

IFT 4030/7030,
Machine Learning for Signal Processing
**Week6: Machine Learning 3,
Classification**

Cem Subakan



UNIVERSITÉ
Laval



- Le devoir 1 est dû la semaine prochaine.
 - ▶ Homework 1 was due this friday, now it's due next friday.
- Aujourd'hui: Classification

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Supervised Learning

- So far we have done unsupervised learning to discover structures.
 - ▶ Jusqu'à maintenant on a juste fait de l'apprentissage non-supervisé.
Le but était de découvrir la structure.
- Now, we will do classification / detection, or supervised learning in other words.
 - ▶ Maintenant on va faire de l'apprentissage supervisé.

A simple example



A simple example



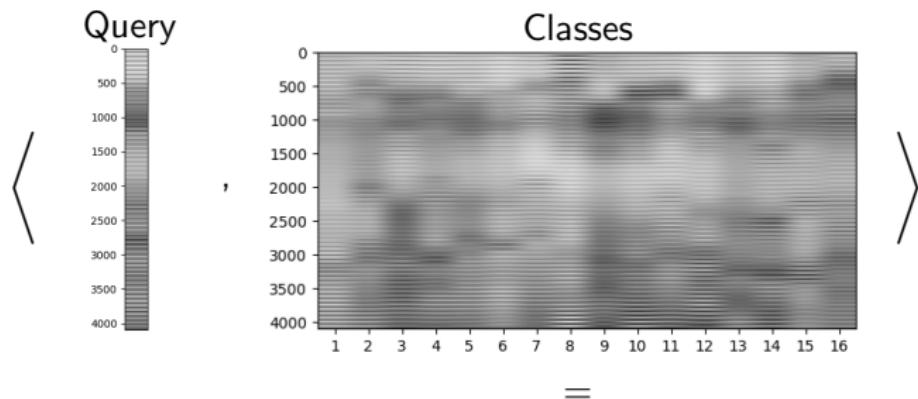
How can we assign the query to a class? / Comment peut-on assigner l'exemple en question à une classe?

We can simply calculate inner products!

We can calculate inner products! / On peut calculer des produits scalaires!

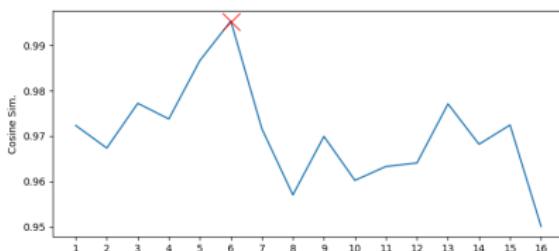
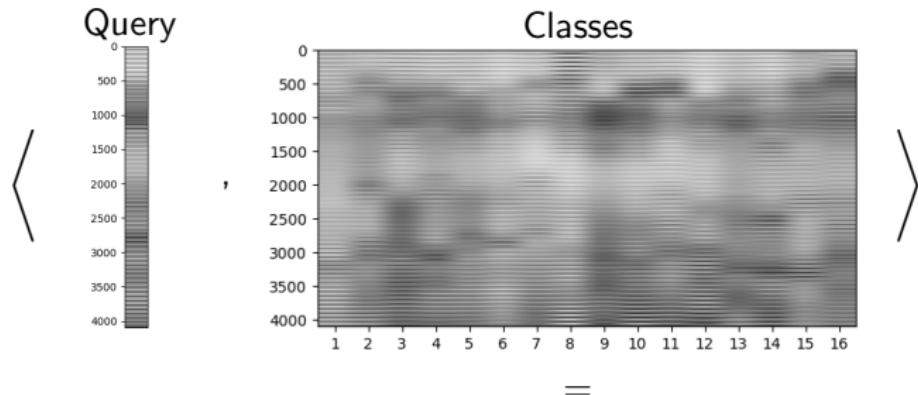
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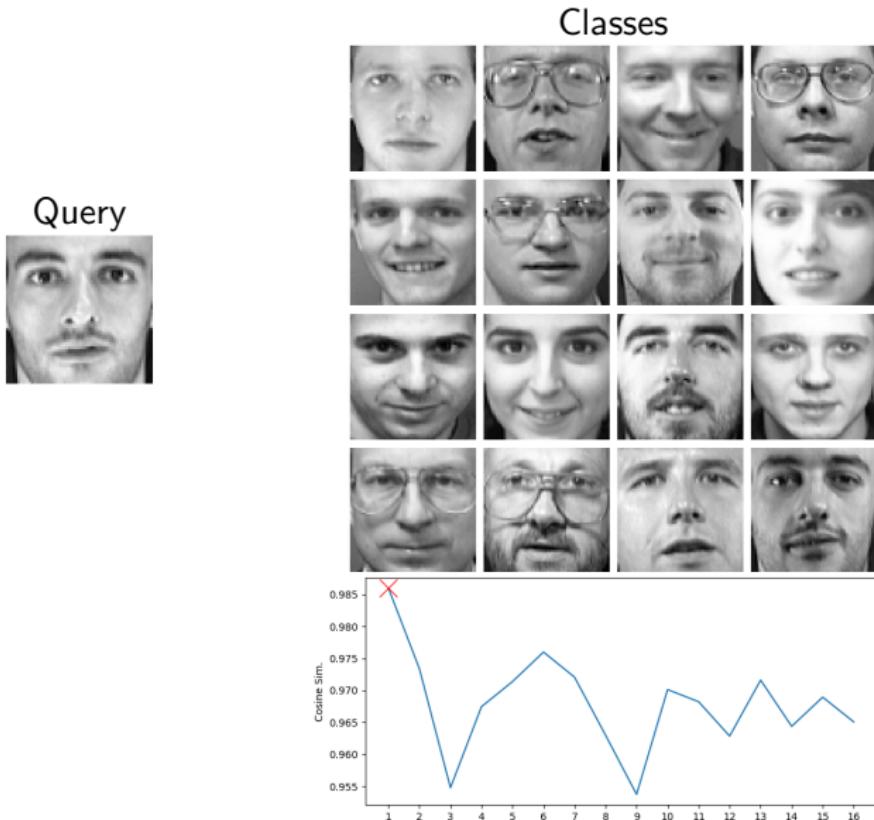
It worked but will it this time?



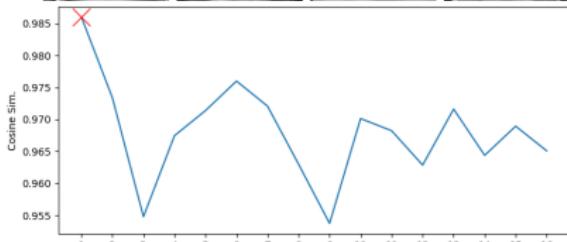
It worked but will it this time?



It worked but will it this time?



It worked but will it this time?



Didn't work this time! / Ça pas fonctionné!

What can we do to fix this?

- We will take a statistical approach (as usual). This will consider the class distributions. / On va prendre une approche de statistique comme d'habitude qui va prendre en compte les distributions de classes.

What can we do to fix this?

- We will take a statistical approach (as usual). This will consider the class distributions. / On va prendre une approche de statistique comme d'habitude qui va prendre en compte les distributions de classes.
- Approach 1: Generative Classification
 - ▶ Approche 1: Classification Générative
 - ▶ We will learn a distribution over samples in the class. / On va fitter une distribution sur les échantillons dans la classe.
- Approach 2: Discriminative Classification
 - ▶ Approche 2: Classification Discriminative
 - ▶ We will learn how the class distribution separate. / On va fitter une fonction pour comprendre comment les classes se séparent.

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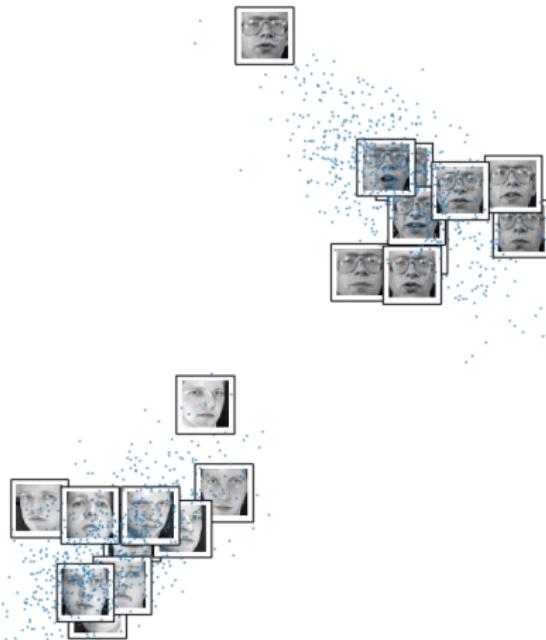
Non-Linear Classification

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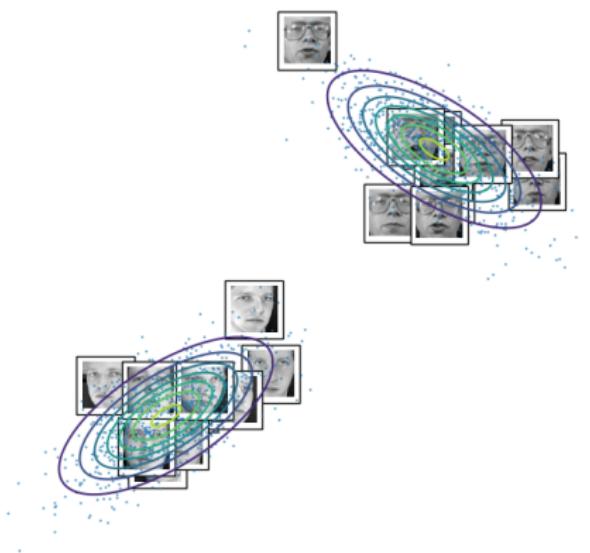
Generative Classification

- Generative Classification fits distributions to each class /
Classification générative fit une distribution à chaque classe.



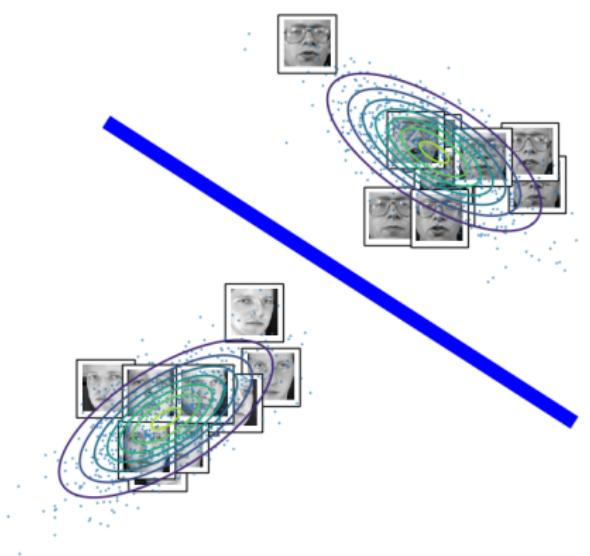
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Generative Classification

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Generative Classification

- Training Time: Fit a distribution $p(x|\theta_k)$ to each class k with maximum likelihood.
 - ▶ L'entraînement: On va fitter une distribution $p(x|\theta_k)$ pour chaque classe k avec maximum likelihood.

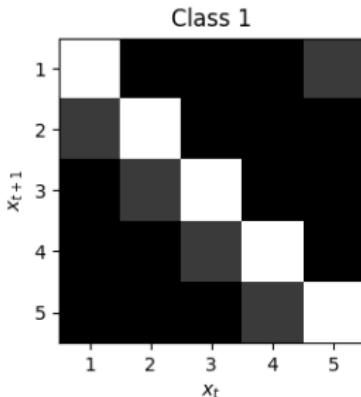
$$\max_{\theta_k} \sum_{n \in \text{class } k} \log p(x_n|\theta_k)$$

- Test Time: Evaluate the likelihood for each model. Assign to the largest likelihood class!
 - ▶ L'entraînement: Evaluez le likelihood pour chaque modèle. Assignez à la classe avec le likelihood plus grand.

$$\hat{c} = \arg \max_k \log p(x_{\text{test}}|\theta_k)$$

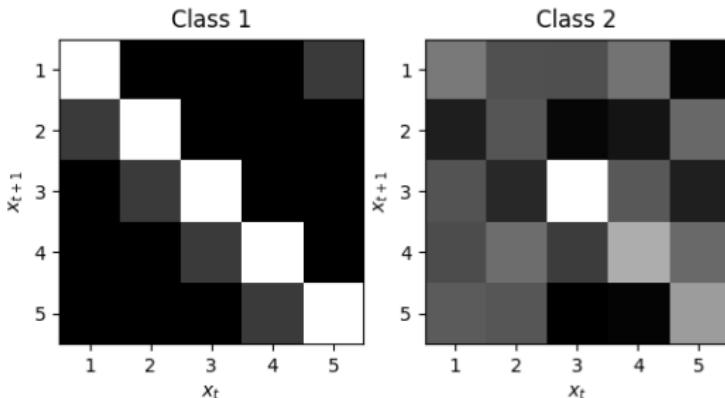
Example Application: Classifying Sequences

- class 1 $\sim \text{Markov}(A_1)$
- class 2 $\sim \text{Markov}(A_2)$



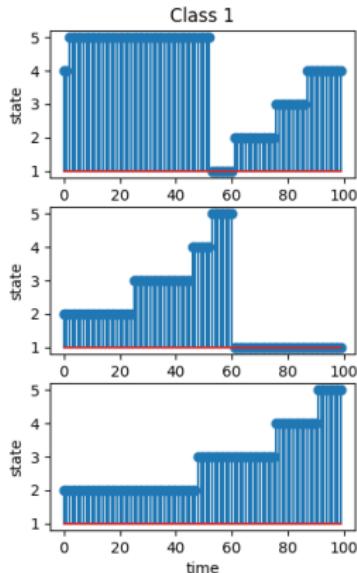
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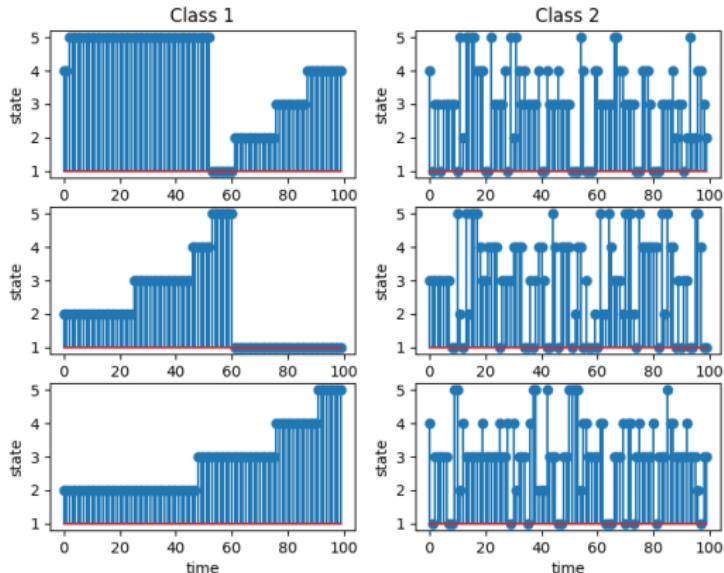
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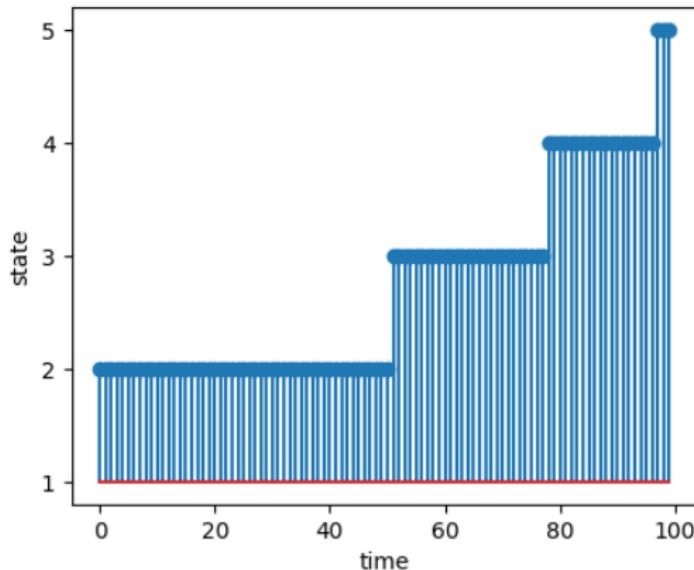
Example Application: Classifying Sequences

- class 1 $\sim \text{Markov}(A_1)$
- class 2 $\sim \text{Markov}(A_2)$



Test time

- Here's a test sequence / Une séquence de test



- Do you think it belongs to which class? / Vous pensez que cette séquence appartient à quelle classe?

Test time

$$\log p(x_{1:T} | A_k, \pi_k) = \log \left(p(x_1 | \pi_k) \prod_{t=2}^T p(x_t | x_{t-1}, A_k) \right)$$

Test time

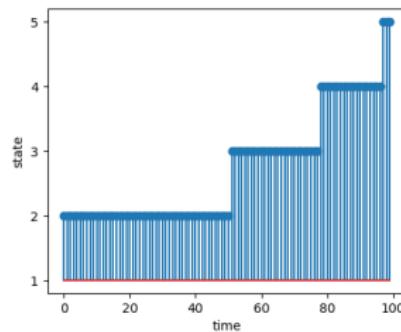
$$\begin{aligned}\log p(x_{1:T} | A_k, \pi_k) &= \log \left(p(x_1 | \pi_k) \prod_{t=2}^T p(x_t | x_{t-1}, A_k) \right) \\ &= \log p(x_1 | \pi_k) + \sum_{t=2}^T \log p(x_t | x_{t-1}, A_k)\end{aligned}$$

Test time

$$\begin{aligned}\log p(x_{1:T} | A_k, \pi_k) &= \log \left(p(x_1 | \pi_k) \prod_{t=2}^T p(x_t | x_{t-1}, A_k) \right) \\ &= \log p(x_1 | \pi_k) + \sum_{t=2}^T \log p(x_t | x_{t-1}, A_k) \\ &= \log \pi_k + \sum_{t=2}^T \log A_k(x_t, x_{t-1})\end{aligned}$$

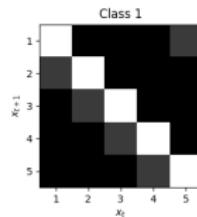
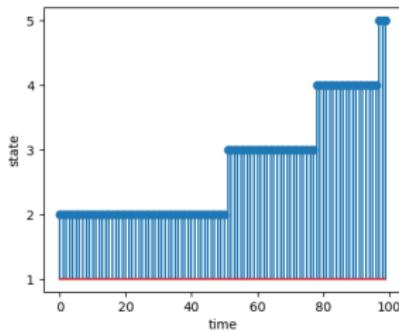
Now, let's calculate

■ Observation sequence / Séquence observé



Now, let's calculate

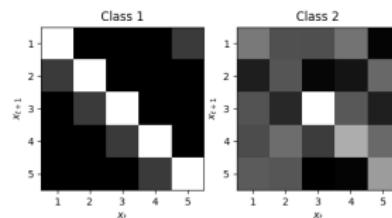
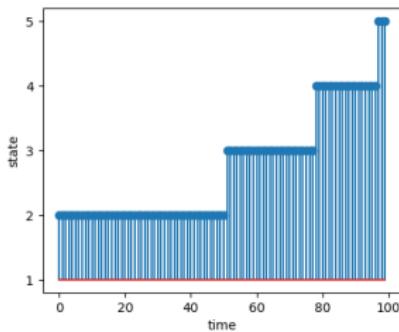
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$$\begin{aligned}\log p(x_{1:T} | A_1, \pi_1) &= \log \left(\pi_1 \prod_{t=2}^T A_1(x_t, x_{t-1}) \right) \\ &= \log \left(\frac{1}{5} \cdot 0.95 \cdot 0.95 \cdot 0.95 \dots \right) \\ &= -15.52\end{aligned}$$

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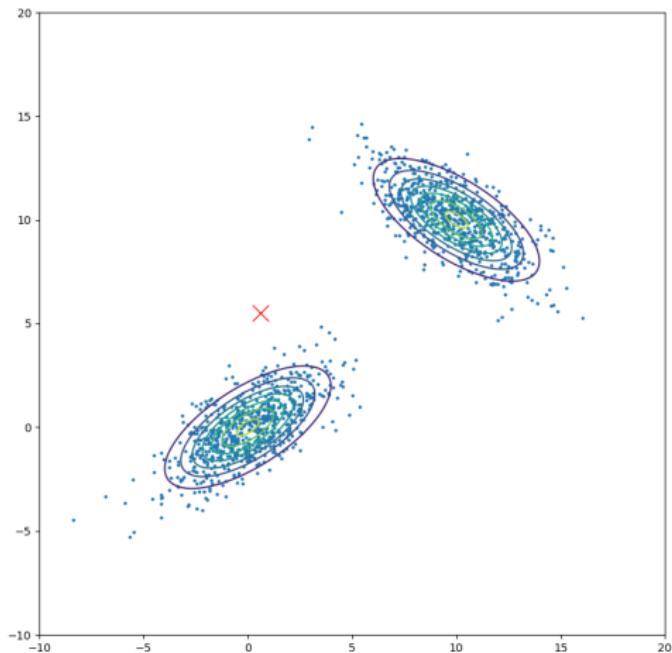


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$$\begin{aligned}\log p(x_{1:T} | A_2, \pi_2) &= \log \left(\pi_1 \prod_{t=2}^T A_2(x_t, x_{t-1}) \right) = \log \left(\frac{1}{5} \cdot 0.21 \cdot 0.21 \cdot 0.21 \dots \right) \\ &= -117.93\end{aligned}$$

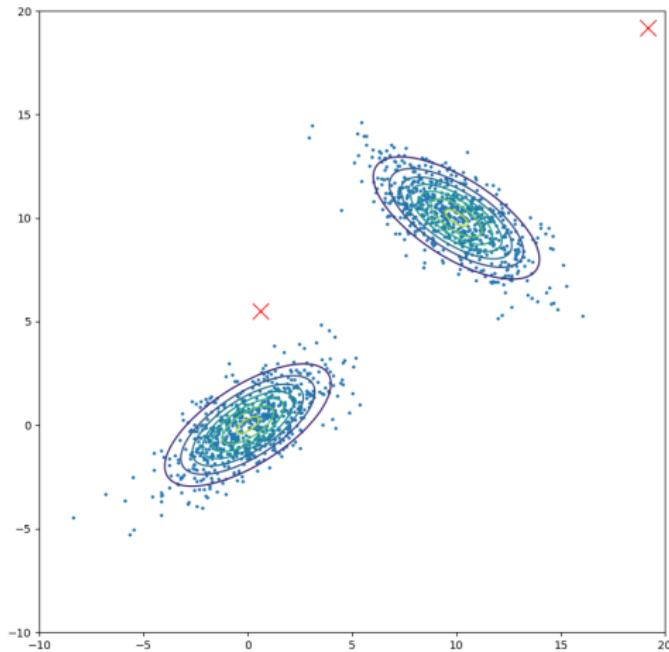
Let's do something simpler

Which class should we assign the red point? / Quel classe doit-on assigner au point rouge?



How about this?

How about this second point? / Et le deuxième point?



The principled way to derive the decision boundaries

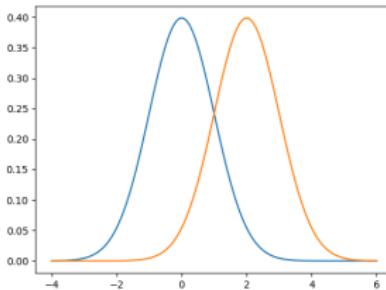
- The discriminant function / La fonction de discrimination

$$d(x) = \log \frac{p(x|c=1)}{p(x|c=2)} = \log \frac{p(x|\theta_1)}{p(x|\theta_2)}$$

- The decision boundary: $d(x) = 0$.
- For Gaussians / simple densities this can be derived analytically.

Deriving the decision boundary

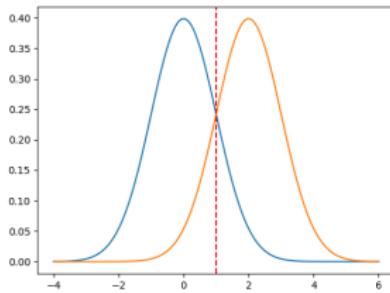
- Derive the boundary in 1d case / Dérivons dans le cas 1d.



$$d(x) = \log \frac{p(x|\theta_1)}{p(x|\theta_2)} = \log \frac{\mathcal{N}(0; 1)}{\mathcal{N}(2; 1)}$$

Deriving the decision boundary

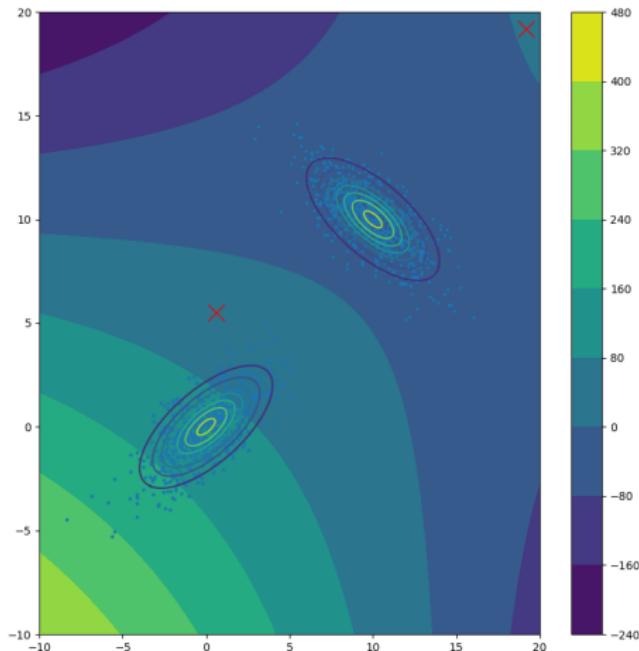
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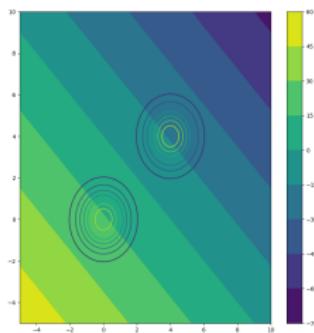
$$\begin{aligned} d(x) &= \log \frac{p(x|\theta_1)}{p(x|\theta_2)} = \log \frac{\mathcal{N}(0; 1)}{\mathcal{N}(2; 1)} \\ &= \log \frac{x^2}{(x - 2)^2} = 0 \\ \rightarrow x^2 &= (x - 2)^2 \rightarrow x = 1 \end{aligned}$$

The 2d decision boundary

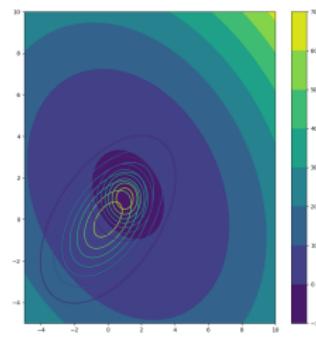
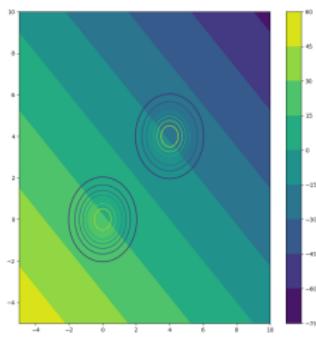
$$\text{Plot } d(x) = \log \frac{\mathcal{N}([x,y]^\top; \mu_1, \Sigma_1)}{\mathcal{N}([x,y]^\top; \mu_2, \Sigma_2)}$$



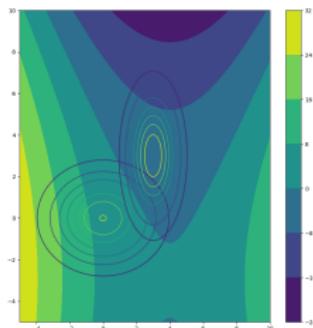
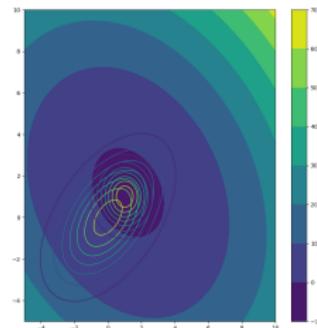
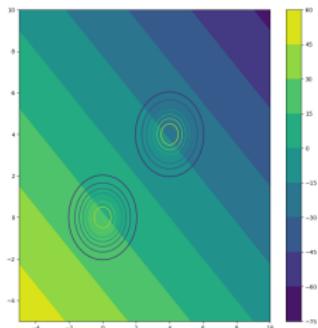
Few other cases



Few other cases



Few other cases



All these cases can be derived analytically! / C'est possible analytiquement
calculer

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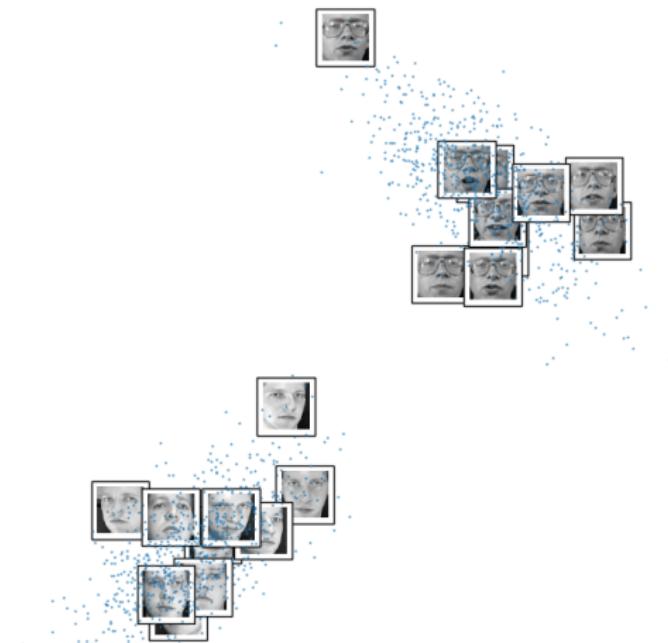
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Discriminative Classification

- Discriminative classification directly learns a decision boundary / Classification discriminative apprend directement un borne de décision.



Discriminative Classification

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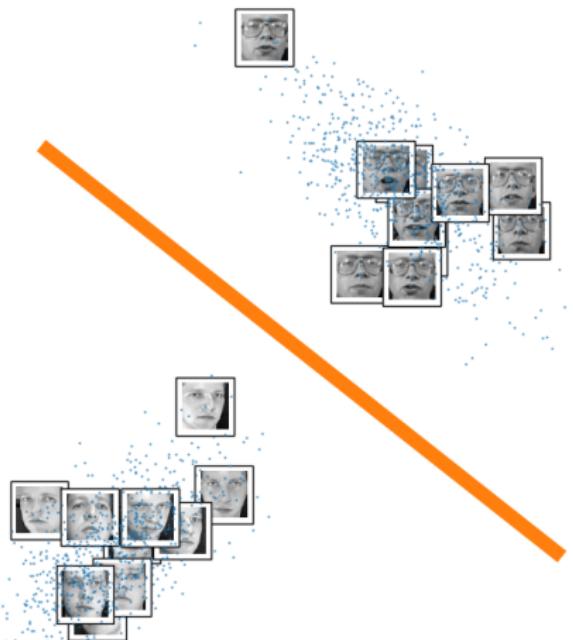


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- More specifically, we want to have (on veut avoir)

$$w^\top x \geq 0 \text{ if } c = 1$$

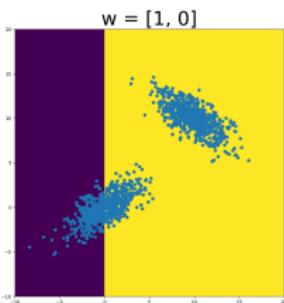
$$w^\top x \leq 0 \text{ if } c = 0$$

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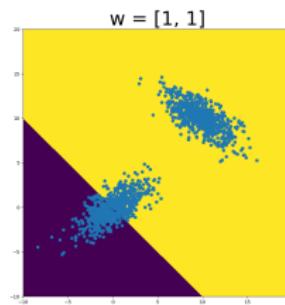
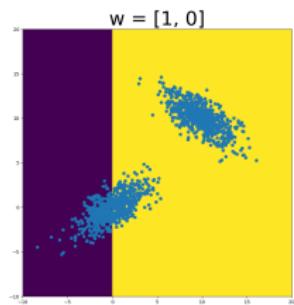


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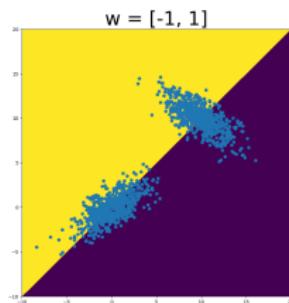
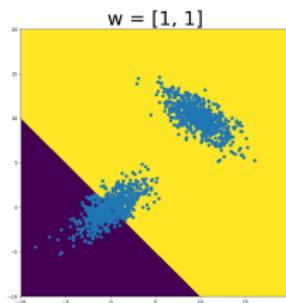
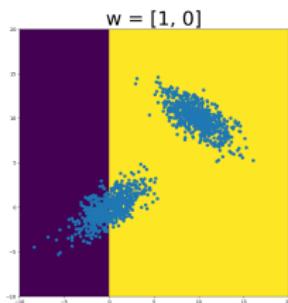


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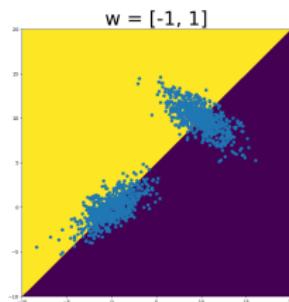
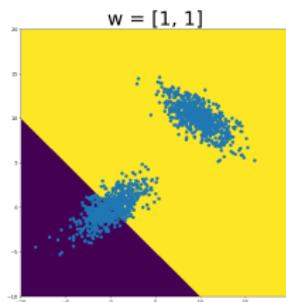
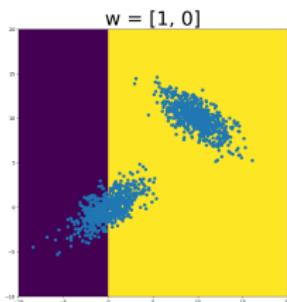


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$$w^\top x \leq 0 \text{ if } c = 0$$



- Btw, we will also add a bias term so that $f(x) = w^\top x + b$ / En passant on va aussi ajouter un biais.

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The perceptron algorithm

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 - ▶ If $\text{sgn}(w^\top x_n) = c_n$ do nothing.
 - ▶ If $\text{sgn}(w^\top x_n) = -c_n$, then $w = w + \eta c_n x_n$. (η is a learning rate)

The perceptron algorithm

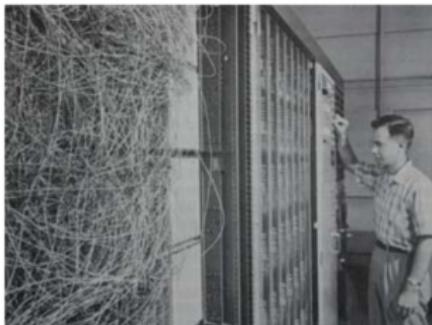
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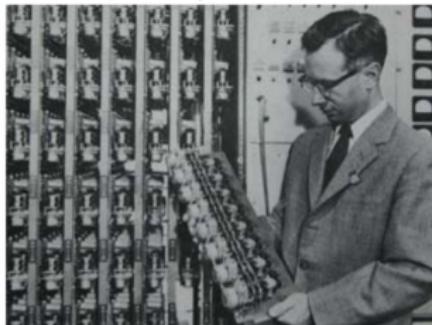
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 - ▶ Do these updates until convergence. / On répète jusqu'à ce que converge.
 - ▶ Note that $c_n \in \{-1, 1\}$. / Notez que $c_n \in \{-1, 1\}$.

The perceptron

Feature extraction processor

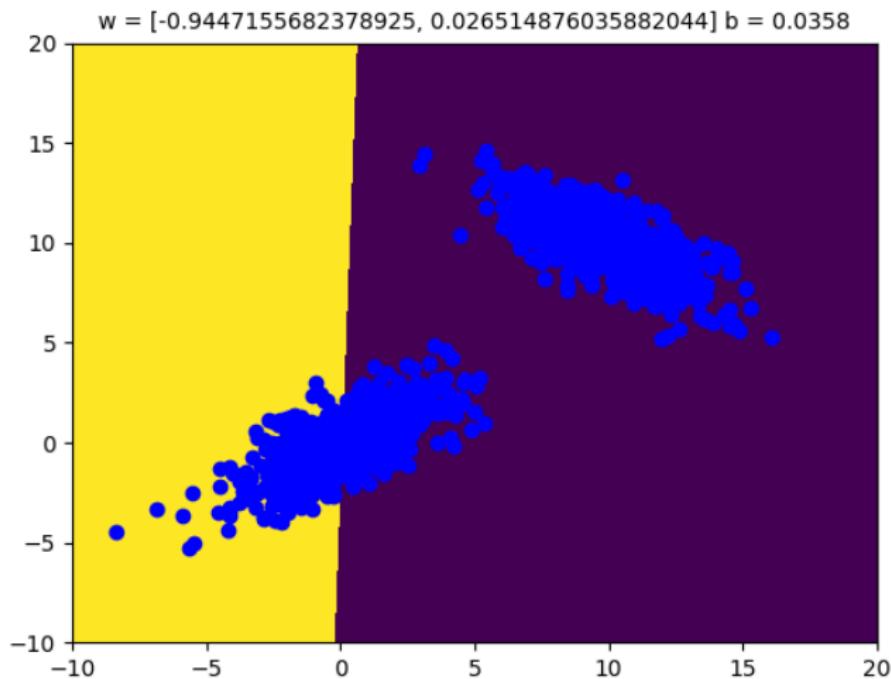


*Perceptron weights
(motor driven potentiometers)*

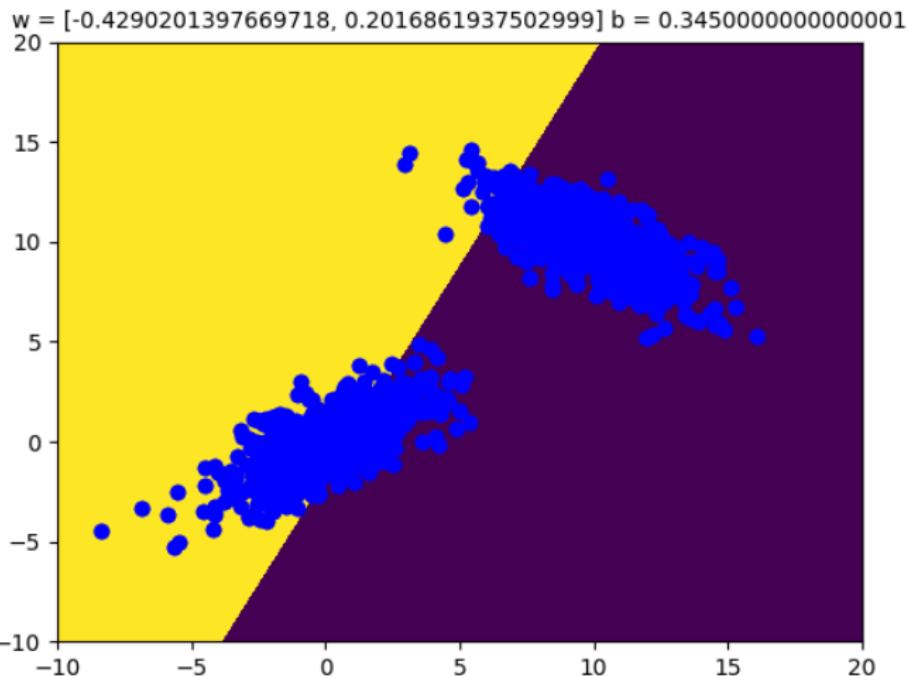


Taken from UIUC MLSP class

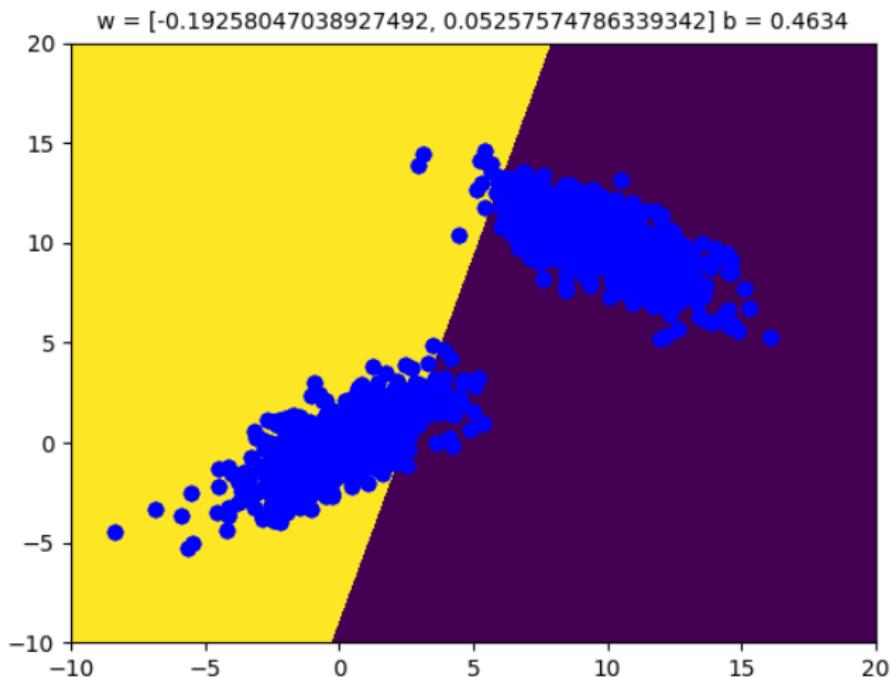
Perceptron Epochs



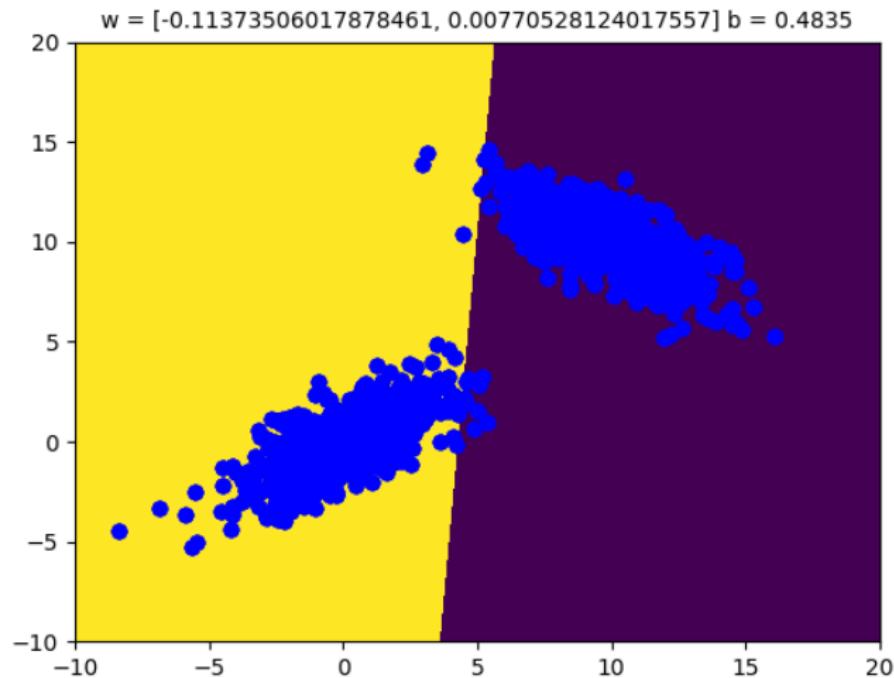
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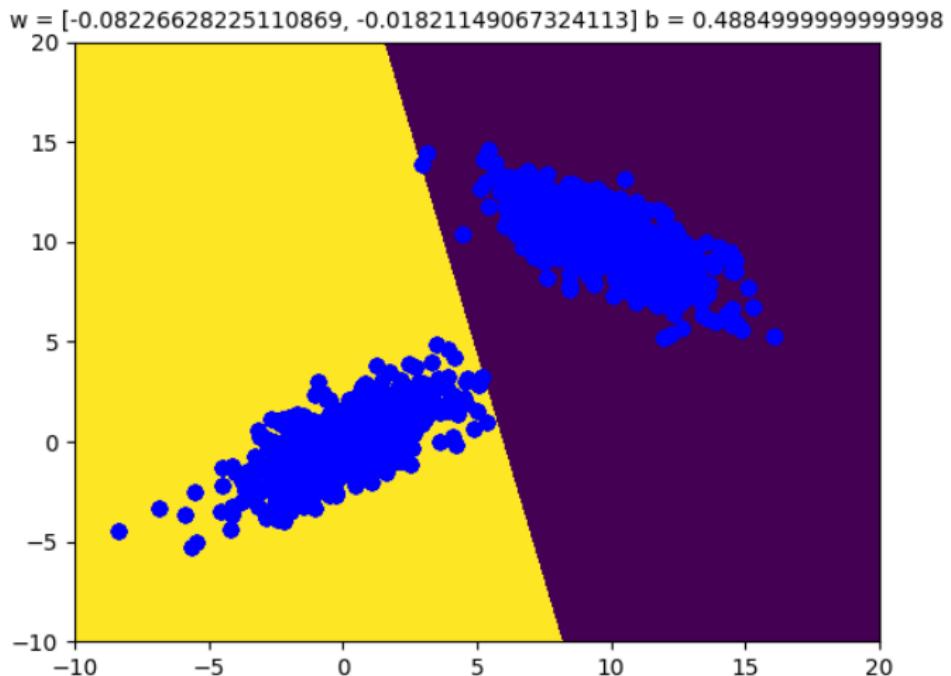
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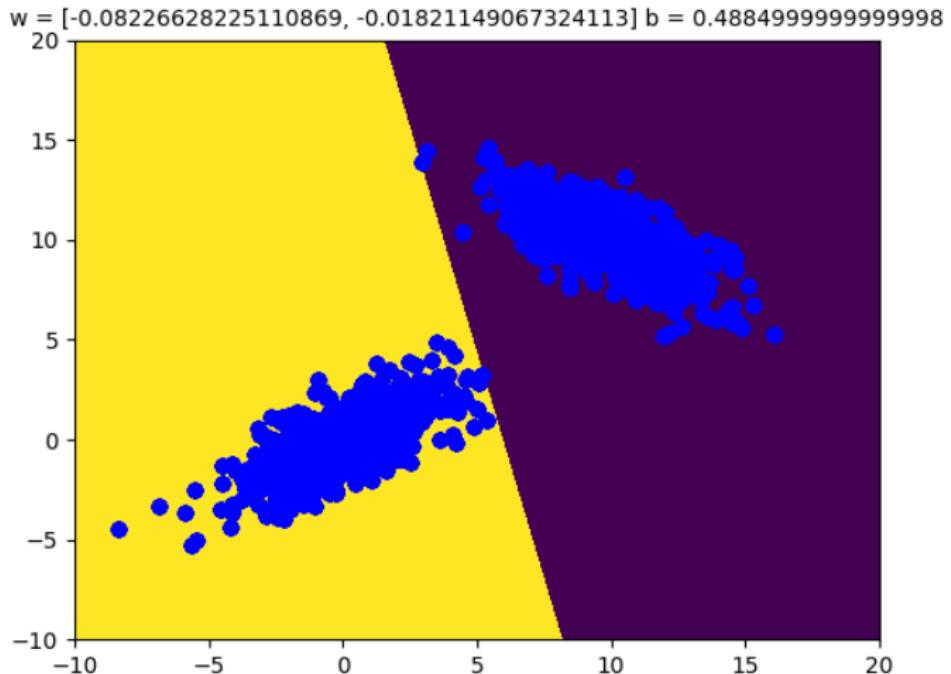
Perceptron Epochs



Perceptron Epochs



Perceptron Epochs



Deriving the perceptron algorithm

- Define a cost function / Définissons une fonction de cout:

$$\mathcal{L}(w, b) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n (w^\top x_n + b)$$

Deriving the perceptron algorithm

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$$\mathcal{L}(w, b) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n (w^\top x_n + b)$$

- Do the gradient descent updates: / Faisons les GD updates:

$$w = w - \eta \nabla_w \mathcal{L}(w, b)$$

$$b = b - \eta \nabla_b \mathcal{L}(w, b)$$

Deriving the perceptron algorithm

- Define a cost function / Définissons une fonction de cout:

$$\mathcal{L}(w, b) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n (w^\top x_n + b)$$

- Do the gradient descent updates: / Faisons les GD updates:

$$w = w - \eta \nabla_w \mathcal{L}(w, b)$$

$$b = b - \eta \nabla_b \mathcal{L}(w, b)$$

- The gradient: / Le gradient:

$$\nabla_w \mathcal{L}(w) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n x_n$$

$$\nabla_b \mathcal{L}(b) = - \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n$$

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- The updates / Les updates: (Same as the perceptron algo! / Les memes updates qu'algo perceptron!)

$$w = w + \eta \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n x_n$$

$$b = b + \eta \sum_{\forall c_n \neq \text{sign}(w^\top x_n + b)} c_n$$

But can we do something better?

- Perceptron is basically / Perceptron est simple:

$$f(x) = u(w^\top x + b)$$

where $u(\cdot)$ is a step function (or $\text{sign}(\cdot)$ depending on how we define c_n) . / où $u(\cdot)$ est un ‘step function’.

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 - ▶ À cause de la step function on ne peut pas utiliser le modèle directement dans le fonction de cout.

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- Because of that we can not directly use above within a loss function, and instead we use a modified objective. (Not all items get updated.)
 - ▶ À cause de la step function on ne peut pas utiliser le modèle directement dans le fonction de cout.
- What if we want to have smooth, differentiable transitions? / Et si on veut avoir des transitions plus smooth?

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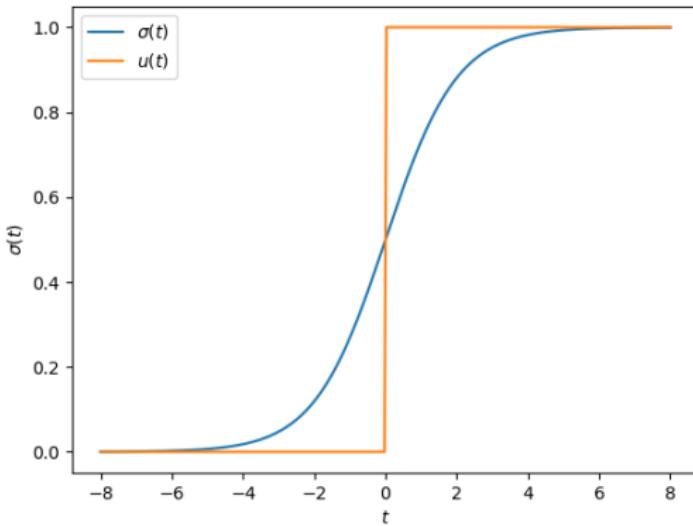
Non-Linear Classification

Kernel Logistic Regression

Neural Network Classification

The logistic function

- We can use the logistic function $\sigma(t) = \frac{1}{1+e^{-t}}$. / On peut utiliser la fonction logistique.



Logistic regression

- Set the estimator as $f(w, b) = \sigma(w^\top x + b)$.

Logistic regression

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- What will be the loss function? / Quelle sera la fonctionne de coût?
- How about the negative Bernoulli log-likelihood?

Bernoulli Distribution

- $\mathcal{BE}(y; \pi) = \pi^y(1 - \pi)^{1-y}, y \in \{0, 1\}.$
- Let's take the log / Prenons le logarithme

$$\log \mathcal{BE}(y; \pi) = y \log \pi + (1 - y) \log(1 - \pi)$$

- We will parametrize the Bernoulli distribution with $\pi = w^\top x + b$ /
On est en train de paramétriser la distribution Bernoulli.

Logistic regression loss function

- Training loss / La fonction de cout pour l'entraînement:

$$\begin{aligned}\mathcal{L}(w, b) &= \sum_n (-y_n \log \pi + (1 - y_n) \log(1 - \pi)) \\ &= \sum_n \left(-y_n \log (\sigma(w^\top x + b)) - (1 - y_n) \log (1 - \sigma(w^\top x + b)) \right)\end{aligned}$$

- Let's calculate the gradient with respect to w . / Calculons le gradient par rapport à w .

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- The chain rule: $\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx} = f'(g(x)) \cdot g'(x)$

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- Few more things: $\frac{\sigma(x)}{dx} = \sigma(x)(1 - \sigma(x))$, $\frac{\log x}{dx} = 1/x$.

The gradient wrt w

- Just do it! / Faisons-le!

$$\begin{aligned}\frac{d\mathcal{L}(w, b)}{dw} &= - \sum_n y_n \frac{\sigma(w^\top x_n + b)}{\sigma(w^\top x_n + b)} (1 - \sigma(w^\top x_n + b)) x_n \\ &\quad + \sum_n (1 - y_n) \frac{1 - \sigma(w^\top x_n + b)}{1 - \sigma(w^\top x_n + b)} \sigma(w^\top x_n + b) x_n\end{aligned}$$

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- Makes a lot of sense! / Ça fait du sens!

The gradient wrt w

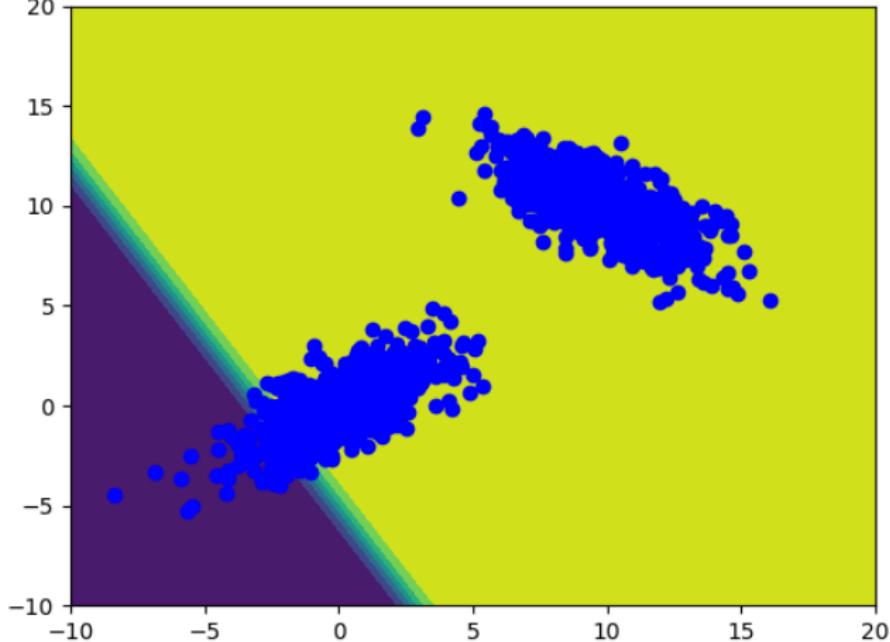
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$$\begin{aligned}\frac{d\mathcal{L}(w, b)}{dw} &= - \sum_n y_n \frac{\sigma(w^\top x_n + b)}{\sigma(w^\top x_n + b)} (1 - \sigma(w^\top x_n + b)) x_n \\ &\quad + \sum_n (1 - y_n) \frac{1 - \sigma(w^\top x_n + b)}{1 - \sigma(w^\top x_n + b)} \sigma(w^\top x_n + b) x_n \\ &= - \sum_n y_n \cancel{\frac{\sigma(w^\top x_n + b)}{\sigma(w^\top x_n + b)}} (1 - \sigma(w^\top x_n + b)) x_n \\ &\quad + \sum_n (1 - y_n) \cancel{\frac{1 - \sigma(w^\top x_n + b)}{1 - \sigma(w^\top x_n + b)}} \sigma(w^\top x_n + b) x_n \\ &= \sum_n (\sigma(w^\top x_n + b) - y_n) x_n\end{aligned}$$

- Makes a lot of sense! / Ça fait du sens!
- torch will automatically do this for us. However, one needs to do this at least once in their life!
 - ▶ torch fait ça automatiquement mais on doit faire cette exercice au moins une fois dans notre vie!

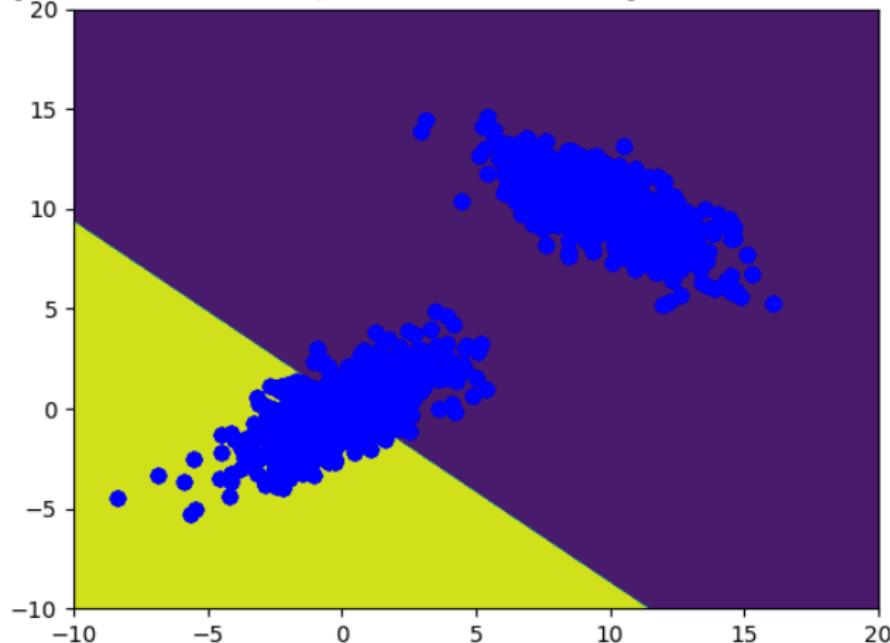
Logistic Regression Epochs

$w = [2.774843454360962, 1.6030927896499634]$ $b = 4.142136573791504$



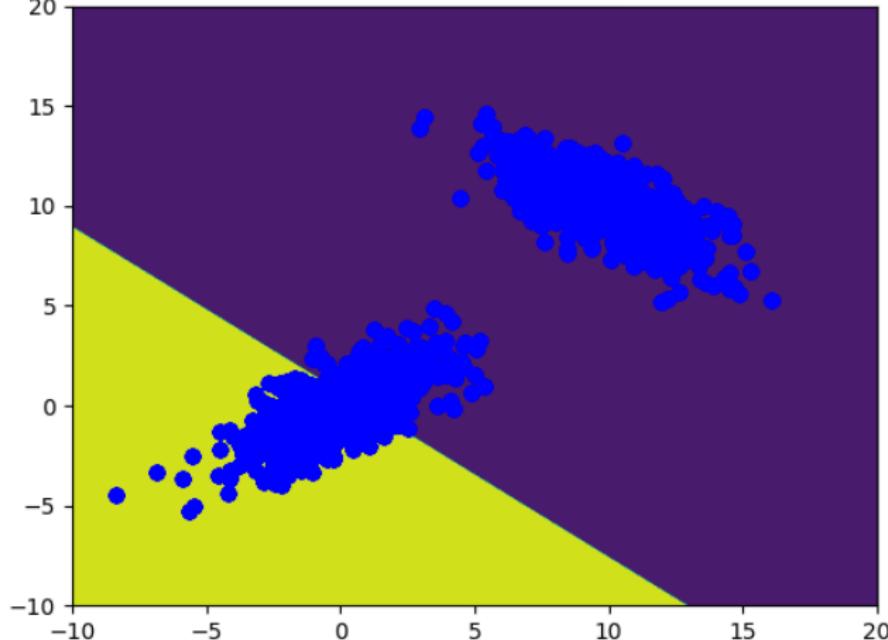
Logistic Regression Epochs

$w = [-64.31231689453125, -71.11492156982422]$ $b = 10.29466533660888$



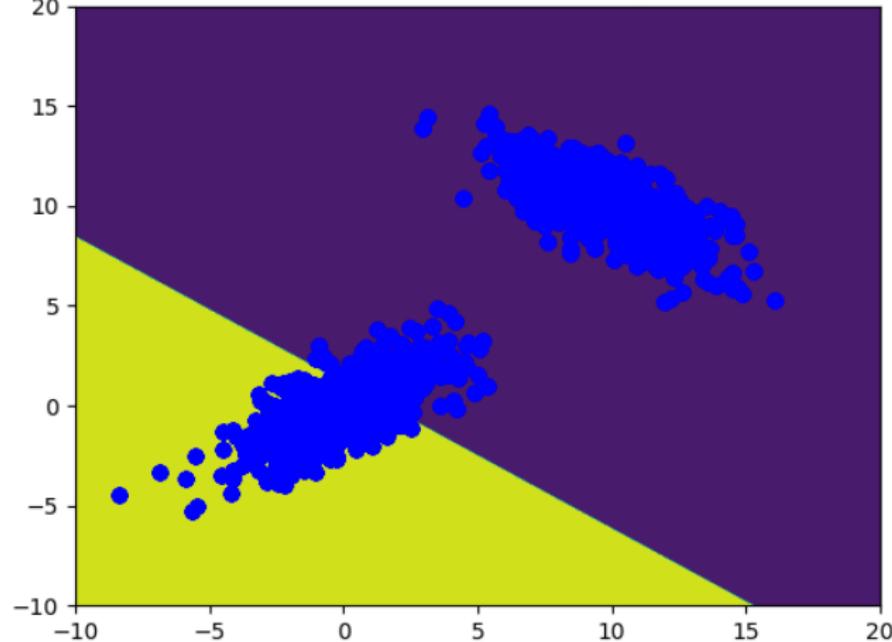
Logistic Regression Epochs

$w = [-57.4975700378418, -69.5396728515625] b = 22.519577026367188$



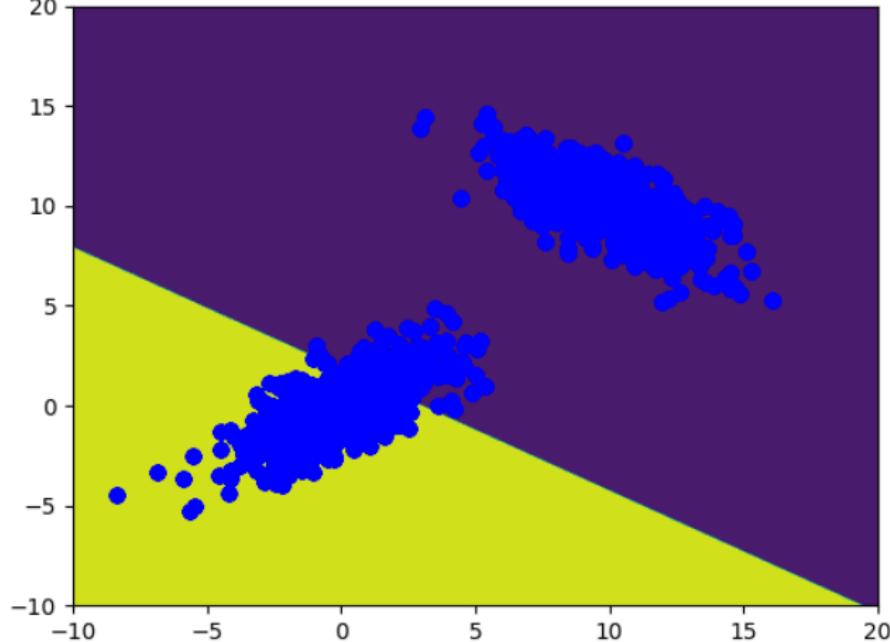
Logistic Regression Epochs

$w = [-46.69367980957031, -63.91728591918945]$ $b = 35.9787139892578$



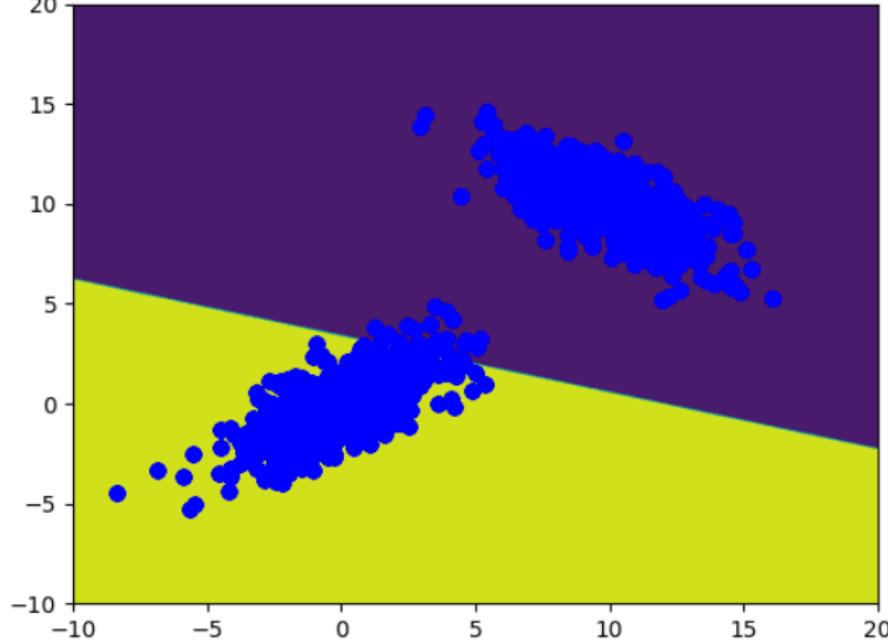
Logistic Regression Epochs

$w = [-32.589866638183594, -53.49893569946289]$ $b = 48.29494476318359$



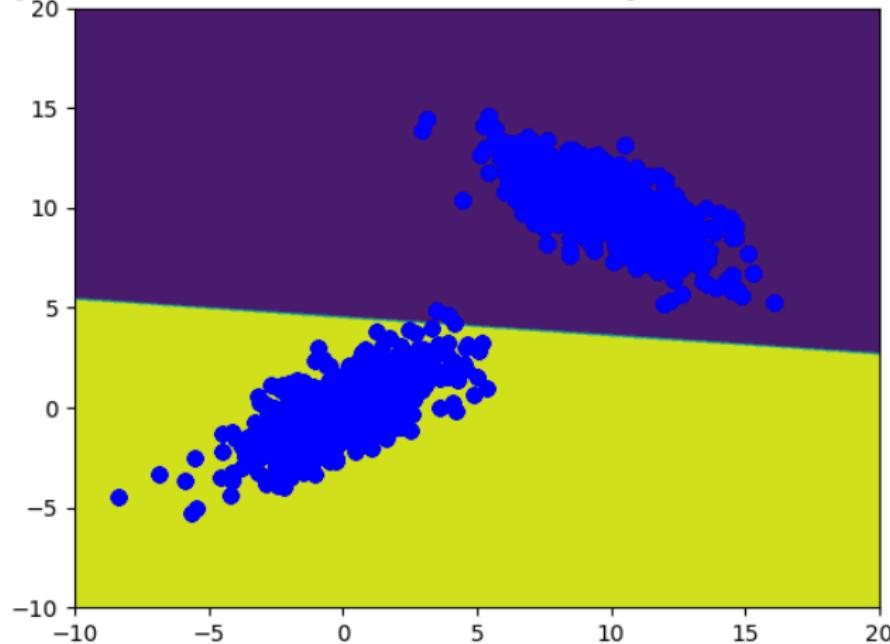
Logistic Regression Epochs

$w = [-9.948322296142578, -35.09688186645508]$ $b = 59.3541374206543$



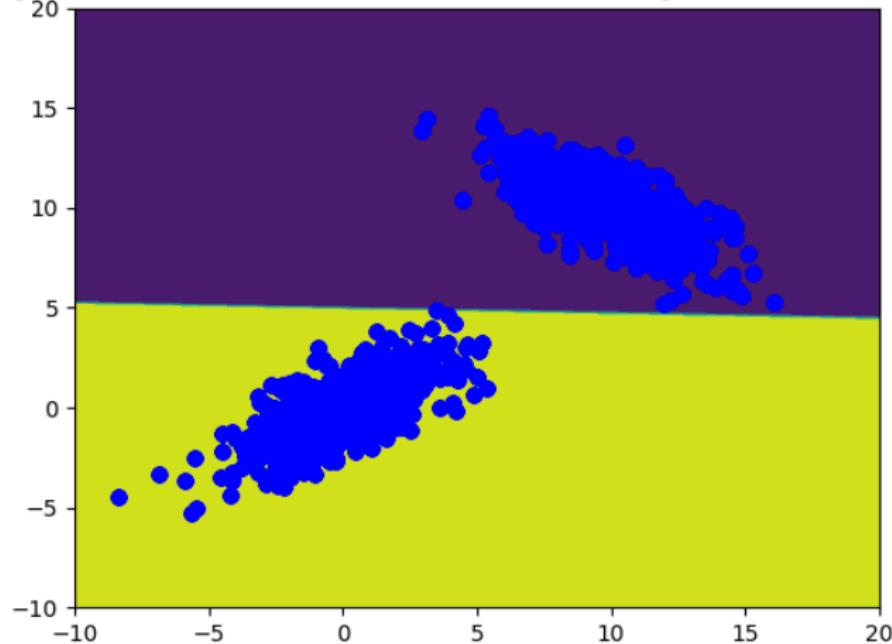
Logistic Regression Epochs

$w = [-2.5054750442504883, -27.3735294342041]$ $b = 61.56295776367187$



Logistic Regression Epochs

$w = [-0.6339982151985168, -24.971982955932617]$ $b = 62.072887420654$



Generalizing to multiway classification

- Define (a new network!) / On définit une nouvelle réseau

$$\pi_k := \exp(w_k^\top x + b_k) / \sum_l \exp(w_l^\top x + b_l)$$

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- The loss function is now going to be the log-likelihood of discrete distribution / La fonction de cout sera maintenant le log-likelihood de la distribution discrète.

$$\mathcal{L}(W, b) = -\log \left(\prod_n \text{Discrete}(y_n; \pi_{1:K}) \right)$$

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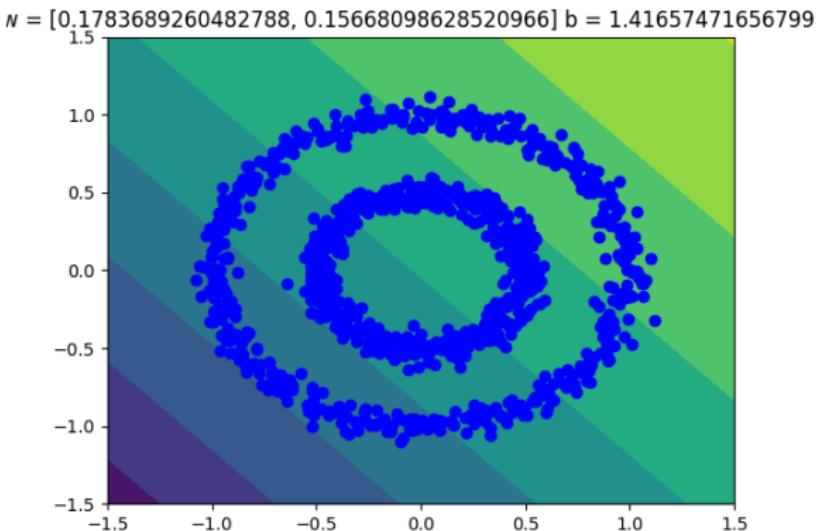
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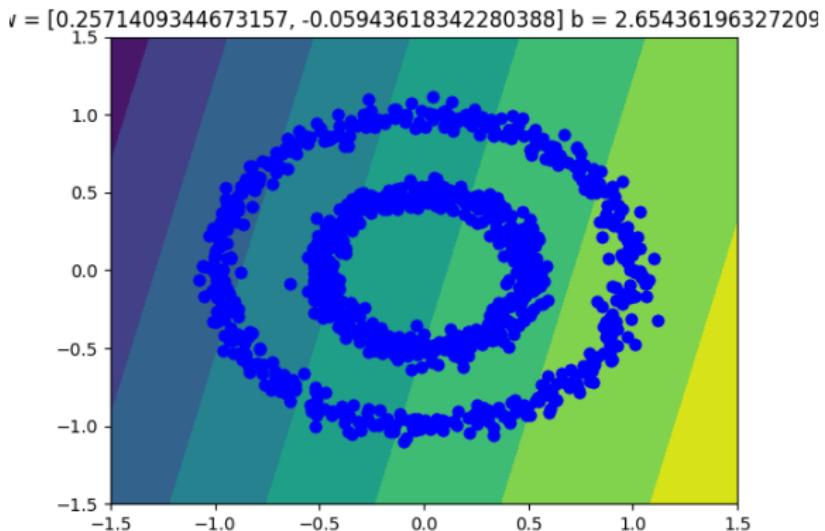
Non-Linear Classification

- What if we have something like this? / Qu'est-ce qu'on fait si on a ça?



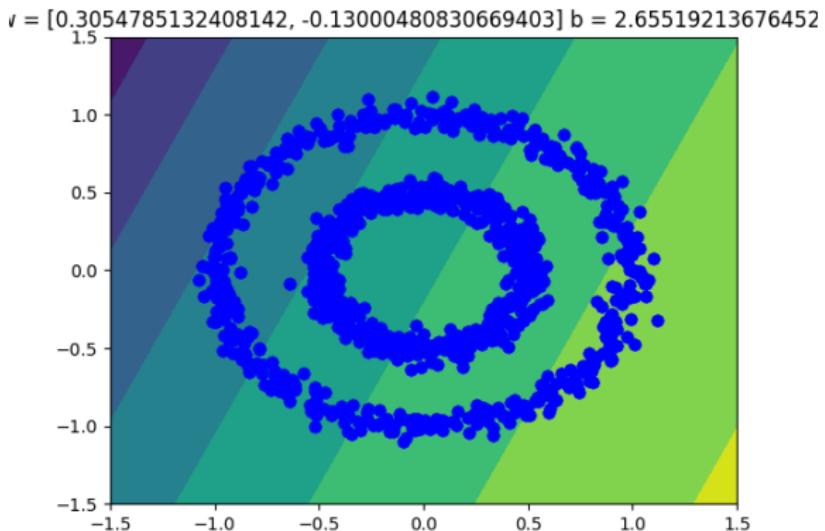
Non-Linear Classification

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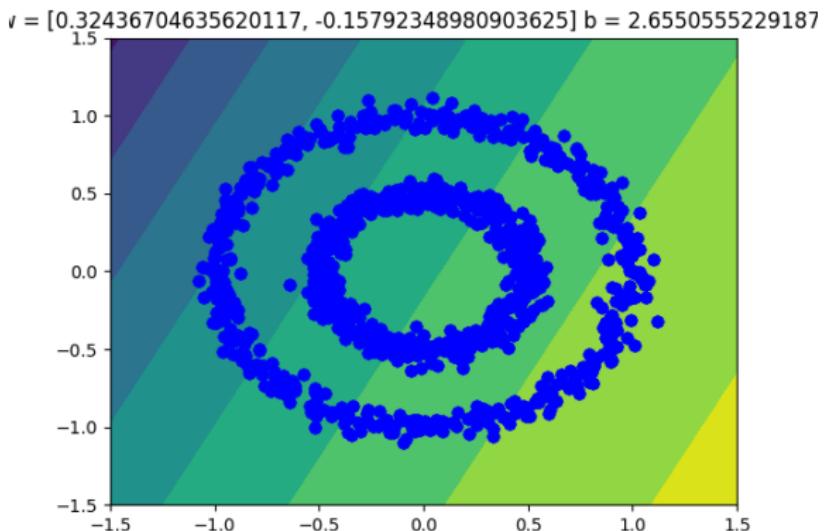
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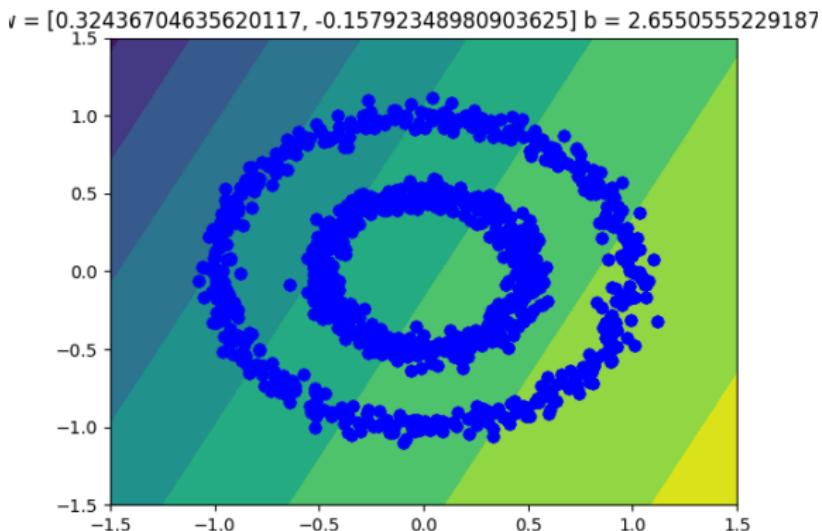
Non-Linear Classification

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Non-Linear Classification

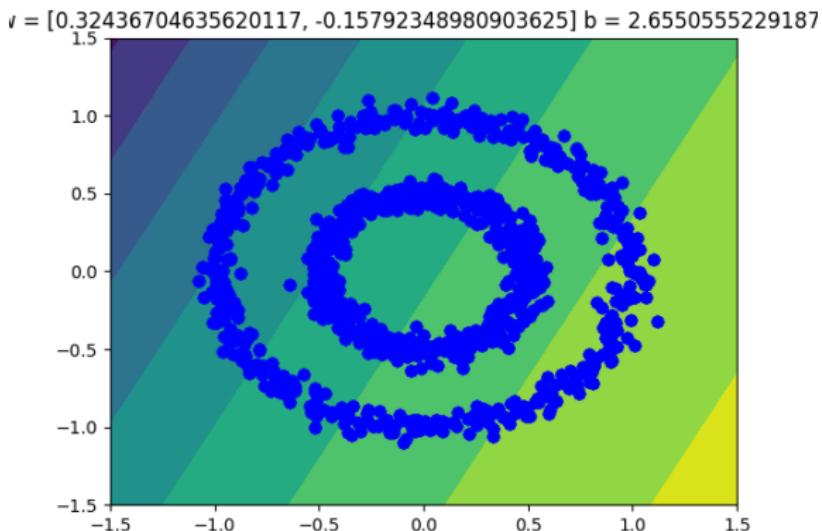
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- Linear classifier is not enough! / Classificateur linéaire n'est pas suffisante!

Non-Linear Classification

- What if we have something like this? / Qu'est-ce qu'on fait si on a ça?



- Linear classifier is not enough! / Classificateur linéaire n'est pas suffisante!
- What can we do? / Qu'est-ce qu'on peut faire?

The Kernel Trick

- We can extend our linear classifier $y = w^\top x \rightarrow y = w^\top \phi(x)$. We introduce a feature transformation $\phi(\cdot)$.
 - ▶ On améliore notre classificateur linéaire en introduisant une transformation de feature $\phi(\cdot)$.

The Kernel Trick

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- As we talked about it last week for Kernel-PCA, introducing $\phi(\cdot)$ is not possible if ϕ is large dimensional. / Comme on en a parlé $\phi(\cdot)$ n'est pas réalisable si on map à une grande nombre de dimensions.

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- **Kernel Trick:** We can substitute/ On substitue $w = \sum_n a_n \phi(x_n)$, so

$$\begin{aligned}f(x) &= \sum_n a_n \phi(x)^\top \phi(x_n) \\&= \sum_n a_n k(x, x_n)\end{aligned}$$

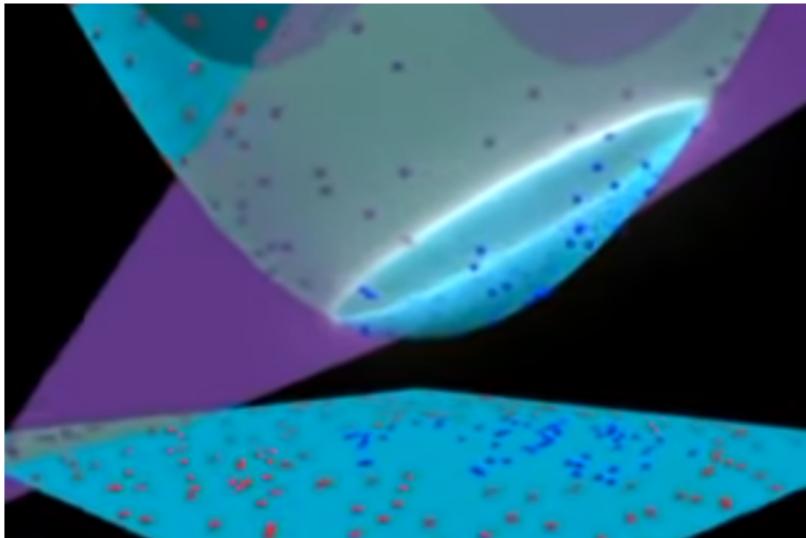
The Kernel Trick

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- We can incorporate this in many places. But let's look at injecting this idea in logistic regression.
 - ▶ On peut incorporer cette idée dans différents place. Mais d'abord essayons d'injecter cette idée dans la regression logistique.

Motivating the Kernel Trick



Let's watch!

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Kernel Logistic Regression

- The coin bias is now estimated with the kernel-trick / Le biais est maintenant estimé avec le kernel trick.

$$\pi = \sum_n a_n k(x, x_n)$$

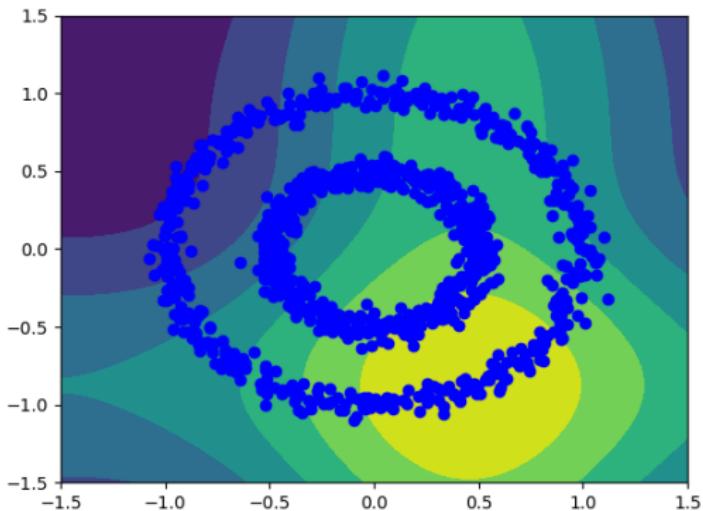
- Then the loss becomes, / le fonctionne de cout devient,

$$\begin{aligned}\mathcal{L}(a) = & \sum_n \left(-y_n \log \left(\sigma \left(\sum_j a_j k(x_n, x_j) \right) \right) \right. \\ & \left. - (1 - y_n) \log \left(1 - \sigma \left(\sum_j a_j k(x_n, x_j) \right) \right) \right)\end{aligned}$$

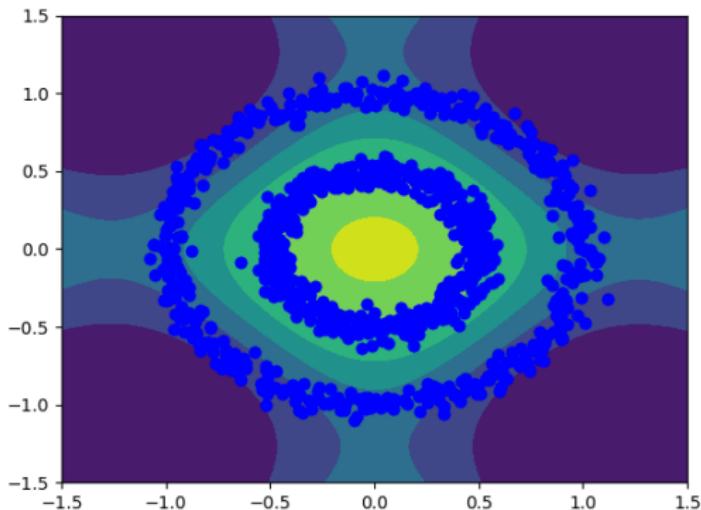
- Kernel functions are chosen from options such as RBF Kernel, Polynomial Kernel,.. / On choisit le kernel entre les options qu'on est habitué.

$$k_{\text{rbf}}(x, x_n) = \exp(-\gamma \|x - x_n\|_2)$$

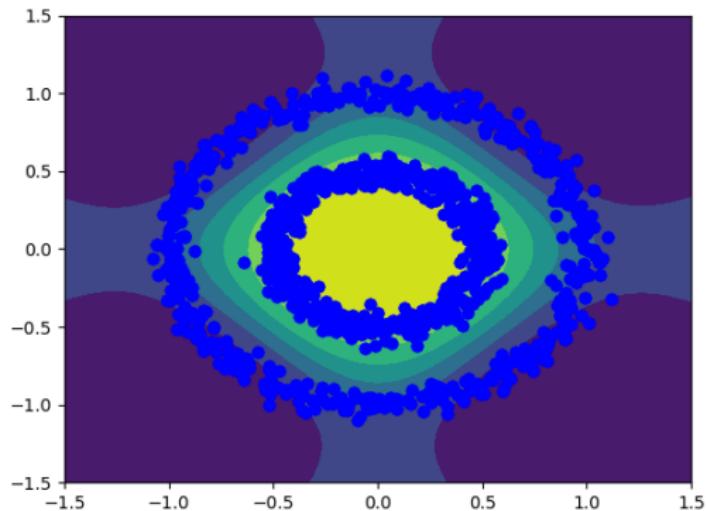
Kernel Logistic Regression Learning Steps



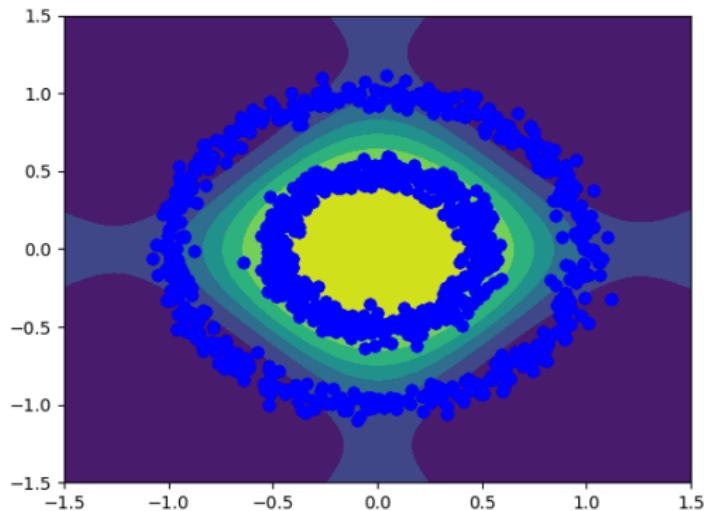
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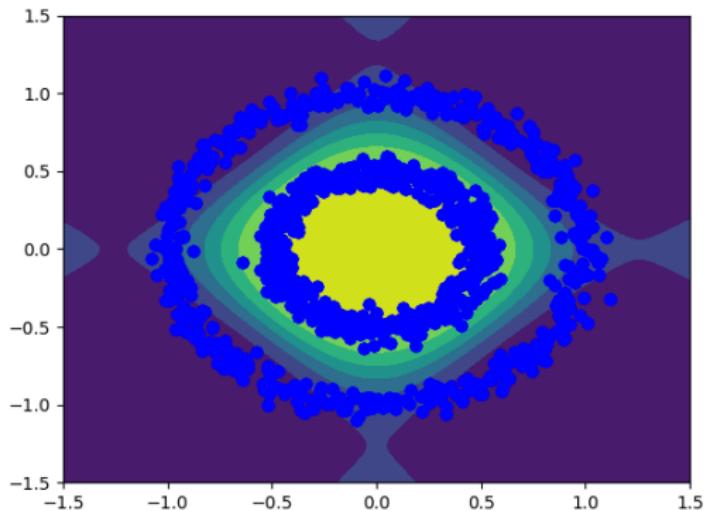
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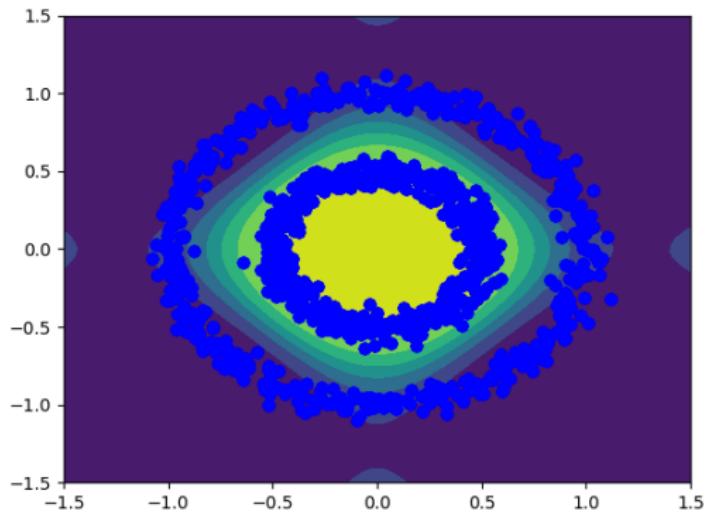
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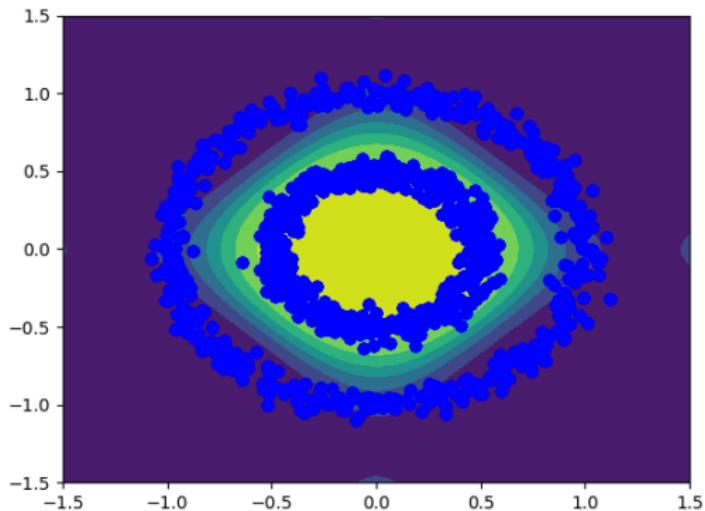
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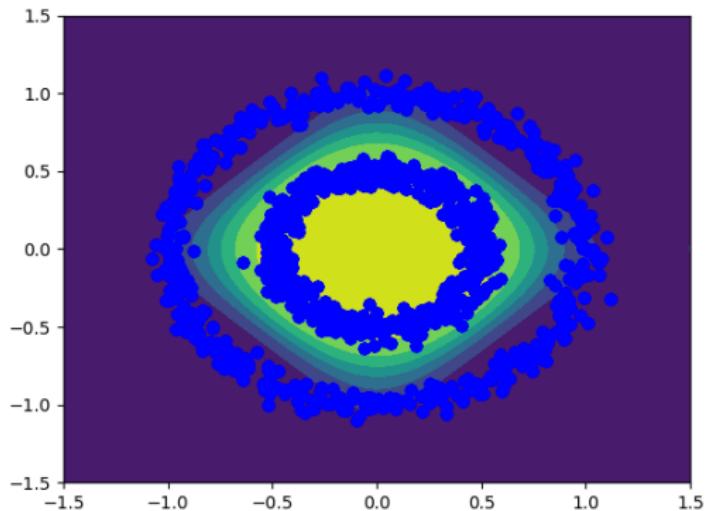
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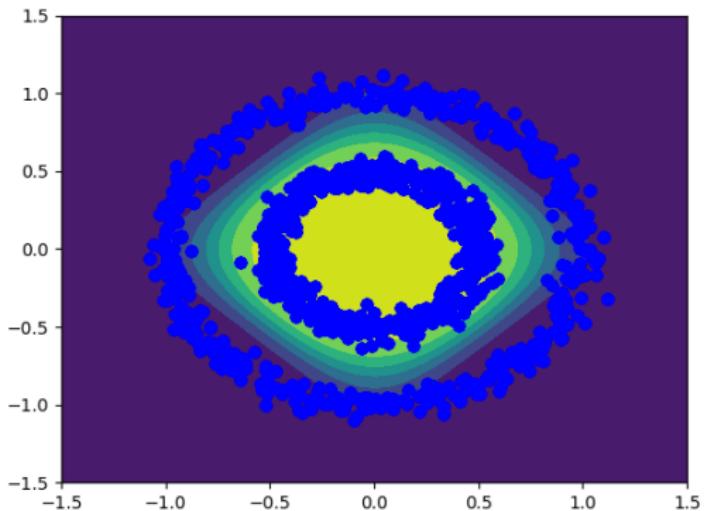
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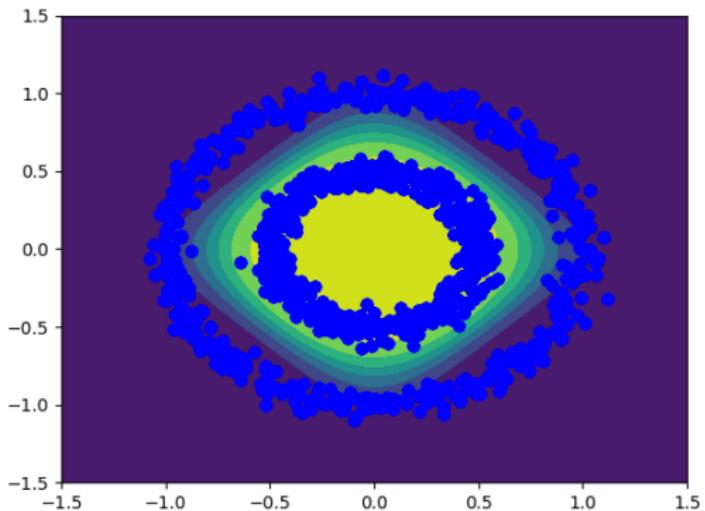


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Neural Network Classification

- The bias parameter can be estimated with a general function $f(x)$.

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- The general function maps x from L dimensions to K_1, K_2, \dots, K_{M-1} dimensions. The last layer w_M is a vector and maps to 1 output dimension. / Chaque couche map à une dimensionnalité différent K_l .

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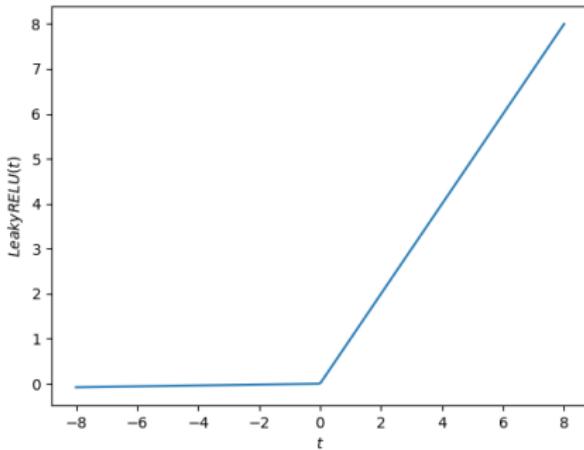
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- The loss function becomes / La fonction de cout devient:

$$\mathcal{L}(\theta) = \sum_n \left(-y_n \log \left(\sigma(f_\theta(x_n)) \right) - (1 - y_n) \log \left(1 - \sigma(f_\theta(x_n)) \right) \right)$$

An Example Neural Network

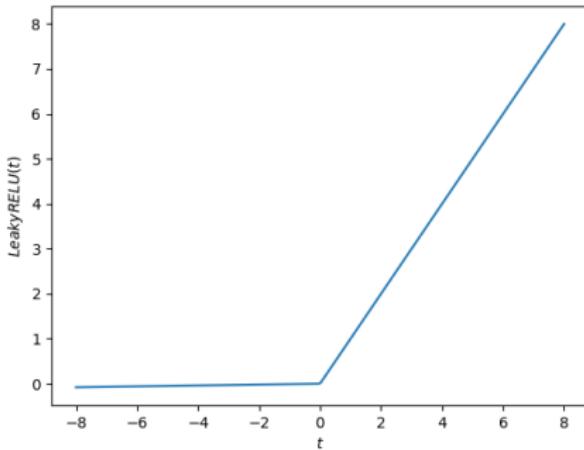
- $f_\theta(x) = w_2 h(W_1^\top x)$, $h(x) = \text{LeakyRELU}(t)$, $W_1 \in \mathbb{R}^{2 \times 100}$, $w_1 \in \mathbb{R}^{100}$.



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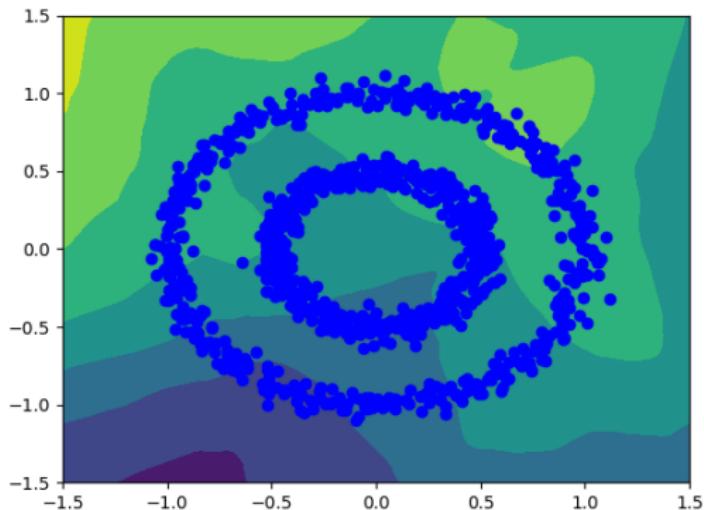
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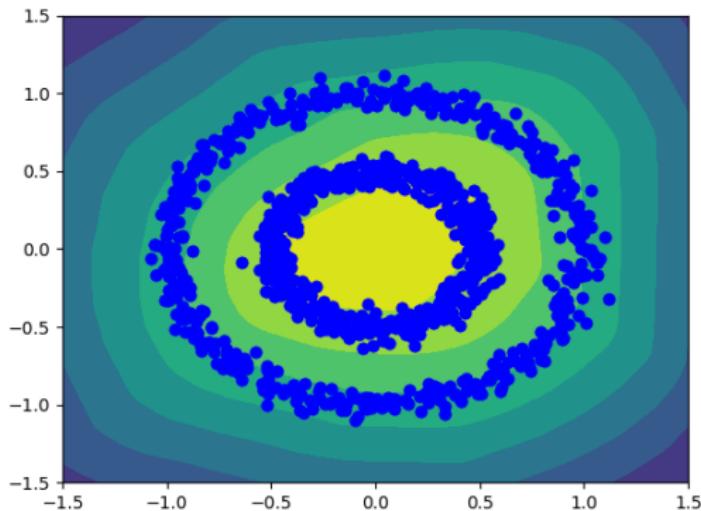


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- Let's say we define $\phi_{W_1}(x) := h(W_1^\top x)$.

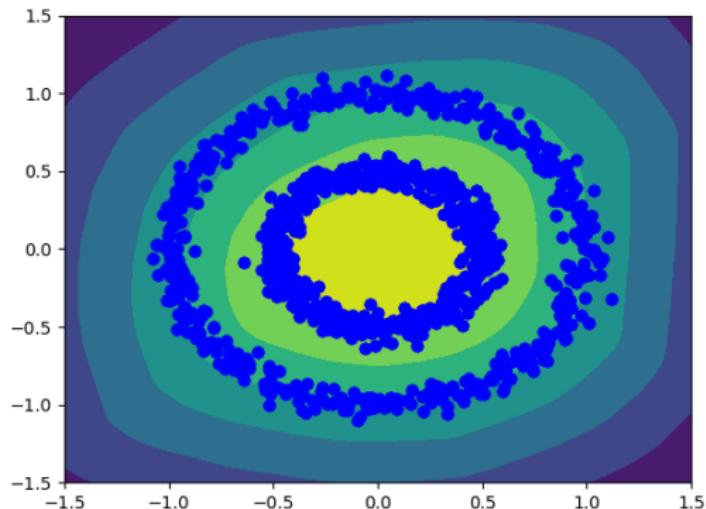
Kernel Logistic Regression Learning Steps



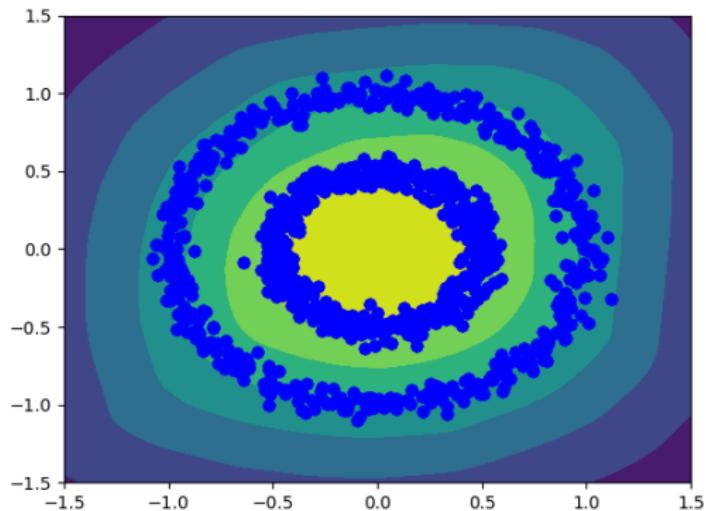
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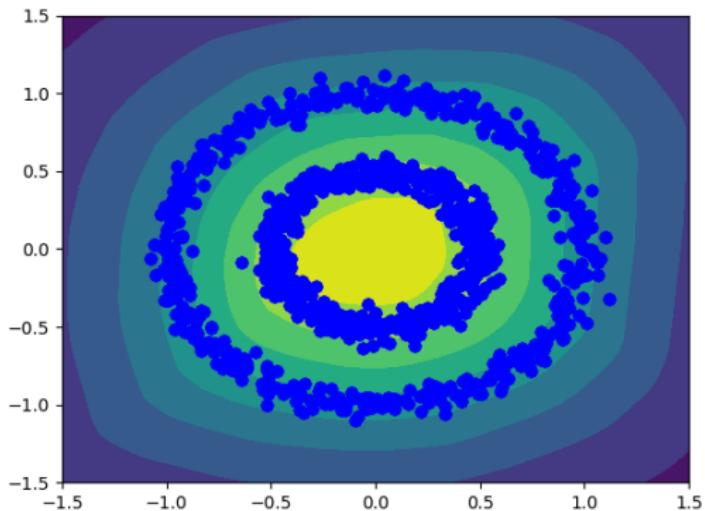
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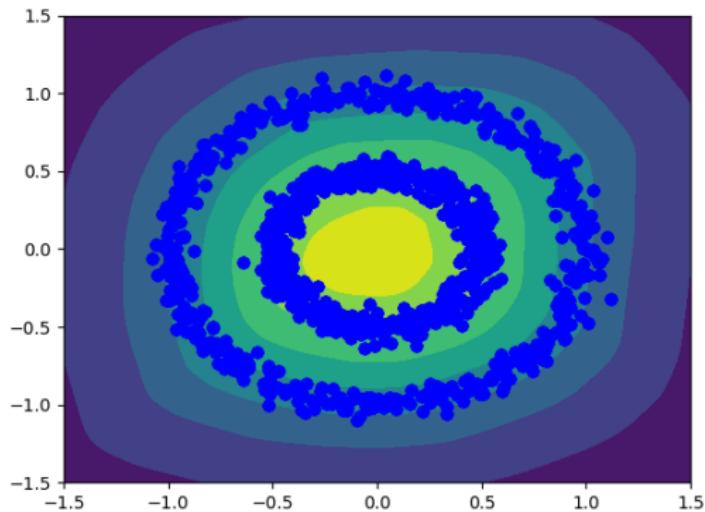
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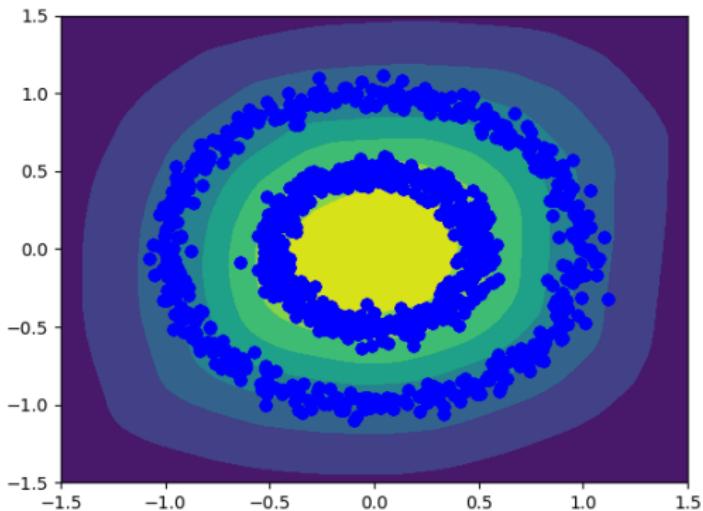
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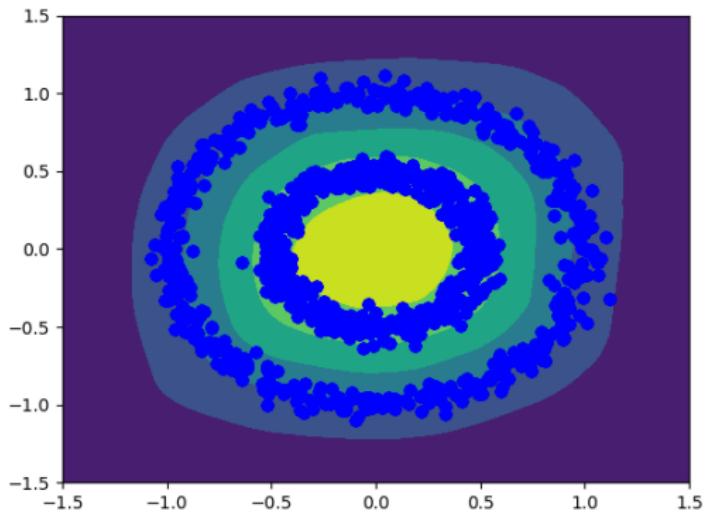
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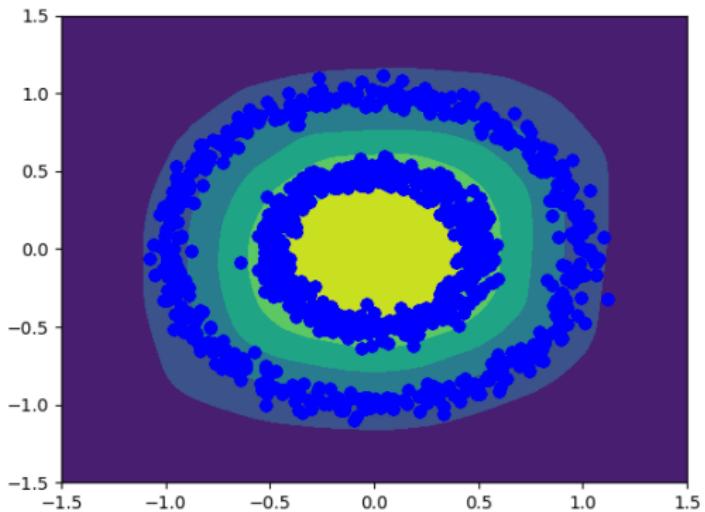
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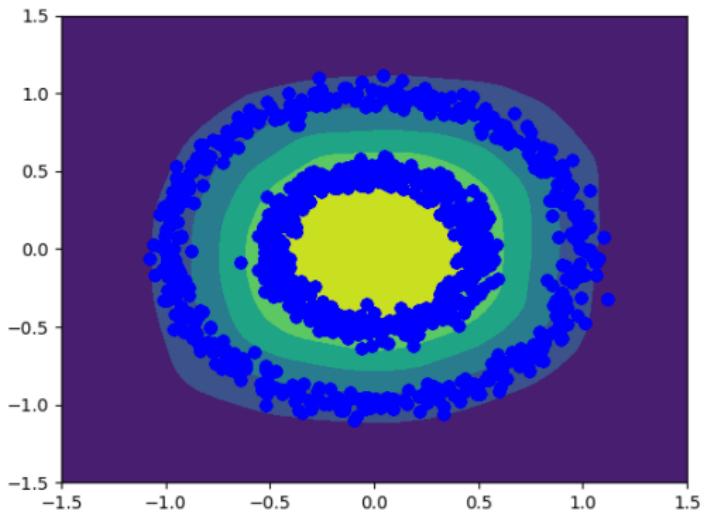
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Kernel Methods vs Neural Networks

- Kernel Methods are not very suitable for large datasets because of the hard dependency on the training set x_1, \dots, x_N . / Les méthodes de noyau ne sont pas très adaptées pour des datasets larges parce qu'ils dépendent sur le training set.
- Kernel Methods come with an interpretability advantage. At the end of the day it's a linear model, and we have the importance weights. / Les méthodes de noyaux sont plus interprétables car ils sont linéaires.
- Neural Nets are more modern because of the way we train them. They are more amenable to train with SGD. / Les réseaux de neurones sont plus adaptés à entraîner avec SGD sur des grands jeux de données.

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- We have not talked about SVMs. It's an important idea to know, but you can get that from many other classes. On n'a pas parlé de SVMs.

Suggested reading

- Bishop, chapters 4, 5, 6

Next week

- More deep learning!