Frequency Analysis of Signals

Textbook pages 230-257; 260-265; 277-297

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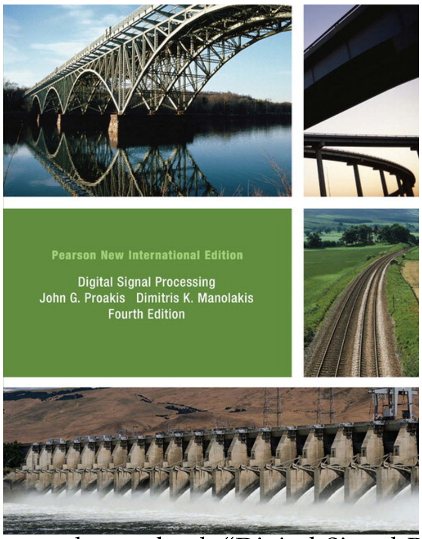
2nd September 2020



This course uses the textbook "Digital Signal Processing: Principles, Algorithms and Applications", New International Edition, by Proakis and Manolakis. Most diagrams are taken from the textbook - these notes are a commentary that should be read in conjunction with the book.

Introduction

This course is structured for hybrid teaching, where you, as the student, are responsible for covering the course material in a reasonable timeframe. Within each topic there will be a set of tasks, with an estimate of the time you should expect to spend on these tasks. Please make sure that you complete the tasks required on a weekly basis to avoid falling behind on the course.

The course headings, like the one above, will indicate whether the subject matter is covered by a video to accompany the notes. Other sections may include material supported by Jupyter notebooks. In all cases you should aim to read through these notes, and work through any derivations that are prsented.

Applications

Signal Analysis has a wide range of applications such as:

Signal detection where a signal may be very small compared with that generated by the background environment. For instance, the detection of gravitational waves requires high sensitivity, and that means the signal is subject to significant degradation due to other sources. Signal analysis may be used to enhance the signal we wish to detect, making the detection process more reliable.

Image processing to identify specific areas of an image, or perhaps to enhance an image, or even as a pre-processing step before identification tasks (e.g. car number plate, or face recognition).

Audio enhancement/recovery. Old recordings, in particular, may suffer from audible clicks which can often be removed following their detection by signal analysis. Signal analysis is also required when a particular component of an audio signal requires enhancement

Forecasting requires a signal to be analysed to identify patterns or trends, and then used to predict future values. One example might be trends in a stock-market where the signal has may different components (dimensions).

Positioning. Many positioning systems rely on measurements that are not particularly accurate or stable. Signal analysis can be used to "clean up" such a signal, and hence improve the position estimate.

Monitoring. An example of this application is the monitoring of wear in a mechanical system. By analysing the vibrations from a mechanical system, subtle changes in the frequency content of the signal can be identified, thus enabling early intervention before system failure occurs.

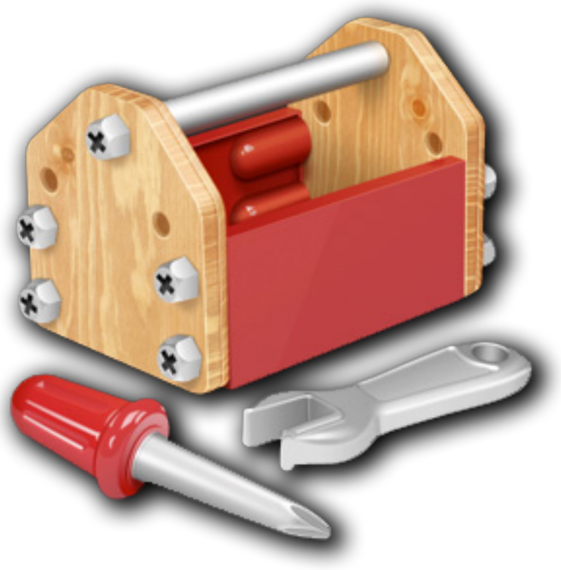
Communications systems are designed to be able to reconstruct a transmitted signal at a receiver. As communication requirements grow, it becomes important to detect signals in less ideal conditions. Signal analysis is a key element of this task.

Simulation is probably the largest application of signal analysis. In order to develop new systems it is important to be able to prototype and test them before going to full-scale manufacture. Simulating the environment, and the behaviour of a proposed system, requires analysis of the environment in which it will operate, as well as reproducing this in a mathematical model for a simulation.

Tools

This course will use examples from some of these areas, with the aim of giving students tools to perform analysis on the future. In particular tools for:

- Frequency analysis
- Analysing the effects of systems, including identifying unknown systems
- Dealing with noise
- Working with sampled data



This final point is important as virtually all analysis systems are implemented on a computer or some form of digital hardware. These cannot easily manage continuous time signals, so most analysis is carried out with sampled time, and sampled frequency domain representations.

The course will cover the underlying theory of signal analysis, with some derivations to show how these can be applied in practice, along with worked examples. Some of these are from the course textbook, and others developed as required. Accompanying the material, all of the Python code used to produce the figures in this course is available through the virtual learning environment, Learn. Students are encouraged to read through the code, and try their own experiments using it.

1 Continuous time signals

In this course, frequency analysis will be based on the decomposition of signals into complex phasor representations, known as Fourier analysis. Other courses have covered Fourier analysis of continuous signals. If you have not studied Fourier analysis, particularly the Fourier transform, the Fourier series and the discrete-time Fourier transform, you are advised to study other sources to learn this.

Recall that the Fourier series of a continuous time periodic signal is discrete in the frequency domain, and the Fourier transform of a continuous time aperiodic signal is continuous in the frequency domain. We will start by revising these relationships, beginning with an aperiodic (which means “not periodic”) time domain signal.

Aperiodic, continuous time signals

Aperiodic in time implies continuous in frequency, thus the Fourier transform is used:

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt \tag{1.30}$$

or, equivalently,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

The complex exponential form of the transform will be used in this lecture course due to its compactness and ease of manipulation. If required, a trigonometric relationship can be derived using the relationship that

$$e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$$

Figure 4.1.8 shows a signal in the time domain and its transform. The signal is defined as:

$$x(t) = \begin{cases} A & |t| \le \frac{\tau}{2} \\ 0 & |t| > \frac{\tau}{2} \end{cases} \tag{1.43}$$

The transform calculation for the aperiodic signal in Figure 4.1.8 is derived as follows:

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt$$

When substituting in for $x(t)$, as it is only non-zero between $-\tau/2$ and $\tau/2$, the limits of the integral change

$$\begin{aligned} &= \int_{-\tau/2}^{\tau/2} Ae^{-j2\pi Ft}dt \\ &= A\left[-\frac{1}{j2\pi F}e^{-j2\pi Ft}\right]_{-\tau/2}^{\tau/2} \\ &= \frac{A}{j2\pi F}\left\{e^{j2\pi F\tau/2} - e^{-j2\pi F\tau/2}\right\} \end{aligned}$$

Using the Euler’s expansion of sin on the formula sheet:

$$= \frac{A}{\pi F}\sin(\pi F\tau)$$

Finally, rearranging the expression in the form of $\sin(\pi x)/\pi x$

$$= A\tau\frac{\sin(\pi F\tau)}{\pi F\tau} = A\tau\,\mathrm{sinc}(F\tau)$$

Note that $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.

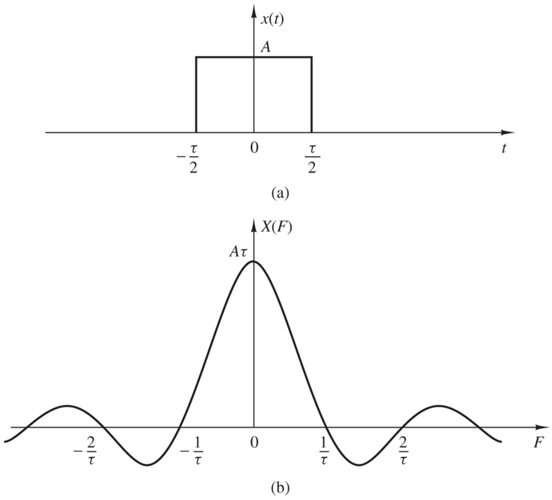


Figure 4.1.8 (a) Rectangular pulse and (b) its Fourier transform.

[1]

The first zero crossing is at $F = \frac{1}{\tau}$, and subsequent zero crossings are at integer multiples of this. Thus, the shorter the pulse (a smaller τ, the wider the response in the frequency domain.

Now let us examine a periodic version of the same signal, where the periodic repetitions are spaced by T_p.

Periodic, continuous time signals

Periodic in time implies sampled in frequency, thus the Fourier Series is used. The sample spacing is $F_0 = \frac{1}{T_P}$ Hz for a period T_P s

$$c_k = \frac{1}{T_P} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt \tag{1.9}$$

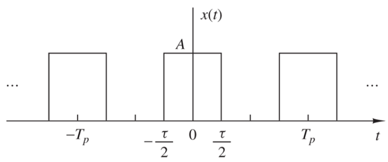


Figure 4.1.3 Continuous-time periodic train of rectangular pulses.

The calculation of the transform can be carried out by examining any period of the signal. Choosing the period covering the pulse centred on the origin:

$$c_k = \frac{1}{T_P} \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi k F_0 t} dt$$

Again, substituting in for $x(t)$ results in the integral limits changing

$$\begin{aligned}
&= \frac{1}{T_P} \int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t} dt \\
&= \frac{A}{T_P} \left[-\frac{1}{j2\pi k F_0} e^{-j2\pi k F_0 t} \right]_{-\tau/2}^{\tau/2} \\
&= \frac{A}{j2\pi k F_0 T_P} \left\{ e^{j2\pi k F_0 \tau/2} - e^{-j2\pi k F_0 \tau/2} \right\} \\
&= \frac{A}{\pi k F_0 T_P} \sin(\pi k F_0 \tau) \\
&= \frac{A\tau}{T_P} \frac{\sin(\pi k F_0 \tau)}{\pi k F_0 \tau} = \frac{A\tau}{T_P} \operatorname{sinc}(k F_0 \tau)
\end{aligned}$$

[1] Selected figures taken from “Digital Signal Processing, New International Edition/4th”, Proakis & Manolakis, ©Pearson Education Limited, 2014. ISBN: 978-1-29202-573-5

When $k = 0$ the function can be evaluated using l'Hôpital's rule (named after the French Mathematician Guillaume de l'Hôpital), resulting in the same value as obtained in the textbook as (4.1.17):

$$\begin{aligned}
c_0 &= \lim_{k\to 0} \frac{A\tau}{T_P} \frac{\sin(\pi k F_0 \tau)}{\pi k F_0 \tau} \\
&= \frac{A\tau}{T_P} \lim_{k\to 0} \frac{\frac{d}{dk}\sin(\pi k F_0 \tau)}{\frac{d}{dk}\pi k F_0 \tau} \\
&= \frac{A\tau}{T_P} \lim_{k\to 0} \frac{\pi F_0 \tau \cos(\pi k F_0 \tau)}{\pi F_0 \tau} \\
&= \frac{A\tau}{T_P} \frac{\pi F_0 \tau}{\pi F_0 \tau} \\
&= \frac{A\tau}{T_P}
\end{aligned}$$

c_k evaluates to zero, i.e. the coefficients cross the horizontal axis, when k is a multiple of $\frac{1}{F_0\tau}$ which is equivalent to $\frac{T_p}{\tau}$. Thus, as τ increases, the shape of the frequency domain narrows. In the three transforms shown in Figure 4.1.5 representing different pulse widths, the first zero crossing occurs at $k = 5$, $k = 10$ and $k = 20$. Thereafter every subsequent zero crossing is spaced by the same number of coefficients.

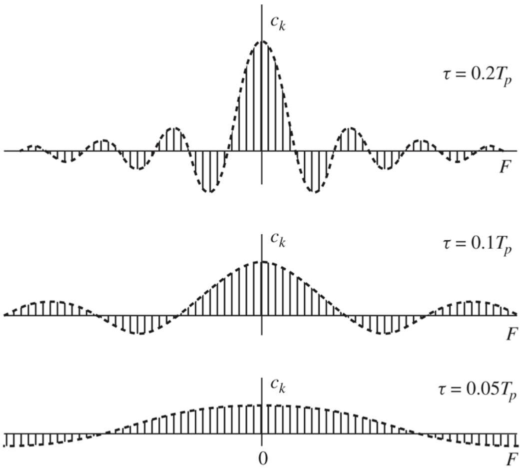


Figure 4.1.5 Fourier coefficients of the rectangular pulse train when T_p is fixed and the pulse width τ varies.

The inverse transform is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \tag{1.8}$$

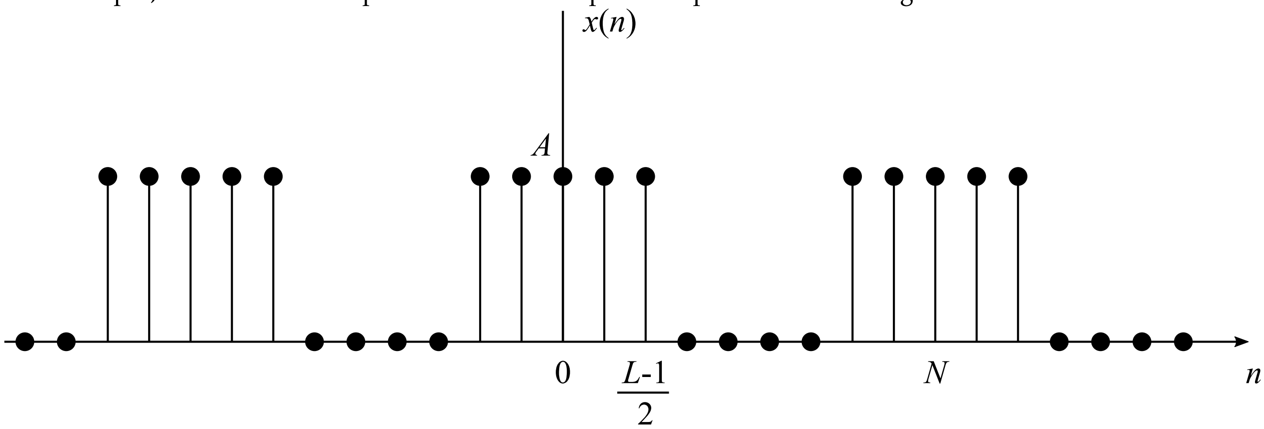
1 Discrete-time periodic signals

In the same way that a signal that is periodic in the time domain is sampled in the frequency domain, a signal that is discrete in the time domain will be periodic in the frequency domain. The spacing of samples in the time domain is the inverse of the spacing of periodic repetitions in the frequency domain. This is the reason that the sampling rate of any digital system should meet the Nyquist criterion, as sampling at a lower rate will result in aliasing.

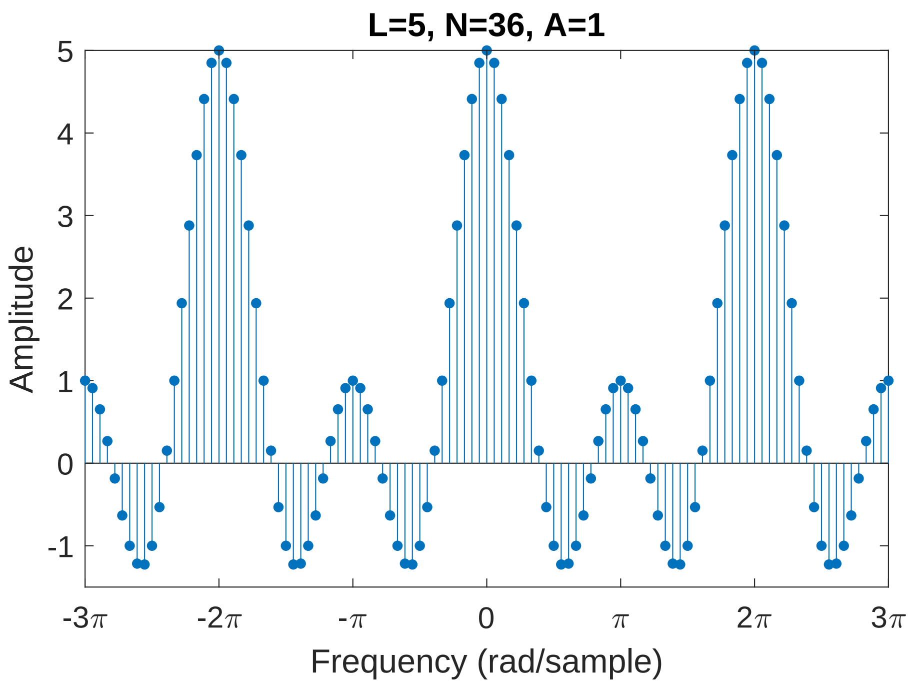
There are various notations, and practices used when referring to the time and frequency domain. The course textbook uses the notation that the sampling frequency is represented as 2π radians. Thus the frequency domain of a periodic signal in time can be defined over the range $(-\pi, \pi)$, or equally $(0, 2\pi)$, noting that it is periodic with period 2π. If it is required to relate a frequency to radians per second, or to Hertz, then the frequency should be scaled by Δt, or $\frac{\Delta t}{2\pi}$ respectively where Δt is the sampling interval measured in seconds.

Thus, a sampled signal that is periodic in the time domain has a frequency domain representation that is periodic and sampled. If it is assumed that the time domain signal has a period of N samples, then the spacing of samples in the frequency domain is $\frac{2\pi}{N}$ radians.

For example, consider the sampled version of the periodic pulse shown in Figure 4.1.3 above:



The transform of this signal is given by:



It is clear that the transform is both sampled (due to the signal being periodic), and periodic (due to the signal being sampled). Changing the values of L and N will alter the details of the transform, but it will remain sampled and periodic. Code that has been used to produce selected figures is available in the on-line course documentation. Students are encouraged to download the code, and alter parameters to see what effect they have.

The Discrete-time Fourier Series (DTFS) is defined by

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \tag{2.8}$$

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \tag{2.7}$$

Sometimes the normalisation of $\frac{1}{N}$ is performed in the synthesis equation (4.2.7) instead of the analysis (4.2.8)

Note the similarity in form between the three transforms presented to this point. Using this form makes it easier to remember them. In the analysis equation, sometimes called the forward transform, the exponent is negative, whilst in the synthesis, or the inverse transform, the exponent is positive. In all transforms, except for the Fourier Transform when expressed in Hertz, there is a scaling coefficient. In the case of the Discrete-time Fourier Series, the scaling coefficient may be in the analysis or the synthesis equation. Provided that the scaling is applied consistently, either option is acceptable.

DTFS example

Consider an example of a sampled, periodic signal. For such a signal, the discrete-time Fourier series should be applied:

$$\begin{aligned}
x(n) &= \cos(\pi n/3) \qquad \Rightarrow N = 6 \\
\Rightarrow c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\
&= \frac{1}{6}\sum_{n=0}^{5} \cos(\pi n/3)e^{-j2\pi kn/6}
\end{aligned}$$

Using Euler's expansion of cos:

$$\begin{aligned}
&= \frac{1}{6}\sum_{n=0}^{5} \frac{1}{2}\left\{e^{j\pi n/3} + e^{-j\pi n/3}\right\}e^{-j2\pi kn/6} \\
&= \frac{1}{12}\sum_{n=0}^{5} \left\{e^{j2\pi(1-k)n/6} + e^{-j2\pi(1+k)n/6}\right\}
\end{aligned}$$

Given that c_k is periodic, with a period of N, the derivation will proceed examining only the values of c_k in the range $(-N/2, N/2)$.

Noting that $e^0 = 1$ we can write:

$$c_k = \frac{1}{12}\begin{cases}
\sum_{n=0}^{5} e^{j2\pi(1-k)n/6} + \sum_{n=0}^{5} 1 & \text{if } k = -1 \\
\sum_{n=0}^{5} 1 + \sum_{n=0}^{5} e^{-j2\pi(1+k)n/6} & \text{if } k = 1 \\
\sum_{n=0}^{5} e^{j2\pi(1-k)n/6} + \sum_{n=0}^{5} e^{-j2\pi(1+k)n/6} & \text{otherwise}
\end{cases}$$

$$\begin{aligned}
\sum_{n=0}^{N-1} e^{j2\pi mn/N} &= \sum_{n=0}^{N-1}\left(e^{j2\pi m/N}\right)^n \\
&= \frac{1-\left(e^{j2\pi m/N}\right)^N}{1-\left(e^{j2\pi m/N}\right)} \\
&= 0 \ \forall m \notin (0, \pm N, \pm 2N, \ldots)
\end{aligned}$$

since $e^{j2\pi m} = 1$ for all integer m.

Therefore, for $k \neq 1$:

$$\sum_{n=0}^{5} e^{j2\pi(1-k)n/6} = \frac{1-e^{j2\pi(1-k)6/6}}{1-e^{j2\pi(1-k)/6}} = \frac{0}{1-e^{j2\pi(1-k)/6}} = 0$$

$$\Rightarrow c_k = \frac{1}{2}\begin{cases} 1 & ;k = \pm 1 \\ 0 & ;k \neq \pm 1 \text{ and } |k| \leq N/2 \end{cases}$$

Thus the DTFS of a sampled cosine is periodic, with period N, and consists of two non-zero frequency components representing the two complex phasors that construct the cosine.

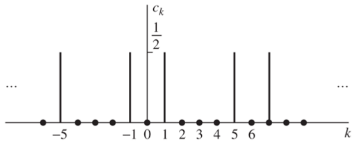


Figure 4.2.1 Spectrum of the periodic signal discussed in Example 4.2.1 (b)

The result is a periodic frequency domain, with a period of six samples. The non-zero samples at $k = \pm 1$ represent the two complex phasors of the Euler expansion of cosine. The other non-zero samples are periodic repetitions of these two.

Compare this with the Fourier series of a continuous time domain cosine signal:

$$\begin{aligned}
x(t) &= \cos(2\pi F_0 t) \\
c_k &= F_0 \int_{-1/2F_0}^{1/2F_0} \cos(2\pi F_0 t) e^{-j2\pi k F_0 t} dt \\
&= \frac{F_0}{2} \int_{-1/2F_0}^{1/2F_0} \left\{ e^{j2\pi F_0 t} + e^{-j2\pi F_0 t} \right\} e^{-j2\pi k F_0 t} dt \\
&= \frac{F_0}{2} \int_{-1/2F_0}^{1/2F_0} \left\{ e^{-j2\pi (k-1) F_0 t} + e^{-j2\pi (k+1) F_0 t} \right\} dt \\
&= \frac{F_0}{2} \left[-\frac{e^{-j2\pi(k-1)F_0 t}}{j2\pi(k-1)F_0} - \frac{e^{-j2\pi(k+1)F_0 t}}{j2\pi(k+1)F_0} \right]_{-1/2F_0}^{1/2F_0} \\
&= \frac{F_0}{2} \left\{ -\frac{e^{-j\pi(k-1)}}{j2\pi(k-1)F_0} - \frac{e^{-j\pi(k+1)}}{j2\pi(k+1)F_0} \right. \\
&\qquad \left. + \frac{e^{j\pi(k-1)}}{j2\pi(k-1)F_0} + \frac{e^{j\pi(k+1)}}{j2\pi(k+1)F_0} \right\} \\
&= \frac{F_0}{2} \left\{ \frac{e^{j\pi(k-1)} - e^{-j\pi(k-1)}}{j2\pi(k-1)F_0} + \frac{e^{j\pi(k+1)} - e^{-j\pi(k+1)}}{j2\pi(k+1)F_0} \right\}
\end{aligned}$$

For integer k, all complex exponentials are one, thus the terms will cancel. However, when $k = \pm 1$ one of the denominators is also zero, so we need to evaluate these. For $k = -1$:

$$\begin{aligned}
c_{-1} &= \lim_{k \to -1} \frac{e^{j\pi(k+1)} - e^{-j\pi(k+1)}}{j4\pi(k+1)} \\
&= \lim_{k \to -1} \frac{\frac{d}{dk}\left\{ e^{j\pi(k+1)} - e^{-j\pi(k+1)} \right\}}{\frac{d}{dk} j4\pi(k+1)} \\
&= \lim_{k \to -1} \frac{j\pi e^{j\pi(k+1)} + j\pi e^{-j\pi(k+1)}}{j4\pi} \\
&= \frac{j2\pi}{j4\pi} = \frac{1}{2}
\end{aligned}$$

Similarly for $k = 1$, $c_1 = \frac{1}{2}$

$$\Rightarrow c_k = \frac{1}{2}\begin{cases} 1 & ;k = \pm 1 \\ 0 & ;\text{otherwise} \end{cases}$$

The Fourier series also has two non-zero terms at $k = \pm 1$, however unlike the DTFS, the Fourier series is not periodic because the time-domain is not sampled.

1.1 Power Density Spectra

The Fourier representation presented above is a transform, which means that the process is fully reversible. It is possible to obtain the time series from the frequency representation, and vice-versa. This is due to the representation of phase using the complex domain in the transform. Phase information represents time shifts of frequency components. This means that the frequency representation changes if the signal is delayed in time, which is often undesirable in an analysis. Where the frequency content of a signal is required, but the phase information is not, then the power density spectrum can be used instead. This gives a measure of the power of individual frequency components of a signal, but does not retain the phase. This means that it is not a transform and is not reversible; however, it is independent of any time delays that the signal experiences.

The derivation of the Power Density Spectrum (PDS) starts with the definition of the power in a signal:

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 \tag{2.10}$$

Noting that $|x(n)|^2 = x(n)x^*(n)$, and expanding one of these using the DTFS, an equivalent expression can be found in the frequency domain such that:

$$P_x = \sum_{k=0}^{N-1}|c_k|^2 \tag{2.11}$$

The summand, $|c_k|^2$, is the PDS that describes the power of a signal in terms of its frequency components. Expanding this, and representing $x(n)$ by its DTFS:

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{k=0}^{N-1}c_k e^{j2\pi kn/N}\right)x^*(n)$$

Switching the order of summation:

$$P_x = \sum_{k=0}^{N-1}c_k\left(\frac{1}{N}\sum_{n=0}^{N-1}x^*(n)e^{j2\pi kn/N}\right) = \sum_{k=0}^{N-1}c_k c_k^*$$

$|c_k|^2$ is the Power Density Spectrum To compute the PDS of a periodic signal, the DTFS, c_k, should be computed (2.8), and then determine its magnitude to find $|c_k|^2$.

Continuing the previous example, the power density spectrum of $x(n) = \cos(\pi n/3)$ is given by:

$$|c_k|^2 = \frac{1}{4}\begin{cases}1 & ; k=-1 \text{ or } k=1 \\ 0 & ;\text{otherwise}\end{cases}$$

which is periodic with period $N = 6$.

Let $x(n)$ be a periodic signal with period N, and defined as:

$$x(n) = \begin{cases}A & ;0 \le n < L \\ 0 & ;L \le n < N\end{cases}$$

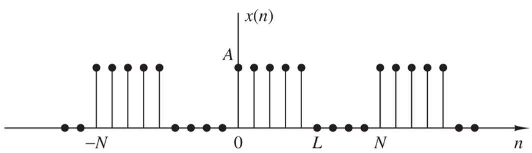


Figure 4.2.2 Discrete-time periodic square-wave signal.

Then the DTFS is given by:

$$\begin{aligned} c_k &= \frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N} \\ &= \frac{1}{N}\sum_{n=0}^{L-1}A.\left(e^{-j2\pi k/N}\right)^n \end{aligned}$$

This is a geometric series of the form:

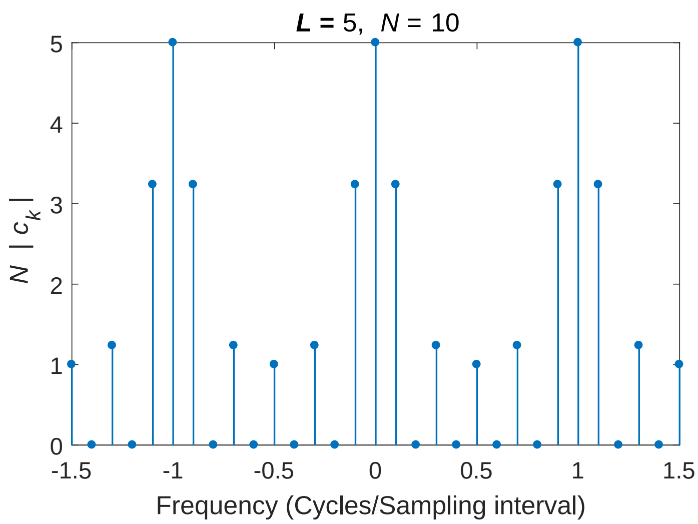
$$\sum_{m=0}^{M-1} g^n = \frac{1-g^M}{1-g}$$

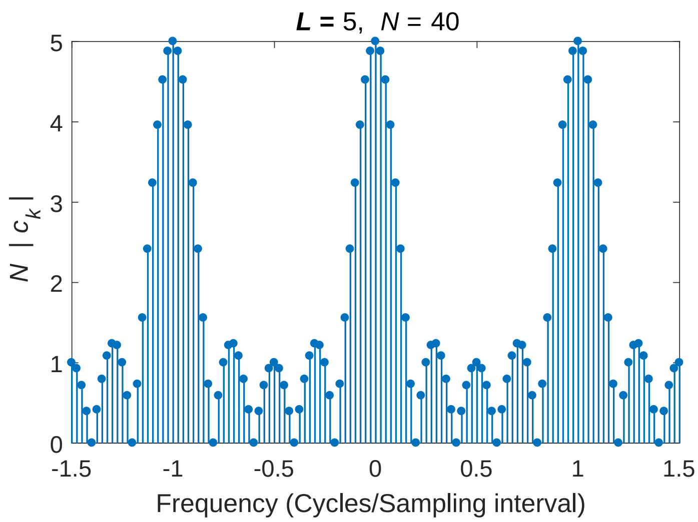
$$\Rightarrow c_k = \frac{A}{N}\frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}}$$

The PDS is given by $|c_k|^2 = c_k c_k^*$, so

$$\begin{aligned}
|c_k|^2 &= \frac{A^2}{N^2}\cdot\frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}}\cdot\frac{1-e^{j2\pi kL/N}}{1-e^{j2\pi k/N}} \\
&= \frac{A^2}{N^2}\cdot\frac{2-e^{-j2\pi kL/N}-e^{j2\pi kL/N}}{2-e^{-j2\pi k/N}-e^{j2\pi k/N}} \\
&= \frac{A^2}{N^2}\cdot\frac{-\left(e^{j2\pi kL/2N}-e^{-j2\pi kL/2N}\right)^2}{-\left(e^{j2\pi k/2N}-e^{-j2\pi k/2N}\right)^2} \\
&= \frac{A^2}{N^2}\cdot\frac{\sin^2(\pi kL/N)}{\sin^2(\pi k/N)}
\end{aligned}$$

which is equivalent to (2.22). (The textbook arrives at the same answer with a slightly different set of intermediate steps). To evaluate the above expression for $k=0$, l'Hôpital's rule can be used. In figure 4.2.3 note that the spacing of the samples in the frequency domain is dependent upon the length of the period, while the envelope of the transform remains the same.





Note that these are corrected versions of Figure 4.2.3 in the textbook. The value of L in the legend of Figure 4.2.3 should be $L=5$, and the textbook has chosen to consider the sin/sin term as the real component and plotted this instead of its magnitude only. If the amplitude were being plotted as all positive quantities, then the phase term will include step changes by π radians.

1 Discrete-time aperiodic signals

Where a signal is aperiodic, that is it is not a periodic signal, then the Fourier series representation is not applicable. A signal that is aperiodic in the time domain is continuous in the frequency domain. By analogy from previously:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{2.23}$$

Because the signal is sampled in the time domain, the frequency representation will be periodic, with a period defined by the reciprocal of the spacing of samples in the time domain. Specifically, the period in the frequency domain, measured in Hertz, is given by:

$$f_s = \frac{1}{\Delta t}$$

where Δt is the interval between samples in the time domain measured in seconds. Expressing the same period in rad/s,

$$\omega_s = \frac{2\pi}{\Delta t}$$

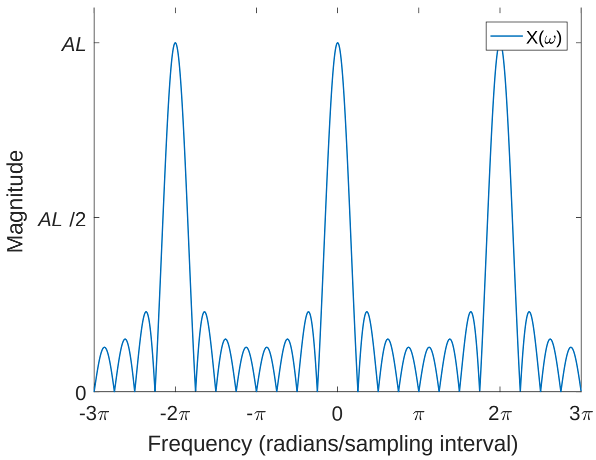
Where the sampling interval is unknown, then the normalised frequency is used by setting $\Delta t = 1$. The frequency axis is then expressed as cycles per sampling interval, or radians per sampling interval when using f and ω respectively.

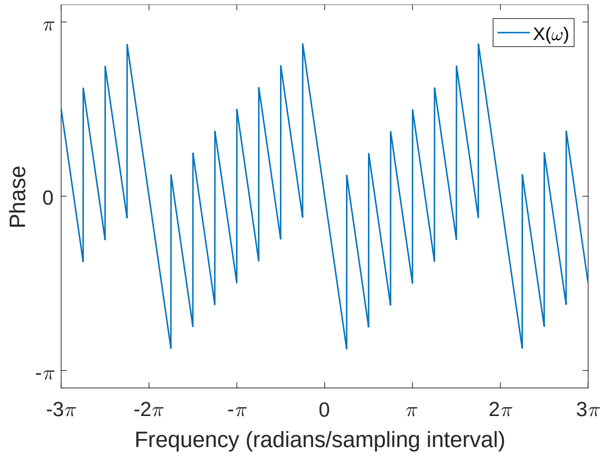
Discrete-time Fourier Transform Example

Let the signal be an aperiodic rectangular pulse:

$$\begin{aligned}
x(n) &= \begin{cases} A & ; 0 \leq n < L \\ 0 & ; n < 0 \text{ or } n \geq L \end{cases} \\
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\
&= \sum_{n=0}^{L-1} Ae^{-j\omega n} = A\frac{1-e^{-j\omega L}}{1-e^{-j\omega}} \\
&= A \cdot \frac{e^{-j\omega L/2}}{e^{-j\omega/2}} \cdot \frac{e^{j\omega L/2} - e^{-j\omega L/2}}{e^{j\omega/2} - e^{-j\omega/2}} \\
&= Ae^{-j\omega(L-1)/2} \cdot \frac{\sin(\omega L/2)}{\sin(\omega/2)}
\end{aligned}$$

For example, if $L = 8$, the transform yields:





Note the step changes of π in the phase plot, and how these correspond to the intersection points with the x-axis. This is a result of the transform having an alternating sign as the numerator transitions between zero crossings.

Discrete-time Aperiodic Signals

The reverse transform is given by

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \tag{2.27}$$

This may be computed over any full period of $X(\omega)$, thus may also be written as:

$$x(n) = \frac{1}{2\pi}\int_{0}^{2\pi} X(\omega)e^{j\omega n}d\omega$$

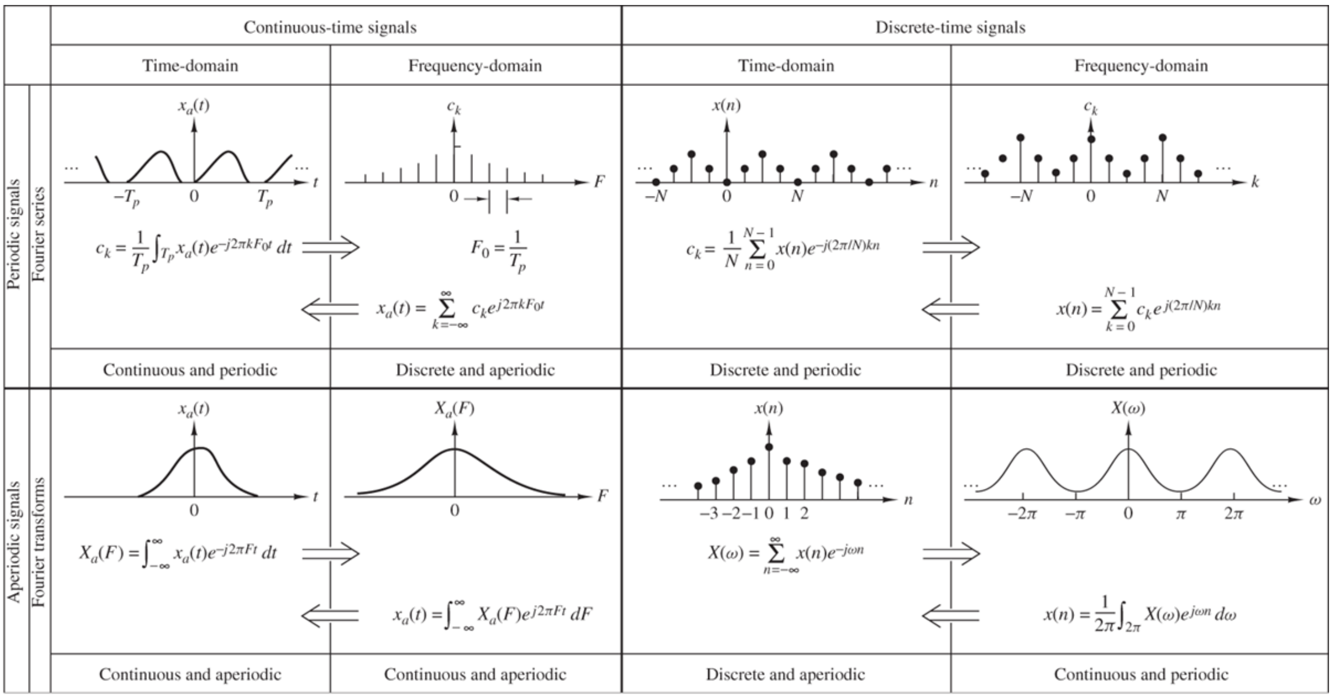


Figure 4.3.1 Summary of analysis and synthesis formulas.

1.1 Energy Density Spectrum

In the same way that the power of a periodic signal can be represented by the power of the constituent frequency components, the energy of an aperiodic signal can be represented in the frequency domain. Using a derivation similar to the one above, it is straightforward to show that:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega \tag{2.41}$$

The energy density spectrum, denoted by $S_{xx}(\omega)$, is then the integrand, $|X(\omega)|^2$. As for the Power Density Spectrum, the Energy Density Spectrum does not retain any phase information, so is non-reversible, and it is also invariant to time delays of the signal. This is a particularly useful property when analysing the frequency content of signals passing through linear systems with non-linear phase responses.

The energy density spectrum is closely related to the power density spectrum - the energy density spectrum should be applied to signals with a finite energy, whilst the power density spectrum is appropriate for signals with a finite power. The two spectra are related by a scaling factor, the duration of time over which the power density spectrum is computed.

Energy Density Spectrum Example

Let $x(n)$ be defined as:

$$x(n) = \begin{cases} a^n & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$$

where $-1 < a < 1$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega} = \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n$$

$$\begin{aligned} X(\omega) &= \frac{1}{1 - a e^{-j\omega}} \\ \Rightarrow S_{xx}(\omega) &= \frac{1}{1 - a e^{-j\omega}} \frac{1}{1 - a e^{j\omega}} \\ &= \frac{1}{1 + a^2 - a\left(e^{j\omega} + e^{-j\omega}\right)} \\ &= \frac{1}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

The resulting spectrum depends upon the parameter a. For $a = 0$, which results in the input being a single non-zero sample, the spectrum is flat.

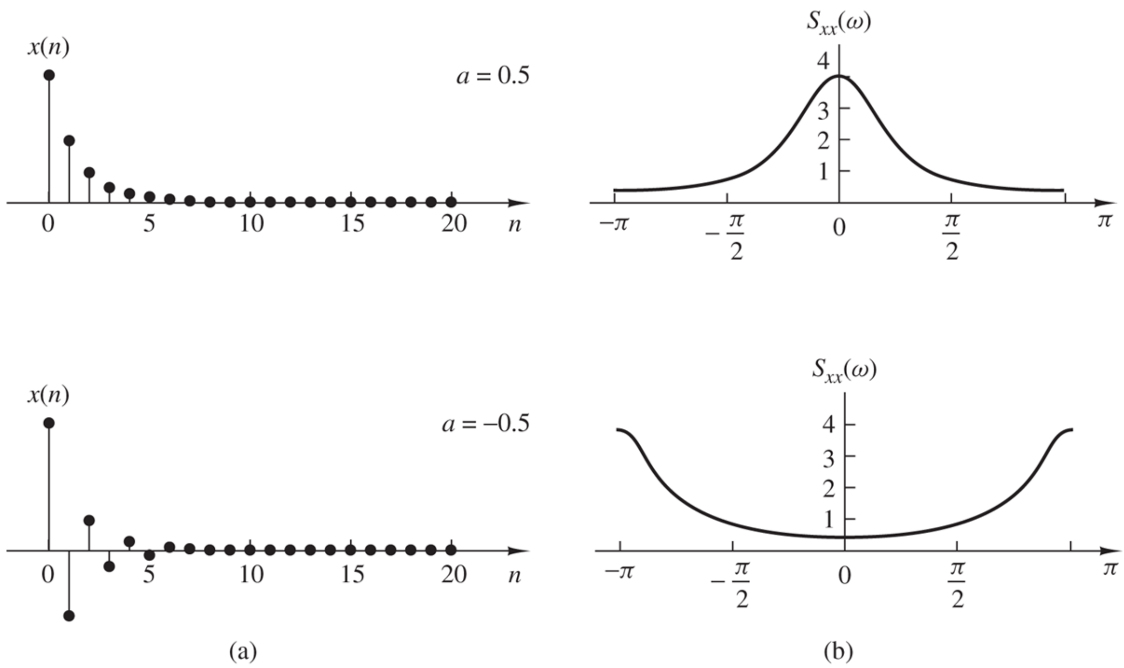


Figure 4.2.6 (a) Sequence $x(n) = (\frac{1}{2})^n u(n)$ and $x(n) = (-\frac{1}{2})^n u(n)$; (b) their energy density spectra.