$f(\underline{y}) \geq f(\underline{x}) + \langle \nabla f(\underline{x}), \underline{y} - \underline{x} \rangle$

$f(\underline{y}) = f(\underline{x}) + \langle \nabla f(\underline{x}), \underline{y} - \underline{x} \rangle + \frac{1}{2} (\underline{y} - \underline{x})^T \nabla^2 f(\underline{x}) (\underline{y} - \underline{x}) + R(\underline{y})$

$\therefore \frac{1}{2} (\underline{y} - \underline{x})^T \nabla^2 f(\underline{x}) (\underline{y} - \underline{x}) + R(\underline{y}) \geq$

$g(t) = f(\underline{x} + t\underline{d}) \qquad \forall \underline{x}, \underline{d}$

$g''(t) = \qquad g(t) =$

$\underset{\geq 0}{?} = \frac{1}{2} (y - x)^T \ldots$

f is convex $\Leftrightarrow$ $f(\underline{x} + t\underline{d})$ is convex.

$\Leftrightarrow \underline{d}^T \nabla^2 f(\underline{x} + t\underline{d}) \underline{d} \geq 0,$

$f(\underline{x} + t\underline{d}) = f(\underline{x}) + t \langle \nabla f(\underline{x}), \underline{d} \rangle + \frac{1}{2} t^2 \underline{d}^T \nabla^2 f(\underline{x}) \underline{d} +$

$\forall \underline{x} \in C, \underline{d} \in \mathcal{D} \,?\, C.$

$\underline{x}^T \nabla^2 f(\underline{x}) \underline{x} \geq 0, \quad \forall \underline{x} \in C.$

$\mathcal{D} = \{ \underline{d} : \exists t > 0 : \underline{x} + t\underline{d} \in C \}$

$$
\begin{aligned}
g(t) &= f(\underline{x} + t\underline{d}) \\
&= f(\underbrace{x_1 + td_1}_{u_1(t)}, \underbrace{x_2 + td_2}_{u_2(t)}, \ldots, \underbrace{x_n + td_n}_{u_n(t)})
\end{aligned}
$$

$\nabla f : \mathbb{R}^n \to \mathbb{R}.$

$\nabla f : \mathbb{R}^n \to \mathbb{R}^n.$

$g'(t) = \sum_{i=1}^n \frac{\partial f}{\partial u_i} \frac{du_i}{dt} = \sum_{i=1}^n \frac{\partial f}{\partial u_i} \cdot d_i = \langle \nabla f(\underline{x} + t\underline{d}), \underline{d} \rangle.$

say $\frac{\partial f}{\partial u_i} = h_i : \mathbb{R}^n \to \mathbb{R}.$

$\langle \nabla f(\underline{x} + t\underline{d}), \underline{d} \rangle = \sum_{i=1}^n \frac{\partial f}{\partial u_i}\Big|_{\underline{x}+t\underline{d}} \cdot d_i$

$$
\begin{aligned}
\therefore \quad & \left(\sum_{i=1}^n \frac{\partial f}{\partial u_i}\Big|_{\underline{x}+t\underline{d}} \cdot d_i \right)' \\
&= \left\{ \sum_{i=1}^n d_i \left(\frac{\partial f}{\partial u_i}\Big|_{x+t\underline{d}} \right)' \right\} = \left\{ \sum_{i=1}^n d_i \frac{d h_i}{dt} (\underline{x} + t\underline{d}) \right\} \\
&= \left[\sum_{i=1}^n d_i \sum_{j=1}^n \frac{\partial h_i}{\partial v_j} \frac{d v_j}{dt} \right] = \sum_{i=1}^n d_i \langle \nabla h_i (\underline{x} + t\underline{d}), \underline{d} \rangle \\
&= d_1 \langle \nabla h_1 (\underline{x} + t\underline{d}), \underline{d} \rangle + \cdots + d_n \langle \nabla h_n (\underline{x} + t\underline{d}), \underline{d} \rangle \\
&=
\end{aligned}
$$

$\underline{d}^T \nabla^2 f(\underline{x} + t\underline{d}) \underline{d}$

$=$

f is convex

$\Leftrightarrow \forall \underline{x} \in C, \underline{d} \in D = \{ \underline{d} : \exists t > 0 : \underline{x} + t\underline{d} \in C \}$

$g(t) = f(\underline{x} + t\underline{d})$ is convex

$\mathrm{dom}(g) = \{ t : \underline{x} + t\underline{d} \in C \}.$

$\Leftrightarrow \forall \underline{x} \in C : \forall \underline{d} \in D : \underline{d}^T \nabla^2 f(\underline{x} + t\underline{d}) \underline{d} \geq 0. \quad \boxed{C \,?\, \mathcal{D}}$

f is convex over $C \Leftrightarrow \forall \underline{x} \in C : \nabla^2 f(x) \succeq 0.$

$\Leftrightarrow \forall \underline{x} \in C : \underline{x}^T \nabla^2 f(\underline{x}) \underline{x} \geq 0.$

$g(t) = f(\underline{x} + t\underline{d})$ $\quad g(t) = f(\underbrace{x_1 + td_1}_{u_1}, \dots, \underbrace{x_n + td_n}_{u_n})$ $\quad f: \mathbb{R}^n \to \mathbb{R}$.

$\nabla f : \mathbb{R}^n \to \mathbb{R}^n \quad \frac{\partial f}{\partial u_i} : \mathbb{R}^n \to \mathbb{R}$

$\therefore g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \underbrace{\frac{du_i}{dt}}_{d_i} = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}(\underline{x} + t\underline{d}) d_i$

Let $h_i = \frac{\partial f}{\partial u_i}$

$\therefore g''(t) = (g'(t))' = \left[\sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial u_i}}_{h_i} (\underline{x} + t\underline{d}) d_i \right]' = \sum_{i=1}^{n} \underbrace{\frac{d h_i}{dt}}(\underline{x} + t\underline{d}) d_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial h_i}{\partial u_j}(\underline{x} + t\underline{d}) d_j d_i$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial u_j \partial u_i}(\underline{x} + t\underline{d}) d_j d_i$

$\frac{\partial f}{\partial_i} : \mathbb{R}^n \to \mathbb{R}$

$\underline{d}^T \nabla^2 f(\cdot) \underline{d}$

$\langle d_1 \cdots d_n \rangle \begin{pmatrix} & \end{pmatrix} \begin{pmatrix} \\ \end{pmatrix}$

$= \langle \quad \rangle \begin{pmatrix} \\ \end{pmatrix}$

$= \_$.

$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial u_1} \\ \vdots \\ \frac{\partial f}{\partial u_n} \end{pmatrix}$ $\quad \nabla f^T = \left(\frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_n}\right)$.

$\nabla^2 f = \left(\nabla\left(\frac{\partial f}{\partial u_1}\right), \dots, \nabla\left(\frac{\partial f}{\partial u_n}\right)\right)$

$= \begin{pmatrix} \frac{\partial^2 f}{\partial u_1^2} & & \\ \frac{\partial^2 f}{\partial u_2 \partial u_1} & \ddots & \\ \vdots & & \\ \frac{\partial^2 f}{\partial u_n \partial u_1} & & \end{pmatrix}$