Steady state HDG

$$\frac{1}{k}\underline{q} + \nabla u = 0 \quad \text{in } \Omega$$
$$\nabla\cdot(\underline{c}u + \underline{q}) = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

▷ Unsteady HDG

$$\frac{1}{k}\underline{q} + \nabla u = 0 \quad \text{in } \Omega \qquad \underline{q} = -k\nabla u$$
$$\frac{\partial u}{\partial t} + \nabla\cdot(\underline{c}u + \underline{q}) = f \quad \text{in } \Omega$$
$$u = g_D \text{ in } \partial\Omega_D$$
$$(\underline{q} + \underline{c}u)\cdot\underline{n} = g_N \text{ in } \partial\Omega_N$$

$$\frac{\partial u}{\partial t} \xrightarrow{HDG} \left(\frac{\partial u}{\partial t}, w\right)_K = \left(\frac{u_h^n - u_h^{n-1}}{\Delta t}, w\right)_K = \left(\frac{u_h^n}{\Delta t}, w\right)_K - \left(\frac{u_h^{n-1}}{\Delta t}, w\right)_K$$

▷ First order backward   still $\hat{q}_h = q_h - \tau(u_h - \hat{u}_h)n$

$$\int_K \frac{1}{k}\underline{q}_h^z\cdot\underline{v} - \int_K u_h^z \nabla\cdot\underline{v} + \int_{\partial K} \hat{u}_h^z\, \underline{v}\cdot\underline{n} = 0$$

$$\int_K \frac{u_h^z}{\Delta t} w + \int_K \nabla\cdot\underline{q}^z\cdot w - \int_K \underline{c}u_h^z\cdot\nabla w + \int_{\partial K} \tau u_h^z\cdot w + \int_{\partial K} \hat{u}^z(\underline{c}\cdot\underline{n} - \tau)\cdot w$$
$$= \int_K f w + \int_K \frac{u_h^{z-1}}{\Delta t} w$$

$$\int_{\partial K\setminus\partial K_D} \hat{u}_h^z(\underline{c}\cdot\underline{n} - \tau)\mu + \int_{\partial K\setminus\partial K_D} \underline{q}_h^z\cdot\underline{n}\mu + \int_{\partial K\setminus\partial K_D} \tau u_h^z \mu = 0$$
$$\int_{\partial K_D} \hat{u}_h^z \mu = \int_{\partial K_D} g_D \mu$$
$$\int_{\partial K_N} \underline{q}_h^z\cdot\underline{n}\cdot\mu = \int_{\partial K_N} g_N\cdot\mu$$

$$\begin{bmatrix} A_x & & B_x & C_x \\ & A_y & B_y & C_y \\ D_x & D_y & E & I \\ N_x & N_y & K & L \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ U \\ \hat{U} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F \\ G \end{bmatrix}$$

Entries to modify

E, N_x, N_y, G, F

$$\int_K \frac{u_h^z}{\Delta t} w = w_m^T @ shap @ diag\left(|J|_q \cdot w_q \cdot \frac{1}{\Delta t}\right) @ shap^T @ u_{hm}^z$$

added to E_K

$$\int_K \frac{u_h^{z-1}}{\Delta t} w = w_m^T @ shap @ diag\left(|J|_q \cdot w_q \cdot \frac{1}{\Delta t}\right) @ shap^T @ u_{hm}^{z-1}$$

added to F_K (only when source exists).

$$\int_{\partial K_N} \underline{q_h^z} \cdot \underline{n} \cdot \mu = \mu_m^T @ shld @ diag(|I|_q \cdot w_q) @ shld^T @ (q_{hx}^z n_x + q_{hy}^z n_y)_m$$

added components to N_x & N_y respectively

$$\int_{\partial K_N} g_N \cdot \mu = \mu_m^T @ shld @ diag(|I|_q \cdot w_q) @ shld^T @ (g_N)_m$$

added to G

$$\frac{1}{k}\cdot\underline{q} + \nabla u = 0 \quad \text{in } \Omega \qquad \underline{q} = -k\nabla u$$

$$\frac{\partial u}{\partial t} + \nabla\cdot(\underline{c}\,u + \underline{q}) = f \quad \text{in } \Omega$$

$$u = g_D \text{ in } \partial\Omega_D$$

$$(\underline{q} + \underline{c}\,u)\cdot\underline{n} = g_N \text{ in } \partial\Omega_N$$

Newton:

$$\frac{1}{k}\cdot\underline{q} + \nabla u = 0 \quad \text{in } \Omega \qquad \underline{q} = -k\nabla u$$

$$\frac{\partial u}{\partial t} + \nabla\cdot(\underline{f(u)} + \underline{q}) = f \quad \text{in } \Omega \qquad \underline{f(u)} = \tfrac{1}{2}u^2\hat{i} + \tfrac{1}{2}u^2\hat{j}$$

$$u = g_D \text{ in } \partial\Omega_D$$

$$(\underline{q} + \underline{c}\,u)\cdot\underline{n} = g_N \text{ in } \partial\Omega_N$$

$$\nabla\cdot(\underline{c}\,u) = \nabla\cdot(c_x u\hat{i} + c_y u\hat{j}) = c_x\frac{\partial u}{\partial x} + c_y\frac{\partial u}{\partial y}$$

$$\nabla\cdot\left(\tfrac{1}{2}u^2\hat{i} + \tfrac{1}{2}u^2\hat{j}\right) = \frac{\partial\left(\frac{1}{2}u^2\right)}{\partial x} + \frac{\partial\left(\frac{1}{2}u^2\right)}{\partial y} = u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y}$$

$$\int_K \frac{1}{k}\underline{q}_h^2\cdot\underline{v} - \int_K u_h^2\nabla\cdot\underline{v} + \int_{\partial K}\hat{u}_h^2\,\underline{v}\cdot\underline{n} = 0 \quad ①$$

$$\int_K \nabla\cdot(\underline{f(u)} + \underline{q})\,w = \int_{\partial K}\underbrace{(\underline{f(u)} + \underline{q})}_{\underline{f}}\cdot w\cdot\underline{n}\,ds - \int_K \nabla w\cdot(\underline{f(u)} + \underline{q})$$

$$\hat{\underline{f}} = \widehat{\underline{f(u)}} + \hat{\underline{q}} = \widehat{\underline{f(u)}} + \underline{q} + \tau(u - \hat{u})\cdot\underline{n}$$

$$\int_{\partial K}\hat{\underline{f}}\cdot w\cdot\underline{n}\,ds = \int_{\partial K}\widehat{\underline{f(u)}}\cdot w\cdot\underline{n} + \int_{\partial K}\underline{q}\cdot\underline{n}\cdot w + \int_{\partial K}\tau u w - \int_{\partial K}\tau\hat{u}w$$

$$\int_K \nabla\cdot(\underline{f(u)} + \underline{q})\,w = \int_{\partial K}\widehat{\underline{f(u)}}\cdot w\cdot\underline{n} + \int_K \nabla\cdot(\underline{q}\,w) + \int_{\partial K}\tau u w - \int_{\partial K}\tau\hat{u}w$$

$$- \int_K \nabla w\cdot\underline{f(u)} - \int_K \nabla w\cdot\underline{q}$$

$$\int_K \nabla\cdot(\underline{f}(u)+\underline{q})w = \int_{\partial K}\hat{\underline{f}(u)}\cdot w\cdot\underline{n} + \int_K \nabla\cdot(\underline{q}\,w) + \int_{\partial K}\tau u w - \int_{\partial K}\tau\hat{u}w - \int_K \nabla w\cdot\underline{f}(u) - \int_K \nabla w\cdot\underline{q}$$

$$= \int_{\partial K}\hat{\underline{f}(u)}\cdot w\cdot\underline{n} + \int_K \nabla\cdot\underline{q}\cdot w + \int_{\partial K}\tau u w - \int_{\partial K}\tau\hat{u}w - \int_K \nabla w\cdot\underline{f}(u)$$

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$$\int_K \frac{u_h^z}{\Delta t}w + \int_K \nabla\cdot\underline{q}^z\cdot w - \int_K \underline{f}(u)\cdot\nabla w + \int_{\partial K}\tau u_h^z w + \int_{\partial K}(\hat{\underline{f}(u)}^z\cdot\underline{n} - \tau\hat{u}^z)w = \int_K f w + \int_K \frac{u_h^{z-1}}{\Delta t}w$$

$$\int_{\partial K\backslash\partial K_B}\hat{\underline{f}}\cdot\underline{n}\,\mu = 0 = \int_{\partial K\backslash\partial K_B}\hat{\underline{f}}(u)\cdot\underline{n}\cdot\mu + \underline{q}\cdot\underline{n}\cdot\mu + \tau u\mu - \tau\hat{u}\mu$$

$$\int_{\partial K\backslash\partial K_B}(\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}_h^z)\mu + \int_{\partial K\backslash\partial K_B}\underline{q}_h^z\cdot\underline{n}\cdot\mu + \int_{\partial K\backslash\partial K_B}\tau u_h^z\mu = 0$$ 3

$$\int_{\partial K_D}\hat{u}_h^z\mu = \int_{\partial K_D} g_D\,\mu$$ 3.2

$$\int_K \frac{1}{k}\underline{q}_h^z\cdot\underline{v} - \int_K u_h^z\nabla\cdot\underline{v} + \int_{\partial K}\hat{u}_h\,\underline{v}\cdot\underline{n} = \underbrace{\underline{0}}_{\underline{f}_1} = \underline{R}_1$$

$$\underbrace{\int_K \nabla\cdot\underline{q}^z\cdot w + \int_K \frac{u_h^z}{\Delta t}w - \int_K \underline{f}(u)\cdot\nabla w + \int_{\partial K}\tau u_h^z w + \int_{\partial K}(\hat{\underline{f}(u)}^z\cdot\underline{n} - \tau\hat{u}^z)w = \int_K f w + \int_K \frac{u_h^{z-1}}{\Delta t}w}_{\underline{f}_2} = \underline{R}_2$$

$$\int_{\partial K\backslash\partial K_B}\underline{q}_h^z\cdot\underline{n}\cdot\mu + \int_{\partial K\backslash\partial K_B}\tau u_h^z\mu + \int_{\partial K\backslash\partial K_B}(\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}_h^z)\mu = \underbrace{\underline{0}}_{\underline{f}_3} = \underline{R}_3$$

$$\int_{\partial K_D}\hat{u}_h^z\mu = \int_{\partial K_D} g_D\,\mu$$

$$\underline{U} = [\,q_x \;\; q_y \;\; u_h \;\; \hat{u}_h\,] \qquad \underline{R}_1(\underline{U}) = \underline{f}_1 = \underline{R}_1(\underline{U}^* + \underline{dU})$$

If linear $\underline{R}_1$, $\underline{R}_1(\underline{U}) = \underline{R}_1(\underline{U}^*) + \underline{R}_1(\underline{dU})$

If nonlinear $\underline{R}_1$, $\underline{R}_1(\underline{U}) \approx \underline{R}_1(\underline{U}^*) + \left.\frac{\partial\underline{R}_1}{\partial\underline{U}}\right|_{\underline{U}^*}\cdot\partial\underline{U}$

$$\underline{R}_1(\underline{U}^{**}) = \underline{R}_1(\underline{U}^*) + J(\underline{U}^*)\cdot\partial\underline{U}$$

$\rightarrow$ additional term to multiply

$$\int_K \frac{1}{k}\underline{q}_h^z\cdot\underline{v} - \int_K u_h^z\,\nabla\cdot\underline{v} + \int_{\partial K}\hat{u}_h^z\,\underline{v}\cdot\underline{n} = \overset{\to f_1}{\underline{0}} = \underline{R}_1$$

$$\partial R_1 = \int_K \frac{1}{k}\underline{v}\cdot\partial\underline{q}_h^z - \int_K \nabla\cdot\underline{v}\cdot\partial u_h^z + \int_{\partial K}\underline{v}\cdot\underline{n}\cdot\partial\hat{u}_h^z$$

$$\int_K \nabla\cdot\underline{q}_h^z\,w + \int_K \frac{u_h^z}{\Delta t}w - \int_K \underline{f}(u)\cdot\nabla w + \int_{\partial K}\tau u_h^z w + \int_{\partial K}(\hat{\underline{f}}(\hat{u})\cdot\underline{n} - \tau\hat{u}^z)w = \int_K f w + \int_K \frac{u_h^{z-1}}{\Delta t}w = \underline{R}_2$$

$$\partial R_2 = \int_K w\cdot\nabla\cdot\partial\underline{q}^z + \int_K \frac{w}{\Delta t}\partial u_h^z - \int_K \nabla w\cdot\left.\frac{\partial \underline{f}}{\partial u}\right|_{u^{*z}}\cdot\partial u_h^z + \int_{\partial K}\tau w\,\partial u_h^z + \int_{\partial K} w\cdot\underline{n}\cdot\left.\frac{\partial\hat{\underline{f}}}{\partial\hat{u}}\right|_{\hat{u}_h^{*z}}\cdot\partial\hat{u}_h$$
$$- \int_{\partial K}\tau w\cdot\partial\hat{u}_h^z$$

- Assume $\tau = 1$ for simplicity

$$\int_{\partial K\setminus\partial K_B}\underline{q}_h^z\cdot\underline{n}\,\mu + \int_{\partial K\setminus\partial K_B}\tau u_h^z\mu + \int_{\partial K\setminus\partial K_B}(\hat{\underline{f}}(u)\cdot\underline{n} - \tau\hat{u}_h)\mu = \underset{\to f_3}{\underline{0}} = \underline{R}_3$$

$$\partial R_3 = \int_{\partial K\setminus\partial K_B}\mu\cdot\underline{n}\cdot\partial\underline{q}_h^z + \int_{\partial K\setminus\partial K_B}\tau\mu\cdot\partial u_h^z + \int_{\partial K\setminus\partial K_B}\mu\cdot\underline{n}\left.\frac{\partial\hat{\underline{f}}}{\partial\hat{u}}\right|_{\hat{u}_h^{*z}}\cdot\partial\hat{u}_h - \int_{\partial K\setminus\partial K_B}\tau\mu\,\partial\hat{u}_h^z$$

$$\underline{f} = \left[\tfrac{1}{2}u^2, \tfrac{1}{2}u^2\right] \quad \hat{\underline{f}} = \left[\tfrac{1}{2}\hat{u}^2, \tfrac{1}{2}\hat{u}^2\right] \quad \frac{\partial f}{\partial \underline{u}} = [u, u] \quad \frac{\partial\hat{f}}{\partial\hat{u}} = [\hat{u}, \hat{u}]$$

$$\underline{R}_1(\underline{U}^{**}) = \underline{R}_1(\underline{U}^*) + J(\underline{U}^*)\cdot\partial\underline{U} \rightarrow J(\underline{U}^*)\cdot\partial\underline{U} = \underline{R}_1(\underline{U}^{**}) - \underline{R}_1(\underline{U}^*)$$

$$\underline{R}_1(U^{**}) = \begin{bmatrix}\underline{f_1}\\ \underline{f_2}\\ \underline{f_3}\end{bmatrix}^{z-1}$$

fixed value

$\underline{U}^* = \underline{U}^{z-1}$ initially

compute $\partial\underline{U}$

then $\underline{U}^* \mathrel{+}= \partial\underline{U}$

compute $\partial\underline{U}$

$\vdots$

until $\text{norm}(\partial\underline{U}) \leq 1e-7$

then $\underline{U}^* \longrightarrow \underline{U}^{**} \longrightarrow \underline{U}^z$

Then repeat for the next time step

★ Collect terms to be altered

→ Residual Evaluation:

$$\underline{R_1} = \int_K \frac{1}{k}\underline{q}_h^z\cdot\underline{v} - \int_K u_h^z \nabla\cdot\underline{v} + \int_{\partial K} \hat{u}_h^z\, \underline{v}\cdot\underline{n} = \underline{0}$$

$$\underline{R_2} = \int_K \nabla\cdot\underline{q}^z w + \int_K \frac{u_h^z}{\Delta t} w \underbrace{- \int_K \underline{f}(u)^z\cdot\nabla w}_{\triangle 1} + \int_{\partial K} \tau u_h^z w + \underbrace{\int_{\partial K} (\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}^z) w}_{\triangle 2} = \int_K f w + \int_K \frac{u_h^{z-1}}{\Delta t} w$$

$$\underline{R_3} = \int_{\partial K\setminus\partial K_B} \underline{q}_h^z\cdot\underline{n}\,\mu + \int_{\partial K\setminus\partial K_B} \tau u_h^z \mu + \underbrace{\int_{\partial K\setminus\partial K_B} (\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}_h^z)\mu}_{\triangle 3} = \underline{0}$$

$\triangle 1$ $\quad -\int_K \underline{f}(u)^z\cdot\nabla w \longrightarrow -\int_K \underline{c}\, u_h\cdot\nabla w \longrightarrow$ E matrix

$\triangle 2$ $\quad +\int_{\partial K} (\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}^z) w \longrightarrow +\int_{\partial K} \hat{u}(\underline{c}\cdot\underline{n} - \tau)\cdot w \longrightarrow$ I matrix

$\triangle 3$ $\quad +\int_{\partial K\setminus\partial K_B} (\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}_h^z)\mu \longrightarrow +\int_{\partial K\setminus\partial K_D} \hat{u}_h(\underline{c}\cdot\underline{n} - \tau)\mu \longrightarrow$ L matrix

$$\begin{aligned}
-\int_K \underline{f}(u)^z\cdot\nabla w = -w_{mj}^T\cdot\Big[& \mathrm{shap}_\xi \,@\, \mathrm{diag}\left(w_{qi}\cdot\tfrac{\partial y}{\partial\eta}\, q_i\right) @\, \mathrm{shap}^T\cdot(f_x^z(u)_m) \\
& - \mathrm{shap}_\eta \,@\, \mathrm{diag}\left(w_{qi}\tfrac{\partial y}{\partial\xi}\, q_i\right) @\, \mathrm{shap}^T\cdot(f_x^z(u)_m) \\
& - \mathrm{shap}_\xi \,@\, \mathrm{diag}\left(w_{qi}\tfrac{\partial x}{\partial\eta}\, q_i\right) @\, \mathrm{shap}^T\cdot(f_y^z(u)_m) \\
& + \mathrm{shap}_\eta \,@\, \mathrm{diag}\left(w_{qi}\tfrac{\partial x}{\partial\xi}\, q_i\right) @\, \mathrm{shap}^T\cdot(f_y^z(u)_m)\Big]
\end{aligned}$$

$$+\int_{\partial K} (\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}^z) w$$

$$= w_{mj\in\partial K}^T\cdot \mathrm{shld} \,@\, \mathrm{diag}(|I|_{q_i}\cdot w_{qi}) \,@\, \mathrm{shld}^T\cdot(\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}^z)_m$$

$$+\int_{\partial K\setminus\partial K_B} (\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}^z)\mu$$

$$= \mu_{mj\in\partial K}^T\cdot \mathrm{shld} \,@\, \mathrm{diag}(|I|_{q_i}\cdot w_{qi}) \,@\, \mathrm{shld}^T\cdot(\hat{\underline{f}}(u)^z\cdot\underline{n} - \tau\hat{u}^z)_m$$

$\rightarrow$ Jacobian Evaluation

$$\partial R_1 = \int_K \frac{1}{k}\underline{v}\cdot d\underline{q_h}^z - \int_K \nabla\cdot\underline{v}\cdot du_h^z + \int_{\partial K} v\underline{n}\cdot\partial\hat{u}_h^z$$

$$\partial R_2 = \int_K w\cdot\nabla\cdot\partial\underline{q}^z + \int_K \frac{w}{\Delta t}\partial u_h^z \underbrace{- \int_K \nabla w\cdot\frac{\partial f}{\partial u}\Big|_{u^{*z}}\cdot\partial u_h^z}_{1} + \int_{\partial K}\tau w\,\partial u_h^z + \int_{\partial K} w\cdot\underline{n}\cdot\frac{\partial\hat{f}}{\partial\hat{u}}\Big|_{\hat{u}_h^{*z}}\cdot\partial\hat{u}_h \underbrace{- \int_{\partial K}\tau w\cdot\partial\hat{u}_h^z}_{2}$$

$$\partial R_3 = \int_{\partial K\backslash\partial K_B}\mu\cdot\underline{n}\cdot\partial q_h^z + \int_{\partial K\backslash\partial K_B}\tau\mu\cdot\partial u_h^z + \underbrace{\int_{\partial K\backslash\partial K_B}\mu\cdot\underline{n}\frac{\partial\hat{f}}{\partial\hat{u}}\Big|_{\hat{u}_h^{*z}}\cdot\partial\hat{u}_h}_{3} - \int_{\partial K\backslash\partial K_B}\tau\mu\,\partial\hat{u}_h^z$$

$$-\int_K \nabla w\cdot\frac{\partial f}{\partial u}\Big|_{u^{*z}}\cdot\partial u_h^z \rightarrow -\int_K \underline{c}\,u_h\cdot\nabla w \longrightarrow E$$

$$\int_{\partial K}\partial\hat{u}_h^z\left(\underline{n}\cdot\frac{\partial\hat{f}}{\partial\hat{u}}\Big|_{\hat{u}_h^{*z}} - \tau\right)w \Rightarrow \int_{\partial K}\hat{u}(\underline{c}\cdot\underline{n}-\tau)\cdot w \longrightarrow I$$

$$\int_{\partial K\backslash\partial K_B}\partial\hat{u}_h^z\left(\underline{n}\cdot\frac{\partial\hat{f}}{\partial\hat{u}}\Big|_{\hat{u}_h^{*z}} - \tau\right)\mu \Rightarrow \int_{\partial K\backslash\partial K_B}\hat{u}_h(\underline{c}\cdot\underline{n}-\tau)\mu \longrightarrow L$$

$$\begin{aligned}
-\int_K \nabla w\cdot\frac{\partial f}{\partial u}\Big|_{u^{*z}}\cdot\partial u_h^z = -w_{mj}^T\cdot\Big[& shap_\xi \;@\; diag\Big(w_{qi}\cdot\frac{\partial y}{\partial\eta}{}_{qi}\Big) \;@\; shap^T\cdot\Big(\frac{\partial f_x}{\partial u}\Big|_{u^{*z}}\cdot\partial u_h^z\Big)_m \\
& - shap_\eta \;@\; diag\Big(w_{qi}\frac{\partial y}{\partial\xi}{}_{qi}\Big) \;@\; shap^T\cdot\Big(\frac{\partial f_x}{\partial u}\Big|_{u^{*z}}\cdot\partial u_h^z\Big)_m \\
& - shap_\xi \;@\; diag\Big(w_{qi}\frac{\partial x}{\partial\eta}{}_{qi}\Big) \;@\; shap^T\cdot\Big(\frac{\partial f_y}{\partial u}\Big|_{u^{*z}}\cdot\partial u_h^z\Big)_m \\
& + shap_\eta \;@\; diag\Big(w_{qi}\frac{\partial x}{\partial\xi}{}_{qi}\Big) \;@\; shap^T\cdot\Big(\frac{\partial f_y}{\partial u}\Big|_{u^{*z}}\cdot\partial u_h^z\Big)_m\Big]
\end{aligned}$$

$$\int_{\partial K}\partial\hat{u}_h^z\left(\underline{n}\cdot\frac{\partial\hat{f}}{\partial\hat{u}}\Big|_{\hat{u}_h^{*z}} - \tau\right)w$$

$$= w_{mj\in\partial K}^T\cdot shld \;@\; diag(|J|_{qi}\cdot w_{qi}) \;@\; shld^T\cdot\left(\partial\hat{u}_h^z\left(\underline{n}\cdot\frac{\partial\hat{f}}{\partial\hat{u}}\Big|_{\hat{u}_h^{*z}} - \tau\right)\right)_m$$

$$\int_{\partial K\backslash\partial K_B}\partial\hat{u}_h^z\left(\underline{n}\cdot\frac{\partial\hat{f}}{\partial\hat{u}}\Big|_{\hat{u}_h^{*z}} - \tau\right)\mu$$

$$= \mu_{mj\in\partial K}^T\cdot shld \;@\; diag(|J|_{qi}\cdot w_{qi}) \;@\; shld^T\cdot\left(\partial\hat{u}_h^z\left(\underline{n}\cdot\frac{\partial\hat{f}}{\partial\hat{u}}\Big|_{\hat{u}_h^{*z}} - \tau\right)\right)_m$$

$$J(\underline{U}^*)\cdot\partial\underline{U} = \underline{R}_1(\underline{U}^{**}) - \underline{R}_1(\underline{U}^*)$$

$$\underbrace{\begin{bmatrix} A_x & 0 & B_x & C_x \\ 0 & A_y & B_y & C_y \\ D_x & D_y & E & I \\ N_x & N_y & K & L \end{bmatrix}}_{J^*} \begin{bmatrix} \partial q_x \\ \partial q_y \\ \partial U_n \\ \partial \hat{U}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ F \\ 0 \end{bmatrix}}_{Z^{-1}} - \underbrace{\begin{bmatrix} A_x & 0 & B_x & C_x \\ 0 & A_y & B_y & C_y \\ D_x & D_y & E & I \\ N_x & N_y & K & L \end{bmatrix}}_{R^*} \begin{bmatrix} q_x^* \\ q_y^* \\ U_n^* \\ \hat{U}_n^* \end{bmatrix}$$

J^* & R^* constructed using $\underbrace{U_n^* \,\&\, \hat{U}_n^*}$

$$F_{tot} = \begin{bmatrix} F_{tot}\,q_x \\ F_{tot}\,q_y \\ F_{tot}\,U_n \\ F_{tot}\,\hat{U}_n \end{bmatrix}$$

$$\begin{bmatrix} A_x & 0 & B_x \\ 0 & A_y & B_y \\ D_x & D_y & E \end{bmatrix} \begin{bmatrix} \partial q_x \\ \partial q_y \\ \partial U_n \end{bmatrix} + \begin{bmatrix} C_x \\ C_y \\ I \end{bmatrix} [\partial \hat{U}_n] = \begin{bmatrix} F_t q_x \\ F_t q_y \\ F_t U_n \end{bmatrix}$$

$$\begin{bmatrix} \partial q_x \\ \partial q_y \\ \partial U_n \end{bmatrix} = \begin{bmatrix} A_x & 0 & B_x \\ 0 & A_y & B_y \\ D_x & D_y & E \end{bmatrix}^{-1} \begin{bmatrix} F_t q_x \\ F_t q_y \\ F_t U_n \end{bmatrix} - \begin{bmatrix} A_x & 0 & B_x \\ 0 & A_y & B_y \\ D_x & D_y & E \end{bmatrix}^{-1} \begin{bmatrix} C_x \\ C_y \\ I \end{bmatrix} [\partial \hat{U}_n]$$

$$[N_x \; N_y \; K] \begin{bmatrix} \partial q_x \\ \partial q_y \\ \partial U_n \end{bmatrix} + [L][\partial \hat{U}_n] = [\mathbf{F}_t \hat{U}_n]$$

$$[N_x \; N_y \; K] \begin{bmatrix} A_x & 0 & B_x \\ 0 & A_y & B_y \\ D_x & D_y & E \end{bmatrix}^{-1} \begin{bmatrix} F_t q_x \\ F_t q_y \\ F_t U_n \end{bmatrix} - [N_x \; N_y \; K] \begin{bmatrix} A_x & 0 & B_x \\ 0 & A_y & B_y \\ D_x & D_y & E \end{bmatrix}^{-1} \begin{bmatrix} C_x \\ C_y \\ I \end{bmatrix} [\partial \hat{U}_n]$$

$$+ [L][\partial \hat{U}_n] = [\mathbf{F}_t \hat{U}_n]$$

$$\left([L] - [N_x \; N_y \; K] \begin{bmatrix} A_x & 0 & B_x \\ 0 & A_y & B_y \\ D_x & D_y & E \end{bmatrix}^{-1} \begin{bmatrix} C_x \\ C_y \\ I \end{bmatrix} \right) [\partial \hat{U}_n] = [\mathbf{F}_t \hat{U}_n] - [N_x \; N_y \; K] \begin{bmatrix} A_x & 0 & B_x \\ 0 & A_y & B_y \\ D_x & D_y & E \end{bmatrix}^{-1} \begin{bmatrix} F_t q_x \\ F_t q_y \\ F_t U_n \end{bmatrix}$$

all matrices jacobian here

dbc should be set to 0 for all $\partial \hat{u}_n$

Back compute ∂q_x ∂q_y ∂u_n

$$\begin{bmatrix} A_x & 0 & B_x \\ 0 & A_y & B_y \\ D_x & D_y & E \end{bmatrix} \begin{bmatrix} \partial q_x \\ \partial q_y \\ \partial u_n \end{bmatrix} + \begin{bmatrix} C_x \\ C_y \\ I \end{bmatrix} [\partial \hat{u}_n] = \begin{bmatrix} F_{tq_x} \\ F_{tq_y} \\ F_{tu_n} \end{bmatrix}$$

$$\begin{bmatrix} \partial q_x \\ \partial q_y \\ \partial u_n \end{bmatrix} = \begin{bmatrix} A_x & 0 & B_x \\ 0 & A_y & B_y \\ D_x & D_y & E \end{bmatrix}^{-1} @ \left(\begin{bmatrix} F_{tq_x} \\ F_{tq_y} \\ F_{tu_n} \end{bmatrix} - \begin{bmatrix} C_x \\ C_y \\ I \end{bmatrix} [\partial \hat{u}_n] \right)$$

Test case for nonlinear for equation.

$$\cancel{\frac{\partial u}{\partial t}} + \nabla \cdot \left(\left\langle \tfrac{1}{2}u^2, \tfrac{1}{2}u^2 \right\rangle - k \nabla u \right) = f \qquad -k\nabla u = \left\langle -k\frac{\partial u}{\partial x}, -k\frac{\partial u}{\partial y} \right\rangle$$

0

solve for steady state solution

$u = \sin(\pi x)\sin(\pi y)$ $\quad \frac{\partial u}{\partial x} = \pi\cos(\pi x)\sin(\pi y)$ $\quad \frac{\partial u}{\partial y} = \pi\sin(\pi x)\cos(\pi y)$

$\frac{\partial^2 u}{\partial x^2} = -\pi^2 \sin(\pi x)\sin(\pi y)$ $\quad \frac{\partial^2 u}{\partial y^2} = -\pi^2 \sin(\pi x)\sin(\pi y)$

$$\frac{1}{2}\frac{\partial u^2}{\partial x} + \frac{1}{2}\frac{\partial u^2}{\partial y} - k\frac{\partial^2 u}{\partial x^2} - k\frac{\partial^2 u}{\partial y^2} = f$$

$$u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} - k\frac{\partial^2 u}{\partial x^2} - k\frac{\partial^2 u}{\partial y^2} = f$$

other nonlinear advection

e^u $\quad \frac{\partial e^u}{\partial x} = e^u \frac{\partial u}{\partial x}$

$\frac{1}{3}u^3$ $\quad \frac{1}{3}3u^2 = u^2$