

IFT 4031/7031,
Machine Learning for Signal Processing
**Week1: Class Intro,
Linear Algebra Refresher**

Cem Subakan



What is this class?

- What do you think this class is?

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- What do you think this class is?
- Is it a Machine Learning class?
- Is it a Signal Processing class?

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- What do you think this class is?
- Is it a Machine Learning class?
- Is it a Signal Processing class?
- What is Machine Learning?
- What is Signal Processing?

Signal Processing

- Here's the wikipedia definition:

Signal processing is an electrical engineering subfield that focuses on analyzing, modifying and synthesizing *signals*, such as *sound*, *images*, *potential fields*, *seismic signals*, *altimetry processing*, and *scientific measurements*.^[1] Signal processing techniques are used to optimize transmissions, *digital storage efficiency*, correcting distorted signals, *subjective video quality* and to also detect or pinpoint components of interest in a measured signal.^[2]

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- Hm, this kinda sounds like machine learning.

How are signals different than data?



- So, signals are just data?
- Yeah-(ish).

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- Yeah-(ish).
- Why are we calling them signals then?

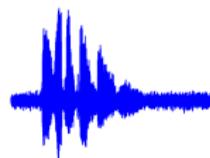
How are signals different than data?



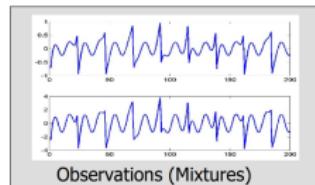
- So, signals are just data?
- Yeah-(ish).
- Why are we calling them signals then?
- When we speak of signals, we refer more to structured data. (Order matters)
- And, saying 'signals', 'signal processing' implies a more Electrical Engineering way to the approach.

Example Signals

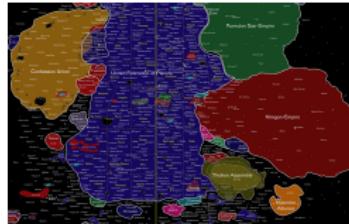
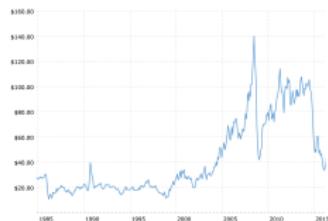
- Images, Audio/Speech



- Brains

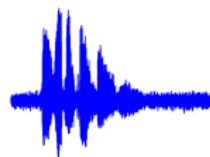


- Financial Time Series, Graphs

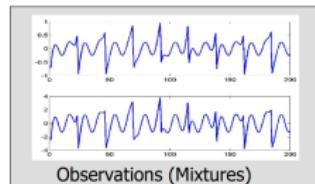
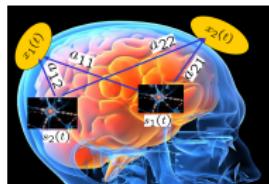


Example Signals

- Images, Audio/Speech



- Brains



- Financial Time Series, Graphs



- More?

But why bother? Isn't ML what's hip now?

- Yes, ML is extremely popular, and we should embrace that.

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But why bother? Isn't ML what's hip now?

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- But, traditional ML isn't very friendly for signals.
- What about signal processing, doesn't that cover what we need?
 - ▶ No!



- ▶ Traditional SP is typically **NOT** statistical, doesn't handle the statistical patterns of the signal well.
- ▶ Traditional SP: Filtering, acquisition, analog-digital-analog conversion, transmission
- ▶ There is statistical signal processing also, but it doesn't go much beyond adaptive filtering.

MLSP: Machine Learning for Signal Processing

- How to build systems that would work with sequences and solve machine intelligence tasks on them?
 - ▶ Various tasks with Speech and Audio: ASR, Speech Enhancement, Music Transcription...

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 - ▶ Understanding Biomedical Sequences

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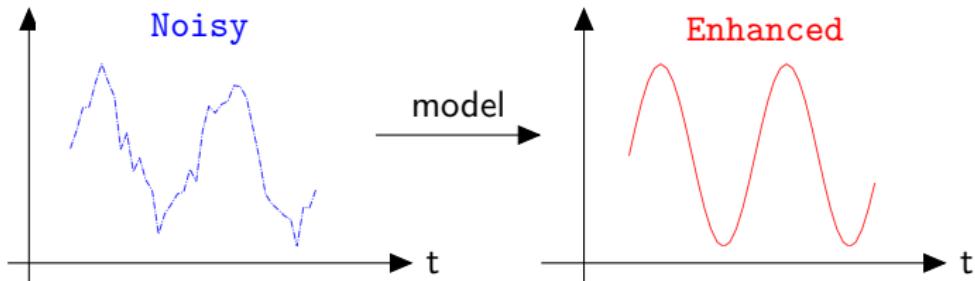
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 - ▶ Generating Videos

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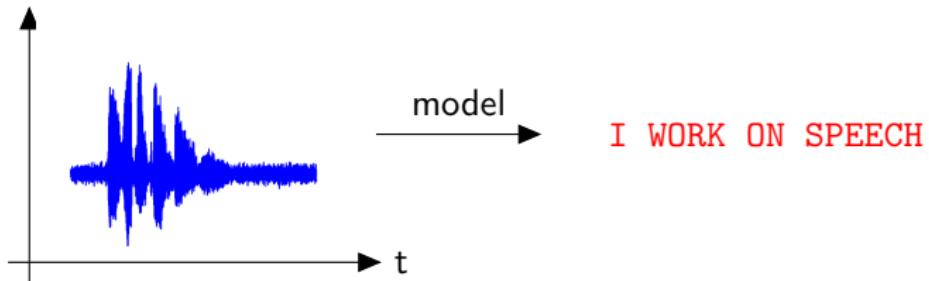
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 - ▶ Understanding Biomedical Sequences
 - ▶ Generating Videos
 - ▶ More...

Speech and Audio Modeling

- Speech Enhancement

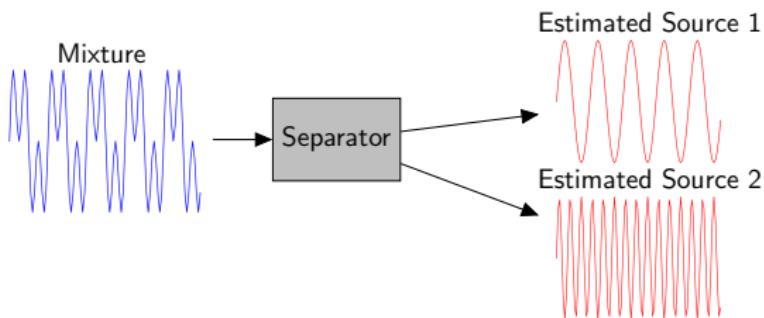


- Speech Recognition

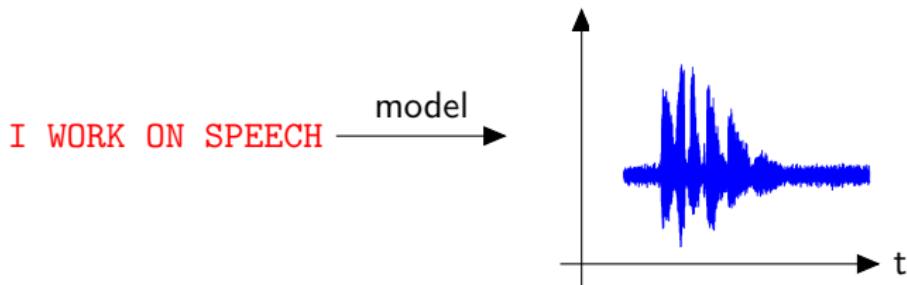


Speech and Audio Modeling

■ Speech Separation

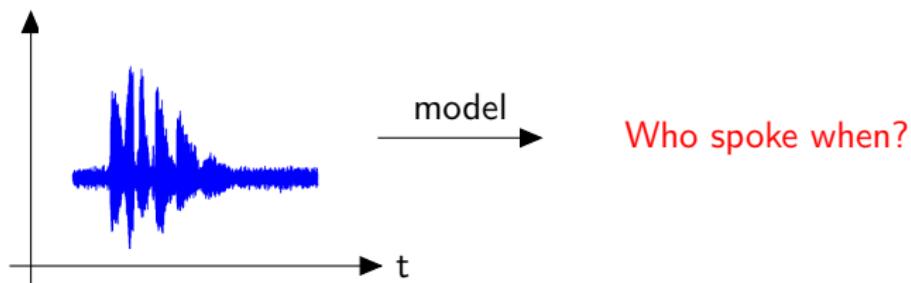


■ Text-to-Speech

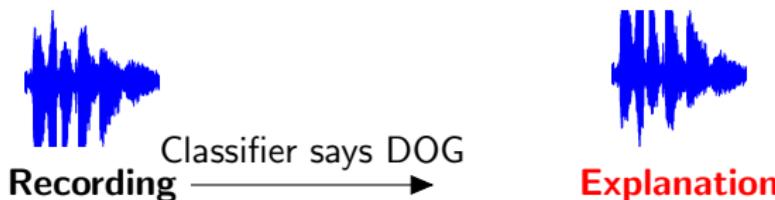


Speech and Audio Modeling

- Speaker Diarization



- Neural Network Explanation



- Other problems: Generating Deep fakes, Detecting deep fakes, Music Source Separation, Music Transcription, Sound Event Detection/Classification...

Speech and Audio Modeling

- Field with huge economic value & job opportunities,
 - ▶ Speech Recognition (e.g. Siri)
 - ▶ Speech Enhancement (e.g. Google meet, Zoom)
 - ▶ Text-to-Speech
 - ▶ Speaker Verification, Spoof Detection(Banks)
 - ▶ Speaker Diarization for Meeting Analysis (Nuance, Microsoft)
 - ▶ Source Separation (e.g. Beatles Rock Band, Meeting Analysis)

Other real-life applications

- Face recognition



Other real-life applications

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- Brain-machine interfaces



Other real-life applications

- Face recognition



- Brain-machine interfaces



- Real time bio-signal analysis, learning generative models for bio/medical signals, condition monitoring (mining machines, production machines), Stock market, many more..

About this class

- This is class heavy on practice. How do we make things that work?
- We do not do deep theory in this class.
 - ▶ We will not prove things.
 - ▶ We will not stay Keras level either.
 - ▶ Our goal is to give useful insights, be useful.
- We go fast, our typical lecture could be a class.

Syllabus: Basics

■ Linear Algebra

- ▶ This class

■ Probability

- ▶ Probability Calculus, Random Variables, Bayesian vs Frequentist Principles

■ Signal Processing

- ▶ Signal Representations, Fourier Transform, Sampling

Syllabus: Machine Learning

■ Decompositions

- ▶ PCA, NMF, Linear Regression, Tensor Decompositions

■ Classification

- ▶ Logistic Regression, Maximum Margin, Kernels, Boosting

■ Deep Learning

- ▶ Deep Learning Essentials, Pytorch

■ Optimization

- ▶ Convex optimization
- ▶ Gradient Descent and friends
- ▶ Non-Convex optimization

■ Clustering

- ▶ Kmeans, Spectral Clustering, DBScan

■ Unsupervised Non-linear learning

- ▶ Manifold Learning, Deep Generative Models

■ Time Series Models

- ▶ HMMs, Kalman Filters

Syllabus: Fun Stuff

- Speech Recognition
- Speech Enhancement/Separation
- Text-to-speech
- Representation Learning Methods for Sequences
- Generative Models for Sequences
- Text prompted models (text prompted image / sound generation)
- Neural Network Interpretation Methods
- Graph Signal Processing / ML

Evaluation

■ Homeworks (45%)

- ▶ 3 homeworks, you need to work on these alone!
- ▶ I would like you to typeset math in \LaTeX . So if you don't know it, start learning it!
- ▶ Do not use Generative AI, if you want to learn!
- ▶ You will need to code. But we will reward good quality presentation of results.

■ Weekly Labs (10%)

- ▶ You will work on hands-on application of the things we talk about. TAs will lead the online sessions.
- ▶ Sessions Fridays 12h30 - 14h20. First hour(ish) will be dedicated to the lab. Then it will turn into an office hour with the TAs.

■ Final Project (45%)

Final project

- This will be a mini-conference.
- Each paper will receive 2/3 peer-reviews (from you). We will evaluate the quality of your reviews (5% of your 45% project grade).
- You will work in teams of 2-3 (no more, no less)
- We will ask who did what in the project. So no freeriding!
- Start making friends!
- Mid-October, proposals are due
- Last 1-2 weeks, paper deadline.

Final project

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- We will ask who did what in the project. So no freeriding!
- Start making friends!
- Mid-October, proposals are due
- Last 1-2 weeks, paper deadline.
 - ▶ We will accept all the papers, and you will make a presentation.
 - ▶ However, you need to do a good job to get a good grade.
 - ▶ If it's a good paper, we can also work together to submit it to a real conference! We can work together towards that.

Communications

- We will have teams page where will have a forum, and you will submit your assignments.
- Be active on the forum, ask questions. Find friends for the project.
- We will do the announcements on teams, so sign-up for it!
- Check <https://ycemsubakan.github.io/mlsp2025.html> for class material.

Instructor: Who am I?

■ Instructor: Cem Subakan

- ▶ cem.subakan@ift.ulaval.ca
- ▶ Assistant Prof. in Computer Science,
Mila Associate Academic Member.
- ▶ Associate member of IEEE MLSP Technical Committee. Was a
general Chair of MLSP 2025 conference.
- ▶ Just send me a message you if you want to meet.

■ I work on machine learning for Speech and Audio.

- ▶ Interpretability, Explainability
- ▶ Speech Separation & Enhancement
- ▶ Multi-Modal Learning
- ▶ Continual Learning
- ▶ Probabilistic Machine/Deep Learning

■ I review for many major conferences, involved in the organization of several workshops.

■ I have written a lot of papers involving MLSP topics, worked with many people, also saw the industry side of things.

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■ Office hours: By appointment. DM me on teams to take an appointment.

You can submit your work to MLSP 2025!







There are other conferences now we can submit!

Who are the TAs?

- Sara Karami
 - ▶ sara.karami.1@ulaval.ca
- Mathieu Bazinet
 - ▶ mathieu.bazinet.2@ulaval.ca
- Benjamin Léger
 - ▶ benjamin.leger.1@ulaval.ca

Who are the TAs?

- Sara Karami
 - ▶ sara.karami.1@ulaval.ca
- Mathieu Bazinet
 - ▶ mathieu.bazinet.2@ulaval.ca
- Benjamin Léger
 - ▶ benjamin.leger.1@ulaval.ca
- TAs will hold the online lab sessions (Fridays 12h30-14h20)
- Advice:
 - ▶ If you need help do not bombard them at the last minute. Seek help early.

Who are you?

■ Name, department, grad/undergrad?

- ▶ What are your interests?
- ▶ Hint: Take notes, and contact the person if something picks your interest.

Table of Contents

Linear Algebra Refresher

Basics

Array Manipulation

More linear algebraic concepts

Decompositions

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Scalars, Vectors, Matrices, Tensors

- Scalar, x ,
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Scalars, Vectors, Matrices, Tensors

- Scalar, x ,
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$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}$$

Scalars, Vectors, Matrices, Tensors

- Scalar, x , just a number.

- Vector, x , of length L

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}$$

- Matrix, x of size $L \times M$

$$\begin{aligned} x &= \begin{bmatrix} x_{1,1} & \dots & x_{1,M} \\ \vdots & \vdots & \vdots \\ x_{L,1} & \dots & x_{L,M} \end{bmatrix} \\ &= [x_1 \ \dots \ x_M] \end{aligned}$$

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- Tensor, x of size $L \times M \times N$

$$x = \begin{bmatrix} x_{1,1,1} & \dots & x_{1,M,1} \\ \vdots & \vdots & \vdots \\ x_{L,1,1} & \dots & x_{L,M,1} \\ \ddots & & \ddots \\ & \begin{bmatrix} x_{1,1,N} & \dots & x_{1,M,N} \\ \vdots & \vdots & \vdots \\ x_{L,1,N} & \dots & x_{L,M,N} \end{bmatrix} \end{bmatrix}$$

Scalars, Vectors, Matrices, Tensors

- Scalar, x , just a number.
0th order tensor.

- Vector, x , of length L
$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}$$
1th order tensor.

- Matrix, x of size $L \times M$
$$x = \begin{bmatrix} x_{1,1} & \dots & x_{1,M} \\ \vdots & \vdots & \vdots \\ x_{L,1} & \dots & x_{L,M} \end{bmatrix} = \begin{bmatrix} x_1 & \dots & x_M \end{bmatrix}$$
2nd order tensor.

- Tensor, x of size $L \times M \times N$

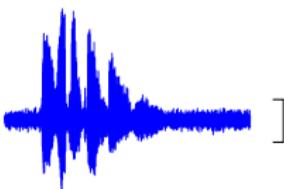
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3rd order tensor.

How do we represent signals as these?

- Sounds, Time Series

$$x^T = [x_1 \quad \dots \quad x_L] = \begin{bmatrix} \text{[sound waveform]} \end{bmatrix}$$



- Images

$$X = \begin{bmatrix} x_{1,1}, & \dots & x_{1,M} \\ \vdots & \vdots & \vdots \\ x_{L,1}, & \dots & x_{L,M} \end{bmatrix} = \begin{bmatrix} \text{[image of Captain Picard]} \end{bmatrix}$$



- Videos as tensors.. and so on..

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Linear Algebra Refresher

Basics

Array Manipulation

More linear algebraic concepts

Decompositions

Index/Array Notation

- We need good ways to communicate operations on these objects.
- **Option 1:** Index Notation
 - ▶ Micro-level and detailed, but not very compact
- **Option 2:** Array Notation
 - ▶ Compact but abstracts away the details

Index Notation

- We define the elements in index form.

- ▶ Element-wise multiplication:

$$c_i = a_i b_i$$

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$$C_{ik} = \sum_j A_{ij} B_{jk}$$

- ▶ Some random tensor operations

$$C_{im} = \sum_{j,l,k} A_{ijlk} B_{mjlk}, \quad c = \sum_{i,j} A_{ij} B_{ij}$$

Array Notation

- We define the elements in index form.

- ▶ Element-wise multiplication:

$$c = a \odot b, c \in \mathbb{R}^L$$

- ▶ Inner product of vectors

$$c = \langle a, b \rangle = a^\top b, c \in \mathbb{R}$$

- ▶ Outer product of vectors

$$c = a \otimes b = ab^\top, c \in \mathbb{R}^{L \times M}$$

- ▶ Matrix-vector product

$$c = Ab, c \in \mathbb{R}^L$$

- ▶ Matrix multiplication

$$C = AB, C \in \mathbb{R}^{L \times M}$$

- ▶ Some random tensor operations

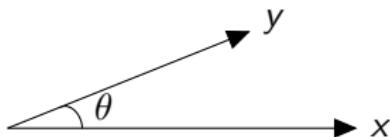
$$C = A \times_{j|k} B, C \in \mathbb{R}^{L \times M} \quad c = A \times_{i,j} B, c \in \mathbb{R}$$

Index vs Array Notation

- Index Notation is very specific, not ambiguous
- But the array notation makes it possible to manipulate the operations with ease. (E.g. gradient calculations)

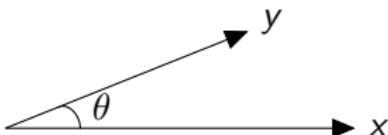
The dot product

■ $c = \sum_i a_i b_i = a^\top b = \|a\| \|b\| \cos\theta$



The dot product

- $c = \sum_i a_i b_i = a^\top b = \|a\| \|b\| \cos\theta$



- Note that,

$$\theta = \arccos \left(\frac{a^\top b}{\|a\| \|b\|} \right)$$

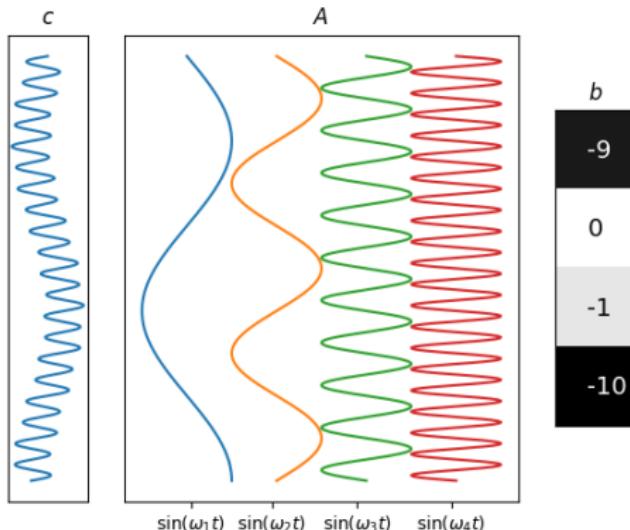
- So, dot product is a great tool to measure similarity.

Matrix-Vector Product

- $c = Ab$, or $c_i = \langle A_{i,:}, c \rangle = \sum_j A_{ij}c_j$. A is a matrix, b is vector. c is a what?

Matrix-Vector Product

- $c = Ab$, or $c_i = \langle A_{i,:}, b \rangle = \sum_j A_{ij}b_j$. A is a matrix, b is vector. c is a what?
- The resulting c vector is a linear combination of columns of A .

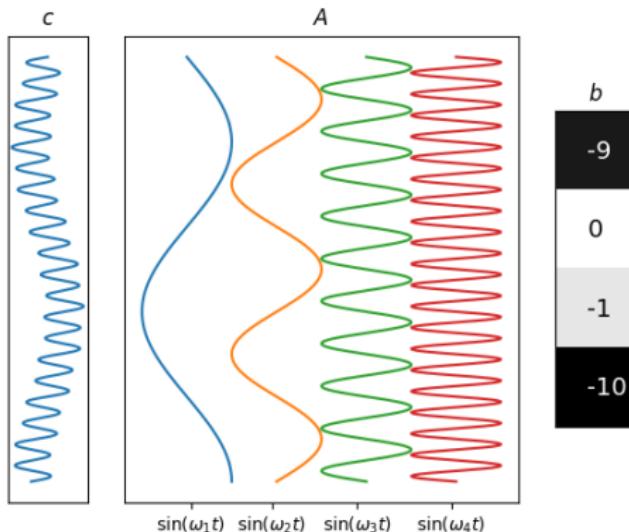


Matrix-Vector Product - 2nd interpretation

- It's a series of dot products.
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Matrix-Matrix Product

- It's a series of Matrix-vector products. (or series of inner products on a grid)
- $C = AB$, or $C_{ij} = \sum_k A_{ik} C_{kj}$, or $C_{ij} = A_{i,:}^\top C_{:,j}$
-

$$C = \begin{bmatrix} A_{1,:}^\top \\ A_{2,:}^\top \\ A_{3,:}^\top \end{bmatrix} [B_{:,1} \quad B_{:,2} \quad B_{:,3}] = \begin{bmatrix} A_1^\top B_1 & A_1^\top B_2 & A_1^\top B_3 \\ A_2^\top B_1 & A_2^\top B_2 & A_2^\top B_3 \\ A_3^\top B_1 & A_3^\top B_2 & A_3^\top B_3 \end{bmatrix}$$

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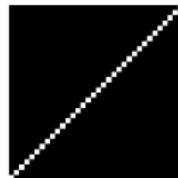
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- Not any pair of two matrices can be multiplied. You need to have equal number of columns from A , number rows from B .
- Master this, it will help! This has to become muscle memory.

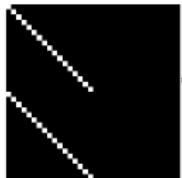
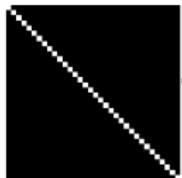
Visualize the matrix product



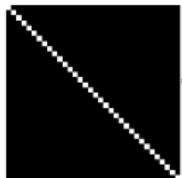
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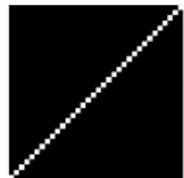
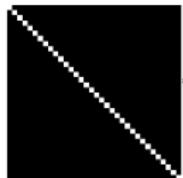
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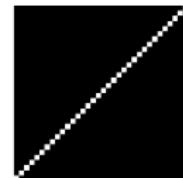
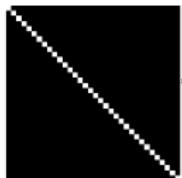
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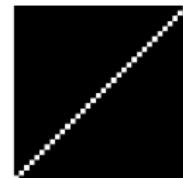
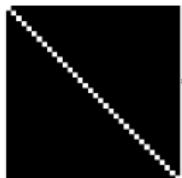
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$$\begin{bmatrix} & & \\ & \times & \\ & & \end{bmatrix}$$



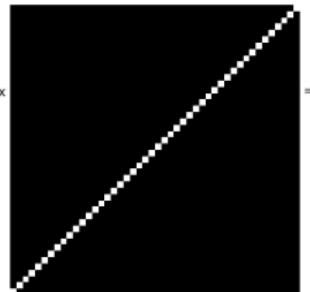
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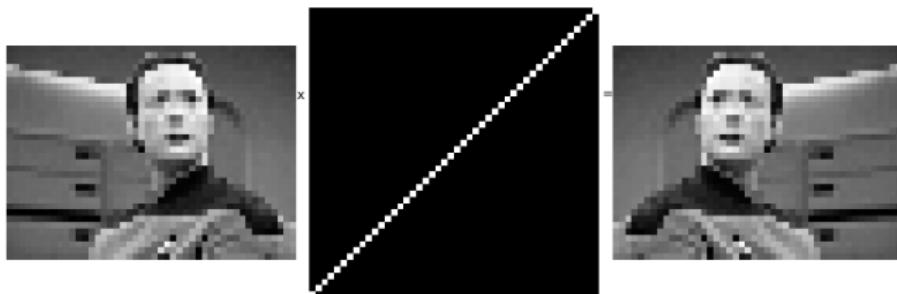
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Multiplying from the other side



Multiplying from the other side



Reversing on the horizontal axis

Einstein Notation

- Let's go beyond matrices!
- $C_{i,j} = \sum_{l,k} A_{i,l,k} B_{l,j,k}$

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- $A_{i,l}, B_{l,j} \rightarrow C_{i,j}$

Let's do more Einstein stuff

- Element-wise multiplication:

$$c = a \odot b, \quad c \in \mathbb{R}^L$$

- Inner product of vectors

$$c = \langle a, b \rangle = a^\top b, \quad c \in \mathbb{R}$$

- Outer product of vectors

$$c = a \otimes b = ab^\top, \quad c \in \mathbb{R}^{L \times M}$$

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$$c = Ab, \quad c \in \mathbb{R}^L$$

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$$C = A \times_{j|k} B, \quad C \in \mathbb{R}^{L \times M} \quad c = A \times_{i,j} B$$

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$$A_{ij|k}, B_{mj|k} \rightarrow C_{im}$$

Implementing Einstein products is easy in Python

■ Batch Matrix Multiplication

$$A_{bij} B_{bjk} \rightarrow C_{bik}$$

```
C = torch.einsum('bij,bjk->bik', A, B)
```

Application of Tensor Operations

- RGB images



- Let us apply a matrix multiplication to each channel, and then average over the channels.

Application of Tensor Operations

- In Index Notation

$$C_{ij} = \sum_{k,c} \underbrace{B_{ik}}_{\text{Matrix image}} \underbrace{A_{kjc}}_{\text{WtOverCh.}} \underbrace{w_c}_{\text{}}$$

- Notice that this notation can handle multilinear operations.

Application of Tensor Operations

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- In Einstein Notation:

$$B_{ik}, A_{kjc}, w_c \rightarrow C_{ij}$$

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Application of Tensor Operations

- First step

$$B_{ik}, A_{kjc} \rightarrow T_{ijc}$$



- Second step

$$T_{ijc} w_c \rightarrow C_{ij}$$



Let's also see some reshaping operations

- Vectorization:

$$\text{vec} \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}$$

- The 'Diag' Operation:

$$\text{Diag} \left(\begin{bmatrix} a_1 & a_2 \end{bmatrix} \right) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

- The 'Reshape' Operation:

$$\text{Reshape}_{32} \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{22} \\ a_{21} & a_{13} \\ a_{12} & a_{23} \end{bmatrix}$$

Kronecker Product

- It's sort of an outer product but has a specific shape,

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- Let's visualize this,

$$\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \otimes \begin{array}{c} \text{Image of a person} \end{array}$$

=



Why Bother?

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- The matrix form could be helpful when calculating gradients, and coming up with efficient implementations.
- Einsum is not as optimized as matrix multiplication.

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Linear Algebra Refresher

Basics

Array Manipulation

More linear algebraic concepts

Decompositions

Matrix inverse

- Let's think about a linear system,

$$Ax = b$$

$$\rightarrow A^{-1}Ax = x = A^{-1}b$$

- Is A^{-1} always defined?

Matrix inverse

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$$\begin{aligned} Ax &= b \\ \rightarrow A^{-1}Ax &= x = A^{-1}b \end{aligned}$$

- Is A^{-1} always defined?
- First, A needs to be square.
- Second, it needs to be full rank. Columns of A need to be linearly independent.

Matrix pseudoinverse

- Let's have the same linear system, but with a rectangular A matrix,

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Matrix pseudoinverse

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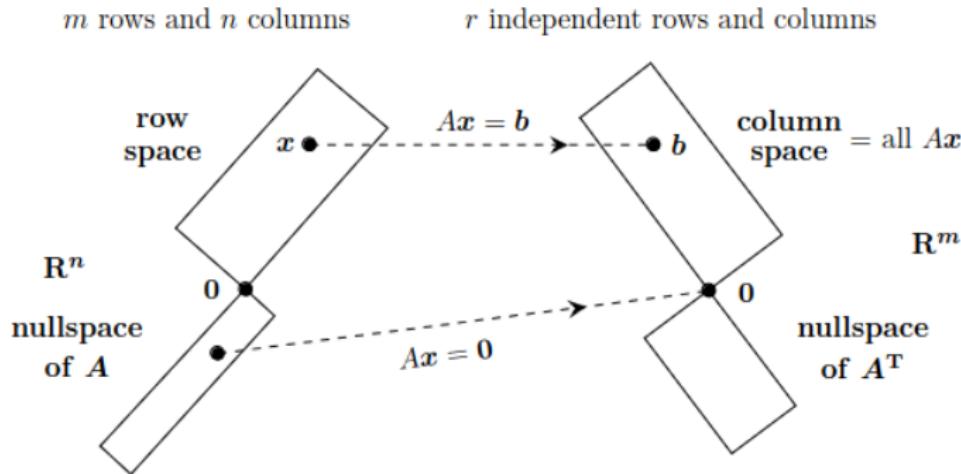
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- $A^\dagger := (A^\top A)^{-1} A^\top$. This is known as the pseudo inverse.
- This is essentially least squares. (We will show that later)

Four Fundamental Subspaces in Linear Algebra



BIG PICTURE OF LINEAR ALGEBRA

row space \perp nullspace

column space of $A \perp$ nullspace of A^T

row rank = column rank = r

Image Taken from Gilbert Strang's 'Introduction to Linear Algebra' book.

Norms, trace

- l_2 norm: $\|x\|_2 = \sqrt{\sum_j x_j^2}$. Also known as Euclidean Norm.
- l_1 norm: $\|x\|_1 = \sum_j |x_j|$.
- l_p norm: $\|x\|_p = \sqrt[p]{\sum_j |x_j|^p}$.
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Do not underestimate this.

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- Frobenius norm: $\|X\|_F = \sqrt{\sum_i \sum_j |X_{ij}|^2} = \sqrt{\text{tr}(XX^\top)}$

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- We are just giving an idea here with simple examples. We will see these more in real action later. (hint: backprop)

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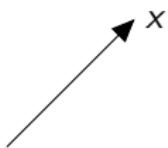
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Decompositions

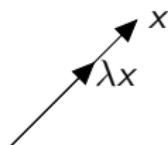
Eigenvalues / Eigenvectors

■ $Ax = \lambda x$



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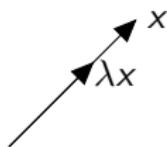
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- Note that x doesn't change its direction.

Eigenvalues / Eigenvectors

- $Ax = \lambda x$



- Note that x doesn't change its direction.
- Eigenvectors are 'characteristic' directions for the system described by A .

Finding the Eigenvectors

- The 'Linear Algebra Class Way':
- Let's have this matrix

$$A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$$

- Calculate the determinant (why?)

$$\det(A - \lambda I) = \begin{vmatrix} 0.8 - \lambda & 0.4 \\ 0.2 & 0.6 - \lambda \end{vmatrix} = \lambda^2 - 1.4\lambda + 0.40$$

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- Then we find vectors in the null space of $A - \lambda I$
- $A - I = \begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix} v = 0$, find a non-zero vector v such that
the equation is satisfied. $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

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- To get all the eigenvectors we can deflate the matrix. Just subtract v , and repeat the process..

Ok, but how is this a decomposition?

- $AV = V\Lambda$, where columns of V are the eigenvectors, and Λ is a diagonal matrix with eigenvalues on the diagonal.

Ok, but how is this a decomposition?

- $AV = V\Lambda$, where columns of V are the eigenvectors, and Λ is a diagonal matrix with eigenvalues on the diagonal.
- And here's the decomposition $A = V\Lambda V^{-1}$.
- But notice that this decomposition is only defined for square matrices.

Singular Value Decomposition

- Let us given a matrix of size X in $\mathbb{R}^{M \times N}$.
- $X = U\Sigma V^\top$, $U \in \mathbb{R}^{M \times M}$ and is orthogonal $U^\top U = I$, $\Sigma \in \mathbb{R}^{M \times N}$ is a matrix with non-zero elements on the main diagonal, and $V \in \mathbb{R}^{N \times N}$, and is orthonal $VV^\top = I$.

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- An alternative way of viewing it is $X = \sum_{k=1}^M \sigma_k u_k v_k^\top$. Note that we can cut the sum short, and keep the biggest singular values! (set $X = \sum_{k=1}^K \sigma_k u_k v_k^\top$, $K \leq M$)

$$X_{M \times N} = U \Sigma V^\top$$

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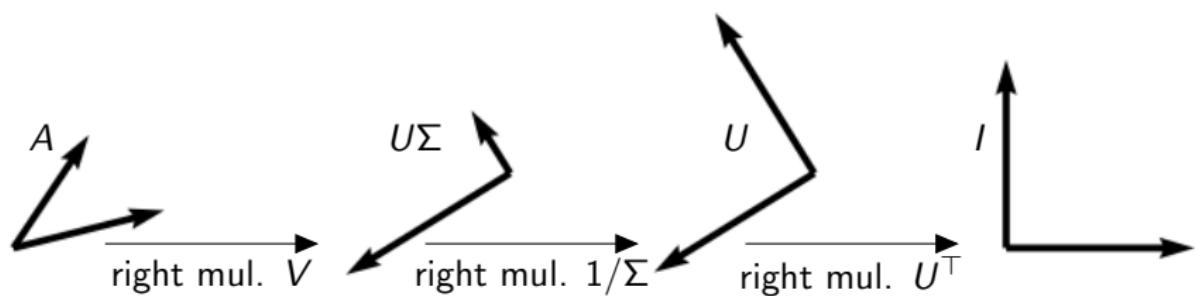
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- Singular vectors U of X , are the eigenvectors of XX^\top .
- Singular values σ_k of X , are the square root of eigenvalues of XX^\top .
- For positive semi-definite matrices, SVD and eigenvalue decomposition are equivalent.

Geometric Interpretation of SVD



List of Decompositions

- **LU decomposition:** $X = LU$, L is lower triangular, U is upper triangular.
- **QR decomposition:** $X = QR$, Q is a matrix with orthonormal columns, R is an upper triangular matrix.
- **Eigenvalue decomposition:** $X = U\Lambda U^{-1}$, columns of U are eigenvalues of X , which is square (diagonalizable) matrix.
- **Singular value decomposition:** $X = U\Sigma V^T$, columns of U , and V have orthonormal columns. Defined for any matrix.

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- There's more, e.g. Cholesky, NMF, CR, ICA, ...

List of special type of matrices we'll see in this class

- **Rotation matrices**
- **Markov matrices** (Probability Transition Matrices)
- **Transform matrices** (Fourier Transform, Convolution,...)
- **Covariance matrices** (Define a Multivariable Random Variable)
- **Adjacency matrices** (Define a Graph)

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- We saw how data/signals can be represented.



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- We saw how the data can be manipulated. (Vector, Matrix, Tensor Operations)



- We took a glimpse into how we can decompose signals.
- We gave a crude summary into what we need from Linear Algebra.

Recommended Reading

- Gilbert Strang, Introduction to Linear Algebra,
https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/video_galleries/video-lectures/,
https://math.mit.edu/~gs/linearalgebra/ila5/indexila5.html
- Trefethen and Bau, Numerical Linear Algebra,
<https://people.maths.ox.ac.uk/trefethen/text.html>
- Matrix Cookbook,
<http://www2.imm.dtu.dk/pubdb/doc/imm3274.pdf>

What's Next

- Probability Calculus, Random Variables, Multi-dimensional Distributions
- Exponential Family Distributions
- Maximum Likelihood, MAP, Bayesian parameter estimation principles
- Labs are starting next week! (first one is Sept. 12)