Frequency-Domain Analysis of LTI Systems

Linear Time-Invariant Systems (pp.308-331)

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Introduction

In previous courses, linear time invariant (LTI) systems have been defined by their impulse response, $h(n)$, transfer function, $H(\omega)$, or z-transform, $H(z)$. In this lecture we will briefly revise this material, and then define relationships with correlations.

1 Transform of LTI system

The z-transform is a frequency representation of a signal, or system. When applied to a system it incorporates information relating to the frequency response, as well as the transient response of the system. The z-transform of an LTI can be written

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

We assume a causal system, so $h(n) = 0 \; \forall n < 0$, and that $h(n)$ are real valued. The frequency response is given by

$$H(\omega) = H(z)|_{z=e^{j\omega}} = \sum_{n=0}^{\infty} h(n) e^{-j\omega n}$$

In other words, the frequency response of an LTI can be determined by evaluating the z-transform over the unit circle. As can be seen, this is equivalent to taking the discrete-time Fourier transform of the impulse response, $h(n)$. Note that for causal systems to be stable, they must have all poles within the unit circle, although zeros are allowed at any location in the z-plane.

If $H(z)$ can be written $H(z) = B(z)/A(z)$, where $B(z)$ has $M+1$ coefficients, and $A(z)$ has $N+1$ coefficients, with $a_0 = 1$, then

$$\begin{aligned} H(z) &= \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \\ &= b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})} \end{aligned}$$

where $\prod$ denotes the product, $\{z_k\}$ are the roots of $B(z)$, and $\{p_k\}$ the roots of $A(z)$. If the impulse response, $h(n)$, is real valued, $\{a_k\}$ and $\{b_k\}$ are real valued. Thus, z_k and p_k are real, or occur in complex-conjugate pairs. Complex conjugate pairs of roots arise from solutions of $az^2 + bz + c = 0$ where $b^2 - 4ac < 0$. In the general case, numerical methods can be used to identify roots.

The density spectrum of the output of a linear system can be derived as follows:

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ S_{yy}(\omega) &= |Y(\omega)|^2 = Y(\omega)Y^*(\omega) \\ &= H(\omega)X(\omega)H^*(\omega)X^*(\omega) \\ \Rightarrow S_{yy}(\omega) &= |H(\omega)|^2 S_{xx}(\omega) \end{aligned} \qquad ((1.34))$$

where $S_{xx}(\omega)$ is the density spectrum of the system input.

It is helpful to re-express $|H(\omega)|^2$ in terms of the z-transform of the system. Since $|H(\omega)|^2 = H(\omega)H^*(\omega)$, and we have already established that $H(\omega) = H(z)|_{z=e^{j\omega}}$, we only need to consider $H^*(\omega)$.

$$
\begin{aligned}
H^*(\omega) &= \left(\frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}\right)^* \\
&= \frac{\sum_{k=0}^{M} b_k^* e^{j\omega k}}{\sum_{k=0}^{N} a_k^* e^{j\omega k}}
\end{aligned}
$$

Since $\{a_k\}$ and $\{b_k\}$ are real

$$
\begin{aligned}
&= \frac{\sum_{k=0}^{M} b_k e^{j\omega k}}{\sum_{k=0}^{N} a_k e^{j\omega k}} \\
H\left(\frac{1}{z}\right) &= \frac{\sum_{k=0}^{M} b_k z^k}{\sum_{k=0}^{N} a_k z^k} \Rightarrow H^*(\omega) = H(z^{-1})|_{z=e^{j\omega}} \\
\Rightarrow |H(\omega)|^2 &= H(z)H(z^{-1})|_{z=e^{j\omega}}
\end{aligned}
$$

This property is a useful one as it defines a relationship with the density spectrum through the use of z-transforms. Exploring further reveals a link with the autocorrelation function.

Relationship with $r_{hh}(m)$

$$
\begin{aligned}
H(z)H(z^{-1}) &= \left\{\sum_{k=-\infty}^{\infty} h(k)z^{-k}\right\}\left\{\sum_{l=-\infty}^{\infty} h(l)z^{l}\right\} \\
&= \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h(k)h(l)z^{-k}z^{l} \\
&= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} h(m+l)h(l)z^{-m} \\
&= \sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(l)h(l+m)z^{-m}
\end{aligned}
$$

Thus, it is then possible to write:

$$
\Rightarrow H(z)H(z^{-1}) = \sum_{m=-\infty}^{\infty} r_{hh}(-m)z^{-m}
$$

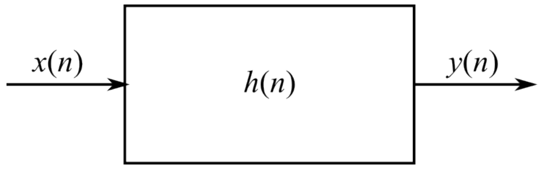
Given this result, showing a linkage between the transfer function and the correlation function, it suggests a similar relationship with the Fourier transform. To establish this, make the substitution $z = e^{j\omega}$, since $|H(\omega)|^2 = H(z)H(z^{-1})|_{z=e^{j\omega}}$ and noting that $r_{hh}(m)$ is symmetric:

$$
|H(\omega)|^2 = \sum_{m=-\infty}^{\infty} r_{hh}(m)e^{-j\omega m}
$$

That is to say, $|H(\omega)|^2$ is the discrete-time Fourier transform of the autocorrelation of the impulse response.

1 Input-output correlation sequences

A linear time-invariant (LTI) system can be represented by its impulse response, $h(n)$. The output, $y(n)$, is then described in terms of its convolution with the input, $x(n)$.



$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

It makes sense to ask the question, what is the correlation between the input of an LTI system, and its output. Using the convolution definition, we can find the crosscorrelation between the system input and the system output. The following derivation follows that in the course textbook, however here the convolution steps are expanded. Note that the autocorrelation of the system impulse response only exists if the system itself is stable.

$$\begin{aligned}
r_{yx}(l) &= \sum_{n=-\infty}^{\infty} y(n)x(n-l) \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h(k)x(n-k)x(n-l) \\
&= \sum_{k=-\infty}^{\infty} h(k) \sum_{n=-\infty}^{\infty} x(n-k)x(n-l) \\
&= \sum_{k=-\infty}^{\infty} h(k) \sum_{m=-\infty}^{\infty} x(m)x(m+k-l) \\
&= \sum_{k=-\infty}^{\infty} h(k)r_{xx}(l-k) = h(l) * r_{xx}(l) \qquad (6.29)
\end{aligned}$$

Thus, the crosscorrelation between the system input and output is given by the convolution of the impulse response with the autocorrelation of the input.

The autocorrelation of the LTI output can be easily found directly, or using the properties of convolution:

$$\begin{aligned}
r_{yy}(l) &= y(l) * y(-l) \\
&= [h(l) * x(l)] * [h(-l) * x(-l)] \\
&= [h(l) * h(-l)] * [x(l) * x(-l)] \\
&= r_{hh}(l) * r_{xx}(l) \qquad (6.30)
\end{aligned}$$

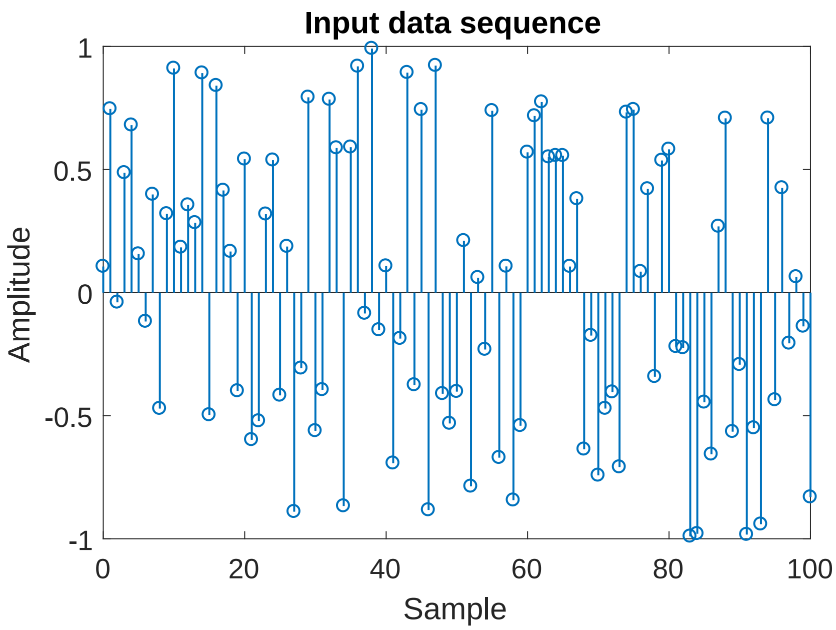
The cross-correlation property can be exploited by practical systems when trying to characterise an unknown LTI system. With a careful choice of input signal, this property can be used to identifying the impulse response of an unknown linear system.

Application for LTI system identification

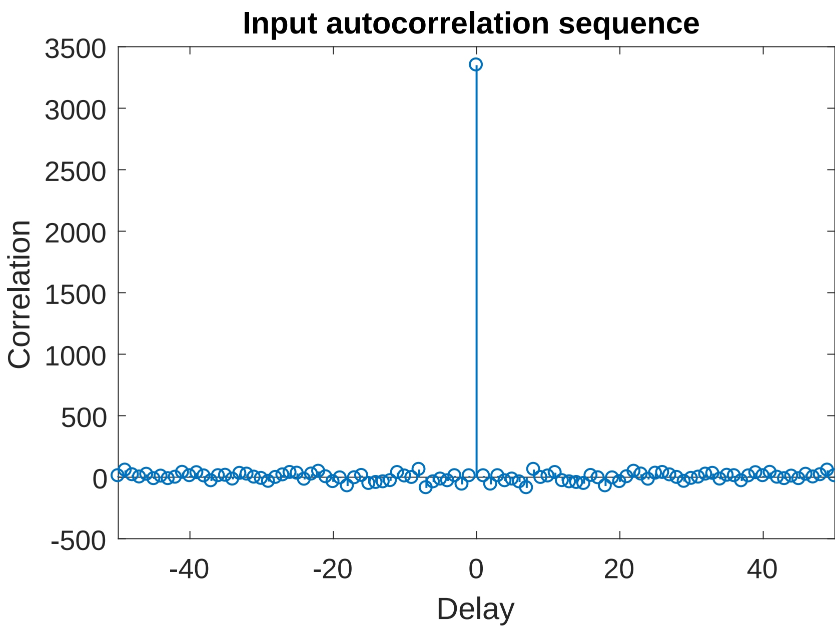
In practical situations it may not be appropriate to use an impulse as an input to an LTI system. Such cases may be, for example, where a system must operate in a covert mode, for example a military radar, or it may be useful to be able to identify the particular input if the system is also subject to other inputs at the same time. One other reason to avoid using an impulse is that it may be more likely to drive a system into a non-linear region of operation, so the output would not be a true estimation of the impulse response.

In such cases, the relationship: $r_{yx}(l) = h(l) * r_{xx}(l)$ may be used. If, instead of making $x(n) = \delta(n)$, we select $x(n)$ such that $r_{xx}(l) = \delta(l)$, then, $r_{yx}(l) = h(l) * \delta(l) = h(l)$.

As an example, $x(n)$ is selected to be a sequence consisting of random numbers between -1 and 1. On applying this input to the system, a sequence $y(n)$ is observed. From these two sequences $r_{yx}(l)$ is determined. The first 100 samples of the input signal, $x(n)$:

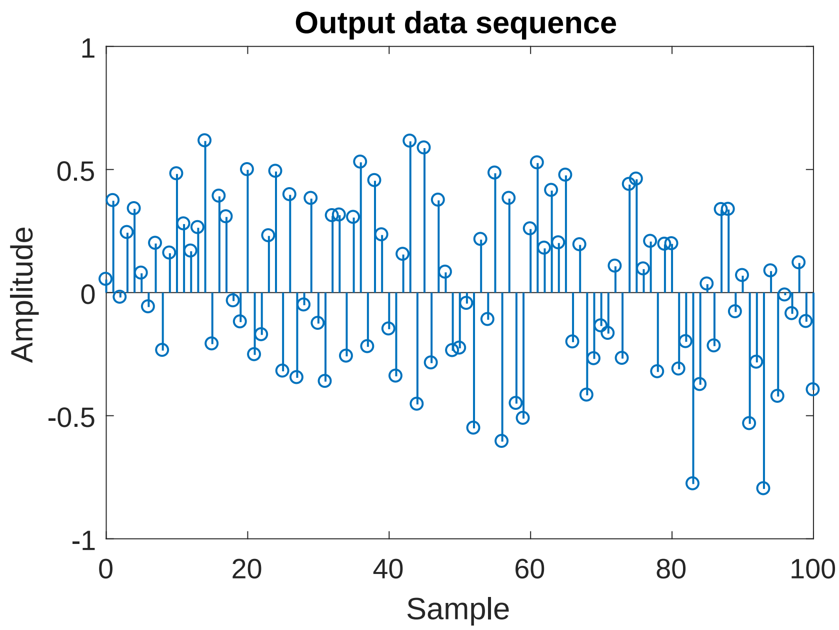


The input signal autocorrelation, $r_{xx}(l)$:

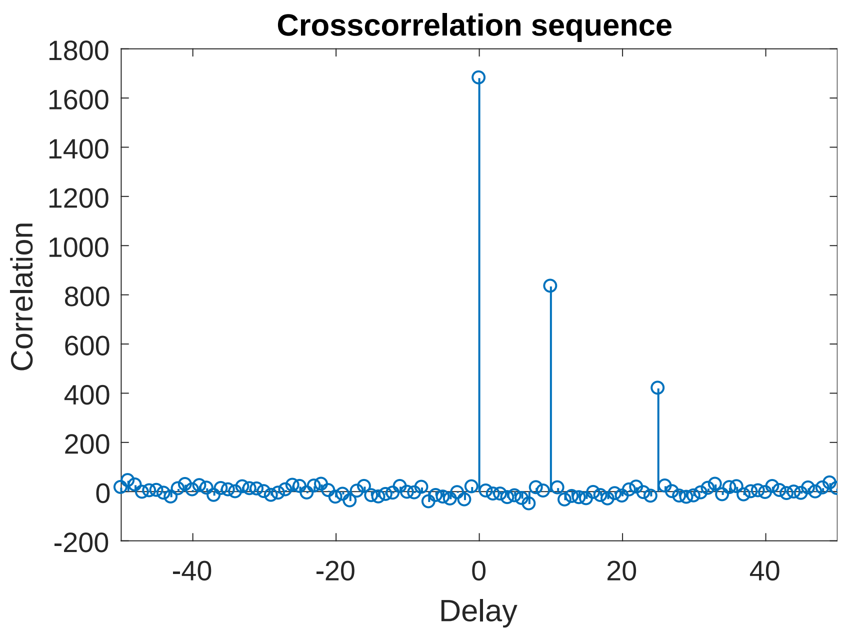


Note that this correlation is not exactly $\delta(n)$, as the terms at non-zero delays are not exactly zero, but it is a sufficiently close approximation for the purpose of identifying the system of interest.

If the signal $x(n)$ is used as the input to an LTI system, and the output of the system $y(n)$ can be observed, then the impulse response can be identified through computation of the crosscorrelation of the two signals. The first 100 samples of the output signal, $y(n)$.



When $r_{yx}(l)$ is calculated, the resulting output is a good estimation of the system impulse response. The crosscorrelation, $r_{yx}(l)$.



The crosscorrelation output then forms an estimate of the system impulse response. The quality of the estimate is dictated by:

- the length of the observation; and
- the degree of approximation to a delta function that the autocorrelation of the input signal, $r_{xx}(l)$ achieves.

The longer that the input and output can be observed, the better the estimate will be as any noise in the observation will be averaged out. There is a disadvantage with this approach though, if the system is a time varying one (for example a practical radar where the targets are moving), then observing for a long time will degrade the estimate, and not improve it!

The autocorrelation of the input should have a high, narrow peak, with low off-peak values. If it is from a periodic source, the period should be longer than the impulse response of the unknown system. Provided these criteria are met, then a good estimate of the impulse response can be obtained.

Example

Consider the finite impulse response filter:

$$y(n) = x(n) - \alpha x(n-1)$$

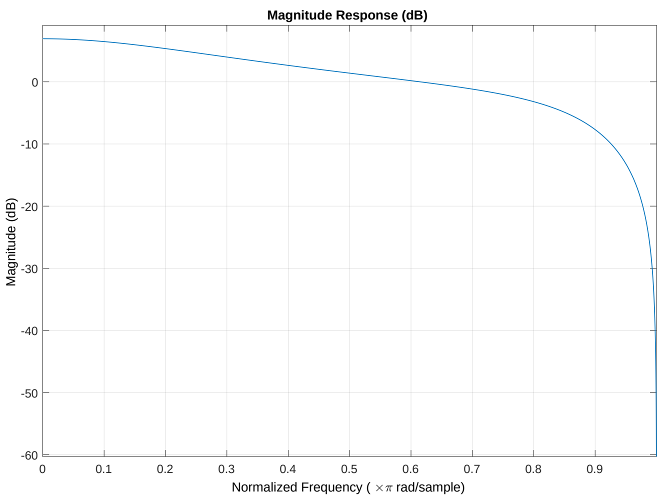
with $|\alpha| \leq 1$. To find its density spectrum, find $H(z)H(z^{-1})$:

$$\begin{aligned} H(z) &= 1 - \alpha z^{-1} \\ \Rightarrow H(z)H(z^{-1}) &= (1 - \alpha z^{-1})(1 - \alpha z) \\ &= 1 + \alpha^2 - \alpha(z + z^{-1}) \\ \Rightarrow |H(\omega)|^2 &= 1 + \alpha^2 - 2\alpha\cos(\omega) \end{aligned}$$

This is the filter frequency response which, for a positive value of α is a high pass filter.

Example 5.2.1

$$\begin{aligned} y(n) &= -0.1y(n-1) + 0.2y(n-2) + \\ & \quad\; x(n) + x(n-1) \\ \Rightarrow H(z) &= \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}} \\ H(z)H(z^{-1}) &= \frac{1 + z^{-1}}{1+0.1z^{-1}-0.2z^{-2}} \frac{1 + z}{1+0.1z-0.2z^{2}} \\ &= \frac{2 + z + z^{-1}}{1.05+0.08(z+z^{-1})-0.2(z^2+z^{-2})} \\ \Rightarrow |H(\omega)|^2 &= \frac{2 + 2\cos(\omega)}{1.05 + 0.16\cos(\omega) - 0.4\cos(2\omega)} \\ \Rightarrow |H(\omega)|^2 &= \frac{2 + 2\cos(\omega)}{1.45 + 0.16\cos(\omega) - 0.8\cos^2(\omega)} \end{aligned}$$



It is worth noting that although $|H(\omega)|^2$ is defined by a specified impulse response, $\{h_n\}$, it is not possible to define a unique impulse response from $|H(\omega)|^2$. This is because it is possible to derive different impulse responses with an identical $|H(\omega)|^2$ by simply inverting poles and zeros in $H(z)$. (Inverting poles and zeros in $H(z)$, when multiplied by the corresponding $H(z^{-1})$, will result in an identical expression). The difference between filters produced in this way is contained within the phase information, which is not present in $|H(\omega)|^2$. In the time domain, the phase changes will lead to frequency components being delayed by different amounts.

Geometric method

It is possible to use the location of poles and zeros in the z domain, to quickly sketch the frequency response of an LTI system. Whilst this involves the same calculations as the direct solution, it provides an intuitive insight into the response of a filter without detailed calculation. We start by re-writing the polynomial expression for $H(\omega)$:

$$\begin{aligned} H(\omega) &= b_0 \frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\omega})} \\ &= b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)} \end{aligned}$$

Each of the terms of the two products is a complex number, whose magnitude represents the distance from a zero, or pole, to a point on the unit circle. The phase is the angle from the zero, or pole, to this point. Thus,

$$|H(\omega)| = |b_0| \frac{\prod_{k=1}^{M}|e^{j\omega} - z_k|}{\prod_{k=1}^{N}|e^{j\omega} - p_k|}$$

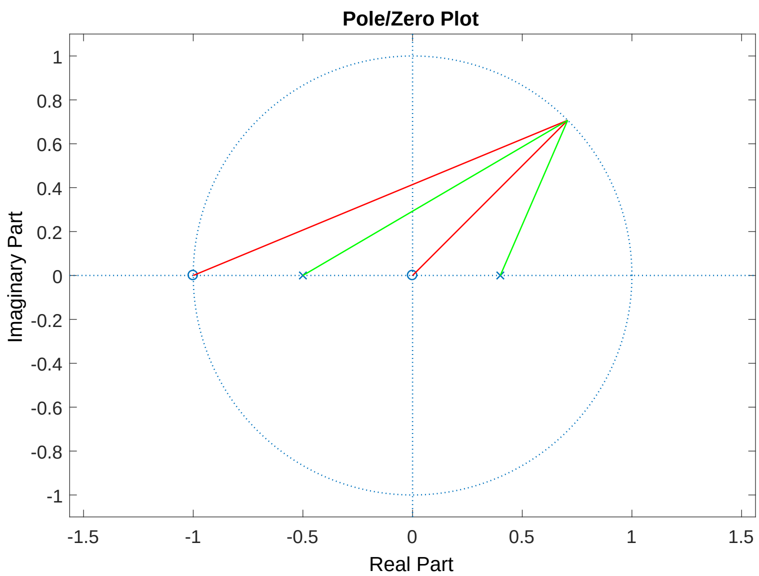
and

$$\measuredangle H(\omega) = \omega(N-M) + \sum_{k=1}^{M} \measuredangle(e^{j\omega} - z_k) - \sum_{k=1}^{N} \measuredangle(e^{j\omega} - p_k)$$

Example 5.2.1

$$\begin{aligned} H(z) &= \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}} \\ \Rightarrow H(z) &= \frac{z(z+1)}{(z+0.5)(z-0.4)} \end{aligned}$$

There are two zeros, $\{z_k\} = \{0, -1\}$, and two poles, $\{p_k\} = \{-0.5, 0.4\}$.



Distance to poles:

- $|e^{j\pi/4} - 0.4| = \sqrt{0.307^2 + 0.707^2} = 0.771$

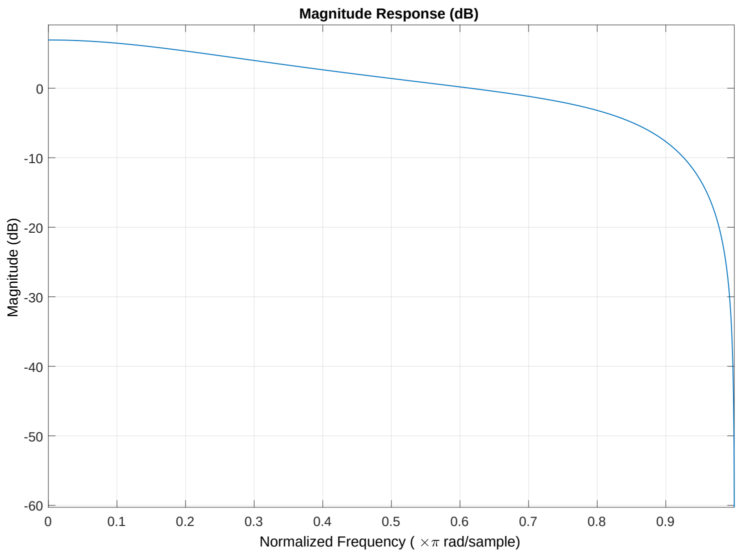
- $|e^{j\pi/4} + 0.5| = \sqrt{1.207^2 + 0.707^2} = 1.399$

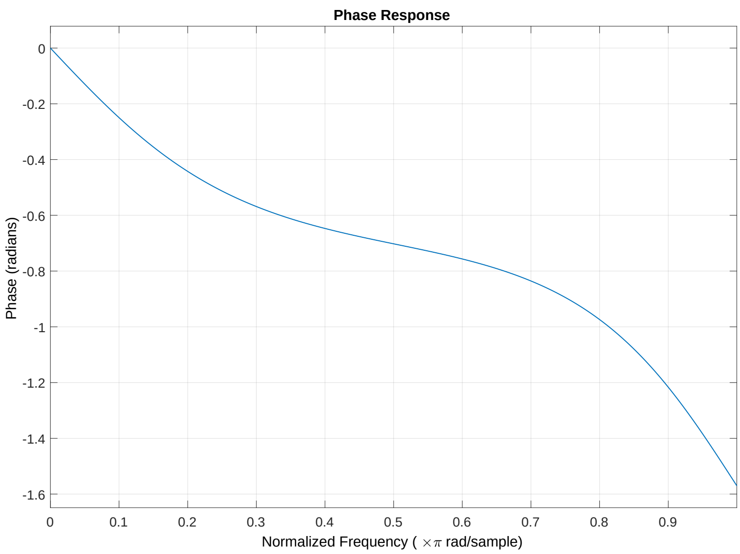
Distance to zero at -1:

- $|e^{j\pi/4} + 1| = \sqrt{1.707^2 + 0.707^2} = 1.848$

Thus, magnitude is equal to $1.848/(0.771 \times 1.399) = 1.71$. Note that the distance to the zero at the origin is always 1. Using a similar technique, the magnitude for other selected points can also be found:

	0	$\pi/4$	$\pi/2$	$3\pi/4$
$z = 0.4$	0.6	0.771	1.077	1.313
$z = -0.5$	1.5	1.399	1.118	0.737
$z = 0$	1	1	1	1
$z = -1$	2	1.848	1.414	0.765
$\|H(\omega)\|$	2.222	1.713	1.174	0.791





Correlation functions and spectra

The material for this section is in the hardcover edition of the textbook, and the New International edition, but not included in the International edition of 2007. Those with the 2007 International edition should use the notes here as the source material for study, and refer to copies in the library.

Input-Output Correlation Functions and Spectra

We have already derived, for an LTI system:

$$r_{yy}(m) = r_{hh}(m) * r_{xx}(m) \tag{3.1}$$

$$r_{yx}(m) = h(m) * r_{xx}(m) \tag{3.2}$$

That is, the autocorrelation of a system output is given by the convolution of the autocorrelation of the input signal and the autocorrelation of the impulse response. Whereas the crosscorrelation between the system output and its input is the convolution of the autocorrelation of the input signal and the channel impulse response. As we have previously observed, this allows the impulse response of an LTI system to be found easily if the input signal has an autocorrelation function that closely approximates an impulse. This was used in the radar example to identify the location and size of targets.

We can now transform the relationship to the frequency domain, as we know from the Wiener-Khintchine relationship that the density spectrum is the Fourier transform of the correlation function. Because of the relationship between convolution in the time domain and multiplication in the frequency domain, this results in a simple relationship.

$$\begin{aligned} S_{yy}(z) &= S_{hh}(z)S_{xx}(z) \\ &= H(z)H(z^{-1})S_{xx}(z) \\ S_{yx}(z) &= H(z)S_{xx}(z) \end{aligned}$$

and thus,

$$\begin{aligned} S_{yy}(\omega) &= |H(\omega)|^2 S_{xx}(\omega) \\ S_{yx}(\omega) &= H(\omega)S_{xx}(\omega) \end{aligned}$$

The cross density spectrum of the output and input is the transfer function multiplied by the density spectrum of the input signal. If $H(\omega)$ is a filter, then the frequency components of $x(n)$, as expressed in the density spectrum, will be attenuated, or amplified by the response of $H(\omega)$. This is a useful result that allows us to reason about the effect of an LTI system on an input signal with a given spectrum.

Example

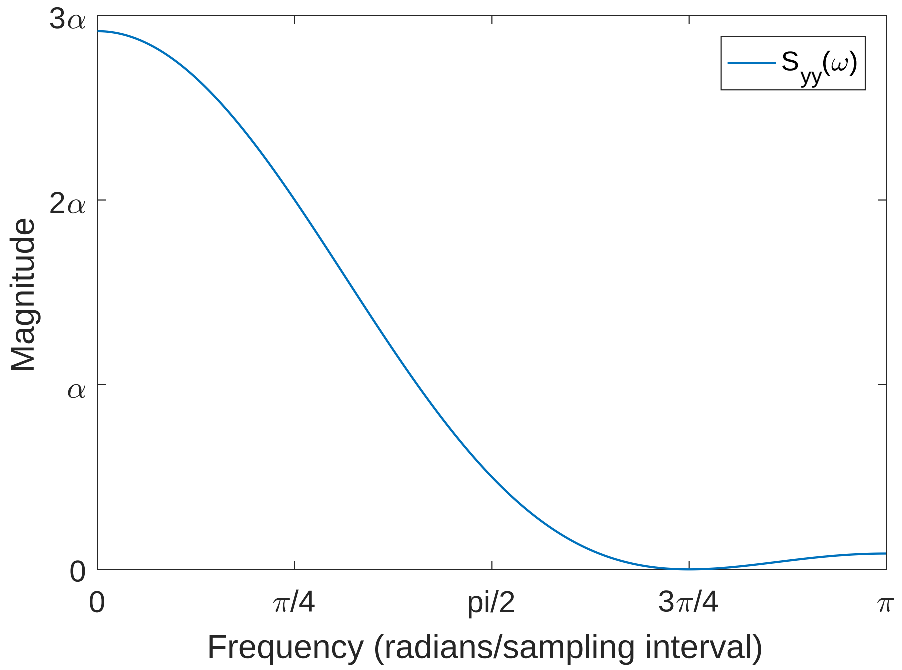
Consider an audio amplifier with a transfer function:

$$H(z) = \frac{1}{2} + \frac{z^{-1}}{\sqrt{2}} + \frac{z^{-2}}{2}$$

driven by a wideband signal specified by $S_{xx}(\omega) = \alpha$. The density spectrum of the output:

$$\begin{aligned} S_{yy}(z) &= H(z)H(z^{-1})S_{xx}(z) \\ &= \left(\frac{1}{2} + \frac{z^{-1}}{\sqrt{2}} + \frac{z^{-2}}{2}\right)\left(\frac{1}{2} + \frac{z}{\sqrt{2}} + \frac{z^2}{2}\right)S_{xx}(z) \\ &= \left(1 + \frac{1}{\sqrt{2}}(z^{-1}+z) + \frac{1}{4}(z^{-2}+z^2)\right)S_{xx}(z) \\ \Rightarrow S_{yy}(\omega) &= \alpha\left(1 + \frac{2\cos(\omega)}{\sqrt{2}} + \frac{\cos(2\omega)}{2}\right) \end{aligned}$$

The resulting density spectrum, when plotted, shows that the output has the majority of its energy or power concentrated around the low frequency components, with attenuation at higher frequencies.



System identification

In the special case that the input signal is chosen to have a flat (constant) density spectrum as in the example above, i.e. $S_{xx}(\omega) = S_x$,

$$\begin{aligned} S_{yx}(\omega) &= H(\omega)S_{xx}(\omega) \\ \Rightarrow H(\omega) &= \frac{S_{yx}(\omega)}{S_x} \\ \Rightarrow h(n) &= \frac{r_{yx}(n)}{S_x} \end{aligned}$$

This means that if we can define the input to an LTI system, $x(n)$, to have a flat spectrum, then it is possible by measuring the cross-density spectrum, $S_{yx}(\omega)$, or the crosscorrelation $r_{yx}(n)$, and the value of S_x, to determine the transfer function or impulse response respectively.

In practice, the quality of the estimate will depend on the degree to which $S_{xx}(\omega)$ covers the frequency range of the system. Any frequency components not contained in $S_{xx}(\omega)$ will not be represented in $S_{yx}(\omega)$, so are not captured in the estimate of $H(\omega)$. This implies that we should aim to select $x(n)$ such that $S_{xx}(\omega)$ is flat, i.e. a constant. By the Wiener-Khintchine relationship, the corresponding autocorrelation function will therefore be a good approximation of a delta function.