$$E[S^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}\Big((X_i-\mu)-(\bar{X}-\mu)\Big)^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}\Big((X_i-\mu)^2-2(\bar{X}-\mu)(X_i-\mu)+(\bar{X}-\mu)^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\frac{2}{n}(\bar{X}-\mu)\sum_{i=1}^{n}(X_i-\mu)+\frac{1}{n}(\bar{X}-\mu)^2\sum_{i=1}^{n}1\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\frac{2}{n}(\bar{X}-\mu)\sum_{i=1}^{n}(X_i-\mu)+\frac{1}{n}\cdot(\bar{X}-\mu)^2\cdot n\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\frac{2}{n}(\bar{X}-\mu)\sum_{i=1}^{n}(X_i-\mu)+(\bar{X}-\mu)^2\right]$$

where $\bar{X}-\mu = \frac{1}{n}\sum_{i=1}^{n}X_i - \mu = \frac{1}{n}\sum_{i=1}^{n}X_i - \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)$

So $E[S^2] = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\frac{2}{n}(\bar{X}-\mu)\cdot n\cdot(\bar{X}-\mu)+(\bar{X}-\mu)^2\right]$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-2(\bar{X}-\mu)^2+(\bar{X}-\mu)^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-(\bar{X}-\mu)^2\right]$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right] - E\left[(\bar{X}-\mu)^2\right]$$

$$= \sigma^2 - E\left[(\bar{X}-\mu)^2\right]$$

prove $E\left[(\bar{X}-\mu)^2\right] = \frac{1}{n}\sigma^2$

we know: $Var(aX_i) = a^2\,Var(X_i)$

$Var\left(\sum_{i=1}^{n}X_i\right) = \sum_{i=1}^{n}Var(X_i)$

$E[aX+bY] = aE(X)+bE(Y)$

$E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n}X_i\right] = \frac{1}{n}\sum_{i=1}^{n}E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$

Hence:

$$E\left[(\bar{X}-\mu)^2\right] = E\left[\left(\bar{X}-E(\bar{X})\right)^2\right]$$

$$= Var(\bar{X})$$

$$= Var\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i)$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

So $E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$

$\frac{n}{n-1}E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = \sigma^2$

故 $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$ 是 unbiased

x' is mean of sample

$$Var(X') = Var\left(\frac{1}{n} Sum(X_i)\right)$$

$$= \frac{1}{n^2} Var(Sum(X_i))$$

$$= \frac{1}{n^2} Sum(Var(X_i))$$

$$= \frac{1}{n^2} Sum(\sigma^2)$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$