Homework 6

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a)

$$x_1 = y_1$$
$$x_2 = y_2$$
$$x_3 = \dot{y}_2$$
$$x_4 = y_3$$
$$x_5 = \dot{y}_3$$

$$\dot{x}_1 + 3x_1 - x_2 = u_1 \Rightarrow \dot{x}_1 = u_1 - 3x_1 + x_2 \quad (1)$$
$$u_2 + 5u_1 = \dot{x}_3 + 2(\dot{x}_1 + x_3 - x_5) + 4(x_2 - x_1)$$
$$\dot{x}_5 + 6x_5 - 2\dot{x}_1 + y_3 = u_2$$

$$\dot{x}_3 = 10x_1 - 6x_2 - 2x_3 + 2x_5 + 3u_1 + u_2 \quad (2)$$

$$\dot{x}_5 = 2u_1 + u_2 - 6x_1 + 2x_2 - x_4 - 6x_5 \quad (3)$$

Then;

$$\dot{X} = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 10 & -6 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ -6 & 2 & 0 & -1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 3 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Given codes were used and edited for the homework. For Full state feedback, general procedure for multiple input system, repeated eigenvalue (k<=r), k=2 n = 5, r = 2 case should be used.

```
In[ ]:= Quit[]
```

b) Full State Feedback Design

In[∘]:=
$$A = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 10 & -6 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ -6 & 2 & 0 & -1 & -6 \end{pmatrix};\ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 3 & 1 \\ 0 & 0 \\ 2 & 1 \end{pmatrix};\ Cm = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix};\ Dm = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};$$

```
n = Dimensions[A][[1]]; r = Dimensions[B][[2]]; m = 3;
Eigenvalues[A] // N
```

Out[∘]= {-5.52097, -2.00948 + 1.62082 ⅈ, -2.00948 - 1.62082 ⅈ, -1.29176, -0.168301}

Find Controllability matrix :

In[∘]:=
```
P = Join[B, A.B, A.A.B, A.A.A.B, A.A.A.A.B, 2];
Print["P = ", MatrixForm[P]]
Print["Rank of P: ", MatrixRank[P]]
If[MatrixRank[P] == n, Print["System is controllable."],
 Print["System is uncontrollable."]]
```

$$P = \begin{pmatrix} 1 & 0 & -3 & 0 & 12 & 1 & -28 & -3 & -16 & -9 \\ 0 & 0 & 3 & 1 & 8 & 0 & -100 & -18 & 532 & 120 \\ 3 & 1 & 8 & 0 & -100 & -18 & 532 & 120 & -2380 & -606 \\ 0 & 0 & 2 & 1 & -18 & -6 & 130 & 37 & -818 & -222 \\ 2 & 1 & -18 & -6 & 130 & 37 & -818 & -222 & 4746 & 1277 \end{pmatrix}$$

Rank of P: 5

System is controllable.

Form X = [I λ - A ⋮ B] matrix and obtain the null space:

In[∘]:=
```
{λ1, λ2, λ3, λ4, λ5} = {-3., -4., -4., -5., -5.};
Imat = IdentityMatrix[n]; Clear[λ];
X = Join[Imat λ - A, B, 2]; Print["X = ", MatrixForm[X]]
U = NullSpace[X]ᵀ; Print["U = ", MatrixForm[U]]
```

$$X = \begin{pmatrix} 3+\lambda & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & \lambda & -1 & 0 & 0 & 0 & 0 \\ -10 & 6 & 2+\lambda & 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & \lambda & -1 & 0 & 0 \\ 6 & -2 & 0 & 1 & 6+\lambda & 2 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} -\frac{1+8\lambda+\lambda^2}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{(9+2\lambda+\lambda^2)(1+6\lambda+\lambda^2)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{3+25\lambda+11\lambda^2+\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{19+117\lambda+41\lambda^2+3\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{3\lambda+25\lambda^2+11\lambda^3+\lambda^4}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{\lambda(19+117\lambda+41\lambda^2+3\lambda^3)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{8+14\lambda+5\lambda^2+\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2\lambda(9+2\lambda+\lambda^2)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{8\lambda+14\lambda^2+5\lambda^3+\lambda^4}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2(9\lambda^2+2\lambda^3+\lambda^4)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Form Ψ and $\mathcal{F}$ matrices by partinioning U :

```
In[•]:= Ψ = Take[U, n]; Print["Ψ = ", MatrixForm[Ψ]]
        ℱ := Take[U, -r];
        Print["ℱ = ", MatrixForm[ℱ]]
```

$$\Psi = \begin{pmatrix} -\frac{1+8\lambda+\lambda^2}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{(9+2\lambda+\lambda^2)(1+6\lambda+\lambda^2)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{3+25\lambda+11\lambda^2+\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{19+117\lambda+41\lambda^2+3\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{3\lambda+25\lambda^2+11\lambda^3+\lambda^4}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{\lambda(19+117\lambda+41\lambda^2+3\lambda^3)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{8+14\lambda+5\lambda^2+\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2\lambda(9+2\lambda+\lambda^2)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{8\lambda+14\lambda^2+5\lambda^3+\lambda^4}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2(9\lambda^2+2\lambda^3+\lambda^4)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \end{pmatrix}$$

$$\mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Form composite matrices $\Omega = [\Psi(\lambda 1)\ \Psi(\lambda 2) \ldots \Psi(\lambda n)]$ and $\Lambda = [\mathcal{F}(\lambda 1)\ \mathcal{F}(\lambda 2) \ldots \mathcal{F}(\lambda n)]$:

```
In[•]:= Ω = Join[Ψ /. λ -> λ1, Ψ /. λ -> λ2, Ψ /. λ -> λ3, Ψ /. λ → λ4, Ψ /. λ → λ5, 2];
        Print["Ω = ", MatrixForm[Ω]]
        Λ = Join[ℱ /. λ -> λ1, ℱ /. λ -> λ2, ℱ /. λ -> λ3, ℱ /. λ -> λ4, ℱ /. λ → λ5, 2] // Simplify
        Λ // MatrixForm
```

$$\Omega = \begin{pmatrix} 0.318182 & 2.18182 & 0.144231 & 1.14423 & 0.144231 & 1.14423 & 0.12963 & 0.888889 & 0.1296 \\ 0. & 1. & -0.144231 & -0.144231 & -0.144231 & -0.144231 & -0.259259 & -0.777778 & -0.2592 \\ 0. & -3. & 0.576923 & 0.576923 & 0.576923 & 0.576923 & 1.2963 & 3.88889 & 1.296 \\ 0.363636 & 1.63636 & 0.307692 & 1.30769 & 0.307692 & 1.30769 & 0.574074 & 2.22222 & 0.5740 \\ -1.09091 & -4.90909 & -1.23077 & -5.23077 & -1.23077 & -5.23077 & -2.87037 & -11.1111 & -2.870 \end{pmatrix}$$

Out[•]= {{0, 1, 0, 1, 0, 1, 0, 1, 0, 1}, {1, 0, 1, 0, 1, 0, 1, 0, 1, 0}}

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Form the G and $\mathcal{J}$ matrices by selecting linearly independent columns, one for each eigenvalue:

```
In[•]:= ColumnChoice = {1, 8, 4, 3, 7};
        G = Ω[[All, ColumnChoice]]; Print["G = ", MatrixForm[G]]
        𝒥 = Λ[[All, ColumnChoice]]; Print["𝒥 = ", MatrixForm[𝒥]]
        Print["|G| = ", Det[G]]
```

$$G = \begin{pmatrix} 0.318182 & 0.888889 & 1.14423 & 0.144231 & 0.12963 \\ 0. & -0.777778 & -0.144231 & -0.144231 & -0.259259 \\ 0. & 3.88889 & 0.576923 & 0.576923 & 1.2963 \\ 0.363636 & 2.22222 & 1.30769 & 0.307692 & 0.574074 \\ -1.09091 & -11.1111 & -5.23077 & -1.23077 & -2.87037 \end{pmatrix}$$

$$\mathcal{J} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$|G| = 0.000849845$

Solve for K :

In[∘]:=
```
K = 𝒯.Inverse[G]; Print["K = ", MatrixForm[K]]
```

$$K = \begin{pmatrix} 5.32907\times10^{-15} & -7.04762 & -2.29524 & -3. & -1. \\ 10. & 35.1429 & 13.8857 & 9. & 5. \end{pmatrix}$$

Test closed-loop eigenvalues :

In[∘]:=
```
Eigenvalues[A - B.K]
```

Out[∘]=
```
{-5., -5., -4., -4., -3.}
```

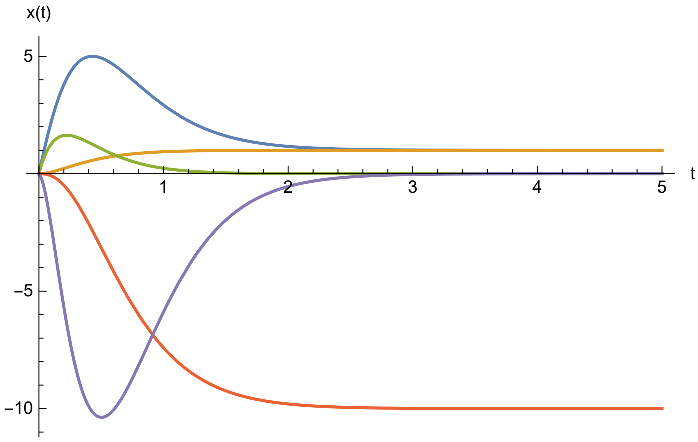
c) State Feedback Simulation

In[∘]:=
```
x[t_] := {x1[t], x2[t], x3[t], x4[t], x5[t]}
u[t_] := {u1[t], u2[t]}
y[t_] := Cm.x[t]
v[t_] := {v1[t], v2[t]};


Cr = Take[Cm, r];
F = Inverse[-Cr.Inverse[A - B.K].B];


EqStateFdbk = Thread[x'[t] == (A - B.K).x[t] + B.F.v[t]] // Chop;
(* We'll use this one to plot. *)
TableForm[EqStateFdbk]
IC = {x1[0] == 0, x2[0] == 0, x3[0] == 0, x4[0] == 0, x5[0] == 0};
```

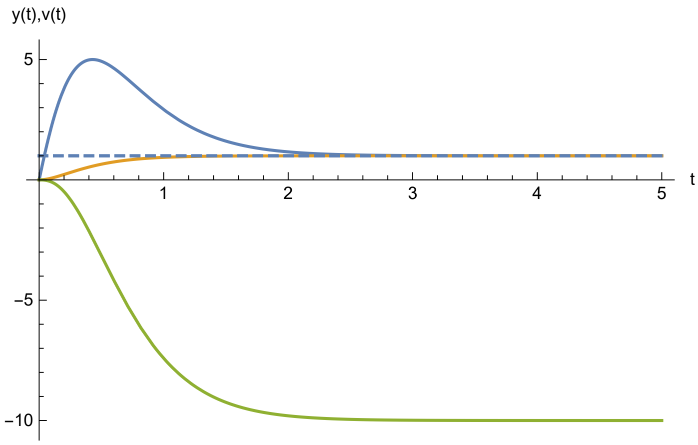
Out[∘]//TableForm=
```
x1′[t] == 60. v1[t] - 35.0476 v2[t] - 3. x1[t] + 8.04762 x2[t] + 2.29524 x3[t] + 3. x4[t] + 1. x5[t
x2′[t] == 1. x3[t]
x3′[t] == 20. v2[t] - 20. x2[t] - 9. x3[t]
x4′[t] == 1. x5[t]
x5′[t] == -60. v1[t] + 55.0476 v2[t] - 16. x1[t] - 19.0476 x2[t] - 9.29524 x3[t] - 4. x4[t] - 9. x5
```

In[∘]:=
```
tmax = 5;
DesInput = {v1[t] → UnitStep[t], v2[t] → UnitStep[t]};
SFResp = NDSolve[{EqStateFdbk /. DesInput, IC}, x[t], {t, 0, tmax}];
Plot[Evaluate[x[t] /. SFResp], {t, 0, tmax},
 AxesLabel → {"t", "x(t)"}, PlotRange → All]
yo = Plot[Evaluate[y[t] /. SFResp], {t, 0, tmax},
   AxesLabel → {"t", "y(t),v(t)"}, PlotRange → All];
yd = Plot[{v1[t], v2[t]} /. DesInput, {t, 0, tmax}, PlotStyle → Dashed, PlotRange → All];
Show[yo, yd]
```

Out[∘]=



Out[∘]=




```
Quit[]
```

d) Output State Feedback

Algorithm for General Procedure multiple input system should be used.

n = 5, r=2, k = 2, D=0

System is controllable (showed in the first part)

```
In[•]:= X = Join[Imat λ - A, B, 2]; MatrixForm[X];
U = NullSpace[X]ᵀ; MatrixForm[U];
Ψ = Take[U, n]; MatrixForm[Ψ];
ℱp = Take[U, -r]; MatrixForm[ℱp];
Ω = Join[Ψ /. λ -> λ1, Ψ /. λ -> λ2, Ψ /. λ -> λ3, Ψ /. λ -> λ4, Ψ /. λ -> λ5, 2];
MatrixForm[Ω] // N;
Ωp = Cm.Ω; Print["Ω'=", MatrixForm[Ωp] // N]
Λp = Join[ℱp /. λ -> λ1, ℱp /. λ -> λ2, ℱp /. λ -> λ3, ℱp /. λ -> λ4, ℱp /. λ -> λ5, 2];
Print["Λ'=", MatrixForm[Λp]]
```

$$\Omega' = \begin{pmatrix} 0.318182 & 2.18182 & 0.144231 & 1.14423 & 0.144231 & 1.14423 & 0.12963 & 0.888889 & 0.12963 \\ 0. & 1. & -0.144231 & -0.144231 & -0.144231 & -0.144231 & -0.259259 & -0.777778 & -0.259259 \\ 0.363636 & 1.63636 & 0.307692 & 1.30769 & 0.307692 & 1.30769 & 0.574074 & 2.22222 & 0.574074 \end{pmatrix}$$

$$\Lambda' = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

```
In[•]:= ColumnChoice2 = {1, 4, 7};
Gp = Ωp[[All, ColumnChoice2]];
MatrixForm[Gp] // N
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0.318182 & 1.14423 & 0.12963 \\ 0. & -0.144231 & -0.259259 \\ 0.363636 & 1.30769 & 0.574074 \end{pmatrix}$$

```
In[•]:= 𝒥p = Λp[[All, ColumnChoice2]]; MatrixForm[𝒥p]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

```
In[•]:= Det[Gp] // N
```

Out[•]= -0.0195464

```
In[•]:= Kstar = 𝒥p.Inverse[Gp] // N; Kstar // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 4.82319 & -6.93333 & -4.22029 \\ -15.7921 & 24.9333 & 16.5681 \end{pmatrix}$$

```
In[•]:= Eigenvalues[A - B.Kstar.Cm] // N
```

Out[•]= {-5., -4., -3.55052, -3., -0.272669}

Only 3 of the desired eigenvalues can be obtained as expected.

d) Output feedback simulation:

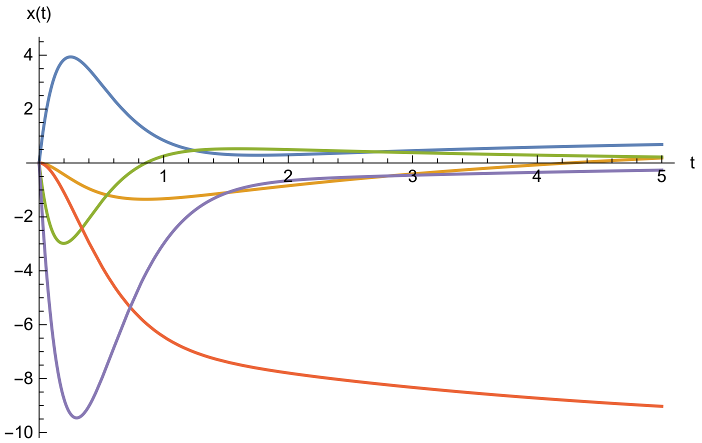
```
In[•]:= Imm = IdentityMatrix[m];
Imr = IdentityMatrix[r];
x[t_] := {x1[t], x2[t], x3[t], x4[t], x5[t]};
Fp = Inverse[-Cr.Inverse[A - B.Kstar.Cm].B];
v[t_] := {v1[t], v2[t]};
```

In[∘]:=
```
EqOutFdbk = Thread[x'[t] == (A - B.Kstar.Cm).x[t] + (B.Fp).v[t]] // Chop;
y[t_] := Cm.x[t]
TableForm[EqOutFdbk]
```

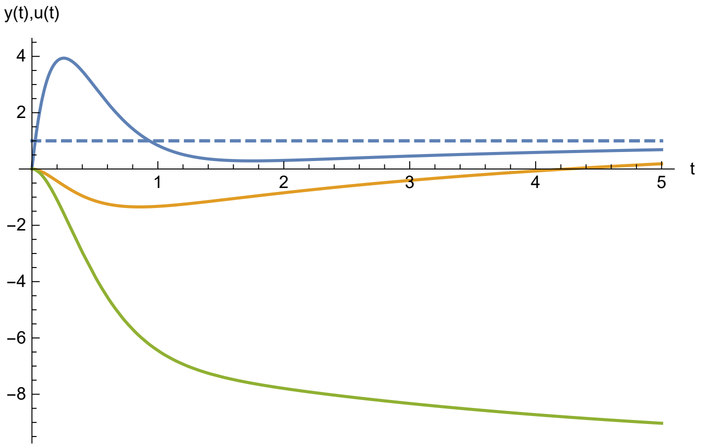
Out[∘]//TableForm=
```
x1′[t] == 88.0087 v1[t] - 45.9159 v2[t] - 7.82319 x1[t] + 7.93333 x2[t] + 4.22029 x4[t]
x2′[t] == 1. x3[t]
x3′[t] == -85.5602 v1[t] + 45.2986 v2[t] + 11.3226 x1[t] - 10.1333 x2[t] - 2. x3[t] - 3.90725 x4[
x4′[t] == 1. x5[t]
x5′[t] == -173.569 v1[t] + 91.2145 v2[t] + 0.145756 x1[t] - 9.06667 x2[t] - 9.12754 x4[t] - 6. x5
```

In[∘]:=
```
OFResp = NDSolve[{EqOutFdbk /. DesInput, IC}, x[t], {t, 0, tmax}];
Plot[Evaluate[x[t] /. OFResp], {t, 0, tmax}, AxesLabel → {"t", "x(t)"}, PlotRange → All]
yo = Plot[Evaluate[y[t] /. OFResp],
   {t, 0, tmax}, AxesLabel → {"t", "y(t),u(t)"}, PlotRange → All];
Show[
 yo,
 yd]
```

Out[∘]=



Out[∘]=



It is not as efficient as full state feedback since unplaced eigenvalues is slower than the desired ones. If I select different columns I can get:

In[•]:=
```
ColumnChoice2 = {1, 4, 10};
Gp = Ωp[[All, ColumnChoice2]];
MatrixForm[Gp] // N
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0.318182 & 1.14423 & 0.888889 \\ 0. & -0.144231 & -0.777778 \\ 0.363636 & 1.30769 & 2.22222 \end{pmatrix}$$

In[•]:=
```
𝒥p = Λp[[All, ColumnChoice2]]; MatrixForm[𝒥p]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

In[•]:=
```
Det[Gp] // N
```

Out[•]= -0.0553613

In[•]:=
```
Kstar = 𝒥p.Inverse[Gp] // N; Kstar // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 4.1614 & -6.93333 & -3.64123 \\ -12.5825 & 24.9333 & 13.7596 \end{pmatrix}$$

In[•]:=
```
Eigenvalues[A - B.Kstar.Cm] // N
```

Out[•]= {-5., -4., -3., -2.31125, -0.85015}

In[•]:=
```
Imm = IdentityMatrix[m];
Imr = IdentityMatrix[r];
x[t_] := {x1[t], x2[t], x3[t], x4[t], x5[t]};
Fp = Inverse[-Cr.Inverse[A - B.Kstar.Cm].B];
v[t_] := {v1[t], v2[t]};
```

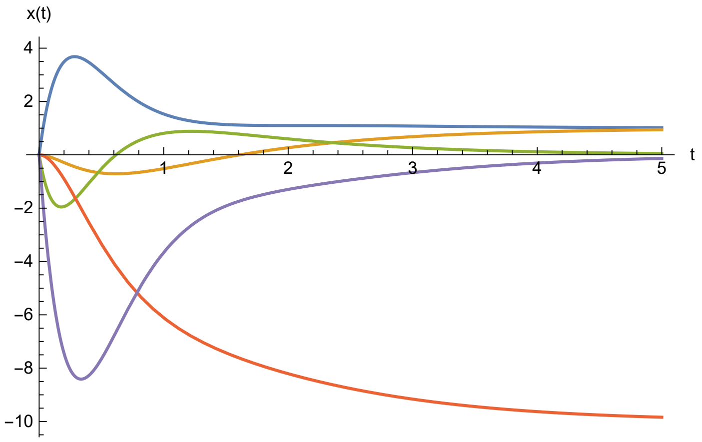
In[•]:=
```
EqOutFdbk = Thread[x'[t] == (A - B.Kstar.Cm).x[t] + (B.Fp).v[t]] // Chop;
y[t_] := Cm.x[t]
TableForm[EqOutFdbk]
```

Out[•]//TableForm=

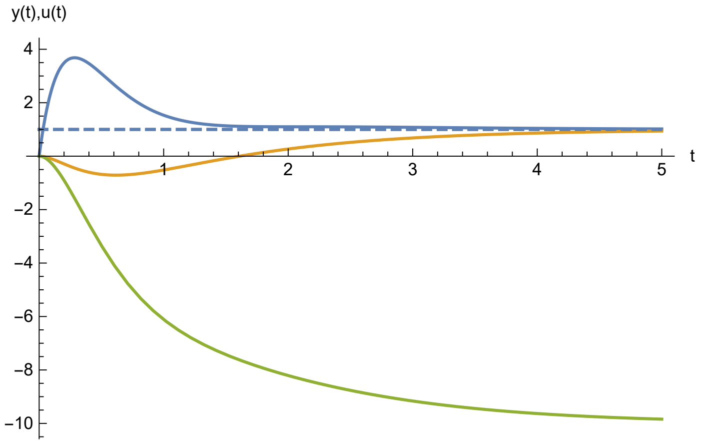
```
x1′[t] == 76.3447 v1[t] - 40.7044 v2[t] - 7.1614 x1[t] + 7.93333 x2[t] + 3.64123 x4[t]
x2′[t] == 1. x3[t]
x3′[t] == -63.9816 v1[t] + 35.657 v2[t] + 10.0982 x1[t] - 10.1333 x2[t] - 2. x3[t] - 2.83596 x4[t
x4′[t] == 1. x5[t]
x5′[t] == -140.326 v1[t] + 76.3614 v2[t] - 1.74035 x1[t] - 9.06667 x2[t] - 7.47719 x4[t] - 6. x5[
```

In[∘]:=
```
OFResp = NDSolve[{EqOutFdbk /. DesInput, IC}, x[t], {t, 0, tmax}];
Plot[Evaluate[x[t] /. OFResp], {t, 0, tmax}, AxesLabel → {"t", "x(t)"}, PlotRange → All]
yo = Plot[Evaluate[y[t] /. OFResp],
   {t, 0, tmax}, AxesLabel → {"t", "y(t),u(t)"}, PlotRange → All];
Show[
 yo,
 yd]
```

Out[∘]=



Out[∘]=



This one gives a better result because of faster eigenvalues. Some column selection may have cause unstability because of positive eigenvectors.

f) Full State Observer

In[∘]:=
```
Q = Join[Cmᵀ, Aᵀ.Cmᵀ, Aᵀ.Aᵀ.Cmᵀ, 2];
Print["Q = ", MatrixForm[Q]]
If[MatrixRank[Q] == n, Print["System is observable."],
 Print["System is uncontrollable."]]
```

$$Q = \begin{pmatrix} 1 & 0 & 0 & -3 & 0 & 0 & 9 & 10 & -6 \\ 0 & 1 & 0 & 1 & 0 & 0 & -3 & -6 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & -6 \end{pmatrix}$$

```
System is observable.
```

In[∘]:=
```
{λo1, λo2, λo3, λo4, λo5} = {-10, -12, -14, -16, -20};
Xo = Join[Imat λ - Aᵀ, Cmᵀ, 2]; MatrixForm[Xo]
Uo = NullSpace[Xo]ᵀ // Simplify; MatrixForm[Uo]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 3+\lambda & 0 & -10 & 0 & 6 & 1 & 0 & 0 \\ -1 & \lambda & 6 & 0 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2+\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 6+\lambda & 0 & 0 & 0 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{2\left(8+6\lambda+3\lambda^2\right)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2\left(5+24\lambda+5\lambda^2\right)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{6+34\lambda+19\lambda^2+8\lambda^3+\lambda^4}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{2\lambda(2+\lambda)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{(3+\lambda)\left(2+13\lambda+8\lambda^2+\lambda^3\right)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2+13\lambda+8\lambda^2+\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{2\lambda}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{(3+\lambda)\left(1+6\lambda+\lambda^2\right)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{1+6\lambda+\lambda^2}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{48+76\lambda+42\lambda^2+11\lambda^3+\lambda^4}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & \frac{2(3+\lambda)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & \frac{2}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{8+12\lambda+5\lambda^2+\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2\lambda(3+\lambda)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2\lambda}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In[∘]:=
```
Ψo = Take[Uo, n]; MatrixForm[Ψo]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{2\left(8+6\lambda+3\lambda^2\right)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2\left(5+24\lambda+5\lambda^2\right)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{6+34\lambda+19\lambda^2+8\lambda^3+\lambda^4}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{2\lambda(2+\lambda)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{(3+\lambda)\left(2+13\lambda+8\lambda^2+\lambda^3\right)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2+13\lambda+8\lambda^2+\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{2\lambda}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{(3+\lambda)\left(1+6\lambda+\lambda^2\right)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{1+6\lambda+\lambda^2}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{48+76\lambda+42\lambda^2+11\lambda^3+\lambda^4}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & \frac{2(3+\lambda)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & \frac{2}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \\ -\frac{8+12\lambda+5\lambda^2+\lambda^3}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2\lambda(3+\lambda)}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} & -\frac{2\lambda}{8+60\lambda+81\lambda^2+43\lambda^3+11\lambda^4+\lambda^5} \end{pmatrix}$$

In[∘]:=
```
ℱo := Take[Uo, -m]; MatrixForm[ℱo]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In[∘]:=
```
Ωo = Join[Ψo /. λ -> λo1, Ψo /. λ -> λo2, Ψo /. λ -> λo3, Ψo /. λ -> λo4, Ψo /. λ -> λo5, 2];
Print["Ωo=", MatrixForm[Ωo] // N]
```

$$\Omega o = \begin{pmatrix} -0.0194571 & 0.0207908 & 0.139887 & -0.00875274 & 0.0103939 & 0.109956 & -0.00469303 \\ 0.00627648 & 0.0900675 & -0.0128668 & 0.00285415 & 0.0781324 & -0.00868138 & 0.0015399 \\ -0.00078456 & -0.0112584 & 0.00160835 & -0.000285415 & -0.00781324 & 0.000868138 & -0.00012832 \\ 0.0975992 & 0.000549192 & -0.000078456 & 0.0821996 & 0.000214061 & -0.0000237846 & 0.0707987 \\ -0.0240075 & 0.00549192 & -0.00078456 & -0.0136048 & 0.00256874 & -0.000285415 & -0.00881776 \end{pmatrix}$$

In[∘]:=
```
Λo = Join[ℱo /. λ -> λo1, ℱo /. λ -> λo2, ℱo /. λ -> λo3, ℱo /. λ -> λo4, ℱo /. λ -> λo5, 2];
Print["Λo=", MatrixForm[Λo]]
```

$$\Lambda o = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In[•]:=
```
ColumnChoiceo = {1, 5, 9, 14, 10};
Go = Ωo[[All, ColumnChoiceo]]; MatrixForm[Go] // N
𝒥o = Λo[[All, ColumnChoiceo]]; MatrixForm[𝒥o]
Det[Go] // N
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -0.0194571 & 0.0103939 & 0.0903683 & 0.00174008 & -0.00280978 \\ 0.00627648 & 0.0781324 & -0.0062146 & 0.0490566 & 0.000925574 \\ -0.00078456 & -0.00781324 & 0.000517883 & -0.00272537 & -0.0000661124 \\ 0.0975992 & 0.000214061 & -9.16607\times 10^{-6} & 0.0000193976 & 0.0621126 \\ -0.0240075 & 0.00256874 & -0.000128325 & 0.000387952 & -0.00619804 \end{pmatrix}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[•]= 1.39926×10^{-8}

In[•]:=
```
L = (𝒥o.Inverse[Go] // Chop)ᵀ;
MatrixForm[L] // FullSimplify // N
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 10.9953 & 1.06857 & -0.00821362 \\ 0.664991 & 29.976 & 0.965242 \\ 17.2498 & 173.43 & 23.1023 \\ -0.710525 & -0.140849 & 20.0287 \\ -12.1896 & 0.730585 & 39.2742 \end{pmatrix}$$

In[•]:=
```
Ac = A - L.Cm; MatrixForm[Ac] // Chop // N
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -13.9953 & -0.0685687 & 0. & 0.00821362 & 0. \\ -0.664991 & -29.976 & 1. & -0.965242 & 0. \\ -7.24984 & -179.43 & -2. & -23.1023 & 2. \\ 0.710525 & 0.140849 & 0. & -20.0287 & 1. \\ 6.18965 & 1.26941 & 0. & -40.2742 & -6. \end{pmatrix}$$

In[•]:=
```
Eigenvalues[A - L.Cm]
```

Out[•]= {-20, -16, -14, -12, -10}

g) Full State Observer Simulation

In[∘]:=
```
xo[t_] := {xo1[t], xo2[t], xo3[t], xo4[t], xo5[t]};
u[t_] := {u1[t], u2[t]};
EqObserver = Thread[xo'[t] == Ac.xo[t] + L.y[t] + B.u[t]] // Chop  // N
```

Out[∘]=
```
{xo1′[t] == u1[t] + 10.9953 x1[t] + 1.06857 x2[t] -
   0.00821362 x4[t] - 13.9953 xo1[t] - 0.0685687 xo2[t] + 0.00821362 xo4[t],
 xo2′[t] == 0.664991 x1[t] + 29.976 x2[t] + 0.965242 x4[t] -
   0.664991 xo1[t] - 29.976 xo2[t] + xo3[t] - 0.965242 xo4[t],
 xo3′[t] == 3. u1[t] + u2[t] + 17.2498 x1[t] + 173.43 x2[t] + 23.1023 x4[t] -
   7.24984 xo1[t] - 179.43 xo2[t] - 2. xo3[t] - 23.1023 xo4[t] + 2. xo5[t],
 xo4′[t] == -0.710525 x1[t] - 0.140849 x2[t] + 20.0287 x4[t] +
   0.710525 xo1[t] + 0.140849 xo2[t] - 20.0287 xo4[t] + xo5[t],
 xo5′[t] == 2. u1[t] + u2[t] - 12.1896 x1[t] + 0.730585 x2[t] + 39.2742 x4[t] +
   6.18965 xo1[t] + 1.26941 xo2[t] - 40.2742 xo4[t] - 6. xo5[t]}
```

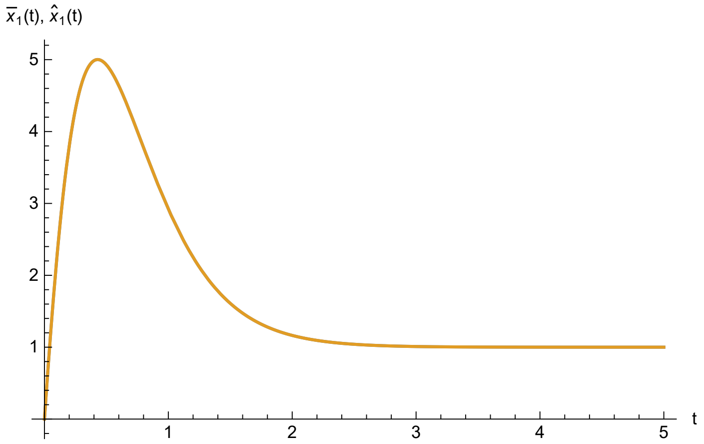
In[∘]:=
```
EqFullObsFdbk = Thread[x'[t] == A.x[t] - B.K.xo[t] + B.F.v[t]];
u[t_] := F.v[t] - K.xo[t]
EqObserver2 = Thread[xo'[t] == Ac.xo[t] + L.y[t] + B.u[t]];
y[t_] := Cm.x[t]
TableForm[{EqFullObsFdbk, EqObserver2} // Flatten // N]
```

Out[∘]//TableForm=
```
x1′[t] == 60. v1[t] - 35.0476 v2[t] - 3. x1[t] + x2[t] - 5.32907 × 10^-15 xo1[t] + 7.04762 xo2[t] + 2
x2′[t] == 0. + x3[t]
x3′[t] == -1.42109 × 10^-12 v1[t] + 20. v2[t] + 10. x1[t] - 6. x2[t] - 2. x3[t] + 2. x5[t] - 10. xo1[t
x4′[t] == 0. + x5[t]
x5′[t] == -60. v1[t] + 55.0476 v2[t] - 6. x1[t] + 2. x2[t] - 1. x4[t] - 6. x5[t] - 10. xo1[t] - 21.6
xo1′[t] == 60. v1[t] - 35.0476 v2[t] + 10.9953 x1[t] + 1.06857 x2[t] - 0.00821362 x4[t] - 13.9953
xo2′[t] == 0.664991 x1[t] + 29.976 x2[t] + 0.965242 x4[t] - 0.664991 xo1[t] - 29.976 xo2[t] + xo3
xo3′[t] == -180. v1[t] + 125.143 v2[t] + 17.2498 x1[t] + 173.43 x2[t] + 23.1023 x4[t] - 17.2498 x
xo4′[t] == -0.710525 x1[t] - 0.140849 x2[t] + 20.0287 x4[t] + 0.710525 xo1[t] + 0.140849 xo2[t]
xo5′[t] == -180. v1[t] + 125.143 v2[t] - 12.1896 x1[t] + 0.730585 x2[t] + 39.2742 x4[t] - 3.81035
```

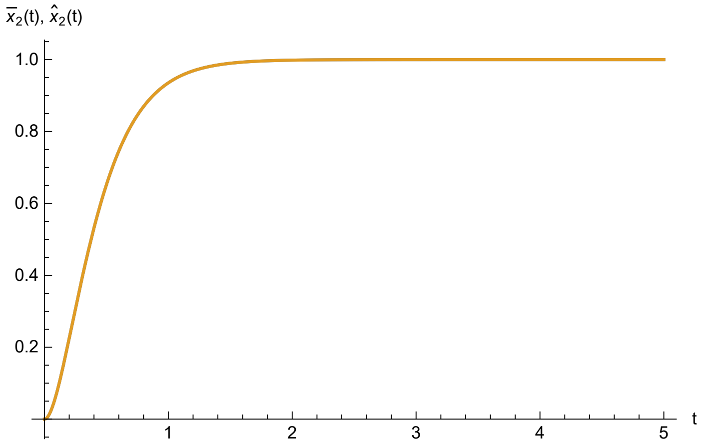
In[∘]:=
```
ICo = {xo1[0] == 0, xo2[0] == 0, xo3[0] == 0, xo4[0] == 0, xo5[0] == 0};
FullObsFdbkResponse =
  NDSolve[{EqFullObsFdbk /. DesInput, IC , EqObserver2 /. DesInput, ICo},
   {x[t], xo[t]} // Flatten, {t, 0, tmax}];
```

In[ ]:=
```
Plot[Evaluate[{x1[t], xo1[t]} /. FullObsFdbkResponse],
 {t, 0, tmax}, AxesLabel → {"t", "x̄1(t), x̂1(t)"}, PlotRange → All]
Plot[Evaluate[{x2[t], xo2[t]} /. FullObsFdbkResponse], {t, 0, tmax},
 AxesLabel → {"t", "x̄2(t), x̂2(t)"}, PlotRange → All]
Plot[Evaluate[{x3[t], xo3[t]} /. FullObsFdbkResponse], {t, 0, tmax},
 AxesLabel → {"t", "x̄3(t), x̂3(t)"}, PlotRange → All]
Plot[Evaluate[{x4[t], xo4[t]} /. FullObsFdbkResponse], {t, 0, tmax},
 AxesLabel → {"t", "x̄4(t), x̂4(t)"}, PlotRange → All]
Plot[Evaluate[{x5[t], xo5[t]} /. FullObsFdbkResponse], {t, 0, tmax},
 AxesLabel → {"t", "x̄5(t), x̂5(t)"}, PlotRange → All]
yo = Plot[Evaluate[y[t] /. FullObsFdbkResponse], {t, 0, tmax},
   AxesLabel → {"t", "ȳ(t), v̄(t)"}, PlotRange → All];
Show[
 yo,
 yd]
```

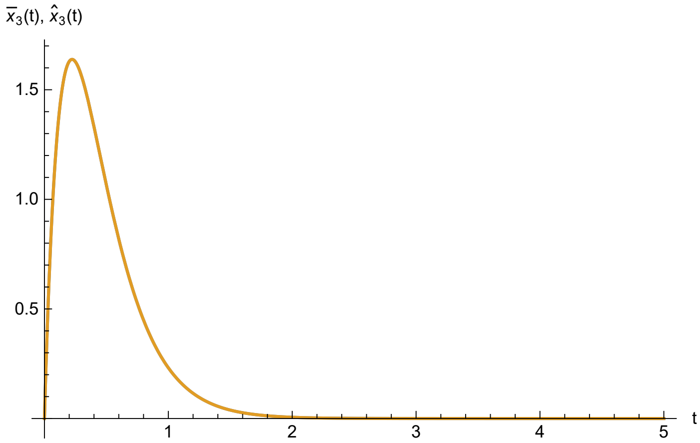
Out[ ]=



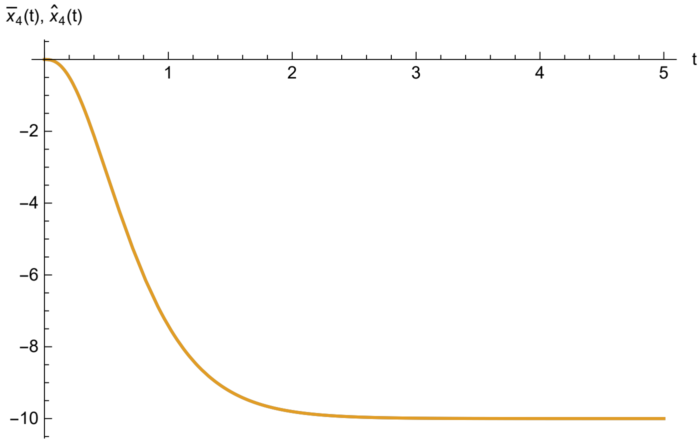
Out[ ]=



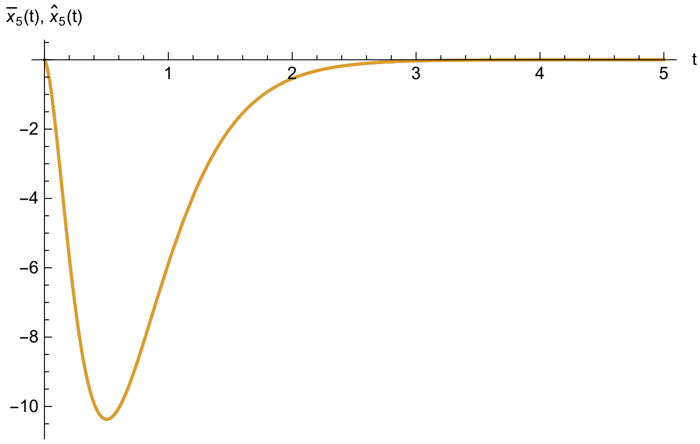
Out[•]=



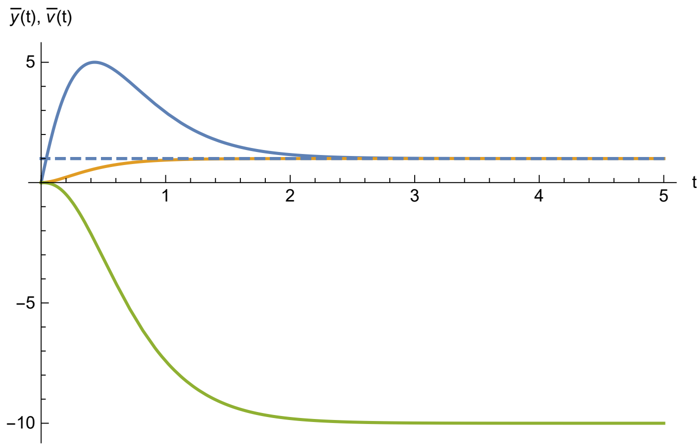
Out[•]=



Out[•]=



Out[•]=



h) Reduced Order Observer Design:

We can measure $m = 3$ states. $n - m$ states that should be estimated :

In[∘]:=
```
{λr1, λr2} = {λo4, λo5} (*-20 and -16 were selected *)
```

Out[∘]=
```
{-16, -20}
```

Partition :

In[∘]:=
```
Print["x = ", MatrixForm[x[t]]]
xs1[t_] := x[t][[{1, 2, 3}]];
xs2[t_] := x[t][[{4, 5}]];
Print["Xs1 = ", MatrixForm[xs1[t]], "\t xs2 = ", MatrixForm[xs2[t]]]
Print["A = ", MatrixForm[A]]
```

$$x = \begin{pmatrix} x1[t] \\ x2[t] \\ x3[t] \\ x4[t] \\ x5[t] \end{pmatrix}$$

$$Xs1 = \begin{pmatrix} x1[t] \\ x2[t] \\ x3[t] \end{pmatrix} \qquad xs2 = \begin{pmatrix} x4[t] \\ x5[t] \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 10 & -6 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ -6 & 2 & 0 & -1 & -6 \end{pmatrix}$$

In[∘]:=
```
A11 = Take[A, m, m]; MatrixForm[A11]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -3 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & -6 & -2 \end{pmatrix}$$

In[∘]:=
```
A12 = Take[A, m, {m + 1, n}]; MatrixForm[A12]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}$$

In[∘]:=
```
A21 = Take[A, {m + 1, n}, m]; MatrixForm[A21]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ -6 & 2 & 0 \end{pmatrix}$$

In[∘]:=
```
A22 = Take[A, {m + 1, n}, {m + 1, n}]; MatrixForm[A22]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix}$$

In[∘]:=
```
B1 = Take[B, m]; MatrixForm[B1]
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 3 & 1 \end{pmatrix}$$

In[∘]:=
```
B2 = B[[{4, 5}, {1, 2}]]; MatrixForm[B2]
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$$

In[∘]:=
```
Ir = IdentityMatrix[n - m];
Xr = Join[Ir λ - A22ᵀ, A12ᵀ, 2]; MatrixForm[Xr];
Ur = NullSpace[Xr]ᵀ // Simplify;
MatrixForm[Ur];
Ψr = Take[Ur, n - m]; MatrixForm[Ψr];
ℱr := Take[Ur, -m]; MatrixForm[ℱr];
Ωr = Join[Ψr /. λ -> λr1, Ψr /. λ -> λr2, 2];
Print["Ωr =", MatrixForm[Ωr]]
Λr = Join[ℱr /. λ -> λr1, ℱr /. λ -> λr2, 2];
Print["Λr =", MatrixForm[Λr]]
```

$$\Omega r = \begin{pmatrix} \frac{2}{161} & 0 & 0 & \frac{2}{281} & 0 & 0 \\ \frac{32}{161} & 0 & 0 & \frac{40}{281} & 0 & 0 \end{pmatrix}$$

$$\Lambda r = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In[∘]:=
```
ColumnChoice = {1, 4};
Gr = Ωr[[All, ColumnChoice]]; MatrixForm[Gr]
𝒥r = Λr[[All, ColumnChoice]]; MatrixForm[𝒥r]
Det[Gr] // N
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{2}{161} & \frac{2}{281} \\ \frac{32}{161} & \frac{40}{281} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Out[∘]= 0.000353662

In[∘]:=
```
Lr = Transpose[𝒥r.Inverse[Gr]];
MatrixForm[Lr]
```
Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -\frac{319}{2} \\ 0 & 0 & 15 \end{pmatrix}$$

In[∘]:=
```
Ar = A22 - Lr.A12; MatrixForm[Ar];
Eigenvalues[Ar]
```
Out[∘]= {-20, -16}

i) Reduced Order Observer Simulation:

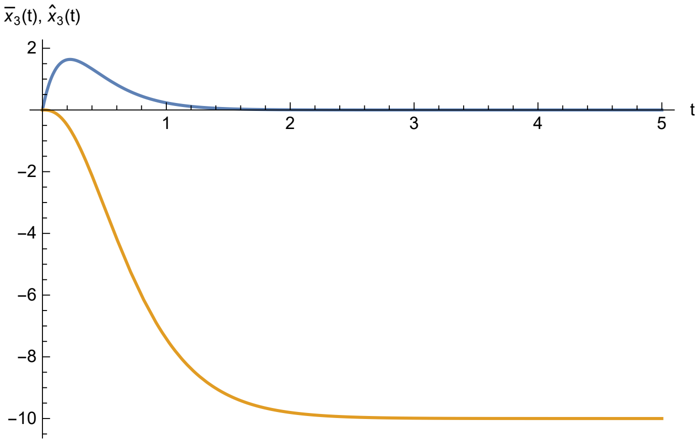
In[∘]:=
```
Clear[u]
xro[t_] := {xro3[t], xro5[t]}
xhat[t_] := {xs1[t], xro[t]} // Flatten
u[t_] := F.v[t] - K.xhat[t]
yr[t_] := xs1'[t] - A11.xs1[t] - B1.u[t]
zr[t_] := A21.xs1[t] + B2.u[t]

EqRedObsFdbk = Thread[x'[t] == A.x[t] - B.K.xhat[t] + B.F.v[t]];
u[t_] := F.v[t] - K.xhat[t]
EqRedObserver = Thread[xro'[t] == Ar.xro[t] + Lr.yr[t] + zr[t]];
ICro = {xro3[0] == 0, xro5[0] == 0};
y[t_] := Cm.x[t]
TableForm[{EqRedObsFdbk, EqRedObserver} // Flatten]
```

Out[∘]//TableForm=

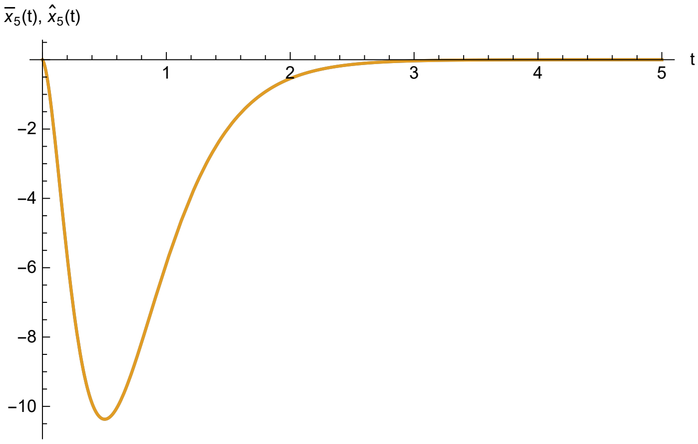
```
x1′[t] == 60. v1[t] - 35.0476 v2[t] - 3. x1[t] + 8.04762 x2[t] + 2.29524 x3[t] + 3. xro3[t] + 1. xr
x2′[t] == 0. + x3[t]
x3′[t] == -1.42109×10^-12 v1[t] + 20. v2[t] + 1.33227×10^-13 x1[t] - 20. x2[t] - 9. x3[t] + 2 x5[t]
x4′[t] == 0. + x5[t]
x5′[t] == -60. v1[t] + 55.0476 v2[t] - 16. x1[t] - 19.0476 x2[t] - 9.29524 x3[t] - x4[t] - 6 x5[t]
xro3′[t] == 320 xro5[t] - 319/2 (180. v1[t] - 125.143 v2[t] - 1.49214×10^-13 x1[t] + 41.1429 x2[t] +
xro5′[t] == -180. v1[t] + 125.143 v2[t] - 16. x1[t] - 33.1429 x2[t] - 13.8857 x3[t] - 10. xro3[t]
```

In[∘]:=
```
RedObsFdbkResponse =
  NDSolve[{EqRedObsFdbk /. DesInput, IC , EqRedObserver /. DesInput, ICro},
   {x[t], xro3[t], xro5[t]} // Flatten, {t, 0, tmax}];

Plot[Evaluate[{x3[t], xro3[t]} /. RedObsFdbkResponse],
 {t, 0, tmax}, AxesLabel → {"t", "x̅3(t), x̂3(t)"}, PlotRange → All]
Plot[Evaluate[{x5[t], xro5[t]} /. RedObsFdbkResponse], {t, 0, tmax},
 AxesLabel → {"t", "x̅5(t), x̂5(t)"}, PlotRange → All]
yro = Plot[Evaluate[y[t] /. RedObsFdbkResponse], {t, 0, tmax},
   AxesLabel → {"t", "y̅(t), v̅(t)"}, PlotRange → All];
Show[
 yro,
 yd]
```

Out[∘]=



Out[∘]=



Out[∘]=