# DSA4/DTSA Frequency Analysis of Signals Exercise Solutions

Please note that these are only *sample* solutions. Other methods may be used to achieve the same result. It is important that the understanding behind the results is gained as well as the process to reach these.

1. The formula for the Fourier transform can be found on the Formula sheet. The Fourier transform is continuous in both time and frequency.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Substituting in for $x(t)$ results in changed limits:

$$\Rightarrow X(\omega) = \int_{0}^{\tau} e^{-j\omega t}dt = \left[\frac{1}{-j\omega}e^{-j\omega t}\right]_0^\tau = \frac{1}{-j\omega}\left\{e^{-j\omega\tau} - 1\right\}$$

To simplify further, we need to re-express the difference in terms of an Euler expansion of sin():

$$\begin{aligned} &= \frac{e^{-j\omega\tau/2}}{-j\omega}\left\{e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right\} = \frac{e^{-j\omega\tau/2}}{-j\omega}(-2j)\sin\left(\frac{\omega\tau}{2}\right) \\ &= \frac{2e^{-j\omega\tau/2}}{\omega}\sin\left(\frac{\omega\tau}{2}\right) \end{aligned}$$

2.

$$\begin{aligned} c_{N-k} &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi(N-k)n/N} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{j2\pi kn/N}e^{-j2\pi n} \end{aligned}$$

As $n$ is an integer, $e^{-j2\pi n} = 1$.

3. The signal is periodic, with period $\tau$. The periodic shape is the first half of a sine wave, thus we can write:

$$x(t) = \sin\left(\frac{\pi t}{\tau}\right) \ ; 0 \le t \le \tau$$

Substituting the signal into the transform we obtain:

$$c_k = \frac{1}{\tau}\int_0^\tau A\sin\left(\frac{t\pi}{\tau}\right)e^{-j2\pi kt/\tau}dt$$

This can be computed using integration by parts, or by using the Euler expansion of sin() as follows:

$$\begin{aligned} c_k &= \frac{A}{\tau}\int_0^\tau \frac{1}{2j}\left\{e^{jt\pi/\tau} - e^{-jt\pi/\tau}\right\}e^{-j2\pi kt/\tau}dt \\ &= \frac{A}{j2\tau}\left[\frac{\tau}{j(\pi - 2\pi k)}e^{j(\pi-2\pi k)t/\tau} + \frac{\tau}{j(\pi+2\pi k)}e^{-j(\pi+2\pi k)t/\tau}\right]_0^\tau \\ &= -\frac{A}{2\pi}\left\{\frac{1}{1-2k}e^{j(\pi-2\pi k)} + \frac{1}{1+2k}e^{-j(\pi+2\pi k)} - \frac{1}{1-2k} - \frac{1}{1+2k}\right\} \end{aligned}$$

As $e^{j\pi} = -1$ and $e^{j2\pi k} = 1$,

$$
\begin{aligned}
&= \frac{A}{2\pi} 2 \left\{ \frac{1}{1-2k} + \frac{1}{1+2k} \right\} \\
&= \frac{2A}{\pi(1-4k^2)} \Rightarrow |c_k^2| = \frac{4A^2}{\pi^2(1-4k^2)^2}
\end{aligned}
$$

4. To begin a proof like this, start with either the frequency or the time domain. For the purposes of this solution, I've started with the time domain:

$$X'(\omega) = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t} dt$$

Now make a variable substitution of $u = -t$, remembering to substitute in for the limits as well as $dt$:

$$\Rightarrow X'(\omega) = -\int_{\infty}^{-\infty} x(u)e^{j\omega u} du = \int_{-\infty}^{\infty} x(u)e^{j\omega u} du$$

Noting that $x(t) = x^*(t)$, since $x(t)$ is real, and taking the complex conjugate:

$$
\begin{aligned}
&\Rightarrow (X'(\omega))^* = \int_{-\infty}^{\infty} x^*(u)e^{-j\omega u} du = \int_{-\infty}^{\infty} x(u)e^{-j\omega u} du = X(\omega) \\
&\Rightarrow X'(\omega) = X^*(\omega)
\end{aligned}
$$

hence $x(-t) \overset{F}{\longleftrightarrow} X^*(\omega)$.

5. The transform should be continuous in time and frequency as the signal is continuous in time, and aperiodic. This implies that we should use the Fourier Transform. Care must be taken because of the modulo operation. Effectively, the integrand is expressed as two different equations, one for positive $t$ and one for negative $t$. To solve, break the integral into sections so that the integrand can be written without using a modulo operation.

$$
\begin{aligned}
x_b(t) &= Ae^{-a|t|} \\
X_b(F) &= \int_{-\infty}^{\infty} Ae^{-a|t|}e^{-j2\pi Ft} dt = \int_{-\infty}^{0} Ae^{-a|t|}e^{-j2\pi Ft} dt + \int_{0}^{\infty} Ae^{-at}e^{-j2\pi Ft} dt \\
&= \int_{-\infty}^{0} Ae^{at}e^{-j2\pi Ft} dt + \int_{0}^{\infty} Ae^{-at}e^{-j2\pi Ft} dt
\end{aligned}
$$

Noting that we have already computed the second integral above:

$$
\begin{aligned}
\Rightarrow X_b(F) &= A\left[\frac{1}{a-j2\pi F}e^{-(a-j2\pi F)t}\right]_0^{\infty} + \frac{A}{a+j2\pi F} \\
&= \frac{A}{a-j2\pi F} + \frac{A}{a+j2\pi F} = \frac{2Aa}{a^2+(2\pi F)^2} \\
\Rightarrow |X_b(F)| &= \frac{2Aa}{a^2+(2\pi F)^2} \qquad \measuredangle X_b(F) = 0
\end{aligned}
$$

The solution is real and even, which could have been predicted since the input is real and even.

6. Starting with the Fourier series, substitute in the signal samples:

$$
\begin{aligned}
c_k &= \frac{1}{N}\sum_{N} x(n)e^{-j2\pi kn/N} \qquad ; N = 6 \\
&= \frac{1}{6}\sum_{n=-2}^{3} x(n)e^{-j\pi kn/3} = \frac{1}{6}e^{j2\pi k/3} + \frac{1}{3}e^{j\pi k/3} + \frac{1}{2} + \frac{1}{3}e^{-j\pi k/3} + \frac{1}{6}e^{-j2\pi k/3}
\end{aligned}
$$

Collecting terms, and using the Euler expansion of cos():

$$\Rightarrow c_k = \frac{1}{2} + \frac{2}{3}\cos\left(\frac{\pi k}{3}\right) + \frac{1}{3}\cos\left(\frac{2\pi k}{3}\right) = \left\{\underset{\uparrow}{\frac{3}{2}}, \frac{2}{3}, 0, \frac{1}{6}, 0, \frac{2}{3}\right\}$$

7. Starting with the definition $x(n) = a(n) + jb(n)$, the transform can be written in terms of real and imaginary components as:

$X(\omega) = \Re\{A(\omega)\} + j\Im\{A(\omega)\} + \Re\{jB(\omega)\} + j\Im\{jB(\omega)\}$

Expanding $B(\omega)$ as the sum of real and imaginary components:

$$= \Re\{A(\omega)\} + j\Im\{A(\omega)\} + \Re\{j\Re\{B(\omega)\} + j^2\Im\{B(\omega)\}\} + j\Im\{j\Re\{B(\omega)\} + j^2\Im\{B(\omega)\}\}$$

Then noting that $\Re\{j\Re\{\cdot\}\} = 0$ and $\Im\{j^2\Im\{\cdot\}\} = 0$ by definition:

$$= \Re\{A(\omega)\} + j\Im\{A(\omega)\} - \Im\{B(\omega)\} + j\Re\{B(\omega)\}$$

A real-valued time domain signal has a frequency response that is Hermitian symmetric, thus:

$$\begin{aligned}
\Re\{A(\omega)\} &= \Re\{A(-\omega)\} \\
\Im\{A(\omega)\} &= -\Im\{A(-\omega)\} \\
\Re\{B(\omega)\} &= \Re\{B(-\omega)\} \\
\Im\{B(\omega)\} &= -\Im\{B(-\omega)\}
\end{aligned}$$

Thus, we can reconstruct $A(\omega)$ as the sum of a real component which is even, and an imaginary component which is odd. Thus, we can write:

$$\begin{aligned}
\Re\{A(\omega)\} &= \frac{\Re\{A(\omega)\} + \Re\{A(-\omega)\}}{2} \\
\Im\{A(\omega)\} &= \frac{\Im\{A(\omega)\} - \Im\{A(-\omega)\}}{2} \\
A(\omega) &= \frac{A(\omega) + A^*(-\omega)}{2}
\end{aligned}$$

where $^*$ denotes complex conjugation.

Applying this formula to $X(\omega)$, then we show that this process will extract $A(\omega)$ from the transform result:

$$\begin{aligned}
\frac{X(\omega) + X^*(-\omega)}{2} &= \frac{1}{2}\left[\Re\{A(\omega)\} + j\Im\{A(\omega)\} - \Im\{B(\omega)\} + j\Re\{B(\omega)\}\right. \\
&\quad \left. + \Re\{A(-\omega)\} - j\Im\{A(-\omega)\} - \Im\{B(-\omega)\} - j\Re\{B(-\omega)\}\right] \\
&= \frac{1}{2}\left[2\Re\{A(\omega)\} + j2\Im\{A(\omega)\}\right] \\
&= A(\omega)
\end{aligned}$$

$B(\omega)$ is contained in the imaginary part of $X(\omega)$, and $jB(\omega)$ has a real component that is odd, and an imaginary component that is even. Using similar logic to above, the transform of $b(n)$ can also be extracted using this symmetry:

$$\begin{aligned}
\frac{X(\omega) - X^*(-\omega)}{j2} &= \frac{1}{j2}\left[\Re\{A(\omega)\} + j\Im\{A(\omega)\} - \Im\{B(\omega)\} + j\Re\{B(\omega)\}\right. \\
&\quad \left. - \Re\{A(-\omega)\} + j\Im\{A(-\omega)\} + \Im\{B(-\omega)\} + j\Re\{B(-\omega)\}\right] \\
&= \frac{1}{j2}\left[-2\Im\{B(\omega)\} + j2\Re\{B(\omega)\}\right] \\
&= B(\omega)
\end{aligned}$$

## An alternative solution

Inspired by the equations to extract even and odd components, we can derive equations to extract $a(n)$ and $b(n)$ from $x(n)$:

$$a(n) = \frac{x(n) + x^*(n)}{2}$$
$$b(n) = \frac{x(n) - x^*(n)}{j2}$$

As $x^*(n) \stackrel{F}{\longleftrightarrow} X^*(-\omega)$:

$$A(\omega) = \frac{X(\omega) + X^*(-\omega)}{2}$$
$$B(\omega) = \frac{X(\omega) - X^*(-\omega)}{j2}$$

# DSA4/DTSA The Discrete Fourier Transform Exercise Solutions

Please note that these are only *sample* solutions. Other methods may be used to achieve the same result. It is important that the understanding behind the results is gained as well as the process to reach these.

1.

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = \sum_{n=0}^{7} x(n)e^{-j\pi kn/4} \\ &= e^{-j0\pi k/4} = 1 \\ Y(k) &= \sum_{n=0}^{7} y(n)e^{-j\pi kn/4} = e^{-j\pi k/4} \end{aligned}$$

The sequences $x(n)$ and $y(n)$ are related by a shift in time, which translates to a linear phase change in frequency. As $X(k)$ is unity, then $Y(k)$ is simply the phase shift.

2. The DFT of a real valued signal has the following properties:

- even components of the signal are transformed to real and even components in the transform
- odd components of the signal are transformed to imaginary and odd components in the transform

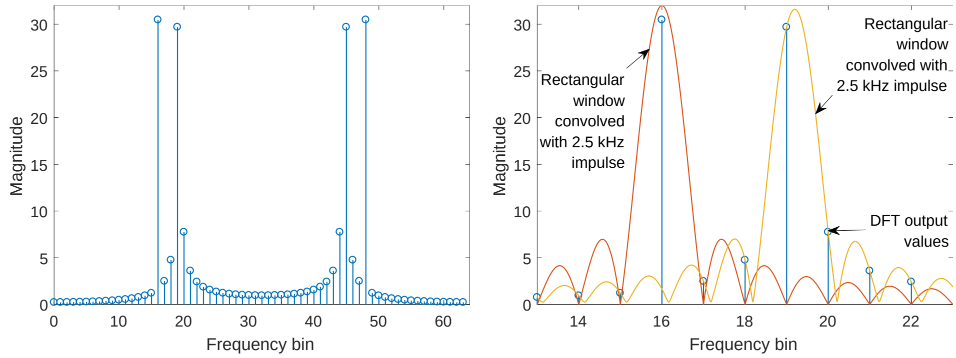
Thus:

$$\begin{aligned} & X(-k) = X^*(k) \\ \Rightarrow\ & X(-1) = 0.125 + j0.3018 \\ & X(-2) = 0 \\ & X(-3) = 0.125 + j0.0518 \end{aligned}$$

Also, for an $N$-point transform: $X(N-k) = X(-k)$, so

$$X(k) = \underset{\uparrow}{0.25}, 0.125 - j0.3018, 0, 0.125 - j0.0518, 0, 0.125 + j0.0518, 0, 0.125 + j0.3018\}$$

3. (a) $f_s = 10\text{kHz} \Rightarrow \Delta t = 0.1$ ms, where $\Delta t$ is the spacing of samples in the time domain. $N = 64$, thus the length of the sequence in time is $N\Delta t = 6.4\ ms$. This results in a spacing of samples in the frequency domain of $\Delta f = \frac{1}{6.4\times10^{-3}} = 156.25$ Hz. (This can be found directly by calculating $f_s/N = \frac{10^4}{64} = 156.25\text{Hz}$).
Now that the spacing of samples has been established, the sample number corresponding to the frequency of interest can be found. A frequency component of 2.5 kHz corresponds to sample number $\frac{2500}{156.25} = 16$. Because of the periodic nature of the transform, it will also appear at sample number $64 - 16 = 48$, being the periodic repetition of sample $-16$.
7 kHz is above the Nyquist frequency of 5 kHz, so aliasing will occur. When sampled, the 7 kHz signal is aliased down to $10\text{kHz} - 7\text{kH} = 3\text{kHz}$. Applying the same process as above to the 3 kHz input, the result is a non-integer value for the sample number: $\frac{3\times10^3}{156.25} = 19.2$. Clearly this frequency component does not fall directly on one of the discrete frequency samples. The result is that leakage will be present. The leakage, and a close-up of the area around bin 19 are shown overleaf.
Note the points at which each $\frac{\sin(\cdot)}{\sin(\cdot)}$ function has a zero value. For the 2.5 kHz sinusoid these coincide with the DFT sample points, so there is no leakage into adjacent bins. Note too the value of bin 16; the combination of the two functions has reduced the height of the DFT, again due to leakage.




(b) A square wave contains only odd harmonics of its fundamental frequency, i.e. 2.5 kHz, 7.5 kHz, 12.5 kHz, etc. So it will appear the same as the 2.5 kHz sinusoid of part (a) above.

4.

$$x_s(n) = x(n)\sin\left(\frac{2\pi k_0 n}{N}\right) \quad ; 0 \le n \le N-1$$

Using the suggestion given in the question:

$$x_s(n) = \frac{x(n)}{2j}\left\{e^{j2\pi k_0 n/N} - e^{-2\pi k_0 n/N}\right) \quad ; 0 \le n \le N-1$$

Now applying the $N$-point DFT we get:

$$\begin{aligned}
X_s(k) &= \sum_{n=0}^{N-1} x_s(n)e^{-j2\pi kn/N} \\
&= \frac{1}{2j}\sum_{n=0}^{N-1} x(n)e^{j2\pi k_0 n/N}e^{-j2\pi kn/N} - \frac{1}{2j}\sum_{n=0}^{N-1} x(n)e^{-j2\pi k_0 n/N}e^{-j2\pi kn/N} \\
&= \frac{1}{2j}\sum_{n=0}^{N-1} x(n)e^{-j2\pi(k-k_0)n/N} - \frac{1}{2j}\sum_{n=0}^{N-1} x(n)e^{-j2\pi(k+k_0)n/N} \\
&= \frac{1}{2j}\{X(k-k_0) - X(k+k_0)\}
\end{aligned}$$

5. (a) To solve, apply the DTFT and simplify:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega} = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

Now take out a complex factor from the numerator and the denominator to express these as Euler expansions of sin():

$$\begin{aligned}
&= \frac{e^{-j\omega N/2}}{e^{-j\omega/2}}\left(\frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}}\right) \\
&= e^{-j\omega(N-1)/2}\left(\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right)
\end{aligned}$$

(b) This can be tackled a number of ways, including solving it directly (as a number of students did in the exam itself). However, it can be solved quickly by noting that this expression is the modulation of $x(n)$ where $N = 4$. Thus:

$$\begin{aligned}
y(n) &= \cos(n\pi/4)x(n) \\
&= \frac{1}{2}\left(e^{jn\pi/4} + e^{-jn\pi/4}\right)x(n) \\
\Rightarrow Y(\omega) &= \frac{1}{2}\left(\sum_{n=0}^{3} x(n)e^{jn\pi/4}e^{-jn\omega} + \sum_{n=0}^{3} x(n)e^{-jn\pi/4}e^{-jn\omega}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{3} x(n)e^{-jn(\omega-\pi/4)} + \sum_{n=0}^{3} x(n)e^{-jn(\omega+\pi/4)}\right) \\
\Rightarrow Y(\omega) &= \frac{1}{2}\left(X\left(\omega - \frac{\pi}{4}\right) + X\left(\omega + \frac{\pi}{4}\right)\right)
\end{aligned}$$

(c) To determine the 4-point DFT, sample the DTFT at the points $\omega = \{0, \pi/2, \pi, 3\pi/2\}$. The calculations can be done directly, or to simplify the process, first find the samples of $X(\omega)$ that will be required:

$$\begin{aligned}
X(\pi/4) &= e^{-j3\pi/8}\frac{\sin(\pi/2)}{\sin(\pi/8)} = \frac{e^{-j3\pi/8}}{\sin(\pi/8)} = 1 - j2.414 \\
X(3\pi/4) &= e^{-j9\pi/8}\frac{\sin(3\pi/2)}{\sin(3\pi/8)} = \frac{e^{-j\pi/8}}{\sin(3\pi/8)} = 1 - j0.414 \\
X(5\pi/4) &= X(-3\pi/4) = X^*(3\pi/4) = \frac{e^{j\pi/8}}{\sin(3\pi/8)} = 1 + j0.414 \\
X(7\pi/4) &= X(-\pi/4) = X^*(\pi/4) = \frac{e^{j3\pi/8}}{\sin(\pi/8)} = 1 + j2.414
\end{aligned}$$

Then, substituting these into the expressions of $Y(\omega)$ at the frequencies of interest gives:

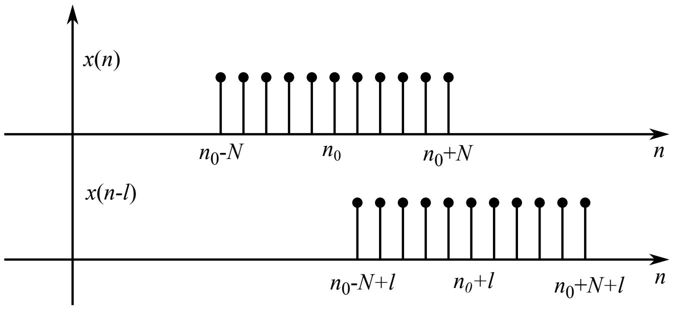
$$\begin{aligned}
Y(0) &= \frac{1}{2}\left(X(-\pi/4) + X(\pi/4)\right) = \frac{1}{2}\left(\frac{e^{j3\pi/8}}{\sin(\pi/8)} + \frac{e^{-j3\pi/8}}{\sin(\pi/8)}\right) = \frac{\cos(3\pi/8)}{\sin(\pi/8)} = 1 \\
Y(1) &= \frac{1}{2}\left(X(\pi/4) + X(3\pi/4)\right) = \frac{1}{2}\left(\frac{e^{-j3\pi/8}}{\sin(\pi/8)} + \frac{e^{-j\pi/8}}{\sin(3\pi/8)}\right) = 1 - j1.414 \\
Y(2) &= \frac{1}{2}\left(X(3\pi/4) + X(5\pi/4)\right) = \frac{1}{2}\left(\frac{e^{-j\pi/8}}{\sin(3\pi/8)} + \frac{e^{j\pi/8}}{\sin(3\pi/8)}\right) = \frac{\cos(\pi/8)}{\sin(3\pi/8)} = 1 \\
Y(3) &= Y^*(1) = \frac{1}{2}\left(\frac{e^{j3\pi/8}}{\sin(\pi/8)} + \frac{e^{j\pi/8}}{\sin(3\pi/8)}\right) = 1 + j1.414
\end{aligned}$$

The answer can either be in numerical or algebraic form.

# DSA4/DTSA Discrete-Time Signals and Systems Exercise Solutions

Please note that these are only *sample* solutions. Other methods may be used to achieve the same result. It is important that the understanding behind the results is gained as well as the process to reach these.

1. In order to more readily work out the summation limits, it can be helpful to sketch a diagram of the signals for a particular value of $l$, the correlation index. For example for $0 < l < 2N$:




From the sketch, it is clear that $x(n)$ is $2N+1$ samples long, and when $|l| \leq 2N$, the two sequences ($x(n)$ and $x(n-l)$) overlap by at least one sample.

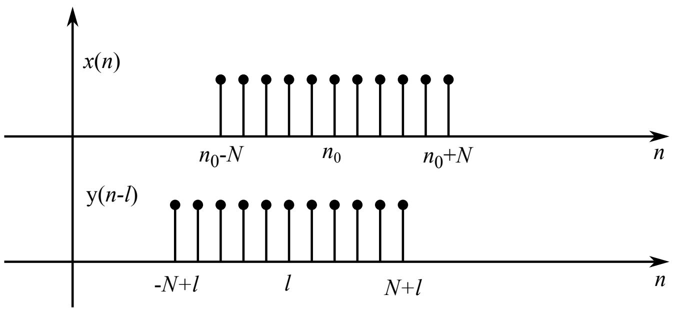
For $0 \leq l \leq 2N$,

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) = \sum_{n_0-N+l}^{n_0+N} 1 \cdot 1 = n_0 + N - (n_0 - N + l) + 1 = 2N - l + 1$$

As $r_{xx}(l)$ is symmetric:

$$r_{xx}(l) = \begin{cases} 2N - |l| + 1 & ; |l| \leq 2N \\ 0 & ; |l| > 2N \end{cases}$$

For the cross-correlation, a similar sketch can be produced:




$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l)$$

The sequences fully overlap when $l = n_0$. For $n_0 - 2N \leq l \leq n_0$:

$$r_{xy}(l) = \sum_{n=n_0-N}^{l+N} 1 \cdot 1 = l + 2N - n_0 + 1$$

For $n_0 \leq l \leq n_0 + 2N$:

$$r_{xy}(l) = \sum_{n=l-N}^{n_0+N} 1 \cdot 1 = n_0 + 2N - l + 1$$

$$\Rightarrow r_{xy}(l) = \begin{cases} 2N + 1 - |l - n_0| & ; n_0 - 2N \leq l \leq n_0 + 2N \\ 0 & ; \text{otherwise} \end{cases}$$

2. As specific values are given for the signals, direct calculation is the simplest approach for the first two parts of the question.

   (a)

$$\begin{aligned} r_{xx}(l) &= \sum_{n=-\infty}^{\infty} x(n)x(n-l) \\ l = 0 :& r_{xx}(0) = 1 \times 1 + 2 \times 2 + 1 \times 1 + 1 \times 1 = 7 \\ l = 1 :& r_{xx}(1) = 2 \times 1 + 1 \times 2 + 1 \times 1 = 5 \\ l = 2 :& r_{xx}(2) = 1 \times 1 + 1 \times 2 = 3 \\ l = 3 :& r_{xx}(3) = 1 \times 1 = 1 \\ \Rightarrow r_{xx}(l) &= \{1, 3, 5, \underset{\uparrow}{7}, 5, 3, 1\} \end{aligned}$$

   (b)

$$\begin{aligned} r_{yy}(l) &= \sum_{n=-\infty}^{\infty} y(n)y(n-l) \\ l = 0 :& r_{yy}(0) = 1 \times 1 + 1 \times 1 + 2 \times 2 + 1 \times 1 = 7 \\ l = 1 :& r_{yy}(1) = 1 \times 1 + 2 \times 1 + 1 \times 2 = 5 \\ l = 2 :& r_{yy}(2) = 2 \times 1 + 1 \times 1 = 3 \\ l = 3 :& r_{yy}(3) = 1 \times 1 = 1 \\ \Rightarrow r_{yy}(l) &= \{1, 3, 5, \underset{\uparrow}{7}, 5, 3, 1\} \end{aligned}$$

   The autocorrelations are the same, and $y(n)$ is a reversal of $x(n)$. It implies that autocorrelation is invariant to sequence reversal. To prove this (note that the proof is not necessary for the question), let $y(n) = x(N - n)$, where $N$ is the length of the sequence $x(n)$. Then:

$$r_{yy}(l) = \sum_{n=-\infty}^{\infty} y(n)y(n-l) = \sum_{n=-\infty}^{\infty} x(N-n)x(N-n-l)$$

   Substitute $u = N - n$

$$r_{yy}(l) = \sum_{u=-\infty}^{\infty} x(u)x(u-l) = r_{xx}(l)$$

3. There are a number of ways to solve this question, including a graphical approach where $x(n)$ and $x(n - l)$ are plotted for each value of $l$. In this solution, the algebraic approach will be used.

   In the range $0 \leq n \leq 7$:

$$x(n) = \begin{cases} A & ; 0 \leq n < 4 \\ -A & ; 4 \leq n < 7 \end{cases}$$

$$r_{xx}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n-l)$$

Substituting in for $x(n)$ we get:

$$r_{xx}(l) = \frac{A}{8}\left\{\sum_{n=0}^{3} x(n-l) - \sum_{n=4}^{7} x(n-l)\right\}$$

It is simplest to substitute in values for $x(n-l)$ at this stage:

$$\begin{aligned}
r_{xx}(0) &= \frac{A}{8}\left\{\sum_{n=0}^{3} A - \sum_{n=4}^{7}(-A)\right\} = A^2 \\
r_{xx}(1) &= \frac{A}{8}\left\{-A + \sum_{n=1}^{3} A - A - \sum_{n=5}^{7}(-A)\right\} = \frac{A^2}{2} \\
r_{xx}(2) &= \frac{A}{8}\left\{-A - A + A + A - A - A + A + A\right\} = 0 \\
r_{xx}(3) &= \frac{A}{8}\left\{\sum_{n=0}^{2}(-A) + A - \sum_{n=4}^{6} A + A\right\} = -\frac{A^2}{2} \\
r_{xx}(4) &= \frac{A}{8}\left\{\sum_{n=0}^{3}(-A) - \sum_{n=4}^{7} A\right\} = -A^2
\end{aligned}$$

The remaining values can by symmetry, thus $r_{xx}(l) = \left\{-\frac{A^2}{2}, 0, \frac{A^2}{2}, \underset{\uparrow}{A^2}, \frac{A^2}{2}, 0, -\frac{A^2}{2}, -A^2\right\}$ which is periodic with a period of 8 samples.

4. (a) Beginning with the definition of autocorrelation:

$$r_{xx}(l) = \sum_{-\infty}^{\infty} x(n)x(n-l)$$

$$\begin{aligned}
r_{xx}(l) &= \sum_{-\infty}^{\infty} \left(s(n) + r_1 s(n-k_1) + r_2 s(n-k_2)\right)\left(s(n-l) + r_1 s(n-l-k_1) + r_2 s(n-l-k_2)\right) \\
&= \sum_{-\infty}^{\infty} s(n)s(n-l) + r_1 \sum_{-\infty}^{\infty} s(n)s(n-l-k_1) + r_2 \sum_{-\infty}^{\infty} s(n)s(n-l-k_2) \\
&\qquad + r_1 \sum_{-\infty}^{\infty} s(n-k_1)s(n-l) + r_1^2 \sum_{-\infty}^{\infty} s(n-k_1)s(n-l-k_1) \\
&\qquad + r_1 r_2 \sum_{-\infty}^{\infty} s(n-k_1)s(n-l-k_2) + r_2 \sum_{-\infty}^{\infty} s(n-k_2)s(n-l) \\
&\qquad + r_1 r_2 \sum_{-\infty}^{\infty} s(n-k_2)s(n-l-k_1) + r_2^2 \sum_{-\infty}^{\infty} s(n-k_2)s(n-l-k_2) \\
&= r_{ss}(l) + r_1 r_{ss}(l+k_1) + r_2 r_{ss}(l+k_2) + r_1 r_{ss}(l-k_1) + r_1^2 r_{ss}(l) + \\
&\qquad r_1 r_2 r_{ss}(l-k_1+k_2) + r_2 r_{ss}(l-k_2) + r_1 r_2 r_{ss}(l+k_1-k_2) + r_2^2 r_{ss}(l) \\
&= (1 + r_1^2 + r_2^2) r_{ss}(l) + r_1(r_{ss}(l-k_1) + r_{ss}(l+k_1)) + r_2(r_{ss}(l-k_2) + r_{ss}(l+k_2)) \\
&\qquad + r_1 r_2 (r_{ss}(l-k_1+k_2) + r_{ss}(l+k_1-k_2))
\end{aligned}$$

(b) In order to be able to find $r_1$, $r_2$, $k_1$ and $k_2$ from observing $r_{xx}(l)$ then $r_{ss}(l) \neq 0$, which implies that the $s(n)$ must be non-zero for one or more values of $n$. It is assumed that $k_1 \neq k_2$ (otherwise there would be only one echo), and then, although the expression is complicated, there should be sufficient information to be able to evaluate all of the parameters, when an ordering of $k_1$ and $k_2$ is assumed (e.g. $k_1 < k_2$).

The process of finding the parameters is to first estimate the delays, $k_1$ and $k_2$, then finding $r_1$ and $r_2$ at $r_{xx}(k_1)$ and $r_{xx}(k_2)$ respectively.
A special case occurs when $k_2 = 2k_1$:

$$r_{xx}(l) = (1+r_1^2+r_2^2)r_{ss}(l)+r_1(1+r_2)(r_{ss}(l-k_1)+r_{ss}(l+k_1))+r_2(r_{ss}(l-k_2)+r_{ss}(l+k_2))$$

but this is also solvable.
Finally, note that the process is made easier if $r_{ss}(l)$ approximates a delta function.

(c) When $r_2 = 0$, then:

$$r_{xx}(l) = (1 + r_1^2)r_{ss}(l) + r_1(r_{ss}(l - k_1) + r_{ss}(l + k_1))$$

thus it is possible to estimate $k_1$, then $r_1$ from the expression. (Estimation of $k_2$ is unnecessary as when $r_2 = 0$, $x(n) = s(n) + r_1 s(n - k_1)$).

# DSA4/DTSA Frequency-Domain Analysis of LTI Systems Exercise Solutions

Please note that these are only *sample* solutions. Other methods may be used to achieve the same result. It is important that the understanding behind the results is gained as well as the process to reach these.

1. To solve this question, start with the definition of the output in terms of the input and the system impulse response:

$$y(n) = \sum_{m=0}^{\infty} h(m)x(n-m)$$

We know the value of $y(0)$, so express this in terms of $h(m)$ and $x(m)$:

$$y(0) = \sum_{m=0}^{\infty} h(m)x(-m)$$

As $x(-m)$ is zero for all $m > 0$, and we know $x(0)$, then:

$$y(0) = h(0)x(0) \Rightarrow h(0) = \frac{y(0)}{x(0)} = 1$$

Taking the same approach for the remaining outputs:

$$\begin{aligned}
y(1) &= \sum_{m=0}^{\infty} h(m)x(1-m) = h(0)x(1) + h(1)x(0) \\
\Rightarrow h(1) &= \frac{y(1) - h(0)x(1)}{x(0)} = -2 \\
y(2) &= \sum_{m=0}^{\infty} h(m)x(2-m) = h(0)x(2) + h(1)x(1) + h(2)x(0) \\
\Rightarrow h(2) &= \frac{y(2) - h(0)x(2) - h(1)x(1)}{x(0)} = 3 \\
y(3) &= h(1)x(2) + h(2)x(1) + h(3)x(0) = h(3) - 1 \Rightarrow h(3) = 0 \\
y(4) &= h(2)x(2) + h(3)x(1) + h(4)x(0) = 6 + h(4) \Rightarrow h(4) = 0 \\
\Rightarrow h(n) &= \{1, -2, 3\}
\end{aligned}$$

Substituting this in to the expression for the crosscorrelation:

$$\begin{aligned}
r_{yx}(l) &= \sum_{n=0}^{\infty} h(n)r_{xx}(n-l) = r_{xx}(l) - 2r_{xx}(l-1) + 3r_{xx}(l-2) \\
r_{xx}(l) &= \sum_{n=-\infty}^{\infty} x(n)x(n-l) \\
r_{xx}(0) &= (x(0))^2 + (x(1))^2 + (x(2))^2 = 6 \\
r_{xx}(1) &= x(0)x(1) + x(1)x(2) = 3 \\
r_{xx}(2) &= x(0)x(2) = 2 \\
\Rightarrow r_{yx}(l) &= \{2, -1, \underset{\uparrow}{6}, 0, 14, 5, 6\}
\end{aligned}$$

Performing the calculation directly:

$$
\begin{aligned}
r_{yx}(l) &= \sum_{n=-\infty}^{\infty} y(n)x(n-l) = \sum_{n=-\infty}^{\infty} y(n+l)x(n) \\
&= y(l) + y(1+l) + 2y(2+l) \\
&= \{2, -1, \underset{\uparrow}{6}, 0, 14, 5, 6\}
\end{aligned}
$$

as above, thereby demonstrating the property.

2. (a) The magnitude response is obtained by evaluating the $z$-transform around the unit circle, i.e. $z = e^{j\omega}$, and finding $|H(\omega)|$.

$$
\begin{aligned}
H(z) &= b_0 + b_1 z^{-D} + b_2 z^{-2D} \\
\Rightarrow H(\omega) &= b_0 + b_1 e^{-j\omega D} + b_2 e^{-j2\omega D}
\end{aligned}
$$

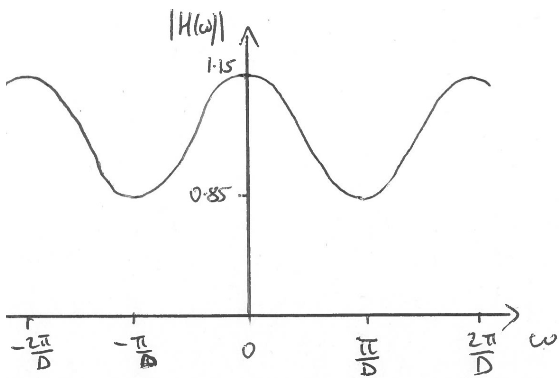
To find the magnitude squared, multiply by its complex conjugate:

$$
\begin{aligned}
\Rightarrow |H(\omega)|^2 &= (b_0 + b_1 e^{-j\omega D} + b_2 e^{-j2\omega D})(b_0 + b_1 e^{j\omega D} + b_2 e^{j2\omega D}) \\
&= b_0^2 + b_1^2 + b_2^2 + (b_0 b_1 + b_1 b_2)\left(e^{j\omega D} + e^{-j\omega D}\right) + b_0 b_2 \left(e^{j2\omega D} + e^{-j2\omega D}\right) \\
&= b_0^2 + b_1^2 + b_2^2 + 2(b_0 b_1 + b_1 b_2)\cos(\omega D) + 2 b_0 b_2 \cos(2\omega D) \\
\Rightarrow |H(\omega)| &= \sqrt{b_0^2 + b_1^2 + b_2^2 + 2(b_0 b_1 + b_1 b_2)\cos(\omega D) + 2 b_0 b_2 \cos(2\omega D)}
\end{aligned}
$$

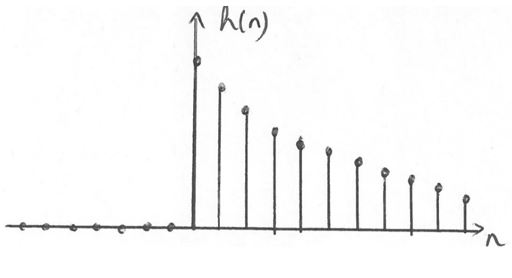
(b) From the given coefficients, work out the terms in the expression for $|H(\omega)|$:

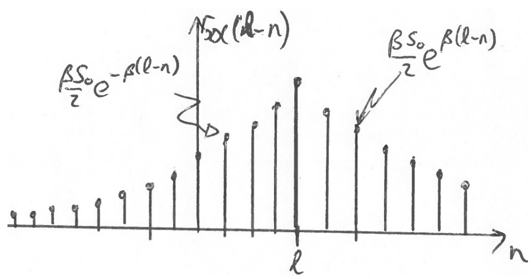
$$
\begin{aligned}
b_0^2 + b_1^2 + b_2^2 &= 1.0125 \\
2b_1(b_0 + b_2) &= 0.3 \\
2b_0 b_2 &= 0.01
\end{aligned}
$$

The $\cos(\omega D)$ term will dominate the result, giving rise to the adjacent sketch:




3. As $r_{xx}(l)$ has a discontinuity at $l = 0$, plotting the signals helps to write down the correct next step. When creating the plots, it is very helpful to write down the equations for each part of the curve. For $r_{xx}(l-n)$ care needs to be taken to make sure that the sign of the exponential is correct.



For $l > 0$, there are two parts to the solution that need to be combined because of the different equations describing the two parts of $r_{xx}(l-n)$. The first part is from $n = 0$ up to $n = l - 1$, and the second from $n = l$ onwards:

$$
\begin{aligned}
r_{xx}(l-n) &= \begin{cases} \frac{\beta S_0}{2} e^{-\beta(l-n)} & ; n < l \\ \frac{\beta S_0}{2} e^{\beta(l-n)} & ; n \geq l \end{cases} \\
r_{yx}(l) &= \sum_{n=0}^{\infty} h(n) r_{xx}(l-n) \\
&= \sum_{n=0}^{l-1} \frac{1}{\alpha} e^{-n/\alpha} \frac{\beta S_0}{2} e^{-\beta(l-n)} + \sum_{n=l}^{\infty} \frac{1}{\alpha} e^{-n/\alpha} \frac{\beta S_0}{2} e^{\beta(l-n)}
\end{aligned}
$$

In order to simplify, both summations must be in the form of the geometric series identity. To achieve this, make the substitution $m = n - l$ for the second summation:

$$
\Rightarrow r_{yx}(l) = \frac{\beta S_0}{2\alpha} e^{-\beta l} \sum_{n=0}^{l-1} e^{(\beta - 1/\alpha)n} + \frac{\beta S_0}{2\alpha} \sum_{m=0}^{\infty} e^{-(m+l)/\alpha} e^{-\beta m}
$$

Using the geometric series identity:

$$
\Rightarrow r_{yx}(l) = \frac{\beta S_0}{2\alpha} \left( e^{-\beta l} \frac{1 - e^{(\beta-1/\alpha)l}}{1 - e^{(\beta - 1/\alpha)}} + e^{-l/\alpha} \frac{1}{1 - e^{-(\beta+1/\alpha)}} \right)
$$

For $l < 0$, $r_{xx}(l-n)$ is described by only one equation, so the calculation is much simpler:

$$
\begin{aligned}
r_{yx}(l) &= \sum_{n=0}^{\infty} \frac{1}{\alpha} e^{-n/\alpha} \frac{\beta S_0}{2} e^{\beta(l-n)} \\
&= \frac{\beta S_0}{2\alpha} e^{\beta l} \frac{1}{1 - e^{-(\beta+1/\alpha)}}
\end{aligned}
$$

So, the final result is:

$$
r_{yx}(l) = \begin{cases} \frac{\beta S_0}{2\alpha} e^{\beta l} \frac{1}{1 - e^{-(\beta+1/\alpha)}} & ; l \leq 0 \\ \frac{\beta S_0}{2\alpha} \left( e^{-\beta l} \frac{1 - e^{(\beta-1/\alpha)l}}{1 - e^{(\beta-1/\alpha)}} + e^{-l/\alpha} \frac{1}{1 - e^{-(\beta+1/\alpha)}} \right) & ; l > 0 \end{cases}
$$

4. This question can be tackled a number of ways. For example, the property $S_{yy}(\omega) = |Y(\omega)|^2$ can be used by finding an expression for $Y(\omega)$ using the shift property of transforms. In this sample solution, the property that $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$ is used. $H(\omega)$ can be found from the impulse response, or via the $z$-transform (which is how I have tackled the question). All of these approaches should result in the same answer.

$$
\begin{aligned}
y(n) &= x(n) - x(n-D) \\
\Rightarrow Y(z) &= X(z)(1 - z^{-D}) \\
\Rightarrow H(z) &= \frac{Y(z)}{X(z)} = 1 - z^{-D} \\
\Rightarrow H(z)H(z^{-1}) &= (1 - z^{-D})(1 - z^{D}) = 2 - (z^{D} + z^{-D}) \\
\Rightarrow |H(\omega)|^2 &= 2 - (e^{j\omega D} - e^{-j\omega D}) = 2(1 - \cos(\omega)) \\
\Rightarrow S_{yy}(\omega) &= 2(1 - \cos(\omega D)) S_{xx}(\omega)
\end{aligned}
$$

# DSA4/DTSA Random Signals, Correlation Functions, and Power Spectra Tutorial Solutions

Please note that these are only *sample* solutions. Other methods may be used to achieve the same result. It is important that the understanding behind the results is gained as well as the process to reach these.

1. These questions can be answered by manipulating the probability distribution function.
   (a) $F(x) = P(X \leq x) \Rightarrow P(X > 0.5) = 1 - F(0.5) = e^{-0.5} = 0.607$
   (b) $P(X \leq 0.25) = F(0.25) = 0.221$
   (c) $P(0.3 < X \leq 0.7) = F(0.7) - F(0.3) = 0.244$

2. (a) $x(n)$ is a function of a random variable, $\theta$, so the expectation should be taken with respect to $\theta$ when performing the integration. As $\theta$ is uniformly distributed between 0 and $\pi$, the height of the density function in this range is $\frac{1}{\pi}$. This is because the area under a density function must be 1.

$$\begin{aligned} E(x(n)) &= E(A\cos(\omega n - \theta)) = A\ E(\cos(\omega n - \theta)) \\ &= A\int_0^{\pi} \cos(\omega n - \theta).\frac{1}{\pi}\ d\theta \\ &= \frac{A}{\pi}\left[-\sin(\omega n - \theta)\right]_0^{\pi} \\ &= -\frac{A}{\pi}\left\{\sin(\omega n - \pi) - \sin(\omega n)\right\} \\ &= \frac{2A}{\pi}\sin(\omega n) \end{aligned}$$

The result shows that the mean value is a function of time, $n$, thus $x(n)$ is non-stationary.

   (b) Using the definition of variance, we first compute $E((x(n))^2)$, and $(E(x(n)))^2$, before finally computing the result. Again, all integrations should be performed with respect to the random variable, $\theta$, over all possible values of the random variable, 0 to $\pi$.

$$\begin{aligned} \mathrm{Var}(x(n)) &= E((x(n))^2) - (E(x(n)))^2 \\ E((x(n))^2) &= E(A^2\cos^2(\omega n - \theta)) = A^2 E\left(\frac{1}{2} + \frac{1}{2}\cos(2(\omega n - \theta))\right) \end{aligned}$$

The expectation of a constant is the constant itself, thus

$$\begin{aligned} E((x(n))^2) &= \frac{A^2}{2} + \frac{A^2}{2}E(\cos(2(\omega - \theta)) = \frac{A^2}{2} + \frac{A^2}{2\pi}\int_0^{\pi}\cos(2(\omega n - \theta))\ d\theta \\ &= \frac{A^2}{2} + \frac{A^2}{2\pi}\left[-\frac{1}{2}\sin(2\omega n - 2\theta)\right]_0^{\pi} \\ &= \frac{A^2}{2} - \frac{A^2}{4\pi}\left\{\sin(2\omega n - 2\pi) - \sin(2\omega n)\right\} = \frac{A^2}{2} \\ (E(x(n)))^2 &= \frac{4A^2}{\pi^2}\sin^2(\omega n) = \frac{2A^2}{\pi^2} - \frac{2A^2}{\pi^2}\cos(2\omega n) \\ \Rightarrow \mathrm{Var}(x(n)) &= \frac{A^2}{2} - \frac{2A^2}{\pi^2} + \frac{2A^2}{\pi^2}\cos(2\omega n) \\ &= A^2\left\{\frac{1}{2} - \frac{2}{\pi^2} + \frac{2\cos(2\omega n)}{\pi^2}\right\} \end{aligned}$$

(c) For the autocorrelation, the same process is applied.

$$\begin{aligned}
r_{xx}(m,n) &= E(x(m)x(n)) \\
&= A^2 E(\cos(\omega m - \theta)\cos(\omega n - \theta)) \\
&= \frac{A^2}{2} E(\cos(\omega(m-n)) + \cos(\omega(m+n) - 2\theta))
\end{aligned}$$

As $\cos(\omega(m-n))$ is not random, $E(\cos(\omega(m-n)) = \cos(\omega(m-n))$

$$\begin{aligned}
&= \frac{A^2\cos(\omega(m-n))}{2} + \frac{A^2}{2}\int_0^{\pi} \cos(\omega(m+n) - 2\theta)\frac{1}{\pi}\, d\theta \\
&= \frac{A^2\cos(\omega(m-n))}{2} + \frac{A^2}{2\pi}\left[-\frac{1}{2}\sin(\omega(m+n) - 2\theta)\right]_0^{\pi} \\
&= \frac{A^2\cos(\omega(m-n))}{2} + \frac{A^2}{4\pi}\{-\sin(\omega(m+n) - 2\pi) + \sin(\omega(m+n))\} \\
&= \frac{A^2\cos(\omega(m-n))}{2}
\end{aligned}$$

It is worthwhile noting that $r_{xx}(0) = E((x(n))^2)$ as expected. Although the autocorrelation is only a function of the time difference, $l = m - n$, and not time varying, the process is still classed as non-stationary as the mean value is a function of time. Note, too, that in this case, as $x(n)$ is periodic, $\lim_{l\to\infty} r_{xx}(l)$ is not equal to $(E(x(n)))^2$. Thus, for a process to be wide-sense stationary, it is not sufficient for the autocorrelation to be time invariant, the mean value must also be time invariant. In this case, the mean value is a function of $n$, so the process is non-stationary.

3. In order to compute the probability that $|y(n)| > 2A$, the effect of the function $\cos(\omega_0 n)$ must be considered. At different values of $n$, the probability that $|y(n)|$ exceeds the threshold will be different. We start, therefore, by defining the condition for $w(n)$, in order to result in $y(n)$ crossing the specified threshold.

$$y(n) = A\cos(\omega_0 n) + w(n)$$

Start by considering the upper threshold

$$P(y(n) > 2A) = P(A\cos(\omega_0 n) + w(n) > 2A)$$

Re-arrange to define the condition in terms of $w(n)$

$$= P(w(n) > A(2 - \cos(\omega_0 n)))$$

and then express in terms of the error function with $\sigma^2 = A/2$ and $\mu = 0$:

$$= \int_{A(2-\cos(\omega_0 n))}^{\infty} \frac{1}{\sqrt{\pi A}} \exp\left(-\frac{x^2}{A}\right) dx$$

Using the substitution $t = \frac{x}{\sqrt{A}}$:

$$= \int_{\sqrt{A}(2-\cos(\omega_0 n))}^{\infty} \frac{1}{\sqrt{\pi}} \exp(-t^2) dt$$

This describes the upper half of the complementary error function:

$$= \frac{1}{2}\mathrm{erfc}\left(\sqrt{A}(2 - \cos(\omega_0 n))\right)$$

and then re-expressing it in terms of the error function:

$$= \frac{1}{2} - \frac{1}{2}\text{erf}\left(\sqrt{A}(2 - \cos(\omega_0 n))\right)$$

Equally,

$$\begin{aligned} P(y(n) < -2A) &= P(A\cos(\omega_0 n) + w(n) < -2A) \\ &= P(w(n) < -A(2 + \cos(\omega_0 n))) \end{aligned}$$

Due to the symmetry of the Gaussian distribution

$$\begin{aligned} &= P(w(n) > A(2 + \cos(\omega_0 n))) \\ &= \frac{1}{2} - \frac{1}{2}\text{erf}\left(\sqrt{A}(2 + \cos(\omega_0 n))\right) \end{aligned}$$

Finally, the probability that $y(n)$ exceeds either threshold is the sum of the two probabilities that have just been determined

$$\Rightarrow P(|y(n)| > 2A) = 1 - \frac{1}{2}\left\{\text{erf}\left(\sqrt{A}(2 - \cos(\omega_0 n))\right) + \text{erf}\left(\sqrt{A}(2 + \cos(\omega_0 n))\right)\right\}$$

Thus the probability is time dependent. This is due to the deterministic nature of $\cos(\omega_0 n)$.

4. To start this question, write down the definition of the autocorrelation of $Z(n)$. As it is a random variable, the autocorrelation is denoted by $\gamma_{zz}(m)$ instead of $r_{zz}(m)$.

$$\begin{aligned} z(n) &= x(n) + x(n+a) \\ \Rightarrow \gamma_{zz}(m) &= E(z(n)z(n-m)) \end{aligned}$$

Now expand $z(n)$ and $z(n-1)$

$$= E((x(n) + x(n+a))(x(n-m) + x(n-m+a)))$$

Multiplying out the brackets, and then noting that the expectation of a sum is the sum of expectations gives

$$\begin{aligned} &= E(x(n)x(n-m)) + E(x(n)x(n-m+a)) + E(x(n+a)x(n-m)) \\ &\quad + E(x(n+a)x(n-m+a)) \end{aligned}$$

As $X(n)$ is stationary the last two terms can be rewritten

$$\begin{aligned} \gamma_{zz}(m) &= E(x(n)x(n-m)) + E(x(n)x(n-m+a)) + E(x(n)x(n-m-a)) \\ &\quad + E(x(n)x(n-m)) \\ &= 2\gamma_{xx}(m) + \gamma_{xx}(m-a) + \gamma_{xx}(m+a) \\ &= 2e^{-|m|} + e^{-|m-a|} + e^{-|m+a|} \end{aligned}$$

5. Starting with the second last line from above, $\gamma_{zz}(m)$ can be expressed in terms of $\gamma_{xx}(m)$, and time shifted versions of this. Thus $\Gamma_{zz}(\omega)$ can be expressed in terms of $\Gamma_{xx}(\omega)$.

$$\Gamma_{zz}(\omega) = \sum_{m=-\infty}^{\infty} \gamma_{zz}(m)e^{-j\omega m}$$

Substituting in for $\gamma_{zz}(m)$ gives

$$= 2\sum_{m=-\infty}^{\infty} \gamma_{xx}(m)e^{-j\omega m} + \sum_{m=-\infty}^{\infty} \gamma_{xx}(m-a)e^{-j\omega m} + \sum_{m=-\infty}^{\infty} \gamma_{xx}(m+a)e^{-j\omega m}$$

Changing the summation variables of the second and third sums gives

$$= 2\sum_{m=-\infty}^{\infty} \gamma_{xx}(m)e^{-j\omega m} + \sum_{k=-\infty}^{\infty} \gamma_{xx}(k)e^{-j\omega(k+a)} + \sum_{l=-\infty}^{\infty} \gamma_{xx}(l)e^{-j\omega(l-a)}$$

Using $e^{\alpha+\beta} = e^{\alpha}e^{\beta}$, and recognising the definition of $\Gamma_{xx}(\omega)$ gives

$$\begin{aligned} &= 2\Gamma_{xx}(\omega) + \Gamma_{xx}(\omega)e^{-ja\omega} + \Gamma_{xx}(\omega)e^{ja\omega} \\ &= 2(1+\cos(a\omega))\Gamma_{xx}(\omega) \end{aligned}$$

Now all that remains is to find $\Gamma_{xx}(\omega)$

$$\Gamma_{xx}(\omega) = \sum_{m=-\infty}^{\infty} e^{-|m|}e^{-j\omega m}$$

Because of the modulo operation, the summation should be split into two parts. We could take the first sum from $-\infty$ to $-1$, however in the next step this will not be the correct form for the Geometric series. So instead we take the sum from $-\infty$ to 0, and then subtract 1 to account for the extra $m = 0$ term.

$$= \sum_{m=-\infty}^{0} e^{m}e^{-j\omega m} + \sum_{m=0}^{\infty} e^{-m}e^{-j\omega m} - 1$$

Rearrange the first sum to get it into the form of a Geometric series, then use the formula from the formula sheet.

$$\begin{aligned} &= \sum_{m=0}^{\infty} e^{(j\omega-1)m} + \sum_{m=0}^{\infty} e^{-(1+j\omega)m} - 1 \\ &= \frac{1}{1-e^{j\omega-1}} + \frac{1}{1-e^{-j\omega-1}} - 1 \end{aligned}$$

Collect terms together over a common denominator

$$= \frac{1-e^{-j\omega-1} + 1 - e^{j\omega-1} - (1-e^{j\omega-1})(1-e^{-j\omega-1})}{(1-e^{j\omega-1})(1-e^{-j\omega-1})}$$

Using $e^{\alpha+\beta} = e^{\alpha}e^{\beta}$, and Euler's formula

$$\begin{aligned} &= \frac{2-2e^{-1}\cos(\omega) - (1-e^{j\omega-1} - e^{-j\omega-1} + e^{-2})}{1-e^{j\omega-1} - e^{-j\omega-1} + e^{-2}} \\ &= \frac{1-e^{-2}}{1-2e^{-1}\cos(\omega)+e^{-2}} \\ \Rightarrow \Gamma_{zz}(\omega) &= \frac{2(1+\cos(a\omega))(1-e^{-2})}{1-2e^{-1}\cos(\omega)+e^{-2}} \end{aligned}$$

6. The autocorrelation is aperiodic, thus the process is not periodic. This means that we can use the property:

$$\lim_{m\to\infty} \gamma_{xx}(m) = (E(X(n)))^2$$

This gives a method of determining the mean. The power is given by $\gamma_{xx}(0)$.

(a)

$$\lim_{m \to \infty} \gamma_{xx}(m) = 25 \Rightarrow E\left(X\left(n\right)\right) = \pm 5$$

(b)

$$\begin{aligned}\mathrm{Var}(X(m)) &= E\left(\left(X\left(n\right)\right)^2\right) - \left(E\left(X\left(n\right)\right)\right)^2 \\ &= \gamma_{xx}(0) - 25 = 4\end{aligned}$$

7. (a) Using Parseval's theorem, which states that the power is defined by the area under the spectral density:

$$\begin{aligned}\text{Total Power} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_{xx}(\omega)\ d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\cos(2\omega) + 16\ d\omega \\ &= \frac{1}{2\pi}\left[2\sin(2\omega) + 16\omega\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\left\{0 + 16\pi + 0 + 16\pi\right\} = 16\end{aligned}$$

(b)

$$\text{Power up to } \frac{\pi}{4} = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} \Gamma_{xx}(\omega)\ d\omega$$

Using the integration result from above with different limits:

$$\begin{aligned}&= \frac{1}{2\pi}\left[2\sin(2\omega) + 16\omega\right]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{2\pi}\left\{2 + 4\pi + 2 + 4\pi\right\} = \frac{2}{\pi} + 4\end{aligned}$$

# DSA4/DTSA Design of Digital Filters, part 1, Tutorial Solutions

Please note that these are only *sample* solutions. Other methods may be used to achieve the same result. It is important that the understanding behind the results is gained as well as the process to reach these.

1. Starting with the properties of the zero locations given in the lecture notes, since $M$ is even, there must be a zero at either $z = 1$ or $z = -1$. As $H_r(0) = 1$, the zero cannot be at $z = 1$, since this corresponds to $H_r(0)$. Thus, one zero must be at $z = -1$. From this point, there are (at least) two ways of proceedig. In method 1, we use transforms to generate a set of simultaneous equations, and solve, and in the second we use the equations defined for FIR design using the frequency method.

   From the table in the lecture notes, this implies that this is a symmetric filter, so $h(0) = h(3)$ and $h(1) = h(2)$.

   Now that these are established, we use the two specified points, $H_r(0)$ and $H_r(\pi/2)$ to set up two simultaneous equations to solve the two unknowns, $h(0)$ and $h(1)$. First, though, an expression for $H_r(\omega)$ should be obtained using the inverse DTFT.

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{3} h(n)e^{-j\omega n} \\
&= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} \\
&= h(0)(1 + e^{-j3\omega}) + h(1)(e^{-j\omega} + e^{-j2\omega})
\end{aligned}$$

   We know that $H(\omega)$ can be expressed as a real function, with a linear phase, so we re-express the terms in brackets as a linear phase multiplying cos terms

$$= e^{-j3\omega/2}\left\{h(0)\left(e^{j3\omega/2} + e^{-j3\omega/2}\right) + h(1)\left(e^{j\omega/2} + e^{-j\omega/2}\right)\right\}$$

   Thus the real valued frequency response can be obtained

$$\Rightarrow H_r(\omega) = 2h(0)\cos\left(\frac{3\omega}{2}\right) + 2h(1)\cos\left(\frac{\omega}{2}\right)$$

   Now, referring to the question, the values of $h(0)$ and $h(1)$ can be solved by substituting in the specified points for $H_r(\omega)$.

$$\begin{aligned}
H_r(0) = 1 &\Rightarrow 2(h(0) + h(1)) = 1 \Rightarrow h(1) = \frac{1}{2} - h(0) \\
H_r\left(\frac{\pi}{2}\right) = \frac{1}{2} &\Rightarrow 2\left(h(0)\cos\left(\frac{3\pi}{4}\right) + h(1)\cos\left(\frac{\pi}{4}\right)\right) = \frac{1}{2} \\
&\Rightarrow h(0)\cos\left(\frac{3\pi}{4}\right) + \left(\frac{1}{2} - h(0)\right)\cos\left(\frac{\pi}{4}\right) = \frac{1}{4} \\
&\Rightarrow h(0) = \frac{\frac{1}{4} - \frac{1}{2}\cos\left(\frac{\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)} = 0.0732 \\
&\Rightarrow h(1) = 0.427 \\
\Rightarrow h(n) &= \{\underset{\uparrow}{0.0732}, 0.427, 0.427, 0.0732\}
\end{aligned}$$

2. Begin with the desired frequency response:

(a)

$$H_d(\omega) = \begin{cases} 1 & ; \ |\omega| \leq \frac{\pi}{6} \\ 0 & ; \ \frac{\pi}{6} < |\omega| \leq \pi \end{cases}$$

For a 25 tap filter, a delay of 12 is required. The impulse response is obtained by taking the inverse DTFT

$$\begin{aligned}
\Rightarrow h_D(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j12\omega} H_d(\omega)\ d\omega = \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} e^{j\omega(n-12)}\ d\omega \\
&= \frac{1}{2\pi}\left[\frac{1}{j(n-12)} e^{j\omega(n-12)}\right]_{-\pi/6}^{\pi/6} = \frac{1}{j2\pi(n-12)}\left\{e^{j\pi(n-12)/6} - e^{-j\pi(n-12)/6}\right\} \\
&= \frac{1}{\pi(n-12)} \sin\left(\frac{\pi(n-12)}{6}\right)
\end{aligned}$$

In this case, as no shaped window is applied, $h(n) = h_D(n)$ for the relevant 25 values of $n$

$$\Rightarrow h(n) = \left\{\underset{\uparrow}{0}, -\frac{1}{22\pi}, -\frac{\sqrt{3}}{20\pi}, -\frac{1}{9\pi}, -\frac{\sqrt{3}}{16\pi}, -\frac{1}{14\pi}, 0, \frac{1}{10\pi}, \dots\right\} \ ; \ 0 \leq n < 25$$

(b) Now that the 25 filter taps have been specified, in order to obtain the frequency response of this finite length impulse response, a discrete-time Fourier transform should be applied.

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{24} h(n) e^{-j\omega n} \\
&= \sum_{n=0}^{24} \frac{1}{\pi(n-12)} \sin\left(\frac{\pi(n-12)}{6}\right) e^{-j\omega n}
\end{aligned}$$

Whilst calculating this by hand would be possible, it is a time consuming process. Python can compute a sampled version of the frequency response by using the function, `fft`, as well as computing the result directly from the equation above. For either approach, care needs to be taken when calculating $h(n)$, as, for $n = 12$, Python will not compute the correct result. Instead we need to compute $h(12)$, and specify this separately:

$$\begin{aligned}
\lim_{n\to 12} h(n) &= \lim_{n\to 12} \frac{\frac{d}{dn}\sin\left(\frac{\pi(n-12)}{6}\right)}{\frac{d}{dn}\pi(n-12)} \\
&= \lim_{n\to 12} \frac{\frac{\pi}{6}\cos\left(\frac{\pi(n-12)}{6}\right)}{\pi} = \frac{1}{6}
\end{aligned}$$

The code computing the above equation directly is shown below:

```
import numpy as np
# We define the impulse response as
# $$\frac{1}{\pi (n-12)}\sin\left(\frac{\pi (n-12)}{6}\right)$$
# This is the ideal impulse response windowed by a rectangular window
# function, and valid for $$-\frac{M}{2}<n<\frac{M}{2}$$

M = 25
mid_point = (M-1) / 2
```

```
n = np.arange(0, M)

distance = n - mid_point

# Avoid division by zero error being flagged
old_settings=np.seterr(divide='ignore', invalid='ignore')

# Calculate the ideal impulse response
h_ideal = np.multiply(np.divide(1, (np.pi*distance)),
                      np.sin(np.pi*distance/6))

# And re-enable the warnings
np.seterr(**old_settings);


#  As we need to evaluate the filter response at the mid point separately,
#  this is computed here.  The window response at the mid-point is 1, so it
#  is just the result of applying de L'Hopital's rule

h_ideal[int(mid_point)] = 1 / 6


# Compute and plot the frequency response using a zero-padded fft, and
# convert to dBs

FFT_length = 4096
f = np.multiply((1/FFT_length), np.arange(0, FFT_length))
H = 20*np.log10(abs(np.fft.fft(h_ideal,FFT_length)))
```

Note that the signal, $h(n)$, being transformed is finite in length. In other words, the FFT does not truncate the signal. Therefore, no shaped window is applied before computing the FFT.

(c) In order to apply the window to the FIR design we use the window function from the formula sheet:

$$\begin{aligned} w(n) &= 0.54 - 0.46\cos\left(\frac{2n\pi}{M-1}\right) \\ &= 0.54 - 0.46\cos\left(\frac{n\pi}{12}\right) \end{aligned}$$

Then multiply the window $w(n)$ with the corresponding values of the desired impulse response $h_D(n)$

$$\Rightarrow h(n) = \frac{1}{\pi(n-12)}\left(0.54 - 0.46\cos\left(\frac{n\pi}{12}\right)\right)\sin\left(\frac{\pi(n-12)}{6}\right)$$

The code to calculate this is:

```
h = np.multiply(h_ideal,np.hanning(M))
```

Once again, as the designed filter, $h(n)$ is finite in length, no windowing is required when computing the FFT.

3. This question is similar in form to the previous one, except that the frequency domain profile is a notch filter. However, the same processing steps apply:

(a)

$$H_d(\omega) = \begin{cases} 1 & ; |\omega| \le \frac{\pi}{6} \\ 0 & ; \frac{\pi}{6} < |\omega| \le \frac{\pi}{3} \\ 1 & ; |\omega| > \frac{\pi}{3} \end{cases}$$

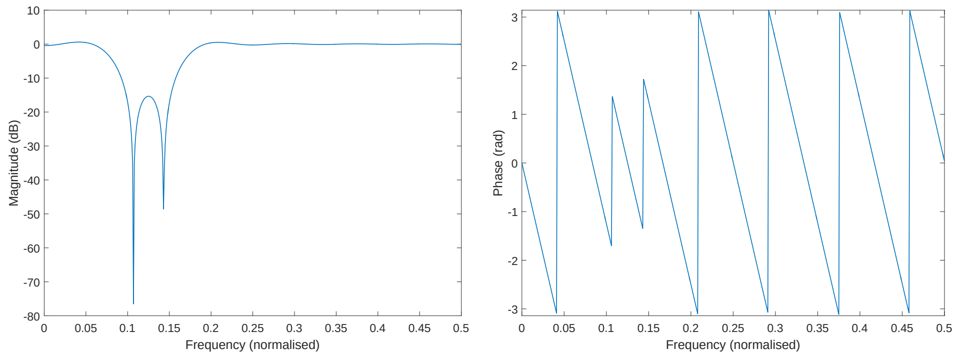
For a 25 tap filter, a delay of 12 is required

$$\begin{aligned}
\Rightarrow h_D(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j12\omega} H_d(\omega) e^{j\omega n}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{-\pi/3} e^{j\omega(n-12)}\, d\omega + \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} e^{j\omega(n-12)}\, d\omega + \frac{1}{2\pi}\int_{\pi/3}^{\pi} e^{j\omega(n-12)}\, d\omega \\
&= \frac{1}{2\pi}\left[\frac{1}{j(n-12)} e^{j\omega(n-12)}\right]_{-\pi}^{-\pi/3} + \frac{1}{2\pi}\left[\frac{1}{j(n-12)} e^{j\omega(n-12)}\right]_{-\pi/6}^{\pi/6} \\
&\qquad + \frac{1}{2\pi}\left[\frac{1}{j(n-12)} e^{j\omega(n-12)}\right]_{\pi/3}^{\pi} \\
&= \frac{1}{j2\pi(n-12)}\Big\{ e^{j\pi(n-12)} - e^{-j\pi(n-12)/3} + e^{j\pi(n-12)/6} - e^{-j\pi(n-12)/6} + \\
&\qquad e^{j\pi(n-12)/3} - e^{-j\pi(n-12)}\Big\} \\
&= \frac{1}{\pi(n-12)}\left\{ \sin\left(\frac{\pi(n-12)}{6}\right) - \sin\left(\frac{\pi(n-12)}{3}\right) + \sin\left(\pi(n-12)\right)\right\} \\
\Rightarrow h(n) &= \left\{ \underset{\uparrow}{0}, -\frac{1-\sqrt{3}}{22\pi}, 0, \frac{1}{9\pi}, \frac{\sqrt{3}}{8\pi}, \frac{1+\sqrt{3}}{14\pi}, 0, \dots \right\}
\end{aligned}$$

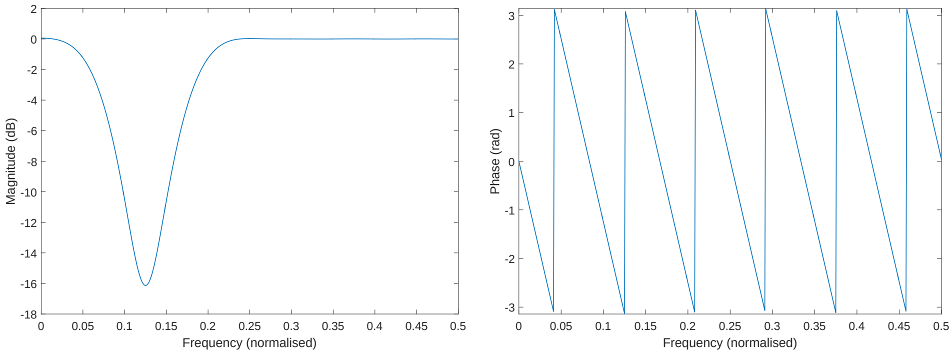
(b) As previously, the value of $h(n)$ at $n = 12$ needs to be evaluated separately using de l'Hôpital's rule in order for Matlab to be able to determine the frequency response. Note that although $\sin(\pi(n-12)) = 0$ for all integer values of $n$, it cannot be ignored when evaluating $h(n)$ for $n = 12$.

$$\begin{aligned}
\lim_{n\to 12} h(n) &= \lim_{n\to 12} \frac{\frac{d}{dn}\left\{\sin\left(\frac{\pi(n-12)}{6}\right) - \sin\left(\frac{\pi(n-12)}{3}\right) + \sin\left(\pi(n-12)\right)\right\}}{\frac{d}{dn}\pi(n-12)} \\
&= \lim_{n\to 12} \frac{\frac{\pi}{6}\cos\left(\frac{\pi(n-12)}{6}\right) - \frac{\pi}{3}\cos\left(\frac{\pi(n-12)}{3}\right) + \pi\cos(\pi(n-12))}{\pi} \\
&= \frac{5}{6}
\end{aligned}$$

The resulting frequency and phase response, when this is substituted into the Python code above, is shown below



(c) The process of applying the window is identical to that in 2c, that is multiplying each term obtained by the corresponding value of $w(n) = 0.54 - 0.46 \cos\left(\frac{n\pi}{12}\right)$. This results in the following frequency and phase responses:




The slower roll-off in the transition band is evident, as is the significantly better attenuation in the stop-band. Unfortunately, this means that the filter doesn't achieve the stopband attenuation. A longer filter would be required in order to reduce the width of the transition band.

4. (a) An FIR may be specified, despite its long length, and hence long latency, as the process may require a linear phase filter. Infinite impulse response designs cannot meet this requirement, but careful design of an FIR can do so.

(b) The specifications indicate that the transition band is 5 kHz, which is equivalent to $\frac{5}{300}2\pi = \frac{\pi}{30}$.

The stopband rejection level is used to select the window. The possible choices are Hamming, Kaiser ($\beta = 6$ or $\beta = 9$) or Blackman. To make the final selection, the filter with the narrowest transition band should be chosen, which implies that the Hamming window is the correct choice.

Finally, the minimum number of coefficients is determined as follows:

$$\frac{6.6\pi}{M} = \frac{\pi}{30}$$
$$\Rightarrow M = 30 \times 6.6 = 198$$

This is an acceptable choice for $M$, allowing the filter to be designed either as a symmetrical one or an anti-symmetrical one as the filter is a bandpass design.

(c) The filter has a passband in the range $\frac{30}{300}2\pi = \frac{\pi}{5}$ to $\frac{40}{300}2\pi = \frac{4\pi}{15}$. Choosing to design a symmetrical filter, the desired frequency response, $H_d(\omega)$ is given by:

$$H_d(\omega) = \begin{cases} 0 & ; |\omega| < \frac{\pi}{5} \\ 1 & ; \frac{\pi}{5} \leq |\omega| < \frac{4\pi}{15} \\ 0 & ; \frac{4\pi}{15} \leq |\omega| \end{cases}$$

As $M = 198$, to make the filter causal, a delay of $\frac{198-1}{2} = 98.5$ samples needs to be added. This can be done in the frequency domain. Then the filter taps should be computed combining the window function and the inverse transform.

$$\begin{aligned}
\Rightarrow h(n) &= w(n)\frac{1}{2\pi}\left\{\int_{-\frac{4\pi}{15}}^{-\frac{\pi}{5}} e^{j\omega(n-98.5)}\ d\omega + \int_{\frac{\pi}{5}}^{\frac{4\pi}{15}} e^{j\omega(n-98.5)}\ d\omega\right\} \\
&= \frac{w(n)}{2\pi}\left\{\left[\frac{1}{j(n-98.5)}e^{j\omega(n-98.5)}\right]_{-\frac{4\pi}{15}}^{-\frac{\pi}{5}} + \left[\frac{1}{j(n-98.5)}e^{j\omega(n-98.5)}\right]_{\frac{\pi}{5}}^{\frac{4\pi}{15}}\right\} \\
&= \frac{w(n)}{2\pi}\frac{1}{j(n-98.5)}\left\{e^{-j\pi(n-98.5)/5} - e^{-j4\pi(n-98.5)/15} + \right. \\
&\qquad \left. e^{j4\pi(n-98.5)/15} - e^{j\pi(n-98.5)/5}\right\} \\
&= \frac{-w(n)}{j2\pi(n-98.5)}\left\{2j\sin\left(\frac{\pi(n-98.5)}{5}\right) - 2j\sin\left(\frac{4\pi(n-98.5)}{15}\right)\right\} \\
&= \frac{w(n)}{\pi(n-98.5)}\left\{\sin\left(\frac{4\pi(n-98.5)}{15}\right) - \sin\left(\frac{\pi(n-98.5)}{5}\right)\right\}
\end{aligned}$$

$w(n)$ is the window which is specified in the formula sheet, and

$$w(n) = 0.54 - 0.46\cos\left(\frac{2n\pi}{197}\right)$$

(d) The first tap is for $n = 0$:

$$\begin{aligned}
w(0) &= 0.54 - 0.46\cos(0) = 0.08 \\
\Rightarrow h(0) &= \frac{0.08}{\pi(-98.5)}\left\{\sin\left(\frac{4\pi(-98.5)}{15}\right) - \sin\left(\frac{\pi(-98.5)}{5}\right)\right\} \\
&= -2.59\times 10^{-4}(-0.743 - 0.809) = 4.01\times 10^{-4}
\end{aligned}$$

# DSA4/DTSA Design of Digital Filters, part 2, Tutorial Solutions

Please note that these are only *sample* solutions. Other methods may be used to achieve the same result. It is important that the understanding behind the results is gained as well as the process to reach these.

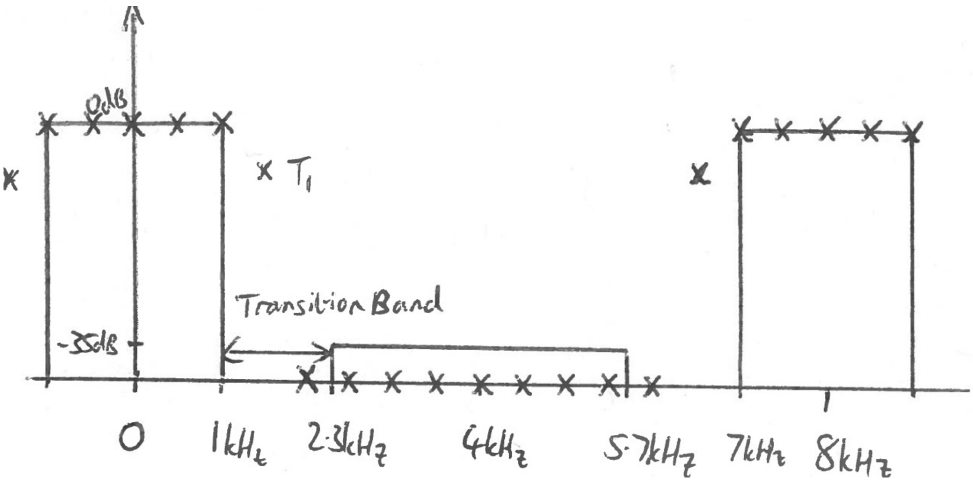
1. As $M = 16$, there are 16 samples across the 8 kHz of the periodic frequency domain transfer function. Thus, the spacing of the sampling points in frequency is 500 Hz. The two possible options for sampled points are: for $\alpha = 0$, {0, 500 Hz, 1 kHz, 1.5 kHz,...}; for $\alpha = 0.5$, the samples start at half of the frequency spacing, and are {250 Hz, 750 Hz, 1.25 kHz,...}.

   The filter is low-pass, therefore $\beta = 0$.

   The passband is defined such that all frequencies up to the cut-off of 1 kHz must not be attenuated, and all frequencies above 2.3 kHz must be attenuated by at least -35 dB. Between the frequencies of 1 kHz and 2.3 kHz the filter response is not constrained by the specification.

   With $\alpha = 0$, the first three sampling points must have a value of 1 as they fall within the passband. In the case of the 1 kHz sampling point, it is on the edge of the passband. For the stopband, all sampling points above 2.3 kHz should be set to zero. However, as there is no sampling point at 2.3 kHz, then the nearest sampling point below this frequency (2 kHz) must also be set to zero. This ensures that all frequencies between 2 kHz and 2.5 kHz will meet the stopband attenuation specified. Otherwise it would not be possible to guarantee that the stopband attenuation would be met between 2.3 kHz and 2.5 kHz.

   The specification, with the sampling points, is shown below



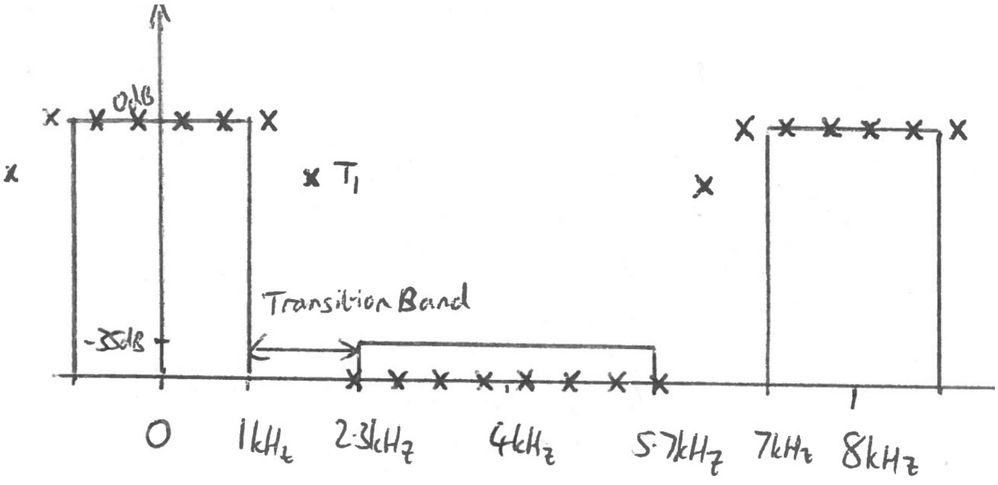

   This leaves a single sampling point, 1.5 kHz, for the transition coefficient, $T_1$. To specify this coefficient, the table on the formula sheet should be used. There are three sampling points within the passband, so BW=3. The table shows that the minimum stopband attenuation with only one transition coefficient is -41.24 dB, so meets the design criteria. Thus the filter specification is met with $T_1 = 0.41001589$.

   The filter, in the frequency domain, is then specified as

$$G(k) = \{1, -1, 1, -0.41001589, 0, 0, 0, 0, 0, 0, 0, 0, 0, -0.41001589, 1, -1\}$$

   For $\alpha = 0.5$, the first two samples (250 Hz and 750 Hz) fall within the passband and must have the value 1. To avoid the filter response rolling off before 1 kHz, the next sample at 1.25 kHz, which is in the transition band, must also be 1. This ensures that all frequencies between 0 Hz and 1 kHz will be passed with a gain of approximately 1 (allowing for filter ripple in the passband). Thus it is the fourth coefficient, $G(3)$, that is the transition coefficient at a frequency of 1.75 kHz. The remaining coefficients from 2.25 kHz and onward are all 0. The first of these coefficients lies just below the stopband edge, so the filter will have all stopband frequencies below the Minimax value quoted on the transition coefficients table.

The specification, with the sampling points, is shown below




As there are three samples that are 1, the third line of the transition coefficients table is used. With just one transition coefficient, for $\alpha = 0.5$, the maximum stopband attenuation is -51.41888444 dB, which clearly satisfies the design criteria. The filter is then specified by:

$$G(k + 0.5) = \{1, -1, 1, -0.30532944, 0, 0, 0, 0, 0, 0, 0, 0, 0.30532944, -1, 1, -1\}$$

Therefore the impulse response is:

$$h(n) = \frac{1}{8}\left\{\sin\left(\frac{\pi}{16}\left(n+\frac{1}{2}\right)\right) - \sin\left(\frac{3\pi}{16}\left(n+\frac{1}{2}\right)\right) + \sin\left(\frac{5\pi}{16}\left(n+\frac{1}{2}\right)\right)\right.$$
$$\left. -0.30532944\sin\left(\frac{7\pi}{16}\left(n+\frac{1}{2}\right)\right)\right\}$$

For the comparable window based design, the window that meets the design criteria should be selected. From the formula sheet the best choice, requiring the fewest taps, is the Hanning window with an attenuation of -44 dB. The transition band specification is given in Hertz, so should be translated to normalised frequency:

$$\frac{1.3 \text{ kHz}}{8 \text{ kHz}}2\pi = \frac{1.3\pi}{4}$$

So, to determine the number of taps:

$$\frac{6.2\pi}{M} \le \frac{1.3\pi}{4} \Rightarrow M \ge 19.08$$

thus the minimum number of taps is 20. This is slightly larger than the designs that have just been created, so is more complex.

2. The minimum number of taps required are defined by the transition band and the stopband attenuation. Based on the given tables it is likely that one transition coefficient will be

sufficient to achieve the required stopband attenuation. As the transition band has been reduced to the range 1.3 kHz to 2.3 kHz, the placement of frequency samples becomes a more challenging issue. With the designs from 1, neither of them were able to place three taps within this range, inclusive. This implies that the spacing of samples in the frequency domain will need to be reduced so that three coefficients lie in the range 1.3 kHz to 2.3 kHz.

The sample points all lie at multiples of $\frac{8\text{ kHz}}{M}$, so either by trial and error, or use of a spreadsheet, or by deriving a suitable equation, the value of $M$ should be increased from 16 until there are three samples within the required range.

For this specification, with $\alpha = 0$, the minimum number of taps is M=18, with the transition band samples falling at frequencies of 1.33 kHz, 1.78 kHz, and 2.22 kHz. For the filter with $\alpha = 0.5$, the minimum number of taps is 20, with the transition band samples lying at 1.4 kHz, 1.8 kHz, and 2.2 kHz. As we do not have transition coefficient tables for $M = 18$ and $M = 20$ the final designs are not required for this tutorial.

For the window design method, the same window selection is still valid as the stopband attenuation criterion has not changed. The only difference is a narrower transition band:

$$\frac{1\text{ kHz}}{8\text{ kHz}}2\pi = \frac{\pi}{4}$$
$$\Rightarrow \frac{6.2\pi}{M} \leq \frac{\pi}{4} \Rightarrow M \geq 24.8$$

Thus, $M = 25$ is the lowest complexity filter we can design using this method.

Presuming that tables for $\alpha = 0$ and $M = 18$ are available, then this is the lowest complexity filter that can be designed for this revised problem.

3. With reference to the transition coefficient tables, in order to achieve a stopband attenuation of -65 dB, at least two transition coefficients will be required. We have tables for $M = 16$ with two transition coefficients, so we will attempt a design with this many taps.

   There are two possible values for $\alpha$, $\alpha = 0$ and $\alpha = 0.5$. The sampling points in the frequency domain are:
   $$\begin{array}{ll} \alpha = 0: & \{0\text{Hz}, 40\text{Hz}, 80\text{Hz}, 120\text{Hz}, 160\text{Hz}, \ldots\} \\ \alpha = 0.5: & \{20\text{Hz}, 60\text{Hz}, 100\text{Hz}, 140\text{Hz}, 180\text{Hz}, \ldots\} \end{array}$$

   There is only one frequency of interest in the passband, 0 Hz, so the bandwidth for both designs is 1. For the first design, with $\alpha = 0$, the transition coefficients will be at 40 Hz and 80 Hz, with the first sample of the stopband being at 120 Hz. This just meets the design specification.

   However, for the second design, with the transition coefficients at 60 Hz, and 100 Hz, the first sample of the stopband region is at 140 Hz. This does not ensure that frequencies between the specified stopband edge, 120 Hz, and 140 Hz will meet the stopband requirements. Thus, this design is not suitable, and can be discarded at this stage.

   Returning to the design with $\alpha = 0$, as $BW = 1$, we use the first line of the table of transition coefficients on the formula sheet and note that the minimax design achieves a stopband attenuation of -71.59789962 dB, which meets our criteria. Thus we use the transition coefficients $T_1 = 0.09225582$ and $T_2 = 0.58621580$ giving the filter as:

   $$G(k) = \{1, -0.58621580, 0.09225582, 0, \ldots, 0.09225582, -0.58621580\}$$

   Using the formula sheet, the impulse response is then given by

   $$h(n) = \frac{1}{16}\left\{1 - 1.1724316\cos\left(\frac{\pi\left(n+\frac{1}{2}\right)}{8}\right) + 0.18451164\cos\left(\frac{\pi\left(n+\frac{1}{2}\right)}{4}\right)\right\}$$

   Find the first three taps by substituting in $n = \{0, 1, 2\}$ into the above equation:

   $$h(0) = 0.001285$$
   $$h(1) = 0.005986$$
   $$h(2) = 0.017376$$

# DSA4/DTSA Power Spectrum Estimation Tutorial Solutions

Please note that these are only *sample* solutions. Other methods may be used to achieve the same result. It is important that the understanding behind the results is gained as well as the process to reach these.

1. The 3 dB bandwidth of a rectangular window is $\frac{1.78\pi}{M}$. The 3 dB bandwidth required is 0.01 cycles per sample, which translates to $0.01 \times 2\pi = 0.02\pi$ radians per sample. This gives rise to the condition:

   $$\frac{1.78\pi}{M} \leq 0.02\pi \Rightarrow M \geq 89$$

   This is the minimum size of $M$ that should be used. In order to make computation efficient, a good choice of FFT length is the nearest power of 2. $2^7 = 128$, so zero-padding the data up to 128 samples (after applying the window to the block of 89 samples) prior to taking each FFT would result in efficient calculations of periodograms.

2. Using the periodogram approach to solve this question, the first issue to be addressed is the improvement of signal to noise ratio (SNR). Beginning with an SNR of 5 dB on the input signal, a final SNR of 35 dB is to be obtained, thus a gain in SNR of 30 dB is required.

   When averaging periodograms together, if the blocks can be assumed to be independent, then the variance is scaled down by the number of samples being averaged. Converting the SNR gain in deciBels to its equivalent linear factor, this implies that the number of blocks that need to be averaged is

   $$K = 10^{\frac{30}{10}} = 1000$$

   Although this is not required for the solution, here is a derivation for why the SNR increases. We start by assuming that $y(n) = x(n) + w(n)$, where $x(n) = X$ is a constant, and $w(n)$ is corrupting white noise. When we average, we are adding terms together, so when we average $L$ values together, we calculate:

   $$a = \frac{1}{L}\sum_{n=0}^{L} y(n) = \frac{1}{L}\sum_{n=0}^{L} x(n) + w(n)$$

   Evaluating the power of $a$ we find:

   $$E(a^2) = \frac{1}{L^2}E\left(\left(\sum_{n=0}^{L} x(n) + w(n)\right)^2\right) = \frac{1}{L^2}E\left(\left(\sum_{n=0}^{L} x(n) + w(n)\right)\left(\sum_{m=0}^{L} x(m) + w(m)\right)\right)$$

   Now if $x(n)$ and $w(n)$ are independent, then $E(x(n)w(m)) = 0$. Additionally, if noise is white, then $E(w(n)w(m)) = 0 \ \forall n \neq m$. However, as $x(n) = X$, $E(x(n)x(m)) = X^2$. By definition, $E(w(n)^2) = \sigma^2$, the noise power. Finally, we find that on expanding the brackets we obtain:

   $$E(a^2) = \frac{1}{L^2}\left[L^2X^2 + L\sigma^2\right] = X^2 + \frac{\sigma^2}{L}$$

   Thus the signal to noise ratio has improved by a factor of $L$.

   The next issue to address is achieving the required sidelobe level. With an output SNR of 35 dB, we should find a window with sidelobe levels lower than -35 dB. This suggests that the Hamming window is most appropriate, with the best resolution for a given length of transform.

   To determine the length of the periodogram, the required resolution should be expressed in terms of normalised frequency, so

   $$\text{Resolution} = \frac{50\text{Hz}}{20\text{kHz}}2\pi = \frac{\pi}{200}$$

Using the formula sheet to determine the resolution of the Hamming window (with a peak sidelobe level of -43 dB), and ensuring this meets the criteria results in the inequality:

$$\frac{2.6\pi}{M} \leq \frac{\pi}{200} \Rightarrow M \geq 520$$

The next power of 2 above 520 is $2^{10} = 1024$, so we should zero pad the blocks up to 1024 samples before applying the FFT.

To determine the total length of data required, we will assume that a 50% overlap is used in the Welch periodogram, thus each periodogram reuses 260 samples from the previous block, and requires another 260 samples from the data stream. Thus, the minimum length of the input data is

$$520 + (1000 - 1) \times 260 = 260,260$$

At a sampling rate of 20 kHz, this equates to just over 13 s of data. Note that this is the minimum length of data required - if the periodograms are not sufficiently uncorrelated, then more blocks will be required to achieve the required reduction in SNR.

3. The first step in forming a minimum variance spectral estimate is to compute $\mathbf{R}_{xx}$. This involves computing $r_{xx}(l)$ for all values of $l$ between 0 and $p$, which is chosen by the designer. Using the method described in the lecture notes for computing $r_{xx}(l)$ is the most efficient technique. The data, of length $N$, is split into $\frac{N}{p+1}$ blocks of length $p+1$, each of which is transformed once using an FFT of length $2p+2$.

   The computational complexity of one FFT is $O\left(\frac{2p+2}{2}\log_2(2p+2)\right)$. Applying this to $\frac{N}{p+1}$ blocks results in a complexity of $O\left(\frac{N}{p+1}\frac{2p+2}{2}\log_2(2p+2)\right) = O\left(N\log_2(2p+2)\right)$ for the FFT processing.

   Each FFT result is multiplied by itself and one other block, resulting in $2\frac{N}{p+1}(2p+2) = 4N$ multiplies, increasing the complexity to $O\left(4N + N\log_2(2p+2)\right) = O(N\log_2(32(p+1)))$.

   The inverse FFT operation is required only once, and is of complexity $O((p+1)\log_2(2p+2))$, so is small by comparison, therefore ignored.

   The next task to perform is the eigenvalue decomposition of $\mathbf{R}_{xx}$, which, as specified in the question, will require a complexity of $O((p+1)^2)$.

   Formation of $\mathbf{u}$ from $\mathbf{\Lambda}$ requires $p+1$ inversions, which is minor in comparison to $O((p+1)^2)$ from the eigenvalue decomposition, so can be ignored. Forming $\mathbf{V}$ from $\mathbf{Q}$ requires $p+1$ FFT operations of length $p+1$, and $p+1$ multiplications. By comparison to the FFT complexity, the multiplications are not significant, so forming $\mathbf{V}$ from $\mathbf{Q}$ has a complexity of $O\left((p+1)\frac{p+1}{2}\log_2(p+1)\right)$. Combined with the eigenvalue decomposition, forming $\mathbf{u}$ and $\mathbf{V}$ from $\mathbf{R}_{xx}$ has a complexity of $O\left(\frac{(p+1)^2}{2}\left(2 + \log_2(p+1)\right)\right)$.

   The final step is the formation of the result, $\frac{1}{\mathbf{Vu}}$, requires $(p+1)^2$ complex multiplies, which when combined with the complexity of forming $\mathbf{u}$ and $\mathbf{V}$ gives a complexity of $O\left(\frac{(p+1)^2}{2}\left(4 + \log_2(p+1)\right)\right) = O\left(\frac{(p+1)^2}{2}\log_2(16(p+1))\right)$ after forming $\mathbf{R}_{xx}$.

   So, the complexity for the full process is $O\left(N\log_2(32(p+1))\right) + O\left(\frac{(p+1)^2}{2}\log_2(16(p+1))\right)$. The resolution of the algorithm is a function of $p$, and the variance of the estimate is a function of $N$. Where high resolution is desired, the algorithm will be dominated by the second term, whereas for large datasets, if $N > (p+1)^2$, the first term will become the dominant one.

   Comparing with the classical techniques, the level of complexity for the minimum variance spectral estimate can be significantly larger, depending upon the desired resolution, however the quality of the final estimate is much better.

4. Through investigation the following observations should be made:

   - The longer the record $x$ is, the better the final variance will be.

- Resolution of the methods is inversely proportional to $M$ and $p$.
- For the periodogram based approaches, altering the length of the FFT alone does not change the resolution, but does show the detail of the spectrum more clearly.
- The sidelobes of the Bartlett periodogram are significant and could make identifying signal components very difficult without a very high resolution.
- The Welch periodogram, while reducing the height of the sidelobes, does result in a wider main lobe, which at low resolutions can obscure closely spaced frequencies.

# DSA4/DTSA Multirate Digital Signal Processing Tutorial Solutions

Please note that these are only *sample* solutions. Other methods may be used to achieve the same result. It is important that the understanding behind the results is gained as well as the process to reach these.

1. (a) In this case, $y_1(n)$ is effectively sub-sampling $x(n)$ by a factor of 2, so the periodic repetition is reduced by the same factor. To show this mathematically:

$$y_1(n) = \frac{1}{2}(x(n) + (-1)^n x(n)) = \frac{1}{2}(x(n) + e^{j\pi n} x(n))$$
$$\Rightarrow Y_1(\omega) = \frac{1}{2}(X(\omega) + X(\omega - \pi))$$

See Figure 1.

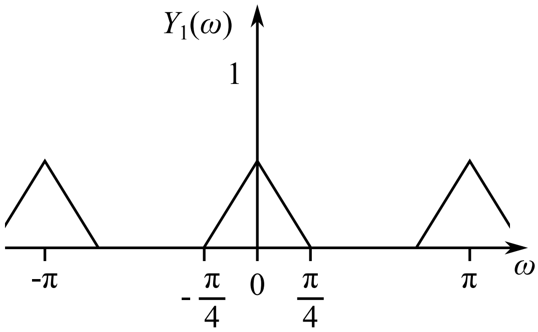



Figure 1: Fourier transform of $y_1(n)$

(b) The same samples appear in $y_2(n)$ as in $y_1(n)$, but the overall sampling rate is reduced by a factor of 2.

$$y_2(n) = x(2n) = y_1(2n)$$
$$Y_2(\omega) = \sum_n y_2(n)e^{-j\omega n} = \sum_n y_1(2n)e^{-j\omega n}$$

Noting that $y_1(2n+1) = 0$, we can make the substitution $m = 2n$

$$Y_2(\omega) = \sum_m y_1(m)e^{-j\omega m/2} = Y_1\left(\frac{\omega}{2}\right) = \frac{1}{2}X\left(\frac{\omega}{2}\right)$$

See Figure 2.

(c) In this case, the sampling rate is increased.

$$y_3(2n) = x(n) \qquad y_3(2n+1) = 0$$
$$Y_3(\omega) = \sum_n y_3(2n)e^{-j\omega 2n} = \sum_n x(n)e^{-j\omega 2n} = X(2\omega)$$

See Figure 3.

2. (a) To achieve a stopband attenuation of -80 dB, a Kaiser window, with $\beta = 9$ is required. This has a transition band of size $\frac{11.4\pi}{M}$. At a sampling frequency of 200 kHz, the desired transition band is $\frac{100\times 2\pi}{2\times 10^5} = \frac{\pi}{1000}$. Thus:

$$\frac{11.4\pi}{M} \leq \frac{\pi}{1000}$$
$$\Rightarrow M \geq 11400$$

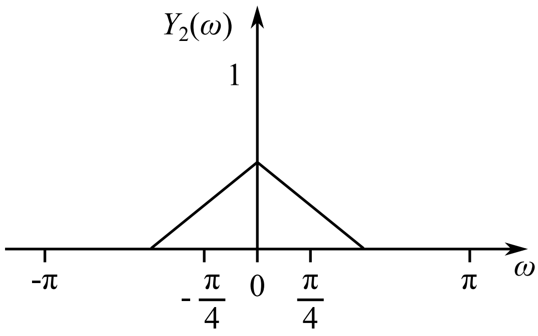



Figure 2: Fourier transform of $y_2(n)$

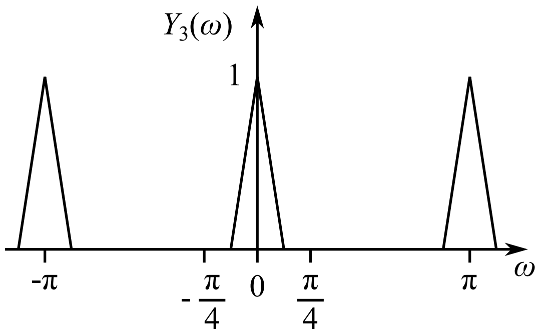



Figure 3: Fourier transform of $y_3(n)$

Thus, at least 11,400 taps are required, making this a very high complexity filter to achieve this task.

(b) There are numerous solutions to this problem, with various degrees of downsampling that can be deployed. In this instance, the factor of 20 for downsampling, as suggested in the lecture notes, will be used, but other choices are equally valid.

As part of the decimator, a low-pass filter is required. At the new sampling rate of 10 kHz, the frequency of interest is repeated between 9 kHz and 11 kHz (and at all multiples of 10 kHz thereafter). This implies that the downsampler needs to attenuate frequencies above 9 kHz to better than -80 dB to avoid any aliased components adding unwanted signals. To ensure better than 80 dB attenuation, the filter in the downsampler should be chosen to reach -80 dB at a frequency of 9 kHz. This would be 8 kHz above the cut-off. The Kaiser window will achieve -80 dB rejection with the smallest number of taps. This results in the decimator having a filter of length:

$$\frac{11.4\pi}{M} \leq 2\pi\frac{8}{200}$$
$$\Rightarrow M \geq 142.5$$

As this is a low-pass filter, a symmetric FIR filter with $M = 143$ is appropriate.

The low pass filter will be implemented at the lower sampling rate, so the design becomes:

| Parameter | Value |
|---|---|
| Sampling frequency | 10 kHz |
| Bandwidth | 1 kHz |
| Transition Band | 100 Hz |
| Stopband attenuation | -80 dB |

As before, the Kaiser window is required, and

$$\begin{aligned}\frac{11.4\pi}{M} &\le 2\pi\frac{100}{1\times10^4}\\ \Rightarrow M &\ge 570\end{aligned}$$

Finally, the upsampler can use the same design of filter as the downsampler, as it should filter out the images with an attenuation of at least -80 dB, so this filter is also 143 taps long. The total number of multiplies required is therefore $2\times143+570 = 856$, and these are computed at the rate of 10 kHz.

Comparing both techniques in terms of multiplies required every second, the first requires a staggering $2.28\times10^9$ multiplies every second, whilst the multirate solution requires only $8.56\times10^6$ multiplies per second, making it over 250 times more efficient!

In this particular case, using the Kaiser window for all of the filtering tasks, the minimum number of operations occurs when the decimation factor is 40, with only $5.225\times10^6$ multiplies per second. Unfortunately, there is no simple method to finding this minimum other than searching over a range of options for the smallest value.

3. Defining the combined dither noise and quantization error in the $z$-domain as $Q(z)$:

$$\begin{aligned}Y(z) &= X(z) - H(z)E(z) + Q(z)\\ E(z) &= Y(z) - (X(z) - H(z)E(z))\\ \Rightarrow E(z) &= \frac{1}{1-H(z)}(Y(z)-X(z))\\ \Rightarrow Y(z) &= X(z) - \frac{H(z)}{1-H(z)}(Y(z)-X(z)) + Q(z)\\ &= \left(1+\frac{H(z)}{1-H(z)}\right)X(z) - \frac{H(z)}{1-H(z)}Y(z) + Q(z)\\ \Rightarrow Y(z)\left(1+\frac{H(z)}{1-H(z)}\right) &= \frac{X(z)}{1-H(z)} + Q(z)\\ \Rightarrow Y(z) &= X(z) + (1-H(z))Q(z)\end{aligned}$$

Let $G(z) = 1-H(z) = 1-2z^{-1}+z^{-2} = (1-z^{-1})^2$. Then

$$\begin{aligned}G(\omega) &= \left(1-e^{-j\omega}\right)^2\\ &= e^{-j\omega}\left(e^{j\omega/2}-e^{-j\omega/2}\right)^2\\ &= e^{-j\omega}\left(2j\sin\left(\frac{\omega}{2}\right)\right)^2\\ &= -4e^{-j\omega}\sin^2\left(\frac{\omega}{2}\right)\end{aligned}$$

The noise density spectrum at the ADC output is given by

$$\Gamma_{nn}(\omega) = |G(\omega)|^2\sigma^2$$

where $\sigma^2$ is the power of the dither and quantization noise

$$\Rightarrow \Gamma_{nn}(\omega) = 16\sigma^2\sin^4\left(\frac{\omega}{2}\right)$$

Thus, at low frequencies, the dither and quantization noise is attenuated.

4. The target SQNR for the different converters can be found using the given equation, setting $b$ to the desired value of $B$, with $O = 1$. This gives

| $B$ | $\text{SQNR}_{\text{req}}$ |
|---|---|
| 8 | 49.92 dB |
| 12 | 74 dB |
| 16 | 98.08 dB |
| 24 | 146.24 dB |
| 32 | 194.4 dB |

For the single bit converter, with $b = 1$, when one bit of dither noise is added, the resulting SQNR is 4.77 dB, as the dither noise reduces the SQNR by 3.01 dB. The gain in SQNR required to meet the values in the table above is achieved through the oversampling and noise shaping term given on the formula sheet: $10(2L + 1)\log_{10}(O)$. Thus, we solve:

$$4.77 + 10(2L+1)\log_{10}(O) = \text{Target SQNR}$$
$$\Rightarrow O = 10^{\frac{\text{Target SQNR}-4.77}{10(2L+1)}}$$

The answers should be rounded up to the nearest integer as we should aim for integer oversampling to keep the decimation process as simple as possible. Calculating values for the table, below, can be made more efficient by using the TABLE function of the standard exam calculator. To do this, substitute the equation for the Target SQNR into the above expression, giving:

$$O = 10^{\frac{6.02b-3.01}{10(2L+1)}}$$

and then use the calculator to determine values for different numbers of bits, or different oversampling levels. If you have lost the manual for your calculator, it can be found at `http://support.casio.com/en/manual/manualfile.php?cid=004009051`. Details for using TABLE are on pages E-24.

| $B$ | No shaping | $1^{\text{st}}$ order | $2^{\text{nd}}$ order | $3^{\text{rd}}$ order |
|---|---|---|---|---|
| 8 | 32735 | 32 | 8 | 5 |
| 12 | 8375293 | 204 | 25 | 10 |
| 16 | $2.14 \times 10^9$ | 1290 | 74 | 22 |
| 24 | $1.40 \times 10^{14}$ | 51960 | 676 | 105 |
| 32 | $9.18 \times 10^{18}$ | 2094113 | 6203 | 512 |

Clearly, for the high accuracy converters, $2^{\text{nd}}$ or $3^{\text{rd}}$ order noise shaping is required to maintain a reasonable oversampling rate.

5. (a) Start with the definition of the matched filter:

$$h(n) = f(n_m - n) = f(N - n); 0 \le n \le N$$

The output of the filter is given by the convolution of its impulse response with the input signal:

$$y(n) = \sum_{m=0}^{N} h(m)f(n-m); 0 \le n \le N$$

Since $f(n) = 0$ for $n < 0$, then

$$= \sum_{m=0}^{n} h(m)f(n-m); 0 \le n \le N$$
$$= \sum_{m=0}^{n} f(N-m)f(n-m); 0 \le n \le N$$

This covers the first $N+1$ outputs. For the next $N$ outputs, the expression changes:

$$y(n) = \sum_{m=0}^{N} h(m)f(n-m); N \leq n \leq 2N$$

Since $f(n) = 0$ for $n > N$, then

$$\begin{aligned}
&= \sum_{m=n-N}^{N} h(m)f(n-m); N \leq n \leq 2N \\
&= \sum_{m=n-N}^{N} f(N-m)f(n-m); N \leq n \leq 2N
\end{aligned}$$

Make the substitution $i = m - n + N$, then

$$\begin{aligned}
&= \sum_{i=0}^{2N-n} f(N-i-n+N)f(n-i-n+N); N \leq n \leq 2N \\
&= \sum_{i=0}^{2N-n} f(2N-i-n)f(N-i); N \leq n \leq 2N
\end{aligned}$$

To show that $y(n)$ is symmetric about point $N$, we need to show that $y(2N-n) = y(n)$. Starting with the first expression:

$$y(2N-n) = \sum_{m=0}^{2N-n} f(N-m)f(2N-n-m); 0 \leq 2N-n \leq N$$

Now re-arrange the limits

$$\begin{aligned}
y(2N-n) &= \sum_{m=0}^{2N-n} f(N-m)f(2N-n-m); -2N \leq -n \leq -N \\
y(2N-n) &= \sum_{m=0}^{2N-n} f(N-m)f(2N-n-m); N \leq n \leq 2N
\end{aligned}$$

This is identical to the second derived expression, thus proving that $y(2N-n) = y(n)$ as required.

(b) If $f(n)$ has even symmetry about $\frac{N}{2}$, then we can write $f(N-n) = f(n)$. The matched filter, $h(n)$ is defined as:

$$h(n) = f(N-n) = f(n)$$

, thus for a symmetric signal, the matched filter is the signal itself.