In[∘]:=
```
Quit[]
```

Question 1) c)

In[∘]:=

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 0.5 & 0.5 \\ 0 & -1 & 1 & 1 & -2 & 1 \\ 2 & 1 & -3 & 1 & 1 & -2 \end{pmatrix};\ B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};\ Cm = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix};\ Dm = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

```
x[t_] := {x1[t], x2[t], x3[t], x4[t], x5[t], x6[t]};
y[t_] := Cm.x[t];
```

In[∘]:=
```
n = Dimensions[A][[1]];
r = Dimensions[B][[2]];
m = Dimensions[Cm][[1]];
Print["n = ", n, "\t r = ", r, "\t m = ", m]
P = Join[B, A.B, A.A.B, A.A.A.B, A.A.A.A.B, A.A.A.A.A.B, 2];
Print["P = ", MatrixForm[P]]
Print["Rank of P: ", MatrixRank[P]]
If[MatrixRank[P] == n, Print["System is controllable."],
 Print["System is uncontrollable."]]
```

n = 6 r = 1 m = 3

$$P = \begin{pmatrix} 0 & 0. & 0.5 & 0. & -2.5 & 7.5 \\ 0 & 0. & 1. & -2.5 & 4.5 & -6. \\ 0 & 1. & -2. & 2.5 & 0.5 & -9. \\ 0 & 0.5 & 0. & -2.5 & 7.5 & -12. \\ 0 & 1. & -2.5 & 4.5 & -6. & 6.5 \\ 1 & -2. & 2.5 & 0.5 & -9. & 17.5 \end{pmatrix}$$

Rank of P: 6

System is controllable.

In[∘]:=

$$Q = \text{Join}[Cm^T, A^T.Cm^T, A^T.A^T.Cm^T, A^T.A^T.A^T.Cm^T, A^T.A^T.A^T.A^T.Cm^T, A^T.A^T.A^T.A^T.A^T.Cm^T, 2];$$

```
Print["Q = ", MatrixForm[Q]]
Print["rank(Q) = ", MatrixRank[Q]]
If[MatrixRank[Q] == n, Print["System is observable."], Print["System is not observable."]]
```

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0. & 0. & 0. & -1. & 0. & 2. & 2. & 1. & -5. & -1. & -3. & 5. & -5. & 4. & 6. \\ 0 & 1 & 0 & 0. & 0. & 0. & 0. & -1. & 1. & 0. & 3. & -3. & 1. & -7. & 5. & -5. & 14. & -4. \\ 0 & 0 & 1 & 0. & 0. & 0. & 1. & 1. & -3. & -2. & -4. & 8. & 0. & 10. & -10. & 10. & -18. & -2. \\ 0 & 0 & 0 & 1. & 0. & 0. & -1. & 1. & 1. & 1. & -2. & 0. & 0. & 5. & -5. & -1. & -13. & 15. \\ 0 & 0 & 0 & 0. & 1. & 0. & 0.5 & -2. & 1. & -1. & 4.5 & -2.5 & 2.5 & -9.5 & 4.5 & -6.5 & 19. & -6. \\ 0 & 0 & 0 & 0. & 0. & 1. & 0.5 & 1. & -2. & 0. & -2.5 & 2.5 & -2.5 & 4.5 & 0.5 & 7.5 & -6. & -9. \end{pmatrix}$$

rank(Q) = 6

System is observable.

In[∘]:=
```
Eigenvalues[A] // N // Chop
```

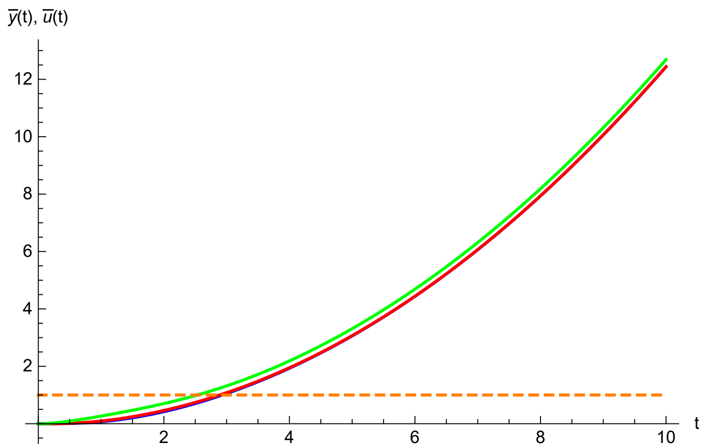
Out[∘]=

$$\{-2., -1.2267 + 1.46771\, i, -1.2267 - 1.46771\, i, -0.546602, 0. + 1.01808 \times 10^{-8}\, i, 0. - 1.01808 \times 10^{-8}\, i\}$$

Open loop step response

In[∘]:=
```
u[t_] = {u1[t]};
EqOpenLoop = Thread[x'[t] == A.x[t] + B.u[t]] // Flatten;
IC = {x1[0] == 0, x2[0] == 0, x3[0] == 0, x4[0] == 0, x5[0] == 0, x6[0] == 0 };
Inputs = {u1[t] → 1}; tmax = 10;
OLResponse = NDSolve[{EqOpenLoop /. Inputs, IC}, x[t], {t, 0, tmax}];
Plot[Evaluate[{y[t] /. OLResponse, u1[t] /. Inputs, u2[t] /. Inputs}],
 {t, 0, tmax}, AxesLabel → {"t", "y̅(t), u̅(t)"}, PlotRange → All,
 PlotStyle → {Blue, Red, Green, {Dashed, Orange}, {Dashed, Purple}}]
```

Out[∘]=



None of the eigenvalues are positive.

However, I see 2 eigenvalues has a real part of zero and imaginary value that is very close to zero $\left(\omega = \sim 0,\ \frac{1}{s^2} \text{ in s – plane, as transfer function}\right)$. System is not going to be damped and value will increase without any damping.

Question 2

a)

In[∘]:= `Quit[]`

In[∘]:=

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 1 & 4 \\ 0 & -1 & 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 & 1 & -2 \end{pmatrix}; B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}; Cm = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}; Dm = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix};$$

```
x[t_] := {x1[t], x2[t], x3[t], x4[t], x5[t], x6[t]}
y[t_] := Cm.x[t]
```

Let's check observability, controllability and stability first!

In[ ]:=
```
n = Dimensions[A][[1]];
r = Dimensions[B][[2]];
m = Dimensions[Cm][[1]];
Print["n = ", n, "\t r = ", r, "\t m = ", m]
P = Join[B, A.B, A.A.B, A.A.A.B, A.A.A.A.B, A.A.A.A.A.B, 2];
Print["P = ", MatrixForm[P]]
Print["Rank of P: ", MatrixRank[P]]
If[MatrixRank[P] == n, Print["System is controllable."],
 Print["System is uncontrollable."]]
```

n = 6 r = 2 m = 4

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 5 & -1 & -15 & 1 & 57 & -5 & -181 & 13 \\ 0 & 0 & 0 & 0 & 2 & 0 & -2 & -1 & -3 & 2 & 43 & -7 \\ 1 & 0 & 0 & 0 & -3 & 0 & 16 & -1 & -54 & 3 & 166 & -10 \\ 0 & 0 & 5 & -1 & -15 & 1 & 57 & -5 & -181 & 13 & 561 & -40 \\ 0 & 0 & 2 & 0 & -2 & -1 & -3 & 2 & 43 & -7 & -206 & 21 \\ 1 & 0 & -3 & 0 & 13 & -1 & -38 & 2 & 112 & -7 & -311 & 19 \end{pmatrix}$$

Rank of P: 6

System is controllable.

In[ ]:=
```
Q = Join[Cm^T, A^T.Cm^T, A^T.A^T.Cm^T,
   A^T.A^T.A^T.Cm^T, A^T.A^T.A^T.A^T.Cm^T, A^T.A^T.A^T.A^T.A^T.Cm^T, 2];
Print["Q = ", MatrixForm[Q]]
Print["rank(Q) = ", MatrixRank[Q]]
If[MatrixRank[Q] == n, Print["System is observable."], Print["System is not observable."]]
```

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & -1 & -1 & -5 & -5 & 2 & 3 & 13 & 13 & -7 & -10 & -40 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & 3 & 3 & -4 & -11 & -11 & -4 & 14 & 40 & 40 & 0 & -42 & -127 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & -5 & -5 & -3 & 5 & 21 & 21 & 5 & -22 & -74 & -74 & 2 & 74 & 241 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 5 & 5 & -2 & -3 & -13 & -13 & 7 & 10 & 40 & 40 & -21 & -29 & -114 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -2 & 1 & 1 & 1 & 5 & -3 & -4 & -4 & -8 & 10 & 20 & 20 & 11 & -28 & -67 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 4 & 1 & -3 & -10 & -10 & 1 & 11 & 36 & 36 & -8 & -32 & -107 & -107 & 41 & 92 & 320 \end{pmatrix}$$

rank(Q) = 6

System is observable.

In[ ]:=
```
Eigenvalues[A] // N // Chop
```

Out[ ]= {-2.74675, -1.93928 + 0.368654 i, -1.93928 - 0.368654 i, 0.950416, -0.515726, 0.190612}

Because of the positive eigenvalues, open loop system is not stable!

Output Controllability matrix :

In[ ]:=
```
Po = Join[Cm.B, Cm.A.B, Cm.A.A.B, Cm.A.A.A.B, Cm.A.A.A.A.B, Cm. A.A.A.A.A.B, 2];
Print["P_o = ", MatrixForm[Po]]
Print["Rank of Po: ", MatrixRank[Po]]
If[MatrixRank[Po] == m, Print["System is output controllable."],
 Print["System is output uncontrollable."]]
```

$$P_o = \begin{pmatrix} 0 & 1 & 0 & 0 & 5 & -1 & -15 & 1 & 57 & -5 & -181 & 13 \\ 0 & 0 & 0 & 0 & 2 & 0 & -2 & -1 & -3 & 2 & 43 & -7 \\ 1 & 0 & 0 & 0 & -3 & 0 & 16 & -1 & -54 & 3 & 166 & -10 \\ 0 & 0 & 5 & -1 & -15 & 1 & 57 & -5 & -181 & 13 & 561 & -40 \end{pmatrix}$$

```
Rank of Po: 4
```

```
System is output controllable.
```

Desired closed - loop eigenvalues :

```
In[∘]:= {λ1, λ2, λ3, λ4} = {-3, -3, -4, -4};
```

Form X = [I λ - A ⋮ B] matrix and obtain the null space:

```
In[∘]:= Imat = IdentityMatrix[n];
        X = Join[Imat λ - A, B, 2] // Simplify; MatrixForm[X]
        U = NullSpace[X]ᵀ // Simplify; MatrixForm[U]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \lambda & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & \lambda & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1+\lambda & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1+\lambda & -1 & -4 & 0 & 0 \\ 0 & 1 & -1 & -1 & 2+\lambda & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 2+\lambda & 1 & 0 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{9+21\lambda+6\lambda^2-9\lambda^3-6\lambda^4-\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{6+17\lambda+15\lambda^2+5\lambda^3}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{5+4\lambda+\lambda^2}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{3+5\lambda+10\lambda^2+2\lambda^3}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)(2+\lambda)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{2-3\lambda^2+7\lambda^3+6\lambda^4+\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)(1+5\lambda+4\lambda^2+\lambda^3)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{\lambda(6+17\lambda+15\lambda^2+5\lambda^3)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{\lambda(5+4\lambda+\lambda^2)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{\lambda(3+5\lambda+10\lambda^2+2\lambda^3)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)^2(2+\lambda)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{1+5\lambda+9\lambda^2+5\lambda^3+3\lambda^4+\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Form Ψ and ℱ ' matrices by partitioning U :

In[∘]:=
```
Ψ = Take[U, n]; MatrixForm[Ψ]
ℱp := Take[U, -r]; MatrixForm[ℱp]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{9+21\lambda+6\lambda^2-9\lambda^3-6\lambda^4-\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{6+17\lambda+15\lambda^2+5\lambda^3}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{5+4\lambda+\lambda^2}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{3+5\lambda+10\lambda^2+2\lambda^3}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)(2+\lambda)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{2-3\lambda^2+7\lambda^3+6\lambda^4+\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)\left(1+5\lambda+4\lambda^2+\lambda^3\right)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{\lambda\left(6+17\lambda+15\lambda^2+5\lambda^3\right)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{\lambda\left(5+4\lambda+\lambda^2\right)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{\lambda\left(3+5\lambda+10\lambda^2+2\lambda^3\right)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)^2(2+\lambda)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{1+5\lambda+9\lambda^2+5\lambda^3+3\lambda^4+\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Form composite matrices Ω = [Ψ (λ1) Ψ (λ2) ... Ψ (λn)] , Ω' = C Ω, and Λ' = [ℱ' (λ1) ℱ ' (λ2) ... ℱ '(λn)] :

In[∘]:=
```
Ω = Join[Ψ /. λ -> λ1, Ψ /. λ -> λ2, Ψ /. λ -> λ3, Ψ /. λ -> λ4, 2];
Ωp = Cm.Ω // Simplify; MatrixForm[Ωp]
Λp = Join[ℱp /. λ -> λ1, ℱp /. λ -> λ2, ℱp /. λ -> λ3, ℱp /. λ -> λ4, 2];
MatrixForm[Λp]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{9}{2} & 0 & \frac{9}{2} & \frac{85}{397} & \frac{142}{397} & \frac{85}{397} & \frac{142}{397} \\ \frac{1}{5} & -\frac{12}{5} & \frac{1}{5} & -\frac{12}{5} & \frac{5}{397} & -\frac{15}{397} & \frac{5}{397} & -\frac{15}{397} \\ \frac{1}{5} & -\frac{29}{10} & \frac{1}{5} & -\frac{29}{10} & \frac{6}{397} & -\frac{18}{397} & \frac{6}{397} & -\frac{18}{397} \\ 1 & -\frac{27}{2} & 1 & -\frac{27}{2} & \frac{57}{397} & -\frac{568}{397} & \frac{57}{397} & -\frac{568}{397} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Form the G' and $\mathcal{J}$ ' matrices by selecting linearly independent columns, one for each eigenvalue, up to *m* :

```
In[ ]:= ColumnChoice = {1, 2, 5, 6};
Gp = Ωp[[All, ColumnChoice]]; MatrixForm[Gp]
𝒥p = Λp[[All, ColumnChoice]]; MatrixForm[𝒥p]
Det[Gp] // N
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{9}{2} & \frac{85}{397} & \frac{142}{397} \\ \frac{1}{5} & -\frac{12}{5} & \frac{5}{397} & -\frac{15}{397} \\ \frac{1}{5} & -\frac{29}{10} & \frac{6}{397} & -\frac{18}{397} \\ 1 & -\frac{27}{2} & \frac{57}{397} & -\frac{568}{397} \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Out[ ]= 0.0309824

Solve for K :

```
In[ ]:= Kstar = 𝒥p.Inverse[Gp] // N // Simplify; MatrixForm[Kstar]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.252033 & 1.22764 & 2.5122 & -0.747967 \\ 3.92683 & -8.19512 & 8.56098 & 0.926829 \end{pmatrix}$$

```
In[ ]:= Eigenvalues[A - B.Kstar.Cm]
```

Out[ ]= {-4., -4., -3., -3., 1.06458, 0.4964}

As expected, 4 eigenvalues (m = 4) were placed at the desired values . Even if question would ask to place more than 4, output feedback method can only be able to place "m" eigenvalues .

In summary, output feedback result is not acceptable . We need state feedback method even though it is computationally expensive and requires additional sensors .

Question 2

b)

Form X = [I λ - A ⋮ B] matrix and obtain the null space:

In[∘]:=
```
{λ1, λ2, λ3, λ4, λ5, λ6} = {-3, -3, -4, -4, -5, -5};
Imat = IdentityMatrix[n];
```

In[∘]:=
```
X = Join[Imat λ - A, B, 2]; MatrixForm[X]
U = NullSpace[X]ᵀ // Simplify; MatrixForm[U]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \lambda & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & \lambda & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1+\lambda & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1+\lambda & -1 & -4 & 0 & 0 \\ 0 & 1 & -1 & -1 & 2+\lambda & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -1 & 2+\lambda & 1 & 0 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{9+21\lambda+6\lambda^2-9\lambda^3-6\lambda^4-\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{6+17\lambda+15\lambda^2+5\lambda^3}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{5+4\lambda+\lambda^2}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{3+5\lambda+10\lambda^2+2\lambda^3}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)\,(2+\lambda)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{2-3\lambda^2+7\lambda^3+6\lambda^4+\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)\,(1+5\lambda+4\lambda^2+\lambda^3)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{\lambda\,(6+17\lambda+15\lambda^2+5\lambda^3)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{\lambda\,(5+4\lambda+\lambda^2)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{\lambda\,(3+5\lambda+10\lambda^2+2\lambda^3)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)^2\,(2+\lambda)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{1+5\lambda+9\lambda^2+5\lambda^3+3\lambda^4+\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Form Ψ and ℱ matrices by partitioning U :

In[∘]:=
```
Ψ = Take[U, n]; MatrixForm[Ψ]
ℱ := Take[U, -r]; MatrixForm[ℱ]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{9+21\lambda+6\lambda^2-9\lambda^3-6\lambda^4-\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{6+17\lambda+15\lambda^2+5\lambda^3}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{5+4\lambda+\lambda^2}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{3+5\lambda+10\lambda^2+2\lambda^3}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)\,(2+\lambda)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{2-3\lambda^2+7\lambda^3+6\lambda^4+\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)\,(1+5\lambda+4\lambda^2+\lambda^3)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{\lambda\,(6+17\lambda+15\lambda^2+5\lambda^3)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{\lambda\,(5+4\lambda+\lambda^2)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{\lambda\,(3+5\lambda+10\lambda^2+2\lambda^3)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \\ \frac{(1+\lambda)^2\,(2+\lambda)}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} & -\frac{1+5\lambda+9\lambda^2+5\lambda^3+3\lambda^4+\lambda^5}{1-3\lambda-12\lambda^2-\lambda^3+10\lambda^4+6\lambda^5+\lambda^6} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Form composite matrices Ω = [Ψ (λ1) Ψ (λ2) ... Ψ (λn)] and Λ = [ℱ (λ1) ℱ (λ2) ... ℱ (λn)] :

In[∘]:=
```
Ω = Join[Ψ /. λ -> λ1, Ψ /. λ -> λ2, Ψ /. λ -> λ3,
   Ψ /. λ -> λ4, Ψ /. λ -> λ5, Ψ /. λ -> λ6, 2] // Simplify;
MatrixForm[Ω]
Λ = Join[ℱ /. λ -> λ1, ℱ /. λ -> λ2, ℱ /. λ -> λ3, ℱ /. λ -> λ4, ℱ /. λ -> λ5, ℱ /. λ -> λ6, 2];
MatrixForm[Λ]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{9}{2} & 0 & \frac{9}{2} & \frac{85}{397} & \frac{142}{397} & \frac{85}{397} & \frac{142}{397} & \frac{277}{1483} & \frac{329}{2966} & \frac{277}{1483} & \frac{329}{2966} \\ \frac{1}{5} & -\frac{12}{5} & \frac{1}{5} & -\frac{12}{5} & \frac{5}{397} & -\frac{15}{397} & \frac{5}{397} & -\frac{15}{397} & \frac{5}{1483} & \frac{11}{1483} & \frac{5}{1483} & \frac{11}{1483} \\ \frac{1}{5} & -\frac{29}{10} & \frac{1}{5} & -\frac{29}{10} & \frac{6}{397} & -\frac{18}{397} & \frac{6}{397} & -\frac{18}{397} & \frac{6}{1483} & \frac{323}{2966} & \frac{6}{1483} & \frac{323}{2966} \\ 1 & -\frac{27}{2} & 1 & -\frac{27}{2} & \frac{57}{397} & -\frac{568}{397} & \frac{57}{397} & -\frac{568}{397} & \frac{98}{1483} & -\frac{1645}{2966} & \frac{98}{1483} & -\frac{1645}{2966} \\ -\frac{3}{5} & \frac{36}{5} & -\frac{3}{5} & \frac{36}{5} & -\frac{20}{397} & \frac{60}{397} & -\frac{20}{397} & \frac{60}{397} & -\frac{25}{1483} & -\frac{55}{1483} & -\frac{25}{1483} & -\frac{55}{1483} \\ -\frac{2}{5} & \frac{34}{5} & -\frac{2}{5} & \frac{34}{5} & -\frac{18}{397} & \frac{451}{397} & -\frac{18}{397} & \frac{451}{397} & -\frac{24}{1483} & \frac{837}{1483} & -\frac{24}{1483} & \frac{837}{1483} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Form the G and 𝒥 matrices by selecting linearly independent columns, one for each eigenvalue :

In[∘]:=
```
ColumnChoice = {1, 2, 6, 7, 9, 12};
G = Ω[[All, ColumnChoice]]; MatrixForm[G]
𝒥 = Λ[[All, ColumnChoice]]; MatrixForm[𝒥]
Det[G] // N
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{9}{2} & \frac{142}{397} & \frac{85}{397} & \frac{277}{1483} & \frac{329}{2966} \\ \frac{1}{5} & -\frac{12}{5} & -\frac{15}{397} & \frac{5}{397} & \frac{5}{1483} & \frac{11}{1483} \\ \frac{1}{5} & -\frac{29}{10} & -\frac{18}{397} & \frac{6}{397} & \frac{6}{1483} & \frac{323}{2966} \\ 1 & -\frac{27}{2} & -\frac{568}{397} & \frac{57}{397} & \frac{98}{1483} & -\frac{1645}{2966} \\ -\frac{3}{5} & \frac{36}{5} & \frac{60}{397} & -\frac{20}{397} & -\frac{25}{1483} & -\frac{55}{1483} \\ -\frac{2}{5} & \frac{34}{5} & \frac{451}{397} & -\frac{18}{397} & -\frac{24}{1483} & \frac{837}{1483} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Out[∘]= -3.39702×10^{-6}

Solve for K :

```
In[]:= Ksf = 𝒥.Inverse[G] // N; MatrixForm[Ksf]
```

Out[]//MatrixForm=

$$\begin{pmatrix} -0.5 & 25.5 & 4.5 & 5. & 14. & 6.5 \\ 7. & 136.4 & 22.6 & -0.4 & 53.6 & -4.4 \end{pmatrix}$$

Test close loop eigenvalues :

```
In[]:= Eigenvalues[A - B.Ksf]
Out[]= {-5., -5., -4., -4., -3., -3.}
```

All eigenvalues are at desired position!

Question 2

c)

Partition the system into *m* measurable states and *n* - *m* states that need to be estimated :

```
In[]:= Print["x = ", MatrixForm[x[t]]]
xv1[t_] := Take[x[t], m];
xv2[t_] := Take[x[t], -(n - m)];
Print["xv1 = ", MatrixForm[xv1[t]], "\t xv2 = ", MatrixForm[xv2[t]]]
Print["A = ", MatrixForm[A]]
A11 = Take[A, m, m];
A12 = Take[A, m, {m + 1, n}];
A21 = Take[A, {m + 1, n}, m];
A22 = Take[A, {m + 1, n}, {m + 1, n}];
Print["A11 = ", MatrixForm[A11], "\t A12 = ", MatrixForm[A12],
 "\t A21 = ", MatrixForm[A21], "\t A22 = ", MatrixForm[A22]]
B1 = Take[B, m];
B2 = Take[B, -(n - m)];
Print["B = ", MatrixForm[B]]
Print["B1 = ", MatrixForm[B1], "\t B2 = ", MatrixForm[B2]]
```

$$x = \begin{pmatrix} x1[t] \\ x2[t] \\ x3[t] \\ x4[t] \\ x5[t] \\ x6[t] \end{pmatrix}$$

$$x_{v1} = \begin{pmatrix} x1[t] \\ x2[t] \\ x3[t] \\ x4[t] \end{pmatrix} \qquad x_{v2} = \begin{pmatrix} x5[t] \\ x6[t] \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1 & 1 & 4 \\ 0 & -1 & 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 1 & 1 & -2 \end{pmatrix}$$

$$A11 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & -1 \end{pmatrix} \qquad A12 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 4 \end{pmatrix} \qquad A21 = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \qquad A22 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Form Xo = [I λ - A_{22}^{T} ⋮ A_{12}^{T}] matrix and obtain the null space:

In[ ]:=
```
{λr1, λr2 } = {-10, -12};
Ir = IdentityMatrix[n - m];
Xr = Join[Ir λ - A22ᵀ, A12ᵀ, 2]; MatrixForm[Xr];
Ur = NullSpace[Xr]ᵀ // Simplify;
MatrixForm[Ur];
Ψr = Take[Ur, n - m]; MatrixForm[Ψr];
ℱr := Take[Ur, -m]; MatrixForm[ℱr];
Ωr = Join[Ψr /. λ -> λr1, Ψr /. λ -> λr2, 2]; Print["Ωr =", MatrixForm[Ωr]]
Λr = Join[ℱr /. λ -> λr1, ℱr /. λ -> λr2, 2];
Print["Λr =", MatrixForm[Λr]]
```

$$\Omega r = \begin{pmatrix} \frac{4}{63} & -\frac{1}{63} & \frac{8}{63} & 0 & \frac{2}{33} & -\frac{1}{99} & \frac{10}{99} & 0 \\ \frac{31}{63} & \frac{8}{63} & -\frac{1}{63} & 0 & \frac{13}{33} & \frac{10}{99} & -\frac{1}{99} & 0 \end{pmatrix}$$

$$\Lambda r = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Form the G and $\mathcal{J}$ matrices by selecting linearly independent columns, one for each eigenvalue :

In[∘]:=
```
ColumnChoice = {2, 5};
Gr = Ωr[[All, ColumnChoice]]; MatrixForm[Gr]
𝒥r = Λr[[All, ColumnChoice]]; MatrixForm[𝒥r]
Det[Gr] // N
```

Out[∘]//MatrixForm=
$$\begin{pmatrix} -\frac{1}{63} & \frac{2}{33} \\ \frac{8}{63} & \frac{13}{33} \end{pmatrix}$$

Out[∘]//MatrixForm=
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Out[∘]= -0.013949

Solve for Lr :

In[∘]:=
```
Lr = Transpose[𝒥r.Inverse[Gr]];
MatrixForm[Lr]
```

Out[∘]//MatrixForm=
$$\begin{pmatrix} 0 & 0 & -\frac{819}{29} & \frac{264}{29} \\ 0 & 0 & \frac{126}{29} & \frac{33}{29} \end{pmatrix}$$

Reduced Order Observer Matrix:

In[∘]:=
```
Clear[u];
Ar = A22 - Lr.A12; MatrixForm[Ar]
```

Out[∘]//MatrixForm=
$$\begin{pmatrix} -\frac{322}{29} & -\frac{208}{29} \\ -\frac{4}{29} & -\frac{316}{29} \end{pmatrix}$$

Test reduced order observer eigenvalues :

In[∘]:=
```
Eigenvalues[Ar]
```

Out[∘]= $\{-12, -10\}$

All eigenvalues are at desired position!

Question 2

d)

State Feedback + Observer simulation

d) Reduced Order Observer Feedback Simulation:

In[∘]:=
```
Clear[u, v, v1, v2]
v[t_] := {v1[t], v2[t]};
xro[t_] = {xro4[t], xro5[t]}
Imr = IdentityMatrix[r];
(*F=Imr*)
Cr = Take[Cm, r];(* required if m>r*)
F = Inverse[-Cr.Inverse[A - B.Ksf].B]
xhat[t] = {xv1[t], xro[t]} // Flatten
u[t_] := F.v[t] - Ksf.xhat[t]
MatrixForm[u[t]]
yr[t_] := xv1'[t] - A11.xv1[t] - B1.u[t]
MatrixForm[yr[t]]
zr[t_] := A21.xv1[t] + B2.u[t]
MatrixForm[zr[t]]
```

Out[∘]= {xro4[t], xro5[t]}

Out[∘]= $\{\{-1.69177\times10^{-16}, 30.\}, \{40., 84.\}\}$

Out[∘]= {x1[t], x2[t], x3[t], x4[t], xro4[t], xro5[t]}

Out[∘]//MatrixForm=

$$\begin{pmatrix} -1.69177\times10^{-16}\, v1[t] + 30.\, v2[t] + 0.5\, x1[t] - 25.5\, x2[t] - 4.5\, x3[t] - 5.\, x4[t] - 14.\, xro4[t] - 6.\text{!} \\ 40.\, v1[t] + 84.\, v2[t] - 7.\, x1[t] - 136.4\, x2[t] - 22.6\, x3[t] + 0.4\, x4[t] - 53.6\, xro4[t] + 4.4\, xro \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} -40.\, v1[t] - 84.\, v2[t] + 7.\, x1[t] + 136.4\, x2[t] + 22.6\, x3[t] - 1.4\, x4[t] + 53.6\, xro4[t] - 4.4\, xro \\ x2'[t] \\ 1.69177\times10^{-16}\, v1[t] - 30.\, v2[t] - 0.5\, x1[t] + 25.5\, x2[t] + 5.5\, x3[t] + 5.\, x4[t] + 14.\, xro4[t] + 6.5 \\ x1[t] - x3[t] + x4[t] + x4'[t] \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} -x2[t] + x3[t] + x4[t] \\ -1.69177\times10^{-16}\, v1[t] + 30.\, v2[t] + 0.5\, x1[t] - 24.5\, x2[t] - 5.5\, x3[t] - 4.\, x4[t] - 14.\, xro4[t] - 6.\text{!} \end{pmatrix}$$

In[ ]:=
```
EqRedObsFdbk = Thread[x'[t] == A.x[t] - B.Ksf.xhat[t] + B.F.v[t] // Chop];
EqRedObserver = Thread[xro'[t] == Ar.xro[t] + Lr.yr[t] + zr[t] // Chop];
ICro = {xro4[0] == -3, xro5[0] == -1};
y[t_] := Cm.x[t]
TableForm[{EqRedObsFdbk, EqRedObserver} // Flatten]
```

Out[ ]//TableForm=
```
x1′[t] == 40. v1[t] + 84. v2[t] - 7. x1[t] - 136.4 x2[t] - 22.6 x3[t] + 1.4 x4[t] - 53.6 xro4[t] + 4.4 ›
x2′[t] == x5[t]
x3′[t] == 30. v2[t] + 0.5 x1[t] - 25.5 x2[t] - 5.5 x3[t] - 5. x4[t] + x6[t] - 14. xro4[t] - 6.5 xro5[t]
x4′[t] == -x1[t] + x3[t] - x4[t] + x5[t] + 4 x6[t]
x5′[t] == -x2[t] + x3[t] + x4[t] - 2 x5[t] + x6[t]
x6′[t] == 30. v2[t] + 0.5 x1[t] - 24.5 x2[t] - 5.5 x3[t] - 4. x4[t] + x5[t] - 2 x6[t] - 14. xro4[t] - 6.
```
$$xro4'[t] == -x2[t] + x3[t] + x4[t] - \frac{322\,xro4[t]}{29} - \frac{208\,xro5[t]}{29} - \frac{819}{29}\left(-30.\ v2[t] - 0.5\ x1[t] + 25.5\ x2[t] + \right.$$
```
xro5′[t] == 30. v2[t] + 0.5 x1[t] - 24.5 x2[t] - 5.5 x3[t] - 4. x4[t] - 14.1379 xro4[t] - 17.3966 xro5
```

In[ ]:=
```
yd[t_] := {y1d[t], y2d[t], y3d[t], y4d[t]};
DesOut = {y1d[t] → 1, y2d[t] → 1, y3d[t] → 1, y4d[t] -> 1}
tmax = 5;
IC2 = {x1[0] == -1, x2[0] == 1, x3[0] == -3, x4[0] == 3, x5[0] == 5, x6[0] == -3}
DesInput = {v1[t] → 1, v2[t] → 1};
RedObsFdbkResponse =
  NDSolve[{EqRedObsFdbk /. DesInput, IC2 , EqRedObserver /. DesInput, ICro},
   {x[t], xro4[t], xro5[t]} // Flatten, {t, 0, tmax}];
```

Out[ ]= {y1d[t] → 1, y2d[t] → 1, y3d[t] → 1, y4d[t] → 1}

Out[ ]= {x1[0] == -1, x2[0] == 1, x3[0] == -3, x4[0] == 3, x5[0] == 5, x6[0] == -3}

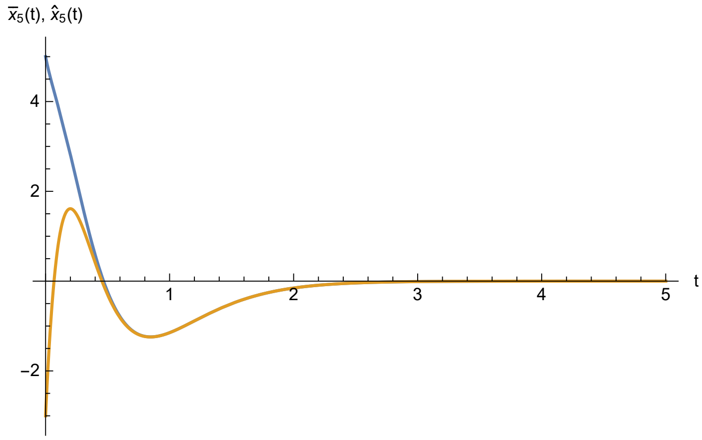
Checking if NDSolve worked

```
In[ ]:= RedObsFdbkResponse

Out[ ]= {{x1[t] → InterpolatingFunction[ $Failed  Domain: {{0., 5.}} Output: scalar ][t],
  x2[t] → InterpolatingFunction[ $Failed  Domain: {{0., 5.}} Output: scalar ][t],
  x3[t] → InterpolatingFunction[ $Failed  Domain: {{0., 5.}} Output: scalar ][t],
  x4[t] → InterpolatingFunction[ $Failed  Domain: {{0., 5.}} Output: scalar ][t],
  x5[t] → InterpolatingFunction[ $Failed  Domain: {{0., 5.}} Output: scalar ][t],
  x6[t] → InterpolatingFunction[ $Failed  Domain: {{0., 5.}} Output: scalar ][t],
  xro4[t] → InterpolatingFunction[ $Failed  Domain: {{0., 5.}} Output: scalar ][t],
  xro5[t] → InterpolatingFunction[ $Failed  Domain: {{0., 5.}} Output: scalar ][t]}}
```

Estimated States x5 and x6 (named xro4, and xro5 to not change the code)

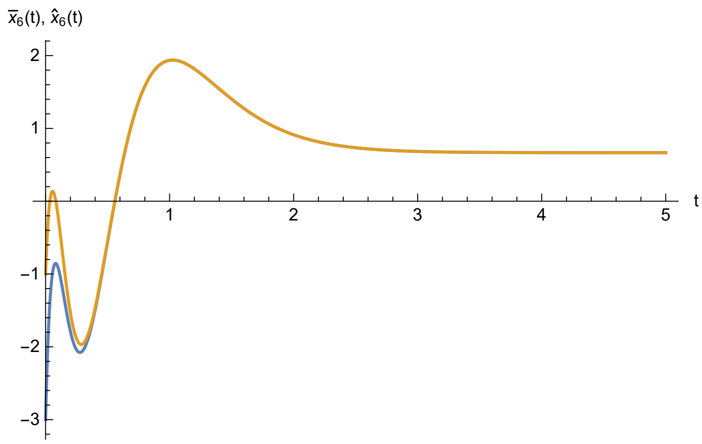
In[∘]:=
```
Plot[Evaluate[{x5[t], xro4[t]} /. RedObsFdbkResponse],
 {t, 0, tmax}, AxesLabel → {"t", "x̄5(t), x̂5(t)"}, PlotRange → All]
```

Out[∘]=



In[∘]:=
```
Plot[Evaluate[{x6[t], xro5[t]} /. RedObsFdbkResponse],
 {t, 0, tmax}, AxesLabel → {"t", "x̄6(t), x̂6(t)"}, PlotRange → All]
```

Out[∘]=

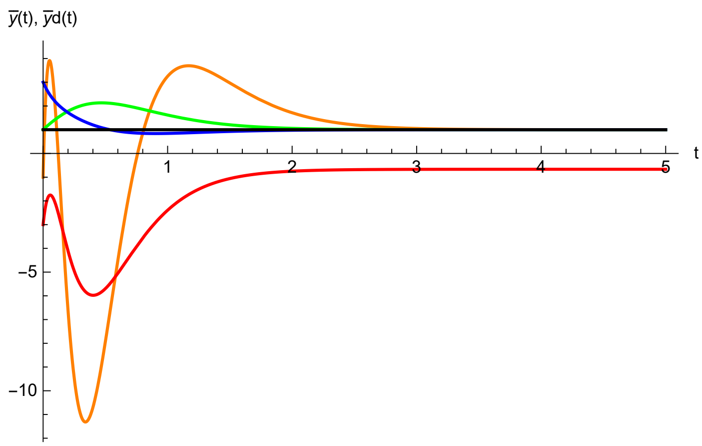


Even though initial guess is incorrect, estimation error yields to zero in time .

Let’s see the output states:

```
In[]:= Plot[Evaluate[{y[t] /. RedObsFdbkResponse, y1d[t] /. DesOut}],
  {t, 0, tmax}, AxesLabel → {"t", "ȳ(t), ȳd(t)"},
  PlotRange → All, PlotStyle → {Orange, Green, Red, Blue , Black}]
```

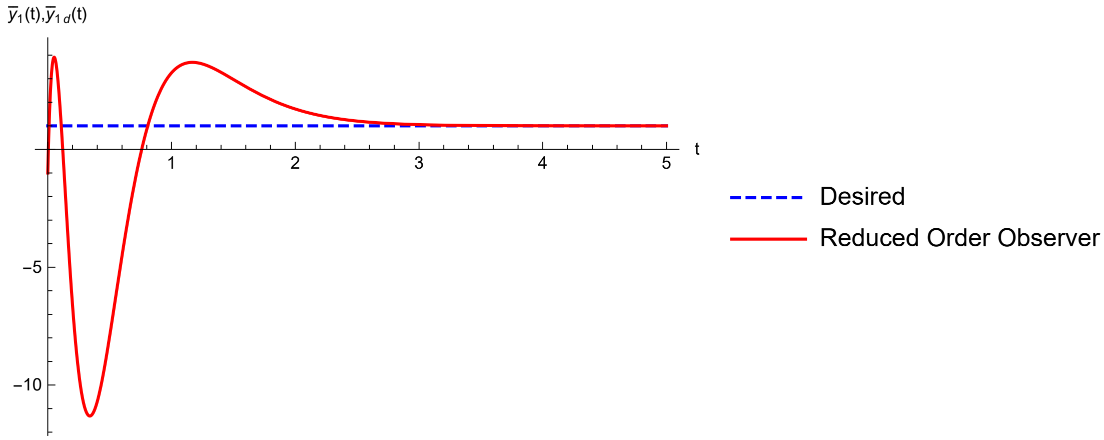
Out[]=



Except for one output, it is possible to reach desired output .

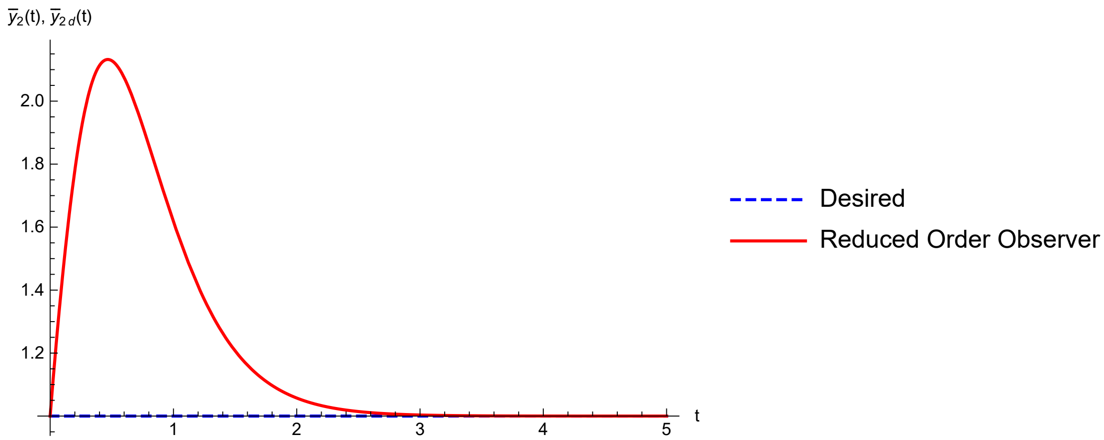
```
In[]:= legend = {"Desired", "Reduced Order Observer"};
style = {{Dashed, Blue}, Red} ;
Plot[{y1d[t] /. DesOut, Evaluate[y[t][[1]] /. RedObsFdbkResponse]},
  {t, 0, tmax}, AxesLabel → {"t", "ȳ1(t),ȳ1d(t)"},
  PlotRange → All, PlotStyle → style , PlotLegends → legend]
Plot[{y2d[t] /. DesOut, Evaluate[y[t][[2]] /. RedObsFdbkResponse]},
  {t, 0, tmax}, AxesLabel → {"t", "ȳ2(t), ȳ2d(t)"},
  PlotRange → All, PlotStyle → style , PlotLegends → legend]
Plot[{y3d[t] /. DesOut, Evaluate[y[t][[3]] /. RedObsFdbkResponse]},
  {t, 0, tmax}, AxesLabel → {"t", "ȳ3(t), ȳ3d(t)"},
  PlotRange → All, PlotStyle → style , PlotLegends → legend]

Plot[{y4d[t] /. DesOut, Evaluate[y[t][[4]] /. RedObsFdbkResponse]},
  {t, 0, tmax}, AxesLabel → {"t", "ȳ4(t), ȳ4d(t)"},
  PlotRange → All, PlotStyle → style , PlotLegends → legend]
```

Out[∘]=



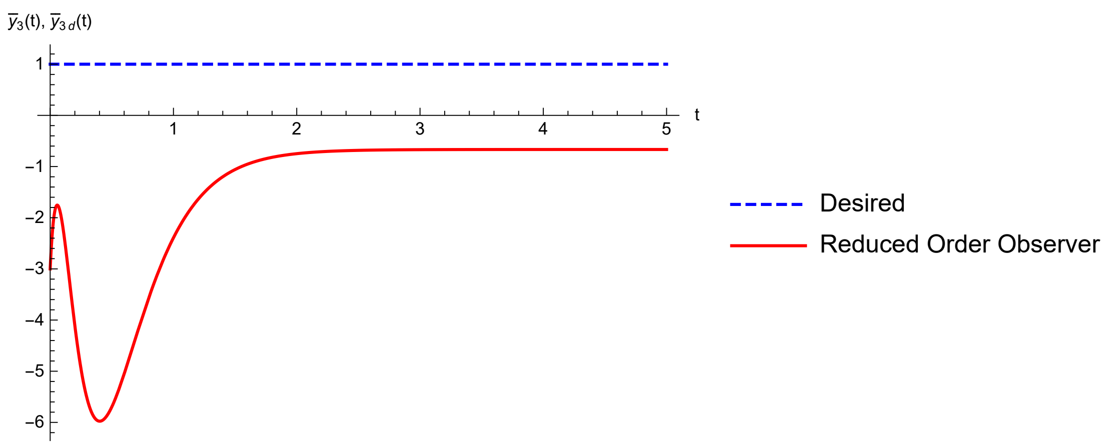
$\overline{y}_1$(t),$\overline{y}_{1d}$(t)
t
Desired
Reduced Order Observer

Out[∘]=



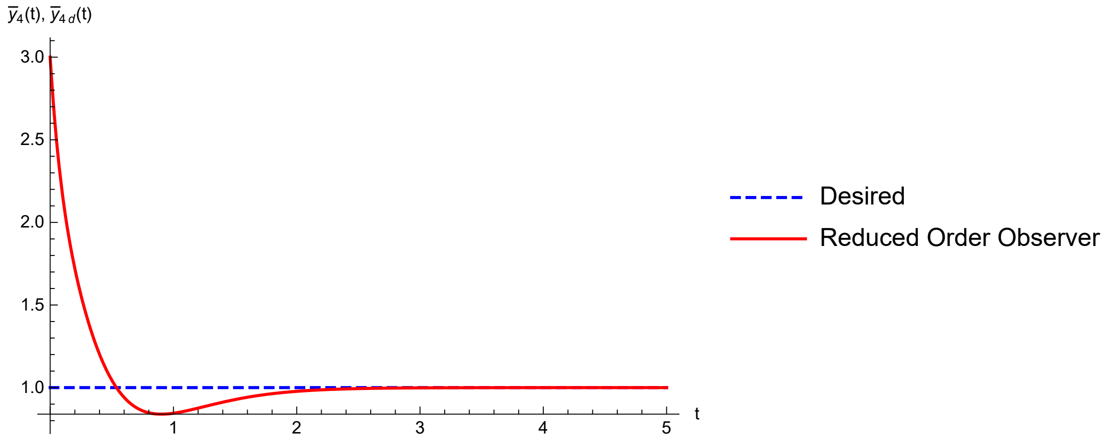
$\overline{y}_2$(t), $\overline{y}_{2d}$(t)
t
Desired
Reduced Order Observer

Out[∘]=



$\overline{y}_3$(t), $\overline{y}_{3d}$(t)
t
Desired
Reduced Order Observer

Out[ ]=



Same with the previous plot but one by one.

Steady state error is zero except for 1 output . Output state 3 has - 2 steady state error .

Controller - Observer shows transient response . This can be reduced with an optimal controller in which gain change over time .