LaTeX Definitions are here.

10 伴随函子

若有函子 $L : \mathsf{D} \xrightarrow{\mathsf{Cat}} \mathsf{C}$
以及函子 $R : \mathsf{C} \xrightarrow{\mathsf{Cat}} \mathsf{D}$, 即

$$\begin{tikzcd}
\mathsf{C} \arrow[loop above, "{}_{:\mathsf{C}}\mathrm{Id}"] \arrow[r, bend right, "R"'] & \mathsf{D} \arrow[loop above, "{}_{:\mathsf{D}}\mathrm{Id}"] \arrow[l, bend right, "L"']
\end{tikzcd}$$

那么规定

伴随函子的第一种定义

- $L \dashv R$ 当且仅当
  对任意 C 中对象 c
  及任意 D 中对象 d
  都有 $(\mathsf{d} \xrightarrow{\mathsf{D}} \mathsf{c}R) \overset{\mathsf{Set}}{\cong} (\mathsf{d}L \xrightarrow{\mathsf{C}} \mathsf{c})$ 。

假如确实有 $L \dashv R$, 那么不难得知

- 这里蕴含着一个二元的自然同构 ϕ_2，见下：

$$\phi_2 : (\_ \xrightarrow{\mathsf{D}} \_R) \xrightarrow{(\mathsf{D}\times\mathsf{C})\longrightarrow\mathsf{Set}} (\_L \xrightarrow{\mathsf{C}} \_)$$
$$(\_\,.\,c)^{\phi_2} : (\_ \xrightarrow{\mathsf{D}} cR) \xrightarrow{\mathsf{D}\overset{\mathsf{C}}{\longrightarrow}\mathsf{Set}} (\_L \xrightarrow{\mathsf{C}} c)$$
$$(d\,.\,\_)^{\phi_2} : (d \xrightarrow{\mathsf{D}} \_R) \xrightarrow{\mathsf{C}\overset{\mathsf{C}}{\longrightarrow}\mathsf{Set}} (dL \xrightarrow{\mathsf{C}} \_)$$

套用反变米田引理我们便可获得

$$\underbrace{\left((\_ \xrightarrow{\mathsf{D}} cR) \xrightarrow{\mathsf{D^{op}}\overset{\mathsf{Cat}}{\longrightarrow}\mathsf{Set}} (\_L \xrightarrow{\mathsf{C}} c)\right)}_{\text{一堆自然变换}} \overset{\mathsf{Set}}{\cong} \underbrace{(cRL \xrightarrow{\mathsf{C}} c)}_{\text{一堆元素}}$$

由反变米田引理的证明可知：对每个左侧集合中的自然同构 $(\_\,.\,c)^{\phi_2}$
右侧集合中都有一个箭头与之对应，即 ${}_{:cR}\mathrm{id}(cR\,.\,c)^{\phi_2} = c^{\varepsilon}$ 。如此

- $\varepsilon : R \overset{\mathsf{Cat}}{\circ} L \xrightarrow{\mathsf{C}\overset{\mathsf{Cat}}{\longrightarrow}\mathsf{C}} {}_{:\mathsf{C}}\mathrm{Id}$ 构成自然变换。

> 考虑任意 $f^{op} : c' \xrightarrow{\mathsf{C}} c$:
>
> $$\begin{tikzcd} cR \xrightarrow{\mathsf{D}} cR \arrow[r, "(cR.c)^{\phi_2}"] \arrow[d, "f^{op}R \xrightarrow{\mathsf{D}} cR"'] & cRL \xrightarrow{\mathsf{C}} c \arrow[d, "f^{op}RL \xrightarrow{\mathsf{C}} c"] \\ c'R \xrightarrow{\mathsf{D}} cR \arrow[r, dashed, "(c'R.c)^{\phi_2}"] & c'RL \xrightarrow{\mathsf{C}} c \\ c'R \xrightarrow{\mathsf{D}} c'R \arrow[r, "(c'R.c')^{\phi_2}"'] \arrow[u, "c'R \xrightarrow{\mathsf{D}} f^{op}R"] & c'RL \xrightarrow{\mathsf{C}} c' \arrow[u, "c'RL \xrightarrow{\mathsf{C}} f^{op}"'] \end{tikzcd}$$
>
> $$\begin{tikzcd} {}_{:cR}\mathrm{id} \arrow[r, maps to, "(cR.c)^{\phi_2}"] \arrow[d, maps to, "f^{op}R \xrightarrow{\mathsf{D}} cR"'] & {}_{:cR}\mathrm{id}(cR.c)^{\phi_2} = c^{\varepsilon} \arrow[d, maps to, "f^{op}RL \xrightarrow{\mathsf{C}} c = c^{(f^{op}RL \overset{\mathsf{C}}{\circ} \_)}"] \\ f^{op}R \arrow[r, dashed, maps to, "(c'R.c)^{\phi_2}"] & f^{op}RL \overset{\mathsf{C}}{\circ} c^{\varepsilon} = c'^{\varepsilon} \overset{\mathsf{D}}{\circ} f^{op} \\ {}_{:c'R}\mathrm{id} \arrow[r, maps to, "(c'R.c')^{\phi_2}"'] \arrow[u, maps to, "c'R \xrightarrow{\mathsf{D}} f^{op}R"] & {}_{:c'R}\mathrm{id}(c'R.c')^{\phi_2} = c'^{\varepsilon} \arrow[u, "c'RL \xrightarrow{\mathsf{D}} f^{op} = (c'RL)^{(\_ \overset{\mathsf{D}}{\circ} f^{op})}"'] \end{tikzcd}$$
>
> 上方右图的第二行的第二个节点说明了一切。这两张图
> 其实就是反变米田引理证明的两个图拼在一起后的结果。

- 这里蕴含着一个二元的自然同构 ϕ_1，见下：

$$\phi_1 : (\_L \xrightarrow{\mathsf{C}} \_) \xrightarrow{(\mathsf{D}\times\mathsf{C})\longrightarrow\mathsf{Set}} (\_ \xrightarrow{\mathsf{D}} \_R)$$
$$(\_\,.\,c)^{\phi_1} : (\_L \xrightarrow{\mathsf{C}} c) \xrightarrow{\mathsf{D}\overset{\mathsf{C}}{\longrightarrow}\mathsf{Set}} (\_ \xrightarrow{\mathsf{D}} cR)$$
$$(d\,.\,\_)^{\phi_1} : (dL \xrightarrow{\mathsf{C}} \_) \xrightarrow{\mathsf{C}\overset{\mathsf{C}}{\longrightarrow}\mathsf{Set}} (d \xrightarrow{\mathsf{D}} \_R)$$

套用协变米田引理我们便可获得

$$\underbrace{\left((dL \xrightarrow{\mathsf{C}} \_) \xrightarrow{\mathsf{C}\overset{\mathsf{Cat}}{\longrightarrow}\mathsf{Set}} (d \xrightarrow{\mathsf{D}} \_R)\right)}_{\text{一堆自然变换}} \overset{\mathsf{Set}}{\cong} \underbrace{(d \xrightarrow{\mathsf{D}} dLR)}_{\text{一堆元素}}$$

由协变米田引理的证明可知：对每个左侧集合中的自然同构 $(d\,.\,\_)^{\phi_1}$
右侧集合中都有一个箭头与之对应，即 ${}_{:dL}\mathrm{id}(d\,.\,dL)^{\phi_1} = d^{\eta}$ 。如此。

- $\eta : {}_{:\mathsf{D}}\mathrm{Id} \xrightarrow{\mathsf{D}\overset{\mathsf{Cat}}{\longrightarrow}\mathsf{D}} L \overset{\mathsf{Cat}}{\circ} R$ 构成自然变换。

> 考虑任意 $g : d \xrightarrow{\mathsf{D}} d'$:
>
> $$\begin{tikzcd} dL \xrightarrow{\mathsf{C}} dL \arrow[r, "(d.dL)^{\phi_1}"] \arrow[d, "dL \xrightarrow{\mathsf{C}} gL"'] & d \xrightarrow{\mathsf{D}} dLR \arrow[d, "d \xrightarrow{\mathsf{D}} gLR"] \\ dL \xrightarrow{\mathsf{C}} d'L \arrow[r, dashed, "(d.d'L)^{\phi_1}"] & d \xrightarrow{\mathsf{D}} d'LR \\ d'L \xrightarrow{\mathsf{C}} d'L \arrow[r, "(d'.d'L)^{\phi_1}"'] \arrow[u, "gL \xrightarrow{\mathsf{C}} d'L"] & d' \xrightarrow{\mathsf{D}} d'LR \arrow[u, "g \xrightarrow{\mathsf{D}} d'LR"'] \end{tikzcd}$$
>
> $$\begin{tikzcd} {}_{:dL}\mathrm{id} \arrow[r, maps to, "(d.dL)^{\phi_1}"] \arrow[d, maps to, "dL \xrightarrow{\mathsf{C}} gL"'] & {}_{:dL}\mathrm{id}(d.dL)^{\phi_1} = d^{\eta} \arrow[d, maps to, "d \xrightarrow{\mathsf{D}} gLR = d^{(\_ \overset{\mathsf{D}}{\circ} gLR)}"] \\ gL \arrow[r, dashed, maps to, "(d.d'L)^{\phi_1}"] & d^{\eta} \overset{\mathsf{D}}{\circ} gLR = g \overset{\mathsf{D}}{\circ} d'^{\eta} \\ {}_{:d'L}\mathrm{id} \arrow[r, maps to, "(d'.d'L)^{\phi_1}"'] \arrow[u, maps to, "gL \xrightarrow{\mathsf{C}} d'L"] & {}_{:d'L}\mathrm{id}(d'.d'L)^{\phi_1} = d'^{\eta} \arrow[u, "g \xrightarrow{\mathsf{D}} d'LR = d'LR^{(g \overset{\mathsf{D}}{\circ} \_)}"'] \end{tikzcd}$$
>
> 上方右图的第二行的第二个节点说明了一切。这两张图
> 其实就是协变米田引理证明的两个图拼在一起后的结果。

对于前面的 ε 和 η 我们有下述交换图成立 :

- $$\begin{tikzcd} L \arrow[r, "\eta \overset{???}{\circ} {:}L^{\iota}"] \arrow[dr, "{:}L^{\iota}"'] & L \overset{\mathsf{Cat}}{\circ} R \overset{\mathsf{Cat}}{\circ} L \arrow[d, "{:}L^{\iota} \overset{???}{\circ} \varepsilon"] \\ & L \end{tikzcd}$$
  $$\begin{tikzcd} \mathsf{C} \\ \mathsf{D} \arrow[u, bend left=60, "L"] \arrow[u, bend right=60, "L"'] \arrow[u, phantom, "\overset{{:}L^{\iota}}{\Rightarrow}"] \\ \mathsf{D} \arrow[u, bend left=60, "{:}_{\mathsf{D}}\mathrm{Id}"] \arrow[u, bend right=60, "L \overset{\mathsf{Cat}}{\circ} R"'] \arrow[u, phantom, "\overset{\eta}{\Rightarrow}"] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C} \\ \mathsf{C} \arrow[u, bend left=60, "R \overset{\mathsf{Cat}}{\circ} L"] \arrow[u, bend right=60, "{:}_{\mathsf{C}}\mathrm{Id}"'] \arrow[u, phantom, "\overset{\varepsilon}{\Rightarrow}"] \\ \mathsf{D} \arrow[u, bend left=60, "L"] \arrow[u, bend right=60, "L"'] \arrow[u, phantom, "\overset{{:}L^{\iota}}{\Rightarrow}"] \end{tikzcd}$$

- $$\begin{tikzcd} R \arrow[r, "{:}R^{\iota} \overset{???}{\circ} \eta"] \arrow[dr, "{:}R^{\iota}"'] & R \overset{\mathsf{Cat}}{\circ} L \overset{\mathsf{Cat}}{\circ} R \arrow[d, "\varepsilon \overset{???}{\circ} {:}R^{\iota}"] \\ & R \end{tikzcd}$$
  $$\begin{tikzcd} \mathsf{D} \\ \mathsf{D} \arrow[u, bend left=60, "{:}_{\mathsf{D}}\mathrm{Id}"] \arrow[u, bend right=60, "L \overset{\mathsf{Cat}}{\circ} R"'] \arrow[u, phantom, "\overset{\eta}{\Rightarrow}"] \\ \mathsf{C} \arrow[u, bend left=60, "R"] \arrow[u, bend right=60, "R"'] \arrow[u, phantom, "\overset{{:}R^{\iota}}{\Rightarrow}"] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{D} \\ \mathsf{C} \arrow[u, bend left=60, "R"] \arrow[u, bend right=60, "R"'] \arrow[u, phantom, "\overset{{:}R^{\iota}}{\Rightarrow}"] \\ \mathsf{C} \arrow[u, bend left=60, "R \overset{\mathsf{Cat}}{\circ} L"] \arrow[u, bend right=60, "{:}_{\mathsf{C}}\mathrm{Id}"'] \arrow[u, phantom, "\overset{\varepsilon}{\Rightarrow}"] \end{tikzcd}$$

伴随函子的第二种定义

假设我们不知道 L 和 R 构成一对伴随函子，并且有自然变换 $\varepsilon : R \overset{\mathsf{Cat}}{\circ} L \xrightarrow{\mathsf{C} \longrightarrow \mathsf{C}} {}_{:\mathsf{C}}\mathrm{Id}$ 和 $\eta : {}_{:\mathsf{D}}\mathrm{Id} \xrightarrow{\mathsf{D} \longrightarrow \mathsf{D}} L \overset{\mathsf{Cat}}{\circ} R$ 能够同时满足本页开头的两幅交换图，那么

-