MT 4113: Computing in Statistics

Computer intensive statistics
Lecture 6: Permutation, randomization, and Monte-Carlo tests

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Oct 10 2018

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Outline

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1. Introduction
   - Revision – Traditional parametric and nonparametric methods
   - Motivating example
2. Permutation methods
   - Permutation test
   - Permutation interval
   - Summary
3. Randomization methods
   - Overview
   - Summary
4. Monte-Carlo tests
   - Overview
   - Summary

Problem statement – inference from one sample

- We have a sample of data $x = x_1, \ldots, x_n$ that are realizations of iid random variables $X = X_1, \ldots, X_n$ with pdf $f()$.
- We wish to make inferences about some population characteristic – say the mean μ of $f()$.

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- We wish to make inferences about some population characteristic – say the mean μ of $f()$.

Parametric methods

- Assume $f()$ is a known function with parameters θ – e.g., $f(\theta) = \mathrm{N}(\mu, \sigma^2)$.
- For a (two-sided) hypothesis test
  - Specify the null hypothesis, H_0 – e.g., $\mu = \mu_0$
  - Construct a test statistic $T(X)$ that has a known distribution under H_0
  - e.g.,

$$T(x) = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

    which has a t-distribution with mean 0, variance 1 and d.f. $= n - 1$, if H_0 is true
  - Calculate $P(|T(X)| \geq |T(x)|; \mu_0)$ – the p-value.
  - If p-value $\leq \alpha$ the result is statistically significant.

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Parametric methods (contd.)

- For (two-sided) confidence intervals on μ
  - For $((1-\alpha)\times 100)\%$ limits, find the scalar values $L(X)$ and $U(X)$ such that

$$P\left(L(X) < \mu \text{ and } U(X) > \mu\right) = 1-\alpha$$

  - Many methods for finding the limits
  - E.g., inverting the test statistic
    - limits are the smallest and largest values of μ_0 that are not rejected under $H_0: \mu = \mu_0$
    - E.g., t-based intervals $\bar{x} \pm t_{\alpha/2,n-1} s/\sqrt{n}$
- So, for parametric methods, you need to specify two distributions:
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Traditional nonparametric methods

- Avoid fully specifying the distribution of $f()$.
- E.g., Wilcoxon signed ranks test
  - $f()$ is symmetric about some median m
  - $H_0 : m = m_0$
  - Test statistic

$$T(x) = \sum_{i:x_i \neq m_0} \text{rank}(|x_i - m_0|)\text{sign}(x_i - m_0)$$

    where rank(x) denotes the rank order of x (smallest has value 1, next smallest 2, etc.) and sign($x - y$) is a sign function, having value 1 if $x > y$ and -1 if $x < y$.
  - Under H_0, T(X) has (approximately) a normal distribution
  - For confidence limits on m, can invert the test statistic

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Computer-intensive statistical methods

- With computer intensive methods, you can avoid the need to fully specify either the distribution of $f()$ and/or $T(X)$
- Examples:
  - Permutation-based approach to Wilcoxon test (avoids the need to specify distribution for $T(X)$; Wilcoxon already avoids $f()$)
  - Monte-Carlo-based version of the t-test (avoids specification of $T(X)$, but not $f(\theta)$)
  - Parametric and non-parametric bootstrap confidence intervals (again avoids $T(X)$; either specify or not $f(\theta)$)

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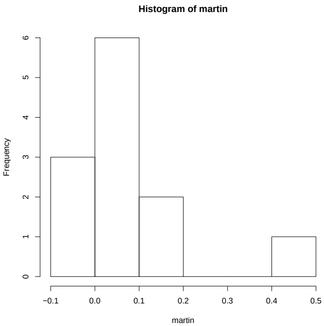
Pine martin habitat preference data



- 0.13, -0.01, -0.01, 0.42, -0.02, 0.01, 0.09, 0.03, 0.04, 0.06, 0.12, 0.03
- Shapiro-Wilk test for normality W = 0.70, p<0.001

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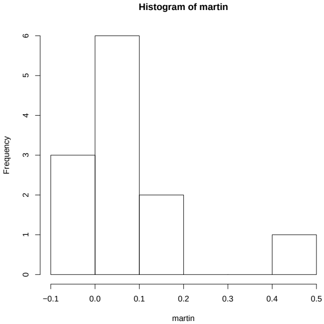

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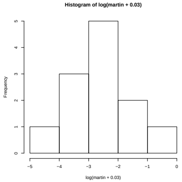
Pine martin example: parametric test

- Assume data come from a normal distribution
  - $H_0 : \mu = 0$
  - t-test: $t = 2.147$, d.f.=11, $p = 0.055$
  - $\bar{x} = 0.074$, 95% t-based CI: (-0.002, 0.150)
- Could try transforming – e.g., $\log(x_i + 0.03)$:



  - $H_0 : \log(\mu + 0.03) = \log(0.03)$
  - t-test: $t = 2.484$, d.f.=11, $p = 0.030$
  - 95% t-based CI (back-transformed): (0.003, 0.095)

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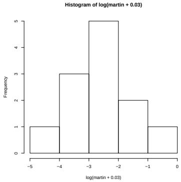

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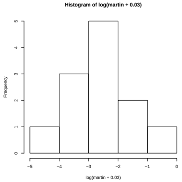

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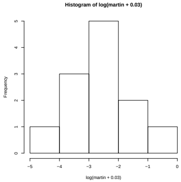

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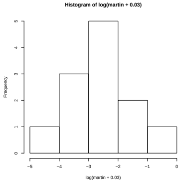

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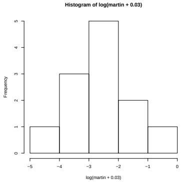

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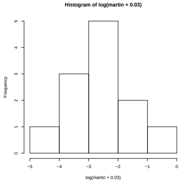

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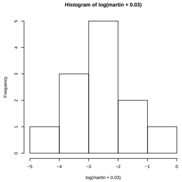

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- Under $H_0 : \mu = \mu_0$, and assuming the distribution of x_i is symmetric about μ_0, then
- $+(x_i - \mu_0)$ is just as likely to occur in the data as $-(x_i - \mu_0)$
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Permutations

- Given 12 data points, there are $2^{12} = 4096$ ways to do this:
  - 0.13, 0.01, 0.01, 0.42, 0.02, 0.01, 0.09, 0.03, 0.04, 0.06, 0.12, 0.03
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- Find what proportion of the permutations have $|\bar{X}|$ greater than the observed $|\bar{x}|$.

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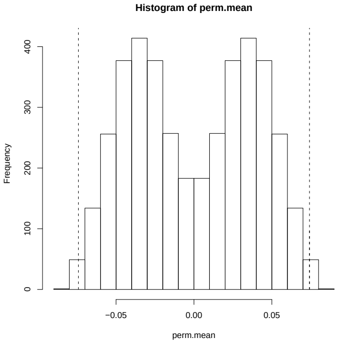

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Permuatation tests - the basic idea

- Pick a null hypothesis H_0 and test statistic $T(X)$
- Under H_0, all possible appropriate data permutations (re-orderings) are equally likely
- So, calculate $T(X)$ for each permutation and see how extreme $T(x)$ is compared with these
    - If $T(x)$ is extreme, reject H_0
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- With permutation-based methods (and many other computer-intensive methods) you are free to choose whatever test statistic you like
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- Invert the test statistic:
  Find the set of values of μ_0 such that $H_0 : \mu = \mu_0$ is not rejected
- For each potential value of μ_0, follow the same procedure as before
  - calculating all permutations of $\pm(x_i - \mu_0)$
  - then seeing how extreme the observed $\bar{x}$ is among the permutations
- Can use stochastic search methods[1] to find the confidence limits
- In our example, 95% CI is (0.013, 0.150) (c.f., t-based CI of (-0.002, 0.150)).

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- Randomization methods are like permutation methods, except that a only random subset of all the possible permutations are generated
- E.g., pine martin example:
  - To generate one randomization, list the set of values $|x_i - \mu_0|$ and randomly assign a sign to each value
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A generalization[2] of randomization tests to include parametric distributions for the data.

- Algorithm:
  1. Set up some distribution for the data, $f()$ or $f(\theta)$, and some H_0
  2. Repeat the following two steps many times:
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A generalization[2] of randomization tests to include parametric distributions for the data.

- Algorithm:
    1. Set up some distribution for the data, $f()$ or $f(\theta)$, and some H_0
    2. Repeat the following two steps many times:
        1. Simulate a data set according to H_0
        2. Calculate $T(X)$ using the simulated data
    3. Add $T(x)$ evaluated from the sample data
    4. Order all of the $T(X)$s
    5. p-value is the proportion of the $T(X)$s as extreme or more extreme than the one from the sample data

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Pine martin example - parametric simulation

- Assume observations are iid $N(\mu, \sigma^2)$, that $H_0 : \mu = 0$, and that $\sigma^2 = s^2$
- Simulate 999 data sets of 12 observations from $N(0, s^2)$.
- Calculate $\bar{x}$ for each of these, and add in the mean from the sample data, to give 1000 values of $\bar{x}$.
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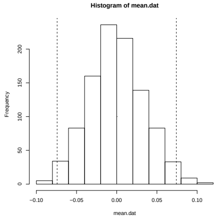

Outline

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  (Randomization tests are a special case)
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Monte Carlo confidence intervals

Confidence intervals can be constructed using similar methods to randomization intervals – but there is a better way – see next lecture...