Affine Geometry

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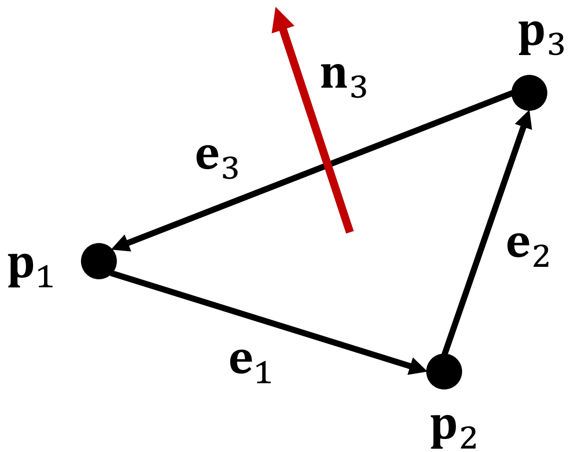
Geometric Programming

- A way of handling geometric entities such as vectors, points, and transforms.
- Traditionally, computer graphics packages are implemented using homogeneous coordinates.
- We will review affine geometry and coordinate-invariant geometric programming.

2D/3D Geometry Representation

- Basic elements to represent 2D/3D geometries
  - **Vectors** to represent directions
  - **Points** to represent locations



Vertices $\mathbf{p}_{1:3}$ are locations

Edges $\mathbf{e}_{1:3}$ are directions

c.f. $\mathbf{e}_\mathbf{i} = \mathbf{p}_{i+1} - \mathbf{p}_i$

Surface normal $\mathbf{n}$ is a direction

c.f. $\mathbf{n} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} = \frac{(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_2)}{|(\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_2)|}$

$$\mathbf{p}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \quad \mathbf{e}_i = \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \\ z_{i+1} - z_i \end{bmatrix} \quad \mathbf{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

bold face

components

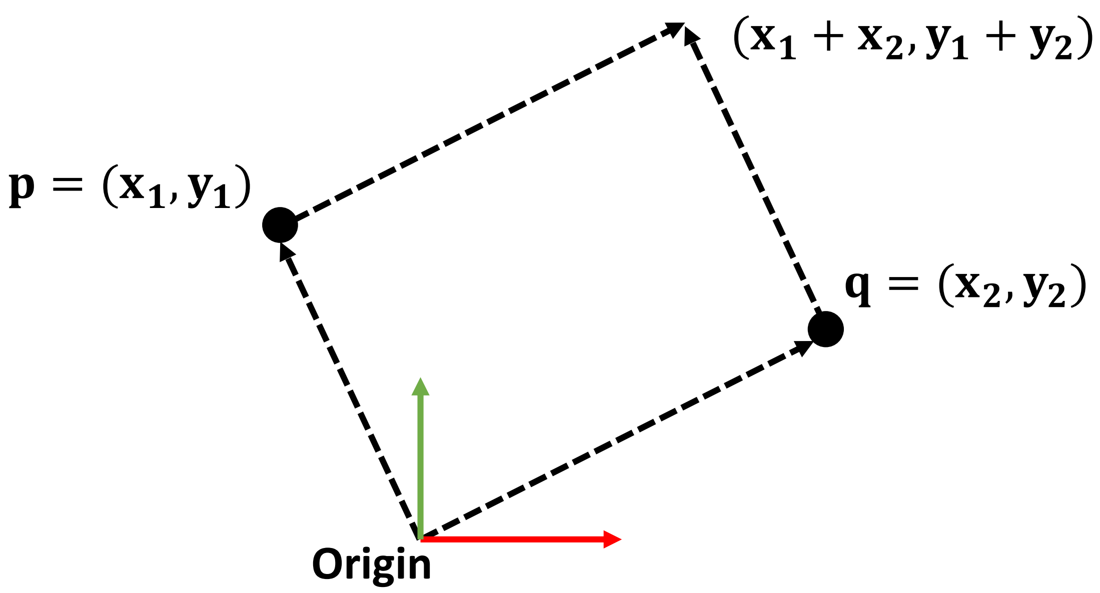
2D/3D Geometry Representation

- Basic elements to represent 2D/3D geometries
  - **Vectors** to represent directions
  - **Points** to represent locations
- We already learned a tool to manipulate vectors, which are the operations applicable in vector spaces
- What is an appropriate tool for manipulating points?

Example of coordinate-dependence

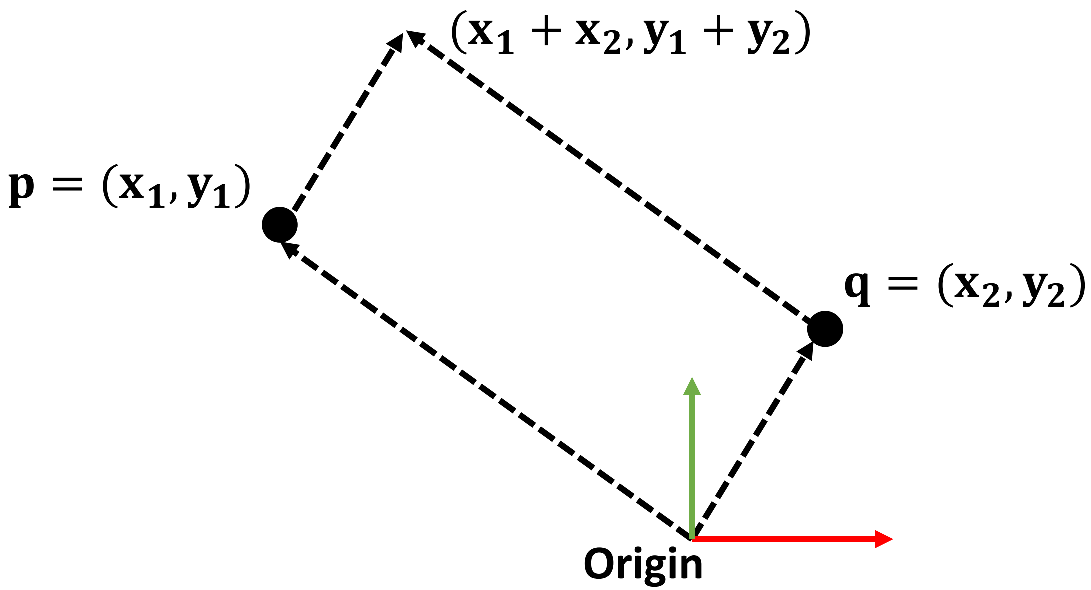
- What is the “sum” of these two positions?

If you assume coordinates, …



- The sum is $(\mathbf{x_1} + \mathbf{x_2}, \mathbf{y_1} + \mathbf{y_2})$
  - $(\mathbf{x_1}, \mathbf{y_1})$ and $(\mathbf{x_2}, \mathbf{y_2})$ are considered as vectors from the **origin** to **p** and **q**, respectively.
  - Is it correct? Is it geometrically meaningful?

If you select a different origin, …



- If you choose a different coordinate system, you will get a different result

Vector and Affine Spaces

- Vector space
  - Includes vectors and related operations
  - No points

- Affine space
  - Includes vectors, points, and related operations
  - Superset of vector space

Points and Vectors

- A ***point*** is a position specified with coordinate values
- A ***vector*** can be specified as the difference between two points
- If an origin is specified, then a point can be represented by a vector from the origin
- But, a point is still not a vector in a coordinate-free point of view

Vector spaces

- A ***vector space*** consists of
  - Set of vectors
  - Two operations
    - Vector addition
    - Scalar multiplication

- A ***linear combination*** of vectors is also a vector

$$\mathbf{u}, \mathbf{v}, \mathbf{w} \in V \quad \Longrightarrow \quad \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} \in V$$

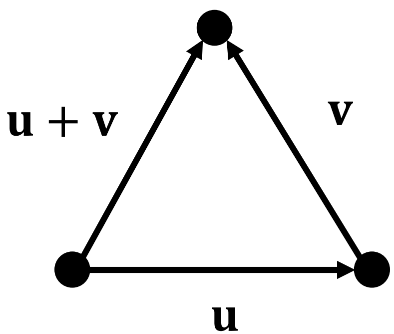
Affine spaces

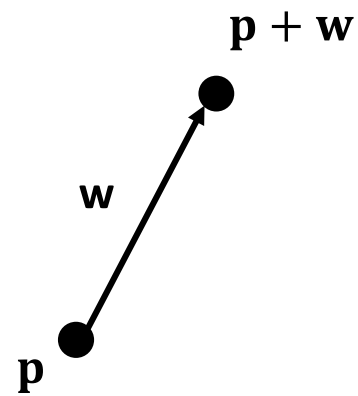
- An ***affine space*** consists of
  - Set of points
  - An associated vector space (vectors, vector operations)
  - Two operations
    - The difference between two points
    - The addition of a vector to a point

Coordinate-Invariant Geometric Operations

- Addition
- Subtraction
- Scalar multiplication
- Linear combination
- Affine combination

Addition



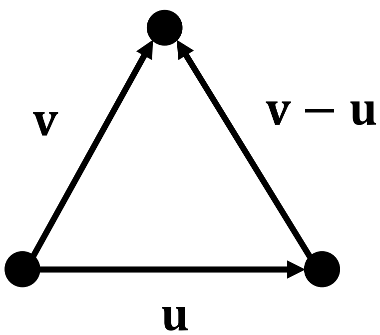


$\mathbf{u} + \mathbf{v}$ is a vector

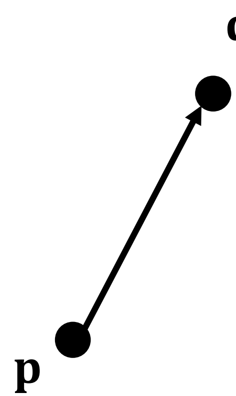
$\mathbf{p} + \mathbf{w}$ is a point

$\mathbf{u}, \mathbf{v}, \mathbf{w}$: vectors
$\mathbf{p}, \mathbf{q}$: points

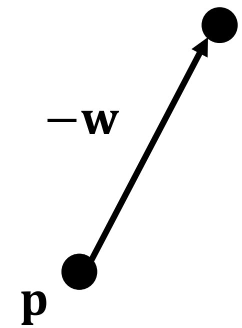
Subtraction



$\mathbf{v} - \mathbf{u}$ is a vector



$\mathbf{q} - \mathbf{p}$ is a vector



$\mathbf{p} - \mathbf{w}$ is a point

$\mathbf{u}, \mathbf{v}, \mathbf{w}$: vectors
$\mathbf{p}, \mathbf{q}$: points

Scalar Multiplication

$$\text{scalar} \cdot \text{vector} = \text{vector}$$

$$1 \cdot \text{point} = \text{point}$$

$$0 \cdot \text{point} = \text{vector}$$

$$\text{c} \cdot \text{point} = (\text{undefined}) \text{ if } (\text{c} \neq 0,1)$$

Linear Combination

- A linear space is spanned by a set of basis vectors
  - Any vector in the space can be represented as a linear combination of basis vectors

$$\sum_{i=0}^{N} c_i \mathbf{v}_i = c_0 \mathbf{v}_0 + c_1 \mathbf{v}_1 + \cdots + c_N \mathbf{v}_N = \mathbf{v}$$

Affine Combination

$$\sum_{i=0}^{N} c_i \mathbf{p}_i = c_0\mathbf{p}_0 + c_1\mathbf{p}_1 + \cdots + c_N\mathbf{p}_N = \mathbf{p}$$

$$= \underbrace{\left(\sum_{i=0}^{N} c_i\right)\mathbf{p}_0}_{\text{A point-like}} + \underbrace{\sum_{i=0}^{N} c_i\,(\mathbf{p}_i - \mathbf{p}_0)}_{\text{A vector}}$$

Point $= \mathbf{p}_0 + \sum c_i(\mathbf{p}_i - \mathbf{p}_o)$ if $\sum c_i = 1$

Vector $= \sum c_i(\mathbf{p}_i - \mathbf{p}_o)$ if $\sum c_i = 0$

$= \text{Undefined}$ otherwise

Examples

- $(\mathbf{p}+\mathbf{q})/2$: mid point of $\mathbf{p}$ and $\mathbf{q}$

- $(\mathbf{p}+\mathbf{q}+\mathbf{r})/3$: center of gravity (point) of $\Delta\mathbf{pqr}$

- $\left(\frac{1}{\sum_N m_i \mathbf{p}_i}\right)\sum_N m_i \mathbf{p}_i$: center of mass (point) of $\boldsymbol{N}$ particles where m_i is the mass of i-th particle

- $(0.5\mathbf{p}+0.5\mathbf{q}-\mathbf{r})$: a vector from $\mathbf{r}$ to the mid point of $\mathbf{p}$ and $\mathbf{q}$

Affine Frame

- A ***frame*** is defined as a set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_N\}$ and a point $\mathbf{o}$
  - Set of vectors are bases of the associated vector space
  - $\mathbf{o}$ is the origin of the frame
  - N is the dimension of the affine space

  - Any point $\mathbf{p}$ can be written as
$$\mathbf{p} = \mathbf{o} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_N\mathbf{v}_N$$

  - Any vector $\mathbf{v}$ can be written as
$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_N\mathbf{v}_N$$

Matrix Representation

- All operations applicable in vector spaces (i.e. linear transformation) can be represented by a matrix

$$\mathbf{w} = \mathrm{A}\mathbf{v}$$

- Similarly, all operations applicable in affine space (i.e. affine transformation) can be written as follows

$$\mathbf{q} = \mathrm{A}\mathbf{v} + \mathbf{p}$$

- Or, the expression can be simplified by introducing an “extra” coordinate

$$\mathbf{q}' = \mathrm{A}'\mathbf{p}'$$

Matrix Representation

- Use an *extra* coordinate
  - **Point** : (x, y, 1) in 2D, (x, y, z, 1) in 3D
  - **Vector** : (x, y, 0) in 2D, (x, y, z, 0) in 3D
- For example

$$\underbrace{(x_1, y_1, z_1, 1)}_{\textbf{point}} + \underbrace{(x_2, y_2, z_2, 1)}_{\textbf{point}} = \underbrace{(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2, 2)}_{\textbf{undefined}}$$

$$\underbrace{(x_1, y_1, z_1, 1)}_{\textbf{point}} - \underbrace{(x_2, y_2, z_2, 1)}_{\textbf{point}} = \underbrace{(x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2, 0)}_{\textbf{vector}}$$

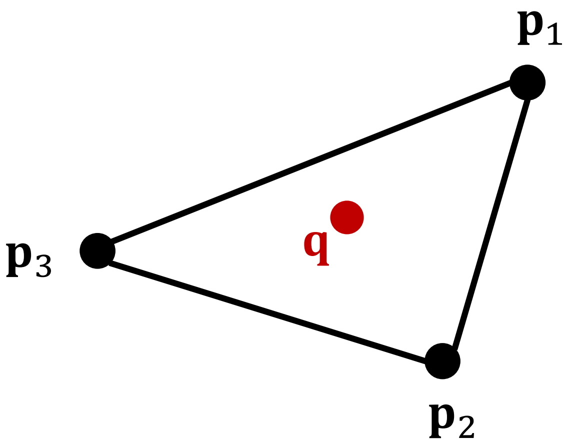
$$\underbrace{(x_1, y_1, z_1, 1)}_{\textbf{point}} + \underbrace{(x_2, y_2, z_2, 0)}_{\textbf{vector}} = \underbrace{(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2, 1)}_{\textbf{point}}$$

Barycentric Coordinate System

- A ***barycentric coordinate system*** is a coordinate system in which the location of a point is specified by reference to a ***simplex***
  - A triangle for the points in 2D space
  - A tetrahedron for the points in 3D space

Barycentric Coordinate System

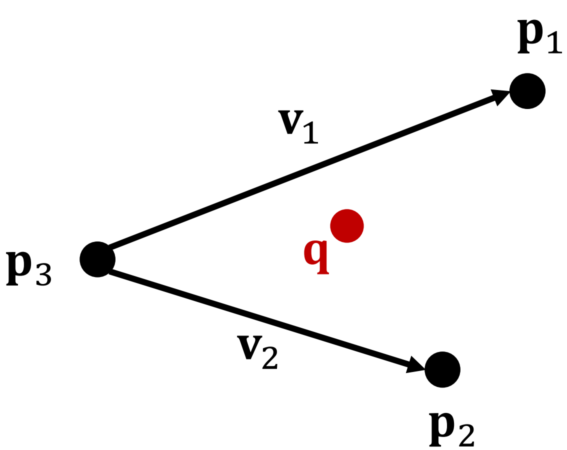
- How can we represent $\mathbf{q}$ by using $\Delta\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$?



$$\mathbf{q} = w_1\mathbf{p}_1 + w_2\mathbf{p}_2 + w_3\mathbf{p}_3$$

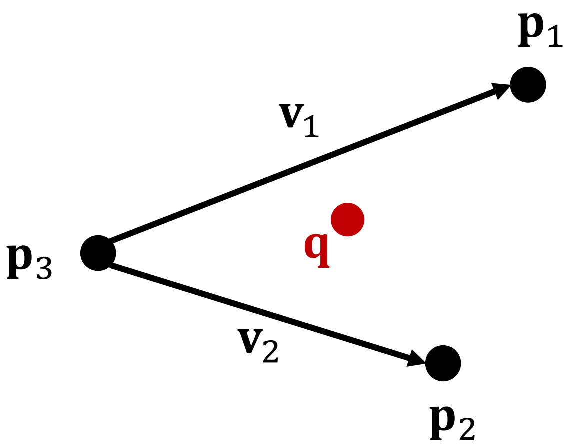
Barycentric Coordinate System

- How can we represent $\mathbf{q}$ by using $\Delta\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$?



$$
\begin{aligned}
\mathbf{q} &= \mathbf{p}_3 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \\
&= \mathbf{p}_3 + c_1(\mathbf{p}_1 - \mathbf{p}_3) + c_2(\mathbf{p}_2 - \mathbf{p}_3) \\
&= \underbrace{c_1}_{w_1}\mathbf{p}_1 + \underbrace{c_2}_{w_2}\mathbf{p}_2 + \underbrace{(1 - c_1 - c_2)}_{w_3}\mathbf{p}_3
\end{aligned}
$$

Barycentric Coordinate System



$$\mathbf{q} = \underbrace{c_1}_{w_1}\mathbf{p}_1 + \underbrace{c_2}_{w_2}\mathbf{p}_2 + \underbrace{(1 - c_1 - c_2)}_{w_3}\mathbf{p}_3$$

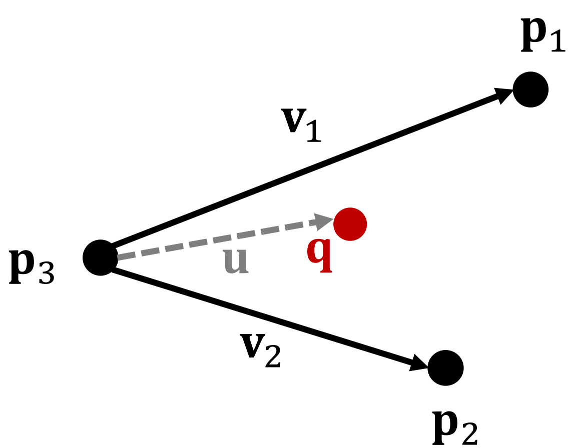
- Properties of barycentric coordinates

$$\sum w_i = 1$$

$0 \le w_i \le 1$ if $\mathbf{q}$ is located inside $\boldsymbol{\Delta}\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3$

Barycentric Coordinate System

- The coordinate values can be computed by solving a linear equation



$$\mathbf{q} = \mathbf{p}_3 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

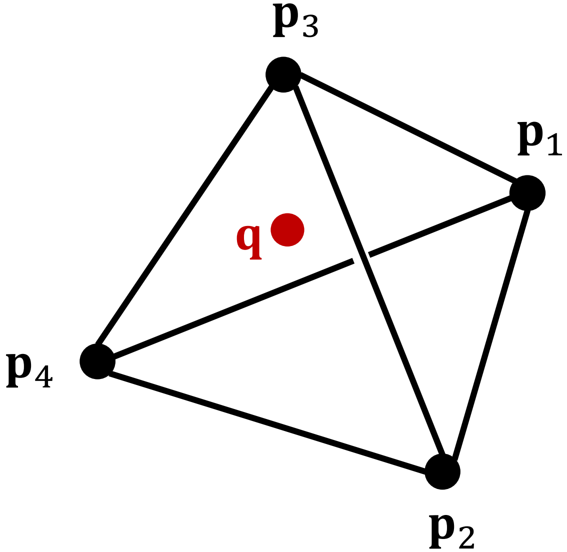
$$\mathbf{u} = \mathbf{q} - \mathbf{p}_3 = [\mathbf{v}_1\ \mathbf{v}_2]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\mathbf{v}_1\ \mathbf{v}_2]^{-1}\mathbf{u}$$

$$w_3 = 1 - c_1 - c_2$$

Barycentric Coordinate System

- How can we represent $\mathbf{q}$ by using a tetrahedron $\mathbf{p}_{1:4}$ in 3D space?



$$\mathbf{q} = w_1\mathbf{p}_1 + w_2\mathbf{p}_2 + w_3\mathbf{p}_3 + w_4\mathbf{p}_4$$

Summary

1. point + point = undefined
2. point − point = vector
3. point ± vector = point
4. vector ± vector = vector
5. scalar • vector = vector
6. Σ scalar • vector = vector
7. scalar • point = point iff scalar = 1
   = vector iff scalar = 0
   = undefined otherwise
8. Σ scalar • point = point iff Σ scalar = 1
   = vector iff Σ scalar = 0
   = undefined otherwise