```
In[]:= Quit[]
```

Problem 1)

Part a)

```
In[]:= Rx[β_] := (1   0        0
                  0   Cos[β]   Sin[β]
                  0   -Sin[β]  Cos[β]);
       Ry[β_] := (Cos[β]  0  -Sin[β]
                  0       1  0
                  Sin[β]  0  Cos[β]);
       Rz[β_] := (Cos[β]   Sin[β]  0
                  -Sin[β]  Cos[β]  0
                  0        0       1);

In[]:= i := {1, 0, 0}; j := {0, 1, 0}; k := {0, 0, 1};

In[]:= {i1, j1, k1} = Rz[-θ1[t]].{i, j, k};

In[]:= {i1, j1, k1} // MatrixForm
Out[]//MatrixForm=
       (Cos[θ1[t]]  -Sin[θ1[t]]  0
        Sin[θ1[t]]  Cos[θ1[t]]   0
        0           0            1)

In[]:= {i2, j2, k2} = Ry[-θ2[t]].{i1, j1, k1};
       {i2, j2, k2} // MatrixForm
Out[]//MatrixForm=
       (Cos[θ1[t]] Cos[θ2[t]]    -Cos[θ2[t]] Sin[θ1[t]]   Sin[θ2[t]]
        Sin[θ1[t]]               Cos[θ1[t]]               0
        -Cos[θ1[t]] Sin[θ2[t]]   Sin[θ1[t]] Sin[θ2[t]]    Cos[θ2[t]])

In[]:= {i3, j3, k3} = Ry[θ3[t]].{i2, j2, k2};
       {i3, j3, k3} // MatrixForm
Out[]//MatrixForm=
       (Cos[θ1[t]] Cos[θ2[t]] Cos[θ3[t]] + Cos[θ1[t]] Sin[θ2[t]] Sin[θ3[t]]   -Cos[θ2[t]] Cos[θ3[t
        Sin[θ1[t]]
        -Cos[θ1[t]] Cos[θ3[t]] Sin[θ2[t]] + Cos[θ1[t]] Cos[θ2[t]] Sin[θ3[t]]   Cos[θ3[t]] Sin[θ1[t

In[]:= vA = {0, 0, 0};
       ω1 = -θ1'[t] k;
       ω2 = -θ2'[t] j;
       ω3 = θ3'[t] j;
```

In[ ]:=
```
rBA = L1 i2 ;
rG1A = L1 / 2 i2;
rG2A = rBA + L2 / 2 i3;
rCA = rBA + L2 i3;

vC = (ω1 + ω2 + ω3) × rCA // FullSimplify (*vA=0*)
```

Out[ ]=
```
{-((L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[θ1[t]] θ1′[t]) -
   (L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) (θ2′[t] - θ3′[t]),
  -Cos[θ1[t]] (L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) θ1′[t],
  Cos[θ1[t]] (L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) (θ2′[t] - θ3′[t])}
```

In[ ]:=
```
vC.j3
```

Out[ ]=
```
-Cos[θ1[t]]^2 (L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) θ1′[t] +
 Sin[θ1[t]] (-((L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[θ1[t]] θ1′[t]) -
    (L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) (θ2′[t] - θ3′[t]))
```

In[ ]:=
```
vC.i3
```

Out[ ]=
```
-Cos[θ1[t]] (L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]])
   (-Cos[θ2[t]] Cos[θ3[t]] Sin[θ1[t]] - Sin[θ1[t]] Sin[θ2[t]] Sin[θ3[t]]) θ1′[t] +
  (Cos[θ1[t]] Cos[θ2[t]] Cos[θ3[t]] + Cos[θ1[t]] Sin[θ2[t]] Sin[θ3[t]])
   (-((L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[θ1[t]] θ1′[t]) -
     (L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) (θ2′[t] - θ3′[t])) +
  Cos[θ1[t]] (L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]])
   (Cos[θ3[t]] Sin[θ2[t]] - Cos[θ2[t]] Sin[θ3[t]]) (θ2′[t] - θ3′[t])
```

In[ ]:=
```
vC.k3
```

Out[ ]=
```
-Cos[θ1[t]] (L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]])
   (Cos[θ3[t]] Sin[θ1[t]] Sin[θ2[t]] - Cos[θ2[t]] Sin[θ1[t]] Sin[θ3[t]]) θ1′[t] +
  (-Cos[θ1[t]] Cos[θ3[t]] Sin[θ2[t]] + Cos[θ1[t]] Cos[θ2[t]] Sin[θ3[t]])
   (-((L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[θ1[t]] θ1′[t]) -
     (L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) (θ2′[t] - θ3′[t])) +
  Cos[θ1[t]] (L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]])
   (Cos[θ2[t]] Cos[θ3[t]] + Sin[θ2[t]] Sin[θ3[t]]) (θ2′[t] - θ3′[t])
```

In[ ]:=
```
Eq1 = vC.j3 == v // FullSimplify
```

Out[ ]=
```
v + (L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) θ1′[t] +
   Sin[θ1[t]] (L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) (θ2′[t] - θ3′[t]) == 0
```

In[ ]:=
```
Eq2 = vC.i3 == 0 // FullSimplify
```

Out[ ]=
```
L1 Cos[θ1[t]] Sin[θ3[t]] (θ2′[t] - θ3′[t]) == 0
```

In[ ]:=
```
Eq3 = vC.k3 == 0 // FullSimplify
```

Out[ ]=
```
Cos[θ1[t]] (L2 + L1 Cos[θ3[t]]) (θ2′[t] - θ3′[t]) == 0
```

In[ ]:=
```
Solve[{Eq1, Eq2, Eq2}, {θ1'[t], θ2'[t], θ3'[t]}]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[ ]= $\left\{\left\{\theta 1'[t] \to -\frac{v}{L1\,\mathrm{Cos}[\theta 2[t]] + L2\,\mathrm{Cos}[\theta 2[t] - \theta 3[t]]},\ \theta 3'[t] \to \theta 2'[t]\right\}\right\}$

Part b)

In[ ]:=
```
aA = {0, 0, 0};
α1 = -θ2''[t] j + (ω1) × (ω2)
```

Out[ ]= $\{-\theta 1'[t]\,\theta 2'[t],\ -\theta 2''[t],\ 0\}$

In[ ]:=
```
α2 = α1 - θ3''[t] k1 + (ω1 + ω2) × (ω3)
```

Out[ ]= $\{-\theta 1'[t]\,\theta 2'[t] + \theta 1'[t]\,\theta 3'[t],\ -\theta 2''[t],\ -\theta 3''[t]\}$

In[ ]:=
```
Ω = ω1 + ω2
```

Out[ ]= $\{0,\ -\theta 2'[t],\ -\theta 1'[t]\}$

In[ ]:=
```
aG1 = aA + α1 × rG1A + ω1 × (Ω × rG1A) // FullSimplify;
Print["aG1 = ", MatrixForm[aG1]]
```

$$a_{G1} = \begin{pmatrix} -\frac{1}{2} L1 \left(\mathrm{Cos}[\theta 1[t]]\,\mathrm{Cos}[\theta 2[t]]\,\theta 1'[t]^2 + \mathrm{Sin}[\theta 2[t]]\,\theta 2''[t]\right) \\ \frac{1}{2} L1\,\theta 1'[t] \left(\mathrm{Cos}[\theta 2[t]]\,\mathrm{Sin}[\theta 1[t]]\,\theta 1'[t] + 2\,\mathrm{Sin}[\theta 2[t]]\,\theta 2'[t]\right) \\ \frac{1}{2} L1\,\mathrm{Cos}[\theta 2[t]] \left(\mathrm{Sin}[\theta 1[t]]\,\theta 1'[t]\,\theta 2'[t] + \mathrm{Cos}[\theta 1[t]]\,\theta 2''[t]\right) \end{pmatrix}$$

In[ ]:=
```
Ω2 = ω1 + ω2 + ω3
```

Out[ ]= $\{0,\ -\theta 2'[t] + \theta 3'[t],\ -\theta 1'[t]\}$

In[ ]:=
```
aG2 = aA + α2 × rG2A + (ω1 + ω2) × (Ω2 × rG2A) // FullSimplify;
Print["aG2 = ", MatrixForm[aG2]]
```

a_{G2} = ($\frac{1}{2}$ (-Cos[θ1[t]] (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) θ1′[t]^2 - Cos[θ1[t]] (2 L1 Cos[θ2[t]] + l
$\frac{1}{2}$ ((2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t
$\frac{1}{2}$ (-((2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]

In[ ]:=
$$IG1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} m1\,L1^2 & 0 \\ 0 & 0 & \frac{1}{12} m1\,L1^2 \end{pmatrix};$$

$$IG2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} m2\,L2^2 & 0 \\ 0 & 0 & \frac{1}{12} m2\,L2^2 \end{pmatrix};$$

In[ ]:=
```
R1 = Ry[-θ2[t]].Rz[-θ1[t]]; (* Rotation for bar 1 *)
R2 = Ry[θ3[t]].Ry[-θ2[t]].Rz[-θ1[t]];(* Rotation for bar 2 *)
```

Need to rotate the inertia matrix for the tilted bar to the global coordinate system:

In[∘]:=
```
IG1r = R1.IG1.R1ᵀ // Simplify;
IG1r // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{1}{12} L1^2\, m1 \left(Cos[\theta 2[t]]^2\, Sin[\theta 1[t]]^2 + Sin[\theta 2[t]]^2\right) & -\frac{1}{24} L1^2\, m1\, Cos[\theta 2[t]]\, Sin[2\,\theta 1[t]] & \\ -\frac{1}{24} L1^2\, m1\, Cos[\theta 2[t]]\, Sin[2\,\theta 1[t]] & \frac{1}{12} L1^2\, m1\, Cos[\theta 1[t]]^2 & \\ \frac{1}{24} L1^2\, m1\, Cos[\theta 1[t]]^2\, Sin[2\,\theta 2[t]] & \frac{1}{24} L1^2\, m1\, Sin[2\,\theta 1[t]]\, Sin[\theta 2[t]] & \frac{1}{12} L1^2\, m1 \end{pmatrix}$$

In[∘]:=
```
IG2r = R2.IG2.R2ᵀ // Simplify;
IG2r // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{1}{12} L2^2\, m2 \left(Cos[\theta 2[t]-\theta 3[t]]^2\, Sin[\theta 1[t]]^2 + Sin[\theta 2[t]-\theta 3[t]]^2\right) & -\frac{1}{24} L2^2\, m2\, Cos[\theta 2[t]-\theta 3[t]] \\ -\frac{1}{24} L2^2\, m2\, Cos[\theta 2[t]-\theta 3[t]]\, Sin[2\,\theta 1[t]] & \frac{1}{12} L2^2\, m2\, Cos[\theta 1[t] \\ \frac{1}{24} L2^2\, m2\, Cos[\theta 1[t]]^2\, Sin\left[2\left(\theta 2[t]-\theta 3[t]\right)\right] & \frac{1}{24} L2^2\, m2\, Sin[2\,\theta 1[t]]\, Sin[\theta 2 \end{pmatrix}$$

In[∘]:=
```
HG1 = IG1.Ω;
Print["HG1 = ", MatrixForm[HG1] // Simplify]
HG2 = IG2r.Ω2;
Print["HG2 = ", MatrixForm[HG2] // Simplify]
```

$$H_{G1} = \begin{pmatrix} 0 \\ -\frac{1}{12} L1^2\, m1\, \theta 2'[t] \\ -\frac{1}{12} L1^2\, m1\, \theta 1'[t] \end{pmatrix}$$

$$H_{G2} = \begin{pmatrix} -\frac{1}{24} L2^2\, m2 \left(Cos[\theta 1[t]]^2\, Sin[2\left(\theta 2[t]-\theta 3[t]\right)]\, \theta 1'[t] + Cos[\theta 2[t]-\theta 3[t]]\, Sin[2\,\theta 1[t] \right. \\ -\frac{1}{24} L2^2\, m2 \left(Sin[2\,\theta 1[t]]\, Sin[\theta 2[t]-\theta 3[t]]\, \theta 1'[t] + 2\, Cos[\theta 1[t]]^2 \left(\theta 2'[t] - \right.\right. \\ \frac{1}{24} L2^2\, m2 \left(-2\left(Cos[\theta 2[t]-\theta 3[t]]^2 + Sin[\theta 1[t]]^2\, Sin[\theta 2[t]-\theta 3[t]]^2\right) \theta 1'[t] + Sin[2\,\theta 1[t]]\, Sin[\theta 2 \right. \end{pmatrix}$$

In[∘]:=
```
Print["dHG1/dt = ", MatrixForm[dHG1 = D[HG1, t] + Ω × HG1 // Simplify]]
Print["dHG2/dt = ", MatrixForm[dHG2 = D[HG2, t] + Ω2 × HG2 // Simplify]]
```

$$\frac{dH_{G1}}{dt} = \begin{pmatrix} 0 \\ -\frac{1}{12} L1^2\, m1\, \theta 2''[t] \\ -\frac{1}{12} L1^2\, m1\, \theta 1''[t] \end{pmatrix}$$

$$\frac{dH_{G2}}{dt} = \begin{pmatrix} -\frac{1}{24} L2^2\, m2 \left(Sin[2\,\theta 1[t]] \left(Sin[\theta 2[t]-\theta 3[t]] - Sin[2\left(\theta 2[t]-\theta 3[t]\right)]\right) \theta 1'[t]^2 - \left(2 - 2\, Cos[2\,\theta 1[t \right.\right. \\ \frac{1}{24} L2^2\, m2 \left(\left(-2\, Cos[2\,\theta 1[t]]\, Sin[\theta 2[t]-\theta 3[t]] + \right.\right. \\ -\frac{1}{24} L2^2\, m2 \left(2\, Sin[2\,\theta 1[t]]\, Sin[\theta 2[t]-\theta 3[t]]^2\, \theta 1'[t]^2 + 2\left(Cos[2\,\theta \right.\right. \end{pmatrix}$$

Part c)

In[∘]:=
```
Print["Weights:"]
W1 = - m1 g k; Print["W1 = ", MatrixForm[W1]]
W2 = - m2 g k; Print["W2 = ", MatrixForm[W2]]
Feq = Thread[- F j3 + W1 + W2 == m aG1 + m aG2 // Simplify];
ColumnForm[Feq]
```

```
Weights:
```

$$W1 = \begin{pmatrix} 0 \\ 0 \\ -g\,m1 \end{pmatrix}$$

$$W2 = \begin{pmatrix} 0 \\ 0 \\ -g\,m2 \end{pmatrix}$$

Out[∘]=

$$-F\,Sin[\theta 1[t]] == -\frac{1}{2}\,m\,\left(Cos[\theta 1[t]]\,\left(3\,L1\,Cos[\theta 2[t]] + L2\,Cos[\theta 2[t] - \theta 3[t]]\right)\,\theta 1'[t]^2 + Cos[\theta 1[t]]\right.$$

$$-F\,Cos[\theta 1[t]] == \frac{1}{2}\,m\,\left(\left(3\,L1\,Cos[\theta 2[t]] + L2\,Cos[\theta 2[t] - \theta 3[t]]\right)\,Sin[\theta 1[t]]\,\theta 1'[t]^2 + 2\,\theta 1'[t]\,\left(\left(3\,L\right.\right.\right.$$

$$-g\,\left(m1 + m2\right) == \frac{1}{2}\,m\,\left(-\left(\left(2\,L1\,Sin[\theta 2[t]] + L2\,Sin[\theta 2[t] - \theta 3[t]]\right)\,\theta 2'[t]\,\left(\theta 2'[t] - \theta 3'[t]\right)\right) - \left(2\,L1\,Co\right.\right.$$

In[∘]:=

```
Meq = Thread[Γ k - MB j3 - MA j2 + rG1A×W1 + rG2A×W2 - rCA × (-F j3) ==
     dHG1 + dHG2 + rG1A× (m aG1) + rG2A× (m aG2) // Simplify];
ColumnForm[
 Meq]
```

Out[∘]=

$$\frac{1}{96}\,\left(48\,\left(-2\,(MA + MB) + g\,L1\,\left(m1 + 2\,m2\right)\,Cos[\theta 2[t]] + g\,L2\,m2\,Cos[\theta 2[t] - \theta 3[t]]\right)\,Sin[\theta 1[t]] - 96\,F\,Cos\right.$$

$$\frac{1}{2}\,Cos[\theta 1[t]]\,\left(-2\,(MA + MB) + g\,L1\,\left(m1 + 2\,m2\right)\,Cos[\theta 2[t]] + g\,L2\,m2\,Cos[\theta 2[t] - \theta 3[t]]\right) + F\,Sin[\theta 1[t]]$$

$$\Gamma + F\,L1\,Cos[\theta 2[t]] + F\,L2\,Cos[\theta 2[t] - \theta 3[t]] + \frac{1}{24}\,\left(2\,L2^2\,m2\,Sin[2\,\theta 1[t]]\,Sin[\theta 2[t] - \theta 3[t]]^2\,\theta 1'[t]^2\right.$$

Expressions are too long since I selected fixed reference frame as the GCS . Please see original mathematica file for results .

Problem 2

Part a)

Velocities of points G1 and G2

In[∘]:=

```
vG1 = vA + Ω × rG1A // Simplify;
vG1 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2}\,L1\,\left(Cos[\theta 2[t]]\,Sin[\theta 1[t]]\,\theta 1'[t] + Sin[\theta 2[t]]\,\theta 2'[t]\right) \\ -\frac{1}{2}\,L1\,Cos[\theta 1[t]]\,Cos[\theta 2[t]]\,\theta 1'[t] \\ \frac{1}{2}\,L1\,Cos[\theta 1[t]]\,Cos[\theta 2[t]]\,\theta 2'[t] \end{pmatrix}$$

In[∘]:=

```
vG2 = vA + Ω2 × rG2A // Simplify;
vG2 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\,\left(-\left(\left(2\,L1\,Cos[\theta 2[t]] + L2\,Cos[\theta 2[t] - \theta 3[t]]\right)\,Sin[\theta 1[t]]\,\theta 1'[t]\right) - \left(2\,L1\,Sin[\theta 2[t]] + L2\,Sin[\theta 2\right.\right. \\ -\frac{1}{2}\,Cos[\theta 1[t]]\,\left(2\,L1\,Cos[\theta 2[t]] + L2\,Cos[\theta 2[t] - \theta 3[t]]\right)\,\theta 1'[t] \\ \frac{1}{2}\,Cos[\theta 1[t]]\,\left(2\,L1\,Cos[\theta 2[t]] + L2\,Cos[\theta 2[t] - \theta 3[t]]\right)\,\left(\theta 2'[t] - \theta 3'\right. \end{pmatrix}$$

Kinetic Energy : 2 Masses

In[∘]:=
```
T1[t] = 1/2 m1 vG1.vG1 + 1/2 Ω.IG1r.Ω // Simplify;
Print["T1 = ", T1[t]]
```

$$T1 = \frac{1}{24} L1^2 m1 \left(\left(4 \text{Cos}[\theta 2[t]]^2 + \text{Sin}[\theta 1[t]]^2 \text{Sin}[\theta 2[t]]^2\right) \theta 1'[t]^2 + \left(\text{Sin}[2 \theta 1[t]] \text{Sin}[\theta 2[t]] + 3 \text{Sin}[\theta 1[t]] \text{Sin}[2 \theta 2[t]]\right) \theta 1'[t] \theta 2'[t] + \left(\text{Cos}[\theta 1[t]]^2 \left(1 + 3 \text{Cos}[\theta 2[t]]^2\right) + 3 \text{Sin}[\theta 2[t]]^2\right) \theta 2'[t]^2\right)$$

In[∘]:=
```
T2[t] = 1/2 m2 vG2.vG2 + 1/2 Ω2.IG2r.Ω2 // Simplify;
Print["T2 = ", T2[t]]
```

$$T2 = \frac{1}{48} m2 \Big(L2^2 \theta 1'[t] \Big(2 \left(\text{Cos}[\theta 2[t] - \theta 3[t]]^2 + \text{Sin}[\theta 1[t]]^2 \text{Sin}[\theta 2[t] - \theta 3[t]]^2\right) \theta 1'[t] + \text{Sin}[2 \theta 1[t]] \text{Sin}[\theta 2[t] - \theta 3[t]] (\theta 2'[t] - \theta 3'[t])\Big) + 6 \Big(\text{Cos}[\theta 1[t]]^2 (2 L1 \text{Cos}[\theta 2[t]] + L2 \text{Cos}[\theta 2[t] - \theta 3[t]])^2 \theta 1'[t]^2 + \Big((2 L1 \text{Cos}[\theta 2[t]] + L2 \text{Cos}[\theta 2[t] - \theta 3[t]]) \text{Sin}[\theta 1[t]] \theta 1'[t] + \left(2 L1 \text{Sin}[\theta 2[t]] + L2 \text{Sin}[\theta 2[t] - \theta 3[t]]\right) (\theta 2'[t] - \theta 3'[t])\Big)^2 + \text{Cos}[\theta 1[t]]^2 (2 L1 \text{Cos}[\theta 2[t]] + L2 \text{Cos}[\theta 2[t] - \theta 3[t]])^2 (\theta 2'[t] - \theta 3'[t])^2\Big) + L2^2 \left(\text{Sin}[2 \theta 1[t]] \text{Sin}[\theta 2[t] - \theta 3[t]] \theta 1'[t] + 2 \text{Cos}[\theta 1[t]]^2 (\theta 2'[t] - \theta 3'[t])\right) (\theta 2'[t] - \theta 3'[t])\Big)$$

In[∘]:=
```
T[t] = T1[t] + T2[t] // Simplify
```

Out[∘]=

$$\frac{1}{48} \Big(2 L1^2 m1 \Big(\left(4 \text{Cos}[\theta 2[t]]^2 + \text{Sin}[\theta 1[t]]^2 \text{Sin}[\theta 2[t]]^2\right) \theta 1'[t]^2 + \left(\text{Sin}[2 \theta 1[t]] \text{Sin}[\theta 2[t]] + 3 \text{Sin}[\theta 1[t]] \text{Sin}[2 \theta 2[t]]\right) \theta 1'[t] \theta 2'[t] + \left(\text{Cos}[\theta 1[t]]^2 \left(1 + 3 \text{Cos}[\theta 2[t]]^2\right) + 3 \text{Sin}[\theta 2[t]]^2\right) \theta 2'[t]^2\Big) + m2 \Big(L2^2 \theta 1'[t] \Big(2 \left(\text{Cos}[\theta 2[t] - \theta 3[t]]^2 + \text{Sin}[\theta 1[t]]^2 \text{Sin}[\theta 2[t] - \theta 3[t]]^2\right) \theta 1'[t] + \text{Sin}[2 \theta 1[t]] \text{Sin}[\theta 2[t] - \theta 3[t]] (\theta 2'[t] - \theta 3'[t])\Big) + 6 \Big(\text{Cos}[\theta 1[t]]^2 \left(2 L1 \text{Cos}[\theta 2[t]] + L2 \text{Cos}[\theta 2[t] - \theta 3[t]]\right)^2 \theta 1'[t]^2 + \Big(\left(2 L1 \text{Cos}[\theta 2[t]] + L2 \text{Cos}[\theta 2[t] - \theta 3[t]]\right) \text{Sin}[\theta 1[t]] \theta 1'[t] + \left(2 L1 \text{Sin}[\theta 2[t]] + L2 \text{Sin}[\theta 2[t] - \theta 3[t]]\right) (\theta 2'[t] - \theta 3'[t])\Big)^2 + \text{Cos}[\theta 1[t]]^2 \left(2 L1 \text{Cos}[\theta 2[t]] + L2 \text{Cos}[\theta 2[t] - \theta 3[t]]\right)^2 (\theta 2'[t] - \theta 3'[t])^2\Big) + L2^2 \left(\text{Sin}[2 \theta 1[t]] \text{Sin}[\theta 2[t] - \theta 3[t]] \theta 1'[t] + 2 \text{Cos}[\theta 1[t]]^2 (\theta 2'[t] - \theta 3'[t])\right) (\theta 2'[t] - \theta 3'[t])\Big)\Big)$$

Potential Energy : 2 Masses

In[∘]:=
```
h1 = rG1A.k;
V1[t] = m1 g h1 // Simplify;
Print["V1 = ", V1[t]]
h2 = rG2A.k;
V2[t] = m2 g h2;
Print["V2 = ", V2[t]]
```

$$V1 = \frac{1}{2} g\, L1\, m1\, Sin[\theta 2[t]]$$

$$V2 = g\, m2 \left(L1\, Sin[\theta 2[t]] + \frac{1}{2} L2 \left(Cos[\theta 3[t]]\, Sin[\theta 2[t]] - Cos[\theta 2[t]]\, Sin[\theta 3[t]]\right)\right)$$

In[ ]:= **V[t] = V1[t] + V2[t]**

Out[ ]= $\frac{1}{2} g\, L1\, m1\, Sin[\theta 2[t]] + g\, m2 \left(L1\, Sin[\theta 2[t]] + \frac{1}{2} L2 \left(Cos[\theta 3[t]]\, Sin[\theta 2[t]] - Cos[\theta 2[t]]\, Sin[\theta 3[t]]\right)\right)$

Part b)

Generalized Coordinates
System has 3 DOF. θ1, θ2, and θ3.

In[ ]:=
```
Print["Point C virtual displacement:\n δrC = ",
  δrC = Coefficient[vC, θ1'[t]] δθ1 +
    Coefficient[vC, θ2'[t]] δθ2 + Coefficient[vC, θ3'[t]] δθ3] // Simplify
```

Point C virtual displacement:

$\delta r_C = \{-\delta\theta 1\, (L1\, Cos[\theta 2[t]] + L2\, Cos[\theta 2[t] - \theta 3[t]])\, Sin[\theta 1[t]] + \delta\theta 2\, (-L1\, Sin[\theta 2[t]] - L2\, Sin[\theta 2[t] - \theta 3[t]]) + \delta\theta 3\, (L1\, Sin[\theta 2[t]] + L2\, Sin[\theta 2[t] - \theta 3[t]]),$
$-\delta\theta 1\, Cos[\theta 1[t]]\, (L1\, Cos[\theta 2[t]] + L2\, Cos[\theta 2[t] - \theta 3[t]]),$
$\delta\theta 2\, Cos[\theta 1[t]]\, (L1\, Cos[\theta 2[t]] + L2\, Cos[\theta 2[t] - \theta 3[t]]) - \delta\theta 3\, Cos[\theta 1[t]]\, (L1\, Cos[\theta 2[t]] + L2\, Cos[\theta 2[t] - \theta 3[t]])\}$

Virtual Work

In[ ]:=
```
Print["Virtual Work from F: δW = ",
 δW = (-F j3).δrC + Γ[t] δθ1 + MA[t] δθ2 + MB[t] δθ3]
```

Virtual Work from F: δW =

$F\, \delta\theta 1\, Cos[\theta 1[t]]^2\, (L1\, Cos[\theta 2[t]] + L2\, Cos[\theta 2[t] - \theta 3[t]]) + \delta\theta 2\, MA[t] + \delta\theta 3\, MB[t] - F\, Sin[\theta 1[t]]\, (-\delta\theta 1\, (L1\, Cos[\theta 2[t]] + L2\, Cos[\theta 2[t] - \theta 3[t]])\, Sin[\theta 1[t]] + \delta\theta 2\, (-L1\, Sin[\theta 2[t]] - L2\, Sin[\theta 2[t] - \theta 3[t]]) + \delta\theta 3\, (L1\, Sin[\theta 2[t]] + L2\, Sin[\theta 2[t] - \theta 3[t]])) + \delta\theta 1\, \Gamma[t]$

I directly added the moments :

Generalized Forces

```
In[ ]:= Print["Generalized forces: "]

Print["Q₁ = ",
 Q1 = Coefficient[δW, δθ1]]

Print["Q₂ = ",
 Q2 = Coefficient[δW, δθ2]]

Print["Q₃ = ",
 Q3 = Coefficient[δW, δθ3]]
```

Generalized forces:

$Q_1 = F\,L1\,\text{Cos}[\theta1[t]]^2\,\text{Cos}[\theta2[t]] + F\,L2\,\text{Cos}[\theta1[t]]^2\,\text{Cos}[\theta2[t]-\theta3[t]] + F\,L1\,\text{Cos}[\theta2[t]]\,\text{Sin}[\theta1[t]]^2 + F\,L2\,\text{Cos}[\theta2[t]-\theta3[t]]\,\text{Sin}[\theta1[t]]^2 + \Gamma[t]$

$Q_2 = MA[t] + F\,L1\,\text{Sin}[\theta1[t]]\,\text{Sin}[\theta2[t]] + F\,L2\,\text{Sin}[\theta1[t]]\,\text{Sin}[\theta2[t]-\theta3[t]]$

$Q_3 = MB[t] - F\,L1\,\text{Sin}[\theta1[t]]\,\text{Sin}[\theta2[t]] - F\,L2\,\text{Sin}[\theta1[t]]\,\text{Sin}[\theta2[t]-\theta3[t]]$

Part c)

```
In[ ]:= Print["Lagrangian Function: ℒ = ", ℒ[t] = T[t] - V[t]]
Print["Lagrange's Equations:"]
Leq1 = D[ ∂θ1'[t] ℒ[t], t] - ∂θ1[t] ℒ[t] == Q1 // Simplify
Leq1 = D[ ∂θ1'[t] ℒ[t], t] - ∂θ2[t] ℒ[t] == Q2 // Simplify
Leq1 = D[ ∂θ3'[t] ℒ[t], t] - ∂θ3[t] ℒ[t] == Q3 // Simplify
```

Lagrangian Function: ℒ =

$$-\frac{1}{2} g\,L1\,m1\,\text{Sin}[\theta2[t]] - g\,m2\left(L1\,\text{Sin}[\theta2[t]] + \frac{1}{2} L2\left(\text{Cos}[\theta3[t]]\,\text{Sin}[\theta2[t]] - \text{Cos}[\theta2[t]]\,\text{Sin}[\theta3[t]]\right)\right) + \frac{1}{48}\Big(2\,L1^2\,m1\,\Big(\left(4\,\text{Cos}[\theta2[t]]^2 + \text{Sin}[\theta1[t]]^2\,\text{Sin}[\theta2[t]]^2\right)\theta1'[t]^2 + \left(\text{Sin}[2\,\theta1[t]]\,\text{Sin}[\theta2[t]] + 3\,\text{Sin}[\theta1[t]]\,\text{Sin}[2\,\theta2[t]]\right)\theta1'[t]\,\theta2'[t] + \left(\text{Cos}[\theta1[t]]^2\left(1 + 3\,\text{Cos}[\theta2[t]]^2\right) + 3\,\text{Sin}[\theta2[t]]^2\right)\theta2'[t]^2\Big) + m2\,\Big(L2^2\,\theta1'[t]\,\Big(2\left(\text{Cos}[\theta2[t]-\theta3[t]]^2 + \text{Sin}[\theta1[t]]^2\,\text{Sin}[\theta2[t]-\theta3[t]]^2\right)\theta1'[t] + \text{Sin}[2\,\theta1[t]]\,\text{Sin}[\theta2[t]-\theta3[t]]\,(\theta2'[t]-\theta3'[t])\Big) + 6\,\Big(\text{Cos}[\theta1[t]]^2\,(2\,L1\,\text{Cos}[\theta2[t]] + L2\,\text{Cos}[\theta2[t]-\theta3[t]])^2\,\theta1'[t]^2 + \Big((2\,L1\,\text{Cos}[\theta2[t]] + L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,\text{Sin}[\theta1[t]]\,\theta1'[t] + \left(2\,L1\,\text{Sin}[\theta2[t]] + L2\,\text{Sin}[\theta2[t]-\theta3[t]]\right)(\theta2'[t]-\theta3'[t])\Big)^2 + \text{Cos}[\theta1[t]]^2\,(2\,L1\,\text{Cos}[\theta2[t]] + L2\,\text{Cos}[\theta2[t]-\theta3[t]])^2\,(\theta2'[t]-\theta3'[t])^2\Big) + L2^2\left(\text{Sin}[2\,\theta1[t]]\,\text{Sin}[\theta2[t]-\theta3[t]]\,\theta1'[t] + 2\,\text{Cos}[\theta1[t]]^2\,(\theta2'[t]-\theta3'[t])\right)(\theta2'[t]-\theta3'[t])\Big)\Big)$$

Lagrange's Equations:

```
Out[•]= 1/48 (2 Sin[2 θ1[t]] (L1^2 m1 Sin[θ2[t]]^2 + L2^2 m2 Sin[θ2[t] - θ3[t]]^2) θ1′[t]^2 +
    θ1′[t] ((L1^2 m1 Sin[2 (θ1[t] - θ2[t])] - 2 L1^2 (7 m1 + 24 m2) Sin[2 θ2[t]] -
          L1^2 m1 Sin[2 (θ1[t] + θ2[t])] - 14 L2^2 m2 Sin[2 (θ2[t] - θ3[t])] -
          L2^2 m2 Sin[2 (θ1[t] + θ2[t] - θ3[t])] - 48 L1 L2 m2 Sin[2 θ2[t] - θ3[t]] +
          L2^2 m2 Sin[2 (θ1[t] - θ2[t] + θ3[t])]) θ2′[t] + 2 L2 m2 (24 L1 Cos[θ2[t]]
           Sin[θ2[t] - θ3[t]] + L2 (7 + Cos[2 θ1[t]]) Sin[2 (θ2[t] - θ3[t])]) θ3′[t]) +
    2 ((6 L1^2 (4 m2 + (m1 + 4 m2) Cos[θ1[t]]) Cos[θ2[t]]^2 Sin[θ1[t]] +
          6 L1^2 m1 Cos[2 θ2[t]] Sin[θ1[t]] + 6 L2^2 m2 Cos[θ2[t] - θ3[t]]^2 Sin[θ1[t]] +
          L1^2 m1 Sin[2 θ1[t]] + L2^2 m2 Sin[2 θ1[t]] + L2^2 m2 Cos[θ2[t] - θ3[t]] Sin[2 θ1[t]] +
          3 L2^2 m2 Cos[θ2[t] - θ3[t]]^2 Sin[2 θ1[t]] + L1 Cos[θ2[t]]
           (L1 m1 Sin[2 θ1[t]] + 12 L2 m2 Cos[θ2[t] - θ3[t]] (2 Sin[θ1[t]] + Sin[2 θ1[t]])) -
          24 L1^2 m2 Sin[θ1[t]] Sin[θ2[t]]^2 - 24 L1 L2 m2 Sin[θ1[t]] Sin[θ2[t]]
           Sin[θ2[t] - θ3[t]] - 6 L2^2 m2 Sin[θ1[t]] Sin[θ2[t] - θ3[t]]^2) θ2′[t]^2 -
       m2 (2 (12 L1^2 + 5 L2^2) Cos[θ1[t]] + 12 L1^2 Cos[θ1[t] - 2 θ2[t]] + 24 L1^2 Cos[2 θ2[t]] +
          12 L1^2 Cos[θ1[t] + 2 θ2[t]] + 3 L2^2 Cos[θ1[t] + 2 θ2[t] - 2 θ3[t]] +
          12 L1 L2 Cos[θ1[t] - θ3[t]] + 12 L2^2 Cos[2 (θ2[t] - θ3[t])] +
          2 L2^2 Cos[θ1[t] + θ2[t] - θ3[t]] + 36 L1 L2 Cos[2 θ2[t] - θ3[t]] +
          12 L1 L2 Cos[θ1[t] + 2 θ2[t] - θ3[t]] + 12 L1 L2 Cos[θ1[t] + θ3[t]] +
          12 L1 L2 Cos[θ1[t] - 2 θ2[t] + θ3[t]] + 2 L2^2 Cos[θ1[t] - θ2[t] + θ3[t]] +
          3 L2^2 Cos[θ1[t] - 2 θ2[t] + 2 θ3[t]]) Sin[θ1[t]] θ2′[t] θ3′[t] +
       m2 (12 L1^2 Cos[θ2[t]]^2 Sin[2 θ1[t]] + 12 L1 L2 Cos[θ2[t]] Cos[θ2[t] - θ3[t]]
            (Sin[θ1[t]] + Sin[2 θ1[t]]) + L2 (L2 Sin[2 θ1[t]] + L2 Cos[θ2[t] - θ3[t]]
               Sin[2 θ1[t]] + 3 L2 Cos[θ2[t] - θ3[t]]^2 (2 Sin[θ1[t]] + Sin[2 θ1[t]]) -
              6 Sin[θ1[t]] Sin[θ2[t] - θ3[t]] (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]])))
        θ3′[t]^2 + 8 L1^2 m1 Cos[θ2[t]]^2 θ1″[t] + 24 L1^2 m2 Cos[θ1[t]]^2 Cos[θ2[t]]^2 θ1″[t] +
       24 L1 L2 m2 Cos[θ1[t]]^2 Cos[θ2[t]] Cos[θ2[t] - θ3[t]] θ1″[t] +
       2 L2^2 m2 Cos[θ2[t] - θ3[t]]^2 θ1″[t] +
       6 L2^2 m2 Cos[θ1[t]]^2 Cos[θ2[t] - θ3[t]]^2 θ1″[t] +
       24 L1^2 m2 Cos[θ2[t]]^2 Sin[θ1[t]]^2 θ1″[t] +
       24 L1 L2 m2 Cos[θ2[t]] Cos[θ2[t] - θ3[t]] Sin[θ1[t]]^2 θ1″[t] +
       6 L2^2 m2 Cos[θ2[t] - θ3[t]]^2 Sin[θ1[t]]^2 θ1″[t] +
       2 L1^2 m1 Sin[θ1[t]]^2 Sin[θ2[t]]^2 θ1″[t] +
       2 L2^2 m2 Sin[θ1[t]]^2 Sin[θ2[t] - θ3[t]]^2 θ1″[t] +
       12 L1 L2 m2 Cos[θ2[t] - θ3[t]] Sin[θ1[t]] Sin[θ2[t]] θ2″[t] +
       L1^2 m1 Sin[2 θ1[t]] Sin[θ2[t]] θ2″[t] +
       3 L1^2 m1 Sin[θ1[t]] Sin[2 θ2[t]] θ2″[t] +
       12 L1^2 m2 Sin[θ1[t]] Sin[2 θ2[t]] θ2″[t] +
       12 L1 L2 m2 Cos[θ2[t]] Sin[θ1[t]] Sin[θ2[t] - θ3[t]] θ2″[t] +
       L2^2 m2 Sin[2 θ1[t]] Sin[θ2[t] - θ3[t]] θ2″[t] +
       3 L2^2 m2 Sin[θ1[t]] Sin[2 (θ2[t] - θ3[t])] θ2″[t] -
       12 L1 L2 m2 Cos[θ2[t] - θ3[t]] Sin[θ1[t]] Sin[θ2[t]] θ3″[t] -
       12 L1^2 m2 Sin[θ1[t]] Sin[2 θ2[t]] θ3″[t] -
       12 L1 L2 m2 Cos[θ2[t]] Sin[θ1[t]] Sin[θ2[t] - θ3[t]] θ3″[t] -
       L2^2 m2 Sin[2 θ1[t]] Sin[θ2[t] - θ3[t]] θ3″[t] -
       3 L2^2 m2 Sin[θ1[t]] Sin[2 (θ2[t] - θ3[t])] θ3″[t])) ==
  F L1 Cos[θ2[t]] + F L2 Cos[θ2[t] - θ3[t]] +
   Γ[
    t]
```

Out[∘]=

$$\frac{1}{48}\Big(24\,g\,L1\,m1\,\text{Cos}[\theta2[t]]+24\,g\,m2\,(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])-$$

$$2\,L1^2\,m1\,\Big(-\frac{1}{2}\,(7+\text{Cos}[2\,\theta1[t]])\,\text{Sin}[2\,\theta2[t]]\,\theta1'[t]^2+$$

$$(6\,\text{Cos}[2\,\theta2[t]]\,\text{Sin}[\theta1[t]]+\text{Cos}[\theta2[t]]\,\text{Sin}[2\,\theta1[t]])\,\theta1'[t]\,\theta2'[t]+$$

$$3\,\text{Sin}[\theta1[t]]^2\,\text{Sin}[2\,\theta2[t]]\,\theta2'[t]^2\Big)-$$

$$2\,m2\,\Big(-6\,\text{Cos}[\theta1[t]]^2\,(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,(2\,L1\,\text{Sin}[\theta2[t]]+$$

$$L2\,\text{Sin}[\theta2[t]-\theta3[t]])\,\theta1'[t]^2-L2^2\,\text{Cos}[\theta1[t]]^2\,\text{Sin}\big[2\,(\theta2[t]-\theta3[t])\big]\,\theta1'[t]^2+$$

$$6\,(-\text{Sin}[\theta1[t]]\,(2\,L1\,\text{Sin}[\theta2[t]]+L2\,\text{Sin}[\theta2[t]-\theta3[t]])\,\theta1'[t]+$$

$$(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,(\theta2'[t]-\theta3'[t]))$$

$$((2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,\text{Sin}[\theta1[t]]\,\theta1'[t]+$$

$$(2\,L1\,\text{Sin}[\theta2[t]]+L2\,\text{Sin}[\theta2[t]-\theta3[t]])\,(\theta2'[t]-\theta3'[t]))+$$

$$L2^2\,\text{Cos}[\theta2[t]-\theta3[t]]\,\text{Sin}[2\,\theta1[t]]\,\theta1'[t]\,(\theta2'[t]-\theta3'[t])-$$

$$6\,\text{Cos}[\theta1[t]]^2\,(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])$$

$$(2\,L1\,\text{Sin}[\theta2[t]]+L2\,\text{Sin}[\theta2[t]-\theta3[t]])\,(\theta2'[t]-\theta3'[t])^2\Big)+$$

$$2\,L1^2\,m1\,\Big(2\,\text{Sin}[2\,\theta1[t]]\,\text{Sin}[\theta2[t]]^2\,\theta1'[t]^2+\big(2\,\text{Cos}[2\,\theta1[t]]\,\text{Sin}[\theta2[t]]-$$

$$(7-3\,\text{Cos}[\theta1[t]]+\text{Cos}[2\,\theta1[t]])\,\text{Sin}[2\,\theta2[t]]\big)\,\theta1'[t]\,\theta2'[t]+$$

$$(6\,\text{Cos}[2\,\theta2[t]]\,\text{Sin}[\theta1[t]]+\text{Cos}[\theta2[t]]\,\text{Sin}[2\,\theta1[t]])\,\theta2'[t]^2+$$

$$8\,\text{Cos}[\theta2[t]]^2\,\theta1''[t]+2\,\text{Sin}[\theta1[t]]^2\,\text{Sin}[\theta2[t]]^2\,\theta1''[t]+$$

$$\text{Sin}[2\,\theta1[t]]\,\text{Sin}[\theta2[t]]\,\theta2''[t]+3\,\text{Sin}[\theta1[t]]\,\text{Sin}[2\,\theta2[t]]\,\theta2''[t]\Big)+$$

$$2\,m2\,\Big(-6\,(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])^2\,\text{Sin}[2\,\theta1[t]]\,\theta1'[t]^2+$$

$$L2^2\,\text{Sin}[2\,\theta1[t]]\,\text{Sin}[\theta2[t]-\theta3[t]]^2\,\theta1'[t]^2+$$

$$12\,\text{Cos}[\theta1[t]]^2\,(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,\theta1'[t]$$

$$(-2\,L1\,\text{Sin}[\theta2[t]]\,\theta2'[t]-L2\,\text{Sin}[\theta2[t]-\theta3[t]]\,(\theta2'[t]-\theta3'[t]))+$$

$$6\,\text{Cos}[\theta1[t]]\,(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,\theta1'[t]$$

$$((2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,\text{Sin}[\theta1[t]]\,\theta1'[t]+$$

$$(2\,L1\,\text{Sin}[\theta2[t]]+L2\,\text{Sin}[\theta2[t]-\theta3[t]])\,(\theta2'[t]-\theta3'[t]))+$$

$$6\,\text{Sin}[\theta1[t]]\,(-2\,L1\,\text{Sin}[\theta2[t]]\,\theta2'[t]-L2\,\text{Sin}[\theta2[t]-\theta3[t]]\,(\theta2'[t]-\theta3'[t]))$$

$$((2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,\text{Sin}[\theta1[t]]\,\theta1'[t]+$$

$$(2\,L1\,\text{Sin}[\theta2[t]]+L2\,\text{Sin}[\theta2[t]-\theta3[t]])\,(\theta2'[t]-\theta3'[t]))+$$

$$2\,L2^2\,\text{Cos}[2\,\theta1[t]]\,\text{Sin}[\theta2[t]-\theta3[t]]\,\theta1'[t]\,(\theta2'[t]-\theta3'[t])-$$

$$L2^2\,\text{Sin}\big[2\,(\theta2[t]-\theta3[t])\big]\,\theta1'[t]\,(\theta2'[t]-\theta3'[t])+$$

$$L2^2\,\text{Sin}[\theta1[t]]^2\,\text{Sin}\big[2\,(\theta2[t]-\theta3[t])\big]\,\theta1'[t]\,(\theta2'[t]-\theta3'[t])+$$

$$L2^2\,\text{Cos}[\theta2[t]-\theta3[t]]\,\text{Sin}[2\,\theta1[t]]\,(\theta2'[t]-\theta3'[t])^2+$$

$$L2^2\,\theta1'[t]\,\big(\text{Sin}[2\,\theta1[t]]\,\text{Sin}[\theta2[t]-\theta3[t]]^2\,\theta1'[t]+$$

$$\text{Cos}[\theta1[t]]^2\,\text{Sin}\big[2\,(\theta2[t]-\theta3[t])\big]\,(-\theta2'[t]+\theta3'[t])\big)+$$

$$6\,\text{Cos}[\theta1[t]]^2\,(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])^2\,\theta1''[t]+$$

$$2\,L2^2\,(\text{Cos}[\theta2[t]-\theta3[t]]^2+\text{Sin}[\theta1[t]]^2\,\text{Sin}[\theta2[t]-\theta3[t]]^2)\,\theta1''[t]+$$

$$6\,(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,\text{Sin}[\theta1[t]]$$

$$\big(\text{Cos}[\theta1[t]]\,(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,\theta1'[t]^2+\text{Sin}[\theta1[t]]$$

$$\theta1'[t]\,(-2\,L1\,\text{Sin}[\theta2[t]]\,\theta2'[t]-L2\,\text{Sin}[\theta2[t]-\theta3[t]]\,(\theta2'[t]-\theta3'[t]))+$$

$$(2\,L1\,\text{Cos}[\theta2[t]]\,\theta2'[t]+L2\,\text{Cos}[\theta2[t]-\theta3[t]]\,(\theta2'[t]-\theta3'[t]))\,(\theta2'[t]-\theta3'[t])+$$

$$(2\,L1\,\text{Cos}[\theta2[t]]+L2\,\text{Cos}[\theta2[t]-\theta3[t]])\,\text{Sin}[\theta1[t]]\,\theta1''[t]+$$

$$(2\,L1\,\text{Sin}[\theta2[t]]+L2\,\text{Sin}[\theta2[t]-\theta3[t]])\,(\theta2''[t]-\theta3''[t])\big)+$$

```
L2^2 Sin[2 θ1[t]] Sin[θ2[t] - θ3[t]] (θ2''[t] - θ3''[t]))) ==
MA[t] + F Sin[θ1[t]] (L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]])
```

Out[ ]=

```
F Sin[θ1[t]] (L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) +
 1/24 m2 (-12 g L2 Cos[θ2[t] - θ3[t]] - 2 L2^2 Cos[2 θ1[t]] Sin[θ2[t] - θ3[t]] θ1'[t]^2 -
   6 L2 Cos[θ1[t]]^2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[θ2[t] - θ3[t]] θ1'[t]^2 -
   L2^2 Cos[θ1[t]]^2 Sin[2 (θ2[t] - θ3[t])] θ1'[t]^2 +
   6 (-2 L1 Cos[θ2[t]] θ2'[t] - L2 Cos[θ2[t] - θ3[t]] (θ2'[t] - θ3'[t]))
    ((2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[θ1[t]] θ1'[t] +
      (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) (θ2'[t] - θ3'[t])) + 2 L2^2 Sin[2 θ1[t]] θ1'[t]
    (θ2'[t] - θ3'[t]) + 6 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]])^2 Sin[2 θ1[t]]
   θ1'[t] (θ2'[t] - θ3'[t]) - 12 Cos[θ1[t]]^2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]])
    (-2 L1 Sin[θ2[t]] θ2'[t] - L2 Sin[θ2[t] - θ3[t]] (θ2'[t] - θ3'[t])) (θ2'[t] - θ3'[t]) -
   6 L2 Cos[θ1[t]]^2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[θ2[t] - θ3[t]]
    (θ2'[t] - θ3'[t])^2 - 6 ((2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[θ1[t]] θ1'[t] +
      (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) (θ2'[t] - θ3'[t]))
    (L2 Sin[θ1[t]] Sin[θ2[t] - θ3[t]] θ1'[t] + L2 Cos[θ2[t] - θ3[t]] (-θ2'[t] + θ3'[t])) -
   L2^2 Sin[2 θ1[t]] Sin[θ2[t] - θ3[t]] θ1''[t] + 6 (-2 L1 Sin[θ2[t]] - L2 Sin[θ2[t] - θ3[t]])
    (Cos[θ1[t]] (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) θ1'[t]^2 +
      Sin[θ1[t]] θ1'[t] (-2 L1 Sin[θ2[t]] θ2'[t] - L2 Sin[θ2[t] - θ3[t]] (θ2'[t] - θ3'[t])) +
      (2 L1 Cos[θ2[t]] θ2'[t] + L2 Cos[θ2[t] - θ3[t]] (θ2'[t] - θ3'[t])) (θ2'[t] - θ3'[t]) +
      (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[θ1[t]] θ1''[t] +
      (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) (θ2''[t] - θ3''[t])) -
   2 L2^2 Cos[θ1[t]]^2 (θ2''[t] - θ3''[t]) - 6 Cos[θ1[t]]^2
    (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]])^2 (θ2''[t] - θ3''[t])) == MB[t]
```

Bonus Attempt: Problem 2

Newton-Euler and Lagrange methods have certain advantage and disadvantages to one another. Lagrange method is based on energy. Using Lagrange method helps us to ignore forces at pins which becomes internal forces at the system. Using newton-euler as a system also does that. Pin forces should be calculated for the regular newton-euler, which makes problem more complicated. However, if we want to find certain reaction forces or moments to a pin, point or joint for design purposes, Newton-Euler method tells us important informations. Also, if there is friction in the system, friction will also do virtual work in Lagrange method. This means that we should consider reaction forces which we usually do not compute in Lagrange method. In this case, maybe it is easier to do just Newton-Euler.

Bonus Attempt Problem 1

Free Body Diagrams are given in the paper. Please refer that.

FBD 1:

```
In[ ]:= FB = FBx i3 + FBy j3 + FBz k3;
        MBr = -MB j3 + MBx i3 + MBz k3;
```

In[∘]:=
```
Print["Force equation:"]
Feq = -F j3 + FB + W2 == m2 aG2 // FullSimplify
```

Force equation:

Out[∘]=
$$\{(-F+FBy)\,\mathrm{Sin}[\theta1[t]]+\mathrm{Cos}[\theta1[t]]\,(FBx\,\mathrm{Cos}[\theta2[t]-\theta3[t]]-FBz\,\mathrm{Sin}[\theta2[t]-\theta3[t]]),$$
$$(-F+FBy)\,\mathrm{Cos}[\theta1[t]]+\mathrm{Sin}[\theta1[t]]\,(-FBx\,\mathrm{Cos}[\theta2[t]-\theta3[t]]+FBz\,\mathrm{Sin}[\theta2[t]-\theta3[t]]),$$
$$-g\,m2+FBz\,\mathrm{Cos}[\theta2[t]-\theta3[t]]+FBx\,\mathrm{Sin}[\theta2[t]-\theta3[t]]\}==$$
$$\{-\frac{1}{2}m2\,(2\,L1\,\mathrm{Cos}[\theta2[t]]+L2\,\mathrm{Cos}[\theta2[t]-\theta3[t]])$$
$$(\mathrm{Cos}[\theta1[t]]\,\theta2'[t]\,(\theta2'[t]-\theta3'[t])+\mathrm{Sin}[\theta1[t]]\,\theta3''[t]),$$
$$\frac{1}{2}m2\,((2\,L1\,\mathrm{Sin}[\theta2[t]]+L2\,\mathrm{Sin}[\theta2[t]-\theta3[t]])\,\theta1'[t]\,\theta2'[t]-$$
$$\mathrm{Cos}[\theta1[t]]\,(2\,L1\,\mathrm{Cos}[\theta2[t]]+L2\,\mathrm{Cos}[\theta2[t]-\theta3[t]])\,\theta3''[t]),$$
$$-\frac{1}{2}m2\,(2\,L1\,\mathrm{Sin}[\theta2[t]]+L2\,\mathrm{Sin}[\theta2[t]-\theta3[t]])\,\theta2'[t]\,(\theta2'[t]-\theta3'[t])\}$$

In[∘]:=
```
rBG1 = -L2/2 i3;
rCG1 = L2/2 i3;
```

In[∘]:=
```
Print["Moment equation about center of mass:"]
Meq = rBG1 × FB + rCG1 × (-F j3) == IG2r.α2 + Ω2 × (IG2r.Ω2) // FullSimplify
```

Moment equation about center of mass:

Out[∘]=
$$\{\frac{1}{24}L2\,(12\,(FBz\,\mathrm{Sin}[\theta1[t]]+(F+FBy)\,\mathrm{Cos}[\theta1[t]]\,\mathrm{Sin}[\theta2[t]-\theta3[t]])+$$
$$L2\,m2\,(\mathrm{Sin}[2\,\theta1[t]]\,\mathrm{Sin}[\theta2[t]-\theta3[t]]\,\theta1'[t]^2-\mathrm{Sin}[2\,\theta1[t]]\,\mathrm{Sin}[\theta2[t]-\theta3[t]]$$
$$(\theta2'[t]-\theta3'[t])^2+2\,\theta1'[t]\,(2\,\mathrm{Cos}[\theta1[t]]^2\,\mathrm{Sin}[\theta2[t]-\theta3[t]]^2\,\theta2'[t]+$$
$$(-\mathrm{Cos}[\theta1[t]]^2+\mathrm{Cos}[\theta2[t]-\theta3[t]]^2+\mathrm{Sin}[\theta1[t]]^2\,\mathrm{Sin}[\theta2[t]-\theta3[t]]^2)\,\theta3'[t])+$$
$$\mathrm{Cos}[\theta1[t]]^2\,\mathrm{Sin}[2\,(\theta2[t]-\theta3[t])]\,\theta3''[t])),$$
$$-\frac{1}{24}L2\,(-12\,FBz\,\mathrm{Cos}[\theta1[t]]+12\,(F+FBy)\,\mathrm{Sin}[\theta1[t]]\,\mathrm{Sin}[\theta2[t]-\theta3[t]]+$$
$$L2\,m2\,(\mathrm{Cos}[\theta1[t]]^2\,\mathrm{Sin}[2\,(\theta2[t]-\theta3[t])]\,\theta1'[t]^2+\mathrm{Cos}[\theta2[t]-\theta3[t]]$$
$$\mathrm{Sin}[2\,\theta1[t]]\,\theta1'[t]\,\theta3'[t]-\mathrm{Sin}[2\,\theta1[t]]\,\mathrm{Sin}[\theta2[t]-\theta3[t]]\,\theta3''[t])),$$
$$\frac{1}{24}L2\,(\mathrm{Cos}[\theta2[t]-\theta3[t]]\,(-12\,(F+FBy)-L2\,m2\,\mathrm{Sin}[2\,\theta1[t]]\,(\theta2'[t]-\theta3'[t])^2)+$$
$$L2\,m2\,\mathrm{Cos}[\theta1[t]]^2\,\mathrm{Sin}[2\,(\theta2[t]-\theta3[t])]\,\theta1'[t]\,(2\,\theta2'[t]-\theta3'[t])+$$
$$2\,L2\,m2\,(\mathrm{Cos}[\theta2[t]-\theta3[t]]^2+\mathrm{Sin}[\theta1[t]]^2\,\mathrm{Sin}[\theta2[t]-\theta3[t]]^2)\,\theta3''[t])\}==\{0,0,0\}$$

In[ ]:= **sol2 = Solve[{Feq}, {FBx, FBy, FBz}] // Simplify**

```
Out[ ]= {{FBx → 1/2 m2 (2 g Sin[θ2[t] - θ3[t]] -
      Cos[θ2[t] - θ3[t]] Sin[θ1[t]] (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) θ1′[t] θ2′[t] -
      (Cos[θ1[t]]^2 (L2 + 2 L1 Cos[θ3[t]]) +
         Sin[θ1[t]]^2 Sin[θ2[t] - θ3[t]] (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]])) θ2′[t]^2 +
      (Cos[θ1[t]]^2 (L2 + 2 L1 Cos[θ3[t]]) + Sin[θ1[t]]^2 Sin[θ2[t] - θ3[t]]
          (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]])) θ2′[t] θ3′[t]),
  FBy → 1/4 (4 F + 2 m2 Cos[θ1[t]] (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) θ1′[t] θ2′[t] -
      m2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[2 θ1[t]] θ2′[t]^2 +
      m2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[2 θ1[t]] θ2′[t] θ3′[t] -
      2 m2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) θ3′′[t]),
  FBz → 1/4 m2 (4 g Cos[θ2[t] - θ3[t]] + 2 Sin[θ1[t]] Sin[θ2[t] - θ3[t]]
       (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) θ1′[t] θ2′[t] -
      (4 L1 Cos[θ2[t] - θ3[t]] Sin[θ1[t]]^2 Sin[θ2[t]] +
         L2 Sin[θ1[t]]^2 Sin[2 (θ2[t] - θ3[t])] + 4 L1 Cos[θ1[t]]^2 Sin[θ3[t]]) θ2′[t]^2 +
      (4 L1 Cos[θ2[t] - θ3[t]] Sin[θ1[t]]^2 Sin[θ2[t]] + L2 Sin[θ1[t]]^2
          Sin[2 (θ2[t] - θ3[t])] + 4 L1 Cos[θ1[t]]^2 Sin[θ3[t]]) θ2′[t] θ3′[t])}}
```

In[ ]:= **Meq2 = Meq /. sol2**

```
Out[ ]= {{1/24 L2 (L2 m2 (Sin[2 θ1[t]] Sin[θ2[t] - θ3[t]] θ1′[t]^2 - Sin[2 θ1[t]] Sin[θ2[t] - θ3[t]]
            (θ2′[t] - θ3′[t])^2 + 2 θ1′[t] (2 Cos[θ1[t]]^2 Sin[θ2[t] - θ3[t]]^2 θ2′[t] +
               (-Cos[θ1[t]]^2 + Cos[θ2[t] - θ3[t]]^2 + Sin[θ1[t]]^2 Sin[θ2[t] - θ3[t]]^2) θ3′[t]) +
          Cos[θ1[t]]^2 Sin[2 (θ2[t] - θ3[t])] θ3′′[t]) +
      12 (1/4 m2 Sin[θ1[t]] (4 g Cos[θ2[t] - θ3[t]] + 2 Sin[θ1[t]] Sin[θ2[t] - θ3[t]]
                (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) θ1′[t] θ2′[t] -
               (4 L1 Cos[θ2[t] - θ3[t]] Sin[θ1[t]]^2 Sin[θ2[t]] + L2 Sin[θ1[t]]^2
                   Sin[2 (θ2[t] - θ3[t])] + 4 L1 Cos[θ1[t]]^2 Sin[θ3[t]]) θ2′[t]^2 +
               (4 L1 Cos[θ2[t] - θ3[t]] Sin[θ1[t]]^2 Sin[θ2[t]] + L2 Sin[θ1[t]]^2
                   Sin[2 (θ2[t] - θ3[t])] + 4 L1 Cos[θ1[t]]^2 Sin[θ3[t]]) θ2′[t] θ3′[t]) +
          Cos[θ1[t]] Sin[θ2[t] - θ3[t]] (F + 1/4 (4 F + 2 m2 Cos[θ1[t]]
                    (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) θ1′[t] θ2′[t] - m2
                    (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[2 θ1[t]] θ2′[t]^2 + m2
                    (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[2 θ1[t]] θ2′[t]
                    θ3′[t] - 2 m2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) θ3′′[t])))),
  -1/24 L2 (-3 m2 Cos[θ1[t]] (4 g Cos[θ2[t] - θ3[t]] + 2 Sin[θ1[t]] Sin[θ2[t] - θ3[t]]
           (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) θ1′[t] θ2′[t] -
          (4 L1 Cos[θ2[t] - θ3[t]] Sin[θ1[t]]^2 Sin[θ2[t]] +
             L2 Sin[θ1[t]]^2 Sin[2 (θ2[t] - θ3[t])] + 4 L1 Cos[θ1[t]]^2 Sin[θ3[t]]) θ2′[t]^2 +
```

```
(4 L1 Cos[θ2[t] - θ3[t]] Sin[θ1[t]]^2 Sin[θ2[t]] + L2 Sin[θ1[t]]^2
    Sin[2 (θ2[t] - θ3[t])] + 4 L1 Cos[θ1[t]]^2 Sin[θ3[t]]) θ2′[t] θ3′[t]) +
L2 m2 (Cos[θ1[t]]^2 Sin[2 (θ2[t] - θ3[t])] θ1′[t]^2 + Cos[θ2[t] - θ3[t]]
   Sin[2 θ1[t]] θ1′[t] θ3′[t] -
  Sin[2 θ1[t]] Sin[θ2[t] - θ3[t]] θ3″[t]) + 12 Sin[θ1[t]] Sin[θ2[t] - θ3[t]]
 (F + 1/4 (4 F + 2 m2 Cos[θ1[t]] (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) θ1′[t] θ2′[t] -
     m2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[2 θ1[t]] θ2′[t]^2 +
     m2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[2 θ1[t]] θ2′[t] θ3′[t] -
     2 m2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) θ3″[t]))),

1/24 L2 (L2 m2 Cos[θ1[t]]^2 Sin[2 (θ2[t] - θ3[t])] θ1′[t] (2 θ2′[t] - θ3′[t]) +
   2 L2 m2
    (Cos[θ2[t] - θ3[t]]^2 + Sin[θ1[t]]^2 Sin[θ2[t] - θ3[t]]^2)
    θ3″[t] +
   Cos[θ2[t] - θ3[t]] (-L2 m2 Sin[2 θ1[t]] (θ2′[t] - θ3′[t])^2 -
     12 (F + 1/4 (4 F + 2 m2 Cos[θ1[t]] (2 L1 Sin[θ2[t]] + L2 Sin[θ2[t] - θ3[t]]) θ1′[t]
          θ2′[t] - m2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[2 θ1[t]] θ2′[t]^2 + m2
          (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) Sin[2 θ1[t]] θ2′[t] θ3′[t] - 2
          m2 (2 L1 Cos[θ2[t]] + L2 Cos[θ2[t] - θ3[t]]) θ3″[t]))))} == {0, 0, 0}}
```

```
In[•]:= sol3 = Solve[Meq2, {MBx, MBy, MB}] // Simplify
Out[•]= $Aborted
```

It took forever to run this code (sol 3). I aborted. I put the code to find reaction forces at point A to show I attempted :

```
MAr = -MA j1 + MAx i1 + Γ k1;
FA = FAx i1 + FAy j1 + FAz k1;
Print["Force equation:"]
Feq2 = FA - FB + W1 == m1 aG1 // FullSimplify


Print["Moment equation about center of mass:"]
Meq2 = -rG1A × FA + rBG1 × (-FB) - MBr + Γ k == IG1r.α1 + Ω × (IG1r.Ω) // FullSimplify

sol4 = Solve[{Feq2, Meq2}, {FAx, FAy, FAz, MAx, Γ, MA}]
```