DSA4/DTSA Frequency Analysis of Signals Exercises

Questions 3, 5 and 6 are taken from "Digital Signal Processing, New International Edition/4th", Proakis & Manolakis, ©Pearson Education Limited, 2014. ISBN: 978-1-29202-573-5.

1. Show that the Fourier transform of the signal $x(t)$ where

$$x(t) = \begin{cases} 1 & ; 0 \leq t < \tau \\ 0 & ; t < 0 \text{ or } t > \tau \end{cases}$$

is given by

$$X(\omega) = \frac{2e^{-j\omega\tau/2}}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$$

2. Using the definition of the Discrete Time Fourier Series (DTFS) (4.2.8), show that $c_{-k} = c_{N-k}$.

3. Consider the full-wave rectified sinusoid in Fig. 1. Determine its spectrum, $|c_k|^2$.

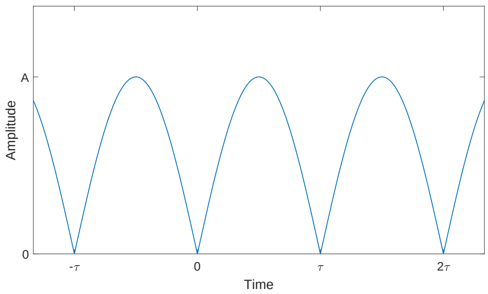


Figure 1: Full-wave rectified sinusoid

4. Show that, for a real signal $x(t)$, the following property holds:

$$x(-t) \overset{F}{\longleftrightarrow} X^*(\omega)$$

5. Compute the magnitude and phase spectra for $x_b(t) = Ae^{-a|t|}$ where $a > 0$.

6. Consider the following periodic signal, with a period of 6 samples:

$$x(n) = \{\ldots, 1, 0, 1, 2, \underset{\uparrow}{3}, 2, 1, 0, 1, \ldots\}$$

Determine the spectrum of the signal $x(n)$. Note that $\uparrow$ denotes the sample $x(0)$.

7. The even component of a signal, $x_e(n)$, and the odd component, $x_o(n)$, can be determined by:

$$x_e(n) = \frac{x(n) + x(-n)}{2}$$
$$x_o(n) = \frac{x(n) - x(-n)}{2}$$

Hence, or otherwise, given the signal $x(n) = a(n) + jb(n)$, show that it is possible to determine the discrete-time Fourier transform of $a(n)$ and $b(n)$ from the discrete-time Fourier transform of $x(n)$.

DSA4/DTSA The Discrete Fourier Transform Exercises

Questions 2, and 4 are taken from "Digital Signal Processing, New International Edition/4th", Proakis & Manolakis, ©Pearson Education Limited, 2014. ISBN: 978-1-29202-573-5.

1. Compute the eight-point discrete Fourier transforms (DFTs) of the sequences $x(n) = \{1, 0, 0, 0, 0, 0, 0, 0\}$, and $y(n) = \{0, 1, 0, 0, 0, 0, 0, 0\}$. Comment on the result in terms of the properties of Fourier transforms.

2. The first five points of the eight-point DFT of a real-valued sequence are $\{0.25, 0.125 - j0.3018, 0, 0.125 - j0.0518, 0\}$. Determine the remaining three points.

3. A 64-point DFT is performed on a sequence sampled at 10kHz. Sketch the general shape of the measured DFT when:

   (a) the input is two sinusoids of equal amplitude, one at 2.5kHz and the other at 7kHz

   (b) the input is a unity mark-space ratio square wave (i.e. a 50% duty cycle square pulse) with a period of 0.4ms.

   State the effect of windowing the data prior to performing the DFT calculations on these signals.

4. If $X(k)$ is the N-point DFT of the sequence $x(n)$, determine the N-point DFT of the sequence:

$$x_s(n) = x(n) \sin\left(\frac{2\pi k_0 n}{N}\right) \;;0 \leq n \leq N-1$$

   in terms of $X(k)$, when k_0 is an integer. Note that you may wish to use Euler's formula to expand the input sequence.

5. **Exam question, 2008, B1, parts (c)-(e)**

   (a) Show that the discrete-time Fourier transform (DTFT) of

$$x(n) = \begin{cases} 0 & ;n < 0 \\ 1 & ;0 \leq n < N \\ 0 & ;n \geq N \end{cases}$$

   is given by

$$X(\omega) = \exp\left(-j(N-1)\omega/2\right) \frac{\sin(N\omega/2)}{\sin(\omega/2)}$$

   (3 marks).

   (b) Hence, or otherwise, calculate the DTFT of $y(n)$ consisting of four samples such that

$$y(n) = \begin{cases} 0 & ;n < 0 \\ \cos(n\pi/4) & ;0 \leq n < 4 \\ 0 & ;n > 4 \end{cases}$$

   (4 marks)

   (c) Using your answer to part (b), determine the result of a 4-point DFT on

$$z(n) = \cos(n\pi/4)$$

   (3 marks)

DSA4/DTSA Discrete-Time Signals and Systems Exercises

Questions 1, 2 and 4 are taken from "Digital Signal Processing, New International Edition/4th", Proakis & Manolakis, ©Pearson Education Limited, 2014. ISBN: 978-1-29202-573-5.

1. Compute the correlation sequences $r_{xx}(l)$ and $r_{xy}(l)$ for the following signal sequences

$$
\begin{aligned}
x(n) &= \begin{cases} 1 & ; n_0 - N \leq n \leq n_0 + N \\ 0 & ; \text{otherwise} \end{cases} \\
y(n) &= \begin{cases} 1 & ; -N \leq n \leq N \\ 0 & ; \text{otherwise} \end{cases}
\end{aligned}
$$

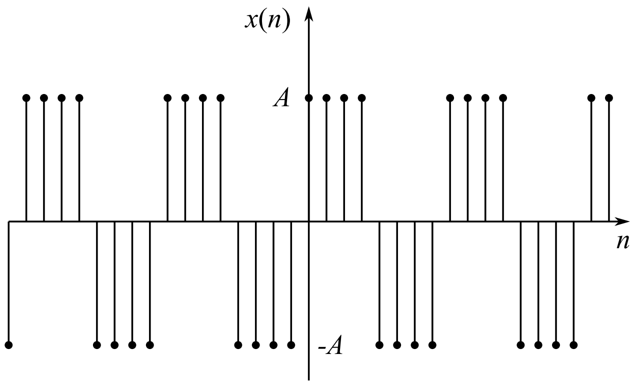
2. Determine the autocorrelation sequences of the following signals:

   (a) $x(n) = \{\underset{\uparrow}{1}, 2, 1, 1\}$

   (b) $y(n) = \{\underset{\uparrow}{1}, 1, 2, 1\}$

   What can you conclude from this?

3. Find the autocorrelation of the asymmetrical square wave shown below:



4. An audio signal, $s(t)$, generated by a loudspeaker is reflected at two different walls with reflection coefficients, r_1 and r_2. The signal $x(t)$, recorded by a microphone close to the loudspeaker, after sampling, is:

$$x(n) = s(n) + r_1 s(n - k_1) + r_2 s(n - k_2)$$

   where k_1 and k_2 are the delays of the two echoes.

   (a) Determine the autocorrelation, $r_{xx}(l)$, of the signal $x(n)$.

   (b) Can we obtain r_1, r_2, k_1 and k_2 by observing $r_{xx}(l)$?

   (c) What happens if $r_2 = 0$?

DSA4/DTSA Frequency-Domain Analysis of LTI Systems Tutorial Exercises

Questions 1 and 2 are taken from "Digital Signal Processing, New International Edition/4th", Proakis & Manolakis, ©Pearson Education Limited, 2014. ISBN: 978-1-29202-573-5.

1. Determine the impulse response of a causal LTI system which produces the response:

$$y(n) = \{\underset{\uparrow}{1}, -1, 3, -1, 6\}$$

when excited by the input signal

$$x(n) = \{\underset{\uparrow}{1}, 1, 2\}$$

Hint: Determine an expression for $y(0)$ *first, remembering that* $x(n) = 0\;; n < 0$ or $n > 2$.
Using your result, show that

$$r_{yx}(l) = \sum_{n=0}^{\infty} h(n) r_{xx}(l-n)$$

2. Consider a system with impulse response

$$h(n) = b_0\delta(n) + b_1\delta(n-D) + b_2\delta(n-2D)$$

   (a) Determine the magnitude response of the system.
   (b) Plot $|H(\omega)|$ for $b_0 = 0.1$, $b_1 = 1$, and $b_2 = 0.05$.

3. A causal low pass filter, $h(n)$, is described by its impulse response as:

$$h(n) = \frac{1}{\alpha}\exp\Big(-\frac{n}{\alpha}\Big) \;\; ; n \geq 0$$

Find the crosscorrelation between the output and the input, $r_{yx}(l)$, for this filter if the input autocorrelation function is defined as

$$r_{xx}(l) = \frac{\beta S_0}{2}\exp(-\beta|l|)$$

4. A signal, $x(n)$, has a power density spectrum defined as $S_{xx}(\omega)$. Find the power density spectrum of $y(n) = x(n) - x(n-D)$.

DSA4/DTSA Random Signals, Correlation Functions, and Power Spectra Tutorial

1. A particular random variable has a probability distribution function given by

$$F(x) = \begin{cases} 0 & ; -\infty < x < 0 \\ 1 - e^{-x} & ; 0 \leq x < \infty \end{cases}$$

   Find:

   (a) the probability that $X > 0.5$
   (b) the probability that $X \leq 0.25$
   (c) the probability that $0.3 < X \leq 0.7$

2. Consider the following time function

$$x(n) = A\cos(\omega n - \theta)$$

   The phase angle, θ, is assumed to be a random variable whose probability density function is given as

$$p(\theta) = \frac{1}{\pi}\ ;\ 0 \leq \theta < \pi$$

   Find

   (a) the mean of $x(n)$;
   (b) the variance of $x(n)$; and
   (c) the autocorrelation of $x(n)$.

   From your answer, state whether the random variable is stationary or not.

3. A signal, $x(n)$, is corrupted by additive white Gaussian noise, $w(n)$, such that at the input to an amplifier,

$$y(n) = x(n) + w(n)$$

   Let $x(n)$ be defined as

$$x(n) = A\cos(\omega_0 n)$$

   and $w(n)$ has a variance, $\sigma^2 = \frac{A}{2}$. Expressing your answer in terms of the error function (erf), what is the probability that the amplifier input, $y(n)$, exceeds $\pm 2A$? Note that as $y(n)$ is non-stationary, the answer will be a function of n.

4. A random variable, $Z(n)$, is defined to be

$$Z(n) = X(n) + X(n + a)$$

   where $X(n)$ is a sample function from a stationary random process whose autocorrelation function is

$$\gamma_{xx}(m) = e^{-|m|}$$

   Derive an expression for the autocorrelation of the random process $Z(n)$.

5. Using your answer to Problem 4, find the power density spectrum of $Z(n) = X(n) + X(n + a)$.

6. Given that the autocorrelation function of a certain stationary process is

$$\gamma_{xx}(m) = 25 + \frac{4}{1 + 6m^2}$$

   find

(a) the mean value of the process, $X(m)$

(b) the variance of the process $X(m)$

7. A stationary random process has a bi-lateral power density spectrum given by:

$$\Gamma_{xx}(\omega) = 4\cos(2\omega) + 16$$

(a) Find the total power of this random process

(b) Find the power of this random process in the range $-\frac{\pi}{4}$ to $\frac{\pi}{4}$.

DSA4/DTSA Design of Digital Filters, part 1, Tutorial

Questions 1, 2 and 3 are taken from "Digital Signal Processing, New International Edition/4th", Proakis & Manolakis, ©Pearson Education Limited, 2014. ISBN: 978-1-29202-573-5.

1. Determine the unit sample response $\{h(n)\}$ of a linear-phase FIR filter of length $M = 4$ for which the frequency response at $\omega = 0$ and $\omega = \pi/2$ is specified as:

$$H_r(0) = 1, \;\; H_r\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

2. Design an FIR linear-phase, digital filter approximating the ideal frequency response

$$H_d(\omega) = \begin{cases} 1 & |\omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} < |\omega| < \pi \end{cases}$$

   (a) Determine the coefficients of a 25-tap filter based on the window method with a rectangular window.

   (b) Using Jupyter notebook, "FIR filter design via the impulse response" as a basis for the code required, determine and sketch the magnitude and phase response of the filter. (You may first want to save the notebook using a different name to avoid your changes being overwritten by any updates if you refresh your installation).

   (c) Repeat parts (a) and (b) using the Hamming window.

3. Repeat Problem 2 for a bandstop filter having the ideal response

$$H_d(\omega) = \begin{cases} 1 & |\omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} < |\omega| < \frac{\pi}{3} \\ 1 & \frac{\pi}{3} \le |\omega| < \pi \end{cases}$$

4. **Exam question, 2009, B2, parts (a)-(d)**

   (a) Give reasons why a Finite Impulse Response (FIR) filter may be specified, despite having a long impulse response, as part of the input processing of a digital system. (3 marks)

   (b) An input to a digital processor is corrupted by additive white noise. The main components of the digital signal are present in the frequency band from 30 kHz to 40 kHz, so all components below 30 kHz, and all those above 40 kHz can safely be removed. If the sampling rate is 300 kHz, select a window and the appropriate number of taps for an FIR filter that meets the following design parameters:

Parameter	**Value**
Sidelobe rejection	50 dB
Transition band	5 kHz

   (5 marks)

   (c) Using your answer from part (b), derive an equation to describe the coefficient weights of the FIR filter. (7 marks)

   (d) Calculate the coefficients for the first tap of this filter, showing your working. (2 marks)

DSA4/DTSA Design of Digital Filters, part 2, Tutorial

1. A linear filter is required for the following problem:

Parameter	Value
Sampling frequency	8kHz
Filter type	Low pass
Cut off frequency	1 kHz
Stop band lower frequency	2.3 kHz
Minimum stop band attenuation	-35 dB

   Thus, all frequencies below 1 kHz should experience a gain of 1, but all frequencies above 2.3 kHz should be attenuated by at least -35 dB. Using the frequency design method, design two 16-tap filters that meet this requirement. (Note that tables of transition coefficients are supplied overleaf).

   Compare the complexity (number of taps) required to complete this design using the windowed impulse response method. Note that you are not required to complete this design; you only need to calculate its complexity.

2. If the specification in Problem 1 is altered to set the pass band cut off to 1.3 kHz, what changes, if any, are required to the three designs of Problem 1. (Two designs using the frequency design method, and one using the impulse response method). Again, compare the complexity of the three solutions.

3. A low pass filter is required for a signal to estimate its d.c. value. The filter is required to attenuate all frequencies above 120 Hz by at least -65 dB, where the sampling frequency is 640 Hz. Design an FIR filter for this problem using the frequency design method.

   For the filter you design, compute the first three taps: $h(0)$, $h(1)$, and $h(2)$.

Transition coefficients

Please note that transition coefficients for $M = 16$ $\alpha = 0$ are listed on the formula sheet.

$M = 16$ $\alpha = 0.5$

BW	Minimax	T_1
1	-53.96356520	0.25378977
2	-51.51438074	0.29471820
3	-51.41888444	0.30532944
4	-53.42378299	0.30116938
5	-58.05526291	0.28356855
6	-75.46617335	0.23382832

$M = 16$ $\alpha = 0.5$

BW	Minimax	T_1	T_2
1	-85.81838430	0.03967402	0.38345785
2	-84.58318972	0.04964418	0.44209940
3	-87.01830665	0.04936662	0.45185357
4	-98.39624028	0.03953840	0.43084339
5	-128.15181090	0.02312418	0.38034532

$M = 16$ $\alpha = 0.5$

BW	Minimax	T_1	T_2	T_3
1	-123.54141856	0.00383981	0.08532602	0.46234368
2	-125.98039914	0.00449150	0.10429832	0.52810238
3	-139.70014215	0.00327721	0.09683591	0.52842276
4	-185.45862453	0.00122351	0.07000531	0.48812431

$M = 32$ $\alpha = 0$

BW	Minimax	T_1
1	-42.99268747	0.42986907
2	-42.39687891	0.41047172
3	-42.37220349	0.40156510
4	-42.41709785	0.39664461
5	-42.45628207	0.39371423
6	-42.46396668	0.39201060
7	-42.43158257	0.39117897
8	-42.35406156	0.39105197
9	-42.19182583	0.39171690
10	-41.98399524	0.39296641
11	-41.89936742	0.39402197
12	-42.17807821	0.39379506
13	-43.56654215	0.38930604
14	-49.53615321	0.37129352

$M = 32$ $\alpha = 0.5$

BW	Minimax	T_1
1	-53.17518428	0.25767054
2	-50.11403588	0.30310979
3	-48.89173732	0.32069379
4	-48.28059910	0.32943587
5	-47.96095581	0.33420776
6	-47.81859669	0.33675001
7	-47.81239120	0.33776696
8	-47.94057259	0.33749498
9	-48.23935640	0.33585178
10	-48.81144899	0.33237340
11	-49.93630956	0.32583856
12	-52.58783539	0.31286147
13	-57.64949607	0.28998694
14	-75.27683880	0.23684948

DSA4/DTSA Power Spectrum Estimation Tutorial

Question 1 is taken from "Digital Signal Processing, New International Edition/4th", Proakis & Manolakis, ©Pearson Education Limited, 2014. ISBN: 978-1-29202-573-5.

1. The Bartlett method is used to estimate the power spectrum of a signal $x(n)$. We know that the power spectrum consists of a single peak with a 3 dB bandwidth of 0.01 cycles per sample, but we do not know the location of the peak.

   Assuming that the number of samples within $x(n)$, N, is large, determine the length of the transform block $M = N/K$ so that the spectral window is narrower than the peak.

2. Design a spectral estimator for the following problem:

Parameter	Value
Input signal to noise ratio	5 dB
Sampling frequency	20 kHz
Desired resolution	50 Hz
Output signal to noise ratio	35 dB

   In your design, you should state how long a data record, N, is required to achieve this performance, as well as explaining the processing steps involved.

3. Calculation of the eigenvalues and eigenvectors of a Toepliz matrix of size $n \times n$ can be performed in $O(n^2)$ time. (In other words, the complexity increases as a function of the matrix dimension squared). Using this notation, identify the complexity of each step in producing a minimum variance spectral estimate (MVSR) of an input signal. Compare your answer to the complexity of producing an equivalent estimate using classical techniques.

4. The Jupyter notebook, Spectral Density Estimation, includes code to calculate the Bartlett, the Welch, and the Minimum Variance Spectral Estimations. Using this code compare the performance of the approaches for an input signal defined as

   $$x(n) = 10\sin(0.4\pi n) + 2\sin(0.41\pi n)$$

   When comparing performance, vary the parameters `p`, `M` and `fs` and note their effect.

DSA4/DTSA Final Tutorial

Question 1 is taken from "Digital Signal Processing, New International Edition/4th", Proakis & Manolakis, ©Pearson Education Limited, 2014. ISBN: 978-1-29202-573-5.

1. From a discrete-time signal $x(n)$ with Fourier transform $X(\omega)$, shown in Fig. 1, determine and sketch the Fourier transform of the following signals:

   (a) $y_1(n) = \begin{cases} x(n) & ; n \text{ even} \\ 0 & ; n \text{ odd} \end{cases}$

   (b) $y_2(n) = x(2n)$

   (c) $y_3(n) = \begin{cases} x(n/2) & ; n \text{ even} \\ 0 & ; n \text{ odd} \end{cases}$

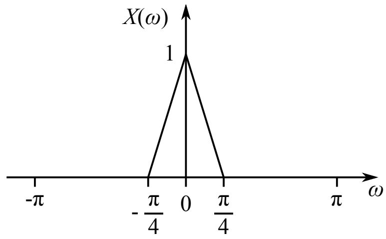


Figure 1: Fourier transform of $x(n)$

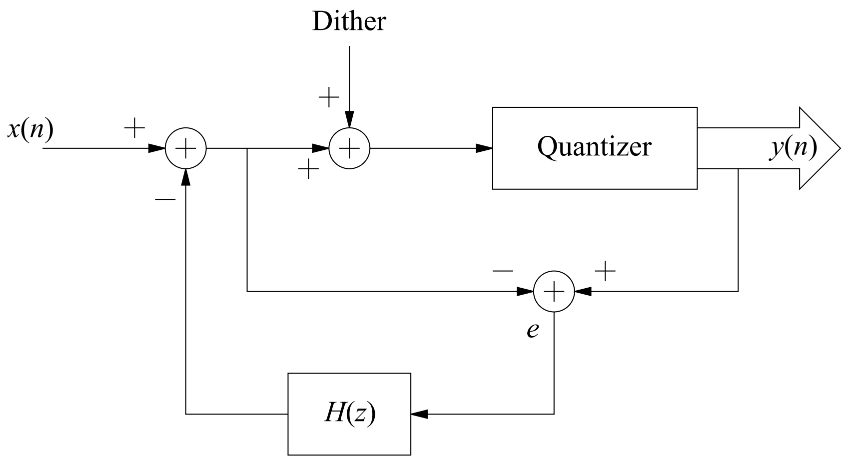
2. A low-pass filter is to be designed to meet the following specification:

Parameter	Value
Sampling frequency	200 kHz
Bandwidth	1 kHz
Transition Band	100 Hz
Stopband attenuation	-80 dB

   (a) Determine the length of filter required to meet this specification using the window method of filter design.

   (b) Using the multirate sampling structure shown in the lecture handout, determine the specification of a less computationally expensive structure to achieve the same result.

3. An analogue to digital converter that employs noise shaping as part of its quantisation process is shown below.



Show that the noise at the output may be given by the expression

$$Y_N(z) = Q(z)\{1 - H(z)\}$$

where $Q(z)$ is the sum of the dither noise and quantization error terms.

By calculating the magnitude of the frequency response of the combined dither noise and quantization error transfer function when

$$H(z) = 2z^{-1} - z^{-2}$$

show how the noise power is distributed in the power spectral density of the noise at the output.

4. It can be shown that an N bit converer, with the largest possible sinusoidal input, has a SNR given by:

$$\mathrm{SNR}_{\mathrm{max}}(\mathrm{dB}) = 6.02N + 1.76 + 10(2L+1)\log_{10}(O)$$

Construct a table showing the oversampling factor, O, required to achieve B-bit resolution from a single bit converter, where $B \in \{8, 12, 16, 24, 32\}$. In your table include results for no noise shaping, and noise shaping of first, second and third orders.

5. A given real-valued signal waveform, $f(n)$, exists over the time interval $(0, N)$ in the presence of white noise. A matched filter for $f(n)$ is chosen to be realizable with minimum delay. Show that:

   (a) the output of the matched filter has even symmetry about the point $n = N$; and

   (b) the impulse response of the matched filter is equal to the signal waveform if $f(n)$ has even symmetry about the point $n = N/2$.