The fiber-full scheme

arXiv: https://arxiv.org/abs/2108.13986

slides: https://ycid.github.io/ycid/cv.pdf

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   - We want a stratification of the Hilbert schemes in terms of these new fiber-full schemes.
2. Review some applications that stem from the existence of the full fiber scheme.
3. From a technical side: results on sheaf/local cohomology and a flattening stratification theorem.

A a Noetherian commutative ring and $S = \mathrm{Spec}(A)$. $R = A[x_0, \ldots, x_r]$ a standard graded commutative ring and $\mathbb{P}^r_A = \mathrm{Proj}(R)$. $P \in \mathbb{Q}[m]$ a numerical polynomial.

Hilbert scheme (Grothendieck, 1961)

$\mathrm{Hilb}^P_{\mathbb{P}^r_A}$ parametrizes closed subschemes $Z \subset \mathbb{P}^r_A$ with Hilbert polynomial P:

$$\left\{\text{closed } Z \subset \mathbb{P}^r_A \mid Z \text{ is flat over } S \text{ and } Z_{\mathfrak{p}} \text{ has Hilbert polynomial } P \text{ for all } \mathfrak{p} \in S\right\},$$

which is the same as

$$\left\{ Z = \mathrm{Proj}(R/I) \subset \mathbb{P}^r_A \;\middle|\; \begin{array}{l} I \subset R \text{ homogeneous ideal,} \\ I = I : (x_0, \ldots, x_r)^\infty, \\ {[R/I]}_\nu \text{ is a locally free } A\text{-module} \\ \text{of constant rank } P(\nu) \text{ for all } \nu \gg 0 \end{array} \right\}.$$

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So, if $[Z] = [\mathrm{Proj}(R/I)] \in \mathrm{Hilb}^P_{\mathbb{P}^r_A}$, then $R/I \otimes_A \kappa(\mathfrak{p})$ has Hilbert polynomial P for all $\mathfrak{p} \in S$, where $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

Example: Twisted cubics (Piene - Schlessinger, 1985)

A *twisted cubic* $C \subset \mathbb{P}^3_{\Bbbk}$ ($\Bbbk$ alg. closed and $\mathrm{char}(\Bbbk) = 0$) is a rational smooth curve of degree 3. Any such C is projectively equivalent to $C_0 = \phi(\mathbb{P}^1_{\Bbbk})$ where $\phi : \mathbb{P}^1_{\Bbbk} \to \mathbb{P}^3_{\Bbbk}$, $(u : v) \mapsto (u^3 : u^2v : uv^2 : v^3)$.

Theorem (Piene - Schlessinger, 1985)

$\mathrm{Hilb}^{3m+1}_{\mathbb{P}^r_{\Bbbk}} = H \cup H'$ (two smooth irreducible components), where H parametrizes twisted cubics and H' parametrizes a plane cubic union an isolated point.

Remark

With the Hilbert scheme compactification of the space of twisted cubics we obtain the "extraneous" component H'.

If $[Z] \in H - H \cap H'$, then $h^0(Z, \mathcal{O}_Z(\nu)) = \begin{cases} 3\nu + 1 \text{ if } \nu \geq 0 \\ 0 \qquad \text{if } \nu \leq -1 \end{cases}$

and $h^1(Z, \mathcal{O}_Z(j)) = h^0(Z, \mathcal{O}_Z(\nu)) - (3\nu + 1)$.

If $[Z] \in H'$, then $h^0(Z, \mathcal{O}_Z(\nu)) = \begin{cases} 3\nu + 1 \text{ if } \nu \geq 1 \\ 2 \qquad \text{if } \nu = 0 \\ 1 \qquad \text{if } \nu \leq -1 \end{cases}$

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Stratification of in cohomological terms

$\mathrm{Hilb}^{3m+1}_{\mathbb{P}^r_{\Bbbk}} = (H - H \cap H') \cup H'$. Thus, $h^i_Z : \mathbb{Z} \to \mathbb{N}$, $h^i_Z(\nu) := \dim_{\Bbbk}\left(\mathrm{H}^i(Z, \mathcal{O}_Z(\nu))\right)$ is the same for any $[Z] \in H - H \cap H'$ and is the same for any $[Z] \in H'$.

Guiding goals

(1) Can we do this kind of cohomological stratification for any Hilbert scheme in terms of **locally closed subschemes**? (2) If so, we want to provide a unified and systematic treatment.

Formal definition of Hilbert and Quot schemes

$S = \mathrm{Spec}(A)$ with A Noetherian, $X \subset \mathbb{P}^r_A$ closed subscheme, $\mathcal{F}$ coherent sheaf on X. For any S-scheme $T = \mathrm{Spec}(B)$, let $\mathcal{F}_T = \mathcal{F} \otimes_A B$. $P \in \mathbb{Q}[m]$ numerical polynomial.

Quot functor $\underline{\mathcal{Quot}}^P_{\mathcal{F}/X/S}$

$$\underline{\mathcal{Quot}}^P_{\mathcal{F}/X/S}(T) = \left\{ \text{coherent } \mathcal{F}_T \twoheadrightarrow \mathcal{G} \;\middle|\; \begin{array}{l} \mathcal{G} \text{ if flat over } T \text{ and } G_t \text{ has} \\ \text{Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}.$$

Theorem (Grothendieck, 1961)

$\underline{\mathcal{Quot}}^P_{\mathcal{F}/X/S}$ is represented by a projective S-scheme $\mathrm{Quot}^P_{\mathcal{F}/X/S}$.

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There is a **universal sheaf** $\mathcal{W}^P_{\mathcal{F}/X/S} \in \underline{\mathit{Quot}}^P_{\mathcal{F}/X/S}\left(\mathrm{Quot}^P_{\mathcal{F}/X/S}\right)$ such that for any $\mathcal{G} \in \underline{\mathit{Quot}}^P_{\mathcal{F}/X/S}(T)$ there is a **unique classifying S-morphism** $g_{\mathcal{G}} : T \to \mathrm{Quot}^P_{\mathcal{F}/X/S}$ such that $\mathcal{G} = \left(\mathcal{W}^P_{\mathcal{F}/X/S}\right)_T = (1_X \times_S g_G)^* \mathcal{W}^P_{\mathcal{F}/X/S}$.

The functor we need to study

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Let $\mathbf{h} = (h_0, \ldots, h_r) : \mathbb{Z}^{r+1} \to \mathbb{N}^{r+1}$ be a tuple of functions.

Defintion: the fiber-full functor $\underline{\mathcal{Fib}}^{\mathbf{h}}_{\mathcal{F}/X/S}$

$$\underline{\mathcal{Fib}}^{\mathbf{h}}_{\mathcal{F}/X/S}(T) = \left\{ \text{coherent } \mathcal{F}_T \twoheadrightarrow \mathcal{G} \;\middle|\; \begin{array}{l} \mathrm{H}^i\left(X_T, \mathcal{G}(\nu)\right) \text{ is a locally free } B\text{-module} \\ \text{of constant rank equal to } h_i(\nu) \\ \text{for all } 0 \le i \le r, \nu \in \mathbb{Z} \end{array} \right\}$$

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It is a functor because of the following base change result.

Lemma

Assume $\mathcal{F}$ is S-flat and $\mathrm{H}^i(X, \mathcal{F})$ are A-flat for all $0 \le i \le r$. Then
$$\mathrm{H}^i(X, \mathcal{F}) \otimes_A B \xrightarrow{\cong} \mathrm{H}^i(X_T, \mathcal{F}_T) \text{ for all } 0 \le i \le r.$$
In particular, all $\mathrm{H}^i(X_T, \mathcal{F}_T)$ are B-flat.

- $\mathcal{F}$ is S-flat $\iff$ $\mathrm{H}^0(X, \mathcal{F}(\nu))$ is A-flat for all $\nu \gg 0$.
- The Hilbert polynomial coincides with the Euler characteristic.

Relation between $\underline{\mathcal{F}ib}^{P}_{\mathcal{F}/X/S}$ and $\underline{Quot}^{P}_{\mathcal{F}/X/S}$

Let $P_{\mathbf{h}} = \sum_{i=0}^{r} (-1)^i h_i$. For any S-scheme $T = \mathrm{Spec}(B)$, we have the inclusion

$$\underline{\mathcal{F}ib}^{P}_{\mathcal{F}/X/S}(T) \subset \underline{Quot}^{P}_{\mathcal{F}/X/S}(T).$$

Therefore, $\underline{\mathcal{F}ib}^{P}_{\mathcal{F}/X/S}$ is a **subfunctor** of $\underline{Quot}^{P}_{\mathcal{F}/X/S}$.

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Our main question!

- Is the fiber-full functor $\underline{\mathcal{Fib}}^P_{\mathcal{F}/X/S}$ representable?
- If so, its representing scheme would grant us all our objectives. This scheme would control the entire cohomological data.

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When is $\mathrm{H}^i(X, \mathcal{F})$ a flat A-module?

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$S = \mathrm{Spec}(A)$, $X = \mathrm{Proj}(R) \subset \mathbb{P}^r_A$ closed subscheme, $\mathcal{F}$ coherent sheaf on X, $\mathfrak{m} = [R]_+ \subset R$, and M a finitely generated graded R-module such that $\mathcal{F} = \widetilde{M}$.

Reminder

$0 \to \mathrm{H}^0_{\mathfrak{m}}(M) \to M \to \bigoplus_{\nu \in \mathbb{Z}} \mathrm{H}^0(X, \mathcal{F}(\nu)) \to \mathrm{H}^1_{\mathfrak{m}}(M) \to 0$ is exact, and $\mathrm{H}^{i+1}_{\mathfrak{m}}(M) \cong \bigoplus_{\nu \in \mathbb{Z}} \mathrm{H}^i(X, \mathcal{F}(\nu))$ for $i \geq 1$.

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It is then equivalent to address the A-flatness of all $\mathrm{H}^i_{\mathfrak{m}}(M)$. This problem has been studied before: Hochster-Roberts (1976), Kollár (2014), Smith (2018), Chardin-CR-Simis (2020).

Theorem (Hochster-Roberts, 1976)

Assume A is a domain. There exists $0 \neq a \in A$ such that $\mathrm{H}^i_{\mathfrak{m}}(M \otimes_A A_a)$ is a locally free A_a-module for all i.

Fiber-full modules

Motivated by work of Kollár-Kovács on the flatness of the cohomologies of a relative dualizing complex (also of Dao-De Stefani-Ma), Varbaro obtained the following:

Theorem (Varbaro, 2021)

Let $A = \Bbbk[t]$, R a fin. gen. A-algebra, M a fin. gen. R-module. Assume M is A-flat and the natural map $\mathrm{Ext}^i_R(M/tM, R) \to \mathrm{Ext}^i_R(M/t^qM, R)$ is injective $\forall\ i, q$. Then, $\mathrm{Ext}^i_R(M, R)$ is flat over A $\forall\ i$.

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We want to address the following general setup:

$(B, \mathfrak{b})$ a Noetherian local ring, R fin. gen. pos. graded B-algebra. M a fin. gen. graded R-module.

Definition

We say M is **fiber-full over B** if M is B-free and the natural map

$$\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{b}^qM) \to \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{b}M) \text{ is surjective } \forall\ i, q.$$

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Theorem (CR)

Assume M is a free B-module. The following six conditions are equivalent:

1. $\mathrm{H}^i_{\mathfrak{m}}(M)$ is a free B-module $\forall\, 0 \leq i \leq r$.

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5. $\mathrm{H}^i_{\mathfrak{m}}\big(M/\mathfrak{b}^q M\big) \to \mathrm{H}^i_{\mathfrak{m}}\big(M/\mathfrak{b} M\big)$ is surjective $\forall\, 0 \leq i \leq r,\ q \geq 1$.

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6. $\mathrm{Ext}^i_{T/\mathfrak{b}^q T}(M/\mathfrak{b}M, \omega_{T/\mathfrak{b}^q T}) \to \mathrm{Ext}^i_{T/\mathfrak{b}^q T}(M/\mathfrak{b}^q M, \omega_{T/\mathfrak{b}^q T})$ is injective $\forall\, 0 \leq i \leq r,\ q \geq 1$.

Moreover, when any of these conditions is satisfied, we have $\mathrm{H}^i_{\mathfrak{m}}(M) \otimes_B C \xrightarrow{\cong} \mathrm{H}^i_{\mathfrak{m}}(M \otimes_B C)$, $\mathrm{Ext}^i_T(M, T) \otimes_B C \xrightarrow{\cong} \mathrm{Ext}^i_{T \otimes_B C}(M \otimes_B C, T \otimes_B C)$ and $\mathrm{H}^i_{\mathfrak{m}}(M) \cong {}^*\mathrm{Hom}_B\left(\mathrm{Ext}^{r-i}_T(M, T(-\delta))\right)$ where $\delta = \deg(x_1) + \cdots + \deg(x_r)$.

Break?

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Corollary

The following conditions are equivalent:

1. $\mathcal{F}$ is fiber-full over S.
2. $\mathcal{F}$ is a locally free $\mathcal{O}_S$-module and $\mathrm{H}^i\left(X \times_S \mathrm{Spec}(\frac{A_\mathfrak{p}}{\mathfrak{p}^q A_\mathfrak{p}}), \mathcal{F}(\nu) \otimes_A \frac{A_\mathfrak{p}}{\mathfrak{p}^q A_\mathfrak{p}}\right)$ is a free $\frac{A_\mathfrak{p}}{\mathfrak{p}^q A_\mathfrak{p}}$-module $\forall\, \mathfrak{p} \in S, i, q, \nu$.
3. $\mathcal{F}$ is a locally free $\mathcal{O}_S$-module and the natural map
$$\mathrm{H}^i\left(X \times_S \mathrm{Spec}(\tfrac{A_\mathfrak{p}}{\mathfrak{p}^q A_\mathfrak{p}}), \mathcal{F}(\nu) \otimes_A \tfrac{A_\mathfrak{p}}{\mathfrak{p}^q A_\mathfrak{p}}\right) \to \mathrm{H}^i\left(X \times_S \mathrm{Spec}(\kappa(\mathfrak{p})), \mathcal{F}(\nu) \otimes_A \kappa(\mathfrak{p})\right)$$
is surjective for all $\forall\, \mathfrak{p} \in S, i, q, \nu$.

Fiber-full functor (again)

The fiber-full functor $\underline{\mathcal{Fib}}_{\mathcal{F}/X/S}$

$\underline{\mathcal{Fib}}_{\mathcal{F}/X/S}(T) = \{\text{coherent } \mathcal{F}_T \twoheadrightarrow \mathcal{G} \mid \mathcal{G} \text{ is fiber-full over } S\}$

Remark

When T is connected, we have the have the decomposition

$$\underline{\mathcal{Fib}}_{\mathcal{F}/X/S}(T) \quad = \quad \bigsqcup_{\mathbf{h}:\mathbb{Z}^{r+1}\to\mathbb{N}^{r+1}} \underline{\mathcal{Fib}}^{\mathbf{h}}_{\mathcal{F}/X/S}(T).$$

Therefore, if all the subfunctors $\underline{\mathcal{Fib}}^{\mathbf{h}}_{\mathcal{F}/X/S}$ are representable, then $\underline{\mathcal{Fib}}_{\mathcal{F}/X/S}$ is also representable (and a disjoint union of them).

Our main result

A Noetherian, $S = \mathrm{Spec}(A)$, $X \subset \mathbb{P}^r_A$ closed subscheme, $\mathcal{F}$ coherent sheaf on X.

Theorem (CR - Ramkumar)

Let $\mathbf{h} = (h_0, \ldots, h_r) : \mathbb{Z}^{r+1} \to \mathbb{N}^{r+1}$ be a tuple of functions. Assume $P_{\mathbf{h}} = \sum_{i=0}^r (-1)^i h_i \in \mathbb{Q}[m]$ is a numerical polynomial. Then, there is a quasi-projective S-scheme $\mathrm{Fib}^{\mathbf{h}}_{\mathcal{F}/X/S}$ that represents the functor $\underline{\mathcal{F}ib}^{\mathbf{h}}_{\mathcal{F}/X/S}$ and that is a locally closed subscheme of the Quot scheme $\mathrm{Quot}^{P_{\mathbf{h}}}_{\mathcal{F}/X/S}$.

We call $\mathrm{Fib}^{\mathbf{h}}_{\mathcal{F}/X/S}$ the **fiber-full scheme**. When $\mathcal{F} = \mathcal{O}_X$, we simply write $\mathrm{Fib}^{\mathbf{h}}_{X/S} \subset \mathrm{Hilb}^{\mathbf{h}}_{X/S}$, instead of $\mathrm{Fib}^{\mathbf{h}}_{\mathcal{O}_X/X/S}$.

General ideal of the proof

Reminder

The proof of the existence of the Quot scheme consists of two steps:

1. One embeds the Quot functor into a Grassmannian functor (not so deep, but it contains some tricky computations: Castelnuovo-Mumford regularity, etc...). The Grassmannian scheme represents the Grassmannian functor.
2. One applies a flattening stratification over the universal sheaf of the Grassmannian (this is the deeper part of the proof).

Theorem (Grothendieck, Mumford)

S locally Noetherian scheme, $X \subset \mathbb{P}_S^r$ closed subscheme, $\mathcal{F}$ coherent sheaf on X. $P \in \mathbb{Q}[m]$ a numerical polynomial. There is a locally closed subscheme $\iota : V_{\mathcal{F}}^P \hookrightarrow S$ such that for any morphism $g : T = \mathrm{Spec}(B) \to S$, $\mathcal{F}_T$ is T-flat with Hilbert polynomial P **if and only if** g can be factored as

$$T \to V_{\mathcal{F}}^P \xrightarrow{\iota} S.$$

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Theorem (CR - Ramkumar)

Assume $\mathcal{F}$ is flat over S. There is a locally closed subscheme $\iota : \mathrm{FDir}^{\mathbf{h}}_{\mathcal{F}} \hookrightarrow S$ such that for any morphism $g : T = \mathrm{Spec}(B) \to S$, $\mathrm{H}^i(X_T, \mathcal{F}_T(\nu))$ is a locally free B-module of rank $h_i(\nu)$ $\forall\, i, \nu$ **if and only if** g can be factored as

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Proof of the existence of $\mathrm{Fib}^{\mathbf{h}}_{\mathcal{F}/X/S}$

1. We already have $\underline{\mathcal{F}ib}^{\mathbf{h}}_{\mathcal{F}/X/S} \hookrightarrow \underline{\mathcal{Q}uot}^{P_{\mathbf{h}}}_{\mathcal{F}/X/S}$. If $\mathcal{G} \in \underline{\mathcal{Q}uot}^{P_{\mathbf{h}}}_{\mathcal{F}/X/S}(T)$, then $\mathcal{G} = \left(\mathcal{W}^{P_{\mathbf{h}}}_{\mathcal{F}/X/S}\right)_T = (1_X \times_S g_G)^* \mathcal{W}^{P_{\mathbf{h}}}_{\mathcal{F}/X/S}$ where $g_{\mathcal{G}} : T \to \mathrm{Quot}^{P_{\mathbf{h}}}_{\mathcal{F}/X/S}$.
2. The theorem above yields that, $\mathrm{H}^i(X_T, \mathcal{G}(\nu))$ is a locally free B-module of rank $h_i(\nu)\ \forall\, i, \nu$ **if and only if** $g_{\mathcal{G}}$ can be factored as
$$T \to \mathrm{FDir}^{\mathbf{h}}_{\mathcal{W}^{P_{\mathbf{h}}}_{\mathcal{F}/X/S}} \xrightarrow{\iota} \mathrm{Quot}^{P_{\mathbf{h}}}_{\mathcal{F}/X/S}.$$
Therefore, we have $\mathrm{Fib}^{\mathbf{h}}_{\mathcal{F}/X/S} := \mathrm{FDir}^{\mathbf{h}}_{\mathcal{W}^{P_{\mathbf{h}}}_{\mathcal{F}/X/S}}$.

A Noetherian, $S = \mathrm{Spec}(A)$, R fin. gen. graded A-algebra. M fin. gen. graded R-module.

Theorem (CR - Ramkumar)

Assume M is flat over A. There is a locally closed subscheme $\iota : \mathrm{FLoc}_M^{\mathbf{h}} \hookrightarrow S$ such that for any morphism $g : T = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$, $\left[\mathrm{H}_{\mathfrak{m}}^i(M \otimes_A B)\right]_\nu$ is a locally free B-module of rank $h_i(\nu)$ $\forall\, i, \nu$ **if and only if** g can be factored as

$$T \to \mathrm{FLoc}_{\mathcal{F}}^{\mathbf{h}} \xrightarrow{\iota} S.$$

Step 0 in the proof of our stratification theorem

Lemma (Grothendieck's complex)

A Noetherian, $S = \mathrm{Spec}(A)$, $X \subset \mathbb{P}^r_A$ closed subscheme, $\mathcal{F}$ coherent sheaf on X that is flat over S. There is a complex $K^\bullet$ of finitely generated free A-modules such that

$$\mathrm{H}^i(X, \mathcal{F} \otimes_A N) \cong \mathrm{H}^i(K^\bullet \otimes_A N)$$

for any A-module N.

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Lemma (Jouanolou's complex)

A Noetherian, $R = A[x_1, \ldots, x_r]$, M a finitely generated graded R-module that is flat over A. Let $F_\bullet : \cdots \to F_1 \to F_0$ be a graded free resolution of M by modules of finite rank. Consider the complex $L_\bullet = \mathrm{H}^r_{\mathfrak{m}}(F_\bullet)$ (Note: each graded strand $[L_\bullet]_\nu$ is a complex of finitely generated free A-modules). Then

$$\mathrm{H}^i_{\mathfrak{m}}(M \otimes_A N) \cong \mathrm{H}_{r-i}(L_\bullet \otimes_A N)$$

for any A-module N.

Example (Twisted cubics)

$\mathrm{Hilb}^{3m+1}_{\mathbb{P}^3_{\Bbbk}} = \mathrm{Fib}^{\mathbf{h}}_{\mathbb{P}^3_{\Bbbk}} \sqcup \mathrm{Fib}^{\mathbf{g}}_{\mathbb{P}^3_{\Bbbk}}$, where $\mathrm{Fib}^{\mathbf{h}}_{\mathbb{P}^3_{\Bbbk}} = H - H \cap H'$ is open and $\mathrm{Fib}^{\mathbf{g}}_{\mathbb{P}^3_{\Bbbk}} = H'$ is closed (we explicitly saw $\mathbf{h}$ and $\mathbf{g}$).

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Example (Points)

Let $\mathbf{h} = (c, 0, \ldots, 0) : \mathbb{Z}^{r+1} \to \mathbb{N}^{r+1}$ and so $P_{\mathbf{h}} = c \in \mathbb{Q}[m]$. Then, we have

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Example (Smooth Hilbert schemes)

$P \in \mathbb{Q}[m]$ such that $\mathrm{Hilb}^{P}_{\mathbb{P}^r_{\Bbbk}}$ is smooth, $L \subset \Bbbk[\mathbb{P}^r_{\Bbbk}]$ be the corresponding saturated lexicographic ideal and $\mathbf{h} = (h_0, \ldots, h_r)$ with $h_i(\nu) = \dim_{\Bbbk}\left(\mathrm{H}^i(\mathbb{P}^r_{\Bbbk}, \mathcal{O}_{V(L)})\right)$. By using the classification of Skjelnes - Smith (2021), we can prove that

$$\mathrm{Fib}^{\mathbf{h}}_{\mathbb{P}^r_{\Bbbk}} = \mathrm{Hilb}^{P}_{\mathbb{P}^r_{\Bbbk}}.$$

Parametrizing ACM and AG schemes

$Y \subset \mathbb{P}_{\Bbbk}^r$ is said to be **arithmetically Cohen-Macaulay** or **arithmetically Gorenstein** when the homogeneous coordinate ring is Cohen-Macaulay or Gorenstein, resp.

Parametrizing ACM and AG schemes

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$S = \operatorname{Spec}(A)$ with A Noetherian. Let $d \in \mathbb{N}$, and $h_0, h_d : \mathbb{Z} \to \mathbb{N}$ be two functions. As all intermediate cohomologies vanish, we want to consider the functors

$$\underline{\mathcal{ACM}}^{h_0,h_d}_{X/S}(T) \;=\; \left\{ Z \in \underline{\mathcal{F}ib}^{\mathbf{h}}_{X/S}(T) \mid Z_t \text{ is ACM for all } t \in T \right\}$$

and

$$\underline{\mathcal{AG}}^{h_0,h_d}_{X/S}(T) \;=\; \left\{ Z \in \underline{\mathcal{F}ib}^{\mathbf{h}}_{X/S}(T) \mid Z_t \text{ is AG for all } t \in T \right\}$$

where $\mathbf{h} = (h_0, 0, \ldots, 0, h_d, 0 \ldots, 0) : \mathbb{Z}^{r+1} \to \mathbb{N}^{r+1}$.

Theorem (CR - Ramkumar)

$\underline{\mathcal{ACM}}^{h_0,h_d}_{X/S}$ and $\underline{\mathcal{AG}}^{h_0,h_d}_{X/S}$ are represented by open S-subschemes $\mathrm{ACM}^{h_0,h_d}_{X/S}$ and $\mathrm{AG}^{h_0,h_d}_{X/S}$ of $\mathrm{Fib}^{\mathbf{h}}_{X/S}$.

Square-free Gröbner degenerations

Theorem (Conca - Varbaro)

$R = \Bbbk[x_1, \ldots, x_r]$, $>$ monomial order on R and $I \subset R$ homogeneous ideal. If $\mathrm{in}_>(I)$ is square-free, then

$$\dim_{\Bbbk}\left(\left[\mathrm{H}^i_{\mathfrak{m}}(R/I)\right]_\nu\right) \;=\; \dim_{\Bbbk}\left(\left[\mathrm{H}^i_{\mathfrak{m}}(R/\mathrm{in}_>(I))\right]_\nu\right)$$

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for all i, ν.

Corollary

Let $\hom_\omega(I) \subset R[t]$ with special fiber equal to $\mathrm{in}_>(I)$. Let $Z = \mathrm{Proj}(S/I) \subset \mathbb{P}^r_{\Bbbk}$. Let $\mathbf{h} = (h_0, \ldots, h_r) : \mathbb{Z}^{r+1} \to \mathbb{N}^{r+1}$ given by $h_i(\nu) := \dim_{\Bbbk}\left(\mathrm{H}^i(Z, \mathcal{O}_Z(\nu))\right)$. For each $\alpha \in \Bbbk$, let $Z_\alpha = \mathrm{Proj}\left(R[t]/\hom_\omega(I) \otimes_{\Bbbk[t]} \Bbbk[t]/(t-\alpha)\right) \subset \mathbb{P}^r_{\Bbbk}$. Then, we have that

$$Z_\alpha \quad \text{corresponds with a point in} \quad \mathrm{Fib}^{\mathbf{h}}_{\mathbb{P}^r_{\Bbbk}/\Bbbk}$$

for all $\alpha \in \Bbbk$.

Future directions

1. **A compactification of the fiber-full scheme.** Find the most natural compactification of the fiber-full scheme.

2. **Deformation theory on the fiber-full scheme.** For instance, computing the tangent space $T_{[Z]}\mathrm{Fib}^{\mathbf{h}}_{\mathbb{P}^r_{\Bbbk}}$ at $Z \in \mathbb{P}^r_{\Bbbk}$ is equivalent to find all $Z' \subset \mathbb{P}^r_{\Bbbk[\epsilon]}$ such that $Z \cong Z' \times_{\mathrm{Spec}(\Bbbk[\epsilon])} \mathrm{Spec}(\Bbbk)$ and $\mathrm{H}^i(Z', \mathcal{O}_{Z'}(\nu))$ is a $\Bbbk[\epsilon]$-flat $\forall\, i, \nu$, where $\Bbbk[\epsilon] = \Bbbk[t]/(t^2)$.

3. **Understand small neighborhoods of monomial ideals in the fiber-full scheme.**

Thanks!