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Full Information H_∞-Control for Discrete-Time Infinite Markov Jump Parameter Systems*

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In this paper we consider the full information discrete-time H_∞-control problem for the class of linear systems with Markovian jumping parameters. The state-space of the Markov chain is assumed to take values in a countably infinite set. Full information here means that the controller has access to both the state-variables and jump-variables. A necessary and sufficient condition for the existence of a feedback controller that makes the ℓ_2-induced norm of the system less than a prespecified bound is obtained. This condition is written in terms of a set of infinite coupled algebraic Riccati equations.

1. INTRODUCTION

A great deal of attention has been recently given to the class of linear systems subject to abrupt changes in their structures. This is due, at least in part, to the large number of applications found in the literature, for instance, in systems subject to random failures, repairs, or sudden environmental disturbances, abrupt variation of the operating point on a nonlinear plant, etc. Markovian jump linear systems (MJLS) comprise an

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important family of systems subject to abrupt variations. In this case the changes in the structure of the system are assumed to be modeled by a Markov chain, taking values in a countably infinite set, and this takes the system into distinct linear forms in a state-variable representation. Practical motivations as well as some theoretical results for JMLS can be found, for instance, in [1, 3–11, 14, 16, 17, 19–22, 25, 27–30, 32, 33].

The H_∞-control problem was originally formulated as a linear design problem in the frequency domain (see, for instance [15, 37]). In a state-space formulation (cf. [2, 12, 31, 34]), the H_∞-control problem consists of obtaining a controller that stabilizes a linear system and ensures that the ℓ_2-induced norm from the additive input disturbance to the output is less than a prespecified attenuation value. The H_∞-control analysis, within this framework, has been extended to comprise non-linear systems (cf. [18, 35]), infinite dimensional linear systems (see [23]), and the LQG problem (cf. [24]). Regarding the MJLSs, the H_∞-control was previously studied in [11, 30] for the continuous-time and in [17] for the discrete-time problem. The technique in [11, 17] provides sufficient conditions for a solution whereas in [30] the differential game interpretation for the problem is employed.

In this paper we obtain a necessary and sufficient condition for the existence of a state-feedback controller that stabilizes (in a probabilistic sense) a MJLS and ensures that, for any ℓ_2-additive disturbance sequence to the system, the output is smaller than some prespecified bound. In the deterministic set-up, this problem would be equivalent to the H_∞-control problem in the time-domain formulation. Using the concept of stochastic stabilizability and stochastic detectability, see Definitions 1 and 2, the necessary and sufficient condition is derived in terms of a set of infinite coupled algebraic Riccati equations. The technique of proof follows the one used in the literature for H_∞-control in a state-space formulation (see [31, 34]). The proof of necessity relies on a representation result by [36]; for the minimum of quadratic forms within translations of Hilbert subspaces, see Lemma 3. Notice that the sufficient condition presented in [17] is apparently different from the one obtained here but, in fact, after some algebraic manipulation, it can be shown that they are identical, when restricted to the case where the Markov chain takes value in a finite set. However, the methods for deriving this condition are widely distinct. The Hilbert space technique employed here provides some powerful tools for analyzing the H_∞-control problem of MJLS with the Markov chain taking values in a countably infinite set, allowing us to show that the condition derived is not only sufficient but also necessary.

The paper content is as follows. In Section 2 the basic definitions and notations are presented. In Section 3 the probabilistic structure of the problem is settled and in Section 4 the H_∞-control problem for MJLS is precisely defined, together with auxiliary results involving the concepts of stochastic stabilizable and stochastic detectable systems. The main characterization result and the proofs of sufficiency and necessity are presented in Section 5.

2. NOTATION AND ASSUMPTIONS

Throughout this paper $\mathbb{C}$ stands for the set of complex numbers, $\mathbb{C}^n$ is the n-dimensional complex Euclidean space, and $\mathbb{N}^0 = \{0, 1, \ldots\}$, $\mathbb{N} = \{1, 2, \ldots\}$. We denote by $\mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ the normed linear space of all n by m complex matrices and, for simplicity, we shall write $\mathbb{M}(\mathbb{C}^n)$ whenever $n = m$. We set $*$ for conjugate transpose. The notation $L \geq 0$ and $L > 0$ indicates that a self-adjoint matrix is positive semi-definite or positive definite, respectively. We denote $\mathbb{M}(\mathbb{C}^n)^+ = \{L \in \mathbb{M}(\mathbb{C}^n);\ L = L^* \geq 0\}$. Either the uniform induced norm in $\mathbb{M}(\mathbb{C}^n)$ or the standard norm in $\mathbb{C}^n$ is represented by $\|\cdot\|$.

Let $\mathscr{H}_1^{m,n}$ ($\mathscr{H}_{\sup}^{m,n}$ respectively) be the linear space made up of all infinite sequences of complex matrices $H = (H_1, H_2, \ldots)$, $H_i \in \mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$, such that the series $\sum_{i=1}^{\infty} \|H_i\|$ converges ($\sup\{\|H_i\|;\ i = 1, 2, \ldots\} < \infty$). For $h \in \mathscr{H}_1^{m,n}$ ($H \in \mathscr{H}_{\sup}^{m,n}$ respectively) we define a norm in $\mathscr{H}_1^{m,n}$ ($\mathscr{H}_{\sup}^{m,n}$) by

$$\boxed{\|H\|_1 = \sum_{i=1}^{\infty} \|H_i\| \qquad \left(\|H\|_{\sup} = \sup\{\|H_i\|;\ i = 1, 2, \ldots\}\right).}$$

We shall write $\mathscr{H}_1^n$ and $\mathscr{H}_{\sup}^n$ whenever $n = m$, $\mathscr{H}_1^{n+} = \{H \in \mathscr{H}_1^n;\ H_i \in \mathbb{M}(\mathbb{C}^n)^+,\ i = 1, 2, \ldots\}$ and similarly for $\mathscr{H}_{\sup}^{n+}$.

Remark 1. It is easy to verify that $(\mathscr{H}_1^{m,n}, \|\cdot\|_1)$ and $(\ell_1, \|\cdot\|_1)$ (the space of all infinite absolute convergent sequences of complex numbers with the usual ℓ_1-norm) are uniformly homeomorphic. Similarly $(\mathscr{H}_{\sup}^{m,n}, \|\cdot\|_{\sup})$ and $(\ell_{\sup}, \|\cdot\|_{\sup})$ (the space of all infinite uniformly bounded sequences of complex numbers with the sup norm) can be shown to be uniformly homeomorphic. Since $(\ell_1, \|\cdot\|_1)$ and $(\ell_{\sup}, \|\cdot\|_{\sup})$ are Banach spaces, we have that $(\mathscr{H}_1^{m,n}, \|\cdot\|_1)$ and $(\mathscr{H}_{\sup}^{m,n}, \|\cdot\|_{\sup})$ are also Banach spaces.

Remark 2. For $H = (H_1, \ldots)$, $L = (L_1, \ldots)$ in $\mathscr{H}_1^{n+}$ we use the notation $H \leq L$ to indicate that $H_i \leq L_i$ for each i in $\mathbb{N}$. It is clear that if $H \leq L$, H, L in $\mathscr{H}_1^{n+}$ then $\|H\|_1 \leq \|L\|_1$.

Finally for any complex Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ we denote by $\mathbb{B}[\mathbb{X}, \mathbb{Y}]$ the Banach space of all bounded linear transformations of $\mathbb{X}$ into $\mathbb{Y}$ with the uniform induced norm represented by $\|\cdot\|$. For simplicity we set $\mathbb{B}[\mathbb{X}] = \mathbb{B}[\mathbb{X}, \mathbb{X}]$ and, for $\mathscr{L} \in \mathbb{B}(\mathbb{X})$, we denote by $r(\mathscr{L})$ the spectral radius of $\mathscr{L}$.

$\|H\|_1 = \sum_{i=1}^{N} \|H_i\|$

$\|H\|_2 = \left(\sum_{i=1}^{N} tr(H_i H_i^*)\right)^{\frac{1}{2}} \Rightarrow \|H_i\| = \|H_i\|_2 = \sqrt{\lambda_{max}(H_i^* H_i)}$

$= max\{x^T \underline{\underline{H_i^T H_i}} x;\ \|x\|_2 = 1\}$

$\|H\|_{sup} = sup\{\|H_i\|;\ i \in N\}$

3. PROBABILISTIC ASSUMPTIONS

初始概率分布

Consider $(\tilde{\Omega}, \tilde{\mathscr{F}})$ a probabilistic space and set $\mathscr{N}$ the σ-field of all subsets of $\mathbb{N}$. Define $\Omega = \tilde{\Omega} \times \prod_{i=0}^{\infty} \mathbb{N}$, where $\times$ and $\prod$ denote the product space, and $\mathscr{F} = \sigma\{S \times \psi_0 \times \psi_1 \times \ldots;\ S \in \tilde{\mathscr{F}}$ and $\psi_i \in \mathscr{N}$ for each $i \in \mathbb{N}^0\}$, where $\sigma\{\mathfrak{S}\}$ denotes the σ-field generated by the subsets $\mathfrak{S}$. Define also for $k \in \mathbb{N}^0$, $\mathscr{F}_k = \sigma\{S \times \psi_0 \times \psi_1 \times \cdots \times \psi_k \times \prod_{i=k+1}^{\infty} \mathbb{N};\ S \in \tilde{\mathscr{F}}$ and $\psi_i \in \mathscr{N}$ for $i = 0, \ldots, k\}$ so that $\mathscr{F}_k \subset \mathscr{F}$. On the probabilistic space $(\Omega, \mathscr{F}, \{\mathscr{F}_k\}_{k \in \mathbb{N}^0})$ we consider a probability measure $\mathscr{P}$ such that

$$\mathscr{P}(\theta(k+1) = j \,|\, \mathscr{F}_k) = \mathscr{P}(\theta(k+1) = j \,|\, \theta(k)) = p_{\theta(k)j},$$

where $p_{ij} \geq 0$ for $i, j \in \mathbb{N}$, $\sum_{j=1}^{\infty} p_{ij} = 1$ for each $i \in \mathbb{N}$, and for each $k \in \mathbb{N}^0$, $\theta(k)$ is a random variable from Ω to $\mathbb{N}$ defined as $\theta(k)(\omega) = \omega_k$ with $\omega = (\xi, \omega_0, \omega_1, \ldots) \in \Omega$, $\xi \in \tilde{\Omega}$, $\omega_k \in \mathbb{N}$. Clearly $\{\theta(k)\}_{k \in \mathbb{N}^0}$ is a Markov chain taking values in $\mathbb{N}$ and with transition probability matrix $\mathbb{P} = [p_{ij}]$.

Set $C^m = L_2(\Omega, \mathscr{F}, \mathscr{P}, \mathbb{C}^m)$ the Hilbert space made up of all second order $\mathbb{C}^m$-valued random variables with the inner product given by $\prec x; y \succ = E(x^* y)$ for all $x, y \in C^m$, where $E(\cdot)$ stands for the expectation of the underlying scalar valued random variables, and the norm is denoted by $\|\cdot\|_2$. Set $\ell_2(C^m) = \oplus_{k=0}^{\infty} C^m$, the direct sum of countably infinite copies of C^m, which is a Hilbert space made up of $r = (r(0), \ldots)$, $r(k) \in C^m$, $k \in \mathbb{N}^0$ such that $\|r\|_2^2 = \sum_{k=0}^{\infty} E(\|r(k)\|^2) < \infty$, and with the inner product also denoted by $\prec \cdot;\cdot \succ$.

Define $\mathscr{C}^m \subset \ell_2(C^m)$ in the following way: $r = (r(0), \ldots) \in \mathscr{C}^m$ if $r \in \ell_2(C^m)$ and $r(k) \in L_2(\Omega, \mathscr{F}_k, \mathscr{P}, \mathbb{C}^m)$ for each $k \in \mathbb{N}^0$. We have that $\mathscr{C}^m$ is a closed linear subspace of $\ell_2(C^m)$ and therefore a Hilbert space. We also define $\mathscr{C}_k^m = \{r_k = (r(0), \ldots, r(k));\ r(i) \in L_2(\Omega, \mathscr{F}_i, \mathscr{P}, \mathbb{C}^m)$ for each $i = 0, \ldots, k\}$ and Θ_0 the set of all $\mathscr{F}_0$-measurable variables taking values in $\mathbb{N}$.

4. PROBLEM FORMULATION AND AUXILIARY RESULTS

In this paper we shall consider the following class of discrete-time Markovian jump linear systems,

$$\begin{aligned} x(k+1) &= A_{\theta(k)} x(k) + B_{\theta(k)} u(k) + D_{\theta(k)} w(k), \\ x(0) &= x_0 \in \mathscr{C}_0^n,\ \theta(0) = \theta_0 \in \Theta_0,\ k \in \mathbb{N}^0, \end{aligned} \tag{1}$$

where $w = (w(0), w(1), \ldots) \in \mathscr{C}^p$ and $u = (u(0), u(1), \ldots) \in \mathscr{C}^m$. From

(1) and the definition of the probabilistic space in Section 3 above we have that $x_k = (x(0), \ldots, x(k)) \in \mathscr{C}_k^n$ for every $k \in \mathbb{N}^0$. We assume that $A = (A_1, A_2, \ldots) \in \mathscr{H}_{\text{sup}}^n$, $B = (B_1, B_2, \ldots) \in \mathscr{H}_{\text{sup}}^{m,n}$, $D = (D_1, D_2, \ldots) \in \mathscr{H}_{\text{sup}}^{p,n}$. We shall write, whenever necessary, $E_{(\theta_0, x_0)}(\cdot)$ to indicate that the expected value is taken in (1) above with $x(0) = x_0$ and $\theta(0) = \theta_0$.

For $F = (F_1, \ldots) \in \mathscr{H}_{\text{sup}}^n$, $P = (P_1, \ldots) \in \mathscr{H}_1^n$, $V = (V_1, \ldots) \in \mathscr{H}_{\text{sup}}^n$, we define the operators $\mathscr{L}(P) = (\mathscr{L}_1(P), \ldots)$ (which is associated to the second moments of a MJLS; see Proposition 2 of [9]) and $\mathscr{E}(V) = (\mathscr{E}_1(V), \ldots)$ as

$$\mathscr{L}_j(P) = \sum_{i=1}^{\infty} p_{ij} F_i P_i F_i^* \tag{2}$$

$$\mathscr{E}_i(V) = \sum_{j=1}^{\infty} p_{ij} V_j.$$

It is easy to verify that $\mathscr{L} \in \mathbb{B}(\mathscr{H}_1^n)$ and $\mathscr{E} \in \mathbb{B}(\mathscr{H}_{\text{sup}}^n)$. In what follows we shall use the subscript 1 and sup to distinguish the spectral radius and norms in $\mathbb{B}(\mathscr{H}_1^n)$ and $\mathbb{B}(\mathscr{H}_{\text{sup}}^n)$, respectively.

We make the following definitions:

Definition 1. We say that (A, B) is stochastically stabilizable (SS) if there exists some $K \in \mathscr{H}_{\text{sup}}^{n,m}$ such that for any initial condition $x_0 \in \mathscr{C}_0^n$, $\theta_0 \in \Theta_0$, we have that $\sum_{k=0}^{\infty} E_{(\theta_0, x_0)}(\|x(k)\|^2) < \infty$ where $x(k)$ is given by (1) with $u(k) = -K_{\theta(k)} x(k)$ and $w = 0$. In this case we say that K stabilizes (A, B).

Definition 2. Consider $C = (C_1, C_2, \ldots) \in \mathscr{H}_{\text{sup}}^{n,r}$. We say that (C, A) is stochastically detectable (SD) if there exists some $H \in \mathscr{H}_{\text{sup}}^{r,n}$ such that for any initial condition $x_0 \in \mathscr{C}_0^n$, $\theta_0 \in \Theta_0$, we have that $\sum_{k=0}^{\infty} E_{(\theta_0, x_0)}(\|x(k)\|^2) < \infty$ where $x(k)$ is given by

$\hat{x}_{k+1} = A\hat{x}_k + H(y_k - \hat{y}_k)$
$= (A - HC)\hat{x}_k + HCx_k$

$$x(k+1) = (A_{\theta(k)} - H_{\theta(k)} C_{\theta(k)}) x(k), \qquad k \in \mathbb{N}^0,$$

$x_{k+1} = Ax_k$
$x_{k+1} - \hat{x}_{k+1} = (A - HC)(x_k - \hat{x}_k)$

and, in this case, we say that H stabilizes (C, A).

Remark 3. If we set $M_i^{1/2} = (C_i^* C_i)^{1/2}$, $M^{1/2} = (M_1^{1/2}, \ldots) \in \mathscr{H}_{\text{sup}}^{n+}$, then we get that (C, A) is SD if and only if $(M_i^{1/2}, A)$ is SD. This follows from the fact that for some unitary matrix U_i, $C_i = U_i M_i^{1/2}$ for each $i \in \mathbb{N}$.

The following result was shown in [9].

Proposition 1. *(A, B) is SS if and only if there exists some $K \in \mathscr{H}_{\text{sup}}^{n,m}$ such that $r_1(\mathscr{L}) < 1$, where $\mathscr{L}$ is defined as in (2) with $F_i = A_i - B_i K_i$. Similarly, (C, A) is SD if and only if there exists some $H \in \mathscr{H}_{\text{sup}}^{r,n}$ such that $r_1(\mathscr{L}) < 1$, where $\mathscr{L}$ is defined as in (2) with $F_i = A_i - H_i C_i$.*

Consider now Eq. (1) with $u(k) = -K_{\theta(k)}x(k)$ and $\mathscr{L}$ as defined in (2) with $F_i = A_i - B_iK_i$. The following result will be required in the sequel, and is proved in the Appendix.

PROPOSITION 2. *$r_1(\mathscr{L}) < 1$ if and only if $x = (x(0), x(1), \dots) \in \mathscr{C}^n$ for every $w = (w(0), w(1), \dots) \in \mathscr{C}^p$, $x_0 \in \mathscr{C}_0^n$, and $\theta_0 \in \Theta_0$.*

For $C = (C_1, \dots) \in \mathscr{H}_{\sup}^{n,r}$, set $M_i = C_i^* C_i$, $i \in \mathbb{N}$, $M = (M_1, \dots) \in \mathscr{H}_{\sup}^{n+}$. The following result is a straightforward modification of Theorem 1 in [9].

PROPOSITION 3. *Suppose that (A, B) is SS and (C, A) is SD. Then there exists a unique solution $L = (L_1, \dots) \in \mathscr{H}_{\sup}^{n+}$ such that, for each $i \in \mathbb{N}$,*

$$\begin{aligned} L_i &= M_i + A_i^* \mathscr{E}_i(L) A_i - A_i^* \mathscr{E}_i(L) B_i \left(I + B_i^* \mathscr{E}_i(L) B_i\right)^{-1} B_i^* \mathscr{E}_i(L) A_i \\ &= M_i + (A_i - B_iJ_i)^* (\mathscr{E}_i(L))(A_i - B_iJ_i) + J_i^* J_i, \qquad i \in \mathbb{N}, \end{aligned}$$

where

$$J_i = \left(I + B_i^* \mathscr{E}_i(L) B_i\right)^{-1} B_i^* \mathscr{E}_i(L) A_i, \qquad i \in \mathbb{N}.$$

Moreover, $J = (J_1, \dots)$ stabilizes (A, B).

For $(x_0, w, q) \in \mathscr{C}_0^n \oplus \mathscr{C}^p \oplus \mathscr{C}^m$, $\theta_0 \in \Theta_0$, and $K \in \mathscr{H}_{\sup}^{m,n}$, define the linear operator $X_K(\theta_0, \cdot)$ from $\mathscr{C}_0^n \oplus \mathscr{C}^p \oplus \mathscr{C}^m$ to $\mathscr{C}^n$ as

$$X_K(\theta_0, x_0, w, q) = x = (x_0, x(1), \dots), \tag{3}$$

where x is defined as in (1) above with $u(k) = -K_{\theta(k)}x(k) + q(k)$, $q = (q(0), \dots) \in \mathscr{C}^m$. From Proposition 2, it is clear that if $r_1(\mathscr{L}) < 1$, then $x \in \mathscr{C}^n$ and $X_K(\theta_0, \cdot) \in \mathbb{B}[\mathscr{C}_0^n \oplus \mathscr{C}^p \oplus \mathscr{C}^m, \mathscr{C}^n]$. Define $z = (z(0), z(1), \dots)$ as

$$\boxed{z(k) = C_{\theta(k)}x(k) + N_{\theta(k)}u(k),} \tag{4}$$

where $C = (C_1, \dots) \in \mathscr{H}_{\sup}^{n,r}$ and $N = (N_1, \dots) \in \mathscr{H}_{\sup}^{m,r}$ satisfy:

(C1) $C_i^* N_i = 0$ for each $i \in \mathbb{N}$, and

(C2) $N_i^* N_i = I$ for each $i \in \mathbb{N}$.

In the Corollary below, conditions (C1) and (C2) are replaced by $N_j^* N_i > \beta I$ for each $i \in \mathbb{N}$ and some $\beta > 0$. Set $M_i = C_i^* C_i$ for each $i \in \mathbb{N}$, and $M = (M_1, \dots) \in \mathscr{H}_{\sup}^{n+}$. We have that $z \in \mathscr{C}^r$ and the following bounded linear operator $Z_K(\theta_0, \cdot) \in \mathbb{B}[\mathscr{C}_0^n \oplus \mathscr{C}^p \oplus \mathscr{C}^m, \mathscr{C}^r]$ is well defined: $Z_K(\theta_0, x_0, w, q) = z = (z(0), z(1), \dots)$. For the H_∞-optimal control problem we define the operators $X_K^0(\theta_0, \cdot)$ and $Z_K^0(\theta_0, \cdot)$ in $\mathbb{B}[\mathscr{C}_0^p, \mathscr{C}^n]$ and $\mathbb{B}[\mathscr{C}_0^p, \mathscr{C}^r]$, respectively, as $X_K^0(\theta_0, w) = X_K(\theta_0, 0, w, 0)$ and $Z_K^0(\theta_0, w) = Z_K(\theta_0, 0, w, 0)$. We want to solve the following problem: given $\delta > 0$, find K that stabilizes (A, B) and such that

$$\boxed{\sup_{\theta_0 \in \Theta_0} \left\| Z_K^0(\theta_0, \cdot) \right\| < \delta,}$$

$\sup \| Z_K(\theta_0, 0, w, 0) \| < \delta$

so that,

$$\|Z_K^0(\theta_0, w)\|_2^2 = \sum_{k=0}^{\infty} E_{(\theta_0,0)}\left(\|C_{\theta(k)}x(k)\|^2 + \|u(k)\|^2\right)$$
$$< \delta^2 \sum_{k=0}^{\infty} E_{(\theta_0,0)}\left(\|w(k)\|^2\right)$$

for every $w \in \mathscr{C}^p$ different from 0 and $\theta_0 \in \Theta_0$.

5. MAIN RESULTS

5.1. *Main Theorem*

The following Theorem, to be proved in the following subsections, will give a solution to the above problem.

假设

THEOREM. *Suppose that* (C, A) *is SD an consider* $\delta > 0$ *fixed. Then there exists* K *that stabilizes* (A, B) *and such that* $\sup_{\theta_0 \in \Theta_0}\|Z_K^0(\theta_0, .)\| < \delta$ *if and only if there exists* $P = (P_1, \dots) \in \mathscr{H}_{\sup}^{n+}$ *satisfying the following conditions*:

(i) $\delta^2 I - D_i^* \mathscr{E}_i(P) D_i \geq \alpha^2 I$ *for all* $i \in \mathbb{N}$ *and some* $\alpha > 0$.

(ii) $P_i = M_i + (A_i - B_i K_i + (1/\delta) D_i G_i)^* (\mathscr{E}_i(P))(A_i - B_i K_i + (1/\delta) D_i G_i) + K_i^* K_i - G_i^* G_i$, $i \in \mathbb{N}$, *where*

$$K_i = (I + B_i^* \mathscr{E}_i(P) B_i)^{-1} B_i^* \mathscr{E}_i(P)\left(A_i + \frac{1}{\delta} D_i G_i\right), \qquad i \in \mathbb{N},$$

$$G_i = \left(I - \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P) D_i\right)^{-1}\left(\frac{1}{\delta} D_i^*\right) \mathscr{E}_i(P)(A_i - B_i K_i), \qquad i \in \mathbb{N},$$

that is,

$$K_i = \left(I + B_i^* \mathscr{E}_i(P) B_i + \frac{1}{\delta^2} B_i^* \mathscr{E}_i(P) D_i \left(I - \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P) D_i\right)^{-1} D_i^* \mathscr{E}_i(P) B_i\right)^{-1}$$
$$\cdot \left(B_i^* \left(I + \frac{1}{\delta^2} \mathscr{E}_i(P) D_i \left(I - \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P) D_i\right)^{-1} D_i^*\right) \mathscr{E}_i(P) A_i\right)$$

$$= \left(I + B_i^* \mathscr{E}_i(P) \left[I - \frac{1}{\delta^2} D_i D_i^* \mathscr{E}_i(P) \right]^{-1} B_i \right)^{-1}$$

$$\cdot \left(B_i^* \mathscr{E}_i(P) \left[I - \frac{1}{\delta^2} D_i D_i^* \mathscr{E}_i(P) \right]^{-1} A_i \right), \qquad i \in \mathbb{N}$$

$$G_i = \left(I - \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P) D_i \right.$$

$$\left. + \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P) B_i (I + B_i^* \mathscr{E}_i(P) B_i)^{-1} B_i^* \mathscr{E}_i(P) D_i \right)^{-1}$$

$$\cdot \left(\frac{1}{\delta} D_i^* \left(I - \mathscr{E}_i(P) B_i (I + B_i^* \mathscr{E}_i(P) B_i)^{-1} B_i^* \right) \mathscr{E}_i(P) A_i \right)$$

$$= \frac{1}{\delta} \left(I - \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P) [I + B_i B_i^* \mathscr{E}_i(P)]^{-1} D_i \right)^{-1}$$

$$\cdot \left(D_i^* \mathscr{E}_i(P) [I + B_i B_i^* \mathscr{E}_i(P)]^{-1} A_i \right), \qquad i \in \mathbb{N}.$$

(iii) $r_1(\mathscr{L}) < 1$ *where* $\mathscr{L}$ *is as defined in* (2) *with* $F_i = A_i - B_i K_i + (1/\delta) D_i G_i$.

Furthermore, in this case, $\sup_{\theta_0 \in \Theta_0} \| Z_K^0(\theta_0, \cdot) \| < \delta$.

Remark 4. It is not difficult to check that, when restricted to the deterministic case (that is, a Markov chain with a single state), the above result reduces to some known results in the current literature (cf. [31]). In addition, if we take $\delta \to \infty$ in (ii) above, we obtain the set of coupled algebraic Riccati equations presented in Proposition 3, which provides the characterization of the LQ control problem for MJLS. This is analogous to what is found in the deterministic H_∞-control case.

Let us suppose that, instead of conditions (C1) and (C2), we only have that $N_i^* N_i > \beta I$ for each $i \in \mathbb{N}$ and some $\beta > 0$. Set $\overline{A} = (\overline{A}_1, \ldots) \in \mathscr{H}_{\sup}^n$, $\overline{B} = (\overline{B}_1, \ldots) \in \mathscr{H}_{\sup}^{p,n}$, $\overline{C} = (\overline{C}_1, \ldots) \in \mathscr{H}_{\sup}^{n,r}$, and $\overline{M} = (\overline{M}_1, \ldots) \in \mathscr{H}_{\sup}^{n+}$ as

$$\overline{A}_i = A_i - B_i (N_i^* N_i)^{-1} N_i^* C_i,$$

$$\overline{B}_i = B_i (N_i^* N_i)^{-1/2},$$

$$\overline{C}_i = C_i - N_i (N_i^* N_i)^{-1} N_i^* C_i,$$

$$\overline{M}_i = \left(\overline{C}_i^* \overline{C}_i \right).$$

The following Corollary is an immediate consequence of the above Theorem, after applying a preliminary feedback $u(k) = -(N^*_{\theta(k)}N_{\theta(k)})^{-1} N^*_{\theta(k)}C_{\theta(k)}x(k) + (N^*_{\theta(k)}N_{\theta(k)})^{-1/2}v(k)$ to (1) and (4) (with $v(k)$ playing the role of the new control variable), and we shall omit the proof (see [23] for similar results).

COROLLARY. *Suppose that* $(\bar{C}, \bar{A})$ *is SD and consider* $\delta > 0$ *fixed. Then there exists K that stabilizes* (A, B) *and such that* $\sup_{\theta_0 \in \Theta_0}\|Z_K^0(\theta_0, \cdot)\| < \delta$ *if and only if there exists* $P = (P_1, \ldots) \in \mathscr{H}^{n+}_{\sup}$ *satisfying conditions* (i), (ii), *and* (iii) *of the Theorem, replacing* A_i, B_i, C_i, *and* M_i *by respectively,* $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, *and* $\bar{M}_i$. *Furthermore in this case,* $\sup_{\theta_0 \in \Theta_0}\|Z_{\bar{K}}^0(\theta_0, \cdot)\| < \delta$, *where* $\bar{K} = (\bar{K}_1, \ldots) \in \mathscr{H}^{m,n}_{\sup}$ *is given by*

$$\boxed{\bar{K}_i = -\left((N_i^*N_i)^{-1/2}K_i + (N_i^*N_i)^{-1}N_i^*C_i\right), \qquad i \in \mathbb{N}.}$$

5.2. *Proof of Sufficiency*

We prove in this subsection the sufficiency part of the Theorem. Note that stochastic stabilizability of (C, A) is not required now. The proof will require the following propositions:

PROPOSITION 4. *Suppose that* (i), (ii), *and* (iii) *of the Theorem holds. Then* $K = (K_1, \ldots)$ *stabilizes* (A, B).

Proof. Set $\hat{F}_i = A_i - B_iK_i$, $\hat{F} = (\hat{F}_1, \ldots) \in \mathscr{H}^n_{\sup}$. From (ii) we have that for every $i \in \mathbb{N}$,

$$\begin{aligned}
P_i &= M_i + K_i^*K_i - G_i^*G_i \\
&\quad + \left(A_i - B_iK_i + \frac{1}{\delta}D_iG_i\right)^*\mathscr{E}_i(P)\left(A_i - B_iK_i + \frac{1}{\delta}D_iG_i\right) \\
&= M_i + K_i^*K_i + G_i^*\left(I - \frac{D_i^*\mathscr{E}_i(P)D_i}{\delta^2}\right)G_i \\
&\quad + (A_i - B_iK_i)^*\mathscr{E}_i(P)(A_i - B_iK_i) \\
&= \hat{C}_i^*\hat{C}_i + \hat{F}_i^*\mathscr{E}_i(P)\hat{F}_i, \qquad (5)
\end{aligned}$$

where $\hat{C}_i^* = [C_i^* \; G_i^*(I - (1/\delta^2)D_i^*\mathscr{E}_i(P)D_i)^{1/2} \; K_i^*]$ and set $\hat{C} = (\hat{C}_1, \ldots)$. Define also $\hat{H} = (\hat{H}_1, \ldots)$, where $\hat{H}_i = [0 \; -(1/\delta)D_i(I - (1/\delta^2)D_i^*\mathscr{E}_i(P)D_i)^{-1/2} \; 0]$. From assumption (iii) of the Theorem, it is easy to verify that $\hat{H}$ stabilizes $(\hat{C}, \hat{F})$ (see Definition 2) and thus $(\hat{C}, \hat{F})$ is SD. By a straightforward modification of Proposition 7 in [9] we have that this and (5) implies that $r_1(\hat{\mathscr{L}}) < 1$, where $\hat{\mathscr{L}} \in \mathbb{B}(\mathscr{H}_1^n)$ is defined as in (2) replacing F_i by $\hat{F}_i$. ∎

For the remainder of this section we consider, for arbitrary $\theta_0 \in \Theta_0$ and any $w \in \mathscr{C}^p$, $x = (0, x(1), \dots) = X_K^0(\theta_0, w)$, and $z = (0, z(1), \dots) = Z_K^0(\theta_0, w)$.

PROPOSITION 5. *Suppose that* (i), (ii), *and* (iii) *of the Theorem hold. Then*

$$\|z\|_2^2 = \delta^2\left(\|w\|_2^2 - \|r\|_2^2\right), \tag{6}$$

where $r = (r(0), r(1), \dots) \in \mathscr{C}^p$ *is defined for* $k \in \mathbb{N}^0$ *as*

$$r(k) = \left(I - \frac{1}{\delta^2} D_{\theta(k)}^* \mathscr{E}_{\theta(k)}(P) D_{\theta(k)}\right)^{1/2} \left(\frac{1}{\delta} G_{\theta(k)} x(k) - w(k)\right).$$

$\neq 0?$

Proof. From (ii) of the Theorem we have that for every $i \in \mathbb{N}$,

$$\begin{aligned} P_i = {} & M_i + K_i^* K_i + (A_i - B_i K_i)^* \mathscr{E}_i(P)(A_i - B_i K_i) \\ & + G_i^* \left(I - \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P) D_i\right) G_i. \end{aligned} \tag{7}$$

Since

$$x(k+1) = \left(A_{\theta(k)} - B_{\theta(k)K\theta(k)}\right)x(k) + D_{\theta(k)} w(k), \qquad x(0) = 0,$$

we get that

$$\begin{aligned} &\left\| P_{\theta(k+1)}^{1/2} x(k+1) \right\|_2^2 \\ &\quad = E\left(x(k+1)^* P_{\theta(k+1)} x(k+1)\right) \\ &\quad = E\Big(E\Big(\left(\left(A_{\theta(k)} - B_{\theta(k)} K_{\theta(k)}\right)x(k)\right. \\ &\qquad\qquad \left. + D_{\theta(k)} w(k)\right)^* P_{\theta(k+1)}\left(\left(A_{\theta(k)} - B_{\theta(k)} K_{\theta(k)}\right)x(k)\right. \\ &\qquad\qquad\qquad\qquad \left. + D_{\theta(k)} w(k)\right) \mid F_k\Big)\Big) \\ &\quad = E\Big(x(k)^*\left(A_{\theta(k)} - B_{\theta(k)} K_{\theta(k)}\right)^* \mathscr{E}_{\theta(k)}(P)\left(A_{\theta(k)} - B_{\theta(k)} K_{\theta(k)}\right)x(k) \\ &\qquad + w(k)^* D_{\theta(k)}^* \mathscr{E}_{\theta(k)}(P)\left(A_{\theta(k)} - B_{\theta(k)} K_{\theta(k)}\right)x(k) \\ &\qquad + x(k)^*\left(A_{\theta(k)} - B_{\theta(k)} K_{\theta(k)}\right)^* \mathscr{E}_{\theta(k)}(P) D_{\theta(k)} w(k) \\ &\qquad + w(k)^* D_{\theta(k)}^* \mathscr{E}_{\theta(k)} D_{\theta(k)} w(k)\Big). \end{aligned}$$

Using (7) it is straightforward to show that

$$\left\| P_{\theta(k+1)}^{1/2} x(k+1) \right\|_2^2 = \left\| P_{\theta(k)}^{1/2} x(k) \right\|_2^2 - \| z(k) \|_2^2 + \delta^2 \| w(k) \|_2^2 - \delta^2 \| r(k) \|_2^2.$$

Recalling that $x(0) = 0$ and $x(N) \to 0$ as $N \to \infty$ (since K stabilizes (A, B)), we get that as $N \to \infty$,

$$\begin{aligned} 0 \le \sum_{k=0}^{N-1} \left(\left\| P_{\theta(k+1)}^{1/2} x(k+1) \right\|_2^2 - \left\| P_{\theta(k)}^{1/2} x(k) \right\|_2^2 \right) &= \left\| P_{\theta(N)}^{1/2} x(N) \right\|_2^2 \\ \le \| P \|_{\sup} \| x(N) \|_2^2 \to 0, \end{aligned}$$

so that

$$0 = \sum_{k=0}^{\infty} \left(-\| z(k) \|_2^2 + \delta^2 \| w(k) \|_2^2 - \delta^2 \underline{\| r(k) \|_2^2} \right)$$

≥ 0

showing (6). ∎

Define the operator $\tilde{W}(\theta_0, \cdot) \in \mathbb{B}(\mathscr{C}^p)$ as $\tilde{W}(\theta_0, \cdot) = (1/\delta) G X_K^0(\theta_0, \cdot) - I$, that is, for $w \in \mathscr{C}^p$, $\tilde{W}(\theta_0, w) = \tilde{w} = (\tilde{w}(0), \tilde{w}(1), \dots)$ where $\tilde{w}(k) = (1/\delta) G_{\theta(k)} x(k) - w(k)$ for $k \in \mathbb{N}^0$. In the next proposition and proof of Lemma 1 we shall drop, for notational simplicity, the dependence of the operators in θ_0.

Proposition 6. *Suppose that* (i), (ii), *and* (iii) *of the Theorem holds. Then $\tilde{W}$ is invertible.*

Proof. For $w \in \mathscr{C}^p$, define the operator $\tilde{Y}$ as $\tilde{Y}(w) = (\tilde{y}(0), \tilde{y}(1), \dots)$ where

$$\begin{aligned} \tilde{y}(k+1) = \left(A_{\theta(k)} - B_{\theta(k)} K_{\theta(k)} + \frac{1}{\delta} D_{\theta(k)} G_{\theta(k)} \right) \tilde{y}(k) \\ - D_{\theta(k)} w(k), \qquad \tilde{y}(0) = 0. \end{aligned}$$

Note that, since $r_1(\mathscr{L}) < 1$ (condition (iii) of the Theorem), we have from Proposition 2 that $\tilde{Y}(w) \in \mathbb{B}(\mathscr{C}^p, \mathscr{C}^n)$ and $\|\tilde{Y}\| \le a$ for some $a \ge 0$ and all $\theta_0 \in \Theta_0$. Define now $\tilde{W}_{\text{inv}} \in \mathbb{B}(\mathscr{C}^p)$ as $\tilde{W}_{\text{inv}} = (1/\delta) G \tilde{Y} - I$, that is, for $w \in \mathscr{C}^p$, $\tilde{W}_{\text{inv}}(w) = \tilde{s} = (\tilde{s}(0), \tilde{s}(1), \dots)$ where $\tilde{s}(k) = (1/\delta) G_{\theta(k)} \tilde{y}(k) - w(k)$ for $k \in \mathbb{N}^0$. From these definitions it is easy to verify that $\tilde{Y}(w) = X_K^0(\tilde{s})$ and $\tilde{Y}(\tilde{w}) = X_K^0(w)$. Let us show that $\tilde{W} \tilde{W}_{\text{inv}} = \tilde{W}_{\text{inv}} \tilde{W} = I$. Indeed, $\tilde{W} \tilde{W}_{\text{inv}}(w) = \tilde{W}(\tilde{s}) = (1/\delta) G X_K^0(\tilde{s}) - \tilde{s} = (1/\delta) G \tilde{Y}(w) - ((1/\delta) G \tilde{Y}(w) - w) = w$, and $\tilde{W}_{\text{inv}} \tilde{W}(w) = \tilde{W}_{\text{inv}}(\tilde{w}) = (1/\delta) G \tilde{Y}(\tilde{w}) - \tilde{w} = (1/\delta) G X_K^0(w) - ((1/\delta) G X_K^0(w) - w) = w$, showing that $\tilde{W}^{-1} = \tilde{W}_{\text{inv}} \in \mathbb{B}(\mathscr{C}^p)$. ∎

We can now prove the sufficiency of the Theorem.

$\tilde{Y}(\tilde{w}) = \tilde{y}_{k+1} = \Gamma_{\theta(k)} \tilde{y}_k - D_{\theta(k)} \left(\frac{1}{\delta} G_{\theta(k)} x_k - w_k \right)$

$= (A_{\theta(k)} - B_{\theta(k)} K_{\theta(k)}) + D_{\theta(k)} w_k = X_K^0(w)$

LEMMA 1. *Suppose that* (i), (ii), *and* (iii) *of the Theorem hold. Then* K *stabilizes* (A, B) *and* $\sup_{\theta_0 \in \Theta_0} \|Z_K^0(\theta_0, \cdot)\| < \delta$.

Proof. Consider $\alpha > 0$ as in (i), and $\alpha_1 > (\alpha/\delta)$ such that $\|\tilde{W}^{-1}\| \leq \alpha_1$ for every $\theta_0 \in \Theta_0$. Since

$$\frac{1}{\alpha_1}\|w\|_2 \leq \|\tilde{W}^{-1}\|^{-1}\|w\|_2 \leq \left\|\tilde{W}(w)\right\|_2$$

we conclude that

$$\begin{aligned}
\|r\|_2^2 &= \sum_{k=0}^{\infty} E\left(\left(\frac{1}{\delta}G_{\theta(k)}x(k) - w(k)\right)^* \left(I - \frac{1}{\delta^2}D_{\theta(k)}^*\mathscr{E}_{\theta(k)}(P)D_{\theta(k)}\right)\right. \\
&\qquad\qquad \left.\times\left(\frac{1}{\delta}G_{\theta(k)}x(k) - w(k)\right)\right) \\
&\geq \frac{\alpha^2}{\delta^2}\sum_{k=0}^{\infty} E\left(\left\|\frac{1}{\delta}G_{\theta(k)}x(k) - w(k)\right\|^2\right) = \frac{\alpha^2}{\delta^2}\sum_{k=0}^{\infty} E\left(\left\|\tilde{W}(w)(k)\right\|^2\right) \\
&= \frac{\alpha^2}{\delta^2}\left\|\tilde{W}(w)\right\|_2^2 \geq \frac{\alpha^2}{(\delta\alpha_1)^2}\|w\|_2^2
\end{aligned}$$

and therefore, from (6) with $w \neq 0$ in $\mathscr{C}^p$,

$$\begin{aligned}
\|z\|_2^2 = \delta^2\left(\|w\|_2^2 - \|r\|_2^2\right) &\leq \delta^2\left(\|w\|_2^2 - \left(\frac{\alpha}{\delta\alpha_1}\right)^2\|w\|_2^2\right) \\
&= \delta^2\left(1 - \left(\frac{\alpha}{\delta\alpha_1}\right)^2\right)\|w\|_2^2 < \delta^2\|w\|_2^2,
\end{aligned}$$

?

proving the desired result. ∎

5.3. *Proof of Necessity*

We show in this subsection the necessity part of the Theorem, as stated in the next Lemma.

$Z_K^0(\theta_0) = Z_{\bar{K}}(\theta_0, 0, 0, 0)$

LEMMA 2. *Suppose that* (C, A) *is SD and there exits* $\bar{K} = (\bar{K}_1, \ldots) \in \mathscr{H}_{\sup}^{n,m}$ *such that it stabilizes* (A, B) *and* $\sup_{\theta_0 \in \Theta_0}\|Z_{\bar{K}}^0(\theta_0)\| < \delta$ *for some* $\delta > 0$. *Then there exists* $P = (P_1, \ldots) \in \mathscr{H}_{\sup}^{n+}$ *satisfying conditions* (i), (ii), *and* (iii) *of the Theorem.*

From Proposition 3 and the fact that (C, A) is SD and (A, B) SS, there exists a unique $L = (L_1, \ldots) \in \mathscr{H}_{\sup}^{n+}$ such that for $i \in \mathbb{N}$,

$$\begin{aligned}
L_i &= M_i + A_i^*\mathscr{E}_i(L)A_i - A_i^*\mathscr{E}_i(L)B_i\left(I + B_i^*\mathscr{E}_i(L)B_i\right)^{-1}B_i^*\mathscr{E}_i(L)A_i \\
&= M_i + (A_i - B_iJ_i)^*(\mathscr{E}_i(L))(A_i - B_iJ_i) + J_i^*J_i, \qquad i \in \mathbb{N}, \qquad \text{(8a)}
\end{aligned}$$

where

$$J_i = \left(I + B_i^* \mathscr{E}_i(L) B_i\right)^{-1} B_i^* \mathscr{E}_i(L) A_i, \qquad i \in \mathbb{N}, \tag{8b}$$

and $J = (J_1, \dots)$ stabilizes (A, B). For any $(x_0, w, q) \in \mathscr{C}_0^n \oplus \mathscr{C}^p \oplus \mathscr{C}^m$, $\theta_0 \in \Theta_0$, $w = (w(0), \dots)$, $q = (q(0), \dots)$, we set (see (3) and (4)) $x = (x(0), \dots) = X_J(\theta_0, x_0, w, q)$, $z = (z(0), \dots) = Z_J(\theta_0, x_0, w, q)$, and

$$\begin{aligned} \mathscr{J}(\theta_0, x_0, w, q) &= \|Z_J(\theta_0, x_0, w, q)\|_2^2 - \delta^2 \|w\|_2^2 \\ &= \sum_{k=0}^{\infty} E_{(\theta_0, x_0)}\Big(\|C_{\theta(k)} x(k)\|^2 + \|-J_{\theta(k)} x(k) + q(k)\|^2 \\ &\qquad - \delta^2 \|w(k)\|_2^2\Big) \end{aligned}$$

$\|u(k)\|_2^2$

and we want to solve the minimax problem

$$\hat{\mathscr{J}}(\theta_0, x_0) = \sup_{w \in \mathscr{C}^p} \inf_{q \in \mathscr{C}^m} \mathscr{J}(\theta_0, x_0, w, q). \tag{9}$$

$\tilde{\mathcal{J}}(\theta_0, x_0, w)$

We shall first solve the minimization problem,

$$\tilde{\mathscr{J}}(\theta_0, x_0, w) = \inf_{q \in \mathscr{C}^m} \mathscr{J}(\theta_0, x_0, w, q). \tag{10}$$

In order to solve the above problem, we need the following result, proved in [10]. Let $\mathscr{R}(\theta_0, \cdot): \mathscr{C}^p \to \mathscr{C}^n$ and $\mathscr{Q}(\theta_0, \cdot): \mathscr{C}^p \to \mathscr{C}^m$ be the transformations defined for $w = (w(0), w(1), \dots) \in \mathscr{C}^p$, $\mathscr{R}(\theta_0, w) = r = (r(0), r(1), \dots)$, where

1998 jump LQ

$$\begin{aligned} r(k) = \sum_{j=0}^{\infty} E\Bigg(&\boxed{\Bigg(\prod_{\ell=k+j+1} \left(A_{\theta(\ell)} - B_{\theta(\ell)} J_{\theta(\ell)}\right)^*\Bigg)} \\ &L_{\theta(k+j+1)} D_{\theta(k+j)} w(k+j) \mid \mathscr{F}_k\Bigg), \qquad k \geq 0 \end{aligned}$$

and $\mathscr{Q}(\theta_0, w) = \tilde{q} = (\tilde{q}(0), \tilde{q}(1), \dots)$ where

$$\tilde{q}(k) = -\left(I + B_{\theta(k)}^* \mathscr{E}_{\theta(k)}(L) B_{\theta(k)}\right)^{-1} B_{\theta(k)}^* r(k), \qquad k \geq 0.$$

PROPOSITION 7. *Consider $\theta_0 \in \Theta_0$ fixed. Under the assumptions of Lemma 2, $\mathscr{R}(\theta_0, \cdot) \in \mathbb{B}[\mathscr{C}^p, \mathscr{C}^n]$ and $\mathscr{Q}(\theta_0, \cdot) \in \mathbb{B}[\mathscr{C}^p, \mathscr{C}^m]$. Moreover,*

$\leq \mathcal{J}(\theta_0, x_0, w, q)$

$$\tilde{\mathscr{J}}(\theta_0, x_0, w) = \mathscr{J}(\theta_0, x_0, w, \tilde{q}) = \inf_{q \in \mathscr{C}^m} \mathscr{J}(\theta_0, x_0, w, q),$$

$\hat{q}$

where $\tilde{q} = \mathscr{Q}(\theta_0, w)$.

?

(10) → $\tilde{q} = \mathcal{Q}(\theta_0, w)$

$\tilde{q}(k) = -(I + B^*_{\theta(k)} \mathcal{E}_{\theta(k)}(L) B_{\theta(k)})^{-1} B^*_{\theta(k)} r(k)$

$r(k) = \sum_{j=0}^{\infty} E\left(\left(\prod_{l=k+j+1} (A_{\theta(l)} - B_{\theta(l)} J_{\theta(l)})^*\right) L_{\theta(k+j+1)} D_{\theta(k+j)} w(k+j) \mid \mathcal{F}_k\right)$

We shall now move on to the maximization problem, that is,

$$\hat{\mathcal{J}}(\theta_0, x_0) = \tilde{\mathcal{J}}(\theta_0, x_0, \hat{w}) = \sup_{w \in \mathcal{C}^p} \tilde{\mathcal{J}}(\theta_0, x_0, w). \tag{11}$$

We shall need the following result, due to Yakubovich [36], also presented in [23].

LEMMA 3. *Consider $\mathcal{H}$ a Hilbert space and a quadratic form $\mathcal{J}(\zeta) = \langle \mathcal{S}\zeta; \zeta \rangle$, $\zeta \in \mathcal{H}$, and $\mathcal{S} \in \mathbb{B}[\mathcal{H}]$ self adjoint. Let $\mathcal{M}_0$ be a closed subspace of $\mathcal{H}$ and $\mathcal{M}$ a translation of $\mathcal{M}_0$ by an element $m \in \mathcal{H}$ (i.e., $\mathcal{M} = \mathcal{M}_0 + m$). If*

$$\inf \frac{\mathcal{J}(\zeta)}{\langle \zeta ; \zeta \rangle} = \inf_{\zeta \in \mathcal{M}_0} \frac{\langle \mathcal{S}\zeta ; \zeta \rangle}{\langle \zeta ; \zeta \rangle} > 0$$

then there exists a unique element $\hat{\zeta} \in \mathcal{M}$ such that $\mathcal{J}(\hat{\zeta}) = \inf_{\zeta \in \mathcal{M}} \mathcal{J}(\zeta)$, where $\hat{\zeta} = p + m$, with $p = \mathcal{G}m \in \mathcal{M}_0$ for some $\mathcal{G} \in \mathbb{B}[\mathcal{H}]$.

For $\theta_0 \in \Theta_0$ fixed, define the following operators $\tilde{X}(\theta_0, \cdot)$, $\bar{X}(\theta_0, \cdot)$ in $\mathbb{B}[\mathcal{C}_0^n \oplus \mathcal{C}^p, \mathcal{C}^n]$, and $\tilde{Z}(\theta_0, \cdot)$, $\bar{Z}(\theta_0, .)$ in $\mathbb{B}[\mathcal{C}_0^n \oplus \mathcal{C}^p, \mathcal{C}^r]$; for $x_0 \in \mathcal{C}_0^n$, $w \in \mathcal{C}^p$, and $\bar{K}$ as in Lemma 2 (see (3) and (4)),

$\tilde{q}$

$(\bar{K}_{\theta(k)} - J_{\theta(k)})x(k)$
$+ \mathcal{Q}(\theta_0, w)$

$$\begin{aligned}
\tilde{X}(\theta_0, x_0, w) &= X_J(\theta_0, x_0, w, \mathcal{Q}(\theta_0, w)), \\
\tilde{Z}(\theta_0, x_0, w) &= Z_J(\theta_0, x_0, w, \mathcal{Q}(\theta_0, w)), \\
\bar{X}(\theta_0, x_0, w) &= X_{\bar{K}}(\theta_0, x_0, w, 0), \\
\bar{Z}(\theta_0, x_0, w) &= Z_{\bar{K}}(\theta_0, x_0, w, 0).
\end{aligned}$$

Note from Proposition 7 that, for any $q \in \mathcal{C}^m$ and every $(x_0, w) \in \mathcal{C}_0^n \oplus \mathcal{C}^p$,

$\tilde{q}$

$\|Z_J(\theta_0, x_0, w, \mathcal{Q}(\theta_0, w))\|_2^2 - \delta^2\|w\|_2^2$

$$\|\tilde{Z}(\theta_0, x_0, w)\|_2^2 - \delta^2\|w\|_2^2 \leq \|Z_J(\theta_0, x_0, w, q)\|_2^2 - \delta^2\|w\|_2^2$$

and choosing $q = (q(0), \ldots)$ as $q(k) = J_{\theta(k)}x(k) - \bar{K}_{\theta(k)}x(k)$, $k \in \mathbb{N}^0$, we get that $Z_J(\theta_0, x_0, w, q) = \bar{Z}(\theta_0, x_0, w)$ (note that, indeed, $q \in \mathcal{C}^m$) and thus,

$= Z_{\bar{K}}(\theta_0, x_0, w, 0)$

$(J_{\theta(k)} - \bar{K}_{\theta(k)})x(k)$

$$\|\tilde{Z}(\theta_0, x_0, w)\|_2^2 - \delta^2\|w\|_2^2 \leq \|\bar{Z}(\theta_0, x_0, w)\|_2^2 - \delta^2\|w\|_2^2. \tag{12}$$

From Lemma 2, there exists $\alpha > 0$ such that for all $w \in \mathcal{C}^p$,

$$\sup_{\theta_0 \in \Theta_0} \left\{ \frac{\|Z_{\bar{K}}^0(\theta_0, w)\|_2^2}{\|w\|_2^2} \right\} < \delta^2 - \alpha^2 \tag{13}$$

$Z_{\bar{K}}^0(\theta_0, w) = Z_{\bar{K}}(\theta_0, 0, w, 0) = \bar{Z}(\theta_0, 0, w)$

$\|\bar{Z}(\theta_0, 0, w)\|_2^2 < \delta^2\|w\|_2^2 - \alpha^2\|w\|_2^2$

$\rightarrow \delta^2\|w\|_2^2 - \|\bar{Z}(\theta_0, 0, w)\|_2^2 > \alpha^2\|w\|_2^2$

and thus, from (13), for every $w \in \mathscr{C}^p$ and arbitrary $\theta_0 \in \Theta_0$,

$$\alpha^2\|w\|_2^2 \leq \delta^2\|w\|_2^2 - \left\|\bar{Z}(\theta_0, 0, w)\right\|_2^2 \leq \delta^2\|w\|_2^2 - \left\|\tilde{Z}(\theta_0, 0, w)\right\|_2^2, \quad (14)$$

that is,

$$\boxed{\inf_{w \in \mathscr{C}^p} \left\{ \frac{\delta^2\|w(k)\|_2^2 - \left\|\tilde{Z}(\theta_0, 0, w)\right\|_2^2}{\|w\|_2^2} \right\} \geq \alpha^2.} \quad (15)$$

PROPOSITION 8. *Consider $\theta_0 \in \Theta_0$ fixed. Under the hypothesis of Lemma 2, for each $x_0 \in \mathscr{C}_0^n$, there exists a unique element $\hat{w} \in \mathscr{C}^p$ such that $\hat{\tilde{\mathscr{J}}}(\theta_0, x_0) = \tilde{\mathscr{J}}(\theta_0, x_0, \hat{w}) = \sup_{w \in \mathscr{C}^p} \tilde{\mathscr{J}}(\theta_0, x_0, w)$. Moreover, for some $\mathscr{W}(\theta_0, \cdot) \in \mathbb{B}[C_0^n, C^p]$, $\hat{w} = \mathscr{W}(\theta_0, x_0)$.*

Proof. We have that

$$\begin{aligned}
\tilde{\mathscr{J}}(\theta_0, x_0, w) &= \left\|\tilde{Z}(\theta_0, x_0, w)\right\|_2^2 - \delta^2\|w\|_2^2 \\
&= \prec \tilde{Z}^*\tilde{Z}(\theta_0, x_0, w); (x_0, w) \succ - \boxed{\prec (0, \delta^2 w); (x_0, w) \succ} \\
&= \prec \mathscr{S}(\theta_0, x_0, w); (x_0, w) \succ ,
\end{aligned}$$

where $\mathscr{S}(\theta_0, x_0, w) = \tilde{Z}^*\tilde{Z}(\theta_0, x_0, w) - (0, \overset{\delta^2 w}{\cancel{\delta^2 v}}, 0)$. We have that $C_0^n \oplus \mathscr{C}^p$ is a Hilbert space and $\mathscr{S} \in \mathbb{B}[C_0^n \oplus \mathscr{C}^p]$ with $\mathscr{S}(\theta_0, \cdot)$ self adjoint. Define $\mathscr{M}_0 = \{(x_0, w) \in C_0^n \oplus \mathscr{C}^p;\ x_0 = 0\}$ and $\mathscr{M} = \mathscr{M}_0 + m$ with $m = (x_0, 0) \in C_0^n \oplus \mathscr{C}^p$. $\mathscr{M}_0$ is a closed subspace of $C_0^n \oplus \mathscr{C}^p$ and $\mathscr{M}$ a translation of $\mathscr{M}_0$ by the element m. From (15),

$$\inf_{\zeta \in \mathscr{M}_0} \left\{ \frac{- \prec \mathscr{S}\zeta; \zeta \succ}{\|\zeta\|_2^2} \right\} = \inf_{w \in \mathscr{C}^p} \left\{ \frac{\delta^2\|w(k)\|_2^2 - \left\|\tilde{Z}(\theta_0, 0, w)\right\|_2^2}{\|w\|_2^2} \right\} \geq \alpha^2 > 0$$

$\hat{\zeta} = (x_0, \hat{w}) = \underset{(0,\hat{w})}{p} + \underset{(x_0,0)}{\underline{\underline{m}}}$

and invoking Lemma 3, we obtain that there exists a unique element $\hat{\zeta} \in \mathscr{M}$ such that $-\tilde{\mathscr{J}}(\theta_0, \hat{\zeta}) = \inf_{\zeta \in \mathscr{M}} -\tilde{\mathscr{J}}(\theta_0, \zeta)$, where $\hat{\zeta} = p + m$, $p = \mathscr{W}'(\theta_0, m)$ for some $\mathscr{W}'(\theta_0, \cdot) \in \mathbb{B}[C_0^n \oplus \mathscr{C}^p]$. Therefore, $\mathscr{W}'(\theta_0, m) = \mathscr{W}'(\theta_0, x_0, 0) = (0, \hat{w})$ and for some $\mathscr{W} \in \mathbb{B}[C_0^n, \mathscr{C}^p]$, $\hat{w} = \mathscr{W}(\theta_0, x_0)$. ∎

Remark 5. Notice that the same type of arguments in the proof of Proposition 8 would yield a similar existence characterization for $\mathscr{Q}(\theta_0, w)$, namely, for each $x_0 \in \mathscr{C}_0^p$ and $w \in \mathscr{C}^p$, there exists an unique element $\tilde{q} \in \mathscr{C}^m$ such that $\tilde{\mathscr{J}}(\theta_0, x_0, w) = \mathscr{J}(\theta_0, x_0, w, \tilde{q}) = \inf_{q \in \mathscr{C}^m} \mathscr{J}(\theta_0, x_0, w, q)$ and $\tilde{q} = \mathscr{Q}(\theta_0, w)$ for some $\mathscr{Q}(\theta_0, \cdot) \in \mathbb{B}[\mathscr{C}^p, \mathscr{C}^m]$. The expressions for $r(k)$ and $\tilde{q}(k)$ before Proposition 7 represent a step further in the characterization of operator $\mathscr{Q}$.

For $\theta_0 \in \Theta_0$ fixed define the operators

$$? \ \boxed{\hat{X}(\theta_0, \cdot) \in \mathbb{B}[\mathscr{C}_0^n, \mathscr{C}^n],} \qquad \hat{Z}(\theta_0, \cdot) \in \mathbb{B}[\mathscr{C}_0^n, \mathscr{C}^r] \tag{16}$$

as

$$\begin{aligned}
\hat{X}(\theta_0, x_0) &= \tilde{X}(\theta_0, x_0, \underline{\underline{\overset{\hat{w}}{\mathscr{W}(\theta_0, x_0)}}}) \\
&= X_J(\theta_0, x_0, \mathscr{W}(\theta_0, x_0), \underline{\underline{\overset{\tilde{q}}{\mathscr{Q}}(\theta_0, \mathscr{W}(\theta_0, x_0))}}) \\
&= (x_0, \hat{x}(1), \dots) = \hat{x} \\
\hat{Z}(\theta_0, x_0) &= \tilde{Z}(\theta_0, x_0, \mathscr{W}(\theta_0, x_0)) \\
&= Z_J(\theta_0, x_0, \mathscr{W}(\theta_0, x_0), \mathscr{Q}(\theta_0, \mathscr{W}(\theta_0, x_0))) \\
&= (\hat{z}(0), \hat{z}(1), \dots) = \hat{z}
\end{aligned}$$

so that

$$\begin{aligned}
\hat{\mathscr{J}}(\theta_0, x_0) &= \sup_{w \in \mathscr{C}^p} \inf_{q \in \mathscr{C}^m} \mathscr{J}(\theta_0, x_0, w, q) \\
&= \underline{\| \hat{Z}(\theta_0, x_0) \|_2^2} - \delta^2 \| \underline{\underline{\underset{\hat{w}}{\mathscr{W}(\theta_0, x_0)}}} \|_2^2 \\
&= \underline{\prec \hat{Z}(\theta_0, x_0); \hat{Z}(\theta_0, x_0) \succ} - \delta^2 \prec \mathscr{W}(\theta_0, x_0); \mathscr{W}(\theta_0, x_0) \succ \\
&= \prec \underline{(\hat{Z}^*\hat{Z} - \delta^2 \mathscr{W}^*\mathscr{W})(\theta_0, x_0); x_0 \succ} = \prec \mathfrak{P}(\theta_0, x_0); x_0 \succ,
\end{aligned}$$

where $\mathfrak{P}(\theta_0, \cdot) \in \mathbb{B}[\mathscr{C}_0^n]$ is defined as $\mathfrak{P}(\theta_0, \cdot) = (\hat{Z}^*\hat{Z} - \delta^2 \mathscr{W}^*\mathscr{W})(\theta_0, \cdot)$. Since for any $x_0 \in \mathscr{C}_0^n$,

$$\boxed{\begin{aligned}
\hat{\mathscr{J}}(\theta_0, x_0) &= \sup_{w \in \mathscr{C}^p} \inf_{q \in \mathscr{C}^m} \mathscr{J}(\theta_0, x_0, w, q) \\
&= \underline{\sup_{w \in \mathscr{C}^p} \left\{ \| \tilde{Z}(\theta_0, x_0, w) \|_2^2 - \delta^2 \|w\|_2^2 \right\} \geq \| \tilde{Z}(\theta_0, x_0, 0) \|_2^2} \geq \underline{\underline{0}},
\end{aligned}}$$

it follows that $\underline{\mathfrak{P}(\theta_0, \cdot) \geq 0}$. For each $i \in \mathbb{N}$ and $x_0 \in \mathbb{C}^n$, define $P_i \in \mathbb{B}[\mathbb{C}^n]$ such that $P_i x_0 = E(\mathfrak{P}(i, x_0))$. For every $x_0 \in \mathbb{C}^n$,

$$\begin{aligned}
\hat{\mathscr{J}}(i, x_0) &= \prec \mathfrak{P}(i, x_0); x_0 \succ = E(\langle \mathfrak{P}(\underline{I}, x_0); x_0 \rangle) = \langle \underline{\underline{E(\mathfrak{P}(i, x_0))}}; x_0 \rangle \\
&= \langle P_i x_0; x_0 \rangle = x_0^* P_i x_0 \geq 0
\end{aligned}$$

so that $\underline{\underline{P_i \geq 0}}$.

In order to prove that $P = (P_1, \dots)$ as defined above satisfies the conditions of the Theorem, we consider a truncated minimax problem

$$\hat{\mathscr{J}}^k(\theta_0, x_0) = \sup_{w \in \mathscr{C}^{k,p}} \inf_{q \in \mathscr{C}^m} \mathscr{J}(\theta_0, x_0, w, q), \tag{17}$$

where for ι integers,

$$\boxed{\mathscr{C}^{k,\iota} = \{ s = (s(0), s(1), \dots) \in \mathscr{C}^{\iota}, s(i) = 0 \text{ for } i \geq k \}.}$$

Moreover, setting $\varphi^k \in \mathbb{B}[\mathscr{C}^p, \mathscr{C}^{k,p}]$ as $\varphi^k(w) = (w(0), \dots, w(k-1), 0, 0, \dots)$, where $w = (w(0), w(1), \dots) \in \mathscr{C}^p$, we get from (11) and (17) that

$$\tilde{\mathscr{J}}(\theta_0, x_0, \underbrace{\varphi^k(\mathscr{W}(\theta_0, x_0))}_{\varphi^k(\hat{\omega})}) \overset{=\,?}{\leq} \underbrace{\hat{\mathscr{J}}^k(\theta_0, x_0)}_{\varphi^k(\hat{\omega})} \leq \underbrace{\hat{\mathscr{J}}(\theta_0, x_0)}_{\hat{\omega}}.$$

Since $\varphi^k(\mathscr{W}(\theta_0, x_0)) \to \mathscr{W}(\theta_0, x_0)$ as $k \to \infty$ we obtain, from continuity of $\tilde{\mathscr{J}}(\theta_0, x_0, \cdot)$, that $\lim_{k\to\infty} \varphi^k(\hat{\omega}) = \hat{\omega}$

$$\hat{\mathscr{J}}^k(\theta_0, x_0) \to \hat{\mathscr{J}}(\theta_0, x_0) \qquad \text{as } k \to \infty. \tag{18}$$

$\lim_{k\to\infty} \hat{\mathscr{J}}^k(\theta_0, x_0) = \hat{\mathscr{J}}(\theta_0, x_0)$

Furthermore, notice that (17) can be rewritten as

$$\begin{aligned} \hat{\mathscr{J}}^k(\theta_0, x_0) = \sup_{w \in \mathscr{C}^{k,p}} \inf_{q \in \mathscr{C}^{k,m}} \Bigg\{ \sum_{l=0}^{k-1} \Big(&\| C_{\theta(l)} x(l) \|_2^2 \\ &+ \| -J_{\theta(l)} x(l) + q(l) \|_2^2 - \delta^2 \| w(l) \|_2^2 \Big) \\ &+ \underbrace{\| L_{\theta(k)}^{1/2} x(k) \|_2^2}_{LQR} \Bigg\}. \end{aligned} \tag{19}$$

We shall now obtain a solution for (17) in a recursive way. Define the sequences $P^k = (P_1^k, \dots)$, $P_i^k \in \mathbb{M}(\mathbb{C}^n)$, $K^k = (K_1^k, \dots)$, $K_i^k \in \mathbb{M}(\mathbb{C}^n, \mathbb{C}^m)$, and $G^k = (G_1^k, \dots)$, $G_i^k \in \mathbb{M}(\mathbb{C}^n, \mathbb{C}^p)$ as

$$\begin{aligned} P^0 &= (P_1^0, P_2^0, \dots) = L = (L_1, L_2, \dots) \\ P_i^{k+1} &= M_i + \left(A_i - B_i K_i^{k+1} + \frac{1}{\delta} D_i G_i^{k+1} \right)^* (\mathscr{E}_i(P^k)) \\ &\quad \times \left(A_i - B_i K_i^{k+1} + \frac{1}{\delta} D_i G_i^{k+1} \right) \\ &\quad + (K_i^{k+1})^* K_i^{k+1} - (G_i^{k+1})^* G_i, \qquad i \in \mathbb{N}, k \geq 0, \end{aligned}$$

where

$$K_i^{k+1} = \Bigg(I + B_i^* \mathscr{E}_i(P^k) B_i + \frac{1}{\delta^2} B_i^* \mathscr{E}_i(P^k) D_i \left(I - \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P^k) D_i \right)^{-1} D_i^* \mathscr{E}_i(P^k) B_i \Bigg)^{-1}$$
$$\cdot \left(B_i^* \left(I + \frac{1}{\delta^2} \mathscr{E}_i(P^k) D_i \left(I - \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P^k) D_i \right)^{-1} D_i^* \right) \mathscr{E}_i(P^k) A_i \right),$$
$$i \in \mathbb{N},\ k \geq 0$$

$$G_i^{k+1} = \Bigg(I - \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P^k) D_i + \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P^k) B_i \left(I + B_i^* \mathscr{E}_i(P^k) B_i \right)^{-1} B_i^* \mathscr{E}_i(P^k) D_i \Bigg)^{-1}$$
$$\cdot \left(\frac{1}{\delta} D_i^* \left(I - \mathscr{E}_i(P^k) B_i \left(I + B_i^* \mathscr{E}_i(P^k) B_i \right)^{-1} B_i^* \right) \mathscr{E}_i(P^k) A_i \right),$$
$$i \in \mathbb{N},\ k \geq 0.$$

The existence of the above inverses will be established in the proof of the proposition below.

PROPOSITION 9. *Consider $\theta_0 \in \Theta_0$ fixed. Under the hypothesis of Lemma* 2, *for each $k \geq 0$, we have that*

(a) $P^k = (P_1^k, P_2^k, \ldots) \in \mathscr{H}_{\sup}^{n+}$

(b) $\delta^2 I - D_i^* \mathscr{E}_i(P^k) D_i \geq \alpha^2 I$ *for all $i \in \mathbb{N}$, and $\alpha > 0$ as in* (15).

(c) $\hat{\mathscr{J}}^k(\theta_0, x_0) = \mathscr{J}(\theta_0, x_0, \hat{w}^k, \hat{q}^k) = E(x_0^* P_{\theta_0}^{k} x_0)$, *where*

$$\hat{w}^k = \left(\hat{w}^k(0), \ldots, \hat{w}^k(k-1), 0, 0, \ldots \right),$$
$$\hat{q}^k = \left(\hat{q}^k(0), \ldots, \hat{q}^k(k-1), 0, 0, \ldots \right),$$

and

$$\hat{w}^k(l) = \frac{1}{\delta} G_{\theta(l)}^{k-l} \hat{x}^k(l), \qquad l = 0, \ldots, k-1,$$
$$\boxed{\hat{q}^k(l) = \left(J_{\theta(l)} - K_{\theta(l)}^{k-l} \right) \hat{x}^k(l),} \qquad l = 0, \ldots, k-1,$$
$$\hat{x}^k = \left(\hat{x}^k(0), \hat{x}^k(1), \ldots \right) = X_J\left(\theta_0, x_0, \hat{w}^k, \hat{q}^k \right).$$

Proof. Let us apply induction on k. From Proposition 3, (a) and (c) are clearly true for $k = 0$. Let us prove now that (b) is satisfied for $k = 0$. Fix $\theta_0 = i \in \mathbb{N}$ and consider $w(0) \in \mathbb{C}^p$, $w = (w(0), 0, \dots) \in \mathscr{C}^p$. Then, from (14),

$$\alpha^2 \|w\|_2^2 \le \delta^2 \|w\|_2^2 - \left\| \bar{Z}(i, 0, w) \right\|_2^2 \le \delta^2 \|w\|_2^2 - \left\| \tilde{Z}(i, 0, w) \right\|_2^2.$$

$\|\tilde{Z}(\theta_0, x_0, w)\|_2^2 = \|\tilde{Z}(\theta_1, x_1, w)\|_2^2$? $x_0 = 0 \Rightarrow Z_0 = (C_0 + N_0 J_0) x_0 = 0$

But $\|\tilde{Z}(i, 0, w)\|_2^2 = \|Z_J(\theta(1), D_{\theta(1)} w(0), 0)\|_2^2 = w(0)^* D_i^* \mathscr{E}_i(L) D_i w(0)$ and $\|w\|_2^2 = w(0)^* w(0)$, so that, from (14),

$E[x^T(1) L_{\theta(1)} x(1)]$

$\alpha^2 \|w\|_2^2 \le \delta^2 \|w\|^2 - \|\tilde{Z}_J(\theta_0, x_0, w)\|_2^2$

$$w(0)^* \left(\delta^2 I - D_i^* \mathscr{E}_i(L) D_i - \alpha^2 I \right) w(0) \ge 0$$

and since i and $w(0)$ are arbitrary, the result is proved for $k = 0$. Suppose now that the proposition holds for k. For $x_0 \in \mathscr{C}_0^n$ and $\theta_0 \in \Theta_0$, we define

$$\beta^{k+1}(\theta_0, x_0) = \sup_{w_0 \in \mathscr{C}_0^p} \inf_{q_0 \in \mathscr{C}_0^m} \Big(E\Big(\|C_{\theta_0} x_0\|^2 + \| -J_{\theta_0} x_0 + q_0 \|^2$$
$$- \delta^2 \|w_0\|^2 + x(1)^* P_{\theta(1)}^k x(1) \Big) \Big), \quad (20)$$

where

$\ge \inf_{q_0 \in \mathscr{C}_0^m} E\left[\|C_{\theta_0} x_0\|^2 + \|q_0 - J_{\theta_0}\|^2 + \|x(1)\|_{P_{\theta(1)}^k}^2 \right] \ge 0$

$$\boxed{x(1) = A_{\theta_0} x_0 + B_{\theta_0}\left(-J_{\theta_0} x_0 + q_0 \right) + D_{\theta_0} w_0.}$$

Set for each $i \in \mathbb{N}$,

$$W_i^{k+1} = \left(I + B_i^* \mathscr{E}_i(P^k) B_i \right) > 0$$

$$\cancel{N_i^{k+1} = \left(I + B_i^* \mathscr{E}_i(P^k) B_i \right)^{-1} B_i^* \mathscr{E}_i(P^k) A_i}$$

$$Q_i^{k+1} = A_i^* \mathscr{E}_i(P^k) A_i + M_i - \left(N_i^{k+1} \right)^* \left(I + B_i^* \mathscr{E}_i(P^k) B_i \right) N_i^{k+1}$$

$C_i^* C_i$ $\quad A_i^* \mathscr{E}_i(P^k) B_i (W_i^{k+1})^{-1} B_i^* \mathscr{E}_i(P^k) A_i$

$$U_i^{k+1} = D_i^* \left(I - \mathscr{E}_i(P^k) B_i^* \left(I + B_i^* \mathscr{E}_i(P^k) B_i \right)^{-1} B_i^* \right) \mathscr{E}_i(P^k) A_i$$

$$R_i^{k+1} = \delta^2 \left(I - \frac{D_i^* \mathscr{E}_i(P^k) D_i}{\delta^2} \right.$$

$\frac{1}{\delta} G_i^{k+1} = (R_i^{k+1})^{-1} U_i^{k+1}$

$$\left. + \frac{1}{\delta^2} D_i^* \mathscr{E}_i(P^k) B_i^* \left(I + B_i^* \mathscr{E}_i(P^k) B_i \right)^{-1} B_i^* \mathscr{E}_i(P^k) D_i \right) > 0$$

$$\hat{u}_0 = \left(W_{\theta_0}^{k+1} \right)^{-1} \left(B_{\theta_0}^* \mathscr{E}_{\theta_0}(P^k) A_{\theta_0} x_0 + B_{\theta_0}^* \mathscr{E}_{\theta_0}(P^k) D_{\theta_0} w_0 \right)$$
$$= N_{\theta_0}^{k+1} x_0 + \left(W_{\theta_0}^{k+1} \right)^{-1} B_{\theta_0}^* \mathscr{E}_{\theta_0}(P^k) D_{\theta_0} w_0$$
$$\boxed{u_0 = -J_{\theta_0} x_0 + q_0.}$$

简单但冗长

A trivial but somewhat lengthy algebraic manipulation leads to

$$\boxed{\begin{aligned}&\|C_{\theta_0}x_0\|^2 + \| - J_{\theta_0}x_0 + q_0\|^2 - \delta^2\|w_0\|^2 + x(1)^*\mathscr{E}_{\theta_0}(P^k)x(1) \qquad (20)\\ &\quad = x_0^*\Big(Q_{\theta_0}^k + \big(U_{\theta_0}^{k+1}\big)^*\big(R_{\theta_0}^{k+1}\big)^{-1}U_{\theta_0}^{k+1}\Big)x_0\\ &\qquad + (u_0 + \hat{u}_0)^*W_{\theta_0}^{k+1}(u_0 + \hat{u}_0) \qquad \min_{u_0}\|u_0+\hat{u}_0\|^2_{W_{\theta_0}^{k+1}}\\ &\qquad - \Big(w_0 - \big(R_{\theta_0}^{k+1}\big)^{-1}U_{\theta_0}^{k+1}x_0\Big)^*\big(R_{\theta_0}^{k+1}\big)^{-1}\Big(w_0 - \big(R_{\theta_0}^{k+1}\big)^{-1}U_{\theta_0}^{k+1}x_0\Big)\end{aligned}}$$

which shows that the solution of (20) is given by

$$u_0 = -J_{\theta_0}x_0 + q_0 = -\hat{u}_0 \qquad \Rightarrow u_0 = -\hat{u}_0$$

$$= -\Big(I + B_{\theta_0}^*\mathscr{E}_{\theta_0}(P^k)B_{\theta_0}\Big)^{-1}\Big(B_{\theta_0}\mathscr{E}_{\theta_0}(P^k)A_{\theta_0}x_0 + B_{\theta_0}^*\mathscr{E}_{\theta_0}(P^k)D_{\theta_0}w_0\Big)$$

$$w_0 = \big(R_{\theta_0}^{k+1}\big)^{-1}U_{\theta_0}^{k+1}x_0 \underset{?}{=} \frac{1}{\delta}G_{\theta_0}^{k+1}x_0,$$

and therefore

$$q_0 = J_{\theta_0}x_0 - \Big(I + B_{\theta_0}^*\mathscr{E}_{\theta_0}(P^k)B_{\theta_0}\Big)^{-1}B_{\theta_0}\mathscr{E}_{\theta_0}(P^k)\Big(A_{\theta_0} + \frac{1}{\delta}D_{\theta_0}G_{\theta_0}^{k+1}\Big)x_0$$

$$= \Big(J_{\theta_0} - K_{\theta_0}^{k+1}\Big)x_0.$$

We have from (20) and the above that

$$\boxed{\beta^{k+1}(\theta_0, x_0) = E\Big(x_0^*\Big(Q_{\theta_0}^k + \big(U_{\theta_0}^{k+1}\big)^*\big(R_{\theta_0}^{k+1}\big)^{-1}U_{\theta_0}^{k+1}\Big)x_0\Big). \quad (20)}$$

After some algebraic manipulation, we get that

$$Q_i^k + \big(U_i^{k+1}\big)^*\big(R_i^{k+1}\big)^{-1}U_i^{k+1} \Rightarrow \big[(R_i^{k+1})^{-1}U_i^{k+1}\big]^*R_i^{k+1}\big[(R_i^{k+1})^{-1}U_i^{k+1}\big] = (G_i^{k+1})^*\frac{1}{\delta^2}R_i^{k+1}G_i^{k+1}$$

$$\begin{aligned}&= A_i^*\mathscr{E}_i(P^k)A_i + M_i - \big(N_i^{k+1}\big)^*\big(I + B_i^*\mathscr{E}_i(P^k)B_i\big)N_i^{k+1}\\ &\quad + \big(G_i^{k+1}\big)^*\Big(I - \frac{1}{\delta^2}D_i^*\mathscr{E}_i(P^k)D_i\\ &\qquad + \frac{1}{\delta^2}D_i^*\mathscr{E}_i(P^k)B_i^*\big(I + B_i^*\mathscr{E}_i(P^k)B_i\big)^{-1}B_i^*\mathscr{E}_i(P^k)D_i\Big)G_i^{k+1}\\ &= M_i + \Big(A_i - B_iK_i^{k+1} + \frac{1}{\delta}D_iG_i^{k+1}\Big)^*\mathscr{E}_i(P^k)\\ &\quad \times\Big(A_i - B_iK_i^{k+1} + \frac{1}{\delta}D_iG_i^{k+1}\Big)\\ &\quad + \big(K_i^{k+1}\big)^*K_i^{k+1} - \big(G_i^{k+1}\big)^*G_i^{k+1} = P_i^{k+1},\end{aligned}$$

so that

$$\boxed{\beta^{k+1}(\theta_0, x_0) = E\left(x_0^*\left(P_{\theta_0}^{k+1}\right)x_0\right) \geq 0,}$$

and thus, $P_i^{k+1} \geq 0$. Notice now that, by definition, $\hat{\mathscr{J}}^{k+1}(\theta_0, x_0) \leq \beta^{k+1}(\theta_0, x_0)$. On the other hand, consider, for any $q \in \mathscr{C}^{k+1,m}$, $w(q) = (w(q)(0), \ldots, w(q)(k), 0, 0, \ldots) \in \mathscr{C}^{k+1,p}$ as,

$i \geq k+1$ $q(i)=0$?

$$w(q)(l) = \frac{1}{\delta} G_{\theta(l)}^{k+1-l} x(q)(l), \qquad l = 0, \ldots, k,$$

and $\boxed{x(q) = (x(q)(0), x(q)(1), \ldots) = X_J(\theta_0, x_0, w(q), q).}$ We get that

$$\beta^{k+1}(\theta_0, x_0) = \inf_{q \in \mathscr{C}^{k+1,m}} \left\{ \sum_{l=0}^{k} \left(\left\| C_{\theta(l)} x(q)(l) \right\|_2^2 \right. \right.$$
$$\left. + \left\| -J_{\theta(l)} x(q)(l) + q(l) \right\|_2^2 - \delta^2 \left\| w(q)(l) \right\|_2^2 \right)$$
$$\left. + \left\| L_{\theta(k+1)}^{1/2} x(q)(k+1) \right\|_2^2 \right\}.$$

Taking the supremum over $W \in \mathscr{C}^{k+1,p}$ we get from (19) that $\beta^{k+1}(\theta_0, x_0) \leq \hat{\mathscr{J}}^{k+1}(\theta_0, x_0)$, showing (c), that is,

$$\boxed{\hat{\mathscr{J}}^{k+1}(\theta_0, x_0) = \beta^{k+1}(\theta_0, x_0) = E\left(x_0^* P_{\theta_0}^{k+1} x_0\right).}$$

(α)

Let us show now that $P^{k+1} = (P_1^{k+1}, \ldots)$ indeed belongs to $\mathscr{H}_{\text{sup}}^{n+}$. To show this it remains to prove that $\|P^{k+1}\|_{\text{sup}} < \infty$. From (12) and (14), for any $x_0 \in \mathbb{C}^n$,

$$x_0^* P_i^{k+1} x_0 = \hat{\mathscr{J}}^{k+1}(i, x_0) = \sup_{w \in \mathscr{C}^{k+1,p}} \left\{ \left\| \tilde{Z}(i, x_0, w) \right\|_2^2 - \delta^2 \|w\|_2^2 \right\}$$

$q = (J_{\theta(k)} - \bar{K}_{\theta(k)}) x(k) \Rightarrow \tilde{Z}(i, x_0, w) \leq \bar{Z}(i, x_0, w)$

$$\leq \sup_{w \in \mathscr{C}^{k+1,p}} \left\{ \left\| \bar{Z}(i, x_0, w) \right\|_2^2 - \delta^2 \|w\|_2^2 \right\}$$

$$\leq \sup_{w \in \mathscr{C}^{k+1,p}} \left\{ \left\| \bar{Z}(i, x_0, 0) + \bar{Z}(i, 0, w) \right\|_2^2 - \delta^2 \|w\|_2^2 \right\}$$

$\Downarrow \|a+b\| \leq \|a\| + \|b\|$

$$\leq \sup_{w \in \mathscr{C}^{k+1,p}} \left\{ \left(\left\| \bar{Z}(i, x_0, 0) \right\|_2 + \left\| \bar{Z}(i, 0, w) \right\|_2 \right)^2 - \delta^2 \|w\|_2^2 \right\}$$

$\Downarrow \|\bar{Z}(i,0,w)\|^2 \geq (\delta^2 + \alpha^2)\|w\|^2$

$$\leq \sup_{w \in \mathscr{C}^{k+1,p}} \left\{ \left(\left\| \bar{Z}(i, x_0, 0) \right\|_2 + \sqrt{(\delta^2 - \alpha^2)} \|w\|_2 \right)^2 - \delta^2 \|w\|_2^2 \right\}.$$

Recalling that $\bar{K}$ stabilizes (A, B), we have from Proposition 2 that for some $c > 0$, $\|\bar{Z}(i, x_0, 0)\|_2 \le c\|x_0\|_2$ (c independent of i), so that

$\le c$?

$\|H\|^2 = \mathrm{tr}(H^* H)$

$\|A\| = \max_{\|x\|=1}(x^T A x) = \lambda_{max}(A)$

$\le \sup_{\|x\|=1}(x^T A x)$

$$\|P^{k+1}\|_{\sup} = \sup_{i\in\mathbb{N}} \|P_i^{k+1}\| = \sup_{i\in\mathbb{N}} \sup_{\|x_0\|=1} \left\{x_0^* P_i^{k+1} x_0\right\}$$

$\le \sup_{i\in N} \sup_{\|x_0\|=1} \sup_{w\in \mathscr{C}^{k+1,p}} \left\{(c+(\delta^2-\alpha^2)\|w\|_2)^2 - \delta^2\|w\|_2^2\right\}$

$$\le \sup_{\ell} \left\{\left(c + \sqrt{(\delta^2 - \alpha^2)\ell}\right)^2 - \delta^2\ell\right\} < \infty.$$

Finally, let us show (b). Consider $w = (w(0), \ldots, w(k+1), 0, 0, \ldots) \in \mathscr{C}^{k+2,p}$, $\hat{w}^{k+1} = (\hat{w}^{k+1}(0), \ldots, \hat{w}^{k+1}(k)) \in \mathscr{C}^{k+1,p}$, and $q = (q(0), \ldots, q(k+1), 0, 0, \ldots) \in \mathscr{C}^{k+2,m}$, $\hat{q}^{k+1} = (\hat{q}^{k+1}(0), \ldots, \hat{q}^{k+1}(k)) \in \mathscr{C}^{k+1,m}$ given by

$$w(0) = w_0 \in \mathbb{C}^p, \qquad w(l) = \hat{w}^{k+1}(l-1) = \frac{1}{\delta} G_{\theta(l)}^{k+2-l} x(l),\ l = 1, \ldots, k,$$

$$q(0) = 0, \qquad q(l) = \hat{q}^{k+1}(l-1) = \left(J_{\theta(l)} - K_{\theta(l)}^{k+2-l}\right) x(l),\ l = 1, \ldots, k,$$

where $x = (x(0), x(1), \ldots) = X_J(\theta_0, x_0, w, q)$. Then, for $\theta_0 = i$ and $x_0 = 0$,

$\sum_{k=0}^{\infty} E[Z_k^T Z_k] = Z_0^T Z_0 + \sum_{k=1}^{\infty} E[Z_k^T Z_k] = \|Z_J(\theta_1, \underset{D_i w_0}{x_1}, \hat{w}^{k+1}, \hat{q}^{k+1})\|_2^2$

$Z_0 = C_i x_0 + D_i(q_0 - J_i x_0) = 0$

$$\|Z_J(i, 0, w, q)\|_2^2 - \delta^2\|w\|_2^2$$

$\to \sum_{k=0}^{\infty} w_k^T w_k = \sum_{k=1}^{\infty} w_k^T w_k + w_0^T w_0$

$$= \left\|\tilde{Z}\left(\theta(1), D_i w_0, \hat{w}^{k+1}\right)\right\|_2^2 - \delta^2\|\hat{w}^{k+1}\|_2^2 - \delta^2\|w_0\|^2$$

$$= \hat{\mathscr{J}}^{k+1}\left(\theta(1), D_i w_0\right) - \delta^2\|w_0\|^2 = E\left(w_0^* D_i^* P_{\theta(1)}^{k+1} D_i w_0\right) - \delta^2\|w_0\|^2$$

$$= w_0^* D_i^* \mathscr{E}_i(P^{k+1}) D_i w_0 - \delta^2\|w_0\|^2.$$

But, from (12) and (14),

$\tilde{Z}(\theta, x, w) = Z_J(\theta, x, w, \tilde{q})$

$\le Z_J(\theta, x, w, q)$ $q = (J_{\theta_k} - \bar{K}_{\theta_k}) x_k$

$= Z_{\bar{K}}(\theta, x, w, 0)$

$= \bar{Z}(\theta, x, w)$

$$\left\|\tilde{Z}\left(\theta(1), D_i w_0, \hat{w}^{k+1}\right)\right\|_2^2 - \delta^2\|\hat{w}^{k+1}\|_2^2$$

$$\le \left\|\bar{Z}\left(\theta(1), D_i w_0, \hat{w}^{k+1}\right)\right\|_2^2 - \delta^2\|\hat{w}^{k+1}\|_2^2$$

$$= \left\|\bar{Z}(i, 0, w)\right\|_2^2 - \delta^2\|w\|_2^2 + \delta^2\|w_0\|^2$$

$\le -\alpha^2\|w\|^2$

$$\le -\alpha^2\|w_0\|^2 + \delta^2\|w_0\|^2,$$

?

that is, for every $w_0 \in \mathbb{C}^p$

$$w_0^*\left(\delta^2 I - D_i^* \mathscr{E}_i(P^{k+1}) D_i\right) w_0 > \alpha^2\|w_0\|^2$$

showing that for every $i \in \mathbb{N}$,

$$\delta^2 I - D_i^* \mathscr{E}_i(P^{k+1}) D_i \geq \alpha^2 I. \quad \blacksquare$$

We can now proceed to the proof of Lemma 2.

Proof of Lemma 2. Let us show that P_i as defined above satisfies (i), (ii), and (iii) of the Theorem. Since for every $x_0 \in \mathbb{C}^n$ and $i \in \mathbb{N}$,

$$x_0^* P_i^k x_0 = \hat{\mathscr{J}}^k(\theta_0, x_0) \uparrow \hat{\mathscr{J}}(\theta_0, x_0) = x_0^* P_i x_0 \qquad \text{as } k \to \infty,$$

we get that $P_i^k \uparrow P_i$ as $k \uparrow \infty$. Moreover, from the proof of Proposition 9, there exists $\alpha > 0$ such that $\|P^k\|_{\sup} < a$ for every $k \geq 0$, showing that $\|P\|_{\sup} < a$, and thus $P \in \mathscr{H}_{\sup}^{n+}$. Taking the limit as $k \to \infty$ in Proposition 9, we get that P satisfies (i) and (ii) of the Theorem. Moreover from uniqueness of $\hat{w}$ established in Proposition 8, and that $\hat{w}^k$ is a maximizing sequence for $\hat{J}(\theta_0, x_0)$ (see (9)), we can conclude, using the same arguments as in the proof of Proposition 3 in [34], that $\hat{w}^k \to \hat{w}$ as $k \to \infty$. Continuity of $\tilde{X}(\theta_0, x_0, \cdot)$ implies that $\tilde{X}(\theta_0, x_0, \hat{w}^k) \to \tilde{X}(\theta_0, x_0, \hat{w})$ as $k \to \infty$, and thus, $\hat{x}^k = (\hat{x}^k(0), \hat{x}^k(1), \ldots) = \tilde{X}(\theta_0, x_0, \hat{w}^k) \to \tilde{X}(\theta_0, x_0, \hat{w}) = (\hat{x}(0), \hat{x}(1), \ldots) = \hat{x}$. Therefore, for each $l \in \mathbb{N}^0$,

$$\hat{w}^k(l) = (1/\delta) G_{\theta(l)}^{k-l} \hat{x}^k(l) \to (1/\delta) G_{\theta(l)} \hat{x}(l) \qquad \text{as } k \to \infty.$$

Similarly,

$$\mathscr{Q}\left(\theta_0, \hat{w}^k\right) = q^k = \left(\hat{q}^k(0), \hat{q}^k(1), \ldots\right) \to \mathscr{Q}\left(\theta_0, \hat{w}\right) = \hat{q} = \left(\hat{q}(0), \hat{q}(1), \ldots\right) \qquad \text{as } k \to \infty,$$

so that for each $l \in \mathbb{N}^0$,

$$\hat{q}^k(l) = \left(J_{\theta(l)} - K_{\theta(l)}^{k-l}\right)\hat{x}^k(l) \to \left(J_{\theta(l)} - K_{\theta(l)}\right)\hat{x}(l) \qquad \text{as } k \to \infty.$$

This shows that

$$\begin{aligned}
\hat{q} &= \left(\hat{q}(0), \hat{q}(1), \ldots\right) = \mathscr{Q}\left(\theta_0, \mathscr{W}(\theta_0, x_0)\right) \\
&= \left(-\left(J_{\theta_0} - K_{\theta_0}\right)x_0, -\left(J_{\theta(1)} - K_{\theta(1)}\right)\hat{x}(1), -\left(J_{\theta(2)} - K_{\theta(2)}\right)\hat{x}(2), \ldots\right) \\
\hat{w} &= \left(\hat{w}(0), \hat{w}(1), \ldots\right) = \mathscr{W}(\theta_0, x_0) \\
&= \left(\frac{1}{\delta} G_{\theta_0} x_0, \frac{1}{\delta} G_{\theta(1)} \hat{x}(1), \frac{1}{\delta} G_{\theta(2)} \hat{x}(2), \ldots\right)
\end{aligned}$$

and thus,

$$\hat{X}(\theta_0, x_0) = \left(x_0, \hat{x}_1, \hat{x}_2, \ldots\right),$$

where

$$\hat{x}(k+1) = \left(A_{\theta(k)} - B_{\theta(k)}K_{\theta(k)} + \frac{1}{\delta}D_{\theta(k)}G_{\theta(k)}\right)\hat{x}(k),$$

$$\hat{x}(0) = x_0, \quad \theta(0) = \theta_0.$$

Since $\hat{X}(\theta_0, \cdot) \in \mathbb{B}[\mathscr{C}_0^n, \mathscr{C}^n]$ (as seen in (16)), we get that $\hat{X}(\theta_0, x_0) \in \mathscr{C}^n$ for any $\theta_0 \in \Theta_0$ and $x_0 \in \mathscr{C}_0^n$ which implies, from Proposition 2, that $r_1(\mathscr{L}) < 1$. ∎

$x_{k+1} = A_{\theta(k)} x_k + D_{\theta(k)} w_k$

APPENDIX

Proof of Proposition 2. $(\Rightarrow)$ All we have to show is that $x \in \ell_2(C^n)$, since that, as mentioned above, $x_k = (x(0), \dots, x(k)) \in \mathscr{C}_k^n$ for every $k \in \mathbb{N}^0$. We have that

$$x(k) = A_{\theta(k-1)} \dots A_{\theta(0)} x(0) + \sum_{\iota=0}^{k-1} A_{\theta(k-1)} \dots A_{\theta(\iota+1)} D_{\theta(\iota)} w(\iota)$$

and by the triangle inequality in $\mathscr{C}^n$,

$$\|x(k)\|_2 \le \|A_{\theta(k-1)} \dots A_{\theta(0)} x(0)\|_2 + \sum_{\iota=0}^{k-1} \|A_{\theta(k-1)} \dots A_{\theta(\iota+1)} D_{\theta(\iota)} w(\iota)\|_2.$$

Set $W_i(\iota) = E(w(\iota)w(\iota)^* 1_{\{\theta(\iota)=i\}})$, $W(\iota) = (W_1(\iota), \dots) \in \mathscr{H}_1^{p+}$, $DW(\iota)D^* = (D_1 W_1(\iota) D_1^*, \dots) \in \mathscr{H}_1^{n+}$. From Lemma 1 of [9], we get that

$= \sum_{i=1}^{N} \|\mathscr{L}_i^{k-\iota-1}(DW(\iota)D^*)\|$

$$\|A_{\theta(k-1)} \dots A_{\theta(\iota+1)} D_{\theta(\iota)} w(\iota)\|_2^2 \le n \|\mathscr{L}^{k-\iota-1}(DW(\iota)D^*)\|_1$$

$F = A$?

$= tr(A_{\theta(k-1)} \cdots A_{\theta(\iota+1)} (D_{\theta(\iota)} w(\iota) w(\iota)^* D_{\theta(\iota)}^*) A_{\theta(\iota+1)}^* \cdots A_{\theta(k-1)}^*)$

$$\le n \|\mathscr{L}^{k-\iota-1}\|_1 \|DW(\iota)D^*\|_1$$

$\|\mathscr{L}^{k-\iota-1}\|_1 = \|\mathscr{L}^{k-\iota-1}(I)\|_1$

$\|W(\iota)\|_1 \le \|w(\iota)\|_2^2$

$$\le n \|D\|_{\sup}^2 \|\mathscr{L}^{k-\iota-1}\|_1 \|w(\iota)\|_2^2,$$

since $\|W(\iota)\|_1 = \Sigma_{i=1}^{\infty} \|W_i(\iota)\| \le \Sigma_{i=1}^{\infty} E(\|w(\iota)\|^2 1_{\{\theta(\iota)=i\}}) = E(\|w(\iota)\|^2) = \|w(\iota)\|_2^2$. Similarly,

$= (\lambda_{max}[W_i(\iota)^* W_i(\iota)])^{\frac{1}{2}}$ $= E(w(\iota)^* w(\iota) 1_{\{\theta(\iota)=i\}})$ $= \|w(\iota)\|_2^2$?

$= \lambda_{max}(E(w(\iota)w^*(\iota) 1_{\{\theta(\iota)=i\}})]$ $\le$

$$\|A_{\theta(k-1)} \dots A_{\theta(0)} x(0)\|_2^2 \le n \|\mathscr{L}^k\|_1 \|x(0)\|_2^2.$$

From Lemma 1 in [26], there exists $0 < \zeta < 1$ and $\beta \ge 1$ such that $\|\mathscr{L}^k\|_1 \le \beta \zeta^k$, and therefore,

$$\boxed{\|x(k)\|_2 \le \sum_{\iota=0}^{k} \zeta_{k-\iota} \beta_\iota,}$$

where $\zeta_{k-\iota} = (\zeta^{1/2})^{(k-\iota)}$ and $\beta_0 = (n\beta)^{1/2}\|x(0)\|_2$, $\beta_\iota = (n\beta)^{1/2}\|D\|_{\sup}\|w(\iota-1)\|_2$, $\iota \geq 1$. Set $a = (\zeta_0, \zeta_1, \dots)$ and $b = (\beta_0, \beta_1, \dots)$. Since $a \in \ell_1$ (that is, $\sum_{\iota=0}^{\infty}|\zeta_\iota| < \infty$) and $b \in \ell_2$ (that is, $\sum_{\iota=0}^{\infty}|\beta_\iota|^2 < \infty$) it follows that the convolution $c = a * b = (c_0, c_1, \dots)$, $c_k = \sum_{\iota=0}^{k}\zeta_{k-\iota}\beta_\iota$, lies itself in ℓ_2 with $\|c\|_2 \leq \|a\|_1\|b\|_2$ (cf. [13, p. 529]). Hence,

$$\|x\|_2 = \left\{\sum_{k=0}^{\infty} E\left(\|x(k)\|^2\right)\right\}^{1/2} \leq \left\{\sum_{\iota=0}^{k} c_\iota^2\right\}^{1/2} = \|c\|_2 < \infty.$$

($\Leftarrow$) Making $w = 0$, we have from Proposition 1 above that $\sum_{k=0}^{\infty}E(\|x(k)\|^2) < \infty$ for every $x_0 \in \mathscr{C}_0^n$ and $\theta_0 \in \Theta_0$ is equivalent to $r_1(\mathscr{L}) < 1$. ∎

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