MT 4113: Computing in Statistics

Computer intensive statistics
Lecture 7: The Bootstrap

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Outline

Repeated sampling: the basis for inference

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- Confidence intervals: if I could repeat the experiment many times, what interval would contain the population mean a specified proportion of times?

Repeated sampling: the basis for inference

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  - Alternative: exact same data generating process as our data – i.e. with the same true population mean. Can generate a distribution of plausible sample means and assume they represent the distribution of plausible population means.

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Computer-intensive inference

Use properties of the dataset we have to help us *simulate* repeated datasets.

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- Nonparametric: sample from the data to create new datasets.
- Parametric: simulate from a distribution based on the data to create new datasets.

Outline

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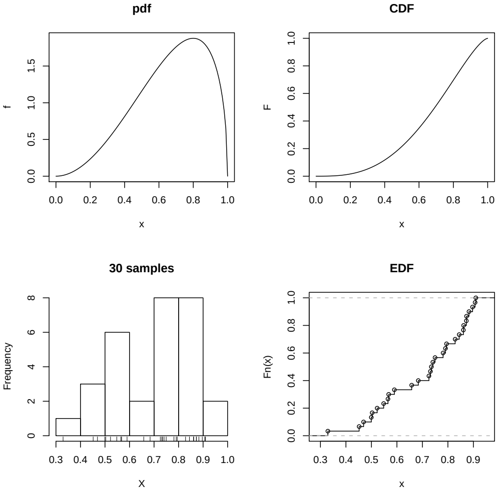
Empirical distribution function (EDF)



Nonparametric bootstrap and the EDF

- Nonparametric bootstrap:
  - generate resamples by simulating from the EDF
  - produces resamples that have the same properties as the data (c.f. previous methods where they had the same properties if *H0* was true)
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  - best approximation we have to sampling again from the original data generating process
- How to sample from the EDF?
  - Resample with replacement from the data

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Pine martin example

Original	0.13	-0.01	-0.01	0.42	-0.02	0.01	0.09	0.03	0.04	0.06	0.12	0.03
Resamples												
1	0.01	-0.01	-0.02	0.04	-0.01	0.13	0.03	-0.01	0.03	0.06	0.13	0.09
2	0.09	0.09	-0.02	-0.01	0.09	0.42	0.03	-0.01	0.04	-0.01	0.03	0.03
3	0.42	0.03	-0.01	0.09	0.06	0.06	0.06	-0.02	-0.01	0.09	0.4	0.09
.												
.												
.												
b	0.03	0.04	0.42	0.09	-0.01	-0.01	0.03	0.04	0.13	0.03	0.09	-0.01

- Notes:
  - number of times each data point occurs in each resample is a random variable
  - in b bootstrap resamples, we expect each data point to appear on average b times
  - can ensure they occur exactly b times with using a *balanced bootstrap* (but usually not worth the extra effort)

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Outline

Bootstrap confidence intervals

- There are many ways to generate a confidence interval via bootstrap resampling[1]
- Today, we will cover the simplest – the "percentile method"

[1] see, e.g., Carpinter, J. and J. Bithell. (2000) Bootstrap confidence intervals: when, which, what? A practical guide for medical statisticians. Statistics in Medicine 19: 1141-1164.

Percentile method

- A useful approximate method for producing CIs
- Besides being relatively simple to understand and implement, it performs well in practice under most commonly-encountered situations
- Very widely used in applied statistics

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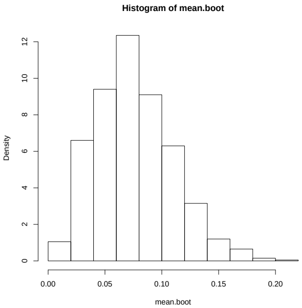
- Example: mean of pine martin data:



Percentile method (contd.)

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- Turns out (under some conditions), the following procedure gives an interval with this property:
  - Order the μ
  - Lower limit is the $\alpha/2(b+1)\%$th value
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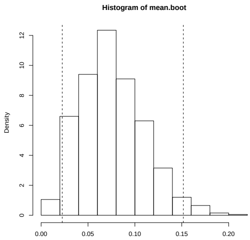

Example

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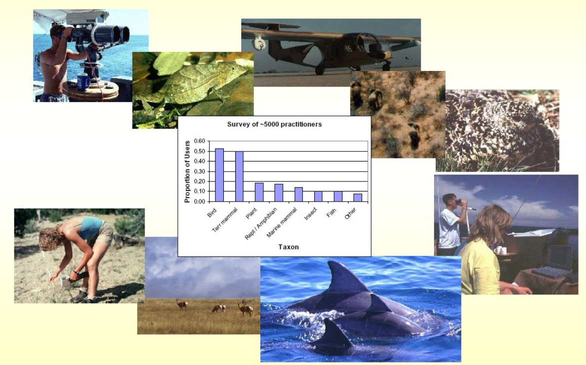
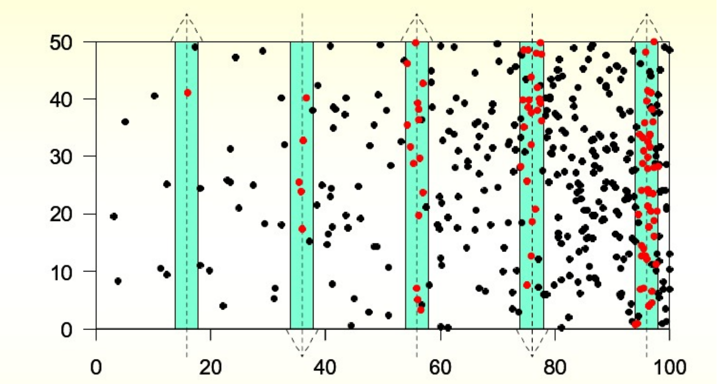
- Example: distance sampling surveys of wildlife populations (other examples in the reading)



Example - distance sampling



- Dots represent groups of animals
- Red dots = detected groups;
  Black dots = undetected groups

Example - distance sampling

- Density

$$\hat{D} = \frac{\text{n seen} \times \text{mean group size}}{\text{area searched} \times \text{p detect}} = \frac{n \times \bar{s}}{2wL \times \hat{p}}$$

- Analytic variance $\hat{\text{var}}(\hat{D}) \approx \hat{D}^2 \left(\frac{var(n)}{n^2} \times \frac{var(\bar{s})}{\bar{s}^2} \times \frac{var(\hat{p})}{\hat{p}^2} \right)$
- Analytic CIs – assume $\hat{D}$ lognormally distributed
- Alternative – nonparametric bootstrap
  - Resample transects with replacement

Outline

1. Introduction and recap
2. Nonparametric bootstrap
   - How to generate resamples
   - Bootstrap confidence interval
   - Summary
3. Parametric bootstrap
   - How to generate resamples
   - Summary

Advantages

- Simple to apply
- General and robust (compared with other general analytic approaches) method of setting CIs
- No need for parametric assumption about $f()$ or quantity of interest

Disadvantages

- Assumes observations/samples are iid
  - *probability integral transform* method when not identically distributed)
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- Generally only asymptotically exact as b and $n \Rightarrow \infty$

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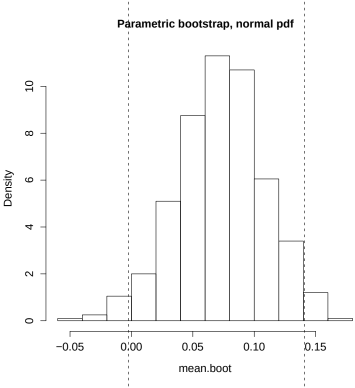
Introduction

- Parametric MC method for obtaining CIs
- Algorithm:
  - Fit a parametric model $f(\theta)$ to the data
  - Generate resamples by simulating from the fitted model
  - Use the resamples to obtain CIs (e.g., by percentile method)

Pine martin example

- 999 resamples generated from $N(\bar{x}, s^2)$



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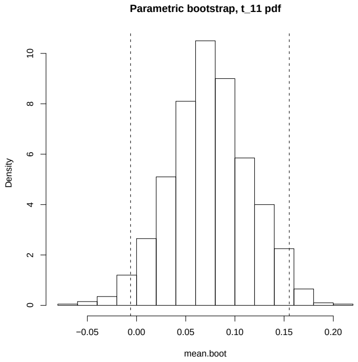
- 999 resamples generated from $t_{11}(\bar{x}, s^2)$



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Advantages

- General and robust (compared with general analytic approaches) method of setting CIs
- Observations don't need to be iid

Disadvantages

- Generally only asymptotically exact as b and $n \Rightarrow \infty$
- Must assume a parametric model for $f(\theta)$