Differential Equations

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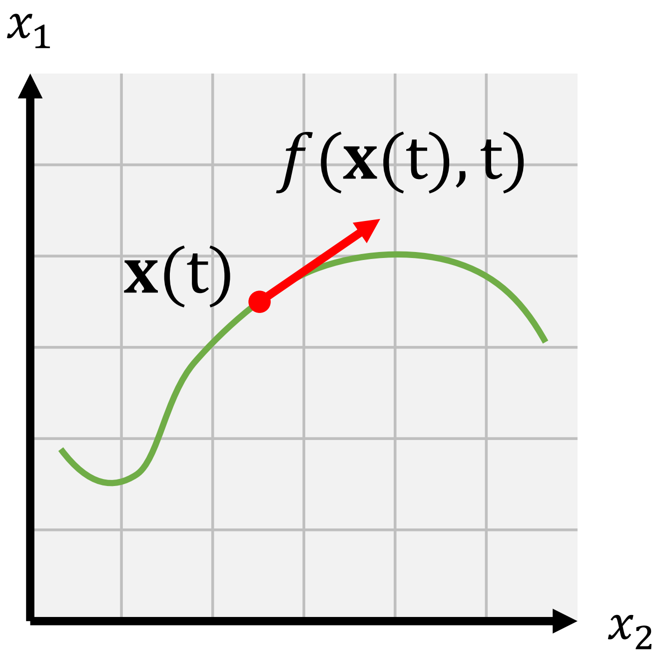
Physically Based Modeling (SIGGRAPH 2001 course) by Andrew Witkin

Differential Equations

- It describes the relation between an unknown function and its (first, second, partial, etc) derivatives

- Many natural phenomena can be explained by using differential equations
  - Newton’s equation (motions of macroscopic objects)
  - Diffusion equation (heat transfer)
  - Maxwell equation (electromagnetism)
  - Navier-Stokes equation (motions of fluids)

A Canonical Differential Equation

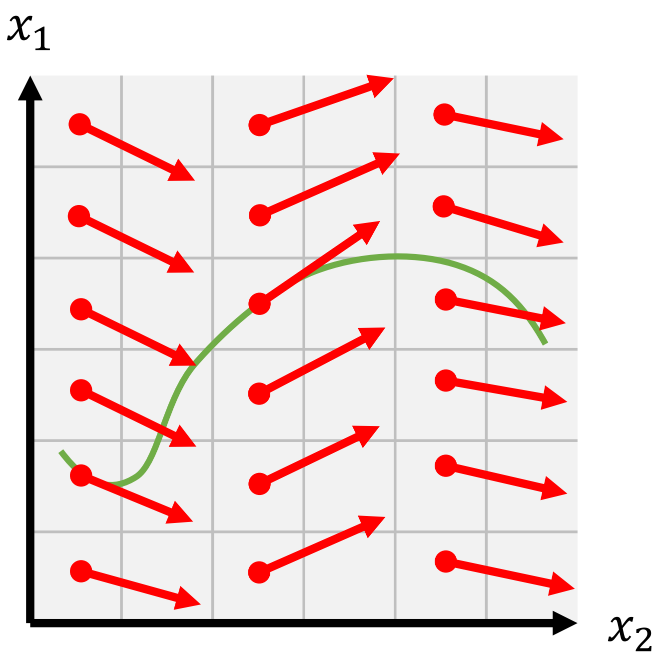


$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t)$$

$\mathbf{x}(t)$: a state of the system

$f(\mathbf{x}(t), \mathrm{t})$: the time derivative of $\mathbf{x}(t)$

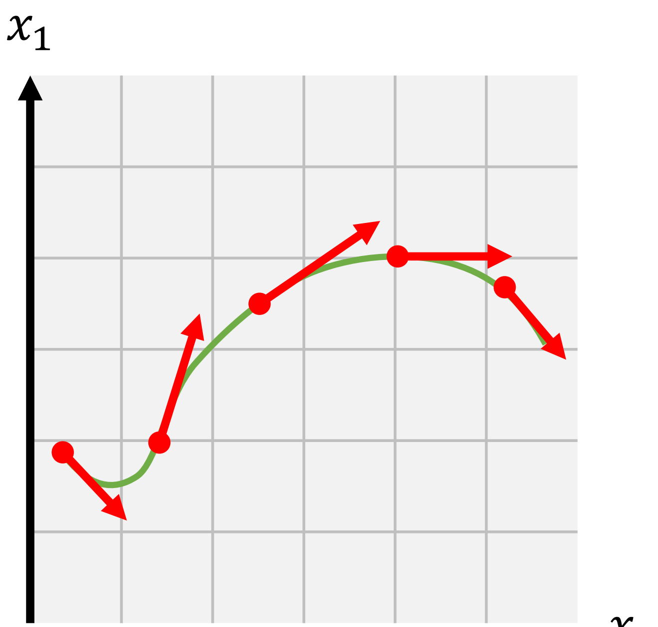
Vector Field



$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t)$$

- The differential equation defines a vector field regarding the velocity
- If $\mathbf{x}(t)$ is a location of moving point $\mathbf{p}$ at time t, the equation tells us that the point should have the same velocity to $f(\mathbf{x}, t)$ given its location $\mathbf{x}(t)$

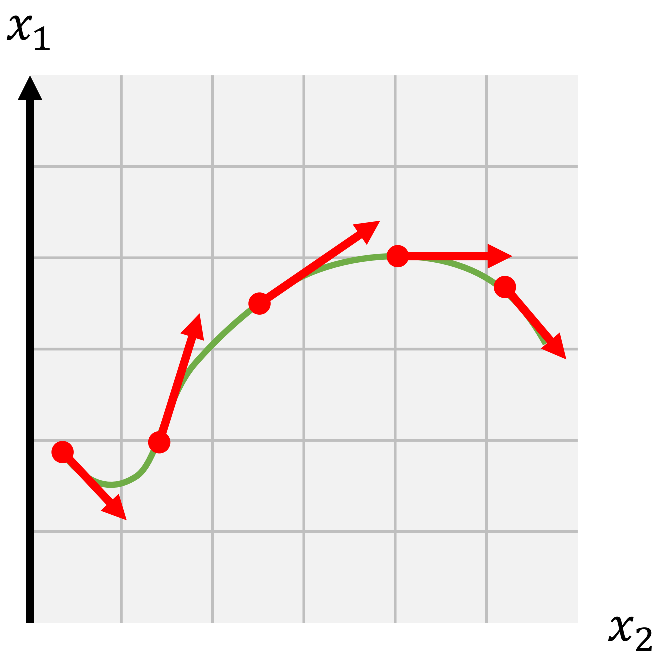
Integral Curves



$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t)$$

- We can think f as driving $\mathbf{p}$ from point to point, like an ocean current
- Once a point $\mathbf{p}$ is dropped at the location $\mathbf{x}$ at time t, all future motion is fully determined by f
- The trajectory swept out by $\mathbf{p}$ through f forms an ***integral curve*** of the vector field

Initial Value Problems



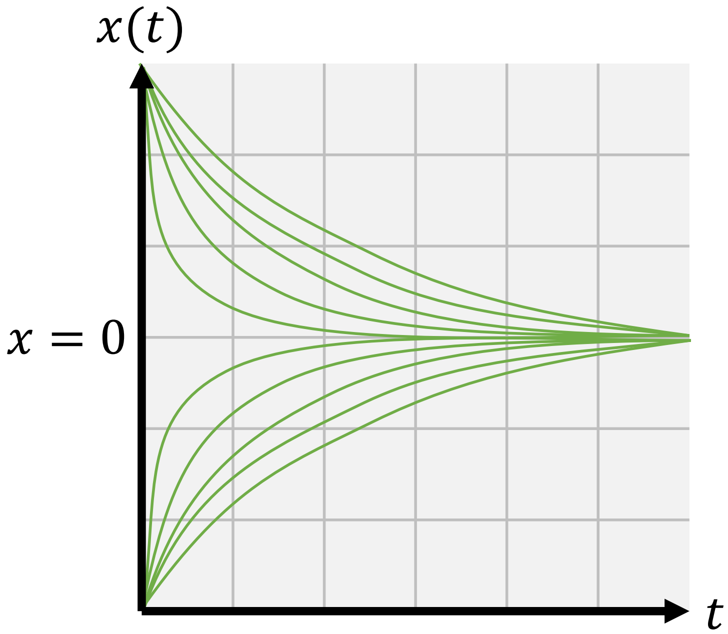
$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t)$$

- Given the starting point, find the integral curve

- Note that ***when to start*** in addition to ***where to start*** could also affects the results if f depends directly on time t

Analytical Methods

- Ways to get analytical solutions
  - Substiution
  - Series expansion
  - Laplace transform



$$\dot{x}(t) = -kx(t)$$

$$\therefore x(t) = e^{-kx(t)}$$

Numerical Methods

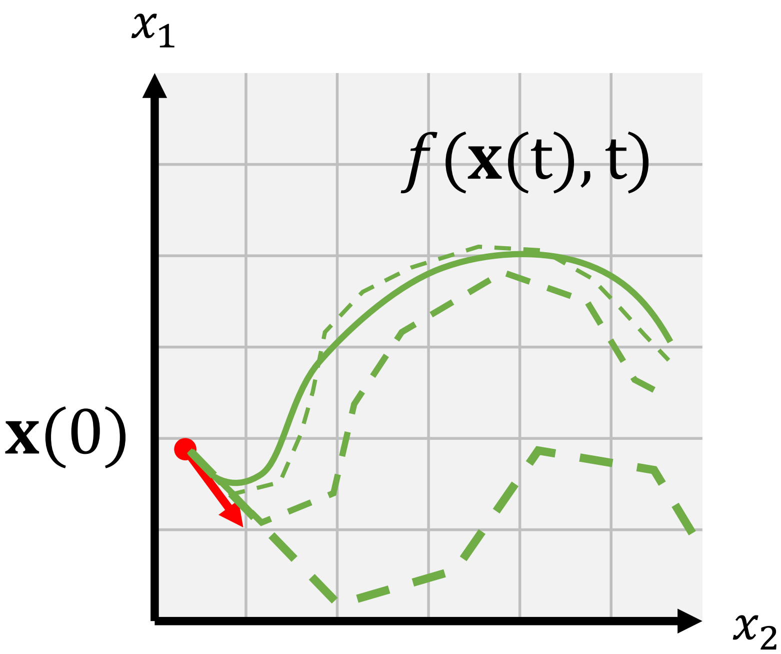
- As f gets complex, it gets harder to find analytical solutions
- Instead, we can use numerical methods to compute approximate integral curves
- The idea is to compute the change of state $\Delta\mathbf{x}$ over the fixed time interval $\mathrm{h} = \Delta t$ by using $\dot{\mathbf{x}} = f$ (i.e. intaneous velocity)

Euler's Method

$$\mathbf{x}(t+h) = \mathbf{x}(t) + \underline{h\dot{\mathbf{x}}}$$

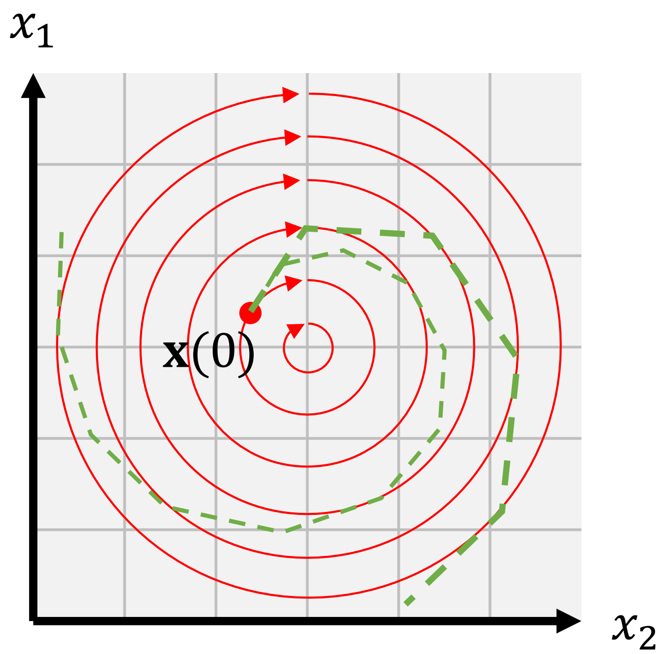
It assumes that the velocity does not change during the timestep h

$$\boxed{\mathbf{x}(t+h) = \mathbf{x}(t) + hf(\mathbf{x}(t), t)}$$



- Simplest numerical method
- Discrete time steps
- Bigger steps, bigger erros
- Smaller steps, larger computation

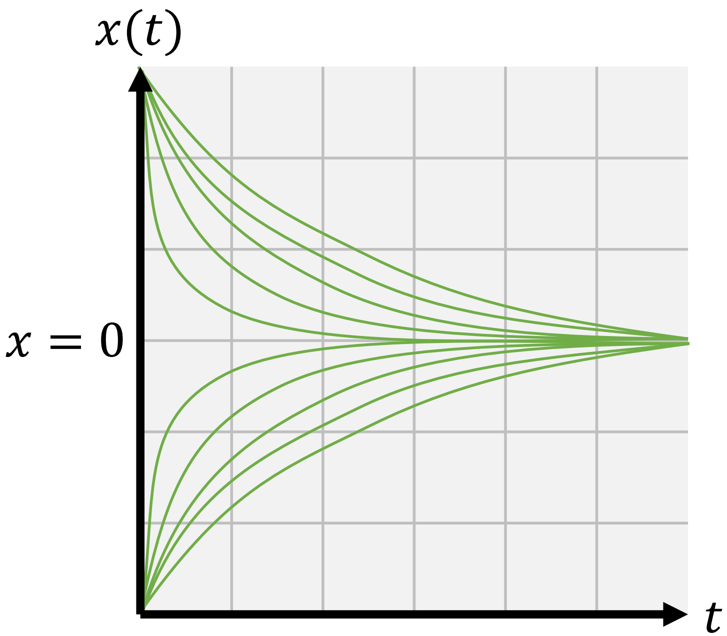
Problem 1: Inaccuracuy



- Suppose we have a vector field rotating around with respect to a point, where any (exact) integral curves should be concentric circles

- When using Euler's method, the radius of interal curve gets larger and larger (i.e. spiral curves) regardless of timestep size

Problem 2: Instability



$$\dot{x}(t) = -kx(t)$$

$$x(t+h) = x(t) + h(-kx(t))$$

- When $h > 1/k$, we have $|\Delta x| = |x|$, so the solution occilates around zero
- When $h \geq 2/k$, the solution does not converge
- Beyond $h > 2/k$, the oscillation diverges, and the system blows up

Euler’s Method: Tayler Series

$$\mathbf{x}(t_0 + h) = \underbrace{\mathbf{x}(t_0) + h\dot{\mathbf{x}}(t_0)}_{\text{Euler's method}} + \underbrace{\frac{h^2}{2!}\ddot{\mathbf{x}}(t_0) + \cdots + \frac{h^n}{n!}\frac{\partial^n \mathbf{x}(\mathrm{t}_0)}{\partial t^n}}_{\text{Error occurred due to linear approximation}}$$

Euler’s method

Error occurred due to linear approximation

$O(h^2)$

- Suppose we take steps of size h/m, this will produces about $(h/m)^2$ error at each timestep, and we need to take m steps more to simulate the orignal duration
- In total, the error is reduced ***linearly*** as we decrese the timestpe

$$(h)^2 \cdot 1 = h^2 \quad \longrightarrow \quad \boxed{\left(\frac{h}{m}\right)^2 \cdot m = \frac{h^2}{m}}$$

The Mid-point Method

$$\mathbf{x}(t_0+h) = \mathbf{x}(t_0) + h\dot{\mathbf{x}}(t_0) + \frac{h^2}{2!}\ddot{\mathbf{x}}(t_0) + \cdots + \frac{h^n}{n!}\frac{\partial^n \mathbf{x}(\mathrm{t}_0)}{\partial t^n}$$

$O(h^3)$

$$\ddot{\mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial f}{\partial t}$$

Complicated to evaluate!

$$f(\mathbf{x}_0 + \Delta\mathbf{x}, t_0 + \Delta t) = f(\mathbf{x}_0, t_0) + \Delta\mathbf{x}\frac{\partial f}{\partial \mathbf{x}} + \Delta\mathrm{t}\frac{\partial f}{\partial t} + \cdots$$

$$\Delta\mathbf{x} = \frac{h}{2}\dot{\mathbf{x}}(t_0) = \frac{h}{2}\frac{\partial \mathbf{x}}{\partial t}$$

$$\Delta\mathrm{t} = \frac{h}{2}$$

$$f\left(\mathbf{x}_0 + \frac{h}{2}f, t_0 + \frac{h}{2}\right) = f(\mathbf{x}_0, t_0) + \frac{h}{2}\underline{\left(\frac{\partial \mathbf{x}}{\partial t}\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial f}{\partial \mathrm{t}}\right)} + O(h^2)$$

Rearrange &
Multiply h both sides

$$\frac{h^2}{2}\ddot{\mathbf{x}}(t) + O(h^3) = h\left(f\left(\mathbf{x}_0 + \frac{h}{2}f, t_0 + \frac{h}{2}\right) - f(\mathbf{x}_0, t_0)\right)$$

The Mid-point Method

$$\boxed{\frac{h^2}{2}\ddot{\mathbf{x}}(t) + O(h^3)} = h\left(f\left(\mathbf{x}_0 + \frac{h}{2}f, t_0 + \frac{h}{2}\right) - f(\mathbf{x}_0, t_0)\right)$$

$$\mathbf{x}(t_0 + h) = \mathbf{x}(t_0) + h\dot{\mathbf{x}}(t_0) + \boxed{\frac{h^2}{2!}\ddot{\mathbf{x}}(t_0) + \cdots + \frac{h^n}{n!}\frac{\partial^n \mathbf{x}(\mathrm{t}_0)}{\partial t^n}}$$

$$\mathbf{x}(t_0 + h) = \mathbf{x}(t_0) + h\cancel{\dot{\mathbf{x}}(t_0)} + h\left(f\left(\mathbf{x}_0 + \frac{h}{2}f, t_0 + \frac{h}{2}\right) - f\cancel{(\mathbf{x}_0, t_0)}\right) + O(h^3)$$

$$\mathbf{x}(t_0 + h) = \mathbf{x}(t_0) + hf\left(\mathbf{x}_0 + \frac{h}{2}f(\mathbf{x}_0, t_0), t_0 + \frac{h}{2}\right) + O(h^3)$$

Mid-point method

The Mid-point Method

- The mid-point method is a 2nd-order method, (i) an euler step is computed, then (ii) the derivative is evaluated again at the step's midpoint which is used to calculate the step

$$\mathbf{x}(t_0 + h) = \mathbf{x}(t_0) + hf\left(\mathbf{x}_0 + \frac{h}{2}f(\mathbf{x}_0, t_0), t_0 + \frac{h}{2}\right)$$

- The error is reduced ***quadratically*** as we decrese the timestpe

$$(h)^3 \cdot 1 = h^3 \quad \Longrightarrow \quad \left(\frac{h}{m}\right)^3 \cdot m = \frac{h^3}{m^2}$$

Higher-order Methods

- By evaluating f a few more times, we can elimatate higher and higher orders of derivatives. The most popular procedure for doing this is a method called ***Runge-Kutta*** of order 4 and has an error per step of $O(h^5)$

$$k_1 = hf(\mathbf{x}_0, t_0)$$

$$k_2 = hf\left(\mathbf{x}_0 + \frac{k_1}{2}, t_0 + \frac{h}{2}\right)$$

$$k_3 = hf\left(\mathbf{x}_0 + \frac{k_2}{2}, t_0 + \frac{h}{2}\right)$$

$$k_4 = hf(\mathbf{x}_0 + k_3, t_0 + h)$$

$$\mathbf{x}(t_0 + h) = \mathbf{x}_0 + \tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4$$

Adaptive Stepsizes

- If we choose a fixed stepsize, we can only proceed as fast as the worst sections of $\mathbf{x}(t)$ will allow. By measuring an error e, we can change stepsize adaptively in runtime

$$e = |\mathbf{x}_a - \mathbf{x}_b|$$

$\mathbf{x}_a$: an estimate by taking an Euler step of size h

$\mathbf{x}_b$: an estimate by taking an Euler step of size $h/2$

- Suppose that we are willing to have an error of as much as 10^{-4} per step, where $\mathbf{x}_a, \mathbf{x}_b$ should differ from each other by $O(h^2)$
  - If the current error is 10^{-8} $\left(\frac{10^{-4}}{10^{-8}}\right)^{1/2} h = 100h$ $\Rightarrow$ Can increase timestep upto 100
  - If the current error is 10^{-3} $\left(\frac{10^{-4}}{10^{-3}}\right)^{1/2} h \approx .316h$ $\Rightarrow$ Should decrease timestep by 0.316 at least

Summary

- We have learned the basics of ordinary differential equations and some numerical methods for solving the equations
- In the upcoming lectures, we will use those methods to simulate many natural phenomena by integrating ODEs
- Euler's method is rarely used in real applications but it provides a foundation of numerical approaches