INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2002; **12**:435–446 (DOI: 10.1002/rnc.631)

Robust $\mathscr{H}_\infty$ filtering for uncertain Markovian jump linear systems[‡]

Carlos E. de Souza and Marcelo D. Fragoso*,†

Department of Systems and Control, Laboratório Nacional de Computação Científica—LNCC/MCT, Av. Getulio Vargas 333, 25651-070, Petrópolis, RJ, Brazil

SUMMARY

This paper investigates the problem of $\mathscr{H}_\infty$ filtering for a class of uncertain Markovian jump linear systems. The uncertainty is assumed to be norm-bounded and appears in all the matrices of the system state-space model, including the coefficient matrices of the noise signals. It is also assumed that the jumping parameter is available. We develop a methodology for designing a Markovian jump linear filter that ensures a prescribed bound on the $\mathscr{L}_2$-induced gain from the noise signals to the estimation error, irrespective of the uncertainty. The proposed design is given in terms of linear matrix inequalities. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: $\mathscr{H}_\infty$ filtering; robust filtering; uncertain systems; Markovian jump linear systems; linear matrix inequalities

1. INTRODUCTION

The $\mathscr{H}_\infty$ filtering approach is concerned with the design of an estimation procedure which ensures that the $\mathscr{L}_2$-induced gain from the noise signals to the estimation error will be less than a prescribed level. In $\mathscr{H}_\infty$ filtering, the noise sources are arbitrary signals with bounded energy, or bounded average power, which renders this approach insensitive to the noise statistics, and thus very appropriate to applications where the statistics of the noise signals are not exactly known; see, e.g. References [1–7] and the references therein. The potential of $\mathscr{H}_\infty$ filtering lies far beyond its insensitivity to the noise statistics. It has been recognized in Reference [6] that the $\mathscr{H}_\infty$ filtering scheme is also less sensitive than its $\mathscr{H}_2$ counterpart to uncertainty in the system parameters.

* Correspondence to: M. D. Fragoso, Department of Systems and Control, Laboratório Nacional de Computação Científica-LNCC/MCT, Av. Getulio Vargas 333, 25651-070, Petrópolis, RJ, Brazil.
† E-mail: frag@lncc.br
‡ A version of this paper was presented at the 35th IEEE Conference on Decision and Control, Kobe, Japan, 1996.

Contract/grant sponsor: Conselho Nacional de Desenvolvimento Científico e Tecnológico-CNPq, Brazil; contract/grant number: 300596/98-7/PQ and 520169/97-2/PQ
Contract/grant sponsor: PRONEX/CNPq; contract/grant number: 66.2015/98-3

Published online 28 January 2002

Received 10 June 2000
Accepted 15 March 2001

Recently, there have been a lot of interest on the problem of $\mathscr{H}_\infty$ filtering for systems with parameter uncertainty. It has been established that the $\mathscr{H}_\infty$ approach can guarantee a prescribed bound to the $\mathscr{L}_2$-induced gain from the noise signals to the estimation error in the presence of norm-bounded parameter uncertainty [8]. Filtering schemes for uncertain linear systems have been developed in References [8–10].

Over the past few years, Markovian jump linear systems have been attracting an increasing attention in the literature. This class of systems is very appropriate to model plants whose structure is subject to random abrupt changes due to, for instance, component and/or interconnections failures, sudden environment changes, modification of the operating point of a linearized model of a nonlinear system, etc. A number of estimation and control problems related to these systems has been analysed by several authors (see, for example, References [11–20] and references therein). In particular with regard to estimation, minimum variance filtering schemes for discrete-time systems have been studied in, for instance, References [11, 12, 19], whereas the $\mathscr{H}_\infty$ filtering is addressed in Reference [14].

In this paper we consider the problem of $\mathscr{H}_\infty$ filtering for a class of uncertain Markovian jump linear systems. This class of systems is described by a nominal Markovian jump linear system subject to real norm-bounded parameter uncertainty, which appears in all the matrices of the system state-space model, including the coefficient matrices of the noise signals. It is assumed that the jumping parameter is available. The problem we address is the design of a Markovian jump linear filter with a mean square stable error dynamics and which provides a prescribed bound on the $\mathscr{L}_2$-induced gain from the noise signals to the estimation error, irrespective of the uncertainty. This problem will be referred to as *robust Markovian jump $\mathscr{H}_\infty$ filtering*. A linear matrix inequality (LMI) approach is developed for solving this robust $\mathscr{H}_\infty$ filtering problem. The results obtained in this paper extend those in Reference [14] to deal with norm-bounded uncertainty in the state-space model of Markovian jump linear systems.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\Re^n$ is the n-dimensional Euclidean space, $\Re^{n\times m}$ is the set of all $n\times m$ real matrices, I_n is the $n\times n$ identity matrix, $\|\cdot\|$ denotes the induced matrix 2-norm, and $\mathscr{L}_2[0,\infty)$ stands for the space of square integrable vector functions over $[0,\infty)$. For a real matrix P, $P>0$ (respectively, $P\geqslant 0$), means that P is symmetric and positive definite (respectively, non-negative definite).

2. PROBLEM FORMULATION

Fix a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and consider the following class of dynamic systems:

$$(\Sigma):\quad \dot{x}(t) = [A(\theta_t) + \Delta A(\theta_t)]x(t) + [B(\theta_t) + \Delta B(\theta_t)]w(t), \quad x(0) = x_0, \quad \theta_0 = i_0 \tag{1}$$

$$y(t) = [C(\theta_t) + \Delta C(\theta_t)]x(t) + [D(\theta_t) + \Delta D(\theta_t)]w(t) \tag{2}$$

$$z(t) = L(\theta_t)x(t) \tag{3}$$

where $x(t)\in\Re^n$ is the state, $x_0\in\Re^n$ is an unknown initial state, $w(t)\in\Re^q$ is the noise signal which is assumed to be an arbitrary signal in $\mathscr{L}_2[0,\infty)$, $y(t)\in\Re^m$ is the measurement, and $z(t)\in\Re^p$ is the signal to be estimated. $\{\theta_t\}$ is a continuous-time Markov process with right continuous trajectories and taking values on a finite set $\phi = \{1, 2, \ldots, N\}$ with stationary transition probabilities:

$$\text{Prob}\{\theta_{t+h} = j \mid \theta_t = i\} = \begin{cases} \lambda_{ij}h + o(h), & i \neq j \\ 1 + \lambda_{ii}h + o(h), & i = j \end{cases}$$

where $h > 0$ and $\lambda_{ij} \geqslant 0$ is the transition rate from the state i to j, $i \neq j$, and

$$\lambda_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{N} \lambda_{ij}. \tag{4}$$

$A(\theta_t)$, $B(\theta_t)$, $C(\theta_t)$, $D(\theta_t)$ and $L(\theta_t)$ are known real matrices for all $\theta_t \in \phi$, and $\Delta A(\theta_t)$, $\Delta B(\theta_t)$, $\Delta C(\theta_t)$ and $\Delta D(\theta_t)$ are unknown matrices representing parameter uncertainties. The admissible uncertainties are assumed to be of the form

$$\Delta A(\theta_t) = H_1(\theta_t)F_1(\theta_t)E_1(\theta_t), \quad \Delta C(\theta_t) = H_2(\theta_t)F_2(\theta_t)E_2(\theta_t) \tag{5}$$

$$\Delta B(\theta_t) = H_3(\theta_t)F_3(\theta_t)E_3(\theta_t), \quad \Delta D(\theta_t) = H_4(\theta_t)F_4(\theta_t)E_4(\theta_t) \tag{6}$$

where $F_i(\theta_t) \in \Re^{m_i \times n_i}$, $i = 1, \ldots, 4$, are unknown real matrices satisfying

$$\| F_i(\theta_t) \| \leqslant 1, \quad \forall t \geqslant 0, \quad i = 1, \ldots, 4 \tag{7}$$

and $H_i(\theta_t)$ and $E_i(\theta_t)$, $i = 1, \ldots, 4$, $\forall \theta_t \in \phi$, are known real matrices of appropriate dimensions.

The set ϕ comprises the various operation modes of system (Σ) and for each possible value of $\theta_t = i$, $i \in \phi$, we will denote the matrices associated with the 'ith mode' by

$$A_i \triangleq A(\theta_t = i), \quad B_i \triangleq B(\theta_t = i), \quad C_i \triangleq C(\theta_t = i), \quad D_i \triangleq D(\theta_t = i), \quad L_i \triangleq L(\theta_t = i)$$

$$E_{ji} \triangleq E_j(\theta_t = i), \quad H_{ji} \triangleq H_j(\theta_t = i), \quad j = 1, \ldots, 4$$

where A_i, B_i, C_i, D_i, L_i and E_{ji}, H_{ji}, $j = 1, \ldots, 4$, are constant matrices for any $i \in \phi$.

It is assumed that no *a priori* estimate of the initial state, x_0, is available and the jumping process, $\{\theta_t\}$, is accessible, i.e. the operation mode of system (Σ) is known for every $t \geqslant 0$.

The filtering problem we will address is to obtain an estimate $\hat{z}(t)$, of $z(t)$ via a causal Markovian jump linear filter which provides a uniformly small estimation error, $z(t) - \hat{z}(t)$, for all $w \in \mathscr{L}_2[0, \infty)$, irrespective of the uncertainty.

In order to put the robust Markovian jump $\mathscr{H}_\infty$ filtering problem in a stochastic setting, we introduce the space $\mathscr{L}_2[\Omega, \mathscr{F}, \mathscr{P}]$ of $\mathscr{F}$-measurable processes, $z(t) - \hat{z}(t)$, for which

$$\| z - \hat{z} \|_2 = \left\{ E\left[\int_0^\infty (z(t) - \hat{z}(t))^T (z(t) - \hat{z}(t))\, \mathrm{d}t \right] \right\}^{1/2} < \infty \tag{8}$$

where $E[\cdot]$ denotes mathematical expectation. For the sake of notation simplification, we shall use indistinctly $\|\cdot\|_2$ to denote the norm either in $\mathscr{L}_2[\Omega, \mathscr{F}, \mathscr{P}]$ or in $\mathscr{L}_2[0, \infty)$, defined by

$$\| w \|_2 \triangleq \left[\int_0^\infty w^{\mathrm{T}}(t) w(t)\, \mathrm{d}t \right]^{1/2}, \quad \text{for} \quad w(t) \in \mathscr{L}_2[0, \infty)$$

Attention will be focused on the design of an n-order filter. Since the jumping process $\{\theta_t\}$ is available for $t \geqslant 0$ and in view of the structure of system (Σ), without loss of generality, the following structure for the Markovian jump linear filter will be adopted:

$$(\Sigma_f): \quad \dot{\hat{x}}(t) = G(\theta_t)\hat{x}(t) + K(\theta_t)y(t), \quad \hat{x}(0) = 0 \tag{9}$$

$$\hat{z}(t) = L(\theta_t)\hat{x}(t) \tag{10}$$

where the matrix functions $G(\theta_t)$ and $K(\theta_t)$ are to be determined.

This paper is concerned with the following robust filtering problem:

Design a Markovian jump filter of the form of (9) *and* (10) *such that the estimation error system is robustly mean square stable and*

$$\| z - \hat{z} \|_2 \leqslant \gamma \left(\| w \|_2^2 + x_0^{\mathrm{T}} R x_0 \right)^{1/2} \tag{11}$$

for all $w(t) \in \mathscr{L}_2[0, \infty)$, $x_0 \in \mathfrak{R}^n$, *and for all possible uncertainties, where* $R > 0$ *is a given weighting matrix for the initial state and* $\gamma \geqslant 0$ *is a given scalar which specifies the level of 'noise' attenuation in the estimation error.*

In the above, *robust mean square stability* is defined in the following sense:

Definition 2.1

Consider the Markovian jump system

$$\dot{x}(t) = [A(\theta_t) + \Delta A(\theta_t)]x(t) + [B(\theta_t) + \Delta B(\theta_t)]w(t), \quad x(0) = x_0, \quad \theta_0 = i_0$$

as in Equation (1). This system is said to be *robust mean square stable* if it is *mean square stable* for all admissible uncertainties, i.e. $E(\| x(t) \|^2) \to c_1$ as $t \to \infty$, where c_1 is a nonnegative real number, for arbitrary initial condition (x_0, θ_0), all admissible $\Delta A(\theta_t)$ an $\Delta B(\theta_t)$ and for any $w(t) \in \mathscr{L}_2[0, \infty)$. It is said to be *internally robust mean square stable*, if it is *internally mean square stable* for all admissible $\Delta A(\theta_t)$ and $\Delta B(\theta_t)$, i.e. if the solution of

$$\dot{x}(t) = [A(\theta_t) + \Delta A(\theta_t)]x(t)$$

is such that $E(\|x(t)\|^2) \to 0$, as $t \to \infty$ for arbitrary initial condition (x_0, θ_0) and for all admissible uncertainties.

We conclude this section with the following fundamental auxiliary result.

Lemma 2.1

The Markovian jump system (1) is *internally robust mean square stable, if and only if* $x(t) \in \mathscr{L}_2[\Omega, \mathscr{F}, \mathscr{P}]$ for any $w(t) \in \mathscr{L}_2[0, \infty)$, i.e.

$$\int_0^\infty E[\| x(t) \|^2] \, \mathrm{d}t = c_1(w, x_0) < c_2 < \infty \quad \text{for any } w(t) \in \mathscr{L}_2[0, \infty) \tag{12}$$

where c_2 is a positive real number.

Proof. An adaptation of Theorem 5.2 in Reference [21].

The weighting matrix R in (11) is a measure of the uncertainty in the initial state, x_0, relative to the uncertainty in w. A 'large' value of R indicates that the initial state is very close to zero. In the case of filtering problems where the effect of the initial state is not taken into account, x_0 is set of zero and the performance measure of (11) is replaced by

$$\|z - \hat{z}\|_2 \leqslant \gamma \|w\|_2 \tag{13}$$

Note that (13) can be viewed as the limit of (11) as the smallest eigenvalue of R approaches infinity.

Remark 2.1

We observe that no 'non-singularity assumption' is imposed to the filtering problem treated in this paper, as is the case with the robust $\mathscr{H}_\infty$ filtering approaches in References [8–10] for uncertain linear systems without jumps.

3. THE ROBUST $\mathscr{H}_\infty$ MARKOVIAN JUMP FILTER

First, note that by defining $\tilde{x} \triangleq x - \hat{x}$, it follows from (1), (2) and (9) that

$$\begin{aligned}\dot{\tilde{x}}(t) = G(\theta_t)\tilde{x} + \{A(\theta_t) + \Delta A(\theta_t) - K(\theta_t)[C(\theta_t) + \Delta C(\theta_t)] - G(\theta_t)\}\, x(t)\\ + \{B(\theta_t) + \Delta B(\theta_t) - K(\theta_t)[D(\theta_t) + \Delta D(\theta_t)]\}\, w(t), \quad \tilde{x}(0) = x_0\end{aligned} \tag{14}$$

Hence, it follows that a state-space model for the estimation error, $z - \hat{z}$, in terms of $\tilde{x}$ and x is given by

$$\begin{aligned}\dot{\xi}(t) = [\hat{A}(\theta_t) + \hat{H}_1(\theta_t)F_1(\theta_t)\hat{E}_1(\theta_t) + \hat{H}_2(\theta_t)F_2(\theta_t)\hat{E}_2(\theta_t)]\, \xi(t)\\ + [\hat{B}(\theta_t) + \hat{H}_3(\theta_t)F_3(\theta_t)\hat{E}_3(\theta_t) + \hat{H}_4(\theta_t)F_4(\theta_t)\hat{E}_4(\theta_t)]\, w(t)\end{aligned} \tag{15}$$

$$z(t) - \hat{z}(t) = \hat{L}(\theta_t)\xi(t) \tag{16}$$

where $\xi = [\tilde{x}^{\mathrm{T}} \ \ x^{\mathrm{T}}]^{\mathrm{T}}$ and

$$\hat{A}(\theta_t) = \begin{bmatrix} G(\theta_t) & A(\theta_t) - K(\theta_t)C(\theta_t) - G(\theta_t) \\ 0 & A(\theta_t) \end{bmatrix}, \quad \hat{B}(\theta_t) = \begin{bmatrix} B(\theta_t) - K(\theta_t)D(\theta_t) \\ B(\theta_t) \end{bmatrix} \tag{17}$$

$$\hat{E}_1(\theta_t) = [0 \quad E_1(\theta_t)], \quad \hat{E}_2(\theta_t) = [0 \quad E_2(\theta_t)], \quad \hat{E}_3(\theta_t) = E_3(\theta_t), \quad \hat{E}_4(\theta_t) = E_4(\theta_t) \tag{18}$$

$$\hat{H}_1(\theta_t) = \begin{bmatrix} H_1(\theta_t) \\ H_1(\theta_t) \end{bmatrix}, \quad \hat{H}_2(\theta_t) = \begin{bmatrix} -K(\theta_t)H_2(\theta_t) \\ 0 \end{bmatrix}, \quad \hat{H}_3(\theta_t) = \begin{bmatrix} H_3(\theta_t) \\ H_3(\theta_t) \end{bmatrix} \tag{19}$$

$$\hat{H}_4(\theta_t) = \begin{bmatrix} -K(\theta_t)H_4(\theta_t) \\ 0 \end{bmatrix}, \quad \hat{L}(\theta_t) = [L(\theta_t) \quad 0] \tag{20}$$

In the sequel, for each possible value of $\theta_t = i$, $i \in \phi$, we denote

$$\hat{A}_i \triangleq \hat{A}(\theta_t = i), \quad \hat{B}_i \triangleq \hat{B}(\theta_t = i), \quad \hat{L}_i \triangleq \hat{L}(\theta_t = i) \tag{21}$$

$$\hat{H}_{ji} \triangleq \hat{H}_j(\theta_t = i), \quad \hat{E}_{ji} \triangleq \hat{E}_j(\theta_t = i), \quad j = 1, \ldots, 4 \tag{22}$$

Our first result deals with the problem of analysing if a given Markovian jump filter of the form of (9) and (10) provides robust mean square stable error dynamics and satisfies the performance constraint of (11) for all admissible uncertainties.

Theorem 3.1

Consider the system (Σ) and let $R > 0$ be a given initial state weighting matrix and $\gamma > 0$ a given scalar. Given a Markovian jump linear filter of the form of (9) and (10), then the corresponding estimation error system is robust mean square stable and

$$\| z - \hat{z} \|_2 \leqslant \gamma \, (\| w \|_2^2 + x_0^{\mathrm{T}} R x_0)^{1/2}$$

for all $w \in \mathscr{L}_2 [0, \infty)$, $x_0 \in \Re^n$, and for all admissible uncertainty, if for all $i \in \phi$ there exist matrices $P_i > 0$ and positive scalars α_i, β_i, δ_i and ε_i satisfying the following inequalities:

$$\begin{bmatrix} Q_i (P_i, \alpha_i, \beta_i, \delta_i, \varepsilon_i) + \sum_{j=1}^{N} \lambda_{ij} P_j & P_i \hat{B}_i \\ \hat{B}_i^T P_i & - S_i (\gamma, \delta_i, \varepsilon_i) \end{bmatrix} < 0, \quad \forall i \in \phi \tag{23}$$

$$[I_n \quad I_n] \, P_{i_0} \begin{bmatrix} I_n \\ I_n \end{bmatrix} - \gamma^2 R \leqslant 0 \tag{24}$$

where i_0 is the state assumed by the jumping process $\{\theta_t\}$ at $t = 0$, and for all $i \in \phi$:

$$\begin{aligned} Q_i (P_i, \alpha_i, \beta_i, \delta_i, \varepsilon_i) = {} & P_i \hat{A}_i + \hat{A}_i^{\mathrm{T}} P_i + \alpha_i \hat{E}_{1i}^{\mathrm{T}} \hat{E}_{1i} + \beta_i \hat{E}_{2i}^{\mathrm{T}} \hat{E}_{2i} + \hat{L}_i^{\mathrm{T}} \hat{L}_i \\ & + P_i \left(\frac{1}{\alpha_i} \hat{H}_{1i} \hat{H}_{1i}^{\mathrm{T}} + \frac{1}{\beta_i} \hat{H}_{2i} \hat{H}_{2i}^{\mathrm{T}} + \frac{1}{\delta_i} \hat{H}_{3i} \hat{H}_{3i}^{\mathrm{T}} + \frac{1}{\varepsilon_i} \hat{H}_{4i} \hat{H}_{4i}^{\mathrm{T}} \right) P_i \end{aligned} \tag{25}$$

$$S_i(\gamma, \delta_i, \varepsilon_i) = \gamma^2 I - \delta_i \hat{E}_{3i}^{\mathrm{T}} \hat{E}_{3i} - \varepsilon_i \hat{E}_{4i}^{\mathrm{T}} \hat{E}_{4i} \tag{26}$$

Before proceeding to the proof of Theorem 3.1. we recall the following well-known inequality:

Lemma 3.1

Let F, X and Y be real matrices of appropriate dimensions with $\| F \| \leqslant 1$. Then for any scalar $\varepsilon > 0$.

$$XFY + Y^{\mathrm{T}} F^{\mathrm{T}} X^{\mathrm{T}} \leqslant \frac{1}{\varepsilon} X X^{\mathrm{T}} + \varepsilon Y^{\mathrm{T}} Y$$

Proof of Theorem 3.1. First note that in view of (23)

$$\begin{aligned} & P_i \hat{A}_i + \hat{A}_i^{\mathrm{T}} P_i + \alpha_i \hat{E}_{1i}^{\mathrm{T}} \hat{E}_{1i} + \beta_i \hat{E}_{2i}^{\mathrm{T}} \hat{E}_{2i} + P_i \left(\frac{1}{\alpha_i} \hat{H}_{1i} \hat{H}_{1i}^{\mathrm{T}} + \frac{1}{\beta_i} \hat{H}_{2i} \hat{H}_{2i}^{\mathrm{T}} \right) P_i \\ & \qquad + \sum_{j=1}^{N} \lambda_{ij} P_j < 0, \quad \forall i \in \phi \end{aligned} \tag{27}$$

Considering Lemma 3.1, it follows from (27) that for any matrices F_{1i} and F_{2i}, $\forall i \in \phi$, of appropriate dimensions and satisfying $\| F_{1i} \| \leqslant 1$ and $\| F_{2i} \| \leqslant 1$:

$$\begin{aligned} & P_i (\hat{A}_i + \hat{H}_{1i} F_{1i} \hat{E}_{1i} + \hat{H}_{2i} F_{2i} \hat{E}_{2i}) + (\hat{A}_i + \hat{H}_{1i} F_{1i} \hat{E}_{1i} + \hat{H}_{2i} F_{2i} \hat{E}_{2i})^{\mathrm{T}} P_i \\ & \qquad + \sum_{j=1}^{N} \lambda_{ij} P_j < 0, \quad \forall i \in \phi \end{aligned}$$

By Theorem 3.1 and Proposition 3.5 in Reference [22], this implies that the error system of (15) is internally mean square stable for all $F_1 (\theta_t)$ and $F_2 (\theta_t)$, satisfying (7), i.e. it is *internally robust mean square stable.*

Now for $T > 0$, introduce the cost function:

$$J(T) = E\left\{\int_0^T [(z(t) - \hat{z}(t))^{\mathrm{T}}(z(t) - \hat{z}(t)) - \gamma^2 w^{\mathrm{T}}(t)w(t)]\,\mathrm{d}t\right\} \tag{28}$$

It is a straightforward exercise to show that $[\xi(t), \theta_t\}$ is a Markov process, and bearing in mind Proposition A.1 in Reference [13] we find easily that its infinitesimal operator is given by

$$\begin{aligned}\mathscr{T} \cdot g(\xi(t), \theta_t) = \{&\xi^{\mathrm{T}}(t)[\hat{A}(\theta_t) + \hat{H}_1(\theta_t)F_1(\theta_t)\hat{E}_1(\theta_t) + \hat{H}_2(\theta_t)F_2(\theta_t)\hat{E}_2(\theta_t)]^{\mathrm{T}} \\ &+ w^{\mathrm{T}}(t)[\hat{B}(\theta_t) + \hat{H}_3(\theta_t)F_3(\theta_t)\hat{E}_3(\theta_t) + \hat{H}_4(\theta_t)F_4(\theta_t)\hat{E}_4(\theta_t)]^{\mathrm{T}}\}\, g_\xi(\xi(t), \theta_t) \\ &+ \sum_{j=1}^{N} \lambda_{\theta_t j}\, g(\xi(t), j)\end{aligned} \tag{29}$$

where $g(\cdot)$ is a real continuous, bounded, functional in $\mathscr{D}(\mathscr{T})$ (the domain of definition of the operator $\mathscr{T}$) with partial derivatives

$$g_\xi \triangleq \left[\frac{\partial g}{\partial \xi_1}, \quad \frac{\partial g}{\partial \xi_2}, \ldots, \frac{\partial g}{\partial \xi_n}\right]^{\mathrm{T}}$$

where ξ_j denotes the jth component of ξ.

Adding and subtracting $\mathscr{T}\,[\xi^{\mathrm{T}}(t)P(\theta_t)\xi(t)]$ to (28), where $P(\theta_t) = P_i$ when $\theta_t = i$, $i \in \phi$, and with P_i satisfying (23) and (24), and considering (16) and (29) we have

$$\begin{aligned}J(T) = E\Bigg\{\int_0^T \Bigg\{&\xi^{\mathrm{T}}(t)[(\hat{A}(\theta_t) + \Delta\hat{A}(\theta_t))^{\mathrm{T}} P(\theta_t) + P(\theta_t)(\hat{A}(\theta_t) + \Delta\hat{A}(\theta_t)) \\ &+ \sum_{j=1}^{N} \lambda_{\theta_t j} P_j + \hat{L}^{\mathrm{T}}(\theta_t)\hat{L}(\theta_t)]\,\xi(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \xi^{\mathrm{T}}(t)P(\theta_t)\,[\hat{B}(\theta_t) + \Delta\hat{B}(\theta_t)]\,w(t) \\ &+ w^{\mathrm{T}}(t)\,[\hat{B}(\theta_t) + \Delta\hat{B}(\theta_t)]^{\mathrm{T}}\, P(\theta_t)\xi(t) - \mathscr{T}\cdot[\xi^{\mathrm{T}}(t)P(\theta_t)\xi(t)]\Bigg\}\,\mathrm{d}t\Bigg\}\end{aligned} \tag{30}$$

where

$$\Delta\hat{A}(\theta_t) \triangleq \hat{H}_1(\theta_t)F_1(\theta_t)\hat{E}_1(\theta_t) + \hat{H}_2(\theta_t)F_2(\theta_t)\hat{E}_2(\theta_t)$$
$$\Delta\hat{B}(\theta_t) \triangleq \hat{H}_3(\theta_t)F_3(\theta_t)\hat{E}_3(\theta_t) + \hat{H}_4(\theta_t)F_4(\theta_t)\hat{E}_4(\theta_t)$$

Using Dynkin's formula (see, e.g. Reference [23]) and considering Lemma 3.1, it follows from (30) that

$$\begin{aligned}J(T) \leqslant E\Bigg\{\int_0^T \Bigg\{&\xi^{\mathrm{T}}(t)[Q(\theta_t) + \sum_{j=1}^{N} \lambda_{\theta_t j} P_j]\,\xi(t) + \xi^{\mathrm{T}}(t)P(\theta_t)\hat{B}(\theta_t)w(t) \\ &+ w^{\mathrm{T}}(t)\,\hat{B}^{\mathrm{T}}(\theta_t)P(\theta_t)\xi(t) - w^{\mathrm{T}}(t)S(\theta_t)w(t)\Bigg\}\,\mathrm{d}t\Bigg\} - E\,\{\xi^{\mathrm{T}}(T)P(\theta_T)\xi(T)\} \\ &+ \xi^{\mathrm{T}}(0)P(\theta_0)\xi(0)\end{aligned} \tag{31}$$

with

$$Q(\theta_t) \triangleq P(\theta_t)\hat{A}(\theta_t) + \hat{A}^{\mathrm{T}}(\theta_t)P(\theta_t) + \alpha(\theta_t)\hat{E}_1^{\mathrm{T}}(\theta_t)\hat{E}_1(\theta_t) + \beta(\theta_t)\hat{E}_2^{\mathrm{T}}(\theta_t)\hat{E}_2(\theta_t)$$
$$+ \hat{L}^{\mathrm{T}}(\theta_t)\hat{L}(\theta_t) + P(\theta_t)[\alpha^{-1}(\theta_t)\hat{H}_1(\theta_t)\hat{H}_1^{\mathrm{T}}(\theta_t) + \beta^{-1}(\theta_t)\hat{H}_2(\theta_t)\hat{H}_2^{\mathrm{T}}(\theta_t)$$
$$+ \delta^{-1}(\theta_t)\hat{H}_3(\theta_t)\hat{H}_3^{\mathrm{T}}(\theta_t) + \varepsilon_4^{-1}(\theta_t)\hat{H}_4(\theta_t)\hat{H}_4^{\mathrm{T}}(\theta_t)]P(\theta_t)$$
$$S(\theta_t) \triangleq \gamma^2 I - \delta(\theta_t)\hat{E}_3^{\mathrm{T}}(\theta_t)\hat{E}_3(\theta_t) - \varepsilon(\theta_4)\hat{E}_4^{\mathrm{T}}(\theta_t)\hat{E}_4(\theta_t)$$

where $\alpha(\theta_t)$, $\beta(\theta_t)$, $\delta(\theta_t)$ and $\varepsilon(\theta_t)$ assume positive values α_i, β_i, δ_i and ε_i, respectively, when $\theta_t = i$, $i \in \phi$.

Observe that since $\hat{x}(0) = 0$ and $\theta_0 = i_0$,

$$\xi^{\mathrm{T}}(0)P(\theta_0)\xi(0) = [x_0^{\mathrm{T}} \quad x_0^{\mathrm{T}}]\,P_{i_o}\begin{bmatrix} x_0 \\ x_0 \end{bmatrix}$$
$$= x_0^{\mathrm{T}}[I_n \quad I_n]\,P_{i_o}\begin{bmatrix} I_n \\ I_n \end{bmatrix} x_0 \tag{32}$$

Furthermore, since the system of (15) is internally robust mean square stable and $w \in \mathscr{L}_2[0, \infty)$, it follows from Lemma 2.1 that it is *robust mean square stable*, i.e.

$$\lim_{T\to\infty} E[\xi^T(T)P(\theta_T)\xi(T)] = 0. \tag{33}$$

as well as that $J(T)$, defined by (28), is well defined as $T \to \infty$.

Now, considering (32) and (33), it results from (31) that

$$\|z - \hat{z}\|_2^2 - \gamma^2(\|w\|_2^2 + x_0^{\mathrm{T}}Rx_0) \leqslant E\left\{\int_0^\infty [\xi^{\mathrm{T}} \quad w^{\mathrm{T}}]\,\Psi(\theta_t)\begin{bmatrix} \xi \\ w \end{bmatrix} \mathrm{d}t\right\}$$
$$+ x_0^{\mathrm{T}}\left\{[I_n \quad I_n]\,P_{i_o}\begin{bmatrix} I_n \\ I_n \end{bmatrix} - \gamma^2 R\right\} x_0 \tag{34}$$

where

$$\Psi(\theta_t) = \begin{bmatrix} Q(\theta_t) + \Sigma_{j=1}^N \lambda_{\theta_t j}P_j & P(\theta_t)\hat{B}(\theta_t) \\ \hat{B}^T(\theta_t)P(\theta_t) & -S(\theta_t) \end{bmatrix}$$

Finally, denoting $Q_i(P_i, \alpha_i, \beta_i, \delta_i, \varepsilon_i) \triangleq Q(\theta_t = i)$ and $S_i(\gamma, \delta_i, \varepsilon_i) \triangleq S(\theta_t = i)$, $i \in \phi$, and considering the inequalities (23) and (24), the results follows from (34) □

Remark 3.1

It should be remarked that the conditions of Theorem 3.1 in fact imply a stronger type of stability for the estimation error dynamics than the robust mean square stability. Indeed, in the proof of Theorem 3.1 we have established the internal mean square stability of the estimation error dynamics for all admissible uncertainties via a Lyapunov function which is independent of the uncertain parameters, i.e. an stochastic version of the so-called 'quadratic stability' is in fact used.

The next result presents a solution to the robust $\mathscr{H}_\infty$ filtering problem for Markovian jump linear systems in terms of linear matrix inequalities.

Theorem 3.2

Consider the system (Σ) and let $R > 0$ be a given initial state weighting matrix and $\gamma > 0$ a given scalar. Then there exists a Markovian jump filter of the form of (9) and (10) such that the estimation error system is robustly mean square stable and

$$\| z - \hat{z} \|_2 \leqslant \gamma (\| w \|_2 + x_0^{\mathrm{T}} R x_0)^{1/2}$$

for all $w \in \mathscr{L}_2 [0, \infty)$, $x_0 \in \Re^n$, and for all admissible uncertainties, if for all $i \in \phi$ there exist matrices $X_i > 0$, $Z_i > 0$, Y_i and W_i, and positive scalars α_i, β_i, δ_i and ε_i satisfying the following LMIs:

$$\begin{bmatrix} M_i & X_i A_i - Y_i C_i - W_i & X_i B_i - Y_i D_i & U_i \\ A_i^{\mathrm{T}} X_i - C_i^{\mathrm{T}} Y_i^{\mathrm{T}} - W_i^{\mathrm{T}} & N_i & Z_i B_i & V_i \\ B_i^{\mathrm{T}} X_i - D_i^{\mathrm{T}} Y_i^{\mathrm{T}} & B_i^{\mathrm{T}} Z_i & -S_i & 0 \\ U_i^{\mathrm{T}} & V_i^T & 0 & -J_i \end{bmatrix} < 0, \quad \forall i \in \phi \tag{35}$$

$$X_{i_o} + Z_{i_o} - \gamma^2 R \leqslant 0 \tag{36}$$

where i_0 is the state assumed by $\{\theta_t\}$ at $t = 0$, and for all $i \in \phi$:

$$M_i = W_i + W_i^{\mathrm{T}} + \sum_{j=1}^{N} \lambda_{ij} X_j + L_i^{\mathrm{T}} L_i \tag{37}$$

$$N_i = Z_i A_i + A_i^{\mathrm{T}} Z_i + \sum_{j=1}^{N} \lambda_{ij} Z_j + \alpha_i E_{1i}^{\mathrm{T}} E_{1i} + \beta_i E_{2i}^{\mathrm{T}} E_{2i} \tag{38}$$

$$S_i = \gamma^2 I - \delta_i E_{3i}^{\mathrm{T}} E_{3i} - \varepsilon_i E_{4i}^{\mathrm{T}} E_{4i} \tag{39}$$

$$U_i = [X_i H_{1i} - Y_i H_{2i} \quad X_i H_{3i} - Y_i H_{4i}] \tag{40}$$

$$V_i = [Z_i H_{1i} \quad 0 \quad Z_i H_{3i} \quad 0] \tag{41}$$

$$J_i = \mathrm{diag}\,\{\alpha_i I, \quad \beta_i I, \quad \delta_i I, \quad \varepsilon_i I\}. \tag{42}$$

Moreover, a suitable filter is given by

$$\dot{\hat{x}}(t) = G_i \hat{x}(t) + K_i y(t), \quad \hat{x}(0) = 0 \tag{43}$$

$$\hat{z}(t) = L_i \hat{x}(t) \tag{44}$$

for $\theta_t = i$, $i \in \phi$, where

$$G_i = X_i^{-1} W_i, \quad K_i = X_i^{-1} Y_i, \quad \forall i \in \phi \tag{45}$$

Proof. First, note that by using Schur's complements the inequality (23) of Theorem 3.1 is equivalent to

$$\begin{bmatrix} \hat{Q}_i + \Sigma_{j=1}^{N} \lambda_{ij} P_j & P_i \hat{B}_i & P_i \hat{H}_i \\ \hat{B}_i^{\mathrm{T}} P_i & -S_i & 0 \\ \hat{H}_i^{\mathrm{T}} P_i & 0 & -J_i \end{bmatrix} < 0, \quad \forall i \in \phi \tag{46}$$

where $\hat{B}_i$, S_i and J_i are as in (21), (39) and (42), respectively, and

$$\hat{Q}_i = P_i\hat{A}_i + \hat{A}_i^{\mathrm{T}} P_i + \alpha_i \hat{E}_{1i}^{\mathrm{T}} \hat{E}_{1i} + \beta_i \hat{E}_{2i}^{\mathrm{T}} \hat{E}_{2i} + \hat{L}_i^{\mathrm{T}} \hat{L}_i$$

$$\hat{H}_i = [\hat{H}_{1i} \quad \hat{H}_{2i} \quad \hat{H}_{3i} \quad \hat{H}_{4i}]$$

where $\hat{A}_i$, $\hat{L}_i$, $\hat{E}_{1i}$, $\hat{E}_{2i}$ and $\hat{H}_{ji}$, $j = 1, \ldots, 4$, are given in (21) and (22)

Let $P_i \triangleq \mathrm{diag}\,\{X_i, Z_i\}$, $\forall i \in \phi$, where X_i and Z_i are symmetric positive definite matrices to be found. Denoting

$$Y_i = X_i K_i, \quad W_i = X_i G, \quad \forall i \in \phi \tag{47}$$

where $K_i \triangleq K(\theta_t = i)$ and $G_i \triangleq G(\theta_t = i)$, $\forall i \in \phi$, and considering (17)–(20) it can be easily shown that (46) is equivalent to (35). Moreover, we also have that (36) ensures condition (24) of Theorem 3.1. Further, note that by (47)

$$G_i = X_i^{-1} W_i, \quad K_i = X_i^{-1} Y_i, \quad \forall i \in \phi$$

Hence, by Theorem 3.1 we conclude that the filter of (43)–(45) solves the robust $\mathscr{H}_\infty$ filtering problem.

Remark 3.2

Theorem 3.2 provides a method for designing a robust $\mathscr{H}_\infty$ Markovian jump linear filter for Markovian jump linear systems subject to norm-bounded parameter uncertainty. The proposed design is given in terms of LMIs which has the advantage that can be solved numerically very efficiently using recently developed algorithms for solving LMIs.

We observe that the problem of finding the robust $\mathscr{H}_\infty$ filter of Theorem 3.2 for the smallest possible $\gamma \geqslant 0$ can be easily solved in terms of the following linear programming problem in γ, X_i, Z_i, Y_i,W_i, α_i, β_i, δ_i and ε_i, $\forall i \in \phi$:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \gamma \geqslant 0, \quad X_i > 0, \quad Z_i > 0, \quad \alpha_i > 0, \quad \beta_i > 0, \quad \delta_i > 0, \\ & \varepsilon_i > 0 \quad (35) \text{ and } \quad (36). \end{array}$$

When the effect of the initial state of system (Σ) is not taken into account, which can be viewed as having $x(0)$ that is certain to be zero, the inequality of (36) will no longer be required as this case corresponds to choosing a sufficient large R (in the sense that its smallest eigenvalue approaches infinity). In such situation, Theorem 3.2 specializes as follows:

Corollary 3.1

Consider system (Σ) with $x(0) = 0$ and let $\gamma > 0$ a given scalar. Then there exists a filter of the form of (9) and (10) such that the estimation error system is robustly mean square stable and

$$\| z - \hat{z} \|_2 \leqslant \gamma \| w \|_2$$

for all $w \in \mathscr{L}_2\,[0, \infty)$ and for all admissible uncertainties, if for all $i \in \phi$ there exist matrices $X_i > 0$, $Z_i > 0$, Y_i and W_i, and positive scalars α_i, β_i, δ_i and ε_i satisfying the LMI (35). Moreover, a suitable filter is given by (43)–(45).

In the case of one mode operation, i.e. there is no jumps in system (Σ), we have $N = 1$, $\phi = \{1\}$ and $\lambda_{11} = 0$. Denoting the matrices of system (Σ) by A, B, C, D, L, E_j, H_j, $j = 1, \ldots, 4$, Theorem 3.2 reduces to the following result:

Corollary 3.2

Consider system (Σ) with no jumps and let $R > 0$ be a given initial state weighting matrix and $\gamma > 0$ a given scalar. Then there exists a causal linear filter such that the estimation error system is asymptotically stable for all admissible uncertainties and

$$\|z - \hat{z}\|_2 \leqslant \gamma\,(\|w\|_2 + x_0^{\mathrm{T}} R x_0)^{1/2}$$

for all $w \in \mathscr{L}_2\,[0, \infty)$, $x_0 \in \Re^n$, and for all admissible uncertainties, if there exist matrices $X > 0$, $Z > 0$, Y and W, and positive scalars α, β, δ and ε satisfying the following LMIs:

$$\begin{bmatrix} W + W^{\mathrm{T}} + L^{\mathrm{T}}L & XA - YC - W & XB - YD & U \\ A^{\mathrm{T}}X - C^{\mathrm{T}}Y^{\mathrm{T}} - W^{\mathrm{T}} & ZA + A^{\mathrm{T}}Z + \alpha E_1^{\mathrm{T}}E_1 + \beta E_2^{\mathrm{T}}E_2 & ZB & V \\ B^{\mathrm{T}}X - D^{\mathrm{T}}Y^{\mathrm{T}} & B^{\mathrm{T}}Z & -(\gamma^2 I - \delta E_3^{\mathrm{T}}E_3 - \varepsilon E_4^{\mathrm{T}}E_4) & 0 \\ U^{\mathrm{T}} & V^{\mathrm{T}} & 0 & -J \end{bmatrix} < 0 \tag{48}$$

$$X + Z - \gamma^2 R \leqslant 0 \tag{49}$$

where

$$U = [XH_1 \quad -YH_2 \quad XH_3 \quad -YH_4], \quad V = [ZH_1 \quad 0 \quad ZH_3 \quad 0] \tag{50}$$

$$J = \mathrm{diag}\,\{\alpha I, \beta I, \delta I, \varepsilon I\}. \tag{51}$$

Moreover, a suitable filter is given by

$$\dot{\hat{x}}(t) = G\hat{x}(t) + Ky(t), \quad \hat{x}(0) = 0$$
$$\hat{z}(t) = L\hat{x}(t)$$

where

$$G = X^{-1}W, \quad K = X^{-1}Y$$

Remark 3.3

Corollary 3.2 provides a methodology for designing robust $\mathscr{H}_\infty$ filters for a class of uncertain linear systems in terms of linear matrix inequalities. This is in contrast with the approach developed in Reference [10] which is based on algebraic Riccati equations and is also restricted to 'non-singular' robust $\mathscr{H}_\infty$ filtering problems.

4. CONCLUSIONS

In this paper we have investigated the problem of robust $\mathscr{H}_\infty$ filtering for a class of Markovian jump linear system with norm-bounded parameter uncertainty and unknown initial state. The $\mathscr{H}_\infty$ filter derived here ensures robust mean square stability for the estimation error and a prescribed bound on the $\mathscr{L}_2$-induced gain from the noise signals to the estimation error irrespective of the uncertainty, while preserves the system feature of having Markovian jumps, i.e. the filter is a Markovian jump linear system as well. The main result is tailored via linear matrix inequalities.

REFERENCES

1. Banavar RN, Speyer JL. A linear-quadratic game approach to estimation and smoothing. *Proceedings of the 1991 American Control Conference*, Boston, MA, June 1991; 2818–2822.
2. Bernstein DS, Haddad WM. Steady-state Kalman filtering with an $\mathscr{H}_\infty$ error bound. *Systems & Control Letters* 1989; **12**:9–16.
3. Grimble MJ. $\mathscr{H}_\infty$ design of optimal linear filters. In *Linear Circuits, Systems and Signal Processing: Theory and Application*, Byrnes CI, Martin CF, Saeks RE, (eds). North-Holland: Amsterdam; 1988; 533–540.
4. Nagpal KM, Khargonekar PP. Filtering and smoothing in an $\mathscr{H}_\infty$ setting. *IEEE Transactions on Automatic Control* 1991; **36**:152–166.
5. Shaked U. $\mathscr{H}_\infty$ minimum error state estimation of linear stationary processes. *IEEE Transactions on Automatic Control* 1990; **35**:554–558.
6. Shaked U, Theodor Y. $\mathscr{H}_\infty$-optimal estimation: A tutorial. *Proceedings of the 31st IEEE Conference on Decision and Control*, Tucson. TX, December 1992; 2278–2286.
7. Yaesh I, Shaked U. Game theory approach to optimal linear estimation in the minimum $\mathscr{H}_\infty$ norm sense. *Proceedings of the 28th IEEE Conference on Decision and Control*, Tampa, FL, December 1989; 421–425.
8. Fu M, de Souza CE, Xie L. $\mathscr{H}_\infty$ estimation for uncertain systems. *International Journal of Robust and Nonlinear Control* 1992; **2**:87–105.
9. de Souza CE, Shaked U, Fu M. Robust $\mathscr{H}_\infty$ filtering for continuous time varying uncertain systems with deterministic input signals. *IEEE Transactions on Signal Processing* 1995; **43**:709–719.
10. Xie L, de Souza CE. On robust filtering for linear systems with parameter uncertainty. *Proceedings of the 34th IEEE Conference on Decision and Control*, New Orleans, LA, December 1995.
11. Blom HAP, Bar-Shalom Y. The interacting multiple model algorithm for systems with Markovian switching coefficients. *IEEE Transactions on Automatic Control* 1988; **33**:780–783.
12. Costa OLV. Linear minimum mean squares error estimation for discrete-time Markovian jump linear systems. *IEEE Transactions on Automatic Control* 1994; **39**:1685–1689.
13. de Souza CE, Fragoso MD. $\mathscr{H}_\infty$ control for linear systems with Markovian jumping parameters. *Control-Theory and Advanced Technology* 1993; **9**:457–466.
14. de Souza CE, Fragoso MD. $\mathscr{H}_\infty$ filtering for Markovian jump linear systems. *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan, December 1996.
15. Fragoso MD. On a partially observable LQG problem for systems with Markovian jumping parameters. *Systems & Control Letters* 1988; **10**:349–356.
16. Ji Y, Chizeck HJ. Controllability, stabilizability, and continuous-time Markovian jumping linear quadratic control. *IEEE Transactions on Automatic Control* 1990; **35**:777–788.
17. Mariton M. Jump linear quadratic control with random state discontinuities. *Automatica* 1987; **23**:237–240.
18. Sworder DD. Feedback control for a class of linear systems with jump parameters. *IEEE Transactions on Automatic Control* 1969; **AC-14**:9–14.
19. Tugnait JK. Detection and estimation for abruptly changing systems. *Automatica* 1982; **18**:607–615.
20. Wonham WH. Random differential equations in control theory. In *Probabilistic Methods in Applied Mathematics*, Bharucha-Reid AT (ed.), vol. 2. Academic Press: New York, 1970; 131–212.
21. Fragoso MD, Costa OLV. A Unified approach for mean square stability of continuous-time linear systems with Markovian jumping parameters and additive disturbances. *39th IEEE Conference on Decision and Control,* Sydney, Australia, December 2000. See also, *LNCC Internal Report No.* 11/99, 1999, http://www.lncc.br/proj-pesq/relpesq-99.html.
22. Feng X, Loparo KA, Ji Y, Chizeck HJ. Stochastic stability properties of jump linear systems. *IEEE Transactions on Automatic Control* 1992; **37**:1884–1892.
23. Ito K, McKean Jr HPM. *Diffusion Processes and Their Sample Paths*. Springer: Berlin, 1965.