Algorithms

S

1 Chained Matrix Multiplication

This lecture discusses the dynamic programming problem for finding the most efficient order to perform chained matrix multiplication.

1.1 Matrix M

To begin the solution, we are going to be constructing an $n*n$ matrix M, containing integers indexed from 1 to n.

For $1 \leq i \leq j \leq n$

$M[i][j] =$ the minimum number of multiplications needed to multiply A_i to A_j, if $i < j$

$M[i][j] = 0$, if $i = j$

1.1.1 Matrix M: Example

Suppose we wish to multiply:

$$A_1 \times A_2 \times A_3 \times A_4 \times A_5 \times A_6$$

With dimensions:

$A_1 = 5 \times 2$
$A_2 = 2 \times 3$
$A_3 = 3 \times 4$
$A_4 = 4 \times 6$
$A_5 = 6 \times 7$
$A_6 = 7 \times 8$

From these dimensions we get input values:

$d_0 = 5,\ d_1 = 2,\ d_2 = 3,\ d_3 = 4,\ d_4 = 6,\ d_5 = 7,\ d_6 = 8$

The value of $M[4][6]$ is the minimum number of multiplications for $A_4 \times A_5 \times A_6$.

This operation can be done in two different orders:

1. $(A_4 \times A_5) \times A_6 = (4 \cdot 6 \cdot 7) + (4 \cdot 7 \cdot 8) = 392$ total multiplications

2. $A_4 \times (A_5 \times A_6) = (6 \cdot 7 \cdot 8) + (4 \cdot 6 \cdot 8) = 528$ total multiplications

So, by taking the minimum of these two calculations, $M[4][6] = 392$

1.1.2 Solving Matrix M

Using the Matrix dimensions from above, we will begin with an empty version of matrix M, whose diagonal values are initialized to zero.

$$M = \begin{bmatrix} 0 & x & x & x & x & x \\ x & 0 & x & x & x & x \\ x & x & 0 & x & x & x \\ x & x & x & 0 & x & x \\ x & x & x & x & 0 & x \\ x & x & x & x & x & 0 \end{bmatrix}$$

Each matrix value can be solved for using the following formula:

$M[i][j] = M[i][k] + M[k+1][j] + d_{i-1} \cdot d_k \cdot d_j$
where k goes from i to $j-1$

Starting with $M[1][2]$,

$i = 1$
$j = 2$
k from 1 to 1

$M[1][2] = M[1][1] + M[2][2] + d_0 \cdot d_1 \cdot d_2$
$= M[1][1] + M[2][2] + 5 \cdot 3 \cdot 2$
$= 0 + 0 + 30 = 30$

Updating M..

$$M = \begin{bmatrix} 0 & 30 & x & x & x & x \\ x & 0 & x & x & x & x \\ x & x & 0 & x & x & x \\ x & x & x & 0 & x & x \\ x & x & x & x & 0 & x \\ x & x & x & x & x & 0 \end{bmatrix}$$

Diagonal 1 is completed in the same manner. This is the diagonal above the 0s

$$M = \begin{bmatrix} 0 & 30 & x & x & x & x \\ x & 0 & 24 & x & x & x \\ x & x & 0 & 72 & x & x \\ x & x & x & 0 & 168 & x \\ x & x & x & x & 0 & 336 \\ x & x & x & x & x & 0 \end{bmatrix}$$

Matrix P is completed while matrix M is being solved. It is also a 6x6 matrix. Each box is indicative of the position at which the arrays are being multiplied. In the example above for M[1][2], the matrices are being multiplied between position 1 and 2 so 1 is placed placed in P[1][2]. Below is the P matrix updated for the first diagonal in matrix M.

$$P = \begin{bmatrix} x & 1 & x & x & x & x \\ x & x & 2 & x & x & x \\ x & x & x & 3 & x & x \\ x & x & x & x & 4 & x \\ x & x & x & x & x & 5 \\ x & x & x & x & x & x \end{bmatrix}$$

Below is an example that is in diagonal 2

$i = 1$
$j = 3$
k from 1 to 2
$d_{i-1} = 5$
$d_j = 4$

k = 1
$M[1][3] = M[1][1] + M[2][3] + 5 \cdot 2 \cdot 4$
$= 0 + 24 + 40 = 64$
The value 24 is given from M[2][3] on the matrix

k = 2
$M[1][3] = M[1][2] + M[3][3] + 5 \cdot 3 \cdot 4$
$= 30 + 0 + 60 = 90$

The minimum of these two values is 64. k = 1 gave us this value so

M[1][3] = 64 P[1][3] = 1

Diagonal 5 is the final answer in this case. This is the upper right box M[1][6]. Let's find this answer

$i = 1$
$j = 6$
k from 1 to 5

k = 1
$M[1][6] = M[1][1] + M[2][6] + 5 \cdot 2 \cdot 8 = 348$

k = 2
$M[1][6] = M[1][2] + M[3][6] + 5 \cdot 3 \cdot 8 = 516$

k = 3
$M[1][6] = M[1][3] + M[4][6] + 5 \cdot 4 \cdot 8 = 684$

k = 4
$M[1][6] = M[1][4] + M[5][6] + 5 \cdot 6 \cdot 8 = 708$

k = 5
$M[1][6] = M[1][5] + M[6][6] + 5 \cdot 7 \cdot 8 = 506$

k = 1 provides a minimum value of 348 for the final answer

Here is the final M matrix

$$M = \begin{bmatrix} 0 & 30 & 64 & 132 & 226 & 348 \\ x & 0 & 24 & 72 & 156 & 268 \\ x & x & 0 & 72 & 198 & 366 \\ x & x & x & 0 & 168 & 392 \\ x & x & x & x & 0 & 336 \\ x & x & x & x & x & 0 \end{bmatrix}$$

Here is the corresponding final P matrix

$$P = \begin{bmatrix} x & 1 & 1 & 1 & 1 & 1 \\ x & x & 2 & 3 & 4 & 5 \\ x & x & x & 3 & 4 & 5 \\ x & x & x & x & 4 & 5 \\ x & x & x & x & x & 5 \\ x & x & x & x & x & x \end{bmatrix}$$