics (IK)
  - The process to compute joint (local) configuration given (partial) global configuration

$$\underbrace{(\theta_1, \theta_2, \dots, \theta_N)}_{\text{Unknown}} = F_i^{-1}\underbrace{(R_i, \mathbf{p}_{\boldsymbol{i}})}_{\text{known}}$$

Forward Kinematics: A Simple Example

- A simple robot arm in 2-dimensional space
  - 3 bodies, 2 revolute (hinge) joints
  - Compute the position of the end-effector (x_e, y_e) given joint angles (θ_1, θ_2) w.r.t. T_{base}

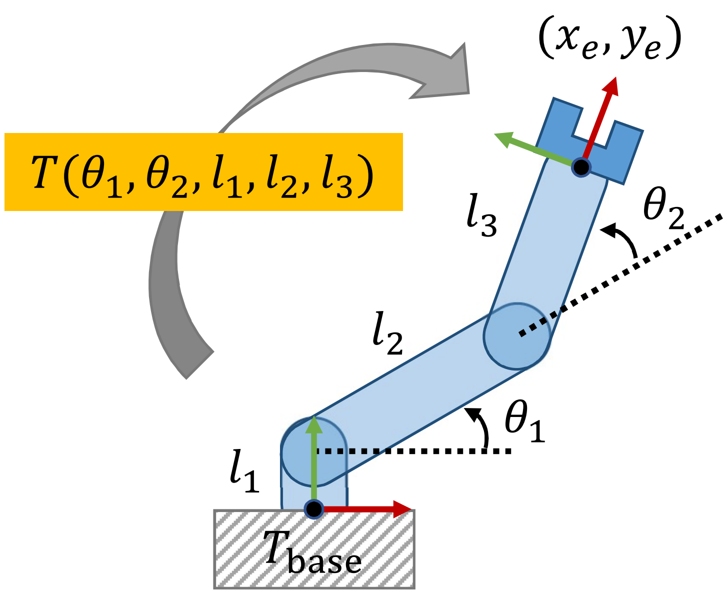


$$x_e = l_2 cos\theta_1 + l_3 \cos(\theta_1 + \theta_2)$$

$$y_e = l_1 + l_2 sin\theta_1 + l_3 \sin(\theta_1 + \theta_2)$$

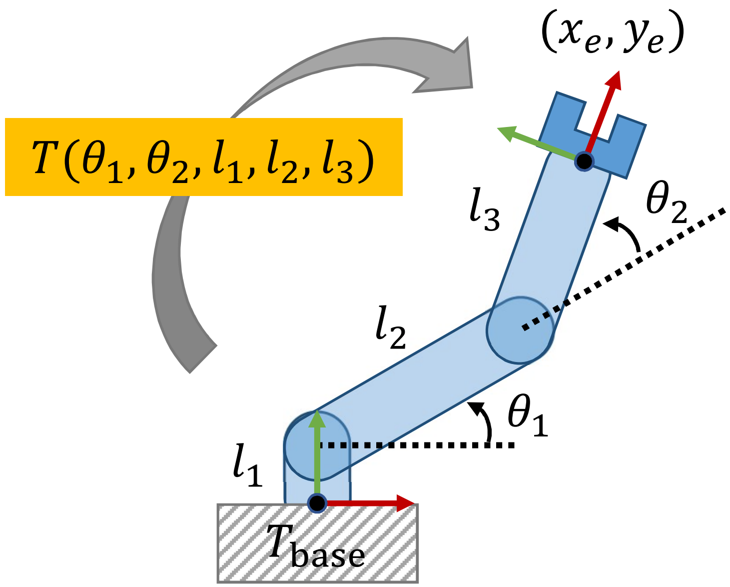
Forward Kinematics: A Simple Example

- Forward kinematics map as a coordinate transformation
  - The body local coordinate system of the end-effector was initially coincide with the reference coordinate system T_{base}
  - Forward kinematics map transforms the position and orientation of the end-effector according to joint angles



$$\begin{bmatrix} x_e \\ y_e \\ 1 \end{bmatrix} = \begin{bmatrix} \\ T(\theta_1, \theta_2, l_1, l_2, l_3) \\ \\ \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A Chain of Transformations

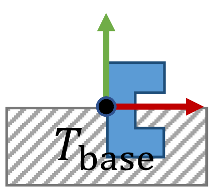


$$\begin{bmatrix} x_e \\ y_e \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & T(\theta_1, \theta_2, l_1, l_2, l_3) & \\ & & \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T = \mathrm{T_y}(l_1) R(\theta_1) \mathrm{T_x}(l_2) R(\theta_2) \mathrm{T_x}(l_3)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & l_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta_1 & -sin\theta_1 & 0 \\ sin\theta_1 & cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta_2 & -sin\theta_2 & 0 \\ sin\theta_2 & cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & l_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Interpretation of Transf. (1)



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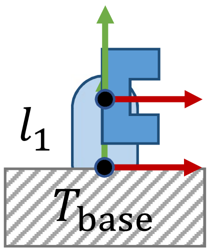

$$T = \boxed{\mathrm{T_y}(l_1)} R(\theta_1)\mathrm{T_x}(l_2)R(\theta_2)\mathrm{T_x}(l_3)$$

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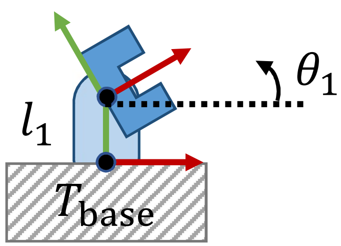

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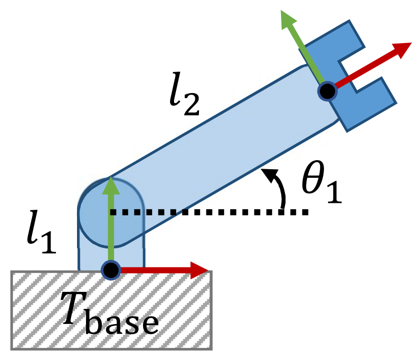

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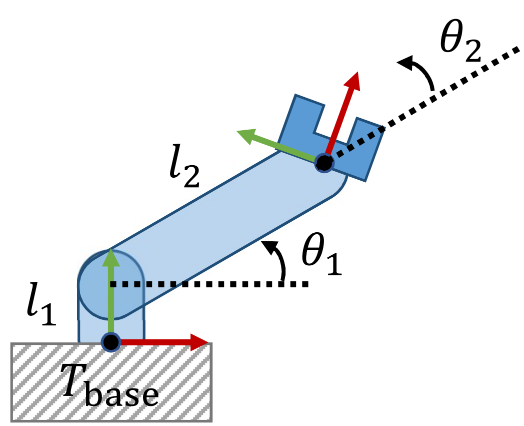

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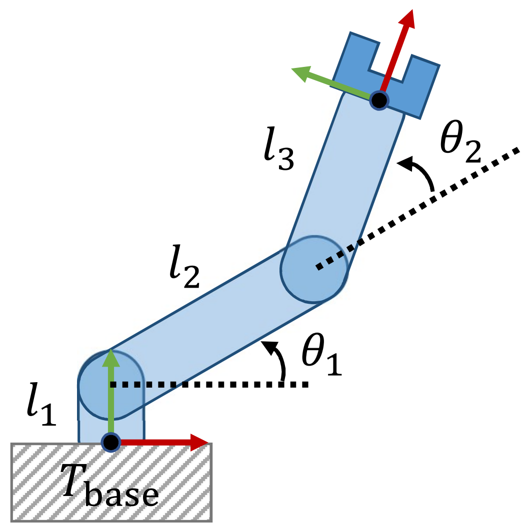

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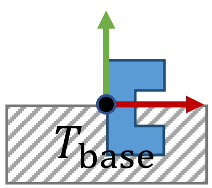
Interpretation of Transf. (2)



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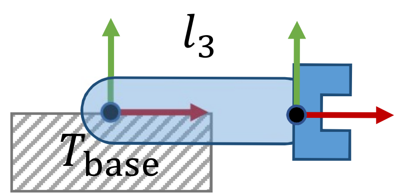

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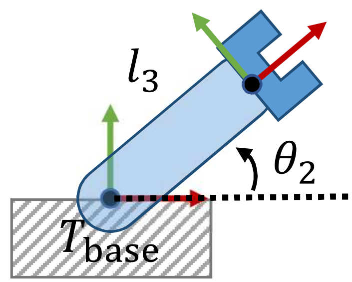

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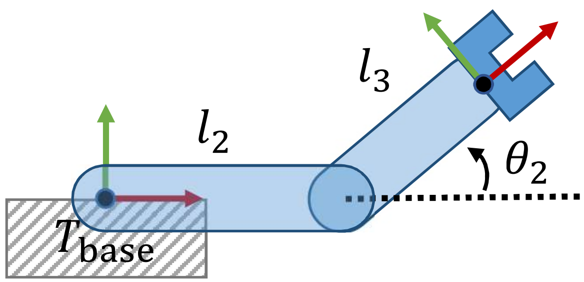

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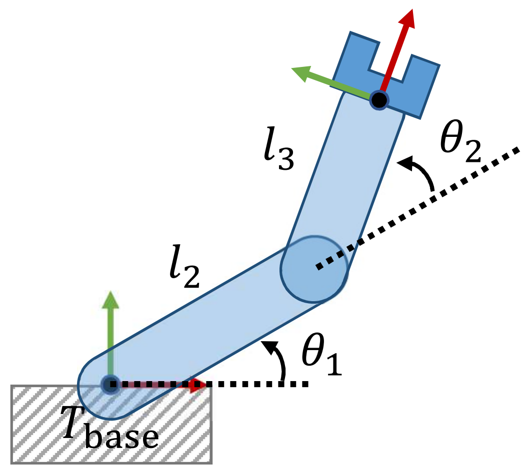

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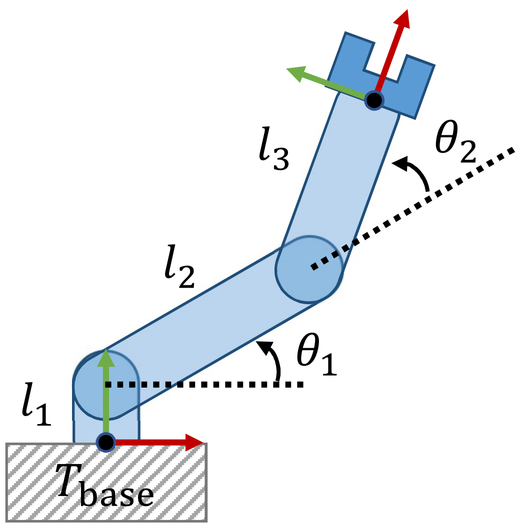

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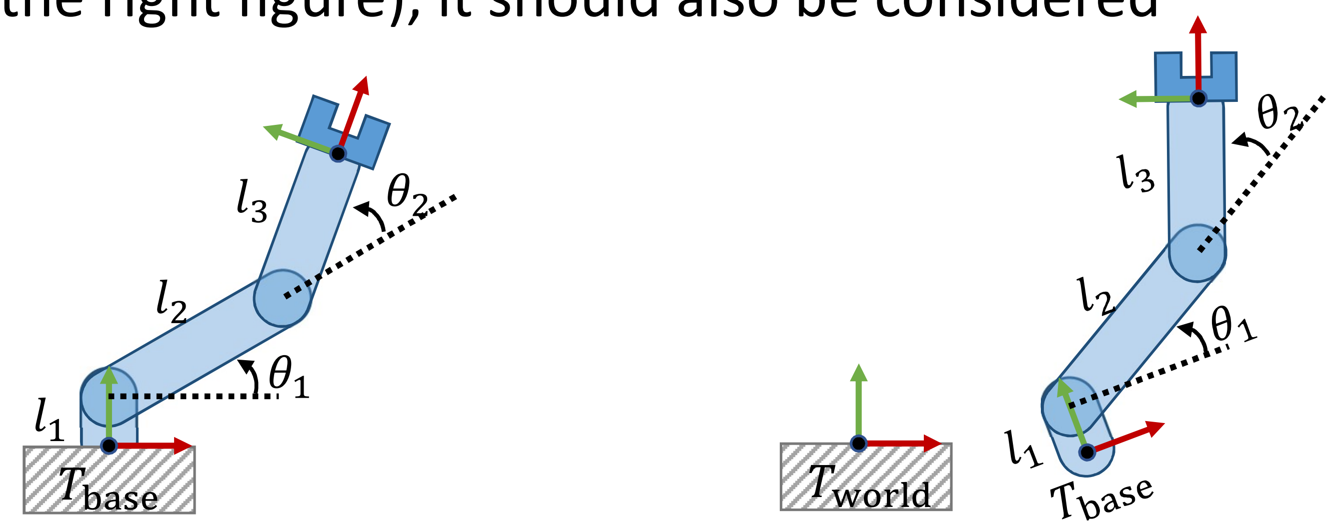
Joint and Link Transformations

- In general, a FK map can be represented as follows

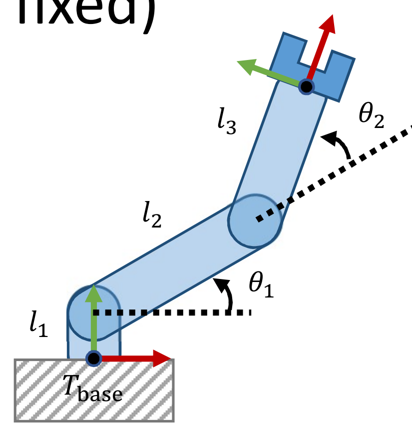
$$T = T_{\text{base}} T_{L1} T_{J1} T_{L2} T_{J2} \cdots$$

- In the previous example, T_{base} was ignored because it is coincided with the world frame. If the base can freely move (e.g., the right figure), it should also be considered



Joint and Link Transformations

- Joint transformations
  - Represent joint movement (variable)
- Link transformations
  - Defines a frame relative to its parent (often fixed)



$$T = \mathrm{T_y}(l_1) R(\theta_1) \mathrm{T_x}(l_2) R(\theta_2) \mathrm{T_x}(l_3)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & l_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta_1 & -sin\theta_1 & 0 \\ sin\theta_1 & cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta_2 & -sin\theta_2 & 0 \\ sin\theta_2 & cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & l_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

How to Represent a Floating Base?

- A free joint (rotation + translation) is used

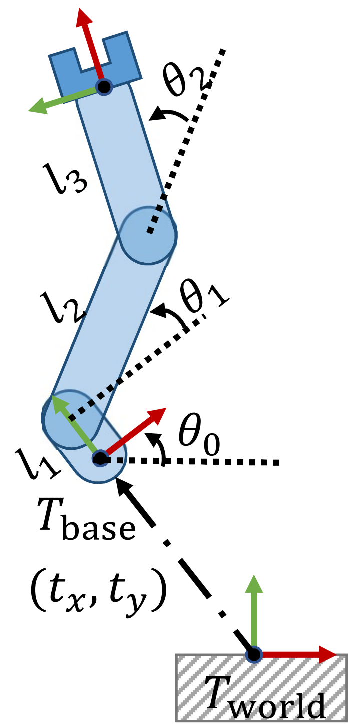
$$T = T_x(t_x)T_y(t_t)R(\theta) \qquad \text{(in 2D space)}$$

$$= \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = T_x(t_x)T_y(t_y)T_z(t_z)R_x(\theta)R_y(\psi)R_z(\phi) \qquad \text{(in 3D space)}$$

$$= \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi & 0 & \sin\psi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\psi & 0 & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

How to Represent a Floating Base?



$$T = \underbrace{T_x(t_x)T_y(t_y)R(\theta_0)}_{T_{\text{base}}}\underbrace{\mathrm{T_y}(l_1)}_{T_{L1}}\underbrace{R(\theta_1)}_{T_{J1}}\underbrace{\mathrm{T_x}(l_2)}_{T_{L2}}\underbrace{R(\theta_2)}_{T_{J2}}\underbrace{\mathrm{T_x}(l_3)}_{T_{L3}}$$

How to Handle Ball-and-Socket Joints (3D Rotation)?

- Three revolute joints whose axes intersect at a point (equivalent to Euler angles)

$$T = R_x(\theta)R_y(\psi)R_z(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi & 0 & \sin\psi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\psi & 0 & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = R_x(\theta)R_z(\phi)R_y(\psi)$$

$$T = R_y(\psi)R_x(\theta)R_z(\phi)$$

$$T = R_y(\psi)R_z(\phi)R_x(\theta)$$

$$T = R_z(\phi)R_x(\theta)R_y(\psi)$$

$$T = R_z(\phi)R_y(\psi)R_x(\theta)$$

$$T = R_x(\theta_1)R_y(\psi)R_x(\theta_2)$$

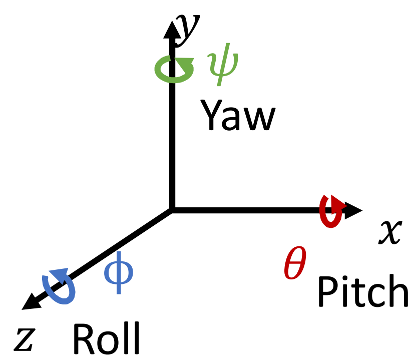
$$T = R_x(\theta_1)R_z(\phi)R_x(\theta_2)$$

$$T = R_y(\psi_1)R_x(\theta)R_y(\psi_2)$$

$$T = R_y(\psi_1)R_z(\phi)R_y(\psi_2)$$

$$T = R_z(\phi_1)R_x(\theta)R_z(\phi_2)$$

$$T = R_z(\phi_1)R_y(\psi)R_z(\phi_2)$$



How to Handle Ball-and-Socket Joints?

- Every rotation in 3D can be expressed by a unit vector $\mathbf{n}$ and a scalar angle α (proved by Leonhard Euler)

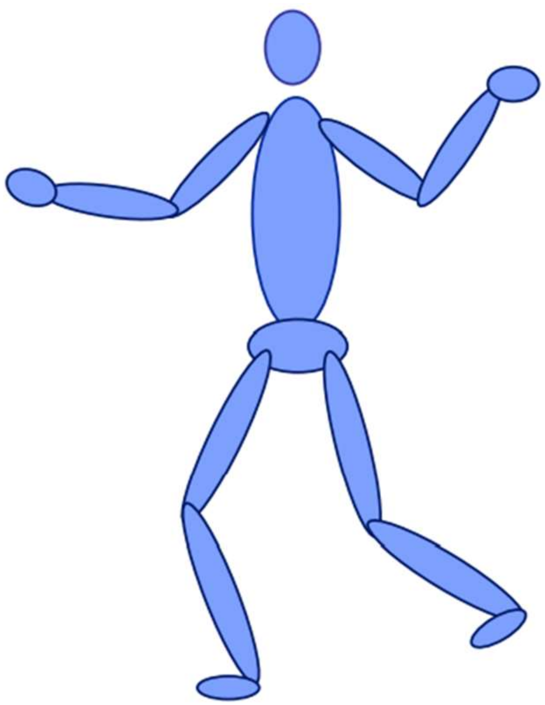
- The corresponding rotation matrix is:

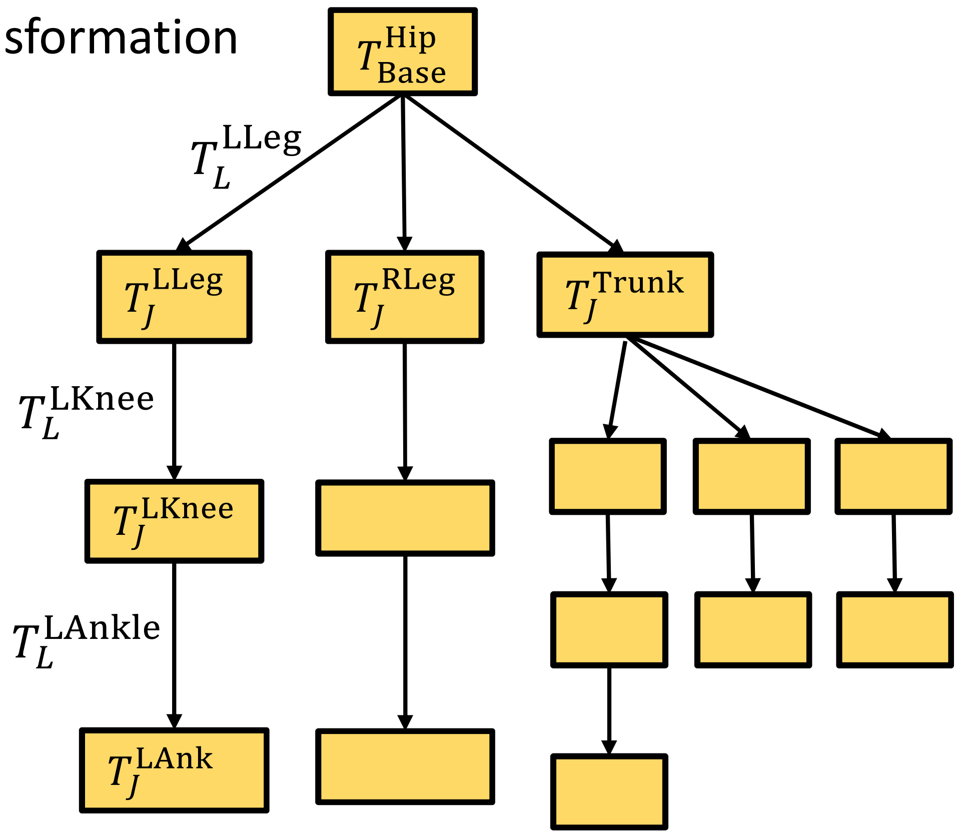
$$T = R(\alpha, \mathbf{n})$$

$$= \begin{bmatrix} \cos\alpha + n_x^2(1-\cos\alpha) & n_x n_y(1-\cos\alpha) - n_z\sin\alpha & n_x n_z(1-\cos\alpha) + n_y\sin\alpha & 0 \\ n_x n_y(1-\cos\alpha) + n_z\sin\alpha & \cos\alpha + n_y^2(1-\cos\alpha) & n_y n_z(1-\cos\alpha) - n_x\sin\alpha & 0 \\ n_x n_z(1-\cos\alpha) - n_y\sin\alpha & n_y n_z(1-\cos\alpha) + n_x\sin\alpha & \cos\alpha + n_z^2(1-\cos\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hierarchical Modeling (Dynamic Objects)

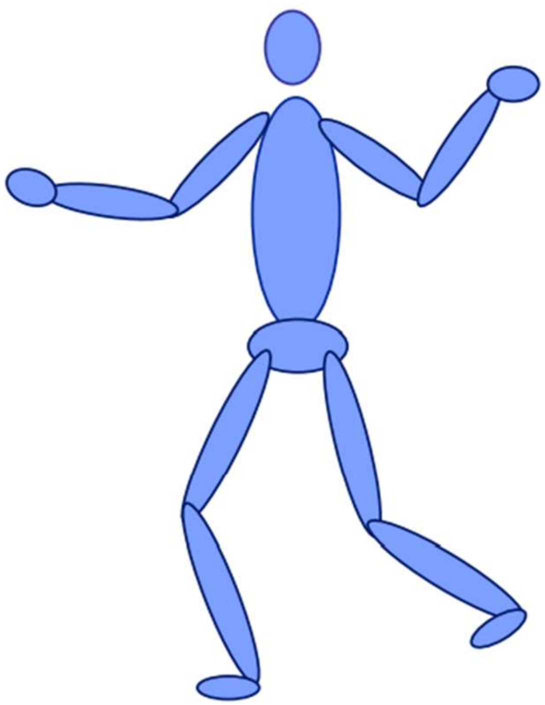
- A tree structure
  - A node contains a joint transformation
  - An edge contains a link transformation

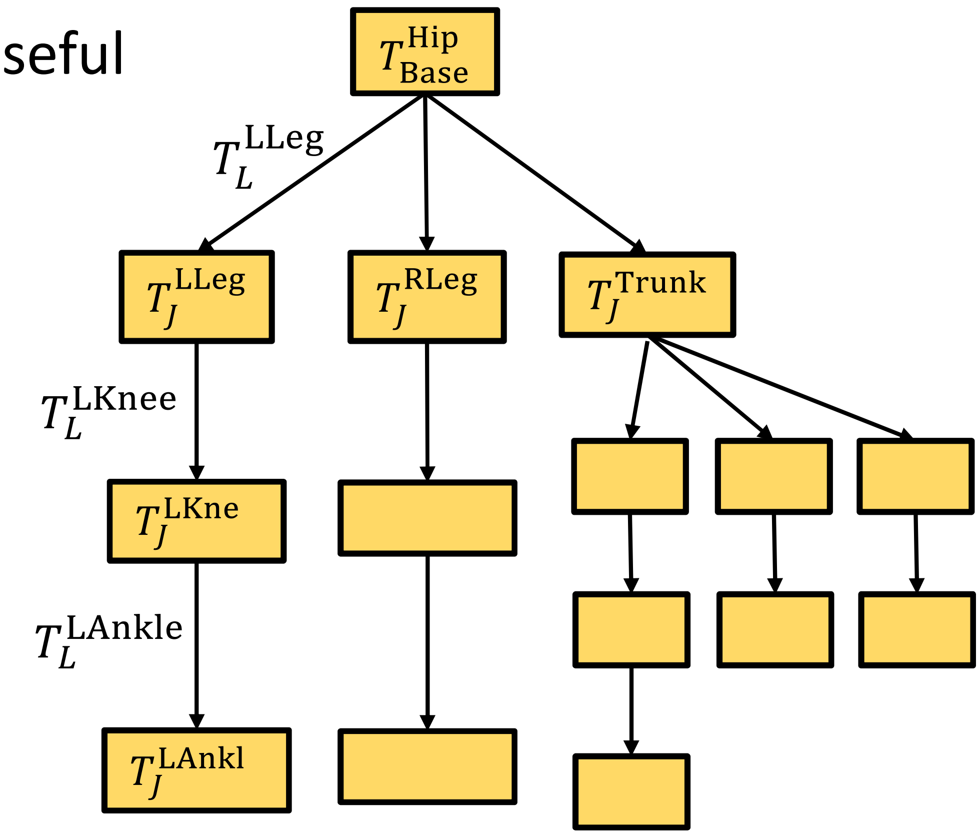




Depth-First Tree Traversal

- Drawing a hierarchical model needs to compute the positions and orientations of all the links in the model
- OpenGL's matrix stack is useful





Summary

- Kinematics is the study of motion of articulated figures
  - Kinematics does not consider physics (forces, mass, …)
- Forward kinematics is straightforward
  - Forward kinematics map is just a series of coordinate transformations
- Inverse kinematics is not trivial
  - There exist many or no solution in many cases