Amplification and Policy Responses with Income-Based Versus Asset-Based Borrowing Constraints *

Yan Chen

December 2022

Abstract

Economies regularly experience episodes during which a significant fraction of agents are borrowing constrained. These constraints give rise to amplification effects, which occasionally generate aggregate demand shortages. This paper analyzes such amplification effects in a stylized model with both asset- and income-based borrowing constraints and investigates how macroeconomic stabilization policies can redress the amplification effects. Income-based borrowing amplifies shocks to net worth when there is an aggregate demand shortage, and asset-based borrowing amplifies shocks to asset prices. A tax on lenders to subsidize borrowers improves the welfare of borrowers and undermines that of lenders when there is no aggregate demand shortage, but can lead to a Pareto improvement when aggregate demand externalities are large. Liquidity operations can lead to a Pareto improvement independent of whether there is an aggregate demand shortage. If both types of borrowing constraints are present, taxing lenders to subsidize asset-constrained agents rather than income-constrained agents can improve welfare more. With either type of borrowing constraint, a macroprudential tax on debt issuance, combined with a lump-sum transfer between borrowers and lenders, will result in constrained efficient allocations.

JEL Classification: E44, G01, G51, H23

Keywords: Financial crisis, Borrowing constraints, Income-based borrowing, Constrained efficiency, Aggregate demand externalities

*Please find the latest version of the paper on my website.

1 Introduction

The 2008 Great Recession originated from shocks to the financial system but transmitted to the economy as a whole via falling asset prices and declining aggregate demand, partly due to household deleveraging. This paper studies how debt in the private sector may exacerbate an economic slump by triggering amplification effects and how macroeconomic stabilization policies can redress the inefficiencies from two financial frictions: asset-based borrowing constraints (ABCs) and income-based borrowing constraints (IBCs).

ABCs are widely incorporated in macroeconomic models with financial frictions.[1] In these models, agents —either households, financial intermediaries, or firms —face a borrowing constraint that restricts the maximum amount they can borrow to a fraction of the liquidation value of their asset holdings. Small and temporary shocks can have large and persistent effects on real variables through asset price feedback loops.

Although asset-based borrowing constraints seem to play an important role in episodes of deleveraging, empirical evidence has shown that income-based borrowing constraints also play a major role and may at times be more important than asset-based borrowing constraints for macroeconomic dynamics. For example, recent studies find only about 20% of non-financial corporate debt in the US is secured by assets. 80% is borrowed against the value of cash flows from firms' continuing operations. Over 80% of cashflow-based borrowing includes income-based covenants in the contract (Lian and Ma, 2021).[2] Given the importance of IBCs, their implications for macroeconomic stabilization policy have not been well explored in the economic literature. An important question then concerns the different macroeconomic implications of the two types of borrowing constraints and the optimal policy responses when both are present during a deleveraging episode such as the Great Financial Crisis.

In this paper, I build a theoretical model to analyze amplification effects with asset-based borrowing constraints, with income-based borrowing constraints, and with both types of constraints on households. I capture the potential for aggregate demand shortages by introducing a zero lower bound (ZLB) on the nominal interest rate.[3] The analytical results of the model with IBC demonstrate the amplification of shocks to wealth through aggregate demand when the debt limit of borrowers is determined by current income. A fall in income will tighten the borrowing constraint, which reduces the amount of debt

[1] Classic macroeconomic models with financial frictions, as in Bernanke and Gertler (1989); Bernanke, Gertler, and Gilchrist (1999); Kiyotaki and Moore (1997); Mendoza (2010).

[2] Covenants are specified in debt contracts and are legally binding. They prevent borrowers' debt capacity from exceeding a multiple of current income, and covenant infringement will directly lead to technical default and negative debt growth. More details in Lian and Ma (2021).

[3] It is sufficient but not necessary to generate demand-driven recessions. An alternative approach is to build a Bewley type of heterogeneous agents with incomplete market model as in Guerrieri and Lorenzoni (2017), but at a cost of analytically tractable results of amplification.

borrowers can take on. When they are more constrained in borrowing, borrowers reduce consumption spending, which lowers aggregate demand and production. Therefore, income falls and tightens the borrowing constraint further.

I consider an economy that starts with loose credit conditions in which agents can easily borrow and accumulate debt. An exogenous constraint on borrowing that depends on either an individual's asset holdings or income then forces borrowers to deleverage, which reflects tightened credit conditions in a slump. Because borrowers' issuance of debt is constrained, the interest rate must fall to induce lenders to hold less debt. Deleveraging will have two countervailing effects on aggregate demand. First, it will directly lower borrowers' demand, thus depressing aggregate demand; second, the endogenous fall in the real interest rate will boost aggregate demand. As long as the economy is away from the ZLB, the fall in interest rate fully counteracts the negative effect of deleveraging on aggregate demand, and there is no aggregate demand shortage. Firms can produce output at the efficient level. Otherwise, if the interest rate hits the ZLB, there will be an aggregate demand shortage. Given the lack of demand, firms are forced to scale down production and wages decline. Since borrowers are constrained by their income, lower income tightens the borrowing constraint and further reduces demand, which results in a negative feedback loop. Borrowers do not take into consideration the adverse effect of their behavior on aggregate demand, which lowers production and wages during deleveraging. This leads to aggregate demand externalities.

When there is no aggregate demand shortage in an IBC model, the fall in interest rates generates wealth redistribution between borrowers and lenders, which renders borrowers better off and lenders worse off, but it does not generate any inefficiencies in the economy. Allocation in an IBC economy when there is no aggregate demand shortage is therefore constrained efficient. In an ABC economy, however, amplification through asset price will cause inefficiencies when there is no aggregate demand shortage. Deleveraging by asset-based borrowers depresses asset prices, which tightens the borrowing constraint.[4] Borrowers are forced to further deleverage, which reduces consumption and depresses asset prices further. This amplification effect through asset price gives rise to pecuniary externalities. The allocation in an ABC economy when there is no aggregate demand shortage is constrained inefficient.[5]

When there is aggregate demand shortage, the IBC economy is constrained inefficient. The inefficiencies originate from the aggregate demand externalities that lower income and tighten the borrowing constraint. The effects of low income and tightened borrowing

[4]The effect of deleveraging on asset price when there is no AD shortage is ambiguous, since lower interest rate drives up asset price, but when the fraction of lenders is much larger than constrained asset-based borrowers in the economy, it tends to lower asset price.

[5]Similar results in Jeanne and Korinek (2010) in an open economy and endowment economy model environment.

constraints reinforce each other, similar to the effects of low asset prices and tightened borrowing constraints in the ABC economy when there is an aggregate demand shortage. Asset prices fall as consumption decreases, which forces borrowers to further deleverage. Deleveraging worsens negative aggregate demand externalities. The resulting lower consumption and lower asset prices are caused by both the pecuniary externalities and aggregate demand externalities.

Next, the paper analyzes policy implications with the two types of borrowing respectively, and calibrates the model with both types of borrowing in one economy. It addresses two major questions: what are the differences in the effects of policy measures with the two types of constraints, and what is the optimal policy in a credit crunch under the two types of borrowing? I analyze the implications of two types of policies that I label fiscal policy and liquidity operations. I model fiscal policy as a transfer across agents during deleveraging. I model liquidity operations as a transfer across time, i.e., policymakers provide liquidity to borrowers in the period in which the constraint is binding, and they pay it back in the following period. This can also be interpreted as the government purchasing assets from borrowers during deleveraging and selling them back in the future.

Fiscal policy that taxes lenders and provides a transfer to borrowers in a crisis will improve the welfare of borrowers and undermine that of lenders when there is no aggregate demand shortage, in both the IBC and ABC economy. In the IBC economy, it also generates wealth redistribution by increasing the interest rate. In the ABC economy, it relaxes the borrowing constraint by boosting asset prices to improve the welfare of borrowers in addition to wealth redistribution due to changes in the interest rate. Lenders are always worse off due to the tax. When there is an aggregate demand shortage, fiscal policy that taxes lenders to provide transfers to borrowers in a crisis can improve the welfare of both borrowers and lenders. When aggregate demand externalities are large enough, such transfers can even lead to a Pareto improvement in both the IBC and ABC economy. Providing a transfer to ABC borrowers can improve welfare more than a transfer to IBC borrowers. The reason is that a lump-sum subsidy to IBC borrowers can reduce their labor supply, lower the amount they borrow, and depress aggregate demand when the interest rate cannot fall further. In contrast, a lump-sum subsidy to ABC borrowers raises asset prices, increases the amount they borrow, and boosts aggregate demand. As a result, income falls for IBC borrowers while it increases for ABC borrowers. And the welfare of ABC borrowers is improved more than that of IBC borrowers.

However, liquidity operations that transfer resources for the same agent across time, such as asset purchases during a deleveraging episode and sales after deleveraging can lead to a Pareto improvement independent of whether there is an aggregate demand shortage, in both the IBC and ABC economy. Since it involves a transfer across time, it improves borrowers' welfare by getting around the borrowing constraint when liquidity is most needed. For lenders, when there is no aggregate demand shortage, it improves their welfare by increasing the interest rate; when there is an aggregate demand shortage, it improves their welfare by increasing income.

The effectiveness of these ex post policies depends on the magnitude of amplification. In a model set-up with separable preferences of households and the wealth effect on labor supply, aggregate demand externalitities might not be large enough such that a fiscal policy as implemented in the previous section achieves such welfare improvements.[6] Therefore, it is important to understand how ex ante macroprudential policies, can be implemented to achieve an efficient outcome. I find that an optimal macroprudential policy can be implemented by either a quantity restriction on debt issuance of borrowers such that there will be no aggregate demand shortage, or a tax on any positive debt issuance, combined with lump-sum transfers between borrowers and lenders.

Literature Review. This paper builds on several strands of the literature. First, it contributes to the literature on macroeconomics with financial frictions. In their seminal work, Kiyotaki and Moore (1997) adopt a collateral constraint on borrowing due to incomplete contracts microfounded by Hart and Moore (1994). In their model, creditor payoff in default and debt capacity are determined by the liquidation value of assets. Amplification arises from fire sales of land from the more productive sector to the less productive sector due to adverse productivity shocks, which depresses land prices and feeds back to net worth, both within a period and dynamically to future asset prices. Other related work studies the pecuniary externalities from asset fire sales, as in Jeanne and Korinek (2010); Bianchi (2011); and Mendoza (2010). My work differs in two respects. First, creditor payoff in default and debt capacity are determined by current earnings instead of the liquidation value of assets; second, shocks are amplified through aggregate demand instead of asset prices.

Second, this paper is closely related to works on aggregate demand-driven recessions. Mian, Rao, and Sufi (2013) and Mian and Sufi (2014) focus on the housing net worth channel through which the fall in the housing net worth of households reduced aggregate demand by direct wealth effects or by tightening households' capability to borrow through a fall in the collateral value. Chaney, Sraer, and Thesmar (2012) and Duchin, Ozbas, and Sensoy (2010) also study the reduction in corporate investment through the fall in collateral value in the Great Recession Theoretically, my work closely follows that of Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2017) who emphasize that deleveraging by borrowers in the economy weighs down on aggregate demand, and Farhi and Werning (2016) and Korinek and Simsek (2016), who highlight the importance of macroprudential policy to address aggregate demand externalities. My work also differs from their papers because I impose an income-based borrowing constraint that generates amplification, rather than an exogenous debt limit.

[6] For example, in Farhi and Werning (2016), the same type of fiscal policy will make lenders worse off.

Third, my work builds on a new strand of the literature that features the significance of an income-based debt limit. Empirical works include Chava and Roberts (2008) and Roberts and Sufi (2009), who study the effect of the violation of debt covenants on borrowers and how lenders will gain rights to influence the financing and investment decisions of the firms; Chodorow-Reich and Falato (2017), who study an earning-based debt limit in the syndicated loan market; and Sufi (2009), who examines the widespread use of cash flow-based financial covenants in bank lines of credit. Ivashina, Laeven, and Moral-Benito (2019) investigate types of commercial credit in general. My theoretical model builds heavily on the comprehensive empirical work of Lian and Ma (2021), who establish the prevalence of cashflow-based borrowing among nonfinancial corporations in the US.

My work is also related to theoretical models that use income-based borrowing constraints to study the macroeconomic effects of debt deleveraging. Goldberg (2010) models income-based borrowing constraint on the firm side, but focuses on the effect of idiosyncratic shocks in a Bewley-Huggett-Aiyagari type of framework. Corbae and Quintin (2015) and Greenwald (2018) both study the importance of a borrowing constraint based on payment-to-income ratio in driving housing prices. The most relevant theoretical work to my paper is by Drechsel (2019), who studies an income-based debt limit in the nonfinancial corporate sector, both empirically and theoretically; incorporates income-based debt limits on firms in a business cycle model; and focuses on firms' response of borrowing to investment shocks. Benigno et al. (2013) incorporate income–based borrowing constraints in open economy models. My work contributes to the literature by studying the interactions of income-based and asset-based borrowing, and the differences in their policy implications.

The rest of the paper is organized as follows. Section 2 introduces the IBC and ABC model set-up. Section 3 characterizes the decentralized equilibrium of the two models and compares the amplification effects. Section 4 conducts comparative statics, Section 5 analyzes the implications of two ex post policies, fiscal policy and liquidity operations. Section 5 analyzes the optimal macroprudential policies. Section 6 introduces a numerical illustration of the model with both types of borrowing, and Section 7 concludes.

2 Model Set-Up

In this section, I will demonstrate and compare the amplification effect with asset- and income-based borrowing constraints on households in a three-period model. The model has an environment that closely follows Korinek and Simsek (2014, 2016), but provides a more generalized framework to incorporate one or more types of borrowing constraints. Moreover, unlike an exogenous debt limit in their paper, the model has an endogenous debt limit dependent on households' asset holdings or current income rather than an exogenous value.

2.1 Environment

There are three discrete time periods $t = 0, 1, 2$. The economy consists of households and firms. Households are of measure one. There are H types of households, indexed by $h \in \mathcal{H}$. In some of our applications, the set of households will consist of only two types, e.g. lenders and borrowers. There can be type a borrowers constrained by asset value when $\mathcal{H} = \{l, a\}$, or type i borrowers constrained by income when $\mathcal{H} = \{l, i\}$. But we will also consider cases with additional heterogeneity. Each type of households has a weight of α^h with $\sum_h \alpha^h = 1$. Borrowers are more impatient than lenders, with the discount factors $\beta^h < \beta^l = 1$, for $h = a, i$, such that in equilibrium borrowers will take on debt. Households own firms and will obtain profits from firm sales. There are two commodities in the economy, a final good for consumption and labor. Households get a transfer of the final good t_t^h in every period.

Preferences. Households preferences are inseparable, following Greenwood, Hercowitz, and Huffman (1988).[7]

$$U^h = u(c_0^h - v(n_0^h)) + \beta^h u(c_1^h - v(n_1^h)) + (\beta^h)^2 u(c_2^h - v(n_2^h)) \tag{2.1}$$

where $u'(\cdot) > 0$, $u(\cdot)$ strictly concave, $\lim_{c\to\ 0} u'(c) = \infty, 0 < v'(\cdot) \leq 1$, $v(\cdot)$ strictly convex, $v'(0) = 0, \lim_{n\to\infty} v'(n) = \infty$.

Technology. The final good is produced competitively by a final good sector using differentiated intermediate goods according to the Dixit-Stiglitz technology:

$$y_t \equiv (\int_0^1 y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj)^{\frac{\epsilon}{\epsilon-1}} \tag{2.2}$$

with ϵ greater than one. $y_t(j)$ the quantity of the intermediate good j produced by a continuum of monopolistic firms indexed by $j \in [0, 1]$. Each firm uses an identical linear technology to produce a differentiated good:

$$y_t(j) = n_t(j) \tag{2.3}$$

where $n_t(j)$ is the aggregate level of labor supplied by all types of households to produce the good j. Firms take household demand and the aggregate price level as given to set

[7]Unlike separable preferences consistent with balanced growth, GHH preference eliminates wealth effects on labor supply, so it will generate more amplification compared to separable preferences as households will not increase labor supply to pay off debt when income falls.

prices in each period. The aggregate price level is defined as:

$$P_t \equiv (\int_0^1 P_t(j)^{1-\epsilon}\, dj)^{\frac{1}{1-\epsilon}}$$

Aggregate price dynamics. In the baseline model, instead of assuming the full staggering pricing dynamics as in Calvo (1983), we assume in the baseline model that none of the monopolistic firms can reset prices due to an infinite price adjustment cost in each time period. Thus, the final good price and the aggregate price level stay constant, $P_t(j) = P_t = P$.

Market structure. Households have equal shares of firms. In each period, they earn labor income at a competitive wage rate and collect profits from firms to consume. There is a credit market in which households can issue a one-period bond at the prevailing real interest rate r_{t+1}[8]. b^h_{t+1} denotes bonds outstanding in period t and needed to be repaid in period $t+1$. Households are also endowed with an asset that yield d^h_t dividend in every period. The dividend is subject to shocks in period 1, but deterministic in period 0 and 2 with $d^h_t = d$. Each household is endowed with $\theta^h_0 = 1$ unit of the asset at the beginning of period 0, and the asset can be traded at a price p_t only within the same type of households. There is no uncertainty in the model, and agents fully anticipate future shocks.

2.2 First-best solution

I characterizes the first-best allocation $\{c^h_t, n^h_t\}_{t=0,1,2}$ as the planner's solution when market imperfections are absent. It serves as a benchmark for the later welfare analysis.

The planner maximizes a weighted sum of utilities subject to the resource constraints. Let γ^h be the Pareto weight of type h agents, with $\sum_h \gamma^h = 1$. The social planner's problem is then given by:

$$\begin{aligned} \max_{\{c^h_t, n^h_t\}_{t=0,1,2}} \quad & \sum_{h\in\mathcal{H}} \sum_t \alpha^h \gamma^h [(\beta^h)^t u(c^h_t - v(n^h_t))] \\ \text{s.t.} \quad & \sum_{h\in\mathcal{H}} \alpha^h c^h_t = y_t + \sum_{h\in\mathcal{H}} \alpha^h (t^h_t + \theta^h_t d_t), \quad \forall t \end{aligned} \tag{2.4}$$

At the optimum, the planner will equate households' marginal rate of substitution in the three periods to the Pareto weights ratio. Denote $u(\tilde{c}^h_t) = u(c^h_t - v(n^h_t))$, for any $h, k \in \mathcal{H}$:

$$\frac{\gamma^h}{\gamma^k} = \frac{u'(\tilde{c}^k_0)}{u'(\tilde{c}^h_0)} = \frac{\beta^k u'(\tilde{c}^k_1)}{\beta^i u'(\tilde{c}^h_1)} = \frac{{\beta^k}^2 u'(\tilde{c}^k_2)}{{\beta^i}^2 u'(\tilde{c}^h_2)} \tag{2.5}$$

[8] r_{t+1} can be pinned down in a model with infinite time horizon. At steady state with borrowers constrained, r_{t+1} is equal to $\frac{1}{\beta^l} - 1$ since lenders are always unconstrained.

Define n^* as the efficient level of labor. Aggregate employment is given by $n_t = y_t$, and is distributed uniformly among households such that $n_t^h = n_t, \forall h$. The first-best allocation for labor is then given by:

$$n_t^h = n^* = v'^{-1}(1)$$

Combine the resource constraints, the efficient labor supply, and Equation 2.5 to obtain the optimal allocation of consumption as a function of the Pareto weights. The Pareto weights will be consistent with the wealth of the households in second-best allocations for them to be comparable. Define the optimal consumption allocation as $\{c_t^{h^{FB}}\}_{t=0,1,2}$, and the corresponding social welfare as U_0^{FB}.

Due to market imperfections from monopolistic competition, firms will exploit a markup of the marginal cost. It is well-known to impose a subsidy τ on firms to correct the distortions from the monopolistic markups. Suppose the monopolistic firms can choose prices to set for now as a frictionless benchmark without price rigidities, and they maximize profit as follows:

$$\begin{aligned} \max_{\{P_t(j), y_t(j), n_t(j)\}_{t=0,1,2}} \quad & \frac{P_t(j)}{P_t} y_t(j) - w_t(1 - \tau(n_t))n_t(j) \\ \text{s.t.} \quad & y_t(j) = n_t(j) \leq (\frac{P_t(j)}{P_t})^{-\epsilon} y_t \end{aligned}$$

The subsidy will be financed by a lump-sum tax $T_t = \tau w_t \int_0^1 n_t(j)dj$ to all households. In equilibrium, the monopolistic firms will set

$$\frac{P_t(j)}{P_t} = w_t \frac{\epsilon}{\epsilon - 1}(1 - \tau) \tag{2.6}$$

where $\tau(n_t)$ is set to $\frac{1}{\epsilon}$ when aggregate employment n_t is lower than or equal to n^*, and zero when aggregate employment n_t is above n^*. As a result of linear production technology, each firm will set the same price for their goods. Define w^* as the efficient level of real wage. When firms can freely adjust price and are appropriately subsidized, w^* will be one. Without the subsidy, households' employment and labor income will be lower.

2.3 Market imperfections

There are two major market imperfections in the model, financial frictions and the lower bound constraint on the real interest rate. Households can borrow against their income and/or against their asset holdings. They face a borrowing constraint with an endogenously determined debt limit in period 1 when issuing bonds. The debt limit is restricted by a fraction of their current income and a fraction of the value of assets they hold. In the baseline model, I focus on either an income-based borrower whose debt limit is determined solely by income, or an asset-based borrower whose debt limit is determined solely by asset value. The extent to which they are constrained by their income or asset is captured by the parameters ϕ^{Ih} or ϕ^{Ah}:

$$b_2^h \geq -\phi^{Ih} e_1^h - \phi^{Ah} \theta_1 p_1, \tag{2.7}$$

where household income e_t^h consists of labor income and profits from the monopolistic firms net of a lump sum tax:

$$e_t^h = w_t n_t^h + \Pi_t - T_t, \tag{2.8}$$

where $\Pi_t = \int_0^1 \Pi_t(j)\, dj$ is profits from firms. This constraint resonates with the empirical findings on the prevalence of income-based and asset-based borrowing. It is also an incentive compatibility condition where it is never optimal for a debtor to default given that creditors can seize a fraction of his or her income, or asset in bankruptcy. In addition, we can define e^* as the efficient level of income using the previously derived n^* and w^*:

$$\begin{aligned} n^* &= v'^{-1}(1) \\ w^* &= 1 \\ e^* &= v'^{-1}(1) \end{aligned}$$

These conditions will serve as an efficient benchmark.

Second, the nominal interest rate will be bounded by a lower bound following Korinek and Simsek (2014). In order to simplify the analytical solution, the lower bound is normalized to zero. With aggregate price level being sticky, the real interest rate will also be bounded by zero.

$$r_{t+1} \geq 0, \quad t = 0, 1 \tag{2.9}$$

The zero lower bound on nominal interest rate is crucial for the result of amplification through aggregate demand in this model, as it will force income to be below the efficient level and determined by aggregate demand. The fall in aggregate demand due to household deleveraging will lower income, tightening the borrowing constraint, which will result in further reduction in aggregate demand and income. This result will still hold if I relax the assumption of price rigidity. Indeed, the result from relaxing this assumption will be in line with the "perverse" proposition brought up by Eggertsson and Krugman (2012) that increasing price flexibility makes the real effect of an adverse shock on net worth worse. Therefore, relaxing this assumption will only make amplification greater in the model. I assume an extreme level of price stickiness to simplify the model.

2.4 Strategies

Since firms cannot reset prices in each period, the aggregate price level is completely sticky. Given the preset good prices, the monopolistic firms choose how much to produce and how many workers to hire to maximize profit:

$$\begin{aligned} \max_{\{y_t(j), n_t(j)\}_{t=0,1,2}} \quad & \frac{P_t(j)}{P_t} y_t(j) - w_t(1-\tau(n_t))n_t(j) \\ \text{s.t.} \quad & y_t(j) = n_t(j) \leq (\frac{P_t(j)}{P_t})^{-\epsilon} y_t \end{aligned} \tag{2.10}$$

where $P_t = P$ is constant, and $\frac{P_t(j)}{P_t}$ is equal to one by symmetry. In equilibrium, the monopolistic firms will always choose to produce to meet the demand since the marginal product is strictly higher than the marginal cost. Therefore, $y_t(j) = n_t(j) = y_t$. The monopolistic firms' production is essentially determined by the aggregate demand for the final good, which is ultimately determined by the real interest rate. Since price is fixed, production is determined by monetary policy that sets the nominal interest rate. Let r^* be the real interest rate at which production and employment are at the frictionless benchmark level. A constrained efficient monetary policy is set according to[9]:

$$i_{t+1} = r_{t+1} = \max(0, r^*) \quad \forall t \tag{2.11}$$

Households' maximization problem is given by:

$$\begin{aligned} \max_{\{c_t^h, n_t^h, \theta_t^h, b_1^h, b_2^h\}_{t=0,1,2}} \quad & u(c_0^h - v(n_0^h)) + \beta^h u(c_1^h - v(n_1^h)) + (\beta^h)^2 u(c_2^h - v(n_2^h)) \\ \text{s.t.} \quad & \frac{b_1^h}{1+r_1} + c_0^h = e_0^h + \theta_0^h d_0^h + (\theta_0^h - \theta_1^h)p_0 + b_0^h, \\ & \frac{b_2^h}{1+r_2} + c_1^h = e_1^h + \theta_0^h d_1^h + (\theta_1^h - \theta_2^h)p_1 + b_1^h, \\ & c_2^h = e_2^h + \theta_2^h d_2^h + b_2^h, \\ & b_2^h \geq -\phi^{Ih} e_1^h - \phi^{Ah}\theta_1 p_1. \end{aligned} \tag{2.12}$$

with $e_t^h = w_t n_t^h + \Pi_t - T_t = w_t n_t^h + n_t - w_t n_t$. Note that profits of firms net of the lump-sum tax will be positive if the real wage is below the efficient level, and will be zero if it is at the efficient level.

Definition 1 *A decentralized equilibrium is a set of prices* $\{w_0, w_1, w_2, r_1, r_2\}$*, real allocations* $\{c_t^h, n_t^h, e_t^h, y_t\}_{t=0,1,2,h\in\{a,i,l\}}$*, asset allocations* $\{\theta_t^h\}_{t=0,1,2,h\in\{a,i\}}$*, bond holdings*

[9] There is a discussion of the constrained efficiency of the monetary policy with or without commitment power in Korinek and Simsek (2016).

$\{b_t^h\}_{t=0,1,2,h\in\{a,i,l\}}$, *and profits and taxes* $\{\Pi_t, T_t\}$ *such that households maximize utility as in* (2.12)*; the final good sector produces according to* (2.2)*; intermediate goods are produced by monopolistic competitive firms that maximize profits according to* (2.10) *given fixed intermediate goods price; the interest rates are set according to* (2.11)*, and all markets clear.*

3 Solving the Decentralized Equilibrium

The decentralized equilibrium will depend on the type of borrowers in the economy. I will first consider the case when $H = 2$, $\mathcal{H} = \{l, i\}$, and $\phi^{Ai} = 0$, where borrowers are constrained by their income. Next I will consider when $H = 2$, $\mathcal{H} = \{l, a\}$, and $\phi^{Ia} = 0$, where borrowers are constrained by the value of their asset holdings. The borrowing constraints can be binding or not binding in equilibrium. I will focus on the equilibrium when they are binding, since it is more relevant for policy interventions.

3.1 The decentralized equilibrium with IBCs

The model can be solved via backward induction. Period 2 consumption and labor choices are intratemporal decisions given b_2^h at the beginning of period 2. Because assets can only be traded among the same type of households, both income-based borrowers and lenders in the economy will have no incentive to trade assets. They hold the one unit of asset endowed in peiod 0 in equilibrium. By market clearing condition, lenders' bond holdings will be $\alpha^l b_t^l = -\alpha^i b_t^i$, where $b_2^i = -\phi^{Ii} w_1 n_1^i$ when borrowers are constrained in equilibrium. Since monetary policy attempts to replicate the efficient level of employment for lenders, the real wage is one. Let net consumption be $\tilde{c}_t^h$, which is equal to $c_t^h - v(n_t^h)$; let λ^i be the Lagrangian multiplier associated with the IBCs; given b_1^i, the equilibrium is pinned down by:

$$u'(\tilde{c}_1^i) = \beta^i(1 + r_2)u'(\tilde{c}_2^i) + \lambda^i(1 + r_2) \tag{3.1}$$

$$u'(\tilde{c}_1^i)(w_1 - v'(n_1^i)) + \phi^{Ii} w_1 \lambda^i = 0 \tag{3.2}$$

$$u'(\tilde{c}_1^l) = \beta^l(1 + r_2)u'(\tilde{c}_2^l) \tag{3.3}$$

$$\alpha^l b_1^l = -\alpha^i b_1^i \tag{3.4}$$

The first Euler equation indicates that higher current consumption makes borrowers less tempted to borrow, so the IBCs will be less tight. The second labor supply decision equation of the borrowers implies that although working more can relax the IBCs, it reduces welfare due to disutility from working, and the marginal benefit of work needs to be balanced out by the marginal cost. By substitution using the bonds market clearing condition and the budget constraints, the decentralized equilibrium can be reduced to the labor supply choice of the borrowers and the Euler equation of the lenders as follows:

$$(w_1 + \frac{\phi^{Ii} w_1}{1+r_2}) u'(\tilde{c}_1^i) = v'(n_1^i) u'(\tilde{c}_1^i) + \beta^i \phi^{Ii} w_1 u'(\tilde{c}_2^i) \tag{3.5}$$

$$u'(\tilde{c}_1^l) = \beta^l (1+r_2) u'(\tilde{c}_2^l) \tag{3.6}$$

Note that since borrowers can and are willing to work more hours to relax the borrowing constraint, their labor supply in equilibrium will be higher than the "efficient" level n^*, i.e., they tend to overwork whenever they are constrained in borrowing. Equation (3.5) implies that the marginal benefit of working an additional hour should be matched with the marginal cost of working an additional hour. It is also a debt supply equation linking the borrowers' labor choice which determines the quantity of debt issuance, to the interest rate. Higher labor supply of the borrowers is associated with a lower interest rate when ϕ^{Ii} is relatively small. To see this, define X_b^{in} as:

$$X_b^{in} = -\frac{\phi^{Ii} w_1}{(1+r_2)^2} [1 + \frac{\beta^i \phi^{Ii} w_1 n_1^i}{(u'(\tilde{c}_1^i))^2} u''(\tilde{c}_1^i) u'(\tilde{c}_2^i)] < 0$$

where "in" denotes income-based borrowing and no AD shortage, and "b" denotes borrowers. This restriction can be approximated as:

$$\phi^{Ii} < \sigma \frac{\tilde{c}_1^i}{w_1 n_1^i}$$

where σ is the elasticity of intertemporal substitution[10]. The net consumption of the borrowers is always higher when they increase the labor supply when the interest rate falls. The intuition is in some way similar to the case where borrowers are unconstrained: lower interest rate induces borrowers to issue more debt which raises net consumption. This relation is demonstrated as the IB curve in Figure 1. Equation (A.7) can be viewed as a bond demand equation that indicates higher interest is associated with higher bond demand as higher interest rate discourage lenders from consuming today, which is shown from the AD curve in Figure 1.

Consider higher leveraging in period 1 that leads to a lower b_1^i. This corresponds to loose credit conditions during economic booms. If borrowers cannot work more hours, the

[10]Derivations are in Appendix A.1. The restriction on ϕ^{Ii} indicates that borrowers may increase labor supply when the interest rate increases if ϕ^{Ii} is too large. This anomaly originates from the assumption that borrowers are always constrained. If ϕ^{Ii} is large enough, the amount of debt borrowers carries assuming they are constrained might be greater than that of they being unconstrained, which is impractical. And if the interest rate rises when borrowers increase labor supply, their net consumption could decrease. Another interpretation of the restriction is to think of $\sigma \tilde{c}_1^i$ as the inverse of risk aversion. Borrowers need to be relatively less risk averse, or the curvature of their utility is small, to issue more debt as the interest rate falls.

interest rate has to rise such that they will consume less with higher debt repayments, whereas for lenders the interest rate will fall for them to consume more with higher debt payments (the effects are shown in Figure 1). As long as ϕ^{Ii} is small enough that borrowers are tightly constrained by the amount they can borrow, the interest rate will eventually fall with more labor supplied by the borrowers. If borrowers are highly leveraged, deleveraging in period 2 can make the interest rate fall to the zero lower bound. Since prices are fixed at the preset level, the real interest rate will determine the demand and therefore how much firms produce. When the real interest rate cannot fall further to boost demand and clear the goods market, aggregate demand falls, which lowers production. Firms' demand for labor is reduced and the real wage will fall, resulting in higher markups. Output, falling below the natural level, will be determined by the aggregate demand at the zero interest rate. This threshold level of b_1^i is defined as $\underline{b}_1^i$, and the derivation of $\underline{b}_1^i$ is in Appendix A.1.

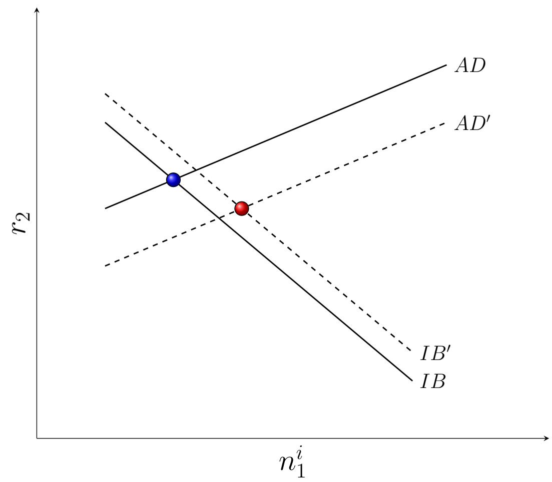

Figure 1: Effect of lower b_1^i on borrowers' employment and interest rate, no AD shortage

Lemma 1 *The decentralized equilibrium in period 1 given that borrowers are constrained is determined by* b_1^i*,*

- *when* $b_1^i \geq \underline{b}_1^i$*, the negative effect of deleveraging on aggregate demand is completely buffered by the fall in interest rate, and firms produce efficiently at* w^**, with lenders' employment* $n_1^l = n^*$ *and borrowers' employment* $n_1^i > n^*$*; there is no aggregate demand shortage;*

- *when* $b_1^i < \underline{b}_1^i$*, there is an aggregate demand shortage, since further fall in interest rate that could have recovered households' demand is circumscribed by the zero lower bound. Firms produce and earn an economic profit at* $w_1 < w^*$*, with lenders' employment* $n_1^l < n^*$*.*

When there is an aggregate demand shortage. If real interest rate is constrained by the lower bound when massive deleveraging triggers an aggregate demand shortage, wage will be below the efficient level. The decentralized equilibrium will be pinned down by the debt supply and demand equation at zero interest rate. Since lenders are unconstrained and their employment is given by $v'(n_1^l) = w_1$, which is an increasing transformation of the real wage, the two equations can be solved from either w_1 and n_1^i, or n_1^l and n_1^i. Note that the real wage will be below the efficient level and firms will earn positive profit with an aggregate demand shortage. I assume lenders and borrowers each obtain what they produce as their total income[11]. Thus households' income is given by $e_1^h = n_1^h$.

$$w_1 - v'(n_1^i) + \phi^{Ii} w_1 = \beta^i \phi^{Ii} \frac{u'(\tilde{c}_2^i)}{u'(\tilde{c}_1^i)} \tag{3.7}$$

$$u'(\tilde{c}_1^l) = \beta^l u'(\tilde{c}_2^l) \tag{3.8}$$

and with $\beta^l = 1$, Equation (3.8) can be rewritten as:

$$n_1^l = 2\frac{\alpha^i}{\alpha^l}\phi^{Ii} n_1^i + v(n_1^l) + \frac{\alpha^i}{\alpha^l} b_1^i + (d_2^l - d_1^l) + (e^* - v(e^*)) \tag{3.9}$$

Since output is determined by aggregate demand, for borrowers, the tighter the borrowing constraint, the higher wage is to increase labor supply. Thus, the wage is increasing in borrowers' employment based on borrowers' labor supply decision (as in Equation (3.7) and the IB curve in Figure 2). The more hours borrowers work, the greater amount lenders will lend out today and get repaid tomorrow, which raises the marginal utility of consumption of today and decreases that of tomorrow. Since the interest rate is stuck at the lower bound, the wage will increase to induce lenders to work more so that lenders can increase their income and consumption. Thus, the wage is also increasing in borrowers' employment from the lenders' intertemporal consumption choice or bond demand (as in Equation (3.8) and the AD curve in Figure 2).[12]

Amplification. Next, consider a comparative static when borrowers take on more debt in period 0 (lower b_1^i). Since the economy is in a liquidity trap, higher leveraging will result in a greater demand shortage, which lowers the labor demand of the firms and dampens the real wage. From lenders' perspective, they will reduce labor supply. Since lenders get more debt repayments in period 1, and their consumption demand is fixed at the current interest rate, they need less labor income to consume (a rightward shift of the AD curve as in Figure 2). On the borrowers' side, accumulating more debt in period

[11]This is an assumption that makes the decentralized equilibrium analytically tractable. The standard way is to compute total income as the sum of labor income and profits from firms.

[12]There is a reinforcing effect of wage on employment for Equation (3.7) and (3.8). For them to have a unique and well-defined solution, some restrictions need to be imposed. Derivations of the restrictions are in Appendix A.1.

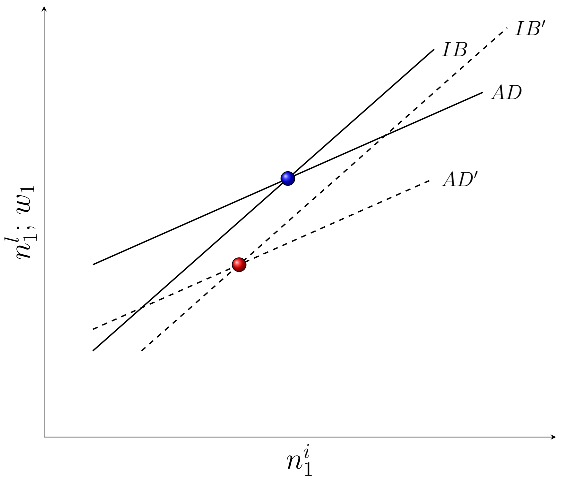

Figure 2: Effect of lower b_1^i on borrowers' employment and interest rate, with AD shortage

0 worsens deleveraging in period 1, tightening the borrowing constraint and increasing borrowers' labor supply (a rightward shift of the IB curve). The new equilibrium wage and employment of all households will be lower if the borrowing constraint is sufficiently tight, i.e., ϕ^{Ii} is sufficiently small. As labor income falls, borrowers become more constrained in borrowing, which further lowers their consumption demand and reduces production. An initial small change in wealth can lead to a large change in wage and income by affecting aggregate demand. Borrowers do not take into consideration the negative effect of debt accumulation in the present on aggregate demand in the future, resulting in worse deleveraging and aggregate demand externalities.

Note that the requirement on ϕ^{Ii} is not critical in obtaining the amplification result. The key mechanism of amplification with IBCs hinges on aggregate demand instead of the individual labor supply decision of borrowers. I derive the decentralized equilibrium when borrowers are constrained by the aggregate income instead of the individual income in the Appendix. It better captures the amplification effect from aggregate demand and provides an analytically tractable solution of the multiplier. The tighter the borrowing constraint is, i.e., the smaller ϕ^{Ii} is, the greater amplification will be generated with IBCs.

Nevertheless, allocations from the decentralized equilibrium when there is no AD shortage are constrained efficient due to the individual labor supply decision of borrowers. Because borrowers are constrained in labor income, they will choose to work more to borrow more until they can consume at the optimal level. This leads to constrained efficient allocations. With AD shortages, consumption can no longer be optimal due to aggregate demand externalities. Although borrowers will still choose to work more, labor income and consumption are sub-optimal due to lower wages. The resulting allocations are inefficient.

3.2 The decentralized equilibrium with ABCs

Similar to the model with IBCs, the decentralized equilibrium can be solved backward. A symmetric equilibrium indicates $\theta_t^a = 1$ for all t. A general form of the asset pricing equation is given by:

$$p_1 = \frac{u'(\tilde{c}_2^a)}{u'(\tilde{c}_1^a)}\beta^a d_2^a$$

Asset price is determined by the present discounted value of future cash flows.[13] There also exists a threshold level of b_1^a such that:

Lemma 2 *The decentralized equilibrium in period 1 given that borrowers are constrained is determined by* b_1^a*,*

- *when* $b_1^a \geq \underline{b}_1^a$*, the negative effect of deleveraging on aggregate demand is completely buffered by the fall in interest rate, and firms produce efficiently at* w^**, with lenders' employment* $n_1^l = n^*$ *and borrowers' employment* $n_1^a > n^*$*; there is no aggregate demand shortage;*

- *when* $b_1^a < \underline{b}_1^a$*, there is an aggregate demand shortage, since a further fall in the interest rate that could have recovered households' demand is circumscribed by the zero lower bound. Firms produce and earn an economic profit at* $w_1 < w^*$*, with lenders' employment* $n_1^l < n^*$*.*

When borrowers are constrained, the interest rate must fall to induce lenders to hold less debt in equilibrium. Thus, the more borrowers are forced to deleverage in period 1, the lower the interest rate will be. As borrowers deleverage, the interest rate may hit the zero lower bound, which may lead to aggregate demand shortages.

When there is no aggregate demand shortage. The constrained equilibrium when $b_2^a = -\phi^{Aa}p_1$ and when there is no aggregate demand shortage is pinned down by the asset pricing equation and the Euler equation of the lenders:

$$p_1 = \frac{u'(e^* + d_2^a - \phi^{Aa}p_1 - v(e^*))}{u'(e^* + d_1^a + b_1^a + \frac{\phi^{Aa}p_1}{1+r_2} - v(e^*))}\beta^a d_2^a \tag{3.10}$$

$$u'(e^* + d_1^l + b_1^l - \frac{\alpha^a}{\alpha^l}\frac{\phi^{Aa}p_1}{1+r_2} - v(e^*)) = \beta^l(1+r_2)u'(e^* + d_2^l + \frac{\alpha^a}{\alpha^l}\phi^{Aa}p_1 - v(e^*)) \tag{3.11}$$

Assets in the model play two major roles: agents who hold the assets can get a dividend in the future which can increase consumption; assets can be used as collateral to borrow. The first role indicates that asset prices will be high when current consumption is high

[13] Due to the beginning-of-period asset sale, asset price in period 1 does not contain the Lagrangian multiplier associated with the borrowing constraint. This simplifies the derivations of the equilibrium and policy analysis in later sections, and does not affect the analytical results.

or expected future consumption is low. According to Equation (3.10), when the interest rate rises, asset prices fall because it lowers the value of bonds, which reduces the amount borrowers can borrow and thus current consumption. The inverse relation is captured by the AP curve in Figure 3.

Consider a comparative static with a fall in the net worth of the borrowers in period 1 will lead to lower consumption. If borrowers are constrained, it will depress asset prices as the demand for assets falls with lower current consumption and the higher marginal utility of current consumption. On the one hand, since borrowers are constrained, further deleveraging will induce a fall in the real interest rate r_2: $\frac{dr_2}{dp_1} \geq 0$, such that lenders are discouraged to hold debt, which tends to shift lenders‘ consumption to the current period.

On the other hand, lower asset prices will make borrowers more constrained, which further decreases consumption and lower asset prices, resulting in a feedback loop. The new decentralized equilibrium is shown in Figure 3, with lower interest rates and lower asset prices. Unlike in the model with an income-based borrowing constraint, this mechanism does not involve any fall in borrowers' or lenders' income as the income is at the efficient level. To have a unique equilibrium, the partial derivative of the right hand side of Equation (3.10) with respect to p_1 must be less than 1. This condition is satisfied if ϕ^{Aa} is small and satisfy:

$$Z_b^{an} = 1 + \frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}(u'(\tilde{c}_1^a)u''(\tilde{c}_2^a) + \frac{u''(\tilde{c}_1^a)u'(\tilde{c}_2^a)}{(1+r_2)}) > 0 \tag{3.12}$$

which simplifies to:

$$\phi^{Aa} < \sigma(\frac{\tilde{c}_1^a}{d_2^a} + \frac{\tilde{c}_2^a}{d_2^a}) \tag{3.13}$$

Note that since Z_b^{an} is less than one, a unit change in wealth of borrowers will cause $\frac{1}{1-Z_b^{an}}$ unit change in asset prices considering the partial equilibrium. Therefore, there is an amplification effect from the asset pricing equation.

When there is an aggregate demand shortage. The equilibrium will be pinned down by the asset pricing equation and the aggregate demand equation at the zero lower bound:

$$p_1 = \frac{u'(e^* + d_2^a - \phi^{Aa}p_1 - v(e^*))}{u'(e_1 + d_1^a + b_1^a + \phi^{Aa}p_1 - v(e_1))}\beta^a d_2^a \tag{3.14}$$

$$e_1 = 2\frac{\alpha^a}{\alpha^l}\phi^{Aa}p_1 + v(e_1) + \frac{\alpha^a}{\alpha^l}b_1^a + (d_2^l - d_1^l) + (e^* - v(e^*)) \tag{3.15}$$

For the asset pricing equation to have a unique and well-defined solution, it is necessary that $Z_b^{an} > 0$ at $r_2 = 0$. Let

$$X_b^{aa} = 1 - v'(e_1)$$

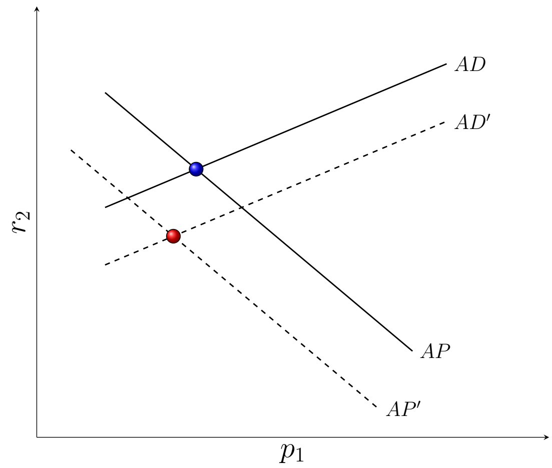

Figure 3: Effect of lower b_1^i on borrowers' employment and interest rate, no AD shortage

For the aggregate demand equation to have a unique and well-defined solution, X_b^{aa} needs to be less than one, which is equivalent to $v'(e_1) < 1$.[14] Decreasing the net worth of the borrowers now will not only depress asset prices through the feedback loop via the borrowing constraint, but also through the amplification mechanism by aggregate demand. That is, the lower consumption level that gives rise to falling asset prices is a result of both the asset-based borrowing constraint and the aggregate demand externalities due to the lower bound on the interest rate. As in Figure 4, a reduced wealth of borrowers will shift the AP curve to the left as it depresses asset prices, and it will shift the AD curve to the right as it lowers income. As a result, both income and asset prices are lower in the new equilibrium. This result is in line with the literature on fire sales and amplification effects from asset-based borrowing.

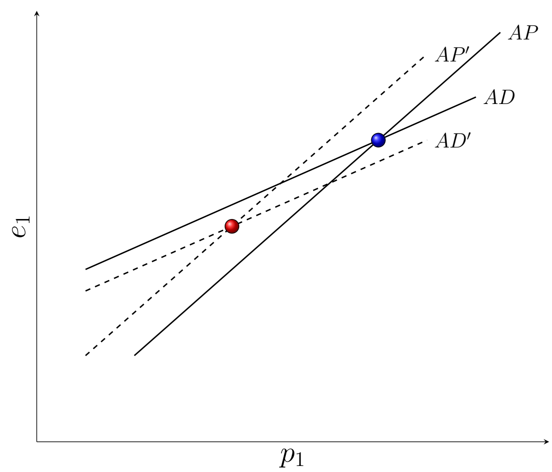

Figure 4: Effect of lower b_1^i on borrowers' employment and wage, with AD shortage

[14]The first "a" in the notation "aa" denotes asset-based borrowing, and the second one denotes aggregate demand shortage.

4 Comparative Statics

Fiscal Policy, and Liquidity Operations In this section, I will first consider the comparative statics of two different types of shocks on income, the interest rate, asset prices, and welfare, a shock on t_1^l and t_2^l to capture a shock on lenders' liquid wealth or a tax on lenders; and on d_1^i/d_1^a and d_2^i/d_2^a, to capture the shock on borrowers' liquid wealth or asset dividend or a subsidy on borrowers. Next, I will analyze the effect of two ex-post policies on welfare, fiscal policy, defined as a transfer across agents within period; and liquidity operations, defined as a transfer across time. I focus on households' welfare after deleveraging in period 1 and period 2, which is defined as the sum of the discounted utility of households in period 1 given by $V^h = u(\tilde{c}_1^h) + \beta^h u(\tilde{c}_2^h)$.

A. a shock on t_1^l and t_2^l

Income-based borrowing with no AD shortage. When there is no aggregate demand shortage, both types of shocks will not have any impact on the real wage and production is at an efficient level. Lenders supply labor given the efficient level of wage. Borrowers, constrained in borrowing by their labor income, will increase labor supply if the demand for bonds is greater. t_1^l and t_2^l can indirectly affect welfare through the interest rate. Higher t_1^l or lower t_2^l of the lenders will induce them to save more and boost their demand for bonds, which lowers the interest rate. A lower interest rate improves the welfare of the borrowers. Borrowers will work more and thus have higher labor income, given a lower interest rate, but it does not affect their welfare since wage is constant[15]. Therefore, the welfare of both borrowers and lenders is affected through interest rate as in (4.3) and (4.4).

$$\frac{dn_1^i}{dt_1^l} = \frac{\frac{u''(\tilde{c}_1^l)}{X_l^{in}}}{\frac{Z_b^{in}}{X_b^{in}} - \frac{Z_l^{in}}{X_l^{in}}} > 0 \qquad \frac{dr_2}{dt_1^l} = \frac{\frac{u''(\tilde{c}_1^l)}{X_l^{in}}}{\frac{Z_b^{in}}{X_b^{in}} - \frac{Z_l^{in}}{X_l^{in}}} \frac{Z_b^{in}}{X_b^{in}} < 0 \tag{4.1}$$

$$\frac{dn_1^i}{dt_2^l} = \frac{-\frac{\beta^l(1+r_2)u''(\tilde{c}_2^l)}{X_l^{in}}}{\frac{Z_b^{in}}{X_b^{in}} - \frac{Z_l^{in}}{X_l^{in}}} < 0 \qquad \frac{dr_2}{dt_2^l} = \frac{-\frac{\beta^l(1+r_2)u''(\tilde{c}_2^l)}{X_l^{in}}}{\frac{Z_b^{in}}{X_b^{in}} - \frac{Z_l^{in}}{X_l^{in}}} \frac{Z_b^{in}}{X_b^{in}} > 0 \tag{4.2}$$

$$\frac{\partial V^i}{\partial t_1^l} = -u'(\tilde{c}_1^i)\frac{\phi^{Ii} w_1 n_1^i}{(1+r_2)^2}\frac{dr_2}{dt_1^l} > 0 \qquad \frac{\partial V^l}{\partial t_1^l} = u'(\tilde{c}_1^l)(1 + \frac{\alpha^i}{\alpha^l}\frac{\phi^{Ii} w_1 n_1^i}{(1+r_2)^2}\frac{dr_2}{dt_1^l}) > 0 \tag{4.3}$$

$$\frac{\partial V^i}{\partial t_2^l} = -u'(\tilde{c}_1^i)\frac{\phi^{Ii} w_1 n_1^i}{(1+r_2)^2}\frac{dr_2}{dt_2^l} < 0 \qquad \frac{\partial V^l}{\partial t_2^l} = u'(\tilde{c}_1^l)(1 + \frac{\alpha^i}{\alpha^l}\frac{\phi^{Ii} w_1 n_1^i}{(1+r_2)^2}\frac{dr_2}{dt_2^l}) > 0 \tag{4.4}$$

where $\frac{Z_b^{in}}{X_b^{in}}$ is the slope of borrowers' labor supply equation, and $\frac{Z_l^{in}}{X_l^{in}}$ is the aggregate

[15]Also by the envelope theorem, changes in optimal labor supply does not directly affect welfare.

demand equation with

$$
\begin{aligned}
Z_b^{in} &= v''(n_1^i) - \frac{\beta^i \phi^{Ii} w_1}{(u'(\tilde{c}_1^i))^2}[\phi^{Ii} w_1 u'(\tilde{c}_1^i) u''(\tilde{c}_2^i) + (w_1 + \frac{\phi^{Ii} w_1}{1+r_2} - v'(n_1^i)) u''(\tilde{c}_1^i) u'(\tilde{c}_2^i)] > 0 \\
X_b^{in} &= -\frac{\phi^{Ii} w_1}{(1+r_2)^2}[1 + \frac{\beta^i \phi^{Ii} w_1 n_1^i}{(u'(\tilde{c}_1^i))^2} u''(\tilde{c}_1^i) u'(\tilde{c}_2^i)] < 0 \\
Z_l^{in} &= -\frac{\alpha^i}{\alpha^l} \phi^{Ii} w_1 [\frac{u''(\tilde{c}_1^l)}{1+r_2} + \beta^l (1+r_2) u''(\tilde{c}_2^l)] > 0 \\
X_l^{in} &= \beta^l u'(\tilde{c}_2^l) - \frac{\alpha^i}{\alpha^l} u''(\tilde{c}_1^l) \frac{\phi^{Ii} w_1 n_1^i}{(1+r_2)^2} > 0
\end{aligned}
$$

The notation of the slopes follows that the first superscript "i" or "a" denotes the type of borrowing, the second superscript "n" or "a" denotes no AD shortage or AD shortage, and the subscript "b" or "l" denotes borrowers or lenders. In the IBC model, type "b" indicates the equation of borrowers' labor supply choice; in the ABC model, type "b" indicates the asset pricing equation. In both the IBC and ABC model, type "l" indicates the Euler equation of the lenders for $t = 1, 2$. $X_b^{in} < 0$ is derived under previous restriction.

Income-based borrowing with AD shortage. When there is an aggregate demand shortage, a positive shock on t_1^l has a similar effect as a negative shock on t_2^l: they both lower households' income. The decrease in income results from the binding constraint on the interest rate. A higher t_1^l or lower t_2^l makes lenders more willing to save, which should lower the interest rate. However, since the interest rate cannot fall further, the bonds market does not clear with an interest rate too high. In response, lenders save more than they should, which lowers demand. As a result, firms hire fewer workers, and scale down production, which decreases the wage rate. Falling income reduces borrowers' debt capacity, which reduces demand further, leading to a feedback loop[16]. With an AD shortage, the wage is below the efficient level, $w_1 = v'(n_1^l) < 1$, welfare of both borrowers and lenders is undermined due to lower income as in (4.6).

$$
\frac{de_1^i}{dt_1^l} = -\frac{de_1^i}{dt_2^l} = -\frac{\frac{1}{X_l^{ia}}}{\frac{Z_b^{ia}}{X_b^{ia}} - \frac{Z_l^{ia}}{X_l^{ia}}} < 0 \qquad \frac{de_1^l}{dt_1^l} = -\frac{de_1^l}{dt_2^l} = -\frac{\frac{1}{X_l^{ia}}}{\frac{Z_b^{ia}}{X_b^{ia}} - \frac{Z_l^{ia}}{X_l^{ia}}} \frac{Z_b^{ia}}{X_b^{ia}} < 0 \tag{4.5}
$$

$$
\frac{\partial V_1^i}{\partial t_1^l} = \frac{v'(n_1^i)}{v'(n_1^l)}(1-w_1) u'(\tilde{c}_1^i)) \frac{de_1^i}{dt_1^l} < 0 \qquad \frac{\partial V_1^l}{\partial t_1^l} = u'(\tilde{c}_1^l) + (1-w_1) u'(\tilde{c}_1^l) \frac{de_1^l}{dt_1^l} < 0 \tag{4.6}
$$

$$
\frac{\partial V_1^i}{\partial t_2^l} = \frac{v'(n_1^i)}{v'(n_1^l)}(1-w_1) u'(\tilde{c}_1^i)) \frac{de_1^i}{dt_2^l} > 0 \qquad \frac{\partial V_1^l}{\partial t_2^l} = \beta^l u'(\tilde{c}_2^l) + (1-w_1) u'(\tilde{c}_1^l) \frac{de_1^l}{dt_2^l} > 0 \tag{4.7}
$$

where $\frac{Z_b^{ia}}{X_b^{ia}}$ is the slope of borrowers' labor supply equation, and $\frac{Z_l^{ia}}{X_l^{ia}}$ is the aggregate

[16] The GHH preference precludes the positive effect on labor supply when consumption falls and thus there is more amplification.

demand equation with

$$Z_b^{ia} = v''(n_1^i) - \frac{\beta^i \phi^{Ii} w_1}{(u'(\tilde{c}_1^i))^2}[\phi^{Ii} u'(\tilde{c}_1^i)u''(\tilde{c}_2^i) + (1 + \phi^{Ii} - v'(n_1^i))u''(\tilde{c}_1^i)u'(\tilde{c}_2^i)] > 0$$

$$X_b^{ia} = (1 + \phi^{Ii} - \beta^i \phi^{Ii} \frac{u'(\tilde{c}_2^i)}{u'(\tilde{c}_1^i)})v''(n_1^l) > 0$$

$$Z_l^{ia} = 2\frac{\alpha^i}{\alpha^l}\phi^{Ii} > 0$$

$$X_l^{ia} = 1 - v'(n_1^l) > 0$$

To have a well-defined equilibrium, the slopes of the two equations are restricted such that $\frac{Z_l^{ia}}{X_l^{ia}} < \frac{Z_b^{ia}}{X_b^{ia}}$ (can be satisfied when ϕ^{Ii} is small). Note that the amplification effect is captured by the multiplier $(1 - w_1)\frac{\frac{1}{X_l^{ia}}}{\frac{Z_b^{ia}}{X_b^{ia}} - \frac{Z_l^{ia}}{X_l^{ia}}}\frac{Z_b^{ia}}{X_b^{ia}} = \frac{1}{1 - \frac{Z_l^{ia}}{X_l^{ia}} / \frac{Z_b^{ia}}{X_b^{ia}}} > 1$ with $\frac{Z_l^{ia}}{X_l^{ia}} < \frac{Z_b^{ia}}{X_b^{ia}}$ for the lenders. Moreover, the income of the lenders are affected more than the borrowers since borrowers will increase labor supply when consumption falls due to lower income, as they are constrained in borrowing by labor income, which counteracted the impact of higher t_1^l, that is $\frac{Z_b^{ia}}{X_b^{ia}} > 1$[17].

Lemma 3 *A change in t_1^l or t_2^l has the opposite impact on an income-based borrowing economy when there is no aggregate demand shortage and when there is an aggregate demand shortage. An increase in t_1^l or a decrease in t_2^l makes the households better off when the interest rate is above the lower bound $\frac{\partial V^h}{\partial t_1^l} > 0$, whereas it makes the households worse-off when the interest rate is stuck at the lower bound $\frac{\partial V^h}{\partial t_1^l} < 0$.*

As the output is aggregate demand determined when prices are sticky, the interest rate will determine consumption demand and thus output. An increase in wealth will boost consumption of the lenders through a fall in the interest rate, leaving income at the optimal level when the interest rate is still flexible to move. The welfare of the borrowers is improved due to lower interest rate while that of the lenders is improved due to the direct effect of higher consumption dominating the adverse effect of lower interest rates. When the interest rate is at the lower bound, however, the demand shortage will be worsened by excessive savings of the lenders, which depresses production. The resulting lower wage and employment reduces income, further tightening the borrowing constraint when the debt limit is determined by income. The welfare of both types of households will be undermined as income decreases.

Asset-based borrowing with no AD shortage. When there is no AD shortage, higher t_1^l or lower t_2^l to the lenders will increase lenders' demand for bonds, lowering the interest rate, and since lenders become more willing to hold debt, the collateral

[17]See proof in the Appendix.

that the borrowers need for borrowing becomes more valuable, which boosts asset price. Therefore, the constraint on borrowers will be relaxed with higher collateral value. Both borrowers and lenders' income stay constant with production and wage at the efficient level. Households earn the same level of income, and there is no heterogeneity in income. The welfare of the borrowers is improved by higher asset price that relaxes their borrowing constraint and lower interest rate. Lenders, similar to lenders in the IBC economy with no AD shortage, are also better off due to the direct effect of higher consumption from greater wealth dominating the welfare loss from lower interest rate. The marginal effects on the interest rate, asset price and welfare are given by:

$$\frac{dp_1}{dt_1^l} = \frac{\frac{u''(\tilde{c}_1^l)}{X_l^{an}}}{\frac{Z_b^{an}}{X_b^{an}} - \frac{Z_l^{an}}{X_l^{an}}} > 0 \qquad \frac{dr_2}{dt_1^l} = \frac{\frac{u''(\tilde{c}_1^l)}{X_l^{an}}}{\frac{Z_b^{an}}{X_b^{an}} - \frac{Z_l^{an}}{X_l^{an}}} \frac{Z_b^{an}}{X_b^{an}} < 0 \tag{4.8}$$

$$\frac{dp_1}{dt_2^l} = \frac{-\frac{\beta^l(1+r_2)u''(\tilde{c}_2^l)}{X_l^{an}}}{\frac{Z_b^{an}}{X_b^{an}} - \frac{Z_l^{an}}{X_l^{an}}} < 0 \qquad \frac{dr_2}{dt_2^l} = \frac{-\frac{\beta^l(1+r_2)u''(\tilde{c}_2^l)}{X_l^{an}}}{\frac{Z_b^{an}}{X_b^{an}} - \frac{Z_l^{an}}{X_l^{an}}} \frac{Z_b^{an}}{X_b^{an}} > 0 \tag{4.9}$$

$$\frac{\partial V^a}{\partial t_1^l} = -u'(\tilde{c}_1^a)\frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_1^l} + \frac{\phi^{Aa}}{1+r_2}\frac{dp_1}{dt_1^l}[u'(\tilde{c}_1^a) - \beta^a(1+r_2)u'(\tilde{c}_2^a)] > 0 \tag{4.10}$$

$$\frac{\partial V^l}{\partial t_1^l} = u'(\tilde{c}_1^l)(1 + \frac{\alpha^a}{\alpha^l}\frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_1^l}) > 0 \tag{4.11}$$

$$\frac{\partial V^a}{\partial t_2^l} = -u'(\tilde{c}_1^a)\frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_2^l} + \frac{\phi^{Aa}}{1+r_2}\frac{dp_1}{dt_2^l}[u'(\tilde{c}_1^a) - \beta^a(1+r_2)u'(\tilde{c}_2^a)] < 0 \tag{4.12}$$

$$\frac{\partial V^l}{\partial t_2^l} = u'(\tilde{c}_1^l)(1 + \frac{\alpha^a}{\alpha^l}\frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_2^l}) > 0 \tag{4.13}$$

where where $\frac{Z_b^{an}}{X_b^{an}}$ is the slope of borrowers' labor supply equation, and $\frac{Z_l^{an}}{X_l^{an}}$ is the aggregate demand equation with

$$Z_b^{an} = 1 + \frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}(u'(\tilde{c}_1^a)u''(\tilde{c}_2^a) + \frac{u''(\tilde{c}_1^a)u'(\tilde{c}_2^a)}{(1+r_2)}) > 0$$

$$X_b^{an} = \frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}u''(\tilde{c}_1^a)u'(\tilde{c}_2^a) < 0$$

$$Z_l^{an} = -\frac{\alpha^a}{\alpha^l}\phi^{Aa}[\frac{u''(\tilde{c}_1^l)}{1+r_2} + \beta^l(1+r_2)u''(\tilde{c}_2^l)] > 0$$

$$X_l^{an} = \beta^l u'(\tilde{c}_2^l) - \frac{\alpha^a}{\alpha^l}u''(\tilde{c}_1^l)\frac{\phi^{Aa}p_1}{(1+r_2)^2} > 0$$

Lemma 4 *A change in t_1^l and t_2^l has similar effects on an income-based borrowing economy and an asset-based borrowing economy when there is no aggregate demand shortage. An* *increase in t_1^l will improve welfare of both types of households: $\frac{\partial V^h}{\partial t_1^l} > 0$. In an income-based borrowing economy, it is achieved via a fall in the interest rate; in an asset-based borrowing economy, it is achieved through not only a fall in the interest rate, but also an increase in the asset price which affects the welfare of the borrowers, not lenders, and*

(a) the decrease in the interest rate generates a redistribution of wealth between borrower and lenders; however, it does not generate any inefficiencies;

(b) the increase in asset price alleviates the pecuniary externalities.

With a positive shock on wealth during deleveraging, the interest rate in both cases will fall as lenders' demand for bonds increases. In the IBC economy, the reduction in interest rate will induce borrowers to work more hours such that they can consume more; similarly, in the ABC economy, it drives up asset prices as higher collateral value enables borrowers to borrow more and consume more. The resulting higher labor supply of the borrowers does not affect welfare whereas higher asset prices can alleviate the pecuniary externalities from the asset price feedback loop when there is no AD shortage.

Lemma 5 *A change in t_1^l and t_2^l has an opposing effect on an income-based borrowing economy when there is an aggregate demand shortage and an asset-based borrowing economy when there is no aggregate demand shortage. An increase in t_1^l undermines welfare with income-based borrowing $(\frac{\partial V^h}{\partial t_1^l})^I_{AD} < 0$, and improves welfare with asset-based borrowing $(\frac{\partial V^h}{\partial t_1^l})^A_{NAD} > 0$.*

An income-based borrowing economy with an AD shortage and an asset-based borrowing economy with no AD shortage can demonstrate the disparate transmission mechanisms of the two types of amplification. With income-based borrowing, shocks are transmitted through aggregate demand, and can be amplified only when wage falls. With asset-based borrowing, it is not necessary to have fluctuating income or wage for shocks to be amplified. Therefore, even when there is no AD shortage and wage is constant at the efficient level, amplification can occur through asset price changes. As t_1^l increases, it lowers income with income-based borrowing, but raises asset price with asset-based borrowing when aggregate demand externalities are absent. Thus, subsidizing lenders in the two economies will have an opposing impact on households' welfare.

Asset-based borrowing with AD shortage. Next, consider a marginal increase in t_1^l and t_2^l when there is an aggregate demand shortage for an asset-based borrowing economy. As with an IBC economy with an AD shortage, higher t_1^l or lower t_2^l leads to excessive saving by lenders, and depresses demand and production. Wage is lower, resulting in lower income for all households. Lower income decreases asset prices, making it harder for borrowers to borrow. With a tighter constraint, borrowers reduce consumption further, which depresses demand and production further, leading to a feedback loop. Unlike in the IBC model, lower aggregate demand and lower asset price reinforce each other. In the IBC model, borrowers will increase working hours in response to lower consumption, which raises wages and tempers the negative effect on income. The marginal effect on income, asset price and welfare are given by:

$$\frac{dp_1}{dt_1^l} = -\frac{\frac{1}{X_l^{aa}}}{\frac{Z_b^{aa}}{X_b^{aa}} - \frac{Z_l^{aa}}{X_l^{aa}}} < 0 \qquad \frac{de_1}{dt_1^l} = -\frac{\frac{1}{X_l^{aa}}}{\frac{Z_b^{aa}}{X_b^{aa}} - \frac{Z_l^{aa}}{X_l^{aa}}}\frac{Z_b^{aa}}{X_b^{aa}} < 0 \tag{4.14}$$

$$\frac{dp_1}{dt_2^l} = \frac{\frac{1}{X_l^{aa}}}{\frac{Z_b^{aa}}{X_b^{aa}} - \frac{Z_l^{aa}}{X_l^{aa}}} > 0 \qquad \frac{de_1}{dt_2^l} = \frac{\frac{1}{X_l^{aa}}}{\frac{Z_b^{aa}}{X_b^{aa}} - \frac{Z_l^{aa}}{X_l^{aa}}}\frac{Z_b^{aa}}{X_b^{aa}} > 0 \tag{4.15}$$

$$\frac{\partial V^a}{\partial t_1^l} = [(1 - v'(e_1))\frac{de_1}{dt_1^l} + \phi^{Aa}\frac{dp_1}{dt_1^l}]u'(\tilde{c}_1^a) \tag{4.16}$$

$$- \beta^a\phi^{Aa}u'(\tilde{c}_2^a)\frac{dp_1}{dt_1^l} < 0 \qquad \frac{\partial V^l}{\partial t_1^l} = u'(\tilde{c}_1^l) + [(1 - v'(e_1))]u'(\tilde{c}_1^l)\frac{de_1}{dt_1^l} < 0 \tag{4.17}$$

$$\frac{\partial V^a}{\partial t_2^l} = [(1 - v'(e_1))\frac{de_1}{dt_2^l} + \phi^{Aa}\frac{dp_1}{dt_2^l}]u'(\tilde{c}_1^a) \tag{4.18}$$

$$- \beta^a\phi^{Aa}u'(\tilde{c}_2^a)\frac{dp_1}{dt_2^l} > 0 \qquad \frac{\partial V^l}{\partial t_2^l} = u'(\tilde{c}_1^l) + [(1 - v'(e_1))]u'(\tilde{c}_1^l)\frac{de_1}{dt_2^l} > 0 \tag{4.19}$$

where $\frac{Z_b^{aa}}{X_b^{aa}}$ is the slope of the asset pricing equation, and $\frac{Z_l^{aa}}{X_l^{aa}}$ is the aggregate demand equation with

$$Z_b^{aa} = 1 + \frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}(u'(\tilde{c}_1^a)u''(\tilde{c}_2^a) + \frac{u''(\tilde{c}_1^a)u'(\tilde{c}_2^a)}{(1 + r_2)}) > 0$$
$$X_b^{aa} = -\frac{\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}(1 - v'(e_1))u''(\tilde{c}_1^a)u'(\tilde{c}_2^a) > 0$$
$$Z_l^{aa} = 2\frac{\alpha^a}{\alpha^l}\phi^{Aa} > 0$$
$$X_l^{aa} = 1 - v'(e_1) > 0$$

Z_b^{aa} is greater than zero under the previous restriction. I also restrict the slope of the asset equation and the aggregate demand equation in order to have a well-defined solution. That is, $\frac{Z_b^{aa}}{X_b^{aa}} > \frac{Z_l^{aa}}{X_l^{aa}}$. Note that the impact of one unit of increase in t_1^l on welfare through the channel of income will be amplified by $\frac{1}{1 - \frac{Z_l^{aa}}{X_l^{aa}}/\frac{Z_b^{aa}}{X_b^{aa}}} > 1$. To capture the reinforcing effect of asset price and aggregate demand, $\frac{Z_b^{ia}}{X_b^{ia}} > \frac{Z_b^{aa}}{X_b^{aa}}$ such that $|\frac{dp_1}{dt_1^l}| > |\frac{dn_1^i}{dt_1^l}|$.

Lemma 6 *A change in lenders' endowment t_1^l and t_2^l has similar effects on an income-based borrowing economy and an asset-based borrowing economy when there is an aggregated* *demand shortage. An increase in t_1^l or a decrease in t_2^l will lower income and undermine the welfare of both types of households: $\frac{\partial V^h}{\partial t_1^l} < 0$. In an asset-based borrowing economy, it affects the welfare of the borrowers through depressing asset prices and tightening the borrowing constraint in addition to the direct effect of lower wages and income; in an income-based economy, it affects the welfare of lenders through lowering income and tightening the borrowing constraint, and the direct effect of lower wage and income. Whether its impact is more pronounced will depend on the responsiveness of income to changes in the asset price $\frac{Z_b^{aa}}{X_b^{aa}}$:*

(a) If $\frac{Z_b^{aa}}{X_b^{aa}} > 1$, the effect of changes in lenders' wealth will be greater in income than asset price for the ABC borrowers, and $\frac{\partial V^a}{\partial t_1^l} > \frac{\partial V^i}{\partial t_1^l}$.

B. a shock on borrowers' dividend d_1^i and d_2^i, or d_1^a and d_2^a

The effects of a shock on borrowers' dividend d_1^i and d_1^i, or d_1^a and d_2^a are equivalent to the effect of a change in t_1^i or t_1^a, so I will use the notation of the transfers instead of the dividends.

Income-based borrowing with no AD shortage. For an income-based borrowing economy, when there is no aggregate demand shortage, an increase in d_1^i or a decrease in d_2^i will increase the consumption of the borrowers. Higher consumption makes borrowers less willing to borrow and therefore less incentivized to work so labor supply decreases, which decreases their debt with lower labor income. Interest rate falls in response to the lower supply of bonds. As with previous results when there is no AD shortage, changes in employment do not affect welfare. The welfare of the borrowers is improved through the direct effect of higher consumption and the reduction in interest rate, while the welfare of lenders is compromised due to lower interest rate. There is again a redistribution effect from interest rate changes, which does not generate any inefficiencies.

$$\frac{dn_1^i}{dt_1^i} = \frac{\frac{J_{b1}^{in}}{X_l^{ia}}}{\frac{Z_b^{in}}{X_b^{in}} - \frac{Z_l^{in}}{X_l^{in}}} < 0 \qquad \frac{dr_2}{dt_1^i} = \frac{\frac{J_{b1}^{in}}{X_l^{ia}}}{\frac{Z_b^{in}}{X_b^{in}} - \frac{Z_l^{in}}{X_l^{in}}} \frac{Z_b^{in}}{X_b^{in}} < 0 \tag{4.20}$$

$$\frac{dn_1^i}{dt_2^i} = \frac{\frac{J_{b2}^{in}}{X_l^{ia}}}{\frac{Z_b^{in}}{X_b^{in}} - \frac{Z_l^{in}}{X_l^{in}}} > 0 \qquad \frac{dr_2}{dt_2^i} = \frac{\frac{J_{b2}^{in}}{X_l^{ia}}}{\frac{Z_b^{in}}{X_b^{in}} - \frac{Z_l^{in}}{X_l^{in}}} \frac{Z_b^{in}}{X_b^{in}} > 0 \tag{4.21}$$

$$\frac{\partial V^i}{\partial t_1^i} = u'(\tilde{c}_1^i)[1 - \frac{\phi^{Ii} w_1 n_1^i}{(1+r_2)^2} \frac{dr_2}{dt_1^i}] > 0 \qquad \frac{\partial V^l}{\partial t_1^i} = u'(\tilde{c}_1^l) \frac{\alpha^i}{\alpha^l} \frac{\phi^{Ii} w_1 n_1^i}{(1+r_2)^2} \frac{dr_2}{dt_1^i} < 0 \tag{4.22}$$

$$\frac{\partial V^i}{\partial t_2^i} = \beta^i (1+r_2) u'(\tilde{c}_2^i) - u'(\tilde{c}_1^i) \frac{\phi^{Ii} w_1 n_1^i}{(1+r_2)^2} \frac{dr_2}{dt_2^i} \qquad \frac{\partial V^l}{\partial t_2^i} = u'(\tilde{c}_1^l) \frac{\alpha^i}{\alpha^l} \frac{\phi^{Ii} w_1 n_1^i}{(1+r_2)^2} \frac{dr_2}{dt_2^i} > 0 \tag{4.23}$$

with $J_{b1}^{in} = -\frac{\beta^i \phi^{Ii} w_1}{(u'(\tilde{c}_1^i))^2} u''(\tilde{c}_1^i) u'(\tilde{c}_2^i) > 0$ and $J_{b2}^{in} = \frac{\beta^i \phi^{Ii} w_1}{(u'(\tilde{c}_1^i))^2} u'(\tilde{c}_1^i) u''(\tilde{c}_2^i) < 0$. By previous restriction, $0 < J_{b1}^{in} < 1$.

Income-based borrowing with an AD shortage. When there is an aggregate demand shortage and the interest rate is at the lower bound, an increase in d_1^i or a decrease in d_2^i will increase the consumption of the borrowers. Higher consumption makes borrowers less willing to borrow and therefore less incentivized to work so labor supply decreases, which decreases their borrowing with lower labor income. Since the interest rate cannot fall to induce lenders to save less, the bonds market does not clear without adjustment of production and wage. Since lenders have excessive savings at the current interest rate, aggregate demand is lower, which decreases production. Firms will hire less and wages fall, reducing the income of households. The welfare of the lenders is undermined due to lower income. The welfare of the borrowers can still be improved by the direct effect of higher consumption.

$$\frac{de_1^i}{dt_1^i} = -\frac{\frac{J_{b1}^{ia}}{X_b^{ia}}}{\frac{Z_b^{ia}}{X_b^{ia}} - \frac{Z_l^{ia}}{X_l^{ia}}} < 0 \qquad \frac{de_1^l}{dt_1^i} = -\frac{\frac{J_{b1}^{ia}}{X_b^{ia}}}{\frac{Z_b^{ia}}{X_b^{ia}} - \frac{Z_l^{ia}}{X_l^{ia}}}\frac{Z_l^{ia}}{X_l^{ia}} < 0 \tag{4.24}$$

$$\frac{de_1^i}{dt_2^i} = -\frac{\frac{J_{b2}^{ia}}{X_b^{ia}}}{\frac{Z_b^{ia}}{X_b^{ia}} - \frac{Z_l^{ia}}{X_l^{ia}}} > 0 \qquad \frac{de_1^l}{dt_2^i} = -\frac{\frac{J_{b2}^{ia}}{X_b^{ia}}}{\frac{Z_b^{ia}}{X_b^{ia}} - \frac{Z_l^{ia}}{X_l^{ia}}}\frac{Z_l^{ia}}{X_l^{ia}} > 0 \tag{4.25}$$

$$\frac{\partial V_1^i}{\partial t_1^i} = u'(\tilde{c}_1^i) + \frac{v'(n_1^i)}{v'(n_1^l)}(1-w_1)u'(\tilde{c}_1^i))\frac{de_1^i}{dt_1^i} > 0 \qquad \frac{\partial V_1^l}{\partial t_1^i} = (1-w_1)u'(\tilde{c}_1^l)\frac{de_1^l}{dt_1^i} < 0 \tag{4.26}$$

$$\frac{\partial V_1^i}{\partial t_2^i} = \beta^i u'(\tilde{c}_2^i) + \frac{v'(n_1^i)}{v'(n_1^l)}(1-w_1)u'(\tilde{c}_1^i))\frac{de_1^i}{dt_2^i} > 0 \qquad \frac{\partial V_1^l}{\partial t_2^i} = (1-w_1)u'(\tilde{c}_1^l)\frac{de_1^l}{dt_2^i} > 0 \tag{4.27}$$

with $J_{b1}^{ia} = -\frac{\beta^i \phi^{Ii} w_1}{(u'(\tilde{c}_1^i))^2}u''(\tilde{c}_1^i)u'(\tilde{c}_2^i) > 0$ and $J_{b2}^{ia} = \frac{\beta^i \phi^{Ii} w_1}{(u'(\tilde{c}_1^i))^2}u'(\tilde{c}_1^i)u''(\tilde{c}_2^i) < 0$. Note that $\frac{\frac{de_1^i}{dt_1^i}}{\frac{de_1^i}{dt_2^i}} = \frac{\frac{de_1^l}{dt_1^i}}{\frac{de_1^l}{dt_2^i}} = \frac{J_{b1}^{ia}}{J_{b2}^{ia}} = -\frac{\tilde{c}_2^i}{\tilde{c}_1^i} < -1$. In addition, J_{b1}^{ia} and J_{b2}^{ia} are relatively small when ϕ^{Ii} is small and both are less than one. Therefore, the effect on income is smaller compared to the case with a change in t_1^l or t_2^l.

Asset-based borrowing with no AD shortage. Consider a marginal increase in d_1^a or a decrease in d_2^a when there is no aggregate demand shortage. An increase in asset dividends will make assets more valuable as it not only boosts the consumption by the borrowers in the current period directly, but relaxes the borrowing constraint as the price of the asset rises, which further increases consumption and inflates asset price. This is the canonical amplification mechanism with the asset-based borrowing constraint. Meanwhile, the interest rate must increase since the supply of bonds rises as the borrowers expand their debt capacity with more valuable collaterals. The welfare of borrowers is improved due to higher asset prices relaxing the borrowing constraint and the direct effect of higher consumption. The welfare of lenders is also improved due to a higher interest rate.

$$\frac{dp_1}{dt_1^a} = \frac{\frac{J_{b1}^{an}}{X_b^{an}}}{\frac{Z_b^{an}}{X_b^{an}} - \frac{Z_l^{an}}{X_l^{an}}} > 0 \qquad \frac{dr_2}{dt_1^a} = \frac{\frac{J_{b1}^{an}}{X_b^{an}}}{\frac{Z_b^{an}}{X_b^{an}} - \frac{Z_l^{an}}{X_l^{an}}} \frac{Z_b^{an}}{X_b^{an}} > 0 \tag{4.28}$$

$$\frac{dp_1}{dt_2^a} = \frac{\frac{J_{b2}^{an}}{X_b^{an}}}{\frac{Z_b^{an}}{X_b^{an}} - \frac{Z_l^{an}}{X_l^{an}}} < 0 \qquad \frac{dr_2}{dt_2^a} = \frac{\frac{J_{b2}^{an}}{X_b^{an}}}{\frac{Z_b^{an}}{X_b^{an}} - \frac{Z_l^{an}}{X_l^{an}}} \frac{Z_b^{an}}{X_b^{an}} < 0 \tag{4.29}$$

$$\frac{\partial V^a}{\partial t_1^a} = u'(\tilde{c}_1^a) - u'(\tilde{c}_1^a) \frac{\phi^{Aa} p_1}{(1+r_2)^2} \frac{dr_2}{dt_1^a} \tag{4.30}$$

$$+ \frac{\phi^{Aa}}{1+r_2} \frac{dp_1}{dt_1^a} [u'(\tilde{c}_1^a) - \beta^a (1+r_2) u'(\tilde{c}_2^a)] > 0 \qquad \frac{\partial V^l}{\partial t_1^a} = u'(\tilde{c}_1^l) \frac{\alpha^a}{\alpha^l} \frac{\phi^{Aa} p_1}{(1+r_2)^2} \frac{dr_2}{dt_1^a} > 0 \tag{4.31}$$

$$\frac{\partial V^a}{\partial t_2^a} = \beta^a (1+r_2) u'(\tilde{c}_2^a) - u'(\tilde{c}_1^a) \frac{\phi^{Aa} p_1}{(1+r_2)^2} \frac{dr_2}{dt_2^a} \tag{4.32}$$

$$+ \frac{\phi^{Aa}}{1+r_2} \frac{dp_1}{dt_2^a} [u'(\tilde{c}_1^a) - \beta^a (1+r_2) u'(\tilde{c}_2^a)] \qquad \frac{\partial V^l}{\partial t_2^a} = u'(\tilde{c}_1^l) \frac{\alpha^a}{\alpha^l} \frac{\phi^{Aa} p_1}{(1+r_2)^2} \frac{dr_2}{dt_2^a} < 0 \tag{4.33}$$

with $J_{b1}^{an} = -\frac{\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2} u''(\tilde{c}_1^a) u'(\tilde{c}_2^a) > 0$ and $J_{b2}^{an} = \frac{\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2} u'(\tilde{c}_1^a) u''(\tilde{c}_2^a) < 0$.

Lemma 7 *A change in t_1^a or t_1^i has different welfare implications for an income-based borrowing economy and an asset-based borrowing economy when there is no aggregate demand shortage. An increase in t_1^a or t_1^i will improve the welfare of borrowers: $\frac{\partial V^a}{\partial t_1^a} > 0$ and $\frac{\partial V^i}{\partial t_1^i} > 0$, and improve the welfare of lenders in the asset-based economy but will undermine welfare of lenders in the income-based economy. The difference in welfare implications originates from the disparate effect on the interest rate:*

(a) with IBC, interest rate falls due to less borrowing with lower labor supply;

(b) with ABC, interest rate rises due to more borrowing with higher asset prices.

Asset-based borrowing with an AD shortage. A marginal increase in d_1^a or a decrease in d_2^a when there is an aggregate demand shortage will increase the consumption of the borrowers. Higher current consumption boosts asset prices, enabling borrowing to take on more debt. Without adjustment of the interest rate, this boosts aggregate demand. Firms hire more labor and produce more, which raises income. Higher income further boosts consumption and asset prices. As a result, assets become more valuable and income is also higher. The welfare of both borrowers and lenders is improved.

$$\frac{dp_1}{dt_1^a} = -\frac{\frac{J_{b1}^{aa}}{X_l^{aa}}}{\frac{Z_b^{aa}}{X_b^{aa}} - \frac{Z_l^{aa}}{X_l^{aa}}} > 0 \qquad \frac{de_1}{dt_1^a} = -\frac{\frac{J_{b1}^{aa}}{X_l^{aa}}}{\frac{Z_b^{aa}}{X_b^{aa}} - \frac{Z_l^{aa}}{X_l^{aa}}} \frac{Z_l^{aa}}{X_l^{aa}} > 0 \tag{4.34}$$

$$\frac{dp_1}{dt_2^a} = \frac{\frac{J_{b2}^{aa}}{X_l^{aa}}}{\frac{Z_b^{aa}}{X_b^{aa}} - \frac{Z_l^{aa}}{X_l^{aa}}} < 0 \qquad \frac{de_1}{dt_2^a} = \frac{\frac{J_{b2}^{aa}}{X_l^{aa}}}{\frac{Z_b^{aa}}{X_b^{aa}} - \frac{Z_l^{aa}}{X_l^{aa}}} \frac{Z_l^{aa}}{X_l^{aa}} < 0 \tag{4.35}$$

$$\frac{\partial V^a}{\partial t_1^a} = [(1 - v'(e_1))\frac{de_1}{dt_1^a} + 1 + \phi^{Aa}\frac{dp_1}{dt_1^a}]u'(\tilde{c}_1^a) \tag{4.36}$$

$$- \beta^a \phi^{Aa} u'(\tilde{c}_2^a)\frac{dp_1}{dt_1^a} > 0 \qquad \frac{\partial V^l}{\partial t_1^a} = [(1 - v'(e_1))]u'(\tilde{c}_1^l)\frac{de_1}{dt_1^a} > 0 \tag{4.37}$$

$$\frac{\partial V^a}{\partial t_2^a} = [(1 - v'(e_1))\frac{de_1}{dt_2^a} + \phi^{Aa}\frac{dp_1}{dt_2^a}]u'(\tilde{c}_1^a) \tag{4.38}$$

$$- \beta^a \phi^{Aa} u'(\tilde{c}_2^a)\frac{dp_1}{dt_2^a} + \beta^a u'(\tilde{c}_2^a) \qquad \frac{\partial V^l}{\partial t_2^a} = [(1 - v'(e_1))]u'(\tilde{c}_1^l)\frac{de_1}{dt_2^a} < 0 \tag{4.39}$$

Lemma 8 *A change in t_1^a or t_1^i has different welfare implications for an income-based borrowing economy and an asset-based borrowing economy when there is an aggregate demand shortage. An increase in t_1^a or t_1^i makes all households better off in an asset-based borrowing economy: $\frac{\partial V^h}{\partial t_1^a} > 0$, whereas it can make lenders worse off in an income-based borrowing economy when aggregate demand externalities are large. The difference in welfare implications originates from the disparate effect on aggregate demand:*

(a) with IBC, aggregate demand falls due to less borrowing with lower labor supply;

(b) with ABC, aggregate demand increases due to more borrowing with higher asset prices.

This result will hold if the asset-based borrowing constraint is in the form $b_1^a \geq \phi^{Aa}\theta_2 p_1$ instead of $b_1^a \geq \phi^{Aa}\theta_1 p_1$ as in the current model. Subsidizing the ABC borrowers to increase consumption will also make them less incentivized to borrow, which lowers asset price, but as long as $\phi^{Aa} islessthanone$, the direct positive effect of higher current consumption on asset price will dominate. The smaller ϕ^{Aa} is, the greater asset price increases given the subsidy[18].

5 Ex Post Policies

During a deleveraging episode, how policymakers can boost income and asset prices to improve welfare? I consider the welfare effects of two types of ex-post policies, fiscal

[18] See proof in the Appendix.

policy and liquidity operations. The fiscal policy I focus on is defined as taxing lenders to subsidize borrowers in a lump-sum manner during the deleveraging period $t = 1$, and the government budget constraint is given by[19]:

$$\alpha^l t_1^l = \alpha^h t_1^h, \forall h \in \{a, i\}$$

Liquidity operation is defined as a lump-sum transfer financed by borrowing from lenders to purchase assets from borrowers in $t = 1$, and selling assets to the borrowers to pay back to lenders at $t = 2$. In practice, when the economy is in a liquidity trap, those liquidity provisions can be carried out at zero cost. Government budget constraints are given by:

$$\begin{aligned}\alpha^l t_1^l &= \alpha^h t_1^h, \\ \alpha^l t_2^l &= \alpha^h t_2^h \forall h \in \{a, i\}\end{aligned}$$

where $t_1^h = t_2^h$. I will assume $\alpha^i = \alpha^a = 0.5$ in each economy for simplicity. The superscript notation denotes the type of borrowing "i" or "a" and whether there is an AD shortage: "n" for no AD shortage or "a" for AD shortage; the subscript notation denotes the type of agents: "b" for borrowers or "l" for lenders. First, consider an economy with no aggregate demand shortage.

Income-based borrowing. The fiscal policy that transfers from lenders to borrowers will increase interest rate[20]. The increase in interest rate will have a redistribution effect in wealth from borrowers to lenders, but it does not generate any inefficiencies. Borrowers are still better off and lenders are worse off due to the direct effect on consumption.

$$\begin{aligned}FP_b^{in} &= -\frac{\partial V_1^i}{\partial t_1^l} + \frac{\partial V_1^i}{\partial t_1^i}\frac{\alpha^l}{\alpha^i} \\ &= \frac{\alpha^l}{\alpha^i}u'(\tilde{c}_1^i) - \frac{\phi^{Ii} n_1^i}{(1+r_2)^2}u'(\tilde{c}_1^i))(\frac{\alpha^l}{\alpha^i}\underbrace{\frac{dr_2}{dt_1^i}}_{-} - \underbrace{\frac{dr_2}{dt_1^l}}_{-}) > 0\end{aligned} \tag{5.1}$$

$$\begin{aligned}FP_l^{in} &= -\frac{\partial V_1^l}{\partial t_1^l} + \frac{\partial V_1^l}{\partial t_1^i}\frac{\alpha^l}{\alpha^i} \\ &= -u'(\tilde{c}_1^l) + \frac{\alpha^i}{\alpha^l}\frac{\phi^{Ii} n_1^i}{(1+r_2)^2}u'(\tilde{c}_1^l))(\frac{\alpha^l}{\alpha^i}\underbrace{\frac{dr_2}{dt_1^i}}_{-} - \underbrace{\frac{dr_2}{dt_1^l}}_{-}) < 0\end{aligned} \tag{5.2}$$

Liquidity operations have a similar impact on the interest rate, but it can make both

[19] I will also consider another type of fiscal policy that subsidizes labor income of the income-based borrowers by taxing lenders in later sections

[20] $|\frac{dr_2}{dt_1^i}| < |\frac{dr_2}{dt_1^l}|$ when ϕ^{Ii} is small.

borrowers and lenders better off. Since lenders are unconstrained, a transfer across time does not affect welfare directly through consumption. They are better off as a result of higher interest rate. Because borrowers are constrained, a transfer across time can improve welfare directly by relaxing the borrowing constraint.

$$\begin{aligned} LO_b^{in} &= -\frac{\partial V_1^i}{\partial t_1^l} + \frac{\partial V_1^i}{\partial t_1^i}\frac{\alpha^l}{\alpha^i} - \frac{\partial V_1^i}{\partial t_2^i}\frac{\alpha^l}{\alpha^i} + \frac{\partial V_1^i}{\partial t_2^l} \\ &= \frac{\alpha^l}{\alpha^i}(u'(\tilde{c}_1^i) - \beta^i u'(\tilde{c}_2^i)) - \frac{\phi^{Ii} n_1^i}{(1+r_2)^2} u'(\tilde{c}_1^i)[\frac{\alpha^l}{\alpha^i}(\frac{dr_2}{dt_1^i} - \frac{dr_2}{dt_2^i}) + (\frac{dr_2}{dt_2^l} - \frac{dr_2}{dt_1^l})] > 0 \end{aligned} \tag{5.3}$$

$$\begin{aligned} LO_l^{in} &= -\frac{\partial V_1^l}{\partial t_1^l} + \frac{\partial V_1^l}{\partial t_1^i}\frac{\alpha^l}{\alpha^i} - \frac{\partial V_1^l}{\partial t_2^i}\frac{\alpha^l}{\alpha^i} + \frac{\partial V_1^l}{\partial t_2^l} \\ &= \frac{\alpha^i}{\alpha^l}\frac{\phi^{Ii} n_1^i}{(1+r_2)^2} u'(\tilde{c}_1^l))[\frac{\alpha^l}{\alpha^i}(\frac{dr_2}{dt_1^i} - \frac{dr_2}{dt_2^i}) + (\frac{dr_2}{dt_2^l} - \frac{dr_2}{dt_1^l})] > 0 \end{aligned} \tag{5.4}$$

Asset-based borrowing. The impact of fiscal policy on interest rate is similar to that of income-based borrowing: interest rate will increase, which generates a wealth redistribution between borrowers and lenders. However, its impact on asset prices is ambiguous since subsidizing borrowers and lenders both increase asset prices. Given that ϕ^{Aa} is small such that the effect of asset prices on welfare is small, borrowers are still better off. In addition, since α^l can be much larger than α^a as constrained asset-based borrowers are only a small fraction of households, the positive effect on asset prices from a large purchase of asset can dominate the adverse effect on asset price from a small amount of borrowing from lenders. Lenders are worse off due to the direct effect of a reduction in consumption dominating the gain from the higher interest rate.

$$\begin{aligned} FP_b^{an} &= -\frac{\partial V_1^a}{\partial t_1^l} + \frac{\partial V_1^a}{\partial t_1^a}\frac{\alpha^l}{\alpha^a} \\ &= \frac{\alpha^l}{\alpha^a}u'(\tilde{c}_1^a) - u'(\tilde{c}_1^a)\frac{\phi^{Aa} p_1}{(1+r_2)^2}(\frac{\alpha^l}{\alpha^a}\underbrace{\frac{dr_2}{dt_1^a}}_{+} - \underbrace{\frac{dr_2}{dt_1^l}}_{-}) \\ &+ \frac{\phi^{Aa}}{1+r_2}[u'(\tilde{c}_1^a) - \beta^a(1+r_2)u'(\tilde{c}_2^a)](\frac{\alpha^l}{\alpha^a}\underbrace{\frac{dp_1}{dt_1^a}}_{+} - \underbrace{\frac{dp_1}{dt_1^l}}_{+}) > 0 \end{aligned} \tag{5.5}$$

$$\begin{aligned} FP_l^{an} &= -\frac{\partial V_1^l}{\partial t_1^l} + \frac{\partial V_1^l}{\partial t_1^a}\frac{\alpha^l}{\alpha^a} \\ &= -u'(\tilde{c}_1^l) + u'(\tilde{c}_1^l)\frac{\alpha^a}{\alpha^l}\frac{\phi^{Aa} p_1}{(1+r_2)^2}(\frac{\alpha^l}{\alpha^a}\frac{dr_2}{dt_1^a} - \frac{dr_2}{dt_1^l}) < 0 \end{aligned} \tag{5.6}$$

Liquidity operations when there is no AD shortage will improve the welfare of both borrowers and lenders in the asset-based borrowing economy as in the income-based borrowing economy.

$$
\begin{aligned}
LO_b^{an} &= -\frac{\partial V_1^a}{\partial t_1^l} + \frac{\partial V_1^a}{\partial t_1^a}\frac{\alpha^l}{\alpha^a} - \frac{\partial V_1^a}{\partial t_2^a}\frac{\alpha^l}{\alpha^a} + \frac{\partial V_1^a}{\partial t_2^l} \\
&= \frac{\alpha^l}{\alpha^a}(u'(\tilde{c}_1^a) - \beta^a u'(\tilde{c}_2^a)) - u'(\tilde{c}_1^a)\frac{\phi^{Aa}p_1}{(1+r_2)^2}[\frac{\alpha^l}{\alpha^a}(\frac{dr_2}{dt_1^a} - \frac{dr_2}{dt_2^a}) + (\frac{dr_2}{dt_2^l} - \frac{dr_2}{dt_1^l})] \\
&+ \frac{\phi^{Aa}}{1+r_2}[u'(\tilde{c}_1^a) - \beta^a(1+r_2)u'(\tilde{c}_2^a)][\frac{\alpha^l}{\alpha^a}(\frac{dp_1}{dt_1^a} - \frac{dp_1}{dt_2^a}) + (\frac{dp_1}{dt_1^l} - \frac{dp_1}{dt_2^l})] > 0
\end{aligned}
\tag{5.7}
$$

$$
\begin{aligned}
LO_l^{an} &= -\frac{\partial V_1^l}{\partial t_1^l} + \frac{\partial V_1^l}{\partial t_1^a}\frac{\alpha^l}{\alpha^a} - \frac{\partial V_1^l}{\partial t_2^a}\frac{\alpha^l}{\alpha^a} + \frac{\partial V_1^l}{\partial t_2^l} \\
&= u'(\tilde{c}_1^l)\frac{\alpha^a}{\alpha^l}\frac{\phi^{Aa}p_1}{(1+r_2)^2}[\frac{\alpha^l}{\alpha^a}(\frac{dr_2}{dt_1^a} - \frac{dr_2}{dt_2^a}) + (\frac{dr_2}{dt_2^l} - \frac{dr_2}{dt_1^l})] > 0
\end{aligned}
\tag{5.8}
$$

Proposition 1 *A fiscal policy that taxes lenders to subsidize borrowers in a crisis will improve the welfare of the borrowers and undermine the welfare of the lenders when there is no aggregate demand shortage, in both the IBC and ABC economy.*

(a) In the IBC economy, it only generates a wealth redistribution by increasing the interest rate;

(b) In the ABC economy, it can relax the borrowing constraint by boosting asset prices to further improve the welfare of the borrowers in addition to a wealth redistribution.

Next consider the IBC and ABC economy when there is an aggregate demand shortage.

Income-based borrowing. When there is an aggregate demand shortage, a fiscal policy that taxes the lenders to subsidize the borrowers during the deleveraging period at $t = 1$ will have an impact on households as follows:

$$
\begin{aligned}
FP_b^{ia} &= -\frac{\partial V_1^i}{\partial t_1^l} + \frac{\partial V_1^i}{\partial t_1^i}\frac{\alpha^l}{\alpha^i} \\
&= \frac{\alpha^l}{\alpha^i}u'(\tilde{c}_1^i) + (1 - v'(n_1^i))u'(\tilde{c}_1^i)(\frac{\alpha^l}{\alpha^i}\underbrace{\frac{de_1^i}{dt_1^i}}_{-} - \underbrace{\frac{de_1^i}{dt_1^l}}_{-}) + \phi^{Ii}[u'(\tilde{c}_1^i) - \beta^i u'(\tilde{c}_1^i)](\frac{\alpha^l}{\alpha^i}\underbrace{\frac{de_1^i}{dt_1^i}}_{-} - \underbrace{\frac{de_1^i}{dt_1^l}}_{-}) > 0
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
FP_l^{ia} &= -\frac{\partial V_1^l}{\partial t_1^l} + \frac{\partial V_1^l}{\partial t_1^i}\frac{\alpha^l}{\alpha^i} \\
&= -u'(\tilde{c}_1^l) + (1-w_1)u'(\tilde{c}_1^l))(\frac{\alpha^l}{\alpha^i}\underbrace{\frac{de_1^l}{dt_1^i}}_{-} - \underbrace{\frac{de_1^l}{dt_1^l}}_{-}) > 0
\end{aligned} \tag{5.10}
$$

The impact of fiscal policy on the income of lenders and borrowers is ambiguous since subsidizing the borrowers lowers income through aggregate demand as analyzed before. To have a positive net effect on income, first ϕ^{Ii} need to be small (to temper the negative effect of lower borrowing on aggregate demand and income) such that $\frac{J_{b1}^{ia}}{X_b^{ia}} < \frac{1}{X_l^{ia}}$ and thus $|\frac{de_1^i}{dt_1^i}| < |\frac{de_1^i}{dt_1^l}|$; second, the amount of lump-sum transfer to the IBC borrowers need to be small if there are both ABC and IBC borrowers in the economy. Higher income will improve the welfare of the borrowers by directly boosting net consumption and relaxing the borrowing constraint. It can improve the welfare of the lenders by directly boosting net consumption. Note that this result will depend on the magnitude of the amplification effect as well. The multiplier effect on welfare from lower t_1^l is given by $\frac{1}{1-\frac{Z_l^{ia}}{X_l^{ia}}/\frac{Z_b^{ia}}{X_b^{ia}}} > 1$.

Liquidity operations that borrow from lenders to purchase assets from income-based borrowers at $t = 1$, and sell assets to income-based borrowers to pay back to lenders at $t = 2$, will affect the welfare of the households:

$$
\begin{aligned}
LO_b^{ia} &= -\frac{\partial V_1^i}{\partial t_1^l} + \frac{\partial V_1^i}{\partial t_1^i}\frac{\alpha^l}{\alpha^i} - \frac{\partial V_1^i}{\partial t_2^i}\frac{\alpha^l}{\alpha^i} + \frac{\partial V_1^i}{\partial t_2^l} \\
&= \frac{\alpha^l}{\alpha^i}(u'(\tilde{c}_1^i) - \beta^i u'(\tilde{c}_2^i)) + (1 - v'(n_1^i))u'(\tilde{c}_1^i)[\frac{\alpha^l}{\alpha^i}(\frac{de_1^i}{dt_1^i} - \frac{de_1^i}{dt_2^i}) + (\frac{de_1^i}{dt_2^l} - \frac{de_1^i}{dt_1^l})] \\
&= \phi^{Ii}[u'(\tilde{c}_1^i) - \beta^i u'(\tilde{c}_1^i)][\frac{\alpha^l}{\alpha^i}(\frac{de_1^i}{dt_1^i} - \frac{de_1^i}{dt_2^i}) + (\frac{de_1^i}{dt_2^l} - \frac{de_1^i}{dt_1^l})] > 0
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
LO_l^{ia} &= -\frac{\partial V_1^l}{\partial t_1^l} + \frac{\partial V_1^l}{\partial t_1^i}\frac{\alpha^l}{\alpha^i} - \frac{\partial V_1^l}{\partial t_2^i}\frac{\alpha^l}{\alpha^i} + \frac{\partial V_1^l}{\partial t_2^l} \\
&= (1-w_1)u'(\tilde{c}_1^l))[\frac{\alpha^l}{\alpha^i}(\frac{de_1^l}{dt_1^i} - \frac{de_1^l}{dt_2^i}) + (\frac{de_1^i}{dt_2^l} - \frac{de_1^i}{dt_1^l})] > 0
\end{aligned} \tag{5.12}
$$

Similarly to the impact of liquidity operations for an ABC economy, liquidity operations for an IBC economy can also lead to a welfare improvement for both borrowers and lenders.

Asset-based borrowing. When there is an aggregate demand shortage, a fiscal policy that taxes the lenders to subsidize the borrowers during the deleveraging period at

$t = 1$ will have an impact on households as follows:

$$
\begin{aligned}
FP_b^{aa} &= -\frac{\partial V_1^a}{\partial t_1^l} + \frac{\partial V_1^a}{\partial t_1^a}\frac{\alpha^l}{\alpha^a} \\
&= \frac{\alpha^l}{\alpha^a}u'(\tilde{c}_1^a) + (1 - v'(e_1))u'(\tilde{c}_1^a))(\frac{\alpha^l}{\alpha^a}\underbrace{\frac{de_1}{dt_1^a}}_{+} - \underbrace{\frac{de_1}{dt_1^l}}_{-}) \\
&+ \phi^{Aa}[u'(\tilde{c}_1^a) - \beta^a u'(\tilde{c}_2^a)](\frac{\alpha^l}{\alpha^a}\underbrace{\frac{dp_1}{dt_1^a}}_{+} - \underbrace{\frac{dp_1}{dt_1^l}}_{-}) > 0
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
FP_l^{aa} &= -\frac{\partial V_1^l}{\partial t_1^l} + \frac{\partial V_1^l}{\partial t_1^a}\frac{\alpha^l}{\alpha^a} \\
&= -u'(\tilde{c}_1^l) + (1 - v'(e_1))u'(\tilde{c}_1^l))(\frac{\alpha^l}{\alpha^a}\frac{de_1}{dt_1^a} - \frac{de_1}{dt_1^l}) > 0
\end{aligned}
\tag{5.14}
$$

Unlike in the IBC model, subsidizing the ABC borrowers will increase asset prices, which reinforces the positive effect on aggregate demand and income. Therefore, a fiscal policy improves the welfare of the borrowers by boosting net consumption from higher income and relaxing the borrowing constraint with higher asset prices. It can also improve welfare of the lenders since the multiplier on income is greater than one and thus the positive effect on net consumption will dominate the negative effect from taxing the lenders.

Liquidity operations will affect the welfare of the households as follows:

$$
\begin{aligned}
LO_b^{aa} &= -\frac{\partial V_1^a}{\partial t_1^l} + \frac{\partial V_1^a}{\partial t_1^a}\frac{\alpha^l}{\alpha^a} - \frac{\partial V_1^a}{\partial t_2^a}\frac{\alpha^l}{\alpha^a} + \frac{\partial V_1^a}{\partial t_2^l} \\
&= \frac{\alpha^l}{\alpha^a}(u'(\tilde{c}_1^a) - \beta^a u'(\tilde{c}_2^a)) + (1 - v'(e_1))u'(\tilde{c}_1^a)[\frac{\alpha^l}{\alpha^a}(\frac{de_1}{dt_1^a} - \frac{de_1}{dt_2^a}) + (\frac{de_1}{dt_2^l} - \frac{de_1}{dt_1^l})] \\
&+ \phi^{Aa}[u'(\tilde{c}_1^a) - \beta^a u'(\tilde{c}_2^a)][\frac{\alpha^l}{\alpha^a}(\frac{dp_1}{dt_1^a} - \frac{dp_1}{dt_2^a}) + (\frac{dp_1}{dt_1^l} - \frac{dp_1}{dt_1^l})] > 0
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
LO_l^{aa} &= -\frac{\partial V_1^l}{\partial t_1^l} + \frac{\partial V_1^l}{\partial t_1^a}\frac{\alpha^l}{\alpha^a} - \frac{\partial V_1^l}{\partial t_2^a}\frac{\alpha^l}{\alpha^a} + \frac{\partial V_1^l}{\partial t_2^l} \\
&= (1 - v'(e_1))u'(\tilde{c}_1^l))[\frac{\alpha^l}{\alpha^a}(\frac{de_1^l}{dt_1^a} - \frac{de_1^l}{dt_2^a}) + (\frac{de_1}{dt_2^l} - \frac{de_1}{dt_1^l})] > 0
\end{aligned}
\tag{5.16}
$$

Liquidity operations will improve the welfare of both borrowers and lenders as previously.

Proposition 2 *A fiscal policy that taxes lenders to subsidize borrowers in a crisis will improve welfare of both borrowers and lenders when there is an aggregate demand shortage, in both the IBC and ABC economy. Subsidizing the ABC borrowers is more effective than* *subsidizing the IBC borrowers:*

(a) in the IBC economy, the sufficient condition for this result to hold is $\frac{1}{1-\frac{Z_l^{ia}}{X_l^{ia}}/\frac{Z_b^{ia}}{X_b^{ia}}} > 1$*;*

(b) in the ABC economy, the sufficient condition for this result to hold is $\frac{1}{1-\frac{Z_l^{aa}}{X_l^{aa}}/\frac{Z_b^{aa}}{X_b^{aa}}} > 1$*;*

(c) if $\frac{Z_b^{ia}}{X_b^{ia}} > \frac{Z_b^{aa}}{X_b^{aa}} > 1$*, fiscal policy improves the welfare of the ABC borrowers more than ABC borrowers.*

Proposition 3 *Liquidity operations that borrow from lenders to purchase assets from borrowers in a crisis, and sell assets to borrowers to pay back to lenders in the future will improve the welfare of both borrowers and lenders when there is no aggregate demand shortage and when there is an aggregate demand shortage, in both the IBC and ABC economy.*

(a) when there is no aggregate demand shortage, it improves lenders' welfare by increasing interest rate;

(b) when there is an aggregate demand shortage, it improves lenders' welfare by increasing wages.

6 Macroprudential Policies

Ex post policies can lead to Pareto improvements when aggregate demand externalities are large. However, it depends on the magnitude of the amplification. In a model set-up with separable preferences of households and the wealth effect on labor supply, aggregate demand externalities might not be large enough such that a fiscal policy as implemented in the previous section achieves such welfare improvements. Therefore, it is important to understand how ex ante policies, such as macroprudential policies, can be implemented to achieve an efficient outcome. I analyze the problem of a constrained planner that faces the same borrowing constraints as households do in the decentralized optimization problem, choosing allocations during the debt accumulation stage.

Let B_{b1} be the aggregate level of debt in the $b \in \{a, i\}$ type of borrowing economy in period 1, and λ_h be the Lagrangian multiplier associated with the type h borrowers. The decentralized problem of the households in period one can be written as:

$$\begin{aligned} V^h(b_1^h, B_{b1}) = \max_{b_2^h, n_1^h} \{ & u(n_1^h(B_{b1}) + d_1^h + b_1^h - \frac{b_2^h}{1 + r_2(B_{b1})} - v(n_1^h(B_{b1}))) \\ & + \beta^h u(n_2^h + d_2^h + b_2^h - v(n_2^h)) + \lambda_h [b_2^h + \phi^{Ih} n_1^h(B_{b1}) + \phi^{Ah} \theta_1 p_1(B_{b1})] \} \end{aligned} \quad (6.1)$$

where $n_1^l(B_{i1}) = 2\frac{\alpha^i}{\alpha^l}\phi^{Ii}n_1^i(b_{i1}) + v(n_1^l(B_{i1})) + \frac{B_{i1}}{\alpha^l} + (d_2^l - d_1^l) + (e^* - v(e^*))$ when there is an AD shortage; $\frac{dn_1^l}{dB_{i1}} = 0$ when there is no AD shortage. $\frac{dn_1^i}{dB_{i1}} > 0$ independent of AD shortage. And $p_1(B_{a1}) = \frac{u'(\tilde{c}_2^a)}{u'(\tilde{c}_1^a)}\beta^a d_2^a$. And $r_2(B_{b1}) = 0$ when there is an AD shortage; $r_2'(B_{b1}) > 0$ when there is no AD shortage. The first-order conditions are given by $u'(c_1^h) = (1+r_2)(\beta^h u'(c_2^h) + \lambda_h)$ and $u'(c_1^h)(1 - v'(n_1^h)) + \lambda_h\phi^{Ih} = 0$. The constrained planner takes into account the impact of aggregate debt on interest rate, aggregate demand, and asset price, so she chooses the aggregate level of debt in period 0 to:

$$\begin{aligned}
\max_{\{c_0^h, n_0^h, B_{b1}\}} \quad & \sum_{h\in\mathcal{H}} \alpha^h\gamma^h[u(c_0^h - v(n_0^h)) + \beta^h V^h(b_1^h, B_{b1})] \\
\text{s.t.} \quad & \sum_{h\in\mathcal{H}} \alpha^h c_0^h = \sum_{h\in\mathcal{H}} \alpha^h(n_0^h + \theta_0^h d_0^h), \\
& B_{i1} = \alpha^i b_1^i = -\alpha^l b_1^l, \quad \text{or} \quad B_{a1} = \alpha^a b_1^a = -\alpha^l b_1^l
\end{aligned} \tag{6.2}$$

The optimality conditions for the constrained planner's problem is given by:

$$v'(n_0^h) = 1 \tag{6.3}$$

$$\gamma^l u'(\tilde{c}_0^l) = \gamma^h u'(\tilde{c}_0^h) \quad \text{for} \quad h \in \{i, a\} \tag{6.4}$$

$$\sum_{h\in\mathcal{H}} \alpha^h\gamma^h\beta^h \frac{\partial V^h(b_1^h, B_{b1})}{\partial B_{h1}} = 0 \tag{6.5}$$

First consider an income-based borrowing economy, i.e., $b = i$. The optimality condition (6.5) can be written as:

$$\begin{aligned}
\gamma^l\beta^l u'(\tilde{c}_1^l) &= \gamma^i\beta^i u'(\tilde{c}_1^i) + \alpha^l\gamma^l\beta^l u'(\tilde{c}_1^l)(1 - v'(n_1^l))\frac{dn_1^l}{dB_{i1}} + \alpha^i\gamma^i\beta^i u'(\tilde{c}_1^i)(1 - v'(n_1^i))\frac{dn_1^i}{dB_{i1}} \\
&\quad + [\alpha^l\gamma^l\beta^l u'(\tilde{c}_1^l)b_2^l + \alpha^i\gamma^i\beta^i u'(\tilde{c}_1^i)b_2^i]\frac{1}{(1+r_2)^2}\frac{dr_2}{dB_{i1}} + \alpha^i\gamma^i\beta^i\phi^{Ii}\frac{dn_1^i}{dB_{i1}}\lambda_i \\
&= \gamma^i\beta^i u'(\tilde{c}_1^i) + \alpha^l\gamma^l\beta^l u'(\tilde{c}_1^l)(1 - v'(n_1^l))\frac{dn_1^l}{dB_{i1}} + [\alpha^l\gamma^l\beta^l u'(\tilde{c}_1^l)b_2^l + \alpha^i\gamma^i\beta^i u'(\tilde{c}_1^i)b_2^i]\frac{1}{(1+r_2)^2}\frac{dr_2}{dB_{i1}}
\end{aligned} \tag{6.6}$$

Note that the planner will never choose a level of aggregate debt B_{i1} which leads to an aggregate demand shortage. The reason is that when there is an AD shortage, $\frac{dn_1^l}{dB_{i1}} = \frac{1+2\alpha^i\phi^{Ii}\frac{dn_1^i}{dB_{i1}}}{\alpha^l(1-v'(n_1^l))}$, which makes the optimality condition of the planner (6.6) impossible to hold with equality. Therefore, the constrained efficient allocations of the planner exist only when $b_1^i \geq \underline{b}_1^i$.

Proposition 4 *In both the IBC economy, a macroprudential policy can be implemented to achieve constrained efficient allocations in the decentralized equilibrium. The macroprudential policy can be implemented as a quantity restriction on any positive debt issuance* *such that $b_1^i \geq \underline{b}_1^i$ combined with a lump-sum transfer between borrowers and lenders.*

Next consider an asset-based borrowing economy, i.e., $b = a$. The optimality condition (6.5) can be written as:

$$\gamma^l \beta^l u'(\tilde{c}_1^l) = \gamma^a \beta^a u'(\tilde{c}_1^a) + \alpha^l \gamma^l \beta^l u'(\tilde{c}_1^l)(1 - v'(n_1^l))\frac{dn_1^l}{dB_{a1}} + \alpha^a \gamma^a \beta^a u'(\tilde{c}_1^a)(1 - v'(n_1^a))\frac{dn_1^a}{dB_{a1}}$$
$$+ [\alpha^l \gamma^l \beta^l u'(\tilde{c}_1^l) b_2^l + \alpha^a \gamma^a \beta^a u'(\tilde{c}_1^a) b_2^a]\frac{1}{(1+r_2)^2}\frac{dr_2}{dB_{a1}} + \alpha^a \gamma^a \beta^a \phi^{Aa}\frac{dp_1}{dB_{a1}}\lambda_a \quad (6.7)$$

Similarly, the planner will never choose a level of aggregate debt B_{a1} which leads to an aggregate demand shortage since when there is an AD shortage, $\frac{dn_1^l}{dB_{a1}} = \frac{1+2\alpha^a \phi^{Aa}\frac{dp_1}{dB_{a1}}}{\alpha^l(1-v'(n_1^l))}$, which makes the optimality condition of the planner (6.7) impossible to hold with equality. Therefore, the constrained efficient allocations of the planner exist only when $b_1^a \geq \underline{b}_1^a$.

Moreover, (6.7) implies the planner will distort the Euler equation of the households whenever the borrowers are constrained, i.e., $\lambda_a > 0$, such that $\frac{u'(\tilde{c}_1^l)}{u'(\tilde{c}_0^l)} > \frac{u'(\tilde{c}_1^a)}{u'(\tilde{c}_0^a)}$. The constrained efficient allocation can be implemented with a tax τ_0^a on bond issuance of the borrowers combined with a lump-sum transfer to the borrowers. Assume the Pareto weights are chosen such that $\frac{\gamma^l}{\gamma^a} = \frac{u'(\tilde{c}_1^a)}{u'(\tilde{c}_1^l)}$ for the equality of wealth distribution. The optimal macroprudential tax τ_0^a is then given by:

$$\tau_0^a = \frac{\alpha^a \beta^a \phi^{Aa}\frac{dp_1}{dB_{a1}}\lambda_a}{\beta^a u'(\tilde{c}_1^a) + \alpha^a \beta^a \phi^{Aa}\frac{dp_1}{dB_{a1}}\lambda_a} \quad (6.8)$$

Proposition 5 *In the ABC economy, a macroprudential policy can be implemented to achieve constrained efficient allocations in the decentralized equilibrium. The macroprudential policy can be implemented as:*

- *a quantity restriction on any positive debt issuance, or*
- *a tax τ_0^a given in* (6.8) *on any positive debt issuance which is rebated to households in a lump-sum manner,*

combined with a lump-sum transfer between borrowers and lenders.

7 An Economy with Two Types of Borrowers

In this section, I will consider the model with additional heterogeneity in which $\mathcal{H} = \{l, i, a\}$, and each type of households has a weight of α^h with $\sum_h \alpha^h = 1$. The model environment is the same as in the previous section. I restrict $\phi^{Ia} = \phi^{Ai} = 0$, and $\phi^{Ii} > 0, \phi^{Aa} > 0$. Firms and households optimization problem is given in (2.10) and

(2.12). One important modification of the model in the numerical illustration is to have aggregate income, instead of individual income, in the income-based borrowing constraint. This modification enables the decentralized equilibrium at $t = 1, 2$ to be reduced to and pinned down by only two endogenous variables, interest rate and asset price when there is no aggregate demand shortage; and aggregate income and asset price when there is an aggregate demand shortage. Comparative statics of changes in t_1^l, t_1^i and t_1^a are similar to those of the model with individual income in the borrowing constraint. However, since borrowers no longer have the incentive to increase labor supply when consumption is low and to decrease labor supply when consumption is high, there will be no adverse impact on aggregate demand when t_1^i increases as seen in the model with individual income in the borrowing constraint when there is an aggregate demand shortage. Therefore, a transfer or subsidy to the IBC borrowers will improve the welfare of households more in the aggregate income model. All the derivations for the decentralized equilibrium and comparative statics are in the Appendix.

In the decentralized equilibrium, income- or asset-based borrowers can be the only type of households who are constrained in borrowing, but I will focus on the decentralized equilibrium in which both types of borrowers are borrowing constrained since it is more relevant for policy consideration. The bonds market clearing condition becomes $b_t^l = -\frac{\alpha^a}{\alpha^l} b_t^a - \frac{\alpha^i}{\alpha^l} b_t^i$.

When there is no aggregate demand shortage, the equilibrium is pinned down by:

$$u'(\tilde{c}_0^i) = \beta^i (1 + r_1) u'(\tilde{c}_1^i) \tag{7.1}$$

$$u'(\tilde{c}_0^a) = \beta^l (1 + r_1) u'(\tilde{c}_1^a) \tag{7.2}$$

$$u'(\tilde{c}_1^l) = \beta^l (1 + r_1) u'(\tilde{c}_1^l) \tag{7.3}$$

$$p_1 = \frac{u'(e^* + d_2^a - \phi^{Aa} p_1 - v(e^*))}{u'(e^* + d_1^a + b_1^a + \frac{\phi^{Aa} p_1}{1+r_2} - v(e^*))} \beta^a d_2^a \tag{7.4}$$

$$u'(e^* + d_1^l + b_1^l - \frac{1}{(1 + r_2)} (\frac{\alpha^a}{\alpha^l} \phi^{Aa} p_1 + \frac{\alpha^i}{\alpha^l} \phi^{Ii} e^*) - v(e^*)) \tag{7.5}$$

$$= \beta^l (1 + r_2) u'(e^* + d_2^l + \frac{\alpha^a}{\alpha^l} \phi^{Aa} p_1 + \frac{\alpha^i}{\alpha^l} \phi^{Ii} e^* - v(e^*)) \tag{7.6}$$

When there is an aggregate demand shortage, the equilibrium is pinned down by:

$$u'(\tilde{c}_0^i) = \beta^i(1+r_1)u'(\tilde{c}_1^i) \tag{7.7}$$

$$u'(\tilde{c}_0^a) = \beta^l(1+r_1)u'(\tilde{c}_1^a) \tag{7.8}$$

$$u'(\tilde{c}_1^l) = \beta^l(1+r_1)u'(\tilde{c}_1^l) \tag{7.9}$$

$$p_1 = \frac{u'(e^* + d_2^a - \phi^{Aa}p_1 - v(e^*))}{u'(e_1 + d_1^a + b_1^a + \phi^{Aa}p_1 - v(e_1))}\beta^a d_2^a \tag{7.10}$$

$$e_1 = 2\frac{\alpha^a}{\alpha^l}\phi^{Aa}p_1 + 2\frac{\alpha^i}{\alpha^l}\phi^{Ii}e_1 + v(e_1) + \frac{\alpha^a}{\alpha^l}b_1^a + \frac{\alpha^i}{\alpha^l}b_1^i + (d_2^l - d_1^l) + (e^* - v(e^*)) \tag{7.11}$$

$$b_1^l = -\frac{\alpha^a}{\alpha^l}b_1^a - \frac{\alpha^i}{\alpha^l}b_1^i \tag{7.12}$$

$$b_2^l = \frac{\alpha^a}{\alpha^l}\phi^{Aa}p_1 - \frac{\alpha^i}{\alpha^l}\phi^{Ii}e_1 \tag{7.13}$$

Illustration: a numerical example. I assume the utility function takes the form of:

$$u(c_t^h, n_t^h) = \frac{1}{1-\frac{1}{\sigma}}(c_t^h - \chi\frac{{n_t^h}^{1+\xi}}{1+\xi})^{1-\frac{1}{\sigma}},$$

where σ is the intertemporal elasticity of substitution and ξ is the frisch elasticity of labor supply. Value of the parameters in the model is calibrated as in Table 1.

elasticity of substitution	σ	0.5	standard value
disutility parameter of labor	χ	1	
frisch elasticity of labor supply	ξ	1	
discount factor of asset-based borrowers	β^a	0.96	standard value
discount factor of income-based borrowers	β^i	0.96	standard value
discount factor of lenders	β^l	1	
fraction of asset-based borrowers	α^a	0.1	the share of borrowing households who have mortgage
fraction of income-based borrowers	α^i	0.15	
fraction of lenders	α^l	0.75	
tightness of the ABC	ϕ^{Aa}	0.3	mortgage debt service payments as a percentage of disposable income
tightness of the IBC	ϕ^{Ii}	0.1	credit card debt as a percentage of GDP
elasticity of substitution	ϵ	0.8	standard value
asset dividend	d_t^h	0.15	average of housing share of US GDP
initial bond holdings of asset-based borrowers	b_0^a	-0.2	household mortgage debt to GDP ratio
initial bond holdings of income-based borrowers	b_0^i	-0.2	household credit card debt to GDP ratio

Table 1: Assumptions on parameters

Following these assumptions on parameters, $e^* = n^* = 1$. The decentralized equilibria are characterized in Figure 5 when there is no AD shortage and in Figure 6 when there is an AD shortage. Both equilibria is unique and well-defined. When there is an AD shortage (given the initial debt of borrowers $b_0^i = -0.28$), there is an equilibrium at which aggregate income is above 1. This equilibrium is not sustainable since firms will earn negative profits if the wage is above one. When there is no AD shortage, a fiscal policy that taxes the lenders to transfer to the asset-based borrowers, will shift the AP and AD curve up, leading to higher asset prices and higher interest rate. When there is an AD shortage, it also shifts up both the AP and AD curve, leading to higher asset prices and aggregate income. With a transfer to the income-based borrowers, there will be no upward shift of the AP curve, and therefore, asset prices and income do not rise as much as subsidizing the asset-based borrowers, which results in a smaller welfare improvement.

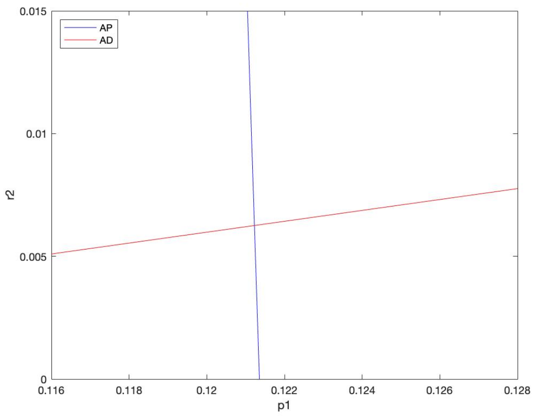

Figure 5: Equilibrium, No AD Shortage

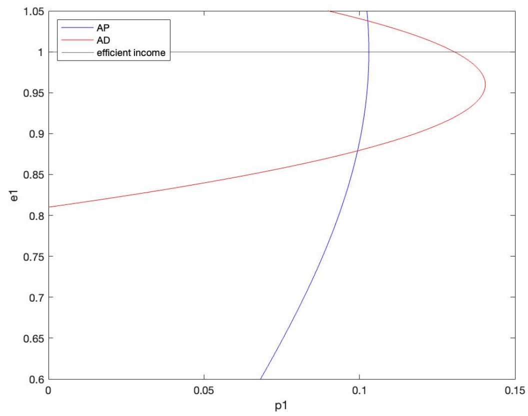

Figure 6: Equilibrium, AD Shortage

Figure 7 and 8 illustrate the marginal welfare gains from the fiscal policy. Fiscal policy does not lead to a Pareto improvement when there is no AD shortage. It incurs a welfare loss for the lenders due to a higher interest rate. However, it leads to a Pareto improvement when there is an AD shortage, since the income of both borrowers and lenders becomes higher, which improves their welfare. Moreover, as the fraction of subsidy given to the asset-based borrowers increases, the marginal gain in the welfare of both types of borrowers increases.

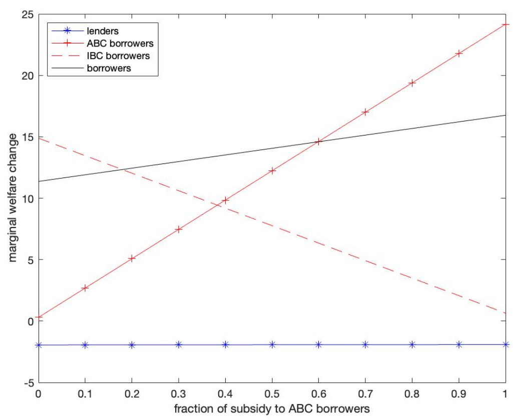

Figure 7: Welfare gains, No AD Shortage

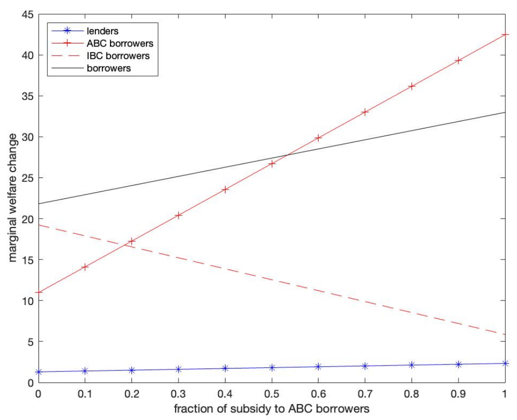

Figure 8: Welfare gains, AD Shortage

8 Conclusion

This paper studies the amplification effects with income-based borrowing constraints versus asset-based borrowing constraints. The effects of shocks are amplified via the pecuniary externalities arising from falling asset prices with the asset-based constraints, whereas they are amplified via the aggregate demand externalities as a result of the binding lower bound on the interest rate with the income-based constraints. The differences in the transmission mechanism of shocks with these types of constraints have different policy implications.

A fiscal policy that taxes lenders to subsidize borrowers in a crisis will improve the welfare of the borrowers and undermine the welfare of the lenders when there is no aggregate demand shortage, in both the IBC and ABC economy. In the IBC economy, it only generates wealth redistribution by increasing the interest rate. In the ABC economy, it can relax the borrowing constraint by boosting asset prices to improve the welfare of the borrowers in addition to wealth redistribution. Lenders are always worse off due to the tax. A fiscal policy that taxes lenders to subsidize borrowers in a crisis can improve the welfare of both borrowers and lenders when there is an aggregate demand shortage, leading to a Pareto improvement when aggregate demand externalities are large in both the IBC and ABC economy. Subsidizing the ABC borrowers in a lump-sum form can improve welfare more than subsidizing the IBC borrowers.

Liquidity operations that borrow from lenders to carry out asset purchases during a deleveraging episode and sales after deleveraging to pay back to lenders can lead to a Pareto improvement independent of whether there is an aggregate demand shortage, in both the IBC and ABC economy. Since it involves a transfer across time, it improves borrowers' welfare by getting around the borrowing constraint. Since lenders are unconstrained, the effect of a current loss in wealth is completely offset by an increase in wealth in the future. When there is no aggregate demand shortage, it improves lenders' welfare by increasing interest rate; when there is an aggregate demand shortage, it improves lenders' welfare by increasing income.

A quantity restriction on debt issuance can achieve constrained efficiency with both IBCs and ABCs. A macroprudential tax on any positive debt issuance combined with a transfer between borrowers and lenders will lead to constrained efficient allocation with ABCs. Due to the form of preferences, it is not feasible to derive an analytical solution of the optimal macroprudential tax with IBCs, which opens up possibilities for future research.

References

G. Benigno, H. Chen, C. Otrok, A. Rebucci, and E. R. Young. Financial crises and macro-prudential policies. *Journal of International Economics*, 89(2):453–470, 2013.

B. Bernanke and M. Gertler. Agency Costs, Net Worth, and Business Fluctuations. *The American Economic Review*, 79(1):14–31, 1989. ISSN 0002-8282. URL `https://www.jstor.org/stable/1804770`. Publisher: American Economic Association.

B. S. Bernanke, M. Gertler, and S. Gilchrist. Chapter 21 The financial accelerator in a quantitative business cycle framework. In *Handbook of Macroeconomics*, volume 1, pages 1341–1393. Elsevier, Jan. 1999. doi: 10.1016/S1574-0048(99)10034-X. URL `https://www.sciencedirect.com/science/article/pii/S157400489910034X`.

J. Bianchi. Overborrowing and systemic externalities in the business cycle. *American Economic Review*, 101(7):3400–3426, 2011.

G. A. Calvo. Staggered prices in a utility-maximizing framework. *Journal of monetary Economics*, 12(3):383–398, 1983.

T. Chaney, D. Sraer, and D. Thesmar. The collateral channel: How real estate shocks affect corporate investment. *American Economic Review*, 102(6):2381–2409, May 2012. doi: 10.1257/aer.102.6.2381. URL `http://www.aeaweb.org/articles?id=10.1257/aer.102.6.2381`.

S. Chava and M. R. Roberts. How does financing impact investment? the role of debt covenants. *The journal of finance*, 63(5):2085–2121, 2008.

G. Chodorow-Reich and A. Falato. The loan covenant channel: How bank health transmits to the real economy. Technical report, National Bureau of Economic Research, 2017.

D. Corbae and E. Quintin. Leverage and the foreclosure crisis. *Journal of Political Economy*, 123(1):1–65, 2015.

T. Drechsel. Earnings-based borrowing constraints and macroeconomic fluctuations. 2019.

R. Duchin, O. Ozbas, and B. A. Sensoy. Costly external finance, corporate investment, and the subprime mortgage credit crisis. *Journal of financial economics*, 97(3):418–435, 2010.

G. B. Eggertsson and P. Krugman. Debt, deleveraging, and the liquidity trap: A fisher-minsky-koo approach. *The Quarterly Journal of Economics*, 127(3):1469–1513, 2012.

E. Farhi and I. Werning. A theory of macroprudential policies in the presence of nominal rigidities. *Econometrica*, 84(5):1645–1704, 2016.

J. Goldberg. A model of deleveraging. *Unpublished manuscript, MIT*, 2010.

D. Greenwald. The mortgage credit channel of macroeconomic transmission. 2018.

J. Greenwood, Z. Hercowitz, and G. W. Huffman. Investment, capacity utilization, and the real business cycle. *The American Economic Review*, pages 402–417, 1988.

V. Guerrieri and G. Lorenzoni. Credit crises, precautionary savings, and the liquidity trap. *The Quarterly Journal of Economics*, 132(3):1427–1467, 2017.

O. Hart and J. Moore. A theory of debt based on the inalienability of human capital. *The Quarterly Journal of Economics*, 109(4):841–879, 1994.

V. Ivashina, L. Laeven, and E. Moral-Benito. Loan types and the bank lending channel. Technical report, Mimeo, 2019.

O. Jeanne and A. Korinek. Excessive volatility in capital flows: A pigouvian taxation approach. *American Economic Review*, 100(2):403–07, 2010.

N. Kiyotaki and J. Moore. Credit Cycles. *Journal of Political Economy*, 105(2):211–248, 1997. ISSN 0022-3808. doi: 10.1086/262072. URL `https://www.jstor.org/stable/10.1086/262072`. Publisher: The University of Chicago Press.

A. Korinek and A. Simsek. Liquidity trap and excessive leverage. Technical report, National Bureau of Economic Research, 2014.

A. Korinek and A. Simsek. Liquidity trap and excessive leverage. *American Economic Review*, 106(3):699–738, 2016.

C. Lian and Y. Ma. Anatomy of Corporate Borrowing Constraints*. *The Quarterly Journal of Economics*, 136(1):229–291, Feb. 2021. ISSN 0033-5533. doi: 10.1093/qje/qjaa030. URL `https://doi.org/10.1093/qje/qjaa030`.

E. G. Mendoza. Sudden Stops, Financial Crises, and Leverage. *American Economic Review*, 100(5):1941–1966, Dec. 2010. ISSN 0002-8282. doi: 10.1257/aer.100.5.1941. URL `https://pubs.aeaweb.org/doi/10.1257/aer.100.5.1941`.

A. Mian and A. Sufi. What explains the 2007–2009 drop in employment? *Econometrica*, 82(6):2197–2223, 2014.

A. Mian, K. Rao, and A. Sufi. Household balance sheets, consumption, and the economic slump. *The Quarterly Journal of Economics*, 128(4):1687–1726, 2013.

M. R. Roberts and A. Sufi. Control rights and capital structure: An empirical investigation. *The Journal of Finance*, 64(4):1657–1695, 2009.

A. Sufi. Bank lines of credit in corporate finance: An empirical analysis. *The Review of Financial Studies*, 22(3):1057–1088, 2009.

Appendix

A.1 Solving the model

Conditions for deleveraging to occur. Borrowers need to be sufficiently more impatient than lenders so that they will choose a level of d_1 greater than $\bar{d}_1$. The Euler equations for households in the initial two time periods are given by:

$$1 + r_1 = \frac{u'(e^* - 1 - \frac{d_1}{1+r_1})}{\beta^l u'((1-\phi)e^* - 1 + d_1)} = \frac{u'(e^* - 1 + \frac{d_1}{1+r_1})}{\beta^b u'((1+\phi)e^* - 1 - d_1)} \tag{A.1}$$

Consider the LHS of Equation (A.1) when r_2 reaches 0. By (A.8), the LHS can be reduced to:

$$(\beta^l)^2(1 + r_1) = u'(e^* - 1 - \frac{\bar{d}_1}{1 + r_1}) \tag{A.2}$$

Observe that r_1 is an increasing function of $\bar{d}_1$, and therefore, the upper bound on c_1^l, which is determined by β^l, determines the upper bound on d_1, $\bar{d}_1$, which defines an upper bound on r_1. Moreover, note that $\frac{d_{(d_1)}}{d_{(1+r_1)}} > 1$. Rewrite the RHS of (A.1):

$$\underline{\beta}^b = \beta^l \, \frac{u'(\bar{c}_1^l - 1)}{u'((1+\phi)e^* - 1 - \bar{d}_1)} \, \frac{u'(e^* - 1 + \frac{\bar{d}_1}{1+\bar{r}_1})}{u'(e^* - 1j - \frac{\bar{d}_1}{1+\bar{r}_1})} \tag{A.3}$$

A higher $\bar{d}_1$ indicates a lower β^l and a higher $\bar{c}_1^l$ due to the strict concavity of $u'(\cdot)$. This will render the first fraction on the RHS of (A.3) less than 1. Similarly, $\bar{r}_1$ increases, and with $\frac{d_{(d_1)}}{d_{(1+r_1)}} > 1$, the second fraction on the RHS of (A.3) will also be less than 1. Equation (A.3) then defines a lower bound for β^b. As long as $\beta^b < \underline{\beta}^b$, borrowers will choose a level of d_1 which is sufficiently high to trigger a demand-driven recession.

Restrictions on ϕ^{Ii} in the IBC model. To see why we need a restriction on ϕ^{Ii}, rewrite Equation (3.5) as:

$$w_1 - v'(n_1^i) + \frac{\phi^{Ii} w_1}{1 + r_2} = \beta^i \phi^{Ii} w_1 \frac{u'(\tilde{c}_2^i)}{u'(\tilde{c}_1^i)} > 0 \tag{A.4}$$

Take derivative with respect to n_1^i with Equation (A.6):

$$\begin{aligned}
&-\frac{\phi^{Ii} w_1}{(1 + r_2)^2}[1 + \frac{\beta^i u''(\tilde{c}_1^i) u'(\tilde{c}_2^i)}{(u'(\tilde{c}_1^i))^2} \phi^{Ii} w_1 n_1^i] \frac{dr_2}{dn_1^i} = \\
&\quad v''(n_1^i) - \frac{\phi^{Ii} w_1 \beta^i}{(u'(\tilde{c}_1^i))^2} \{-u'(\tilde{c}_1^i) u''(\tilde{c}_2^i) \phi^{Ii} w_1 - u''(\tilde{c}_1^i) u'(\tilde{c}_2^i)[w_1 - v'(n_1^i) + \frac{\phi^{Ii} w_1}{1 + r_2}]\}
\end{aligned} \tag{A.5}$$

Since RHS is positive, if

$$1 + \frac{\beta^i u''(\tilde{c}_1^i) u'(\tilde{c}_2^i)}{(u'(\tilde{c}_1^i))^2} \phi^{Ii} w_1 n_1^i > 0,$$

the interest rate will be decreasing when employment of the borrowers increases. Approximate $\beta^i(1 + r_2)u'(\tilde{c}_2^i) \approx \beta^i u'(\tilde{c}_2^i) = u'(\tilde{c}_1^i)$, and the CRRA utility function with σ the elasticity of substitution, the inequality can be rewritten as:

$$\phi^{Ii} < \sigma \frac{\tilde{c}_1^i}{w_1 n_1^i}.$$

The threshold level of b_1^i in the IBC model. The threshold level of b_1^i can be derived from Equation (3.5) and (A.7) by setting the real interest rate to zero and the real wage to 1:

$$w_1 - v'(n_1^i) + \phi^{Ii} w_1 = \beta^i \phi^{Ii} w_1 \frac{u'(e^* + t_2^i + d_2^i - \phi^{Ii} w_1 n_1^i - v(n^*))}{u'(w_1 n_1^i + t_1^i + d_1^i + b_1^i + \phi^{Ii} w_1 n_1^i - v(n_1^i))} \tag{A.6}$$

$$u'(w_1 n_1^l + t_1^l + d_1^l - \frac{\alpha^i}{\alpha^l} b_1^i - \frac{\alpha^i}{\alpha^l} \phi^{Ii} w_1 n_1^i - v(n_1^l)) = \beta^l u'(e^* + t_2^l + d_2^l + \frac{\alpha^i}{\alpha^l} \phi^{Ii} w_1 n_1^i - v(n^*)) \tag{A.7}$$

With lower b_1^i or greater leverage, labor supply of the borrowers is increasing by both of the equations. Define the solution from the system of equations as $\underline{b}_1^i$. Therefore, ϕ^{Ii} has to be sufficiently small so that the interest rate will reach the zero lower bound before borrowers work more hours to be unconstrained by the borrowing limit.

A.2 Aggregate income in the borrowing constraint

When the debt limit is determined by aggregate income with no asset-based borrowing households in the economy. Similarly, the model can be solved via backward induction. Period 2 consumption and labor choices are intratemporal decisions given b_2^h at the beginning of period 2. By market clearing condition, lenders' bond holdings will be $\alpha^l b_2^l = -\alpha^i b_2^i$. Let net consumption be $\tilde{c}_t^h$, which is equal to $c_t^h - v(n_t^h)$. With monetary policy replicating the first-best outcome in every period, the Euler equation of the lenders is then given by:

$$u'(\tilde{c}_1^l) = \beta^l (1 + r_2) u'(e^* + t_2^l + d_2^l - \frac{\alpha^i}{\alpha^l} b_2^i - v(n^*)) \tag{A.8}$$

For a given level of b_2^i that borrowers take on, as r_2 falls, net consumption of the lenders $\tilde{c}_1^l$ will increase. Since prices are fixed, the real interest rate will govern the demand and therefore how much firms produce. As borrowers accumulate debt, the IBC they face in period 1 may force them to deleverage. Deleveraging by the borrowers reduces consumption demand of the borrowers. The interest rate will have to fall to induce lenders to hold less bonds, which boosts lenders' consumption to an extent where firms produces optimally satisfying aggregate demand. However, if debt accumulation is beyond a threshold level, the real interest rate may not fall enough to clear the goods market. Since the intertemporal price cannot adjust, the intratemporal price, the wage rate will fall, reducing labor supply. Output, falling below the optimal, will be determined by the aggregate demand at the zero interest rate.

When there is no aggregate demand shortage. Consider the decentralized equilibrium when there is no demand shortage and all markets clear[21]. Due to the constraint on borrowers' debt, the maximum level of debt they can take on will be $\phi^{Ii}e^*$. This will define the corresponding upper bound on net consumption $\bar{c}_1^l$, $\bar{\bar{c}}_1^l$ when r_2 reaches the lower bound 0

$$\bar{\bar{c}}_1^l = (1 + \frac{\alpha^i}{\alpha^l}\phi^{Ii})e^* + t_2^l + d_2^l - v(n^*)$$

Correspondingly, the upper bound on consumption of the lenders is given by:

$$\bar{c}_1^l = \bar{\bar{c}}_1^l + v(n^*) = (1 + \frac{\alpha^i}{\alpha^l}\phi^{Ii})e^* + t_2^l + d_2^l \tag{A.9}$$

The upper bound on lenders' consumption in period 1 reflects that lenders' demand is constrained by the lower bound on the interest rate. Aggregate demand in period 1 can be written as:

$$\begin{aligned}\alpha^l c_1^l + \alpha^i c_1^i &= \alpha^l c_1^l + \alpha^i(e^* + t_1^i + d_1^i + \frac{\phi^{Ii}e^*}{1+r_2} + b_1^i) \\ &= e^* + \alpha^l(t_1^l + d_1^l) + \alpha^i(t_1^i + d_1^i)\end{aligned} \tag{A.10}$$

If real interest rate is above the lower bound, firms can always operate efficiently, and the efficient level of income is given by $e^* = n^*$, where $n^* = v'^{-1}(1)$, as in the first-best solution. The allocations are constrained efficient, with consumption of the households in period 1 given by:

$$c_1^h = e^* + t_1^h + d_1^h + b_1^h - \frac{\frac{\alpha^i}{\alpha^l}\phi^{Ii}e^*}{1+r_2}$$

When there is an aggregate demand shortage. If real interest rate is constrained by the lower bound, aggregate demand will be below the efficient level. This can be a result of large accumulation of debt in period 0 that triggers massive deleveraging in period 1 by the borrowers. The loss in demand by the borrowers need to be picked up by a fall in the interest rate, which will induce an increase in consumption demand by the lenders, as shown in Equation (A.10). If b_1^i exceeds a certain level, the interest rate will reach the zero lower bound. This threshold of debt is given by:

$$|\bar{b}_1^i| = 2\phi^{Ii}e^* + \frac{\alpha^l}{\alpha^i}(t_2^l + d_2^l - t_1^l - d_1^l) \tag{A.11}$$

Amplification. If $-b_1^i > |\bar{b}_1^i|$, deleveraging by borrowers will trigger a demand-driven recession when income becomes sub-optimal. Lenders' consumption demand cannot reach $\bar{c}_1^l$, but is still maximized at the zero interest rate. Note that since lenders and borrowers' labor supply $n_t^l = n_t^i$, they earn the same level of labor income. In addition, when wage is below the efficient level, firms will earn positive profits, and therefore $e_t^h = e_t$ and $e_1^h = w_1 n_1 + y_1 - w_1 n_1 = y_1 = n_1$ in equilibrium. Household income is then determined by aggregate demand at $r_2 = 0$ and is given by:

$$e_1 + \alpha^l(t_1^l + d_1^l) + \alpha^i(t_1^i + d_1^i) = \alpha^l c_1^l + \alpha^i c_1^i$$

[21] The sufficient conditions for the existence of such an equilibrium are in the Appendix.

$$e_1 = 2\frac{\alpha^i}{\alpha^l}\phi^{Ii}e_1 + v(e_1) + \frac{\alpha^i}{\alpha^l}b_1^i + (t_2^l + d_2^l - t_1^l - d_1^l) + (e^* - v(e^*)) \tag{A.12}$$

Equation (A.12) demonstrates the amplification of shocks through aggregate demand. A fall in borrowers' net worth will reduce borrowers' demand, leading to a fall in income. Lower income can dampen consumption demand by both the lenders and borrowers in period 1, which reduces income further. Equation (A.12) is equivalent to lenders' Euler equation at $r_2 = 0$:

$$u'(e_1+t_1^l+d_1^l-\frac{\alpha^i}{\alpha^l}b_1^i-\frac{\frac{\alpha^i}{\alpha^l}\phi^{Ii}e_1}{1+r_2}-v(e_1)) = \beta^l(1+r_2)u'(e^*+t_2^l+d_2^l+\frac{\frac{\alpha^i}{\alpha^l}\phi^{Ii}e_1}{1+r_2}-v(e^*)). \tag{A.13}$$

When there is an aggregate demand shortage, the equilibrium is completely pinned down by lenders' Euler equation at $r_2 = 0$. This equation also shows how wage has to adjust when the intertemporal price the interest rate is fixed. To have a unique and well-defined equilibrium, it requires that $1 - 2\frac{\alpha^i}{\alpha^l}\phi^{Ii} - v'(e_1)$ to be greater than zero.

Figure 9 illustrates this multiplier-effect result. One unit of decrease in borrowers' net worth can generate $(\frac{\alpha^i}{\alpha^l})\frac{1}{1-2\frac{\alpha^i}{\alpha^l}\phi^{Ii}-v'(e_1)}$ unit of fall in income.

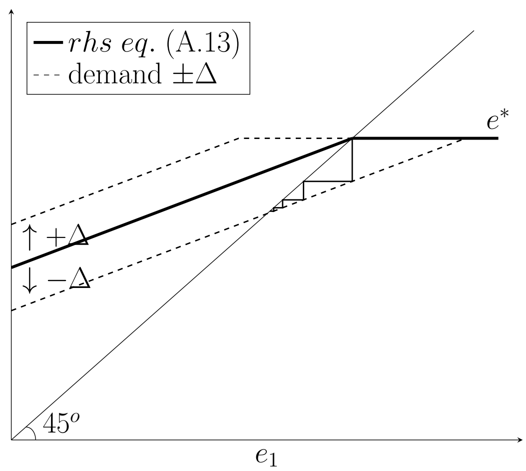

Figure 9: Amplification Through Aggregate Demand

A. a shock on lenders' endowment t_1^l

When there is no aggregate demand shortage, both types of shocks will not have any impact on the aggregate income. However, shocks on lenders' endowment can indirectly affect welfare through interest rate. More endowment of the lenders can boost their demand for bonds and will lower the interest rate, which benefits the borrowers while undermines the lenders. This result follows when the debt limit is determined by individual income: interest rate fall for the same reason, but borrowers will have higher employment and thus higher individual income which further improves welfare.

$$(\frac{de_1}{dt_1^l})_c^I = 0 \tag{A.14}$$

$$(\frac{dr_2}{dt_1^l})_c^I = \frac{u''(\tilde{c}_1^l)}{\beta^l u'(\tilde{c}_2^l) - \frac{\phi^{Ii}e^*}{(1+r_2)^2}u''(\tilde{c}_1^l)} < 0 \tag{A.15}$$

$$(\frac{\partial V^i}{\partial t_1^l})_c^I = -u'(\tilde{c}_1^i)\frac{\phi^{Ii}e^*}{(1+r_2)^2}(\frac{dr_2}{dt_1^l})_c^I > 0 \tag{A.16}$$

$$(\frac{\partial V^l}{\partial t_1^l})_c^I = u'(\tilde{c}_1^l)(1 + \frac{\phi^{Ii}e^*}{(1+r_2)^2}(\frac{dr_2}{dt_1^l})_c^I) \tag{A.17}$$

$$= \frac{\beta^l u'(\tilde{c}_1^l)u'(\tilde{c}_2^l)}{\beta^l u'(\tilde{c}_2^l) - \frac{\phi^{Ii}e^*}{(1+r_2)^2}u''(\tilde{c}_1^l)} > 0 \tag{A.18}$$

When there is an aggregate demand shortage, a unit positive shock on lenders' endowment in period 1 has a similar effect as a negative shock on their endowment in period 2: they both lower households' income by $1 - 2\frac{\alpha^i}{\alpha^l}\phi^{Ii} - v'(e_1)$. The decrease in income results from the limit on lenders' demand. Higher endowment or transfer in period 1 makes lenders less willing to work as their demand is constrained by the lower bound on the interest rate; similarly, the consumption smoothing motive of the lenders prompts them to save more and consume less in period 1 when lower endowment (that is a decrease in t_2^l) increases the marginal utility of consumption in period 2[22]. The resulting lower labor supply decrease production and income, reducing borrowers' debt capacity, which reduces demand further. With individual income in the borrowing constraint, employment of both borrowers and lenders will decrease because of lower wage, which lowers utility.

$$(\frac{de_1}{dt_1^l})_{nc}^I = -(1 - 2\frac{\alpha^i}{\alpha^l}\phi^{Ii} - v'(e_1)) < 0 \tag{A.19}$$

$$(\frac{dr_2}{dt_1^l})_{nc}^I = 0 \tag{A.20}$$

$$(\frac{\partial V_1^i}{\partial t_1^l})_{nc}^I = [(1 - v'(e_1))u'(\tilde{c}_1^i) + \phi^{Ii}(u'(\tilde{c}_1^i) - \beta^i u'(\tilde{c}_2^i))]\frac{de_1}{dt_1^l} < 0 \tag{A.21}$$

$$(\frac{\partial V_1^l}{\partial t_1^l})_{nc}^I = u'(\tilde{c}_1^l) + (1 - v'(e_1))u'(\tilde{c}_1^l)\frac{de_1}{dt_1^l} \tag{A.22}$$

$$= u'(\tilde{c}_1^l)\frac{\phi^{Ii}}{\alpha^l}\frac{de_1}{dt_1^l} < 0 \tag{A.23}$$

A change in lenders' transfer t_1^l has an opposite impact on an income-based borrowing economy when there is no aggregate demand shortage and when there is an aggregate demand shortage. An increase in t_1^l makes the households better-off when interest rate is above the lower bound $\frac{\partial V^h}{\partial t_1^l} > 0$, whereas it makes the households worse-off when interest rate is stuck at the lower bound $\frac{\partial V^h}{\partial t_1^l} < 0$.

As output is aggregate-demand determined when prices are sticky, the interest rate

[22] The GHH preference precludes the positive effect on labor supply when consumption falls and thus there is more amplification.

governs the consumption demand and thus output. An increase in the endowment will boost consumption of the lenders through a fall in the interest rate, leaving income at the optimal level when the interest rate is still flexible to move. Welfare of the borrowers is improved due to lower interest rate while that of the lenders is improved due to the direct effect of higher endowment dominating the adverse of effect of lower interest rate. When the interest rate is at the lower bound, however, the demand shortage will be worsened by the increase in lenders' endowment since lenders do not need to earn that much income to consume the same amount. The resulting lower labor supply reduces income, further tightening the borrowing constraint. Welfare of both types of households will be undermined as income decreases.

Asset-based borrowing. when there is no aggregate demand shortage, a transfer to the lenders will increase lenders' demand for bonds, lower the interest rate, and since lenders become more willing to hold debt, the collateral that the borrowers need for borrowing becomes more valuable. Therefore, asset price will increase and the constraint on borrowers will be relaxed. The marginal increase in lenders' endowment will decrease the interest rate and increase asset price, though households' income stay unchanged as there is no aggregate demand shortage. The effect on welfare is similar to that with the income-based borrowing constraint. Define:

$$M = \frac{(1+r_2)\frac{dp_1}{dt_1^l} - p_1\frac{dr_2}{dt_1^l}}{(1+r_2)^2}$$

The marginal effect on income, interest rate, asset price and welfare is given by:

$$(\frac{de_1}{dt_1^l})_c^A = 0 \tag{A.24}$$

$$(\frac{dr_2}{dt_1^l})_c^A = \frac{u''(\tilde{c}_1^l)}{\beta^l u'(\tilde{c}_2^l) - \frac{\phi^{Aa}p_1}{(1+r_2)^2}u''(\tilde{c}_1^l) + \frac{\phi^{Aa}p_1}{X(1+r_2)^2}\frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}u''(\tilde{c}_1^a)u'(\tilde{c}_2^a)(\frac{u''(\tilde{c}_1^l)}{1+r_2} + \beta^l(1+r_2)u''(\tilde{c}_2^l))} < 0 \tag{A.25}$$

$$(\frac{dp_1}{dt_1^l})_c^A = \frac{\phi^{Aa}\beta^a d_2^a p_1 u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}{X(1+r_2)^2(u'(\tilde{c}_1^a))^2}\frac{dr_2}{dt_1^l} > 0 \tag{A.26}$$

$$(\frac{\partial V^a}{\partial t_1^l})_c^A = -u'(\tilde{c}_1^a)\frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_1^l} + \frac{\phi^{Aa}}{1+r_2}\frac{dp_1}{dt_1^l}[u'(\tilde{c}_1^a) - \beta^a(1+r_2)u'(\tilde{c}_2^a)] > 0 \tag{A.27}$$

$$(\frac{\partial V^l}{\partial t_1^l})_c^A = (1 + \frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_1^l})u'(\tilde{c}_1^l) > 0 \tag{A.28}$$

Take partial derivative with respect to t_1^l to the asset pricing equation and the lenders' Euler equation to get:

$$M = -\frac{\frac{(u'(\tilde{c}_1^a))^2}{\phi^{Aa}\beta^a d_2^a} + u'(\tilde{c}_1^a)u''(\tilde{c}_2^a)}{u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}\frac{dp_1}{dt_1^l} \tag{A.29}$$

$$u''(\tilde{c}_1^l)(1-\phi^{Aa}M) = \beta^l(u'(\tilde{c}_2^l)\frac{dr_2}{dt_1^l} + \phi^{Aa}(1+r_2)u''(\tilde{c}_2^l)\frac{dp_1}{dt_1^l}) \tag{A.30}$$

Let $N = -\frac{\frac{(u'(\tilde{c}_1^a))^2}{\phi^{Aa}\beta^a d_2^a} + u'(\tilde{c}_1^a)u''(\tilde{c}_2^a)}{u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}$ such that $M = N\frac{dp_1}{dt_1^l}$. Equation (A.30) can be simplified to:

$$u''(\tilde{c}_1^l) = \beta^l u'(\tilde{c}_2^l)\frac{dr_2}{dt_1^l} + (\phi^{Aa}u''(\tilde{c}_1^l)N + \phi^{Aa}(1+r_2)u''(\tilde{c}_2^l))\frac{dp_1}{dt_1^l} \tag{A.31}$$

By the definition of M and (A.29),

$$\frac{dr_2}{dt_1^l} = \frac{1+r_2}{p_1}(1-(1+r_2)N)\frac{dp_1}{dt_1^l} \tag{A.32}$$

Plug N into (A.32) to get:

$$\frac{dp_1}{dt_1^l} = \frac{\phi^{Aa}\beta^a d_2^a p_1 u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}{X(1+r_2)^2(u'(\tilde{c}_1^a))^2}\frac{dr_2}{dt_1^l} \tag{A.33}$$

Since $X > 0$ by the previous assumption, $\frac{dr_2}{dt_1^l}$ and $\frac{dp_1}{dt_1^l}$ must be with opposite signs. Given (A.32), $1-(1+r_2)N < 0$ and thus $N > 0$. For (A.31) to be satisfied, $\frac{dr_2}{dt_1^l}$ has to be non-positive and $\frac{dp_1}{dt_1^l}$ has to be non-negative. Therefore, M is also non-negative. To solve for $\frac{dp_1}{dt_1^l}$ and $\frac{dr_2}{dt_1^l}$, plug (A.33) in (A.31).

Since the RHS of (A.30) is negative, $1-\phi^{Aa}M > 0$, which renders $\frac{\partial V^l}{\partial t_1^l} > 0$. And similarly as $1-(1+r_2)N < 0$, $\frac{\partial V^a}{\partial t_1^l} > 0$ is given by:

$$\begin{aligned}
\frac{\partial V^a}{\partial t_1^l} &= \phi^{Aa}Mu'(\tilde{c}_1^a) - \beta^a u'(\tilde{c}_2^a)\phi^{Aa}\frac{dp_1}{dt_1^l} \\
&= \phi^{Aa}\frac{dp_1}{dt_1^l}[Nu'(\tilde{c}_1^a) - \beta^a u'(\tilde{c}_2^a)] \\
&\geq \phi^{Aa}\frac{dp_1}{dt_1^l}[(1+r_2)N\beta^a u'(\tilde{c}_2^a) - \beta^a u'(\tilde{c}_2^a)] \\
&> 0
\end{aligned}$$

To further simplify the expression and to compare it with the welfare effect for the income-based borrowers when there is no aggregate demand shortage, we have:

$$\begin{aligned}
\frac{\partial V^a}{\partial t_1^l} &= \phi^{Aa}\frac{(1+r_2)\frac{dp_1}{dt_1^l} - p_1\frac{dr_2}{dt_1^l}}{(1+r_2)^2}u'(\tilde{c}_1^a) - \beta^a u'(\tilde{c}_2^a)\phi^{Aa}\frac{dp_1}{dt_1^l} \\
&= -u'(\tilde{c}_1^a)\frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_1^l} + u'(\tilde{c}_1^a)\frac{\phi^{Aa}}{1+r_2}\frac{dp_1}{dt_1^l} - \phi^{Aa}\beta^a u'(\tilde{c}_2^a)\frac{dp_1}{dt_1^l} \\
&= -u'(\tilde{c}_1^a)\frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_1^l} + \frac{\phi^{Aa}}{1+r_2}\frac{dp_1}{dt_1^l}[u'(\tilde{c}_1^a) - \beta^a(1+r_2)u'(\tilde{c}_2^a)]
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial V^l}{\partial t_1^l} &= [1-\phi^{Aa}\frac{(1+r_2)\frac{dp_1}{dt_1^l}-p_1\frac{dr_2}{dt_1^l}}{(1+r_2)^2}]u'(\tilde{c}_1^l)+\beta^l u'(\tilde{c}_2^l)\phi^{Aa}\frac{dp_1}{dt_1^l} \\
&= (1+\frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_1^l})u'(\tilde{c}_1^l)+\frac{\phi^{Aa}}{1+r_2}\frac{dp_1}{dt_1^l}[u'(\tilde{c}_1^l)-\beta^a(1+r_2)u'(\tilde{c}_2^l)] \\
&= (1+\frac{\phi^{Aa}p_1}{(1+r_2)^2}\frac{dr_2}{dt_1^l})u'(\tilde{c}_1^l)
\end{aligned}
$$

A change in lenders' endowment t_1^l has similar effects on an income-based borrowing economy and an asset-based borrowing economy when there is no aggregate demand shortage. An increase in t_1^l will improve welfare of both types of households: $\frac{\partial V^h}{\partial t_1^l}>0$. In an income-based borrowing economy, it is achieved via a fall in the interest rate; in an asset-based borrowing economy, it is achieved through not only a fall in the interest rate, but also an increase in the asset price which affects welfare of the borrowers not lenders, and

(a) the decrease in the interest rate $|(\frac{dr_2}{dt_1^l})_c^A|<|(\frac{dr_2}{dt_1^l})_c^I|$;

(b) lenders' welfare increases $(\frac{\partial V^l}{\partial t_1^l})_c^A>(\frac{\partial V^l}{\partial t_1^l})_c^I$; welfare increases are ambivalent to compare between an asset-based borrower and an income-based borrower $(\frac{\partial V^a}{\partial t_1^l})_c^A \lesseqgtr (\frac{\partial V^i}{\partial t_1^l})_c^I$.

Next consider a marginal increase in t_1^l when there is an aggregate demand shortage for an asset-based borrower.

$$
(\frac{de_1}{dt_1^l})_{nc}^A = -\frac{1+\frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}[u'(\tilde{c}_1^a)u''(\tilde{c}_2^a)+u''(\tilde{c}_1^a)u'(\tilde{c}_2^a)]}{(1-v'(e_1))[1+\frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}(u'(\tilde{c}_1^a)u''(\tilde{c}_2^a)+(1+\frac{1}{\alpha^l})u''(\tilde{c}_1^a)u'(\tilde{c}_2^a))]}<0 \tag{A.34}
$$

$$
(\frac{dr_2}{dt_1^l})_{nc}^A = 0 \tag{A.35}
$$

$$
(\frac{dp_1}{dt_1^l})_{nc}^A = \frac{\beta^a d_2^a u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}{(u'(\tilde{c}_1^a))^2[1+\frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}(u'(\tilde{c}_1^a)u''(\tilde{c}_2^a)+(1+\frac{1}{\alpha^l})u''(\tilde{c}_1^a)u'(\tilde{c}_2^a))]}<0 \tag{A.36}
$$

$$
(\frac{\partial V^a}{\partial t_1^l})_{nc}^A = [(1-v'(e_1))\frac{de_1}{dt_1^l}+\phi^{Aa}\frac{dp_1}{dt_1^l}]u'(\tilde{c}_1^a)-\beta^a\phi^{Aa}u'(\tilde{c}_2^a)\frac{dp_1}{dt_1^l}<0 \tag{A.37}
$$

$$
(\frac{\partial V^l}{\partial t_1^l})_{nc}^A = [(1-v'(e_1))\frac{de_1}{dt_1^l}+1-\phi^{Aa}\frac{dp_1}{dt_1^l}]u'(\tilde{c}_1^l)+\beta^l\phi^{Aa}u'(\tilde{c}_2^l)\frac{dp_1}{dt_1^l} \tag{A.38}
$$

$$
= u'(\tilde{c}_1^l)+[(1-v'(e_1))]u'(\tilde{c}_1^l)\frac{de_1}{dt_1^l}<0 \tag{A.39}
$$

Take the partial derivative with t_1^l to the asset pricing equation and the aggregate demand equation:

$$
X\frac{dp_1}{dt_1^l} = -Z\frac{de_1}{dt_1^l} \tag{A.40}
$$

$$
Y\frac{de_1}{dt_1^l} = \phi^{Aa}\frac{dp_1}{dt_1^l}-\alpha^l \tag{A.41}
$$

where $Z = \frac{\beta^a d_2^a (1-v'(e_1))}{(u'(\tilde{c}_1^a))^2} u'(\tilde{c}_2^a) u''(\tilde{c}_1^a) < 0$. Combine (A.40) and (A.41) to obtain:

$$\frac{dp_1}{dt_1^l} = \frac{\alpha^l Z}{XY + \phi^{Aa} Z} \tag{A.42}$$

$$\frac{de_1}{dt_1^l} = -\frac{\alpha^l X}{XY + \phi^{Aa} Z} \tag{A.43}$$

We restrict the slope of the asset equation and the aggregate demand equation in order to have a well-defined solution. That is, $\frac{de_1^{AP}}{dp_1} > \frac{de_1^{AD}}{dp_1}$, where

$$\begin{aligned} \frac{de_1^{AP}}{dp_1} &= -\frac{X}{Z} \\ \frac{de_1^{AD}}{dp_1} &= \frac{\phi^{Aa}}{Y} \end{aligned}$$

With this restriction, $X + \frac{\phi^{Aa} Z}{Y} > 0$ and $\frac{dp_1}{dt_1^l} < 0$ and $\frac{dp_1}{dt_1^l} < 0$. Moreover, note that the slope of the AP equation and AD equation can be greater or less than one. We exclude the circumstance where both slopes are greater than one, as when $\frac{de_1^{AD}}{dp_1}$ is greater than one, $1 - \phi^{Aa} - \alpha^a - \alpha^l v'(e_1)$ will be negative, which contradicts with our assumptions for the income-based borrowing economy when there is an aggregate demand shortage if we set $\alpha^a = \alpha^i$ and $\phi^{Aa} = \phi^{Ii}$.

To compare the change in income and welfare with the income-based borrowing constraint, we redefine Y as $Y = 1 - \alpha^{i/a} - \alpha^l v'(e_1)$, and the marginal change in income with income-based borrowing Equation (A.19) can be written as $|(\frac{de_1}{dt_1^l})_{nc}^I| = \frac{\alpha^l}{Y - \phi^{Ii}}$. By Equation (A.42) and (A.42), we can rewrite $(\frac{dp_1}{dt_1^l})_{nc}^A$ and $(\frac{de_1}{dt_1^l})_{nc}^A$ as:

$$\begin{aligned} (\frac{de_1}{dt_1^l})_{nc}^A &= -\frac{\alpha^l X}{XY + \phi^{Aa} Z} \\ &= -\frac{\alpha^l X}{X(Y - \phi^{Aa}) + \phi^{Aa}(X + Z)} \\ &= -\frac{\alpha^l}{Y - \phi^{Aa}} (\frac{X}{X + \phi^{Aa} \frac{X+Z}{Y - \phi^{Aa}}}) \end{aligned} \tag{A.44}$$

$$\begin{aligned} (\frac{dp_1}{dt_1^l})_{nc}^A &= -\frac{Z}{X} (\frac{de_1}{dt_1^l})_{nc}^A \\ &= -\frac{\alpha^l}{Y - \phi^{Aa}} (\frac{-Z}{X + \phi^{Aa} \frac{X+Z}{Y - \phi^{Aa}}}) \end{aligned} \tag{A.45}$$

Consider first when $1 \geq \frac{de_1^{AP}}{dp_1} \geq \frac{de_1^{AD}}{dp_1}$, it renders $X \leq -Z$ and $Y > \phi^{Aa}$, and $\frac{-Z}{X + \phi^{Aa} \frac{X+Z}{Y-\phi^{Aa}}} \geq \frac{X}{X + \phi^{Aa} \frac{X+Z}{Y-\phi^{Aa}}} \geq 1$. Therefore, $|(\frac{dp_1}{dt_1^l})_{nc}^A| \geq |(\frac{de_1}{dt_1^l})_{nc}^A| \geq |(\frac{de_1}{dt_1^l})_{nc}^I|$. By Equation (A.21), (A.23), (A.37) and (A.39), we have $|(\frac{\partial V^a}{\partial t_1^l})_{nc}^A| \geq |(\frac{\partial V^i}{\partial t_1^l})_{nc}^I|$, and $|(\frac{\partial V^l}{\partial t_1^l})_{nc}^A| \geq |(\frac{\partial V^l}{\partial t_1^l})_{nc}^I|$.

when $\frac{de_1^{AP}}{dp_1} \geq 1 \geq \frac{de_1^{AD}}{dp_1}$, it renders $X \geq -Z$ and $Y > \phi^{Aa}$, and $\frac{-Z}{X + \phi^{Aa} \frac{X+Z}{Y-\phi^{Aa}}} \leq$

$\frac{X}{X+\phi^{Aa}\frac{X+Z}{Y-\phi^{Aa}}} \leq 1$. Therefore, $|(\frac{dp_1}{dt_1^l})^A_{nc}| \leq |(\frac{de_1}{dt_1^l})^A_{nc}| \leq |(\frac{de_1}{dt_1^l})^I_{nc}|$. By Equation (A.21), (A.23), (A.37) and (A.39), we have $|(\frac{\partial V^a}{\partial t_1^l})^A_{nc}| \leq |(\frac{\partial V^i}{\partial t_1^l})^I_{nc}|$, and $|(\frac{\partial V^l}{\partial t_1^l})^A_{nc}| \leq |(\frac{\partial V^l}{\partial t_1^l})^I_{nc}|$. A change in lenders' endowment t_1^l has similar effects on an income-based borrowing economy and an asset-based borrowing economy when there is an aggregate demand shortage. An increase in t_1^l will lower income and undermine the welfare of both types of households: $\frac{\partial V^h}{\partial t_1^l} < 0$. In an asset-based borrowing economy, it affects welfare of the borrowers through depressing asset price in addition to lowering income as in an income-based economy. In both economies, it affects the welfare of lenders only through lowering income. Whether its impact is more pronounced will depend on the responsiveness of income to changes in the asset price:

(a) If $1 \geq \frac{de_1^{AP}}{dp_1} \geq \frac{de_1^{AD}}{dp_1}$,

(i) $|(\frac{dp_1}{dt_1^l})^A_{nc}| \geq |(\frac{de_1}{dt_1^l})^A_{nc}| \geq |(\frac{de_1}{dt_1^l})^I_{nc}|$;

(ii) $|(\frac{\partial V^a}{\partial t_1^l})^A_{nc}| \geq |(\frac{\partial V^i}{\partial t_1^l})^I_{nc}|$, and $|(\frac{\partial V^l}{\partial t_1^l})^A_{nc}| \geq |(\frac{\partial V^l}{\partial t_1^l})^I_{nc}|$.

(b) If $\frac{de_1^{AP}}{dp_1} \geq 1 \geq \frac{de_1^{AD}}{dp_1}$,

(i) $|(\frac{dp_1}{dt_1^l})^A_{nc}| \leq |(\frac{de_1}{dt_1^l})^A_{nc}| \leq |(\frac{de_1}{dt_1^l})^I_{nc}|$;

(ii) $|(\frac{\partial V^a}{\partial t_1^l})^A_{nc}| \leq |(\frac{\partial V^i}{\partial t_1^l})^I_{nc}|$, and $|(\frac{\partial V^l}{\partial t_1^l})^A_{nc}| \leq |(\frac{\partial V^l}{\partial t_1^l})^I_{nc}|$.

B. a shock on borrowers' dividend d_1^i or d_1^a

Income-based borrowing. For an income-based borrowing economy, when there is no aggregate demand shortage, shocks on asset dividend do not even have any effect on the interest rate if borrowers are constrained. They only affect borrowers' welfare by direct wealth effect.

$$(\frac{de_1}{dd_1^i})^I_c = 0 \tag{A.46}$$

$$(\frac{dr_2}{dd_1^i})^I_c = 0 \tag{A.47}$$

$$(\frac{\partial V^i}{\partial d_1^i})^I_c = u'(\tilde{c}_1^i) > 0 \tag{A.48}$$

$$(\frac{\partial V^l}{\partial d_1^i})^I_c = 0 \tag{A.49}$$

Interest rate is unaffected because higher dividend boosts demand and thus income, which lowers interest rate as borrowers are less constrained by income. The reduction in interest rate is offset by a monetary policy that has to raise interest rate to maintain the optimal level of output and prevent an overheating economy.

when there is an aggregate demand shortage and the interest rate is at the lower bound, the shock on d_1^i does not influence income as in Equation (A.12), despite the negative effect on borrowers' demand. Income is left unaffected, and the welfare of the households similarly responds to the shock as with the case when there is no aggregate demand shortage.

$$(\frac{de_1}{dd_1^i})_{nc}^I = 0 \tag{A.50}$$

$$(\frac{dr_2}{dd_1^i})_{nc}^I = 0 \tag{A.51}$$

$$(\frac{\partial V^i}{\partial d_1^i})_{nc}^I = u'(\tilde{c}_1^i) > 0 \tag{A.52}$$

$$(\frac{\partial V^l}{\partial d_1^i})_{nc}^I = 0 \tag{A.53}$$

Asset-based borrowing. Next consider a marginal increase in d_1^a when there is no aggregate demand shortage. An increase in asset dividend will make asset more valuable as it not only boosts consumption by the borrowers in the current period directly, but relaxes the borrowing constraint as the price of the asset rises, which further increases consumption and inflates the asset price. This is the canonical amplification mechanism with the asset-based borrowing constraint. Meanwhile, the interest rate must increase since the supply of bonds rises as the borrowers expand their debt capacity with more valuable collaterals.

$$(\frac{de_1}{dd_1^a})_c^A = 0 \tag{A.54}$$

$$(\frac{dr_2}{dd_1^a})_c^A = Q\frac{dp_1}{dd_1^a} > 0 \tag{A.55}$$

$$(\frac{dp_1}{dd_1^a})_c^A = \frac{1}{p_1Q - (1+r_2) - \frac{(u'(\tilde{c}_1^a))^2}{\phi^{Aa}\beta^a d_2^a u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)} - \frac{u'(\tilde{c}_1^a)u''(\tilde{c}_2^a)}{u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}} > 0 \tag{A.56}$$

$$(\frac{\partial V^a}{\partial d_1^a})_c^A = u'(\tilde{c}_1^a)(1+\phi^{Aa}M) - \beta^a u'(\tilde{c}_2^a)\phi^{Aa}\frac{dp_1}{dd_1^a} > 0 \tag{A.57}$$

$$(\frac{\partial V^l}{\partial d_1^a})_c^A = -\phi^{Aa}Mu'(\tilde{c}_1^l) + \beta^a u'(\tilde{c}_2^l)\phi^{Aa}\frac{dp_1}{dd_1^a} > 0 \tag{A.58}$$

Take partial derivative with respect to d_1^a to the asset pricing equation and the lenders' Euler equation to get:

$$\frac{dp_1}{dd_1^a} = -\frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}[u'(\tilde{c}_1^a)u''(\tilde{c}_2^a)\frac{dp_1}{dd_1^a} + u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)(1+M)] \tag{A.59}$$

$$-u''(\tilde{c}_1^l)\phi^{Aa}M = \beta^l(u'(\tilde{c}_2^l)\frac{dr_2}{dd_1^a} + \phi^{Aa}(1+r_2)u''(\tilde{c}_2^l)\frac{dp_1}{dd_1^a}) \tag{A.60}$$

Simplifying (A.60) to get an expression for $\frac{dp_1}{dd_1^a}$ and $\frac{dr_2}{dd_1^a}$:

$$[\frac{\phi^{Aa}p_1u''(\tilde{c}_1^l)}{(1+r_2)^2} - \beta^l u'(\tilde{c}_2^l)]\frac{dr_2}{dd_1^a} = [\phi^{Aa}\beta^l(1+r_2)u''(\tilde{c}_2^l) + \frac{\phi^{Aa}u''(\tilde{c}_1^l)}{1+r_2}]\frac{dp_1}{dd_1^a} \tag{A.61}$$

according to which we can write $\frac{dr_2}{dd_1^a} = Q\frac{dp_1}{dd_1^a}$ where $Q = \frac{\phi^{Aa}\beta^l(1+r_2)u''(\tilde{c}_2^l)+\frac{\phi^{Aa}u''(\tilde{c}_1^l)}{1+r_2}}{\frac{\phi^{Aa}p_1u''(\tilde{c}_1^l)}{(1+r_2)^2}-\beta^l u'(\tilde{c}_2^l)} > 0$.
Combine the definition of M and (A.59) to get

$$-X\frac{dp_1}{dd_1^a} = (1 - \frac{p_1\frac{dr_2}{dd_1^a}}{(1+r_2)^2})\frac{\phi^{Aa}\beta^a d_2^a u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}{(u'(\tilde{c}_1^a))^2} \tag{A.62}$$

Since $Q > 0$ and $X > 0$, $\frac{dp_1}{dd_1^a}$ and $\frac{dr_2}{dd_1^a}$ have to be both positive for (A.62) to be satisfied. Thus $1 - \frac{p_1\frac{dr_2}{dd_1^a}}{(1+r_2)^2} > 0$. Combine (A.62) and (A.61) to get:

$$\frac{dp_1}{dd_1^a} = \frac{\phi^{Aa}\beta^a d_2^a u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}{p_1Q\phi^{Aa}\beta^a d_2^a u'(\tilde{c}_2^a)u''(\tilde{c}_1^a) - (1+r_2)\phi^{Aa}\beta^a d_2^a u'(\tilde{c}_2^a)u''(\tilde{c}_1^a) - \phi^{Aa}\beta^a d_2^a u''(\tilde{c}_2^a)u'(\tilde{c}_1^a) - (u'(\tilde{c}_1^a))^2} \tag{A.63}$$

To see how welfare changes, note that $u'(\tilde{c}_1^a) > \beta^a(1+r_2)u'(\tilde{c}_2^a)$ and $1 - \frac{p_1\frac{dr_2}{dd_1^a}}{(1+r_2)^2} > 0$.

ABC not clear. A marginal increase in d_1^a when there is an aggregate demand shortage.

$$(\frac{de_1}{dd_1^a})^A_{nc} = -\frac{\phi^{Aa}\beta^a d_2^a u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}{(u'(\tilde{c}_1^a))^2[1+\frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}(u'(\tilde{c}_1^a)u''(\tilde{c}_2^a)+(1+\frac{1}{\alpha^l})u''(\tilde{c}_1^a)u'(\tilde{c}_2^a))]} > 0 \tag{A.64}$$

$$(\frac{dr_2}{dd_1^a})^A_{nc} = 0 \tag{A.65}$$

$$(\frac{dp_1}{dd_1^a})^A_{nc} = -\frac{(1-v'(e_1))u'(\tilde{c}_2^a)u''(\tilde{c}_1^a)}{(u'(\tilde{c}_1^a))^2[1+\frac{\phi^{Aa}\beta^a d_2^a}{(u'(\tilde{c}_1^a))^2}(u'(\tilde{c}_1^a)u''(\tilde{c}_2^a)+(1+\frac{1}{\alpha^l})u''(\tilde{c}_1^a)u'(\tilde{c}_2^a))]} > 0 \tag{A.66}$$

$$(\frac{\partial V^a}{\partial d_1^a})^A_{nc} = [(1-v'(e_1))\frac{de_1}{dd_1^a} + 1 + \phi^{Aa}\frac{dp_1}{dd_1^a}]u'(\tilde{c}_1^a) - \beta^a\phi^{Aa}\frac{dp_1}{dd_1^a}u'(\tilde{c}_2^a) > 0 \tag{A.67}$$

$$(\frac{\partial V^l}{\partial d_1^a})^A_{nc} = [(1-v'(e_1))\frac{de_1}{dd_1^a} - \phi^{Aa}\frac{dp_1}{dd_1^a}]u'(\tilde{c}_1^l) + \beta^l\phi^{Aa}\frac{dp_1}{dd_1^a}u'(\tilde{c}_2^l) > 0 \tag{A.68}$$

Take the partial derivative with d_1^a to the asset pricing equation and the aggregate demand equation:

$$X\frac{dp_1}{dt_1^l} = -Z\frac{dr_2}{dt_1^l} - \frac{Z}{1-v'(e_1)} \tag{A.69}$$

$$Y\frac{dr_2}{dt_1^l} = \phi^{Aa}\frac{dp_1}{dt_1^l} \tag{A.70}$$

Combine (A.69) and (A.70) to obtain:

$$\frac{dp_1}{dt_1^l} = -\frac{YZ}{(1-v'(e_1))(XY+\phi^{Aa}Z)}$$

$$\frac{de_1}{dt_1^l} = -\frac{\phi^{Aa}Z}{(1-v'(e_1))(XY+\phi^{Aa}Z)}$$

Again, with the restrictions on the slope of the asset equation and the aggregate demand equation that $\frac{de_1^{AP}}{dp_1} > \frac{de_1^{AD}}{dp_1}$, $X + \frac{\phi^{Aa}Z}{Y} > 0$ and $\frac{dp_1}{dt_1^l} > 0$ and $\frac{dp_1}{dt_1^l} > 0$.