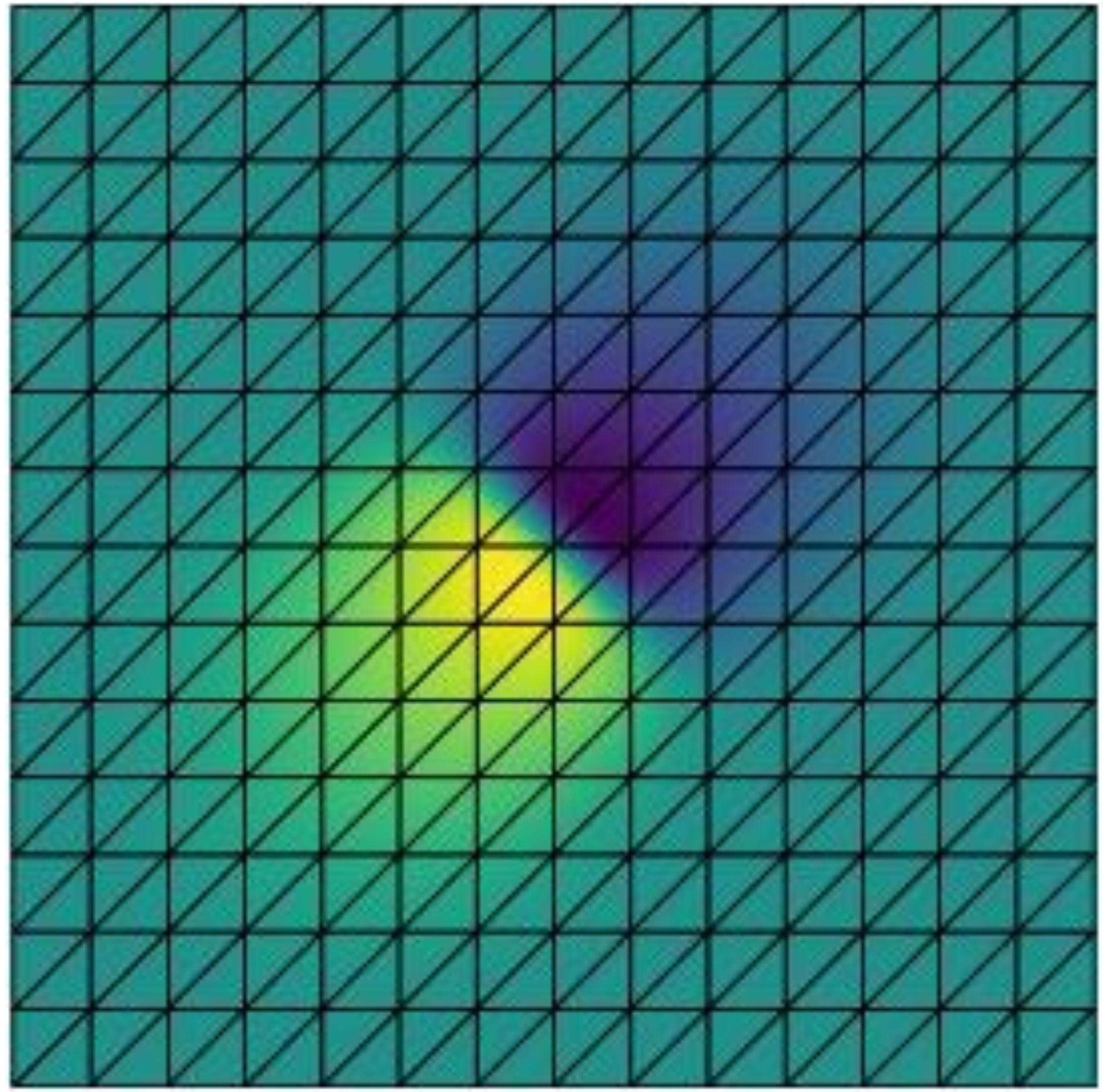
HDG Extension to Unsteady 2d Burgers' Equation

16.930 Final Project

Yang Chen



Unsteady 2d Burgers' Equation

- Equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{f}(\boldsymbol{u}) - \kappa \cdot \nabla u) = s \qquad in\ K$$
$$u = g_D \qquad on\ \partial K$$

s: $Source\ \ Term$
$\boldsymbol{f}(\boldsymbol{u})$: $Nonlinear\ Advection\ Term$

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- *Example Case*

$$\boldsymbol{f}(\boldsymbol{u}) = [0.5u^2, 0.5u^2]$$
$$\frac{d\boldsymbol{f}(\boldsymbol{u})}{du} = [u, u]$$

X-direction advection speed

Y-direction advection speed

Unsteady 2d Burgers' Equation

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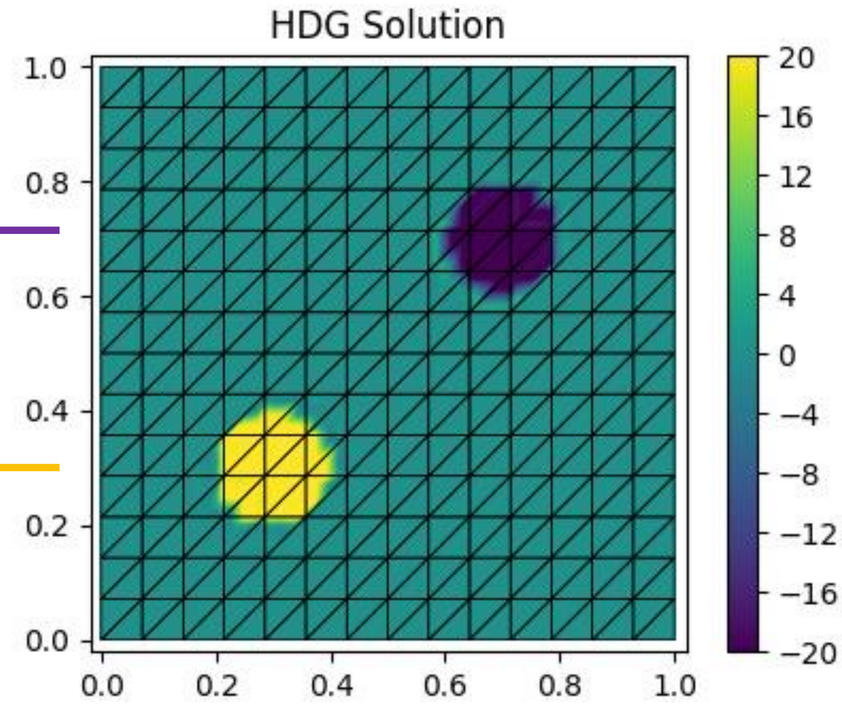
$$\frac{d\boldsymbol{f}(\boldsymbol{u})}{du} = [u, u]$$
$$g_D = 0, \quad s = 0$$

Purple region advected at speed [-20, -20] to lower left

Yellow region advected at speed [20, 20] to upper right



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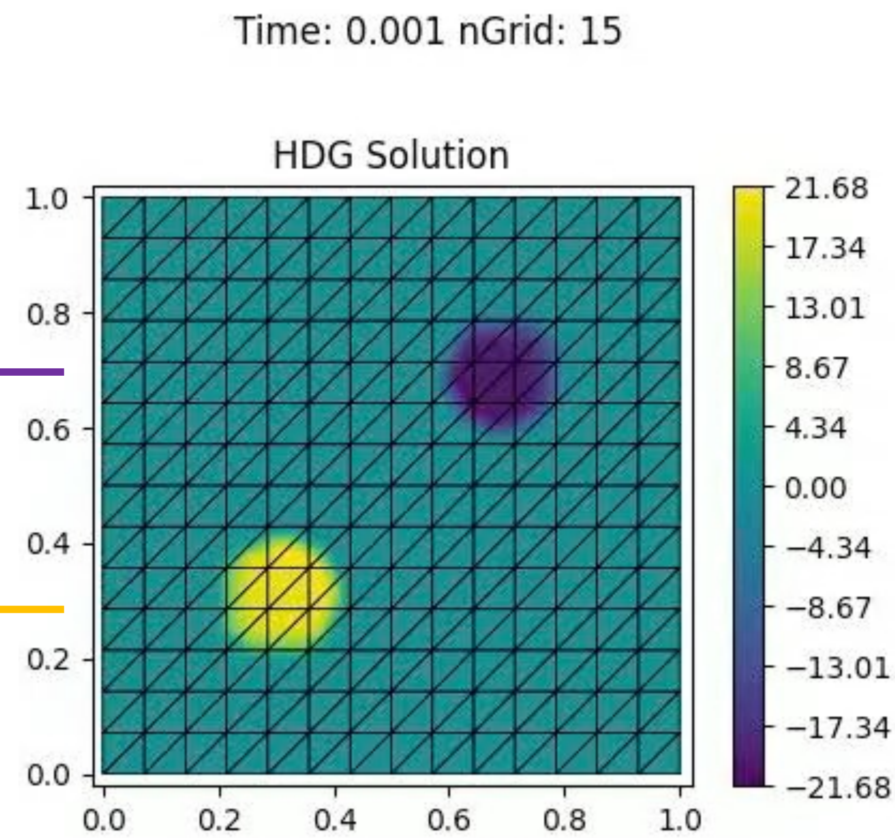
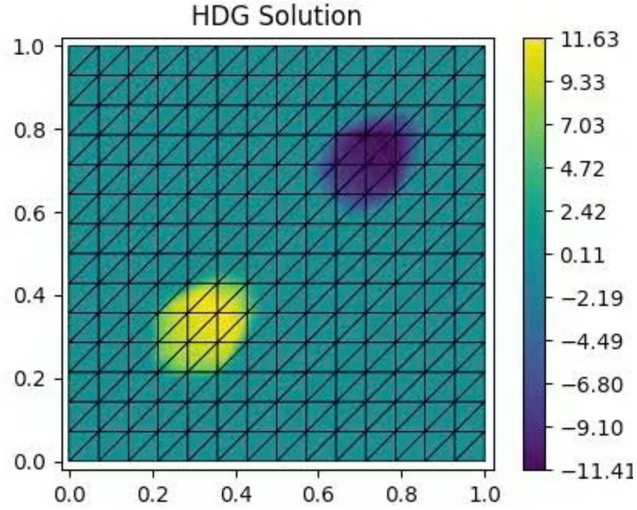


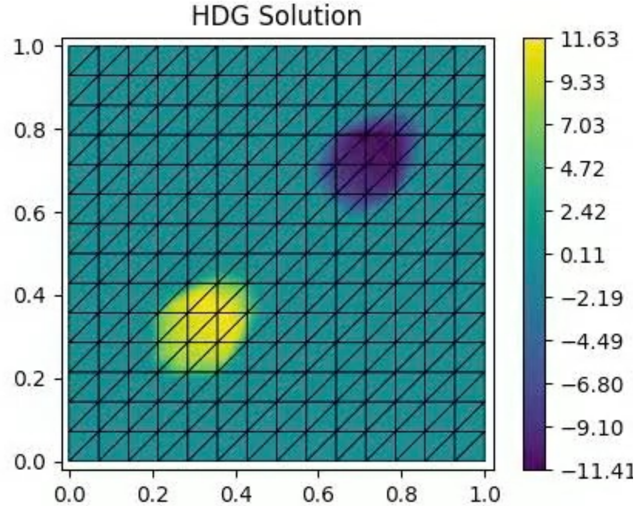
Other Examples

$$\boldsymbol{f}(\boldsymbol{u}) = [1/3u^3, 1/3u^3] \;\&\; \frac{d\boldsymbol{f}(\boldsymbol{u})}{du} = [u^2, u^2]$$

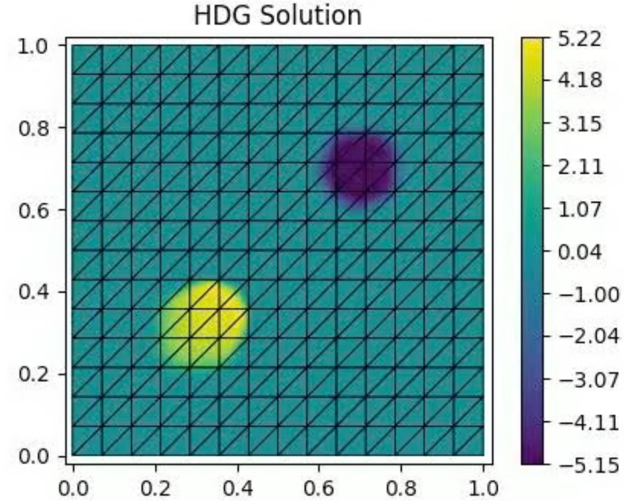


Other Examples

$\boldsymbol{f}(\boldsymbol{u}) = [1/3u^3, 1/3u^3]\ \&\ \frac{df(\boldsymbol{u})}{du} = [u^2, u^2]$



$\boldsymbol{f}(\boldsymbol{u}) = [e^u, e^u]\ \&\ \frac{df(\boldsymbol{u})}{du} = [e^u, e^u]$



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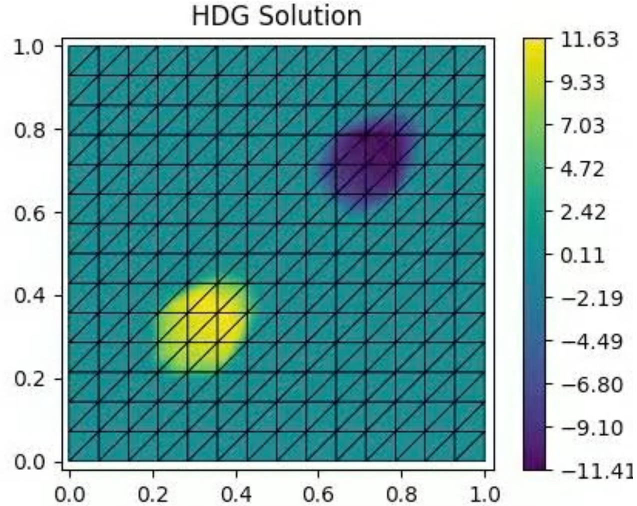

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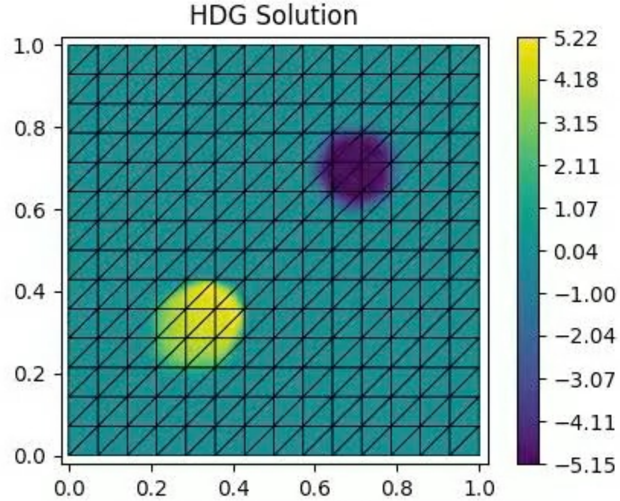
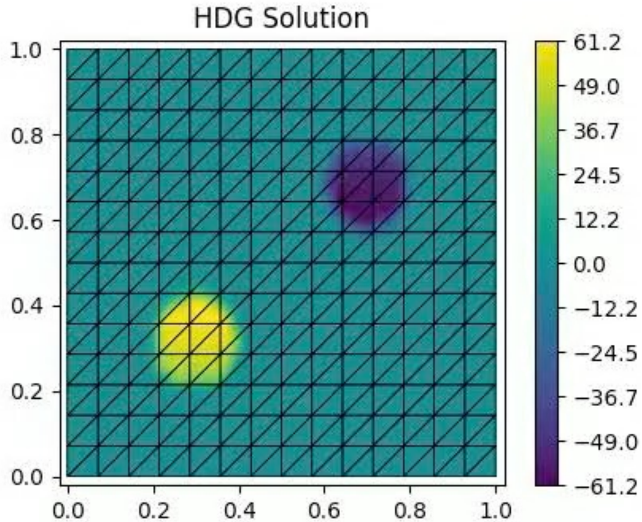


$\boldsymbol{f}(\boldsymbol{u}) = [0, 1/2u^2]$ & $\frac{d\boldsymbol{f}(\boldsymbol{u})}{du} = [0, u]$



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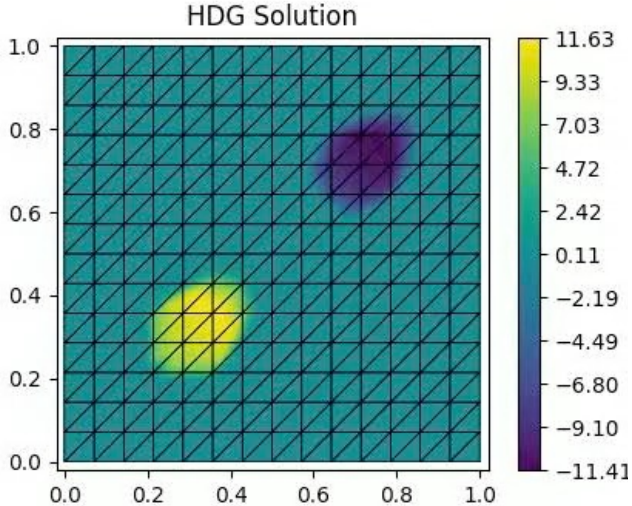

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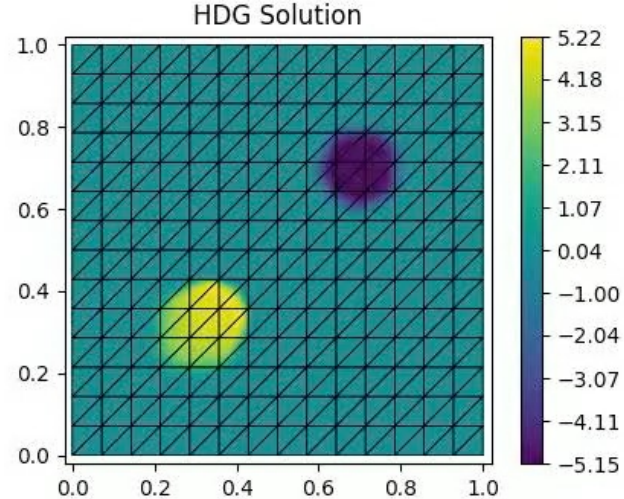

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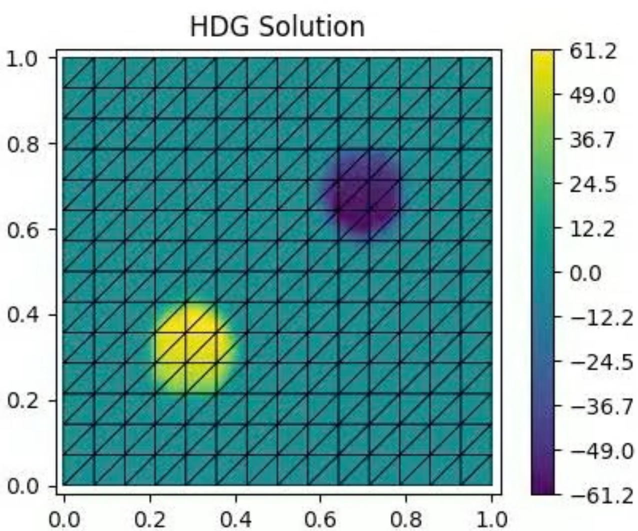
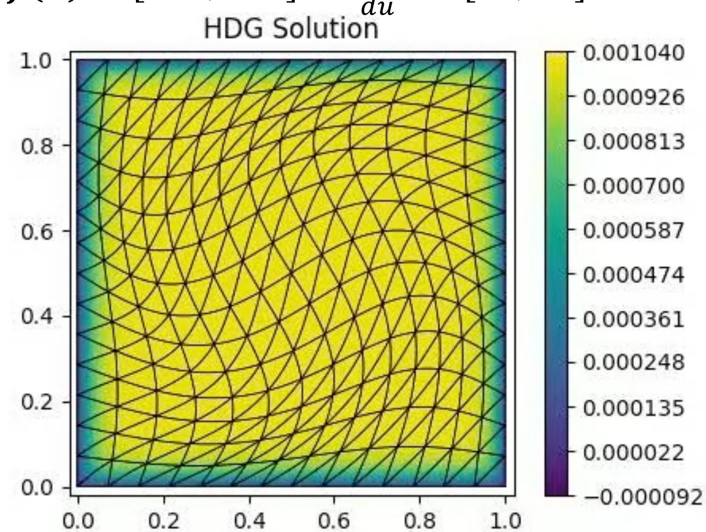


$\boldsymbol{f}(\boldsymbol{u}) = [10u, 10u]$ & $\frac{d\boldsymbol{f}(\boldsymbol{u})}{du} = [10, 10]$



Formulation

- $1^{st} Order\ Form$

$$\frac{1}{\kappa}\boldsymbol{q} + \nabla u = 0 \qquad in\ K$$
$$\frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{f}(\boldsymbol{u}) + \boldsymbol{q}) = f \quad in\ K$$
$$u = g_D \qquad on\ \partial K$$

- $Weak\ Form$

$$R_1 = \left(\frac{1}{\kappa}\boldsymbol{q_h^t}, \boldsymbol{v}\right)_K - (u_h^t, \nabla \cdot \boldsymbol{v})_K + \left\langle \widehat{u_h^t}, \boldsymbol{v} \cdot \boldsymbol{n} \right\rangle_{\partial K} = 0 = f_1$$

$$R_2 = \left(\nabla \cdot \boldsymbol{q_h^t}, w\right)_K + \left(\frac{u_h^t}{\Delta t}, w\right)_K - (\boldsymbol{f}(\boldsymbol{u_h})^{\boldsymbol{t}}, \nabla w)_K + \left\langle \tau \cdot u_h^t, w \right\rangle_{\partial K} + \left\langle \left(\boldsymbol{f}\left(\widehat{\boldsymbol{u_h^t}}\right) \cdot \boldsymbol{n} - \tau \cdot \widehat{u_h^t}\right), w \right\rangle_{\partial K}$$
$$= (s, w)_K + \left(\frac{u_h^{t-1}}{\Delta t}, w\right)_K = f_2$$

$$R_3 = \left\langle \boldsymbol{q_h^t} \cdot \boldsymbol{n}, \mu \right\rangle_{\partial K_{in}} + \left\langle \tau \cdot u_h^t, \mu \right\rangle_{\partial K_{in}} + \left\langle \left(f\left(\widehat{u_h^t}\right) \cdot n - \tau \cdot \widehat{u_h^t}\right), \mu \right\rangle_{\partial K_{in}} = 0 = f_3$$

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Time derivative: 1st order upwind
Previous time step to the RHS as source

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Nonlinear Jump: $\widehat{\boldsymbol{f}(\boldsymbol{u}_{\boldsymbol{h}}) + \boldsymbol{q}_h} \equiv \boldsymbol{f}(\widehat{\boldsymbol{u}_{\boldsymbol{h}}}) + \boldsymbol{q}_{\boldsymbol{h}} + \tau(u_h - \widehat{u_h}) \cdot \boldsymbol{n}$

Formulation into "matrices"

- $Weak\ Form$

$$R_1 = \left(\frac{1}{\kappa}\boldsymbol{q_h^t}, \boldsymbol{v}\right)_K - (u_h^t, \nabla\cdot\boldsymbol{v})_K + \left\langle \widehat{u_h^t}, \boldsymbol{v}\cdot\boldsymbol{n}\right\rangle_{\partial K} = 0 = f_1$$

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- $Matrix\ Form$

$$R = \begin{bmatrix} A_x & 0 & B_x & C_x \\ 0 & A_y & B_y & C_y \\ D_x & D_y & E & I \\ N_x & N_y & K & L \end{bmatrix} \cdot \begin{bmatrix} q_{hx}^{n-1} \\ q_{hy}^{n-1} \\ u_h^{n-1} \\ \widehat{u_h^{n-1}} \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_2 \\ f_3 \end{bmatrix}$$

t: Physical Time Step 0, 0+dt, 0+2dt…
n: Newton Iteration Step within each Time Step dt

Formulation into “matrices”

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Only submatrices E, I, L, f_3 need to be changed WRT previous linear HDG code

Deal with Nonlinear term

- $Matrix\ Form$

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$$= \begin{bmatrix} A_x & 0 & B_x & C_x \\ 0 & A_y & B_y & C_y \\ D_x & D_y & E_{lin} & I_{lin} \\ N_x & N_y & K & L_{lin} \end{bmatrix} \cdot \begin{bmatrix} q_{hx}^{n-1} \\ q_{hy}^{n-1} \\ u_h^{n-1} \\ \widehat{u_h^{n-1}} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_{non} & I_{non} \\ 0 & 0 & 0 & L_{non} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_2 \\ f_3 \end{bmatrix}$$

Deal with Nonlinear term

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- Linear Based Matrix
  - Never updated
- Nonlinear Addon Matrix
  - Updated every **Newton iteration (inner loop)**
- Source vector that has the t-1 term inside
  - Updated every **time iteration (outer loop)**

Jacobian Matrix

- *Matrix Form*

$$R = \begin{bmatrix} A_x & 0 & B_x & C_x \\ 0 & A_y & B_y & C_y \\ D_x & D_y & E & I \\ N_x & N_y & K & L \end{bmatrix} \cdot \begin{bmatrix} q_{hx}^{n-1} \\ q_{hy}^{n-1} \\ u_h^{n-1} \\ \widehat{u_h^{n-1}} \end{bmatrix}$$

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- Jacobian matrix **dR/dU** share the **<u>same based matrix</u>** but only a different Nonlinear Addon Matrix
  - $-\left(\frac{\boldsymbol{df^t}}{\boldsymbol{du}} du_h^t, \nabla w\right)_K \rightarrow -(\boldsymbol{f}(\boldsymbol{u_h})^{\boldsymbol{t}}, \nabla w)_K \rightarrow E_{non}$
  - $\left\langle\left(\frac{\boldsymbol{df^t}}{\boldsymbol{du}} \widehat{du_h^t} \cdot \boldsymbol{n} - \tau \cdot \widehat{du_h^t}\right), w\right\rangle_{\partial K} \rightarrow \left\langle\left(\boldsymbol{f}(\widehat{\boldsymbol{u_h^t}}) \cdot \boldsymbol{n} - \tau \cdot \widehat{u_h^t}\right), w\right\rangle_{\partial K} \rightarrow I_{non}$
  - $\left\langle\left(\frac{\boldsymbol{df^t}}{\boldsymbol{du}} \widehat{du_h^t} \cdot \boldsymbol{n} - \tau \cdot \widehat{du_h^t}\right), \mu\right\rangle_{\partial K_{in}} \rightarrow \left\langle\left(\boldsymbol{f}(\widehat{\boldsymbol{u_h^t}}) \cdot \boldsymbol{n} - \tau \cdot \widehat{u_h^t}\right), \mu\right\rangle_{\partial K_{in}} \rightarrow L_{non}$

Newton Iteration

- $Matrix\ Form$

$$[Jacobian]^{n-1} \cdot [dU] = [Source]^{t-1} - [Residual]^{n-1} \cdot [U]^{n-1}$$

- Outer Loop: t = t+dt
- Inner Loop: $[U]^{n-1} = [U]^{n-1} + \alpha \cdot [dU]$
  - Ended when mean(abs(dU)) < 1e-14 for example
  - α: Relaxation factor, Reduce from 1 when dU increases (overshoot)

Newton Iteration

- $HDG\ Matrix\ Form$

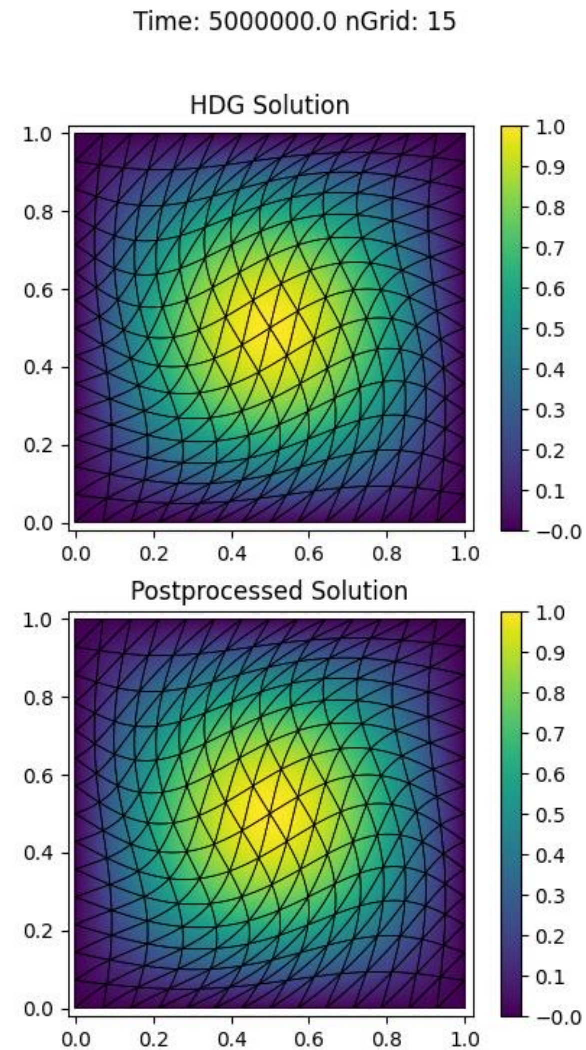
$$\left([dL] - [dN_x \quad dN_y \quad dK]\begin{bmatrix} dA_x & 0 & dB_x \\ 0 & dA_y & dB_y \\ dD_x & dD_y & dE \end{bmatrix}^{-1}\begin{bmatrix} dC_x \\ dC_y \\ dI \end{bmatrix}\right)[d\hat{u}_h]$$

$$= [F_{tot\widehat{u_h}}] - [dN_x \quad dN_y \quad dK]\begin{bmatrix} dA_x & 0 & dB_x \\ 0 & dA_y & dB_y \\ dD_x & dD_y & dE \end{bmatrix}^{-1}\begin{bmatrix} F_{totq_x} \\ F_{totq_y} \\ F_{totu_h} \end{bmatrix}$$

$$\begin{bmatrix} F_{totq_x} \\ F_{totq_y} \\ F_{totu_h} \\ F_{tot\widehat{u_h}} \end{bmatrix} = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_2 \\ f_3 \end{bmatrix} - \begin{bmatrix} A_x & 0 & B_x & C_x \\ 0 & A_y & B_y & C_y \\ D_x & D_y & E & I \\ N_x & N_y & K & L \end{bmatrix} \cdot \begin{bmatrix} q_{hx}^{n-1} \\ q_{hy}^{n-1} \\ u_h^{n-1} \\ \widehat{u_h^{n-1}} \end{bmatrix}$$

Convergence

- $Test\ Solution: u = \sin(\pi x)\sin(\pi y)$



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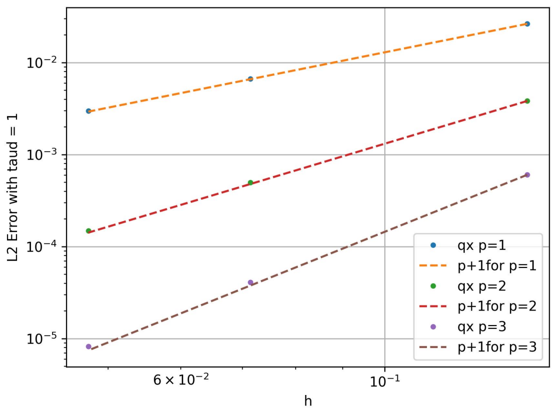

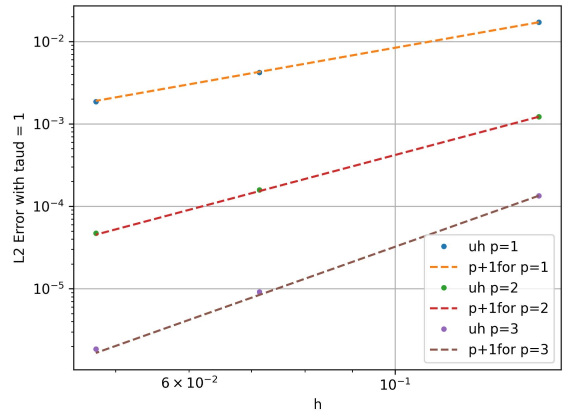

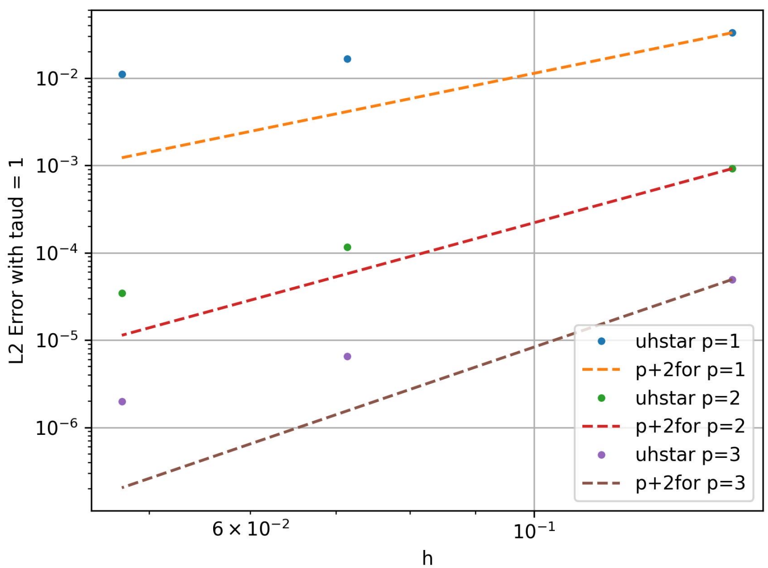

Don’t know what happened to post-processing