Design of Digital Filters

Textbook pages 670-694

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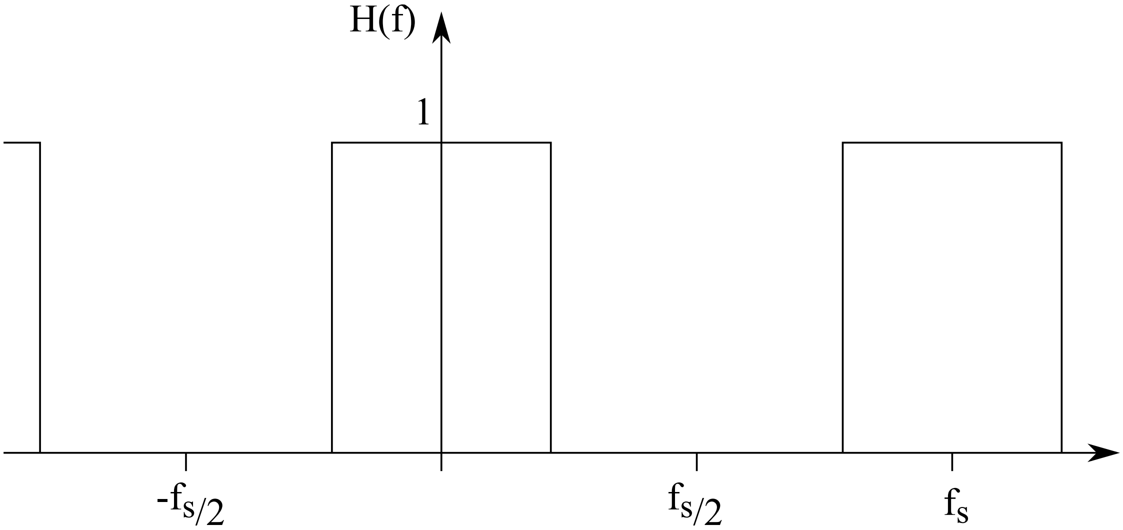
Ideal filter response

The ideal filter response for a low pass filter is one whose frequency response is 1 in the passband and 0 otherwise. This is also known as a brick wall filter, and is shown below.



Unfortunately, constructing such a filter in reality has two fundamental problems. To see why, first consider the inverse Fourier transform of an ideal filter:

$$\begin{aligned}
H(\omega) &= \begin{cases} 1 & ;|\omega| \le \omega_c \\ 0 & ;\omega_c < |\omega| \le \pi \end{cases} \\
h(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}\,d\omega \\
&= \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\right]_{-\omega_c}^{\omega_c} \\
&= \frac{1}{\pi n}\frac{1}{2j}\left\{e^{j\omega_c n} - e^{-j\omega_c n}\right\} \\
&= \frac{1}{\pi n}\sin(\omega_c n) \\
&= \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n}
\end{aligned}$$

The impulse response, $h(n)$, is zero only when n is an integer multiple of $2\pi/\omega_c$, (except for $n = 0$ where $h(n) = \omega_c/\pi$ from de l'Hôpital's rule), thus has an infinite number of non-zero values. That implies that there is a delay of an infinite length before the output can be obtained. In addition to this, there are values of $n < 0$ for which $h(n) \neq 0$. Thus, the filter requires future values of the input in order to determine the current output, i.e. it is not causal.

These two issues result in the ideal filter not being able to be realised, and thus we can, at best, only approximate the ideal filter response.[1]

[1] Selected figures taken from "Digital Signal Processing, New International Edition/4th", Proakis & Manolakis, ©Pearson Education Limited, 2014. ISBN: 978-1-29202-573-5

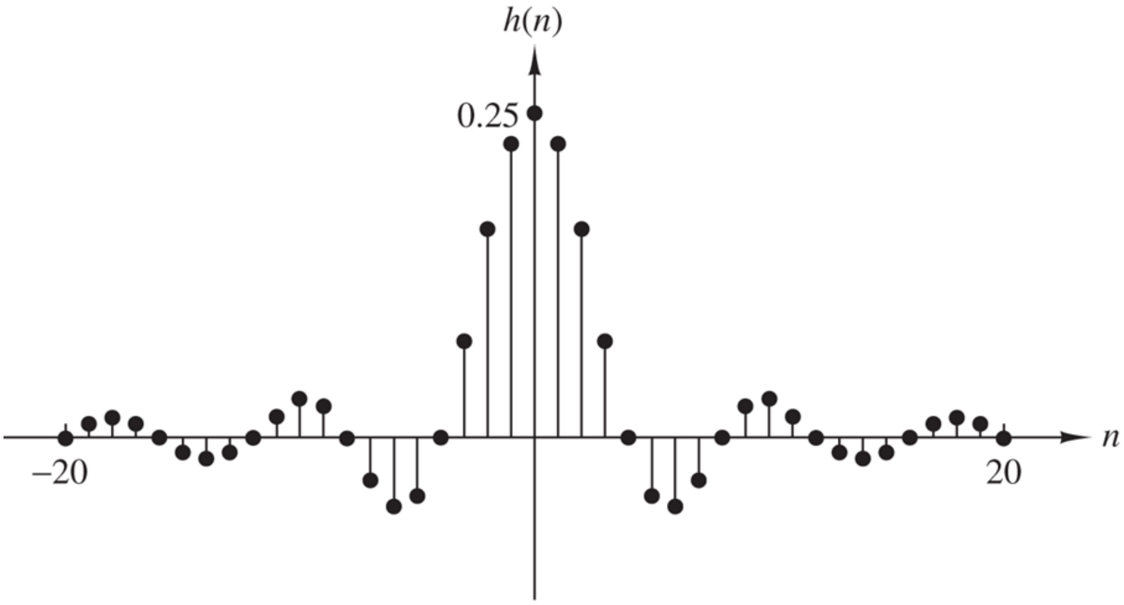


Figure 10.1.1 Unit sample response of an ideal lowpass filter.

Practical filter response

A practical filter will differ from the ideal with a frequency response that cannot be zero in the stopband, and will have a transition band. There may, or may not, be ripple in the passband.

The transition band is defined as the range of frequencies between the passband edge and the stopband edge. The passband edge is the frequency where the filter response drops below the cut-off, e.g. -3 dB. The stopband edge is the frequency at which the response drops to the desired stopband height. For a fixed length of filter, reducing δ_1, δ_2 or the width of the transition band results in an increase in the other parameters, and vice versa. Filter design then reduces to balancing the competing parameters.

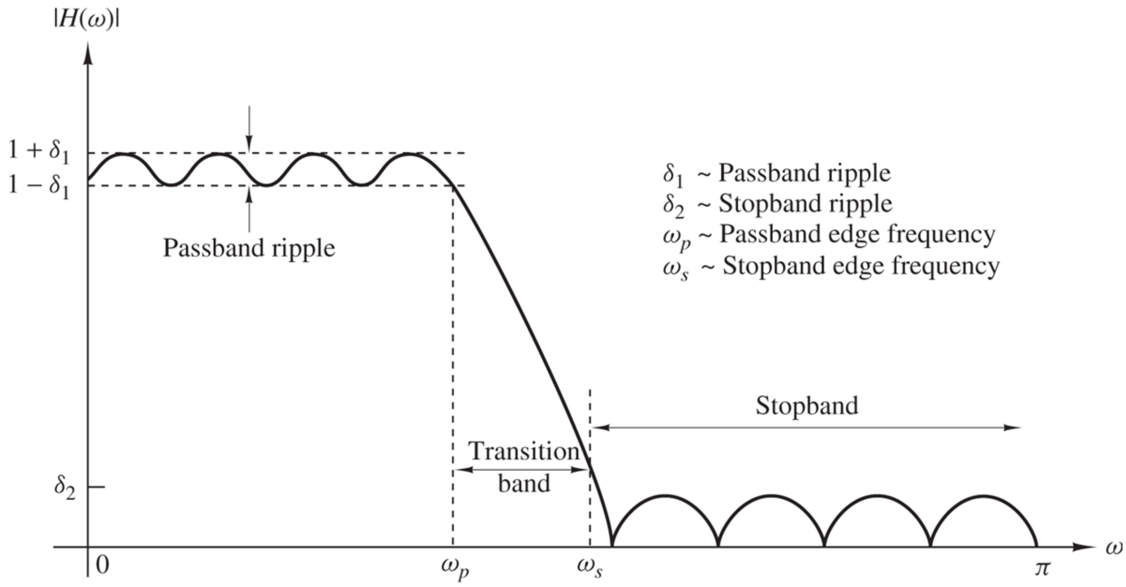


Figure 10.1.2 Magnitude characteristics of physically realizable filters.

Linear phase filters

A linear phase filter is one whose delays can be represented by a single value of group delay across all frequencies. In other words, all of the frequency components experience the same time delay from input to output of the filter.

This is a highly desirable property for many applications, such as communications, and signal processing. Because this is such an important class of filter, this module will focus exclusively on the design of linear phase filters. Note that not all linear filters have a linear phase response, for example infinite impulse response (IIR) filters are linear filters, but their phase is not linear, so they are not suitable for use in certain applications.

We begin by examining the properties required to ensure that the filters will be linear phase, and can be expressed only with real coefficients. (Use of only real coefficients simplifies the construction of the filter multiply operations).

Start with a filter with purely real coefficients, then

$$H'(\omega) = H'^{*}(-\omega)$$

A simple way to ensure linear phase is to define a purely real, or imaginary frequency response:

$$H'(\omega) = H_R(\omega) \text{ or } H'(\omega) = jH_I(\omega)$$

If $H'(\omega) = H_R(\omega)$, since the filter coefficients, $h'(n)$, are real, then $h'(n) = h'(-n)$, and the filter is symmetric. Likewise if $H'(\omega) = jH_I(\omega)$, then $h'(n) = -h'(-n)$, and the filter is anti-symmetric. In the case that $H'(\omega) = H_R(\omega)$, the filter is symmetric, whilst it is anti-symmetric if $H'(\omega) = jH_I(\omega)$. This is obtained from the properties of the Fourier transform.

If $h'(n)$ is a linear phase filter, then $h(n) = h'(n-L)$ is also linear phase with group delay, L.

Symmetric and Antisymmetric FIR filters

We will use the properties above to design four sets of filters comprising those with odd and even number of taps, and those with symmetrical or antisymmetrical coefficients. The choice of structure largely depends on whether the response should be zero at half the sampling frequency and/or at a frequency of 0.

FIR design

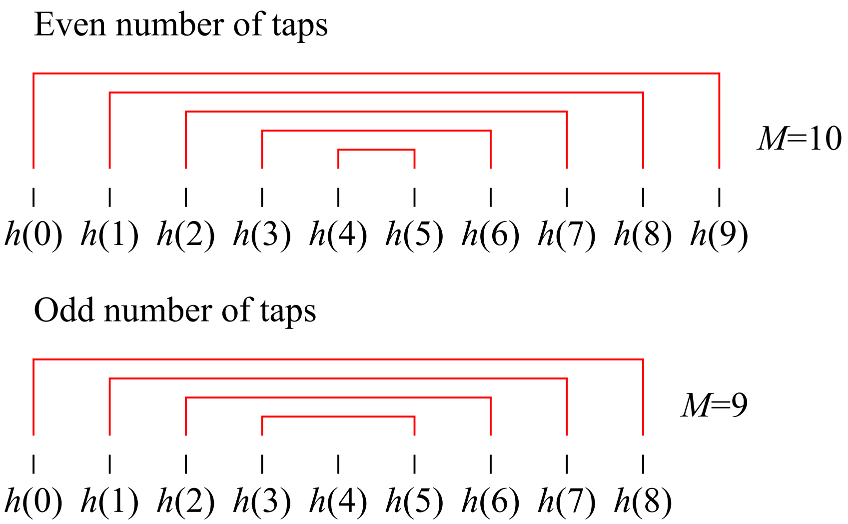
Let

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

where $h(n)$ is defined as the impulse response of a causal filter. The filter is causal as $h(n) = 0$ for all $n < 0$. We will also specify that $h(n)$ are real coefficients such that $h(n) = h'\left(n - \frac{M-1}{2}\right)$ where $h'(n)$ is symmetric or anti-symmetric The symmetry property can be re-expressed in terms of the causal filter as:

$$h(n) = \pm h(M-1-n), \; n = 0, 1, \dots, M-1$$

which is a symmetrical, or an antisymmetrical, filter with a group delay of $(M-1)/2$ The figure below identifies the symmetrical filter values for an even, and an odd number of taps, M.



Due to the properties of polynomials with real coefficients, the roots of $H(z)$ occur in complex conjugate pairs. Additionally, as the phase response must be linear, the zeros also occur in reciprocal pairs. Thus, any zero that is real, and does not lie on $z = 1$ or $z = -1$ must have one other zero that is its reciprocal. In figure 10.2.1 this is represented by z_2. Equally, complex zeros, not lying on the unit circle, occur in groups of four.

To show this property, consider the following transfer functions. Firstly, an impulse response with complex conjugate zeros, z_0 and z_0^*:

$$\begin{aligned} H(z) &= (1 - z_0 z^{-1})(1 - z_0^* z^{-1}) \\ &= 1 - (z_0 + z_0^*)z^{-1} + |z_0|^2 z^{-2} \\ \Rightarrow h(n) &= \delta(n) - (z_0 + z_0^*)\delta(n-1) + |z_0|^2 \delta(n-2) \end{aligned}$$

It is clear that the filter coefficients are all real as a result of the complex conjugate pairs. However, as the impulse response is not symmetric ($h(2) \neq h(0)$), then the frequency response cannot be formed from a purely real, or purely imaginary function with a linear phase.

In order to achieve linear phase, the reciprocals of the complex conjugate zeros need to be added:

$$H(z) = (1 - z_0 z^{-1})\left(1 - \frac{1}{z_0} z^{-1}\right)(1 - z_0^* z^{-1})\left(1 - \frac{1}{z_0^*} z^{-1}\right)$$

Expanding this and setting $z_0 = a + jb$, then:

$$\begin{aligned} H(z) = 1 - 2a\left(1 + \frac{1}{a^2+b^2}\right)z^{-1} + \left(a^2 + b^2 + \frac{1+4a^2}{a^2+b^2}\right)z^{-2} - \\ 2a\left(1 + \frac{1}{a^2+b^2}\right)z^{-3} + z^{-4} \end{aligned}$$

This does have a symmetric impulse response, and therefore corresponds to a real transfer function with a linear phase.

Thus, in order to have real coefficients for $h(n)$, *and* a linear phase response, the zeros must occur in sets that include complex conjugates and reciprocals. Zeros occurring on the real axis only require reciprocals to be present, and zeros occurring on the unit circle only require complex conjugate zeros to be present. Zeros occurring at $z = \pm 1$ can occur individually.

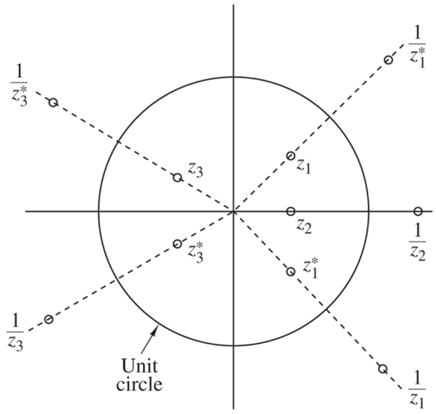


Figure 10.2.1 Symmetry of zero locations for a linear-phase FIR filter.

In this figure, the filter contains eight complex zeros, in two groups of four conjugate zeros and their reciprocals, and two real zeros in a reciprocal pair. By definition, there are $M-1$ zeros in an M-tap FIR filter, thus this filter has an odd number of coefficients. To satisfy the requirement for reciprocal pairs of zeros, where M is even, and there is therefore an odd number of zeros, one zero is present either at $z = -1$ (half the sampling frequency), or at $z = 1$ (zero frequency, i.e. d.c.) depending upon whether the filter is symmetrical or antisymmetrical.

Frequency response characteristics

For a symmetrical filter, the frequency response can be represented by:

$$H(\omega) = H_R(\omega)e^{-j\omega(M-1)/2}$$

and for an antisymmetrical filter:

$$H(\omega) = H_R(\omega)e^{-j[\omega(M-1)/2-\pi/2]}$$

We only need to consider $H_R(\omega)$.

Types of FIR filter

Except for symmetric filters with odd M, a zero, or zeros are automatically present:

	Even M	Odd M
Symmetric	$z = -1$	—
Antisymmetric	$z = 1$	$z = -1$ and $z = 1$

A zero at $z = -1$ is not suitable for a high-pass filter, likewise a zero at $z = 1$ is not suitable for a low-pass filter.

The symmetry properties can be used to reduce the computation involved by manipulating the z transform:

$$H(z) = \sum_{n=0}^{M-1} h(n)z^{-n}$$

by collecting symmetric terms. Where M is even, then $h\left(\frac{M}{2}-1\right) = \pm h\left(\frac{M}{2}\right)$, $h\left(\frac{M}{2}-2\right) = \pm h\left(\frac{M}{2}+1\right)$, and so on. For odd values of M, the centre term, $h\left(\frac{M-1}{2}\right)$ is the centre of symmetry, and $h\left(\frac{M-3}{2}\right) = \pm h\left(\frac{M+1}{2}\right)$, $h\left(\frac{M-5}{2}\right) = \pm h\left(\frac{M+3}{2}\right)$, etc. (It is worth noting that for an antisymmetric filter, and an odd value of M, $h\left(\frac{M-1}{2}\right) = 0$ by definition as otherwise $h\left(\frac{M-1}{2}\right) \neq -h\left(\frac{M-1}{2}\right)$).

It is straightforward to show that symmetric filters enforce a zero at $z = -1$, and likewise antisymmetric filters a zero at $z = 1$. For a symmetrical filter:

$$H(z) = \begin{cases} \sum_{n=0}^{(M/2)-1} h(n)[z^{-n} + z^{1+n-M}] & ;M \text{ even} \\ h((M-1)/2)z^{(1-M)/2} + \\ \qquad \sum_{n=0}^{(M-3)/2} h(n)[z^{-n} + z^{1+n-M}] & ;M \text{ odd} \end{cases}$$

Where M is even,

$$
\begin{aligned}
H(z)|_{z=-1} &= \sum_{n=0}^{(M/2)-1} h(n)[(-1)^{-n} + (-1)^{1+n-M}] \\
&= \sum_{n=0}^{(M/2)-1} h(n)[(-1)^{n} + (-1)^{1-M}(-1)^{n}] \\
&= \sum_{n=0}^{(M/2)-1} h(n)[(-1)^{n} - (-1)^{n}] = 0
\end{aligned}
$$

Similarly, for an antisymmetric filter:

$$
H(z) = \begin{cases} \sum_{n=0}^{(M/2)-1} h(n)[z^{-n} - z^{1+n-M}] & ;M \text{ even} \\ \sum_{n=0}^{(M-3)/2} h(n)[z^{-n} - z^{1+n-M}] & ;M \text{ odd} \end{cases}
$$

Here we note that $h((M-1)/2) = 0$ when M is odd. Where M is even,

$$
H(z)|_{z=1} = \sum_{n=0}^{(M/2)-1} h(n)[1^{-n} - 1^{1+n-M}] = 0
$$

Whilst, for odd values of M:

$$
\begin{aligned}
H(z)|_{z=-1} &= \sum_{n=0}^{(M-3)/2} h(n)[(-1)^{-n} - (-1)^{1+n-M}] \\
&= \sum_{n=0}^{(M-3)/2} h(n)[(-1)^{n} - (-1)^{n}] = 0 \\
H(z)|_{z=1} &= \sum_{n=0}^{(M-3)/2} h(n)[1^{-n} - 1^{1+n-M}] = 0
\end{aligned}
$$

Design Process

Having established the position of any zeros dictated by the choice of M and filter symmetry, the filter itself can be designed. The desired filter response, $H_{RD}(\omega)$ defines the idealised filter. Generally this is impossible to construct as we saw at the start of this lecture.

The idealised filter will either be symmetric, or antisymmetric ($H_{RD}(\omega) = \pm H_{RD}(-\omega)$).

To make the final filter of length M causal, a delay of $\frac{M-1}{2}$ samples is required. This can be performed in the frequency domain, by multiplying by $e^{-j(M-1)\omega/2}$.

Now we define the idealised impulse response, $h_D(n)$, as

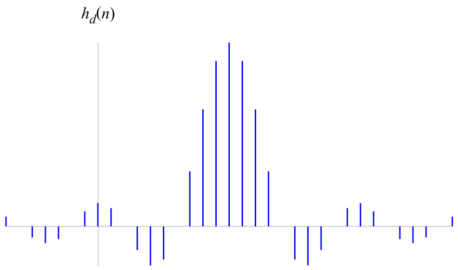
$$h_D(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j(M-1)\omega/2} H_{RD}(\omega) e^{j\omega n}\, d\omega$$

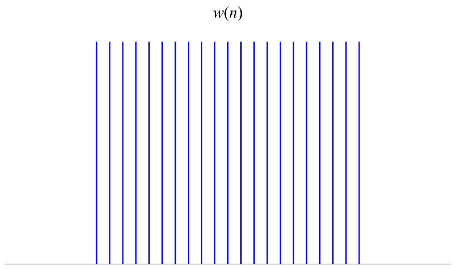
At this point, the magnitude of the frequency response of $h_D(n)$ is identical to $|H_{RD}(\omega)|$. The next step involves truncating $h_D(n)$ to have only M samples. The truncation process will have an impact on the frequency response, and is analogous to the truncation of samples in the discrete Fourier transform. We will use windows to control the effects of the truncation.

Window based design

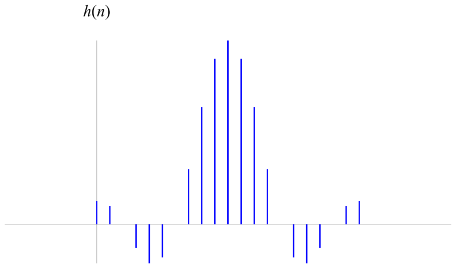
Having specified the types of filters available, and their restrictions in terms of zero placement, we now move on to approximate an ideal filter response using a windowing technique. The design is based on truncating the ideal filter impulse response to have only M 'taps', i.e. M coefficients.

To turn the ideal filter impulse response into a practical design it must be finite in length. Simply truncating the ideal response is equivalent to multiplying the response by a rectangular window.

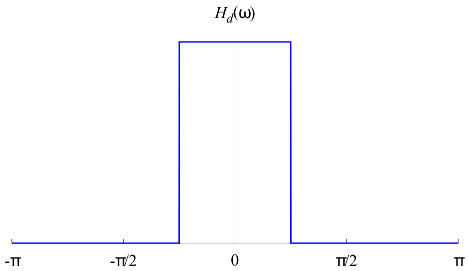


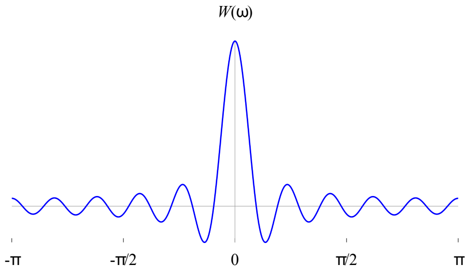


MULTIPLY

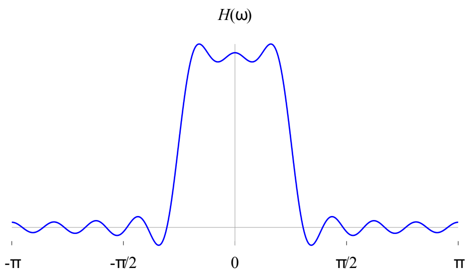


In the frequency domain this becomes:





CONVOLVE



Thus, simply truncating the ideal impulse response, $h_d(n)$, to M samples is equivalent to multiplying the impulse response by a rectangular window. In the frequency domain this will introduce ripples in the passband and in the stopband.

Rectangular window

Using a rectangular window with $M = 61$:

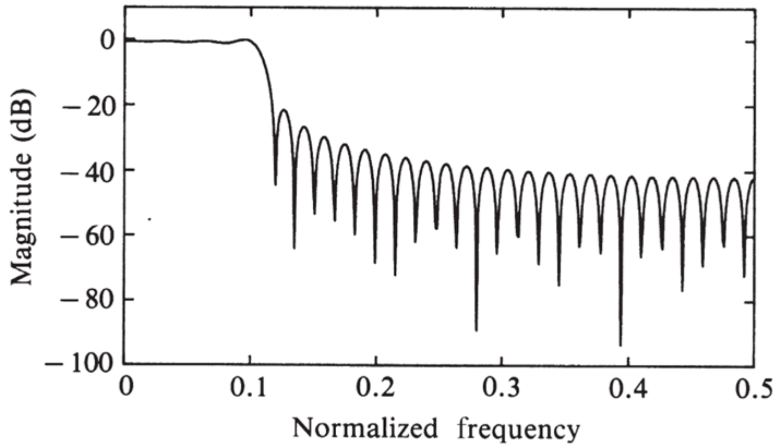


Figure 10.2.8 Lowpass FIR filter designed with rectangular window ($M = 61$).

A different window can be selected to achieve a different ripple levels, as appropriate to the application. The most important characteristics are the transition band and the stopband attenuation. The passband ripple may also be important depending upon the application.

Window parameters

	Max stopband	Transition BW
Rectangular	-21 dB	$1.8\pi/M$
Hanning	-44 dB	$6.2\pi/M$
Hamming	-53 dB	$6.6\pi/M$
Blackman	-74 dB	$11\pi/M$

The approach to designing a filter using this technique is to first establish what the maximum stopband value should be. This is dictated by the eventual application. If, for example, the noise floor of a system is -50 dB, then a Hamming window may be most appropriate as the stopband would be below the noise floor. The Blackman window also meets this criterion, however it requires more taps, and therefore increases complexity and introduces a larger processing delay unnecessarily. Once the window has been selected, then the number of taps, M, is defined by the width of the transition band.

Example

Design a lowpass filter with the following requirements:

Cut-off frequency	0.1π
Stopband attenuation	-40 dB
Transition bandwidth	0.02π

First select the window; the Hanning window meets the stopband attenuation requirement with the fewest taps, so

$$w(n) = \frac{1}{2}\left(1 - \cos\left(\frac{2\pi n}{M-1}\right)\right)$$

Then calculate M from the transition band requirement:

$$\frac{6.2\pi}{M} \leq 0.02\pi \Rightarrow M \geq 310$$

To begin with, we will select a value of M that is odd, as it can model any filter, either high or lowpass. So we set $M = 311$, which implies that we should add a delay of $\frac{M-1}{2} = 155$ samples to ensure that the final filter is causal. Next, compute the ideal impulse response, $h_d(n)$:

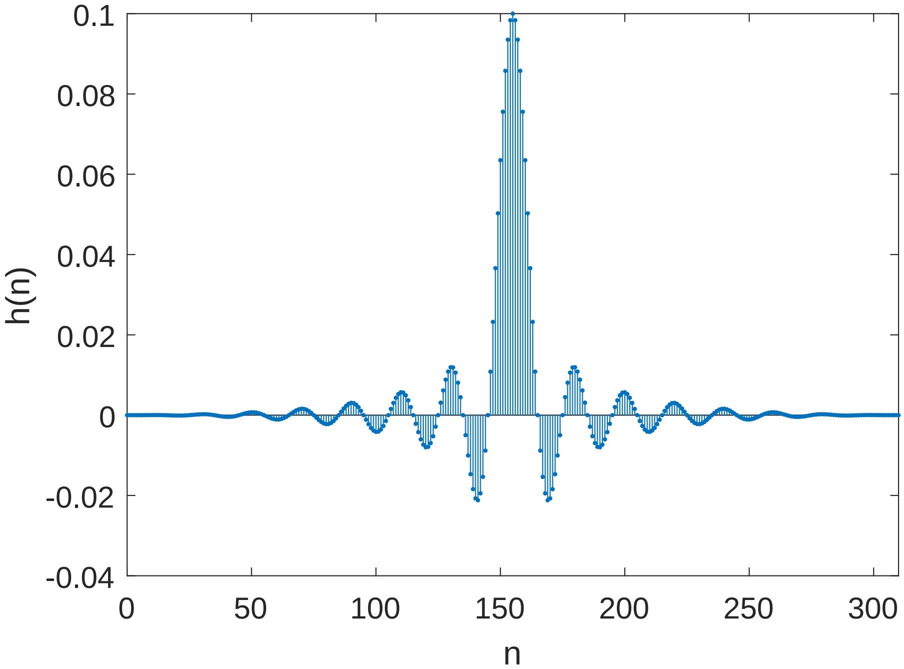
$$H_{RD}(\omega) = \begin{cases} 1 & ;|\omega| < 0.1\pi \\ 0 & ;\text{otherwise} \end{cases}$$

$$\begin{aligned} \Rightarrow h_d(n) &= \frac{1}{2\pi}\int_{-0.1\pi}^{0.1\pi} e^{-j155\omega} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi}\left[\frac{1}{j(n-155)} e^{j\omega(n-155)}\right]_{-0.1\pi}^{0.1\pi} \\ &= \frac{1}{\pi(n-155)}\sin(0.1\pi(n-155)) \end{aligned}$$

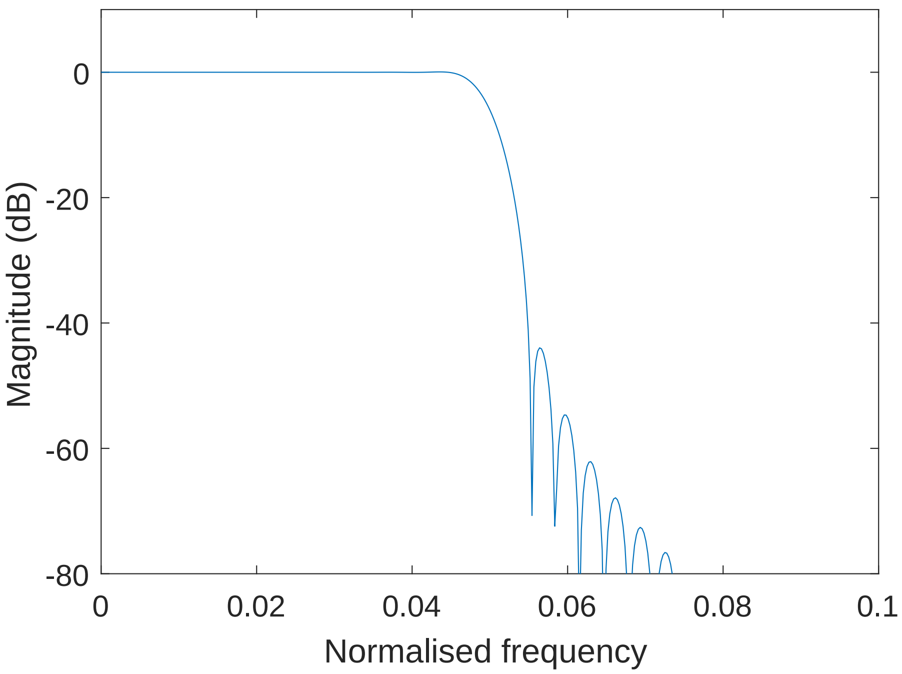
$h_d(n)$ needs to be truncated to length M, and made causal by applying the window which is defined over $n \in \{0,1,\ldots 310\}$. The FIR filter is thus given by:

$$\begin{aligned} h(n) &= w(n)h_d(n)\ ; 0 \le n < 311 \\ &= \frac{1}{2\pi(n-155)}\left(1-\cos\left(\frac{2\pi n}{310}\right)\right). \\ &\qquad \sin(0.1\pi(n-155))\ ; 0 \le n < 311 \end{aligned}$$

The impulse response is:



and the frequency response:



We may instead specify a value of M that is even, and for this example achieve a filter with one fewer taps. In our case, for $M = 310$:

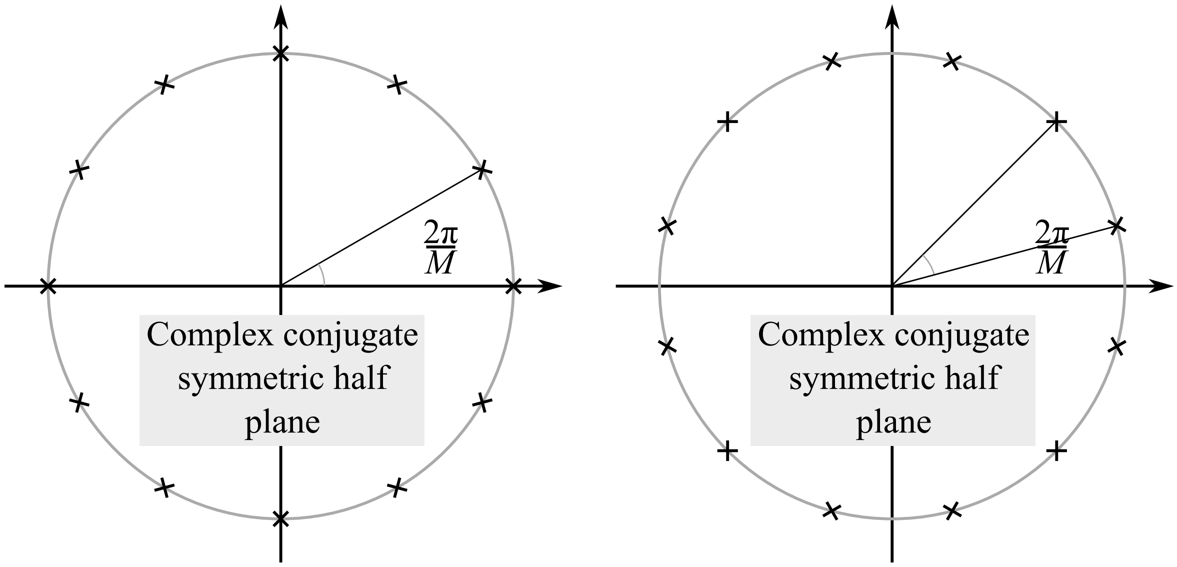
$$
\begin{aligned}
H_{RD}(\omega) &= \begin{cases} 1 & ;|\omega| < 0.1\pi \\ 0 & ;\text{otherwise} \end{cases} \\
\Rightarrow h_d(n) &= \frac{1}{2\pi}\int_{-0.1\pi}^{0.1\pi} e^{j\omega(n-154.5)}d\omega \\
&= \frac{1}{2\pi}\left[\frac{1}{j(n-154.5)}e^{j\omega(n-154.5)}\right]_{-0.1\pi}^{0.1\pi} \\
&= \frac{1}{\pi(n-154.5)}\sin(0.1\pi(n-154.5)) \\
\Rightarrow h(n) &= \frac{1}{2\pi(n-154.5)}\left(1-\cos\left(\frac{2\pi n}{309}\right)\right)\cdot \\
&\qquad \sin(0.1\pi(n-154.5))\ ; 0 \le n < 310
\end{aligned}
$$

Matlab code is available in Learn to generate figures used in lectures. `FIR_design.m` includes both designs above, and shows that the resulting filters have very similar frequency responses. Students are encouraged to download, run, and modify the Matlab code to explore the concepts in more detail.

For a low-pass filter design, a symmetric response is required as an anti-symmetric filter places a zero at $z = 1$. For a high-pass design, a symmetric or an antisymmetric impulse response can be used provided that M is chosen such that a zero is not placed at $z = -1$. To design with an antisymmetric impulse response, the frequency response should be defined by $H(\omega) = jH_I(\omega)$, with $h(0) = 0$, and $H_I(\omega)$ is an odd function.

1 Frequency sampling method

The window design method defines the frequency response over a continuous ω in the passband and stopband. For complicated transfer functions, a sampled frequency approach can be taken. Here, instead of specifying the filter over a continuous ω, it is specified only at specified sample points that are spaced equally in frequency. As the designed filter should comprise only real valued coefficients for $h(n)$, zeros should occur in complex conjugate pairs in the z-transform. Thus, there are two possible sampling sets for an M^{th} order filter, with the first sampling point at $\omega_0 = 0$ or $\omega_0 = \pi/M$.



1.1 Naïve approach

The naïve approach to this design technique is to simply invert the frequency response at the sampling points to obtain an impulse response. As we will discover, this has problems for frequencies that are not at the sampling points near the filter edge.

Method

The method begins with a sampled frequency response, $H(k+\alpha)$, where $\alpha \in \left\{0, \frac{1}{2}\right\}$ and $k = 0, 1, \ldots, M-1$. Based on an inverse discrete Fourier transform, it can be shown that:

$$h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{j2\pi(k+\alpha)n/M} \; ; n \in [0, M-1] \tag{2.33}$$

Because of symmetry, as $h(n)$ is real:

$$H(k+\alpha) = H^*(M-k-\alpha) \tag{2.34}$$

Only $U = \left\lfloor \frac{M-1}{2} \right\rfloor$ values of k need to be computed. The notation $\lfloor \cdot \rfloor$ represents the "floor" operation, which returns the largest integer not bigger than its argument. (In other words, it rounds down to the nearest integer). $H(k+\alpha)$ is defined by the desired frequency response $H_d(\omega)$ by

$$H(k+\alpha) = H_d\left(\frac{2\pi}{M}(k+\alpha)\right) \tag{2.32}$$

Note that the textbook drops the subscript d in (2.31) and subsequent equations.

Causal filter

Instead of using (2.32) directly, which would result in a non-causal filter, a delay, γ needs to be added to make the filter causal.

$$H(k+\alpha) = H_r\left(\frac{2\pi(k+\alpha)}{M}\right) e^{j\gamma}$$

where

$$\gamma = \frac{\beta\pi}{2} - \frac{2\pi(k+\alpha)}{M} \frac{(M-1)}{2} \,.$$

γ consists of two components, the first term, $\frac{\beta\pi}{2}$ is required to control whether the filter has a frequency response that is real and symmetric ($\beta = 0$), or imaginary and anti-symmetric ($\beta = 1$). These are the only two options allowed that maintain a real-valued impulse response $h(n)$. It achieves this by adding a $\pi/2$ phase shift to the transfer function when $\beta = 1$, and leaving it unmodified if $\beta = 0$.

The second term of γ is a linear phase shift that is applied to the frequency response. Referring to previous lectures, a linear phase shift in frequency corresponds to a shift in the time domain. In this case, the shift that is required is a $\frac{M-1}{2}$ sample delay. $\frac{2\pi(k+\alpha)}{M}$ is the frequency at the sampling points of the transfer function. The product of the two terms achieves the required linear phase shift that ensures that the filter is causal.

Although this expression can be used directly to obtain the final result, it is also possible to rearrange the expression to allow the filter taps to be specified without using any complex notation. To do this, the expression for $H(k+\alpha)$ is rearranged as follows:

$$\begin{aligned}
H(k+\alpha) &= H_r\left(\frac{2\pi(k+\alpha)}{M}\right) \\
&\qquad \exp\left(j\frac{\beta\pi}{2} - j\frac{2\pi(k+\alpha)(M-1)}{2M}\right) \\
&= H_r\left(\frac{2\pi(k+\alpha)}{M}\right) \\
&\qquad \exp\left(j\frac{\beta\pi}{2} - j\pi(k+\alpha) + j\frac{2\pi(k+\alpha)}{2M}\right) \\
&= (-1)^k H_r\left(\frac{2\pi(k+\alpha)}{M}\right) \\
&\qquad \exp\left(j\frac{\beta\pi}{2} - j\pi\alpha + j\frac{2\pi(k+\alpha)}{2M}\right)
\end{aligned}$$

Then a new set of frequency samples is defined:

$$G(k+\alpha) = (-1)^k H_r\left(\frac{2\pi(k+\alpha)}{M}\right)$$

Substituting this in to (2.33) gives

$$\begin{aligned}
h(n) &= \frac{1}{M}\sum_{k=0}^{M-1} G(k+\alpha)\exp\left(j\frac{\beta\pi}{2} - j\pi\alpha + j\frac{2\pi(k+\alpha)}{2M} + \right. \\
&\qquad \left. j2\pi(k+\alpha)n/M\right) \; ; n \in [0, M-1] \\
&= \frac{1}{M}\sum_{k=0}^{M-1} G(k+\alpha)\exp\left(j\pi\left(\frac{\beta}{2} - \alpha\right)\right) \\
&\qquad \exp\left(j\frac{2\pi(k+\alpha)}{M}\left(n+\frac{1}{2}\right)\right) \; ; n \in [0, M-1]
\end{aligned}$$

The first exponential term is dependent upon whether the filter is symmetric or antisymmetric, and whether the frequency samples are offset by π/M or not. It evaluates to one of $\{0, j, -1, -j\}$. The second exponential term, along with the symmetry of $G(k+\alpha)$ produces sin and cos terms only when simplified. The result is Table 10.3 in the textbook. The table can be simplified as follows:

$\alpha = 0$ $\beta = 0$	$h(n) = \frac{1}{M}\left\{G(0) + 2\sum_{k=1}^{U} G(k)\cos\left(\frac{2\pi k}{M}\left(n+\frac{1}{2}\right)\right)\right\}$
$\alpha = \frac{1}{2}$ $\beta = 0$	$h(n) = \frac{2}{M}\sum_{k=0}^{U} G(k+\alpha)\sin\left(\frac{2\pi(k+\alpha)}{M}\left(n+\frac{1}{2}\right)\right)$
$\alpha = 0$ $\beta = 1$	$h(n) = \frac{1}{M}\left\{(-1)^{n+1}G(M/2) - 2\sum_{k=1}^{V} G(k)\sin\left(\frac{2\pi k}{M}\left(n+\frac{1}{2}\right)\right)\right\}$
$\alpha = \frac{1}{2}$ $\beta = 1$	$h(n) = \frac{2}{M}\sum_{k=0}^{V} G(k+\alpha)\cos\left(\frac{2\pi(k+\alpha)}{M}\left(n+\frac{1}{2}\right)\right)$
$G(k+\alpha) = (-1)^k H_r\left(\frac{2\pi(k+\alpha)}{M}\right)$ $U = \left\lfloor\frac{M-1}{2}\right\rfloor$ $V = \left\lceil\frac{M-1}{2}\right\rceil$	

This table is reproduced on the formula sheet supplied in the exam.

Example

Let

$$H_r(\omega) = \begin{cases} 1 & |\omega| < 0.1\pi \\ 0 & \text{otherwise} \end{cases}$$

and $M = 64$.

$$\begin{aligned} G(k+\alpha) &= (-1)^k H_r\left(\frac{2\pi(k+\alpha)}{M}\right) \\ &= \begin{cases} (-1)^k & k < \frac{M}{20} - \alpha \text{ or } k > \frac{19M}{20} - \alpha \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

For a low-pass response, the filter must be symmetric, so $\beta = 0$. This arises as a low-pass filter must not have a zero at $z = 1$. An anti-symmetric filter response forces a zero to occur at this location, thus it is unsuitable for this application.

We can test both $\alpha = 0$ and $\alpha = \frac{1}{2}$ to see which gives the best results.

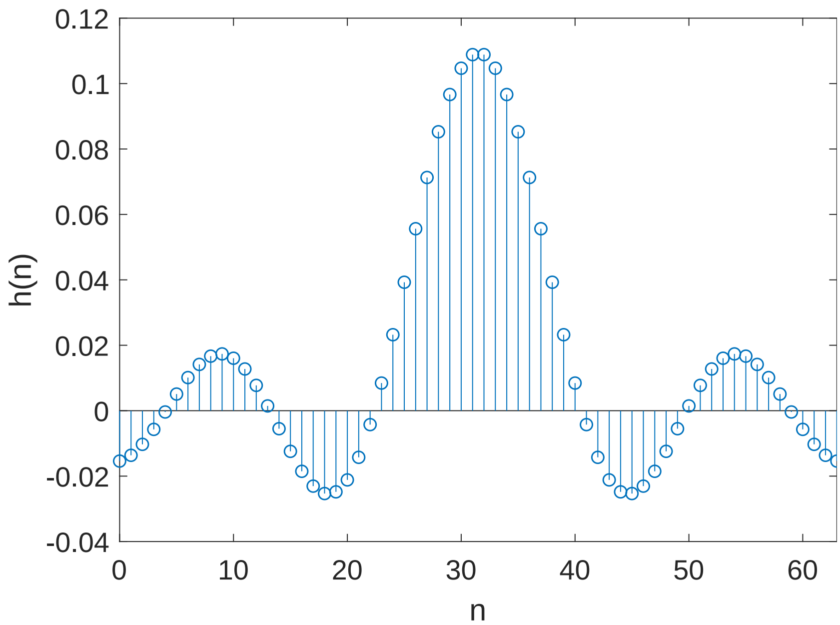
For $\alpha = 0$:

$$h(n) = \frac{1}{64}\left\{1 + 2\sum_{k=1}^{3}(-1)^k \cos\left(\frac{2\pi k}{64}\left(n + \frac{1}{2}\right)\right)\right\}$$

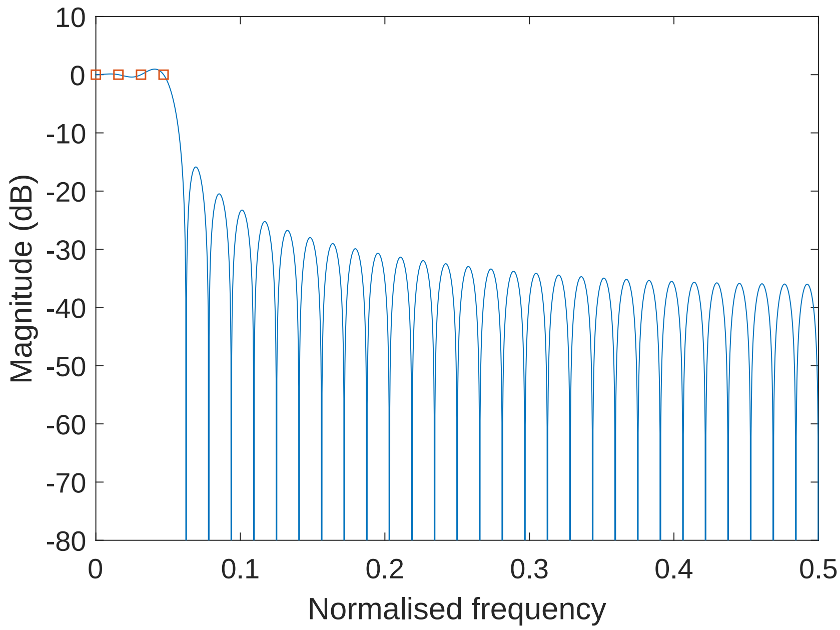
For $\alpha = \frac{1}{2}$:

$$h(n) = \frac{1}{32}\sum_{k=0}^{2}(-1)^k \sin\left(\frac{2\pi(k+\alpha)}{64}\left(n + \frac{1}{2}\right)\right)$$

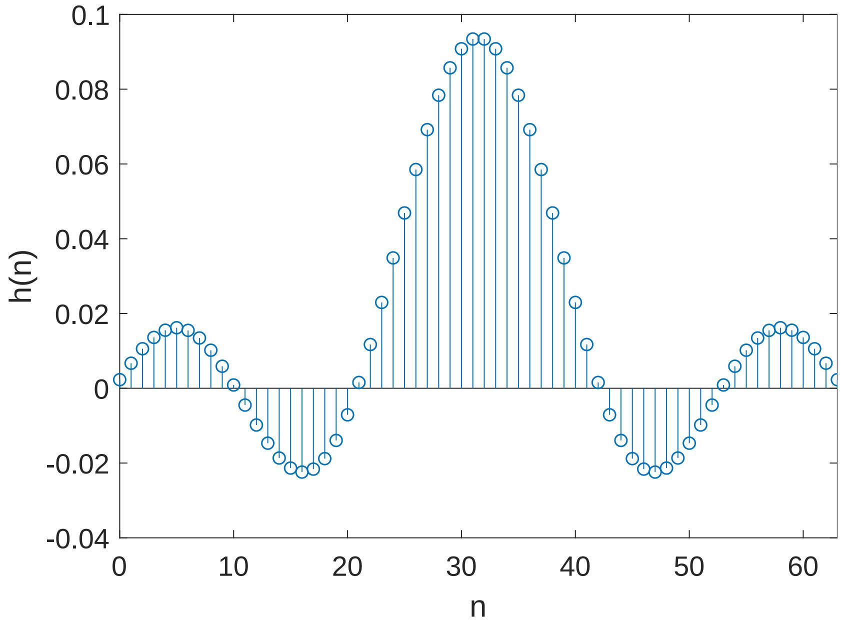
For $\alpha = 0$, $h(n)$ is:



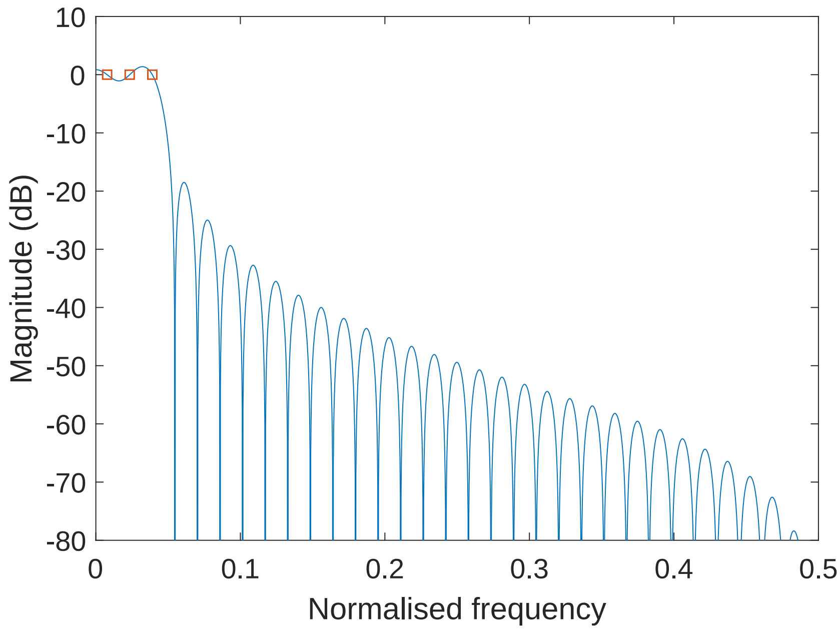
For $\alpha = 0$, $H(\omega)$ is:



For $\alpha = \frac{1}{2}$, $h(n)$ is:



For $\alpha = \frac{1}{2}$, $H(\omega)$ is:



Note that in the plots of the transfer function, the sample points of $H(\omega)$ are marked. In the case of $\alpha = 0$, four samples have a value of 0 dB, and the rest are all at $-\infty$ dB. For $\alpha = 0.5$, three of the samples have a value of 0 dB. The variation from the ideal response occurs between the sample points.

For both values of α, the frequency response has high sidelobe levels at the edge of the stopband, though the sidelobe height for $\alpha = 0.5$ is marginally lower than that for $\alpha = 0$. Conversely, the passband ripple is higher for $\alpha = 0.5$. Making a decision as to which filter to use depends upon the specific problem that is being addressed as there is a trade-off between passband ripple and stopband attenuation in this case.

1 Requirement for conjugate and reciprocal zeros

To start the analysis, we will assume that $h(n)$ are real, and that $H(\omega)$ is real. This means that the filter will be non-causal as the two assumptions ensure that $h(n)$ is symmetric.

1.1 Conjugate pairs

To show that zeros in an expression for $H(z)$ need to be complex conjugate pairs, we need only take its inverse transform. In this analysis we assume that z_1 is complex, thus $\mathrm{Im}(z_1) \neq 0$:

$$\begin{aligned} H(z) &= \frac{(z - z_1)(z - z_2)}{z} \\ &= \frac{z^2 - (z_1 + z_2)z + z_1 z_2}{z} \\ &= z - (z_1 + z_2) + z_1 z_2 z^{-1} \\ \Rightarrow h(n) &= \{1, -\underset{\uparrow}{(z_1 + z_2)}, z_1 z_2\} \end{aligned}$$

For this expression to be real, $\mathrm{Im}(z_1) = -\mathrm{Im}(z_2)$ and $\mathrm{Im}(z_1 z_2) = 0$, which is satisfied if $z_2 = z_1^*$. Therefore, the zeros occur in complex conjugate pairs.

1.2 Reciprocal zeros

Now we define a five tap filter, exploiting the result for complex conjugate zeros above, and then ensure that $H(\omega)$ is real. Here we make the assumption that z_1 is not on the unit circle, thus $z_1 \neq \frac{1}{z_1^*}$:

$$\begin{aligned} H(z) &= \frac{(z - z_1)(z - z_1^*)(z - z_2)(z - z_2^*)}{z^2} \\ \Rightarrow H(\omega) &= \frac{(e^{j\omega} - z_1)(e^{j\omega} - z_1^*)(e^{j\omega} - z_2)(e^{j\omega} - z_2^*)}{e^{j2\omega}} \\ \Rightarrow H^*(\omega) &= \frac{(e^{-j\omega} - z_1^*)(e^{-j\omega} - z_1)(e^{-j\omega} - z_2^*)(e^{-j\omega} - z_2)}{e^{-j2\omega}} \\ &= \frac{(1 - e^{j\omega} z_1^*)(1 - e^{j\omega} z_1)(1 - e^{j\omega} z_2^*)(1 - e^{j\omega} z_2)}{e^{j2\omega}} \\ &= z_1^* z_1 z_2^* z_2 \frac{\left(\frac{1}{z_1^*} - e^{j\omega}\right)\left(\frac{1}{z_1} - e^{j\omega}\right)\left(\frac{1}{z_2^*} - e^{j\omega}\right)\left(\frac{1}{z_2} - e^{j\omega}\right)}{e^{j2\omega}} \\ H(\omega) = H^*(\omega) &\Rightarrow z_2 = \frac{1}{z_1} \Rightarrow |z_1|^2 |z_2|^2 = 1 \end{aligned}$$

1 Transformation of finite impulse response filters

Once a finite impulse response filter, $h(n)$, has been defined, then it is possible to change the type of filter (low-pass, bandpass, highpass, etc.) through simple transformations. Here we will deal with the basic transformations to turn a low-pass filter into a high-pass and a bandpass filter, with the same bandwidth as the low-pass design.

1.1 z-transform perspective

To understand the transformations, it is helpful to consider the z-transform of the filter, and examine the location of zeros. A low-pass filter will have the zeros located on the unit circle within the stopband area, which is towards the left hand side of the z-transform diagram. The corresponding z-transform for a highpass filter would be the mirror-image of the lowpass one, reflected in the y-axis. Figure 1 shows the pole-zero map for a 311 tap filter designed using the window design method and a Hanning window.

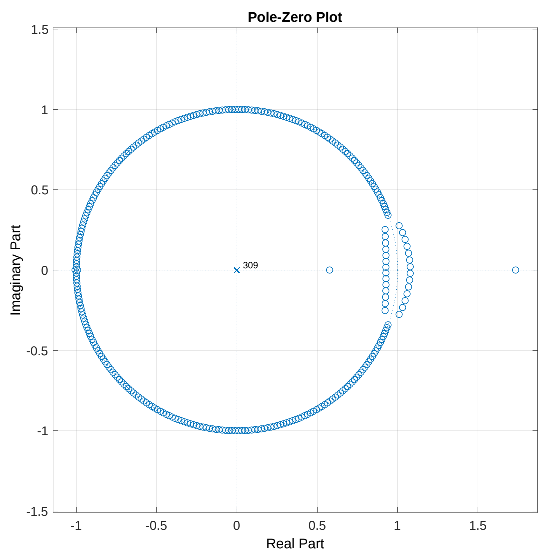


Figure 1: Low pass filter zero locations

1.2 Highpass filter

Reflection in the imaginary axis can easily be achieved by simply setting $H_{hp}(z) = H_{lp}(-z)$, where $H_{hp}(z)$ is the z-transform of the highpass filter, and $H_{lp}(z)$ the transform of the lowpass filter. Then:

$$H_{hp}(z) = \sum_{n=0}^{M-1} b_n(-z)^{-n} \tag{1}$$

$$= \sum_{n=0}^{M-1} b_n(-1)^{-n}z^{-n} \tag{2}$$

$$= \sum_{n=0}^{M-1} ((-1)^n b_n)z^{-n} \tag{3}$$

Thus $h_{hp}(n) = (-1)^n h_{lp}(n)$. This produces a highpass filter with a bandwidth equivalent to the lowpass filter without the need for any more calculation. Figure 2 shows the resulting zero locations.

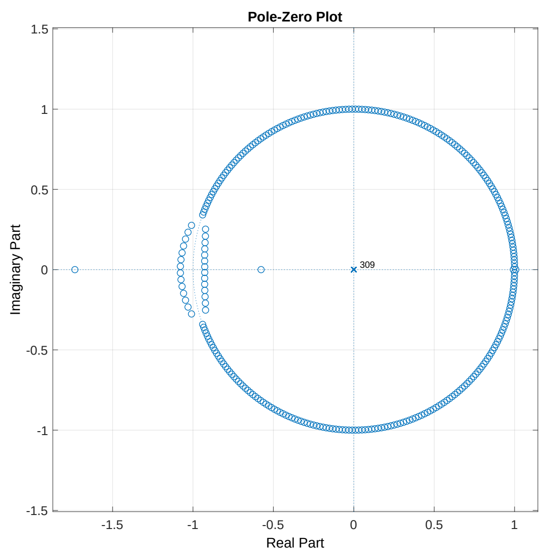


Figure 2: High pass filter zero locations

1.3 Bandpass filter

To generate a bandpass filter, instead of reflecting the zeros, we need to rotate them in the z-transform domain. A rotation is achieved by replacing z with $e^{j\omega_c}z$, which rotates the filter response to be centred on the frequency ω_c. However, this will result in a complex impulse reesponse, and a complex filter output as the frequency response will no longer be complex conjugate symmetric. To solve this problem, we also need to rotate the filter in the opposite direction, and sum the results. Thus: $H_{bp} = H_{lp}(e^{j\omega_c}z) + H_{lp}(e^{-j\omega_c}z)$. Expanding the z-transform we find the following:

$$\begin{aligned} H_{bp}(z) & \sum_{n=0}^{M-1} b_n(e^{j\omega_c}z)^{-n} + \sum_{n=0}^{M-1} b_n(e^{-j\omega_c}z)^{-n} && (4) \\ &= \sum_{n=0}^{M-1} e^{-j\omega_c n} b_n z^{-n} + e^{j\omega_c n} b_n z^{-n} && (5) \\ &= \sum_{n=0}^{M-1} 2\cos(\omega_c n) b_n z^{-n} && (6) \end{aligned}$$

Thus a bandpass filter, with a centre frequency of ω_c can be easily produced as $h_{bp}(n) = 2\cos(\omega_c n)h_{lp}(n)$. Note that the bandwidth of this filter will be twice the cut-off frequency of the lowpass filter. Figure 3 shows the final locations. Note that due to the factor of 2 required, the zeros located on the real axis are scaled in size.

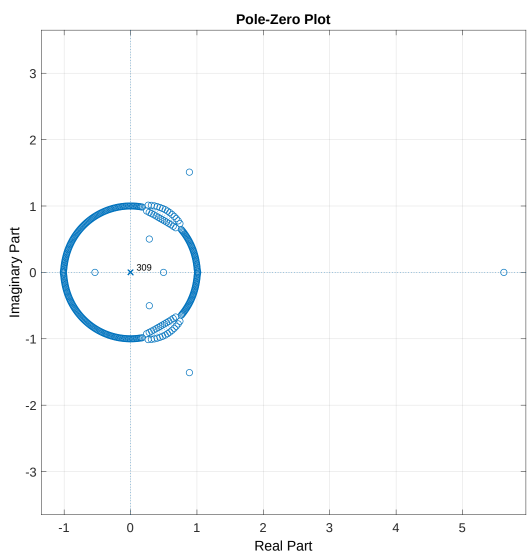


Figure 3: Bandpass filter zero locations