# Ski Rental

Yanze Song

The LOG Reading Group @ University of Alberta

October 31, 2023

# Outline

# Outline

# Problem Setup

**Definition (*Ski Rental Problem*)**

- Rent skis: \$1 per day
- Buy skis: $\$B > 1$
- Total $T$ days of skiing (unknown)
- On each day $t$, a player will be told if she'll ski or not, and then she'll decide to buy or to rent.

Goal: Develop an algorithm that costs as little money as possible in all cases

# Problem Setup

Define the optimization variables as follows:

$$x = \begin{cases} 1, \text{ if buy skis} \\ 0, \text{ else} \end{cases} \qquad z_t = \begin{cases} 1, \text{ if rent on day } t \\ 0, \text{ else} \end{cases}$$

Then the offline problem can be formulated as follows:

$$\begin{aligned} \min_{x, z_i}: \quad & B \cdot x + \sum_{i=1}^{T} z_i \\ \text{s.t.:} \quad & x + z_i \geq 1, \ \forall\, i \\ & x \in \{0, 1\} \\ & z_i \in \{0, 1\}, \ \forall\, i \end{aligned}$$

# Outline

# Performance Metric

## Definition (*Competitive Ratio*)

- Compare with the offline optimal under all inputs
- A deterministic online algorithm is $\alpha$-competitive if

$$\alpha = \max_{\sigma} \frac{\mathsf{c}(\mathsf{Alg}, \sigma)}{\mathsf{c}(\mathsf{Opt}, \sigma)}$$

- A randomized online algorithm is $\alpha$-competitive if

$$\alpha = \max_{\sigma} \frac{\mathbb{E}[\mathsf{c}(\mathsf{ALG}, \sigma)]}{\mathsf{c}(\mathsf{Opt}, \sigma)}$$

# Outline

# Algorithms: Deterministic

**Algorithm ($Alg_B$)**

Rent up to day $B-1$, and then buy skis on day $B$

- $(2-\frac{1}{B})$-competitive

Two cases to consider:

- $T < B$: $\frac{c(Alg_B)}{c(Opt)} = \frac{T}{T} = 1$
- $T \geq B$: $\frac{c(Alg_B)}{c(Opt)} = \frac{2B-1}{B} = 2 - \frac{1}{B}$

$\alpha = \max_{\sigma} \frac{c(Alg,\sigma)}{c(Opt,\sigma)} = \max\{1, 2-\frac{1}{B}\} = 2 - \frac{1}{B}$

# Algorithms: Deterministic

**Theorem** (*Optimality of $Alg_B$*)

$2 - \frac{1}{B}$ is the optimal C.R. among all deterministic algorithms.

All deterministic algorithms have the form:

$$Alg_t\text{: Rent up to day } t-1\text{, and buy on day } t$$

Then two cases to consider:

- $t > B$: Let $T > t > B$, then $\frac{c(Alg_t)}{c(Opt)} = \frac{t-1+B}{B} \geq \frac{B+B}{B} > 2 - \frac{1}{B}$
- $t < B$: Let $t \leq T \leq \frac{t-1+B}{2-\frac{1}{B}} < B$, then $\frac{c(Alg_t)}{c(Opt)} = \frac{t-1+B}{T} \geq 2 - \frac{1}{B}$

# Algorithms: Randomized

A toy example ($B = 4$):

| C.R. | T=1 | T=2 | T=3 | T=4 | T=5 | T=6 | ... |
|---|---|---|---|---|---|---|---|
| $Alg_1$ | 4/1 | 4/2 | 4/3 | 4/4 | 4/4 | 4/4 | |
| $Alg_2$ | 1/1 | 5/2 | 5/3 | 5/4 | 5/4 | 5/4 | |
| $Alg_3$ | 1/1 | 2/2 | 6/3 | 6/4 | 6/4 | 6/4 | |
| $Alg_4$ | 1/1 | 2/2 | 3/3 | 7/4 | 7/4 | 7/4 | |
| $Alg_5$ | 1/1 | 2/2 | 3/3 | 4/4 | 8/4 | 8/4 | |
| $Alg_6$ | 1/1 | 2/2 | 3/3 | 4/4 | 5/4 | 9/4 | |
| ... | | | | | | | |

Each algorithm has its own expertise, i.e., is optimal w.r.t. a certain input $T$. In fact, the expertise of $\{Alg_t\}_{t=1}^{4}$ spans all possibilities of $T$, in other words, $\{Alg_t\}_{t\geq 5}$ are somehow useless (informally).

# Algorithms: Randomized

**Idea**

We give each algorithm a chance to be used - we construct a probability distribution $\{p_t\}_{t=1}^{\infty}$ on $\{Alg_t\}_{t=1}^{\infty}$.

Recall that, a randomized algorithm is $\alpha$-competitive if

$$\alpha := \max_{\sigma} \frac{\mathbb{E}[c(ALG, \sigma)]}{c(Opt, \sigma)}$$

$$\Rightarrow \sum_{t=1}^{\infty} c(Alg_t, \sigma) \cdot p_t \leq \alpha \cdot Opt(\sigma), \ \forall \sigma$$

# Algorithms: Randomized

W.r.t. the toy example, if we want to achieve $\alpha$-competitiveness, we need

$$\begin{aligned}
4p_1 + 1p_2 + 1p_3 + 1p_4 + 1p_5 + 1p_6 + \ldots &\leq 1\alpha \\
4p_1 + 5p_2 + 2p_3 + 2p_4 + 2p_5 + 2p_6 + \ldots &\leq 2\alpha \\
4p_1 + 5p_2 + 6p_3 + 3p_4 + 3p_5 + 3p_6 + \ldots &\leq 3\alpha \\
4p_1 + 5p_2 + 6p_3 + 7p_4 + 4p_5 + 4p_6 + \ldots &\leq 4\alpha \\
4p_1 + 5p_2 + 6p_3 + 7p_4 + 8p_5 + 5p_6 + \ldots &\leq 4\alpha \\
&\vdots
\end{aligned}$$

Two observations:

- $p_t = 0, \forall\, t > B$ - subtract the $t^{th}$ from the $t+1^{th}$ inequality
- After the $B^{th}$ inequality, they're useless - same as the $B^{th}$

# Algorithms: Randomized

After two simplifications:

- Keep the $1^{st}$ inequality
- Subtract the $t^{th}$ from the $t+1^{th}$ inequality

we then have

$$\begin{aligned} Bp_1 + p_2 + p_3 + \cdots + p_B &\leq \alpha \\ Bp_2 + p_3 + \cdots + p_B &\leq \alpha \\ Bp_3 + \cdots + p_B &\leq \alpha \\ &\vdots \\ Bp_B &\leq \alpha \end{aligned}$$

# Algorithms: Randomized

We're to minimize $\alpha$ s.t. the above constraints, that is, we're to solve the following problem:

$$\begin{aligned}
\min_{p_t}: \quad & \alpha \\
\text{s.t.:} \quad & Bp_1 + p_2 + p_3 + \cdots + p_B \leq \alpha \\
& Bp_2 + p_3 + \cdots + p_B \leq \alpha \\
& Bp_3 + \cdots + p_B \leq \alpha \\
& \vdots \\
& Bp_B \leq \alpha \\
& p_t \in [0, 1], \forall\, t \\
& \sum_{t=1}^{\infty} p_t = 1
\end{aligned}$$

# Algorithms: Randomized

We may notice that

$$\alpha \text{ is minimized} \Leftrightarrow \text{all inequalities become equalities.}$$

Proof for "$\Rightarrow$":

- Assume $\alpha$ is minimized, and there exist a strict inequality
- We then can adjust $p_t$ s.t. all become strict inequalities, and then $\alpha$ can be further pushed down - contradicts with the minimality of $\alpha$

Proof for "$\Leftarrow$":

- Assume all are equalities
- It becomes solving a linear system with full rank. There's only one solution, and hence the minimality is achieved

# Algorithms: Randomized

**Algorithm**

Choose $Alg_t$ with $p_t = \frac{\alpha}{B}(\frac{B-1}{B})^{B-t}$, where $\alpha = \frac{1}{1-(1-\frac{1}{B})^B}$

- $\frac{1}{1-(1-\frac{1}{B})^B}$-competitive
- $\frac{1}{1-(1-\frac{1}{B})^B} < \frac{e}{e-1}, \forall B$; $\lim_{B\to\infty} \frac{1}{1-(1-\frac{1}{B})^B} = \frac{e}{e-1} \approx 1.58$

# Algorithms: Randomized

**Theorem**

$\frac{1}{1-(1-\frac{1}{B})^B}$ is the optimal C.R. among all randomized algorithms.

Do we need to prove the optimality of $\alpha$ if we achieve it by solving the above optimization problem?

- No, the optimality of $\alpha$ is guaranteed based on the procedures above
- The set of all deterministic algorithms is clear for this problem
- What if the problem is more complicated - some hidden deterministic algorithms?
- Unrealistic to traverse all randomized algorithms

# Algorithms: Randomized

**Theorem** (*Yao's Principle*)

The expected cost of the best randomized algorithm on the worst-case input is no lower than the expected cost of the deterministic algorithm that performs the best on the worst-case input distribution, i.e., $\forall ALG, \forall X$ :

$$\max_{x \in \mathcal{X}} \mathbb{E}[c(ALG, x)] \geq \min_{d \in D} \mathbb{E}[c(Alg_d, X)],$$

where

- $ALG$: Randomized algorithm, a random variable
- $X$: Input distribution, a random variable
- $\mathcal{X}$: Input set
- $D$: The set of all deterministic algorithms (indicies)

# Algorithms: Randomized

**Theorem** (*Yao's Principle*)

Under the competitive ratio context, we have $\forall ALG, \forall X$ :

$$\max_{x \in \mathcal{X}} \frac{\mathbb{E}[c(ALG, x)]}{c(Opt.x)} \geq \min_{d \in D} \mathbb{E}[\frac{c(Alg_d, X)}{c(Opt, X)}],$$

or equivalently,

$$\underbrace{\min_{ALG} \max_{x \in \mathcal{X}} \frac{\mathbb{E}[c(ALG, x)]}{c(Opt, x)}}_{\text{optimal C.R.}} \geq \underbrace{\max_{X} \min_{d \in D} \mathbb{E}[\frac{c(Alg_d, X)}{c(Opt, X)}]}_{\text{lower bound (tight?)}}$$

We're then to construct the worst-case input distribution $X$.

# Algorithms: Randomized

**Theorem**

For the ski-rental problem,

$$\max_{X} \min_{d \in D} \mathbb{E}[\frac{c(Alg_d, X)}{c(Opt, X)}] = \frac{1}{1 - (1 - \frac{1}{B})^B}$$

# Outline

# OPD Framework

- Primal:

$$\begin{aligned} \min_{x,z_i}: \quad & B \cdot x + \sum_{i=1}^{T} z_i \\ \text{s.t.:} \quad & x + z_i \geq 1, \ \forall i \\ & x \in \{0,1\} \\ & z_i \in \{0,1\}, \ \forall i \end{aligned}$$

$p^*$ is the opt value

- Relaxed Primal:

$$\begin{aligned} \min_{x,z_i}: \quad & B \cdot x + \sum_{i=1}^{T} z_i \\ \text{s.t.:} \quad & x + z_i \geq 1, \ \forall i \\ & x \geq 0 \\ & z_i \geq 0, \ \forall i \end{aligned}$$

$p_r^*$ is the opt value

- Dual:

$$\begin{aligned} \max_{y_i}: \quad & \sum_{i=1}^{T} y_i \\ \text{s.t.:} \quad & \sum_{i=1}^{T} y_i \leq B \\ & 0 \leq y_i \leq 1, \ \forall i \end{aligned}$$

$d^*$ is the opt value

$\underbrace{\hspace{6em}}_{p^* \geq p_r^*, \text{ by relaxation}}$

$\underbrace{\hspace{6em}}_{p_r^* \geq d^*, \text{ by weak duality}}$

$\underbrace{\hspace{6em}}_{d^* \leq p_r^* \leq p^*}$

# OPD Framework

Several clarifications:

- $p_t$: The primal objective value up to time $t$ incurred by our algorithm
- $d_t$: The dual objective value up to time $t$ incurred by our algorithm
- $\alpha := \min_{\sigma} \frac{c(Opt,\sigma)}{c(Alg,\sigma)} \leq 1$, so we are to maximize $\alpha$

So far,

- What we have: $d^* \leq p_r^* \leq p^*$
- Further more, we have $d_T \leq d^* \leq p_r^* \leq p^*$
- What we want: $\alpha p_T \leq p^*$
- The above holds if $\alpha p_T \leq d_T$

That is,

$$\alpha p_T \leq d_T + \text{primal and dual feasibility} \Rightarrow \alpha\text{-competitiveness}$$

# OPD Framework

In the online manner, $\alpha p_T \leq d_T$ is hard to achieve, so we want to achieve some inequality at each step, and then the cumulative effect yields $\alpha p_T \leq d_T$, so

$$\begin{aligned}
& \alpha p_T \leq d_T \\
\Leftrightarrow\ & \alpha\left(\sum_{t=1}^{T}(p_t - p_{t-1}) + p_0\right) \leq \left(\sum_{t=1}^{T}(d_t - d_{t-1}) + d_0\right) \\
\Leftarrow\ & \alpha(p_t - p_{t-1}) \leq d_t - d_{t-1},\ \forall\, t, \text{ if } p_0 = d_0 = 0
\end{aligned}$$

That is, if $p_0 = d_0 = 0$, to achieve $\alpha p_T \leq d_T$, we need

$$\alpha \Delta p_t \leq \Delta d_t,\ \forall\, t$$

which is called the incremental inequality.

# OPD Framework

Previously,

$$\alpha p_T \leq d_T + \text{primal and dual feasibility} \Rightarrow \alpha\text{-competitiveness}$$

Now, if $p_0 = d_0 = 0$,

$$\alpha \Delta p_t \leq \Delta d_t + \text{primal and dual feasibility} \Rightarrow \alpha\text{-competitiveness}$$

Generally, to design an $\alpha$-competitive algorithm, we need:

- Primal and dual feasibility
- Initial conditions ($p_0, d_0$, not always 0) and terminal conditions (e.g. a necessary condition for maximizing $\alpha$)
- Update methods for the primal and dual variables s.t. the incremental inequality holds (depends on the initial and terminal conditions)

# Outline

# Algorithms with OPD: Deterministic

**W.r.t. the relaxed problem**, need to design $x_0, y_0$, and the update methods for $x_t, z_t, y_t$.

**Idea**

Assume we want to use the following (very simple) idea:

- For feasibility and simplicity, choose $x_0 = y_0 = 0$
- Since deterministic, we buy on day $G$, i.e., $x_G = 1$, and keep $x_{t>G} = 1$
- For feasibility and simplicity, keep $x_t \in [0, 1]$, $z_t = 1 - x_t$, so keep $z_{t>G} = 0$
- $\Delta p_{t>G} = 0$, no need for $\Delta d_{t>B} = y_{t>B}$ to leave the room for the primal variables to change, so $y_{t>G} = 0$. Then we fully use, and evenly split $B$ to $y_{t\leq G}$, i.e. $y_{t\leq G} = \frac{B}{G}$
- Due to the previous, $G \geq B$, or otherwise $y_t$ is infeasible

# Algorithms with OPD: Deterministic

For the **fractional** ski rental problem (relaxed primal), here's our design:

- Initial condition:
  $x_0 = y_0 = 0$
- **Terminal condition** (motivated by the optimality of $Alg_B$):
  Buy on day $B$, i.e. $x_B = 1$, and $x_{t>B} = 1$
- Update method of the primal variable - incomplete:
  $x_t \in [0, 1]$, $z_t = 1 - x_t$, so $z_{t>B} = 0$
- Incremental inequality:
  $\alpha \Delta p_t \leq \Delta d_t \Rightarrow \alpha(B(x_t - x_{t-1}) + z_t) \leq y_t$
- Update method of the dual variable:
  $y_{t \leq B} = 1$, and $y_{t>B} = 0$

# Algorithms with OPD: Deterministic

## Algorithm (Draft)

$x_0 = 0$
for each day $t$:
    if $t < B$:

$$\text{update } x_t \text{ s.t. } \begin{cases} \alpha(B(x_t - x_{t-1}) + z_t) \leq 1 \text{ with } \alpha \text{ and } z_t \text{ to be defined} \\ x_B = 1 \\ x_t \in [0,1] \end{cases}$$

$z_t = 1 - x_t$
    else:
        $x_t = 1,\ z_t = 0$

# Algorithms with OPD: Deterministic

Now we solve the (in)equalities:

- $\alpha(B(x_t - x_{t-1}) + (1 - x_t)) \leq 1 \Leftrightarrow \underbrace{x_t \leq (1 + \frac{1}{B-1})x_{t-1} + \frac{1-\alpha}{\alpha(B-1)}}_{\text{update method for } x_t \text{ is here}}$
- $x_t = (1 + \frac{1}{B-1})x_{t-1} + \frac{1-\alpha}{\alpha(B-1)} = \frac{\alpha-1}{\alpha}(1 - (1 + \frac{1}{B-1})^t)$ is a good candidate, increasing
- $x_B = 1 \Rightarrow \alpha = 1 - \frac{1}{(1+\frac{1}{B-1})^B}$

We then check if $\alpha$-competitiveness is guaranteed (obvious):

- Primal and dual feasibility: $x_0 = 0, x_B = 1$, $x_t$ is increasing
- $\alpha(B(x_t - x_{t-1}) + z_t) \leq 1$: Directly holds

# Algorithms with OPD: Deterministic

**Algorithm**

$x_0 = 0$

for each day $t$:

  if $t < B$:

    $x_t = (1 + \frac{1}{B-1})x_{t-1} + \frac{1-\alpha}{\alpha(B-1)}$ with $\alpha = 1 - \frac{1}{(1+\frac{1}{B-1})^B}$

    $z_t = 1 - x_t$

  else:

    $x_t = 1,\ z_t = 0$

- $1 - \frac{1}{(1+\frac{1}{B-1})^B}$-competitive for the **fractional** problem ($\alpha p_{r_T} \leq d_T \leq p_r^*$)
- $1 - \frac{1}{(1+\frac{1}{B-1})^B} > \frac{e-1}{e}, \forall B$; $\lim_{B\to\infty} 1 - \frac{1}{(1+\frac{1}{B-1})^B} = \frac{e-1}{e}$

# Algorithms with OPD: Deterministic

We can derive the algorithm even **without** the intuition of the terminal condition $x_B = 1$:

- Initial condition:
  $x_0 = y_0 = 0$
- Terminal condition:
  Buy on day $G \geq B$, i.e. $x_G = 1$, and $x_{t>G} = 1$
- Update method of the primal variable - incomplete:
  $x_t \in [0, 1]$, $z_t = 1 - x_t$, so $z_{t>G} = 0$
- Incremental inequality:
  $\alpha \Delta p_t \leq \Delta d_t \Rightarrow \alpha(B(x_t - x_{t-1}) + z_t) \leq y_t$
- Update method of the dual variable:
  $y_{t \leq G} = \frac{B}{G}$, and $y_{t>G} = 0$

# Algorithms with OPD: Deterministic

By the incremental inequality,

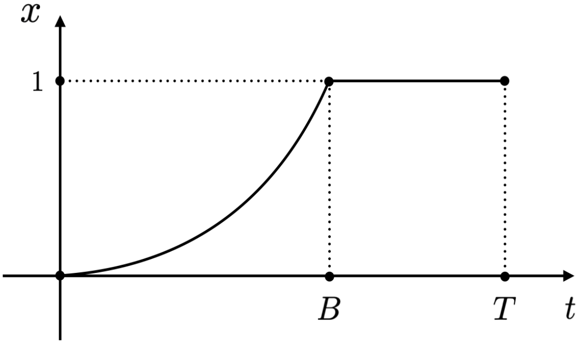
$$\alpha(B(x_t - x_{t-1}) + (1 - x_t)) \leq \frac{B}{G} \Leftrightarrow x_t \leq \frac{B - \alpha G}{\alpha(B-1)G} + (1 + \frac{1}{B-1})x_{t-1}$$

Pick $x_t = \frac{B-\alpha G}{\alpha(B-1)G} + (1 + \frac{1}{B-1})x_{t-1} = \frac{\alpha G - B}{\alpha G}(1 - (1 + \frac{1}{B-1})^t)$, and by $x_G = 1$ we have

$$\alpha = \frac{B(1 - \frac{1}{(1+\frac{1}{B-1})^G})}{G}$$

which is essentially a function $\alpha$ of $G$, where $G \geq B$. Note that it's decreasing, so if we want to maximize $\alpha$, we need $G = B$, and this yields the algorithm above.

# Algorithms with OPD: Deterministic




The "potential" of buying skis is increasing

# Algorithms with OPD: Randomized

W.r.t. the primal problem (un-relaxed),

- $\alpha p_T \leq d_T$ not satisfied (but $\alpha p_{r_T} \leq d_T$ satisfied)
- Dual feasibility satisfied
- Primal feasibility not satisfied

Can we extend this result to the integral problem?

- Buy $x_t \in [0, 1]$ amount of skis, rent $z_t \in [0, 1]$ amount of skis, $x_t + z_t = 1$
- **Fractions** $\Leftrightarrow$ **Expectations** $\Leftrightarrow$ Probabilities
- $x_t = \mathbb{E}[X_t] = \mathbb{P}_t^B$, $z_t = \mathbb{E}[Z_t] = \mathbb{P}_t^R$, where $X_t$, $Z_t$ are standard Bernoulli RVs
- Update $\mathbb{P}_t^B$, $\mathbb{P}_t^R$ instead

# Algorithms with OPD: Randomized

**Algorithm**

$\mathbb{P}_0^B = 0$

for each day $t$:

  if $t < B$:

    $\mathbb{P}_t^B = (1 + \frac{1}{B-1})\mathbb{P}_{t-1}^B + \frac{1-\alpha}{\alpha(B-1)}$ with $\alpha = 1 - \frac{1}{(1+\frac{1}{B-1})^B}$

    $\mathbb{P}_t^R = 1 - \mathbb{P}_t^B$

    Decide to buy or to rent based on the distribution $\{\mathbb{P}_t^B, \mathbb{P}_t^R\}$

  else:

    $\mathbb{P}_t^B = 1$, $\mathbb{P}_t^R = 0$, i.e., buy

# Algorithms with OPD: Randomized

Now the primal feasibility is satisfied, what about $\alpha\mathbb{E}[p_T] \leq d_T$? Actually, we have

$$\mathbb{E}[p_T] = p_{r_T}, \text{ and hence } \alpha\mathbb{E}[p_T] \leq d_T,$$

where the equality can be proved as below:

$$\mathbb{E}[p_t] = \mathbb{E}[X_t + \sum_{i=1}^{t} Z_i] = \mathbb{E}[X_t] + \sum_{i=1}^{t} \mathbb{E}[Z_i] = \mathbb{P}_t^B + \sum_{i=1}^{t} \mathbb{P}_t^R = x_t + \sum_{i=1}^{t} z_i = p_{r_T}.$$

Therefore, $1 - \frac{1}{(1+\frac{1}{B-1})^B}$ can be extended to the integral problem.

# Outline

# Summary

For the ski rental problem,

- Optimal C.R. for the deterministic algorithm: $2 - \frac{1}{B}$
- Optimal C.R. for the randomized algorithm: $\frac{1}{1-(1-\frac{1}{B})^B}$
- We use the OPD framework to derive a $\frac{1}{1-(1-\frac{1}{B})^B}$-competitive deterministic algorithm for the fractional ski rental problem
- Then we extend the $\frac{1}{1-(1-\frac{1}{B})^B}$-competitiveness to the integral problem using randomization

# Outline

# References

# Thank You!