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Optimal Quadratic Control of Jump Linear Systems with Gaussian Noise in Discrete-time

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ABSTRACT

Jump linear systems, which possess randomly jumping parameters, can be used to model systems subject to failures (or structural changes). They can also be used to describe linearized systems changing around operating points. In this paper, an optimal discrete-time Jump Linear Quadratic Gaussian (JLQG) control problem is investigated. The system to be controlled is linear, except for randomly jumping parameters which obey a discrete-time finite state Markov process. A quadratic expected cost is minimized, for systems subject to additive Gaussian input and measurement noise. It is assumed that the system structure (i.e., jumping parameters) is known at each time.

A separation property enables us to design the optimal JLQ controller and optimal x-state estimator separately. Based on the appropriate controllability and observability properties for discrete-time jump linear systems, the infinite time horizon JLQG problem is solved. The optimal infinite time horizon JLQG compensator has a steady state control law but does not not have a steady-state filter. A suboptimal JLQG compensator, using a filter which converges to a steady-state filter, is then constructed.

1. Introduction and problem formulation

We consider jump linear systems described by

$$x_{k+1} = A_k(r_k)x_k + B_k(r_k)u_k + \Gamma_k(r_k)w_k \qquad (1)$$

$$y_k = C(r_k)x_k + D_k(r_k)v_k \qquad (2)$$

where $x \in R^n$ is the x-process state, $u \in R^{n1}$ is the x-process input, $y \in R^m$ is the x-process measurement, and w_k and v_k are mutually independent zero mean Gaussian white noise sequences with unit variance, of dimension n_2 and m_1 respectively. $A_k(\cdot)$, $B_k(\cdot)$, $C_k(\cdot)$, $D_k(\cdot)$ and $\Gamma_k(\cdot)$ are appropriately dimensioned matrices that are functions of the form process $\{r_k\}$, which takes values in the **finite set $M = \{1,2,\ldots,M\}$**; here $k \in Z^+$ is the time index. The different values in M are **forms** (or **modes** or **conditions**) of the system. Process $\{r_k\}$ is a finite state discrete-time Markov chain, with transition probability matrix

$$\underline{P}_k = \Pr\{r_{k+1}=j \mid r_k=i\} = \{p_k(i,j)\} \in R^{M\times M} \qquad (3)$$

where $i,j = 1,2,\ldots,M$.

In (1), the initial x-state is a Gaussian vector independent of $\{w_k\}$ and $\{v_k\}$:

$$x_0 \sim N(\bar{x}_0, P_0) \qquad (4)$$

We assume that the value of form r_k is known at time k, but not in advance. Let us define the information vector Z_k to be

$$Z_k \triangleq \{u_0,\ldots,u_{k-1},y_1,\ldots,y_k,r_0,\ldots,r_k\} \qquad (5)$$

and consider the cost

$$\boxed{J_N = E\{\sum_{k=0}^{N-1}[u_k'R_k(r_k)u_k + x_{k+1}'Q_{k+1}(r_{k+1})x_{k+1}] + x_N'K_T(r_N)x_N\}} \qquad (6)$$

Here the matrices $Q_{k+1}(i)$, $K_T(i)$ are symmetric and positive semi-definite and $R_k(i)$ is positive definite for each i. $E\{\cdot\}$ is the expectation taken over all the underlying random quantities (x_0, w_k, v_k and r_k) under the assumption that r_0 is given. Let U_k be the set of finite R^m-valued functions on Z_k. These (causal) functions are chosen to be the admissible set of control laws. The JLQG optimal control problem can be stated as follows:

Find the control laws $\{u_0,\ldots,u_{N-1}\}$, where $u_k \in U_k$, to minimize (6) subject to (1)-(4).

Note that although form r_k is known at time k, the x-process is not directly observed. The solution of this problem without noise appears in [2,3,4]. The contribution of this paper is the inclusion of input and measurement noise in (1)-(6), which necessitates estimation of $\{x_k\}$.

The organization of this paper is as follows: in Section 2, a separation theorem is used to obtain the optimal JLQG compensator for the finite time horizon problems. In Section 3 the appropriate controllability, observability and duality properties that are needed to solve the infinite time horizon situation are presented. Then the optimal infinite time horizon JLQG problem is solved, obtaining a compensator with a steady-state controller but time-varying filter. In Section 4 a suboptimal compensator that yields a time-invariant version when time horizon tends to infinity is developed. Section 5 contains an example.

2. Finite time horizon problem

It is well known that for the linear quadratic gaussian control problem, the design of the optimal controller and optimal state estimator can be done

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separately [1], and then combined to obtain the optimal LQG compensator. A similar result holds for the jump system case. That is, it can be easily shown [5] that the optimal compensator solving (1)-(6) uses the same control law as the JLQ regulator [2,3,4] (under perfect observation of x), with x_k replaced by the optimal x-state estimate

$$\hat{x}_k \triangleq E\{x_k|Z_k\} \tag{7}$$

Now consider the solution of JLQG problem (1)-(6). By this certainty equivalence principle, we can design the optimal x-state estimator and controller (JLQ controller) separately, and then combine them to obtain the optimal compensator. At time k, r_k is known. Thus for filtering purpose the system (1)-(5) is a time varying linear system; Combining this with the optimal JLQ solution of [2,3,4], we obtain the following:

Theorem 1: For the JLQG problem (1)-(6), the optimal control laws are given by:

$$u_k = -L_k(i)\hat{x}_k \tag{8}$$

for k = N-1, N-2,...,0 and $i \in \mathbf{M}$, where gains $\{L_k(i)\}$ are computed recursively, backwards in time from k = N, by the following set of M coupled matrix difference equations:

$$L_k(i) = [R_k(i) + B_k'(i)\hat{K}_{k+1}(i)B_k(i)]^{-1} B_k'(i)\hat{K}_{k+1}(i)A_k(i) \tag{9}$$

and

$$K_k(i) = A_k'(i)\hat{K}_{k+1}(i)[A_k(i)-B_k(i)L_k(i)] \tag{10}$$

$$\hat{K}_{k+1}(r_k=i) = \sum_{j=1}^{M} p_k(i,j)[K_{k+1}(r_{k+1}=j) + Q_{k+1}(r_{k+1}=j)] \tag{11}$$

with terminal conditions

$$K_N(i) = K_N(i)$$

The optimal (minimum mean square error) estimate $\hat{x}_k$ in (8) is given by the following Kalman Filter:

$$\hat{x}_k = \hat{x}_{k|k-1} + S_k[y_k - C_k(r_k)\hat{x}_{k|k-1}] \tag{12}$$

$$S_k = P_{k|k-1}C_k'(r_k)[C_k(r_k)P_{k|k-1}C_k'(r_k) + D_k(r_k)D_k'(r_k)]^{-1} \tag{13}$$

$$P_k = [I - S_kC_k(r_k)]P_{k|k-1} \tag{14}$$

$$\hat{x}_{k|k-1} = A_{k-1}(r_{k-1})\hat{x}_{k-1} + B_{k-1}(r_{k-1})u_{k-1} \tag{15}$$

$$P_{k|k-1} = A_{k-1}(r_{k-1})P_{k-1}A_{k-1}'(r_{k-1}) + \Gamma_{k-1}(r_{k-1})\Gamma_{k-1}'(r_{k-1}) \tag{16}$$

$$\text{with} \quad \hat{x}_{0|-1} = \bar{x}_0, \quad P_{0|-1} = \bar{P}_0 \tag{17}$$

The optimal cost is

$$J_N^*(r_0,\bar{x},\bar{P}_0) = \bar{x}_0'K_0(r_0)\bar{x}_0 + tr[K_0(r_0)\bar{P}_0] + \sum_{k=0}^{N-1}\{tr[\tilde{K}_k(r_k))P_k] + tr[\Gamma_k'(r_k)\hat{K}_{k+1}(r_k)\Gamma_k(r_k)]\} \tag{18}$$

Here $\hat{K}_{k+1}(r_k)$ is given in (11), $K_k(r_k)$ in (10) and for notational convenience we define

$$\tilde{K}_k(r_k) = A_k'(r_k)\hat{K}_{k+1}(r_k)A_k(r_k) - K_k(r_k)$$

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The optimal controller gains $L_k(r_k)$ (along with $K_k(r_k)$) can be computed off-line, in advance of system operation (backwards in time from k = N). The filter gain and covariance matrices S_k, P_k must be computed on-line, because the system parameters at time k in (1)-(5) are not known until time k. It is important to note that the filter parameters S_k and P_k are form-sample path dependent. It is clear that a precomputed filter, based upon only the a priori form transition probabilities, is not the optimal use of information vector Z_k.

3. Infinite time horizon solution

The optimal compensator given in Theorem 1 is time varying. It is natural to consider the behavior of this solution for time invariant systems as the time horizon becomes infinite. For the JLQ problem (without estimation), necessary and sufficient conditions for the existence of finite cost, steady-state control laws are known [3]. There also exist sufficient conditions based on appropriate definition of controllability [4]. Unfortunately, the underlying jump process makes it impossible to obtain a time-invariant filter. Therefore, if the computational simplicity of a time-invariant compensator is required, a suboptimal filter must be used to estimate x.

The remainder of this paper considers the infinite time horizon problem, for time invariant cases of system (1)-(7); that is, with $p_k(i,j) = p_{ij}$, $A_k(r_k) = A(r_k)$, etc. Before we formulate and solve the infinite time horizon JLQG problem, the appropriate controllability and observability properties must be established. Absolute controllability and observability properties of discrete-time Markovian jump linear systems were investigated in [4]. We review them briefly here.

Consider the noiseless system

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k \tag{19}$$

$$y_k = C(r_k)x_k \tag{20}$$

$$Pr\{r_{k+1}=j|r_k=i\} = \{p_{ij}\} \in R^{M\times M} \tag{21}$$

where i,j = 1,2,...,M.

Definition: **Absolute Controllability** [4].

Consider the jump linear system (19),(21). For every initial state (x_0,r_0) and any choice of target value $\underline{x} \in R^n$, if there exists an admissible control law u such that the time

$$T_{ca} = \inf_{u \in U} \{k>0: x(k,x_0,r_0,u) = \underline{x}\}$$

is finite then the system is **absolutely controllable**.

Conditions for absolute controllability are given in the following [4]:

Lemma: System (19), (21) is absolutely controllable if and only if

(i) for every infinite sample path of form process, there exists a finite time $T_{ca}(w) < \infty$ such that the first $T = T_{ca}(w)$ forms $\{i_0, i_1, \ldots, i_{T-1}\}$ in the sample path yield a controllability matrix C_J with

$$\mathrm{rank}\ [C_J] = \mathrm{rank}\ [B(i_{T-1}) \vdots A(i_{T-1})B(i_{T-2}) \vdots \ldots \vdots \prod_{j=1}^{T-1} A(i_j)B(i_0)] = n$$

(ii) For each x_0, r_0 and $\underline{x}$, the control sequence $U_{T-1} = [u_0' \ \ldots \ u_{T-2}' \ u_{T-1}']'$, where

$$U_{T-1} = [C_J'C_J]^{-1}C_J' \ \{ \ \underline{x} - \prod_{j=0}^{T-1} A(i_j)x_0 \ \}$$

which accomplishes this transition must be causal; that is, u_k can be determined by $\{(r_t, x_t), \ t \le k\}$ from. -*-

Now we consider observability properties of system (19)-(21). Note that observability is not a stochastic concept for jump linear systems without noise, because we have the past values of $\{r_k\}$. We can use the usual observability definition for deterministic systems. Let $y_k(x_0 = x_{\#1})$ denote the output obtained at time k from system (19)-(21) having $x_0 = x_{\#1}$. For any initial form r_0, and any two initial x-states $x_{\#1}$ and $x_{\#2}$, let T be the minimum time such that equivalent outputs $y_k(x_0=x_{\#1}) = y_k(x_0=x_{\#2})$ and known inputs (given the form sequence) during the interval $k_0 \le k \le T$ implies that $x_{\#1} = x_{\#2}$ with certainty. We say that this system **is observable** if this time T is finite. A rank condition for observability of this kind of system is as follows:

Theorem 2: ([5])

System (19)-(21) is observable if and only if for every sample path of the form process, there exists a finite time $T_{oa} < \infty$ such that for the first $T = T_{oa}$ forms $\{i_0, i_1, \ldots, i_{T-1}\}$, the jump observability matrix O_J has

$$\mathrm{rank}[O_J] = \mathrm{rank} \begin{bmatrix} C(i_0) \\ C(i_1)A(i_0) \\ \vdots \\ C(i_{T-1}) \prod_{j=1}^{T-1} A(i_j) \end{bmatrix} = n$$

-*-

The above observability condition depends on the fact that the past form values $\{r_\tau : \tau < t\}$ are known and used. If this assumption is not true (for example, because of memory limitation), then definition of absolute observability [5] is appropriate: $\{C(r_k), A(r_k)\}$ is absolutely observable if and only if $\{A'(r_k), C'(r_k)\}$ is absolutely controllable. Note that $\{A'(r_k), C'(r_k)\}$ being absolutely controllable is not equivalent to the (deterministic) controllability of all the pairs (A_i', C_i') for each $i \in \mathbf{M}$.

Now recall the solution of JLQ problem [3]. For cost function (6) subject to (19) and (20), with $R_k(i) = R(i)$, $Q_k(i) = Q(i)$ and $p_k(i,j) = p_{ij}$, the finite time horizon solution can be given by the formulas of the controller portion in Theorem 1, with the estimate $\hat{x}_k$ replaced by x_k.

For the infinite time horizon case, we have the following result [5]:

Theorem 3: If system (19)-(21) (with $C'(i)C(i) = Q(i)$ in (20)) is observable and absolutely controllable, as $N \to \infty$, the solution of JLQ problem converges to a finite, steady-state set

$$\lim_{N-k\to\infty} L_k(i) = L(i) \quad \text{and} \quad \lim_{N-k\to\infty} K_k(i) = K(i) \ge 0$$

The controlled system is stabilized in the sense that $E\{x_k'x_k\} \to 0$ as $k \to \infty$ -*-

This is proved in [4] for absolute observability, and extended to require only observability in [5].

The optimal filter as in Theorem 1 can be obtained as the controller solution of an algebraic dual problem. Consider the dual system of (19)-(21),

$$x_{t+1}^* = A'(r_t^*)x_t^* + C'(r_t^*)u_t^* \tag{22}$$

$$y_t^* = B'(r_t^*)x_t^* \tag{23}$$

Here the "reverse form process" $\{r_t^*\}$ ($r_t^* = r_{N-k}$) has t running backwards from N to 0. Note that we assume that the values of r_τ^*, for $\tau \le t_1 - 1$, are not available at point t_1. Only the posterior transition probability probability matrix $\underline{P}^*$ is known:

$$\underline{P}^* = \{p^*(r_{t-1}^*=j : r_t^*=i)\} \tag{24}$$

But at time t_1, we know the values of r_s^* exactly for all $s \ge t_1$. Define the sequence $\{x_t^*, \ t = 0,1,\ldots,N)$ as follows:

$$x_{t+1}^* = A'(r_t^*)x_t^* + C'(r_t^*)u_t \tag{25}$$

with initial condition

$$(x_0^*)'A(r_0^*) = a' \tag{26}$$

Here $\{r_t^*\}$ ($r_t^* = r_{N-t}$) is like in (24).

Theorem 4: Duality theorem [5]

The filter (optimal x-state estimator) problem for the time invariant Markovian jump linear system (1)-(3) is equivalent to the optimal control (regulator) problem of algebraic dual system (25-26), with cost function:

$$E\{a'(x_N - \hat{x}_N)\}^2 = (x_N^*)'K_T(r_N^*)x_N^* + \sum_{t=0}^{N-1} [(x_{t+1}^*)'Q(r_{t+1}^*)x_{t+1}^* + u_t'R(r^*_t)u_t] \tag{27}$$

where $Q(r_{t+1}^*) = \Gamma(r_{t+1}^*)\Gamma'(r_{t+1}^*)$

$R(r_t^*) = D(r_t^*)D'(r_t^*)$

and $K_T(r_N^*) = E\{A(r_N^*)x_0x_0'A'(r_N^*)\}$ -*-

Combining these results, we obtain the optimal compensator when the time horizon becomes infinite:

Theorem 5: For time invariant case of JLQG problem (1)-(6), if

(i) $\{A(r_k), B(r_k)\}$ and $\{A(r_k),[\Gamma(r_k)\Gamma'(r_k)]^{1/2}\}$ are absolutely controllable; and

(ii) $\{C(r_k),A(r_k)\}$ and $\{[Q(r_k)]^{1/2}, A(r_k)\}$ are observable;

then as the time horizon $N \to \infty$, we have the following solution:

$$u_k = -L(i)\hat{x}_k \qquad (i \in M) \tag{28}$$

where L(i) are given by the following sets of M coupled matrix equations:

$$L(i) = [R(i) + B'(i)\hat{K}(i)B(i)]^{-1}B'(i)\hat{K}(i)A(i) \tag{29}$$

and

$$K(i) = A'(i)\hat{K}(i)[A(i)-B'(i)L(i)] \tag{30}$$

where

$$\hat{K}(r_k=i) = \sum_{j=1}^{M} p_{ij}[K(r_{k+1}=j) + Q(r_{k+1}=j)] \tag{31}$$

These $K(i) > 0$ are finite and unique. The x-state estimate $\hat{x}_k$ is given by the following Kalman Filter:

$$\hat{x}_k = \hat{x}_{k|k-1} + S_k[y_k - C(r_k)\hat{x}_{k|k-1}] \tag{32}$$

$$S_k = P_{k|k-1}C'(r_k)[C(r_k)P_{k|k-1}C'(r_k) + D(r_k)D'(r_k)]^{-1} \tag{33}$$

$$P_k = [I - S_kC(r_k)]P_{k|k-1} \tag{34}$$

$$\hat{x}_{k|k-1} = A(r_{k-1})\hat{x}_{k-1} + B(r_{k-1})u_{k-1} \tag{35}$$

$$P_{k|k-1} = A(r_{k-1})P_{k-1}A'(r_{k-1}) + \Gamma(r_{k-1})\Gamma'(r_{k-1}) \tag{36}$$

$$\text{with} \quad \hat{x}_{-1} = 0, \quad P_{0|-1} = \alpha I \tag{37}$$

where α is any chosen positive number. The controlled system is Lagrange stable in the sense that $E\{x_k'x_k\} < \infty$ as $k \to \infty$. -*-

The proof of this theorem can be found in [5]. Note that as time advances, the filter quantities P_k and S_k depend less and less on the choice of initial conditions $\hat{x}_{-1}$ and $P_{0|-1}$. For this reason, $\hat{x}_{-1}$ and $P_{0|-1}$ are specified as in (37). If too small an α value is chosen, convergence will be slow and possibly numerical difficulties may arise in computation. In computational examples, $\alpha \geq 100$ works well.

4. Suboptimal steady-state solution

In this section we consider the use of a time invariant (but therefore suboptimal) filter. This allows us to have a completely precomputed compensator (although filter and controller gains still depend on observed r_k values). The main idea is to use a Bayesian estimate, $\hat{P}_{k|k-1}(r_k)$, to replace the extrapolation error covariance matrix, $P_{k|k-1}$. Then the filter gain S_k will only depend on r_k.

Suboptimal JLQ filter:

$$\hat{x}_k = \hat{x}_{k|k-1} + S_k(r_k)[y_k - C(r_k)\hat{x}_{k|k-1}] \tag{38}$$

$$S_k(r_k) = \hat{P}_{k|k-1}(r_k)C'(r_k)[C(r_k)\hat{P}_{k|k-1}(r_k)C'(r_k) + D(r_k)D'(r_k)]^{-1} \tag{39}$$

$$P_k(r_k) = [I - S_k(r_k)C(r_k)]\hat{P}_{k|k-1}(r_k) \tag{40}$$

$$\hat{x}_{k|k-1} = A(r_{k-1})\hat{x}_{k-1} + B(r_{k-1})u_{k-1} \tag{41}$$

$$\hat{P}_{k|k-1}(r_k=i) = \sum_{j=1}^{M} p_{ij}^*[A(j)P_{k-1}(j)A'(j) + \Gamma(j)\Gamma'(j)] \tag{42}$$

$$\text{with} \quad \hat{x}_{-1} = 0, \quad P_{0|-1} = \alpha I \tag{43}$$

Here p_{ij}^* in (42) is the **posterior** form transition probability, which can be obtained from the transition probabilities by using Bayes' theorem:

$$p^*(r_{k-1}=j, r_k=i) = p_{ij}^* = p(j,i)/(\sum_{e=1}^{M} p(e,i)) \tag{43}$$

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Combining this suboptimal filter with the JLQ optimal controller, we have the following result:

Theorem 6: (suboptimal steady-state compensator [5])

For the time invariant case of (1)-(7), consider the suboptimal compensator (the combination of JLQ controller with the filter (38-43)), as $N \to \infty$. If

(i) $\{A(r_k),B(r_k)\}$ and $\{A(r_k), [\Gamma(r_k)\Gamma'(r_k)]^{1/2}\}$ are absolutely controllable;

(ii) $\{C(r_k), A(r_k)\}$ is absolutely observable and $\{(Q(r_k))^{1/2}, A(r_k)\}$ is observable for each $i \in M$;

then the filter gains and covariance matrices will converge as $k \to \infty$. These constant matrices can be solved from the following coupled matrix equations:

$$S(i) = \hat{P}(i)C'(i)[C(i)\hat{P}(i)C'(i) + D(i)D'(i)]^{-1} \quad (46)$$

and

$$P(i) = [I - S(i)C(i)]\hat{P}(i) \quad (47)$$

where

$$\hat{P}(i) = \sum_{j=1}^{M} p^*_{ij}[A(j)P(j)A'(j) + \Gamma(j)\Gamma'(j)] \quad (48)$$

The controller will converge to the constant (in each form) version given by (28-30) in Theorem 5. -*-

5. Example

Consider a two form jump linear system (1)-(5) and (6') with form transition probability matrix:

$$\underline{P} = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}$$

Thus by (43) $\underline{P}^* = \begin{bmatrix} 5/8 & 3/8 \\ 5/12 & 7/12 \end{bmatrix}$

The system parameters are

$$A(1) = \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix},$$

$$B(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\Gamma(1) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad \Gamma(2) = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix},$$

$$C(1) = [0 \ 1], \quad C(2) = [1 \ 1],$$

$$D(1) = 0.1, \quad D(2) = 0.15,$$

$$R(1) = 5, \quad R(2) = 1$$

$$Q(1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q(2) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

Here $\{A(r_k),B(r_k)\}$ is absolutely controllable by the **Lemma**, because

(i) on all the sample paths, we can find C_J with rank 2;
(ii) for each x_0 and r_0, we can find a casual control sequence (of length at most 3) to drive x_T ($T < \infty$) to the origin:

if $r_k = 1$, to drive any x_{k+2} to the origin, we take $u_k = [6 \ \ 12]x_k$, and then we use $u_{k+1} = [-16 \ \ -33]x_k$ if $r_{k+1} = 1$, but $u_{k+1} = [2 \ \ -5]x_k$, if $r_{k+1} = 2$.

This fact leads to the following control sequences:

- for $r_0 = 1$, the above control sequence will drive x_2 to the origin.
- for $r_0 = 2$, we apply the deadbeat control which will make $x_2 = 0$ if $r_1 = 2$. If $r_1 = 1$, then the $r_k = 1$ control sequence above drives x_3 to the origin.

We can factor $\Gamma(1)\Gamma'(1)$ as

$$\Gamma(1)\Gamma'(1) = \begin{bmatrix} 0.04 & 0.02 \\ 0.02 & 0.01 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.1 & 0.1\sqrt{3} \\ 0.05 & 0.05\sqrt{3} \end{bmatrix}}_{\tilde{\Gamma}(1)} \underbrace{\begin{bmatrix} 0.1 & 0.1\sqrt{3} \\ 0.05 & 0.05\sqrt{3} \end{bmatrix}}_{\tilde{\Gamma}'(1)}$$

Since $\tilde{\Gamma}(1)$ is of full rank, this equivalent system (with two inputs) can drive its state to any given point in one step. The same thing can be done for form 2. So $\{A(r_k), [\Gamma(r_k)\Gamma'(r_k)]^{1/2}\}$ is also absolutely controllable.

It is straightforward to show that $\{C(r_k),A(r_k)\}$ is observable, and that $([Q(r_k)]^{1/2},A(r_k))$ is absolutely observable.

Thus by Theorem 6, JLQ controller and suboptimal filter will converge to their corresponding steady-state versions as time horizon tends to infinity. The steady-state suboptimal compensator solution is as follows:

(a) Suboptimal Steady-state (Bayes) filter gains:

$$S(1) = \begin{bmatrix} 0.8429851 \\ 0.9681149 \end{bmatrix}, \quad S(2) = \begin{bmatrix} 0.4804170 \\ 0.4983117 \end{bmatrix}$$

(b) Steady-state JLQ controller gains:

$$L(1) = [\ 3.944999 \quad 5.821748\ ]$$

$$L(2) = [\ 5.308100 \quad 3.647641\ ]$$

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