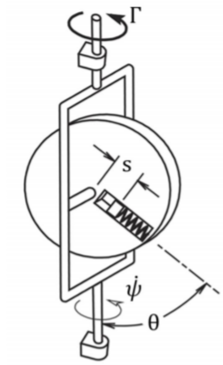
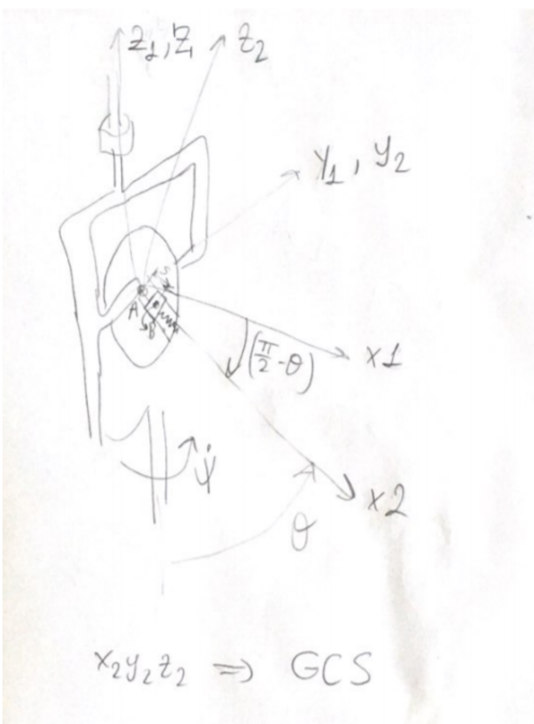
Problem 1. The disk rotates by angle θ relative to the massless gimbal, which precesses about a vertical axis at the angular acceleration $\ddot{\psi}$ because of the torque Γ. The disk has radius *R* and mass *M* and it can be considered thin. A block of mass *m* slides inside a slot within the disk, compressing a spring of stiffness *K* ($s = 0$ corresponds to the un-stretched position of the spring). You can consider the slot to extend to the circumference of the disk and neglect the friction and the rotational inertia of the block.

a) (50p) Derive the Newton-Euler equations of motion governing ψ, θ and s and find an expression for Γ that does not depend on the reaction forces.
Hint: It is easier to use the equations for the entire system

b) (50p) Derive the Lagrange's equations of motion governing ψ, θ and s.





a. Newton Euler

In[ ]:= **Quit[]**

In[∘]:=
```
i2 = {1, 0, 0}; (* i2j2k2 is selected as GCS. See figure *)
j2 = {0, 1, 0};
k2 = {0, 0, 1};
```

$$Rx[\beta\_] := \begin{pmatrix} 1 & 0 & 0 \\ 0 & Cos[\beta] & Sin[\beta] \\ 0 & -Sin[\beta] & Cos[\beta] \end{pmatrix};$$

$$Ry[\beta\_] := \begin{pmatrix} Cos[\beta] & 0 & -Sin[\beta] \\ 0 & 1 & 0 \\ Sin[\beta] & 0 & Cos[\beta] \end{pmatrix};$$

$$Rz[\beta\_] := \begin{pmatrix} Cos[\beta] & Sin[\beta] & 0 \\ -Sin[\beta] & Cos[\beta] & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$\{i1, j1, k1\} = Ry\left[\frac{\pi}{2} - \theta[t]\right]^{\mathsf{T}}.\{i2, j2, k2\} \text{ (*see figure*)}$$

Out[∘]=
```
{{Sin[θ[t]], 0, Cos[θ[t]]}, {0, 1, 0}, {-Cos[θ[t]], 0, Sin[θ[t]]}}
```

In[∘]:=
```
{i, j, k} = Rz[ψ[t]]ᵀ.{i1, j1, k1} // Simplify(* fixed coordinate system *)
```

Out[∘]=
```
{{Cos[ψ[t]] Sin[θ[t]], -Sin[ψ[t]], Cos[θ[t]] Cos[ψ[t]]},
 {Sin[θ[t]] Sin[ψ[t]], Cos[ψ[t]], Cos[θ[t]] Sin[ψ[t]]}, {-Cos[θ[t]], 0, Sin[θ[t]]}}
```

In[∘]:=
```
ω = ψ'[t] k - θ'[t] j1 // Simplify
```

Out[∘]=
```
{-Cos[θ[t]] ψ′[t], -θ′[t], Sin[θ[t]] ψ′[t]}
```

In[∘]:=
```
α = ψ''[t] k + ω × (ψ'[t] k) - θ''[t] j1 + ω × (- θ'[t] k) // Simplify
```

Out[∘]=
```
{Sin[θ[t]] θ′[t]^2 - Sin[θ[t]] θ′[t] ψ′[t] - Cos[θ[t]] ψ″[t],
 -θ″[t], Cos[θ[t]] θ′[t]^2 - Cos[θ[t]] θ′[t] ψ′[t] + Sin[θ[t]] ψ″[t]}
```

In[∘]:=
```
rAB = s[t] i2 (*A is center of disk. B is block. See figure. *)
```

Out[∘]=
```
{s[t], 0, 0}
```

In[∘]:=
```
vA = 0;
vBrel = D[rAB, t]
vB = ω × rAB + vBrel // Simplify
```

Out[∘]=
```
{s′[t], 0, 0}
```

Out[∘]=
```
{s′[t], s[t] Sin[θ[t]] ψ′[t], s[t] θ′[t]}
```

In[∘]:=
```
aA = 0;
aBrel = D[vBrel, t]
```

Out[∘]=
```
{s″[t], 0, 0}
```

In[∘]:=
```
aB = aA + aBrel + α × rAB + (ω) × (ω × rAB) + 2 ω × vBrel  // Simplify
```

Out[∘]=
```
{-s[t] (θ′[t]^2 + Sin[θ[t]]^2 ψ′[t]^2) + s″[t],
 2 Sin[θ[t]] s′[t] ψ′[t] + s[t] (Cos[θ[t]] θ′[t]^2 + Sin[θ[t]] ψ″[t]),
 2 s′[t] θ′[t] + s[t] (-Cos[θ[t]] Sin[θ[t]] ψ′[t]^2 + θ″[t])}
```

In[∘]:= (* Inertia of disk in i2j2k2. j2 aligned. *)

$$\text{Idisk} = \begin{pmatrix} \frac{1}{4} M R^2 & 0 & 0 \\ 0 & \frac{1}{2} M R^2 & 0 \\ 0 & 0 & \frac{1}{4} M R^2 \end{pmatrix}$$

Out[∘]= $\left\{\left\{\frac{M R^2}{4}, 0, 0\right\}, \left\{0, \frac{M R^2}{2}, 0\right\}, \left\{0, 0, \frac{M R^2}{4}\right\}\right\}$

In[∘]:= Hdisk = Idisk.ω

Out[∘]= $\left\{-\frac{1}{4} M R^2 \text{Cos}[\theta[t]]\, \psi'[t], -\frac{1}{2} M R^2 \theta'[t], \frac{1}{4} M R^2 \text{Sin}[\theta[t]]\, \psi'[t]\right\}$

I will crop the system and look at only disk and the block inside the disk. Forces and couples on point A in i2j2k2 :

In[∘]:= FA = FAX i2 + FAY j2 + FAZ k2

Out[∘]= {FAX, FAY, FAZ}

In[∘]:= MA = Mx i1 + My j1 + Γ k1(* I wrote this in i1j1k1 to include Γ easily*)

Out[∘]= $\{-\Gamma \text{Cos}[\theta[t]] + \text{Mx Sin}[\theta[t]], \text{My}, \text{Mx Cos}[\theta[t]] + \Gamma \text{Sin}[\theta[t]]\}$

In[∘]:= WB = - m g k1
ΣM = MA + rAB × WB == Idisk.α + ω × Hdisk // Thread // Simplify

Out[∘]= $\{g\, m\, \text{Cos}[\theta[t]], 0, -g\, m\, \text{Sin}[\theta[t]]\}$

Out[∘]= $\{4\,\Gamma\, \text{Cos}[\theta[t]] + M R^2 \text{Sin}[\theta[t]]\, \theta'[t]^2 == 4\, \text{Mx Sin}[\theta[t]] + M R^2 \text{Cos}[\theta[t]]\, \psi''[t],$
$2\, \text{My} + 2\, g\, m\, s[t]\, \text{Sin}[\theta[t]] + M R^2 \theta''[t] == 0,$
$\text{Mx Cos}[\theta[t]] + \Gamma\, \text{Sin}[\theta[t]] == \frac{1}{4} M R^2 \left(\text{Cos}[\theta[t]]\, \theta'[t]^2 + \text{Sin}[\theta[t]]\, \psi''[t]\right)\}$

In[∘]:= sol1 = Solve [ΣM, {Mx, My, Γ}] // Simplify

Out[∘]= $\left\{\left\{\text{Mx} \to \frac{1}{4} M R^2 \theta'[t]^2, \text{My} \to -g\, m\, s[t]\, \text{Sin}[\theta[t]] - \frac{1}{2} M R^2 \theta''[t], \Gamma \to \frac{1}{4} M R^2 \psi''[t]\right\}\right\}$

I did not add the spring force since it is an internal force when we look at disk – block system. It is between two of them. I did not solve reaction forces since question does not ask.

In[∘]:= WA = -M g k1;
ΣF = FA + WB + WA == m aB // Simplify // Thread

Out[∘]= $\{\text{FAX} + g\,(m + M)\, \text{Cos}[\theta[t]] == m\left(-s[t]\left(\theta'[t]^2 + \text{Sin}[\theta[t]]^2 \psi'[t]^2\right) + s''[t]\right),$
$\text{FAY} == m\left(2\, \text{Sin}[\theta[t]]\, s'[t]\, \psi'[t] + s[t]\left(\text{Cos}[\theta[t]]\, \theta'[t]^2 + \text{Sin}[\theta[t]]\, \psi''[t]\right)\right),$
$\text{FAZ} - g\,(m + M)\, \text{Sin}[\theta[t]] == m\left(2\, s'[t]\, \theta'[t] + s[t]\left(-\text{Cos}[\theta[t]]\, \text{Sin}[\theta[t]]\, \psi'[t]^2 + \theta''[t]\right)\right)\}$

b) Lagrange

In[∘]:= Tdisk[t] = $\frac{1}{2}$ ω.Idisk.ω // Simplify

Out[∘]= $\frac{1}{8} M R^2 \left(2\,\theta'[t]^2 + \psi'[t]^2\right)$

In[∘]:= TB[t] = $\frac{1}{2}$ m vB.vB // Simplify (* Inertia of m is neglected. *)

Out[∘]= $\frac{1}{2} m \left(s'[t]^2 + s[t]^2 \left(\theta'[t]^2 + \mathrm{Sin}[\theta[t]]^2\,\psi'[t]^2\right)\right)$

In[∘]:= V[t] = $\frac{1}{2}$ K s[t]^2 + m g rAB.k1 // Simplify(* Point A is selected as reference pont *)

Out[∘]= $\frac{1}{2} s[t] \left(-2\,g\,m\,\mathrm{Cos}[\theta[t]] + K\,s[t]\right)$

In[∘]:= T[t] = TB[t] + Tdisk[t] // Simplify

Out[∘]= $\frac{1}{8} \left(M R^2 \left(2\,\theta'[t]^2 + \psi'[t]^2\right) + 4\,m \left(s'[t]^2 + s[t]^2 \left(\theta'[t]^2 + \mathrm{Sin}[\theta[t]]^2\,\psi'[t]^2\right)\right)\right)$

```
In[∘]:= ψ[t_] = q1[t];
s[t_] = q2[t];
θ[t_] = q3[t];
(* Generalized Forces *)
Q1 = Γ
Q2 = 0
Q3 = 0
```

Out[∘]= Γ

Out[∘]= 0

Out[∘]= 0

In[∘]:= Print["Lagrangian Function: ℒ = ", ℒ[t] = T[t] - V[t]] // Simplify

Lagrangian Function: ℒ = $-\frac{1}{2} q2[t] \left(-2\,g\,m\,\mathrm{Cos}[q3[t]] + K\,q2[t]\right) + \frac{1}{8} \left(M R^2 \left(q1'[t]^2 + 2\,q3'[t]^2\right) + 4\,m \left(q2'[t]^2 + q2[t]^2 \left(\mathrm{Sin}[q3[t]]^2\,q1'[t]^2 + q3'[t]^2\right)\right)\right)$

In[∘]:=
```
Leq1 = D[ ∂q1'[t] ℒ[t], t] - ∂q1[t] ℒ[t] == Q1 // Simplify
Leq2 = D[ ∂q2'[t] ℒ[t], t] - ∂q2[t] ℒ[t] == Q2 // Simplify
Leq3 = D[ ∂q3'[t] ℒ[t], t] - ∂q3[t] ℒ[t] == Q3 // Simplify
```

Out[∘]= $4\,\Gamma == 8\,m\,q2[t]\,\mathrm{Sin}[q3[t]]^2\,q1'[t]\,q2'[t] + M\,R^2\,q1''[t] + 4\,m\,q2[t]^2\,\mathrm{Sin}[q3[t]]\,\left(2\,\mathrm{Cos}[q3[t]]\,q1'[t]\,q3'[t] + \mathrm{Sin}[q3[t]]\,q1''[t]\right)$

Out[∘]= $q2[t]\,\left(K - m\,\mathrm{Sin}[q3[t]]^2\,q1'[t]^2 - m\,q3'[t]^2\right) + m\,\left(-g\,\mathrm{Cos}[q3[t]] + q2''[t]\right) == 0$

Out[∘]= $m\,q2[t]^2\,\left(\mathrm{Sin}[2\,q3[t]]\,q1'[t]^2 - 2\,q3''[t]\right) == 2\,m\,q2[t]\,\left(g\,\mathrm{Sin}[q3[t]] + 2\,q2'[t]\,q3'[t]\right) + M\,R^2\,q3''[t]$

In[∘]:=
```
Clear[ψ, s, θ]
Plugback = {q1[t] → ψ, q1'[t] → ψ', q1''[t] → ψ'', q2[t] → s,
   q2'[t] → s', q2''[t] → s'', q3[t] → θ, q3'[t] → θ', q3''[t] → θ''};
Simplify[Leq1 /. Plugback, {L > 0, m > 0, m > 0, R > 0 }]
Simplify[Leq2 /. Plugback, {L > 0, M > 0, m > 0, R > 0}]
Simplify[Leq3 /. Plugback, {L > 0, M > 0, m > 0, R > 0}]
```

Out[∘]= $4\,\Gamma == 8\,m\,s\,\mathrm{Sin}[\theta]^2\,s'\,\psi' + M\,R^2\,\psi'' + 4\,m\,s^2\,\mathrm{Sin}[\theta]\,\left(2\,\mathrm{Cos}[\theta]\,\theta'\,\psi' + \mathrm{Sin}[\theta]\,\psi''\right)$

Out[∘]= $K\,s + m\,s'' == m\,\left(g\,\mathrm{Cos}[\theta] + s\,(\theta')^2 + s\,\mathrm{Sin}[\theta]^2\,(\psi')^2\right)$

Out[∘]= $m\,s^2\,\left(\mathrm{Sin}[2\,\theta]\,(\psi')^2 - 2\,\theta''\right) == 2\,g\,m\,s\,\mathrm{Sin}[\theta] + 4\,m\,s\,s'\,\theta' + M\,R^2\,\theta''$