LATEX Definitions are here.

# 伴随函子的 unit 与 counit

$$
\begin{tikzcd}
\mathsf{C} \arrow[loop above, "{}_{:\mathsf{C}}\mathrm{Id}"] \arrow[r, bend right, "R"'] & \mathsf{D} \arrow[loop above, "{}_{:\mathsf{D}}\mathrm{Id}"] \arrow[l, bend right, "L"']
\end{tikzcd}
$$

伴随函子 : 对任意 c 和 d 有 $(\mathsf{d} \xrightarrow{\mathsf{D}} \mathsf{c}R) \overset{\mathsf{Set}}{\cong} (\mathsf{d}L \xrightarrow{\mathsf{C}} \mathsf{c})$ 。如此

不难看出这其实蕴含着一个二元的自然同构 $\phi_2$，见下：

$$\begin{aligned}
\phi_2 &: (_ \xrightarrow{D} _R) \xrightarrow{\overset{C}{(D\times C)} \xrightarrow{C} Set} (_L \xrightarrow{C} _)\\
(_ \,.\, c)^{\phi_2} &: (_ \xrightarrow{D} cR) \xrightarrow{D \xrightarrow{C} Set} (_L \xrightarrow{C} c)\\
(d \,.\, _)^{\phi_2} &: (d \xrightarrow{D} _R) \xrightarrow{C \xrightarrow{C} Set} (dL \xrightarrow{C} _)
\end{aligned}$$

套用反变米田引理我们便可获得

$$\underbrace{\left((\_ \xrightarrow{D} cR) \xrightarrow{D^{op} \xrightarrow{Cat} Set} (\_L \xrightarrow{C} c)\right)}_{\text{一堆自然变换}} \overset{Set}{\cong} \underbrace{(cRL \xrightarrow{C} c)}_{\text{一堆元素}}$$

由反变米田引理的证明可知：对每个左侧集合中的自然同构 $(\_ \,.\, c)^{\phi_2}$
都会有一个右侧集合中的箭头与之相对应，即 $(\mathrm{id}_{:cR})(cR \,.\, c)^{\phi_2} = c^{\varepsilon}$ 。
为何 $\varepsilon$ 构成自然变换呢？下方右图第二行的第二个节点说明了一切。
这两张图这其实就是反变米田引理证明的两个图拼在一起后的结果。

$$\begin{tikzcd}
cR \xrightarrow{D} cR \arrow[r, "(cR.c)^{\phi_2}"] \arrow[d, "f^{op}R \xrightarrow{D} cR"'] & cRL \xrightarrow{C} c \arrow[d, "f^{op}RL \xrightarrow{C} c"] \\
c'R \xrightarrow{D} cR \arrow[r, dashed, "(c'R.c)^{\phi_2}"] & c'RL \xrightarrow{C} c \\
c'R \xrightarrow{D} c'R \arrow[r, "(c'R.c')^{\phi_2}"'] \arrow[u, "c'R \xrightarrow{D} f^{op}R"] & c'RL \xrightarrow{C} c' \arrow[u, "c'RL \xrightarrow{C} f^{op}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_{:cR} \arrow[r, mapsto, "(cR.c)^{\phi_2}"] \arrow[d, mapsto, "f^{op}R \xrightarrow{D} cR"'] & \mathrm{id}_{:cR}(cR.c)^{\phi_2} = c^{\varepsilon} \arrow[d, mapsto, "f^{op}RL \xrightarrow{C} c = c^{(f^{op}RL \overset{C}{\circ} _)}"] \\
f^{op}R & f^{op}RL \overset{C}{\circ} c^{\varepsilon} = c'^{\varepsilon} \overset{D}{\circ} f^{op} \arrow[l, mapsto, "(c'R.c)^{\phi_2}"'] \\
\mathrm{id}_{:c'R} \arrow[r, mapsto, "(c'R.c')^{\phi_2}"'] \arrow[u, mapsto, "c'R \xrightarrow{D} f^{op}R"] & \mathrm{id}_{:c'R}(c'R.c')^{\phi_2} = c'^{\varepsilon} \arrow[u, "c'RL \xrightarrow{D} f^{op} = (c'RL)^{(_ \overset{D}{\circ} f^{op})}"']
\end{tikzcd}$$

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不难看出这其实蕴含着一个二元的自然同构 $\phi_1$，见下：

$$\begin{aligned}
\phi_1 &: (_L \xrightarrow{C} _) \xrightarrow{\overset{C}{(D\times C)} \xrightarrow{C} Set} (_ \xrightarrow{D} _R)\\
(_ \,.\, c)^{\phi_1} &: (_L \xrightarrow{C} c) \xrightarrow{D \xrightarrow{C} Set} (_ \xrightarrow{D} cR)\\
(d \,.\, _)^{\phi_1} &: (dL \xrightarrow{C} _) \xrightarrow{C \xrightarrow{C} Set} (d \xrightarrow{D} _R)
\end{aligned}$$

套用协变米田引理我们便可获得

$$\underbrace{\left((dL \xrightarrow{C} \_) \xrightarrow{C \xrightarrow{Cat} Set} (d \xrightarrow{D} \_R)\right)}_{\text{一堆自然变换}} \overset{Set}{\cong} \underbrace{(d \xrightarrow{D} dLR)}_{\text{一堆元素}}$$

由协变米田引理的证明可知：对每个左侧集合中的自然同构 $(d \,.\, \_)^{\phi_1}$
都会有一个右侧集合中的箭头与之相对应，即 $(\mathrm{id}_{:dL})(d \,.\, dL)^{\phi_1} = d^{\eta}$ 。
为何 $\eta$ 构成自然变换呢？下方右图第二行的第二个节点说明了一切。
这两张图这其实就是协变米田引理证明的两个图拼在一起后的结果。

$$\begin{tikzcd}
dL \xrightarrow{C} dL \arrow[r, "(d.dL)^{\phi_1}"] \arrow[d, "dL \xrightarrow{C} gL"'] & d \xrightarrow{D} dLR \arrow[d, "d \xrightarrow{D} gLR"] \\
dL \xrightarrow{C} d'L \arrow[r, dashed, "(d.d'L)^{\phi_1}"] & d \xrightarrow{D} d'LR \\
d'L \xrightarrow{C} d'L \arrow[r, "(d'.d'L)^{\phi_1}"'] \arrow[u, "gL \xrightarrow{C} d'L"] & d' \xrightarrow{D} d'LR \arrow[u, "g \xrightarrow{D} d'LR"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{id}_{:dL} \arrow[r, mapsto, "(d.dL)^{\phi_1}"] \arrow[d, "dL \xrightarrow{C} gL"'] & \mathrm{id}_{:dL}(d.dL)^{\phi_1} = d^{\eta} \arrow[d, "d \xrightarrow{D} gLR = d^{(_ \overset{D}{\circ} gLR)}"] \\
gL \arrow[r, "(d.d'L)^{\phi_1}"] & d^{\eta} \overset{D}{\circ} gLR = g \overset{D}{\circ} d'^{\eta} \\
\mathrm{id}_{:d'L} \arrow[r, mapsto, "(d'.d'L)^{\phi_1}"'] \arrow[u, "gL \xrightarrow{C} d'L"] & \mathrm{id}_{:d'L}(d'.d'L)^{\phi_1} = d'^{\eta} \arrow[u, "g \xrightarrow{D} d'LR = d'LR^{(g \overset{D}{\circ} _)}"']
\end{tikzcd}$$