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A Detector-Based Approach for the Constrained Quadratic Control of Discrete-Time Markovian Jump Linear Systems

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***Abstract*—This paper considers the quadratic control problem of discrete-time Markov jump linear systems (MJLS) with constraints on the norm of the state and control variables. We assume that the Markov chain parameter is not available and, instead, there is a detector which emits signals providing information on this parameter. It is desired to derive a feedback linear control using the information provided by this detector in order to stochastically stabilize the closed loop system, satisfy the constraints whenever the initial conditions belong to an invariant set, and minimize an upper-bound for the quadratic cost. We show that an LMI (linear matrix inequalities) optimization problem can be formulated in order to obtain a solution for this problem. Two other related problems, one for minimizing the guaranteed quadratic cost considering fixed initial conditions, and the other for maximizing an estimate of the domain of an invariant set, can also be formulated using our LMI approach. The paper is concluded with some numerical simulations.**

***Index Terms*—Stochastic Optimal Control, Markov Processes, Constrained Control, LMIs, Hybrid Systems.**

I. Introduction

Markovian jump linear systems (MJLS) have been receiving lately a great deal of attention since that this class of systems can represent models that are subject to sudden changes in their dynamic behavior. As an example of these changes we can mention abrupt environmental disturbances, component failures or repairs, changes in subsystems interconnections, abrupt variations in the operation point for a non-linear plant, etc. This large number of applications lead to a great interest on this field and several results, regarding applications, stability conditions and optimal control problems, can be found in the current literature (see, for instance, [1] for economic systems, [2], [3] for aircraft control systems, [4] for control of solar thermal central receivers, [5] for robotic manipulator systems, and [6], [7] for active fault-tolerant control systems (AFTCS)). We can mention the books [8], [9], [10], [11], [12], [13] and references therein for a sample of works on the subject and applications in AFTCS.

In this paper we study the constrained quadratic state feedback control problem of a discrete-time MJLS. The unconstrained case has been studied by several authors in the literature (e.g. see the books [8], [9], [11], [12]). For the constrained case we can mention the papers [14], [15], which considers constraints in the first two moments of the processes, [16], which considers an extension of MJLS, where the subsystems are allowed to be nonlinear, and applies notions of invariance and stability for such systems, [17], which follows a Model Predictive Control approach (MPC) and considers polytopic uncertainties both in the system matrices and in the transition probabilities between modes, [18], which considers a robust MPC problem for MJLS with polytopic constraints on the inputs and states, and [19], where LMI optimization problems are derived to obtain a state feedback controller to minimize an upper-bound for the quadratic cost and satisfying some constraints on the state and control variables. In all the above papers it is considered that the Markov parameter is available for the controller. However in many applications the controller does not have direct access to the Markov parameter but, instead, the information on the Markov chain is gleaned from an associated detector. This approach, which closely follows a Hidden Markov model approach, was adopted in [20] under the name of the detector approach and, basically, assumes that $(\theta(k), \hat{\theta}(k))$ is a hidden Markov model ([21]) in which the mode of operation of the system, represented by the hidden component $\theta(k)$ is not directly observed and only an estimation, represented by the observable component $\hat{\theta}(k)$ (given for instance by some failure detector), is available to the controller. As shown in [20], this formulation encompasses several situations regarding the availability of the Markov chain: the complete, cluster, and mode-independent cases. There is by now several works using this framework, such as the H_2 and H_∞ state-feedback control problems studied in [20] and [22], the mixed H_2/H_∞ state-feedback control problem in [23], the H_2 and H_∞ filtering problems tackled in [24] and [25], respectively, and for the continuous-time MJLS, the H_2 and H_∞ state-feedback control considered in [26] and [27], and the H_∞ filtering problem, in [28].

The goal of the present work is to trace a parallel with the results presented in [19], [29], [20], [30], [31] to derive LMI optimization problems capable of handling the quadratic control problem with restrictions on the state and control variables for MJLS, based only on the information coming from the detector $\hat{\theta}(k)$. We provide LMI optimization problems for minimizing the guaranteed quadratic cost and maximizing the estimate of the domain of an invariant set. To our knowledge, there is no other analytical or numerical way of handling this kind of problem with partial information on the jump parameter in the literature.

The paper is organized in the following way. Section II presents the notation that will be used throughout the work. Section III deals with the appropriate notions of stability and stabilizability for MJLS, as well as some auxiliary results. In Section IV we show that the problem of constrained quadratic control for MJLS with partial information on the jump parameter can be stated in terms of an LMI optimization problem, so that convex programming can be used for obtaining an approximation of the optimal solution. Section V presents some numerical simulations to illustrate the developed results. The paper is concluded in Section VI with some final comments.

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II. Notation

For $\mathbb{X}$ and $\mathbb{Y}$ complex Banach spaces, we set $\mathbb{B}(\mathbb{X},\mathbb{Y})$ the Banach space of all bounded linear operator of $\mathbb{X}$ into $\mathbb{Y}$. For simplicity we set $\mathbb{B}(\mathbb{X}) := \mathbb{B}(\mathbb{X},\mathbb{X})$. We denote by $\mathbb{R}^n$ the n-dimensional real space, and set $\mathbb{M}(\mathbb{R}^n,\mathbb{R}^m)$ the normed linear space of all m by n real matrices. Whenever $m = n$ we write $\mathbb{M}(\mathbb{R}^n,\mathbb{R}^n) = \mathbb{M}(\mathbb{R}^n)$ for simplicity. The superscript $'$ will indicate transpose. $L \geq 0$ and $L > 0$ will be used if a self-adjoint matrix is positive semi-definite or positive definite respectively and we write $\mathbb{M}(\mathbb{R}^n)^+ = \{L \in \mathbb{M}(\mathbb{R}^n); L = L' \geq 0\}$. We denote by $\|.\|$ either the induced norm in $\mathbb{M}(\mathbb{R}^n)$ or the standard norm in $\mathbb{R}^n$. We set $\mathrm{diag}\{Q_s\}$ as the matrix in $\mathbb{M}(\mathbb{R}^{Sn})$ formed by $Q_1,\ldots,Q_S$ in the diagonal, and zero elsewhere.

We define $\mathbb{H}^{m,n}$ the linear space made up of all N-sequence of matrices $V = (V_1,\ldots,V_N)$, $V_i \in \mathbb{M}(\mathbb{R}^m,\mathbb{R}^n)$, $i \in \mathbb{N}$. We set $\mathbb{H}^{n,n} = \mathbb{H}^n$ and $\mathbb{H}^{n+} = \{V = (V_1,\ldots,V_N) \in \mathbb{H}^n;\ V_i \in \mathbb{M}(\mathbb{R}^n)^+, i \in \mathbb{N}\}$. For $H = (H_1,\ldots,H_N)$ and $V = (V_1,\ldots,V_N)$ in $\mathbb{H}^{n+}$ the notation $H \leq L$ ($H < L$) indicates that $H_i \leq L_i$ ($H_i < L_i$) for each $i \in \mathbb{N}$.

On a probability space $(\Omega,\mathcal{P},\mathcal{F})$ with filtration $\{\mathcal{F}_k\}$ we define $E(.)$ as the expected value operator and ℓ_2^n as the Hilbert space formed by the sequence of second order random vectors $z = (z(0), z(1),\ldots)$ with $z(k) \in \mathbb{R}^n$ and $\mathcal{F}_k$-measurable for each $k = 0,1,\ldots$, and such that, $\|z\|_2^2 := \sum_{k=0}^{\infty} \|z(k)\|_2^2 < \infty$, where $\|z(k)\|_2^2 := E(\|z(k)\|^2)$.

We have the following results, stated as remarks, which will be useful in the sequel.

Remark 1: $W = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} > 0$ if and only if $R > 0$, $Q > SR^{-1}S'$. For non-strict inequalities this result can be generalized as follows: $W = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$ if and only if $R \geq 0$, $Q \geq SR^{\dagger}S'$, $S(I - RR^{\dagger}) = 0$, where $R^{\dagger}$ denotes the Moore-Penrose inverse of R (see [32]).

Remark 2: If $Q > 0$ then $U + U' - Q \leq U'Q^{-1}U$.

III. Stability Results

We consider in this paper the following controlled discrete-time linear system with Markov jumps on a probabilistic space $(\Omega,\mathcal{P},\mathcal{F})$:

$$x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k), \tag{1}$$
$$x(0) = x_0,\ \ \theta(0) = \theta_0. \tag{2}$$

Here the state variable is given by $x(k) \in \mathbb{R}^n$ and the control variable by $u(k) \in \mathbb{R}^m$. We consider that $\theta(k)$ is a Markov chain taking values in the set $\mathbb{N} = \{1,\ldots,N\}$ with transition probability matrix $\mathbf{P} = [p_{ij}]$.

We assume that $\theta(k)$ *is not directly observed* but, instead, there is a finite set $\mathbb{M} = \{1,\ldots M\}$ such that a signal $\widehat{\theta}(k) \in \mathbb{M}$ is emitted associated to the Markov chain $\theta(k)$, independently of all previous and present values of the other processes. More precisely, let $\widehat{\mathcal{F}}_0$ be the σ-field generated by $\{x(0), u(0), \theta(0)\}$ and $\widehat{\mathcal{F}}_k$ be the σ-field generated by $\{x(0), u(0), \theta(0), \widehat{\theta}(0),\ldots, x(k), u(k-1), \theta(k)\}$ (therefore excluding $\widehat{\theta}(k)$ at time k). We assume that $\widehat{\theta}(k) \in \{1,\ldots,M\}$ is related to $\theta(k)$ in such a way that

$$P(\widehat{\theta}(k) = \ell \mid \widehat{\mathcal{F}}_k) = P(\widehat{\theta}(k) = \ell \mid \theta(k)) = \alpha_{\theta(k)\ell},\ \ \ell \in \mathbb{M}, \tag{3}$$

with $\sum_{\ell=1}^{M} \alpha_{i\ell} = 1$ for each $i \in \mathbb{N}$. Therefore, we have that at each time k we observe the signal $\widehat{\theta}(k)$. We define for each $i \in \mathbb{N}$,

$$\mathcal{I}_i \doteq \{\ell \in \mathbb{M}; \alpha_{i\ell} > 0\} = \{k_1^i,\ldots,k_{\tau^i}^i\}$$

and we assume that $\cup_{i=1}^{N}\mathcal{I}_i = \mathbb{M}$. It will be convenient to define $\tau = \tau^1 + \ldots + \tau^N$. As pointed out in [20], we have 2 extreme situations:

a) $M = N$ and $\alpha_{ii} = 1$, for $i \in \mathbb{N}$, which would correspond to the situation in which $\widehat{\theta}(k) = \theta(k)$, that is, $\theta(k)$ is known. In this case $\mathcal{I}_i = \{i\}$ and $\mathbb{M} = \mathbb{N}$.
b) $M = N$ and $\left[\alpha_{i\ell}\right] = \frac{1}{M}$, for all $i \in \mathbb{N}$ and $\ell \in \mathbb{M}$, which corresponds to the situation in which $\widehat{\theta}(k)$ does not provide any information about $\theta(k)$, that is, $\theta(k)$ is totally unknown.

Remark 3: As pointed out in the introduction, there is a close relationship between the detector-based approach and hidden Markov processes, and thus algorithms for estimating the transition p_{ij} and detector $\alpha_{i\ell}$ parameters for Hidden Markov models could be used in our framework (see, for instance, [33]). However the estimation of the parameters p_{ij} and $\alpha_{i\ell}$, although of great interest, is a major problem on its own and falls outside the scope of this paper.

We will consider state-feedback controls using the observed emitted signal $\widehat{\theta}(k)$ instead of the unknown variable $\theta(k)$, that is, $u(k)$ will be of the following form:

$$u(k) = K_{\widehat{\theta}(k)}x(k), \tag{4}$$

for $K_\ell \in \mathbb{M}(\mathbb{R}^n,\mathbb{R}^m)$, $\ell \in \mathbb{M}$. Associated to a control as in (4) set for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$A_{i\ell} \doteq A_i + B_iK_\ell. \tag{5}$$

We define for each $i \in \mathbb{N}$ the following operators $\mathcal{E}$, $\mathcal{L}$ in $\mathbb{B}(\mathbb{H}^n)$. For $V = (V_1,\ldots,V_N) \in \mathbb{H}^n$, and $i,j \in \mathbb{N}$,

$$\mathcal{E}_i(V) = \sum_{j=1}^{N} p_{ij}V_j, \tag{6}$$
$$\mathcal{L}_i(V) = \sum_{\ell\in\mathcal{I}_i} \alpha_{i\ell}A_{i\ell}'\mathcal{E}_i(V)A_{i\ell}. \tag{7}$$

We recall the following definition of stochastic stabilizability.

Definition 1: We say that System (1) is stochastically stabilizable if there exists $K_\ell \in \mathbb{B}(\mathbb{R}^n,\mathbb{R}^m)$, $\ell \in \mathbb{M}$, such that for $u(k)$ as in (4) we have, for every initial condition x_0 with finite second moment and every initial Markov state θ_0, that

$$\|x\|_2^2 = \sum_{k=0}^{\infty} E(\|x(k)\|^2) < \infty. \tag{8}$$

We denote by $\mathcal{K}$ the set of feedback gains $K = \{K_\ell; \ell \in \mathbb{M}\}$, such that stochastically stabilizes System (1).

The following result presents conditions for stochastic stabilizability of System (1), the proof can be found in [20].

Theorem 1: The following assertions are equivalent:

i) System (1) is stochastically stabilizable.
ii) There exists $K_\ell \in \mathbb{B}(\mathbb{R}^n,\mathbb{R}^m)$, $\ell \in \mathbb{M}$ and $P \in \mathbb{H}^n$, $P > 0$, such that for $A_{i\ell}$ as in (5),

$$P - \mathcal{L}(P) > 0. \tag{9}$$

IV. Constrained Quadratic Control Problem

Consider again System (1) on a probabilistic space $(\Omega,\mathcal{P},\mathcal{F})$, the σ-field $\widehat{\mathcal{F}}_k$, $k = 0,1,\ldots$ as defined in Section III, and the output

$$z(k) = C_{\theta(k)}x(k) + D_{\theta(k)}u(k). \tag{10}$$

We recall that we will consider controllers as in (4). For the set of feedback gains $K = \{K_\ell; \ell \in \mathbb{M}\} \in \mathcal{K}$ define:

$$J(K) \doteq \|z\|_2^2 = \sum_{k=0}^{\infty} E(\|z(k)\|^2) \tag{11}$$
$$= \sum_{k=0}^{\infty} E(\|C_{\theta(k)}x(k) + D_{\theta(k)}u(k)\|^2), \tag{12}$$

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with $z = (z(0), \ldots)$ given by (10) when $u(k) = K_{\widehat{\theta}(k)} x(k)$. Given $\delta > 0$ we want to find $K \in \mathcal{K}$ and a set $\mathcal{D}_0 \subset \mathbb{R}^n \times \mathbb{N}$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that the constraints

$$\|F_\iota x(k) + G_\iota u(k)\| \leq \rho_\iota, \text{ for } k = 0, 1, \ldots, \ \iota = 1, \ldots, t. \quad (13)$$

are satisfied and $J(K) \leq \delta \|x_0\|^2$.

Notice that we have two main elements that we can play with, the upper-bound value δ that we would like to minimize, and the set $\mathcal{D}_0$, which we would like to make as large as possible. For the case without the constraints (13) and with $\mathcal{D}_0 = \mathbb{R}^n \times \mathbb{N}$ the problem of minimizing δ over $K \in \mathcal{K}$ such that $J(K) \leq \delta \|x_0\|^2$ was analyzed in [20] via an LMI optimization problem. In the sequel we will present another approach for this problem taking also into account the constraints (13) (see also Remark 4). The motivation for the restrictions (13) is that many systems are subject to constraints on the manipulated and controlled variables. We notice that (13) comprises the so-called norm bounds and componentwise peak bounds on the inputs $u(k)$, as well as on a system output $y(k) = Hx(k)$. For instance, the norm bounds and componentwise peak bounds on the inputs $u(k)$ (see, for instance, [30], [32]) are given respectively as $\|u(k)\| \leq u_{\max}$ and $|u_\iota(k)| \leq u_{\iota,\max}$ (where $u_\iota(k)$ represents the ι^{th} element of the vector $u(k)$) for $k = 0, 1, \ldots,$ and $1 \leq \iota \leq m$, and fixed positive upper-bound values $u_{\max}$ and $u_{\iota,\max}$. By taking $F_\iota = 0$ in (13) and $G_\iota = I$, $\rho_\iota = u_{\max}$ we recover the norm bounds constraints while by taking $F_\iota = 0$, $G_\iota = e'_\iota$ (where e_ι is the unitary vector formed by 1 at the ι position, 0 elsewhere), and $\rho_\iota = u_{\iota,\max}$, we obtain the componentwise peak bounds constraints. Similarly, norm bounds and constraint and componentwise peak bounds on $y(k)$, given respectively by $\|y(k)\| \leq y_{\max}$ and $|y_\iota(k)| \leq y_{\iota,\max}$ can be, as for the input case, written as in (13). In this sense (13) is more general since it allows constraints for a linear combination of the input and output. As pointed out in [30], constraints on the input are typically hard constraints, since they represent limitations on process equipment (such as valve saturations in industrial processes), and thus cannot be relaxed. On the other hand, constraints on the output are often associated to performance goals in which it is desired to keep the output $y(k)$ within some upper-bounds norm values $y_{\max}$ and/or peak bounds values $y_{\iota,\max}$. In Section V we present an economic example to illustrate the use of (13).

We present next an LMI optimization problem that aims at obtaining a $K \in \mathcal{K}$ which minimizes the upper-bound value δ at the same time that obtains an invariant set $\mathcal{D}_0$ such that whenever $(x_0, \theta_0) \in \mathcal{D}_0$ we have that $(x(k), \theta(k)) \in \mathcal{D}_0$ for all $k = 0, 1, \ldots$ and the constraints (13) are satisfied. Other versions of this problem in which considers the initial condition (x_0, θ_0) fixed or that fixes $\delta > 0$ and aims at finding the largest inner ball inside an invariant set $\mathcal{D}_0$ will be presented in Corollaries 1 and 2. To define the LMI optimization problem, set for $i \in \mathbb{N}$, $\mathbf{\Gamma}_i = [p_{i1}^{1/2} \mathbf{I}_1 \ldots p_{iN}^{1/2} \mathbf{I}_N] \in \mathbb{M}(\mathbb{R}^n, \mathbb{R}^{\tau n})$, where $\mathbf{I}_i$ is an $n \times \tau^i n$ matrix formed by τ^i identity matrices of dimension n, and

$$\operatorname{diag}\{R_{s\zeta}\} \doteq \operatorname{diag}\{R_{1k_1^1}, \ldots, R_{1k_{\tau 1}^1}, \ldots, R_{Nk_1^N}, \ldots, R_{Nk_{\tau N}^N}\},$$

a block-diagonal matrix of dimension $n\tau$ and, for fixed $i \in \mathbb{N}$,

$$\operatorname{diag}\{R_{i\zeta}\} \doteq \operatorname{diag}\{R_{ik_1^i}, \ldots, R_{ik_{\tau i}^i}\},$$

a block-diagonal matrix of dimension $n\tau_i$. Notice that $\operatorname{diag}\{R_{s\zeta}\} = \operatorname{diag}\{\operatorname{diag}\{R_{1\zeta}\}, \ldots, \operatorname{diag}\{R_{N\zeta}\}\}$. We define the following problem:

Problem 1: Find $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, such that

$$\min \delta$$

subject to,

$$\begin{bmatrix} \delta I & \mathbf{I}_i \\ \bullet & \operatorname{diag}\{R_{i\zeta}\} \end{bmatrix} \geq 0, \text{for } i \in \mathbb{N}, \quad (14)$$

$$\begin{bmatrix} U'_\ell + U_\ell - \alpha_{i\ell} R_{i\ell} & (U'_\ell A'_i + Y'_\ell B'_i)\mathbf{\Gamma}_i & [U'_\ell C'_i \quad Y'_\ell D'_i] \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \quad (15)$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$\begin{bmatrix} U'_\ell + U_\ell - Q_i & (U'_\ell A'_i + Y'_\ell B'_i) \\ \bullet & Q_j \end{bmatrix} > 0, \quad (16)$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, j such that $p_{ij} > 0$, and

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota U_\ell + G_\iota Y_\ell \\ \bullet & U'_\ell + U_\ell - Q_i \end{bmatrix} > 0, \quad (17)$$

for $\iota = 1, \ldots, t$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$.

Next we define the invariant set that we will consider in the paper. For $P = (P_1, \ldots, P_N) > 0$ define the function $P(x, i) = x' P_i x$, $i \in \mathbb{N}$, and, for $\gamma > 0$,

$$L_P(\gamma) := \Big\{(x, i) \in \mathbb{R}^n \times \mathbb{N}; x' P_i x \leq \frac{1}{\gamma}\Big\}. \quad (18)$$

The next theorem shows that if there is a solution to the LMI optimization problem (Problem 1) posed above then we can get a stochastically stabilizing controller $K = \{K_\ell; \ell \in \mathbb{M}\} \in \mathcal{K}$ such that the quadratic cost $J(K)$ is upper-bounded by δ and there is an invariant set $L_P(1)$ such that whenever the initial conditions $(x_0, \theta_0) \in L_P(1)$ we have that $(x(k), \theta(k)) \in L_P(1)$ for all $k = 0, 1, \ldots,$ and the constraints (13) are satisfied.

Theorem 2: Suppose there is a solution $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, for Problem 1. Define $K_\ell = Y_l U_\ell^{-1}$, $\ell \in \mathbb{M}$ and $P(x, i) = x' P_i x$, $P_i = Q_i^{-1}$, $i \in \mathbb{N}$. Then the following assertions hold:

i) $K \in \mathcal{K}$,
ii) $J(K) \leq \delta \|x_0\|^2$.

If $(x_0, \theta_0) \in L_P(1)$ then

iii) $(x(k), \theta(k)) \in L_P(1)$ for all $k = 0, 1, \ldots,$
iv) the constraints (13) are satisfied.

Proof: First of all notice that from Remark 2 and (15) we get that

$$\begin{bmatrix} U'_\ell (\alpha_{i\ell} R_{i\ell})^{-1} U_\ell & (U'_\ell A'_i + Y'_\ell B'_i)\mathbf{\Gamma}_i & [U'_\ell C'_i \quad Y'_\ell D'_i] \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \quad (19)$$

so that by pre and post multiplying (19) by $\operatorname{diag}\{(U'_\ell)^{-1}, I, I\}$ and its transpose, it yields to:

$$\begin{bmatrix} (\alpha_{i\ell} R_{i\ell})^{-1} & (A'_i + K'_\ell B'_i)\mathbf{\Gamma}_i & [C'_i \quad K'_\ell D'_i] \\ \bullet & \operatorname{diag}\{R_{s\zeta}\} & 0 \\ \bullet & \bullet & I \end{bmatrix} > 0, \quad (20)$$

and from Remark 1 we get that (20) is equivalent to

$$R_{i\ell}^{-1} > \alpha_{i\ell} \Big\{ (A_i + B_i K_\ell)' \left(\sum_{j=1}^N p_{ij} \Big(\sum_{\zeta \in \mathcal{I}_j} R_{j\zeta}^{-1} \Big) \right) (A_i + B_i K_\ell) + (C_i + D_i K_\ell)'(C_i + D_i K_\ell) \Big\} \quad (21)$$

for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$. Set $V_i = \sum_{\zeta \in \mathcal{I}_i} R_{i\zeta}^{-1}$, $i \in \mathbb{N}$, $V = (V_1, \ldots, V_N)$. From (21) we have that

$$V_i > \sum_{\zeta \in \mathcal{I}_i} \alpha_{i\zeta} \Big\{ (A_i + B_i K_\zeta)' \mathcal{E}_i(V)(A_i + B_i K_\zeta) + (C_i + D_i K_\zeta)'(C_i + D_i K_\zeta) \Big\} \quad (22)$$

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and thus (22) implies that $V - \mathcal{L}(V) > 0$, so that from Theorem 1 we get that $K \in \mathcal{K}$, showing i). Let us now show ii). Following the same steps as in the proof of Proposition 4 in [20] we get from (22) that

$$\begin{aligned}\left\|V_{\theta(k)}^{1/2}x(k)\right\|_2^2 &= E(x(k)'V_{\theta(k)}x(k))\\ &> E(x(k+1)'V_{\theta(k+1)}x(k+1)) + \|z(k)\|_2^2\\ &= \left\|V_{\theta(k+1)}^{1/2}x(k+1)\right\|_2^2 + \|z(k)\|_2^2 . \end{aligned} \tag{23}$$

Summing up (23) from $k = 0$ to ∞, and recalling that $K \in \mathcal{K}$ so that, from the stochastic stability of (1), (4), $\left\|V_{\theta(k)}^{1/2}x(k)\right\|_2^2 \to 0$ as $k \to \infty$, we obtain that

$$J(K) = \|z\|_2^2 = \sum_{k=0}^{\infty} \|z(k)\|_2^2 \leq E(x_0'V_{\theta_0}x_0) \leq \delta\|x_0\|_2^2 \tag{24}$$

where the last inequality follows from (14) since that from Remark 1 we derive that (14) is equivalent to

$$\delta I \geq \sum_{\zeta\in\mathcal{I}_i} R_{i\zeta}^{-1} = V_i. \tag{25}$$

Let us now show iii). From (16) and Remark 2 we have that

$$\begin{bmatrix} U_\ell'Q_i^{-1}U_\ell & (U_\ell'A_i' + Y_\ell'B_i') \\ \bullet & Q_j \end{bmatrix} > 0, \tag{26}$$

so that by pre and post multiplying (16) by $\mathrm{diag}\{(U_\ell')^{-1}, I\}$ and its transpose, we get that

$$\begin{bmatrix} Q_i^{-1} & (A_i' + K_\ell'B_i') \\ \bullet & Q_j \end{bmatrix} > 0. \tag{27}$$

From Remark 1 and setting $P_s = Q_s^{-1}$, we get that (27) yields to

$$P_i > (A_i + B_iK_\ell)'P_j(A_i + B_iK_\ell), \text{ for } p_{ij} > 0. \tag{28}$$

Hence, from (1) and (4), $x(k+1) = (A_{\theta(k)} + B_{\theta(k)}K_{\widehat{\theta}(k)})x(k)$, and from (28) we get that

$$x(k)'P_{\theta(k)}x(k) > x(k+1)'P_{\theta(k+1)}x(k+1). \tag{29}$$

and thus (29) yields to

$$x_0'P_{\theta_0}x_0 > x(k)'P_{\theta(k)}x(k) > x(k+1)'P_{\theta(k+1)}x(k+1). \tag{30}$$

From (30) we have that if $(x_0, \theta_0) \in L_P(1)$ (that is, $x_0'P_{\theta_0}x_0 \leq 1$) then $(x(k), \theta(k)) \in L_P(1)$ for every $k = 0, 1, \ldots$ (since from (30) $x(k)'P_{\theta(k)}x(k) \leq 1$), completing the proof of iii). Let us now show iv). From (17) and Remark 2 again we get that

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota U_\ell + G_\iota Y_\ell \\ \bullet & U_\ell'Q_i^{-1}U_\ell \end{bmatrix} > 0. \tag{31}$$

Pre and post multiplying (31) by $\mathrm{diag}\{I, (U_\ell')^{-1}\}$ and its transpose, we have that

$$\begin{bmatrix} \rho_\iota^2 I & F_\iota + G_\iota K_\ell \\ \bullet & Q_i^{-1} \end{bmatrix} > 0. \tag{32}$$

From (32) and Remark 1 we derive that

$$\rho_\iota^2 I > (F_\iota + G_\iota K_\ell)Q_i(F_\iota + G_\iota K_\ell)', \tag{33}$$

so that, from (33), we conclude that $\|(F_\iota + G_\iota K_\ell)Q_i^{1/2}\|^2 \leq \rho_\iota^2$ for all $\ell \in \mathcal{I}_i$, $i \in \mathbb{N}$. Thus we get that

$$\begin{aligned}&\|F_\iota x(k) + G_\iota u(k)\|^2 = \|(F_\iota + G_\iota K_{\widehat{\theta}(k)})x(k)\|^2\\ &= \|(F_\iota + G_\iota K_{\widehat{\theta}(k)})Q_{\theta(k)}^{1/2}Q_{\theta(k)}^{-1/2}x(k)\|^2\\ &\leq \|(F_\iota + G_\iota K_{\widehat{\theta}(k)})Q_{\theta(k)}^{1/2}\|^2\|Q_{\theta(k)}^{-1/2}x(k)\|^2\\ &\leq \rho_\iota^2\ x(k)'Q_{\theta(k)}^{-1}x(k) = \rho_\iota^2\ x(k)'P_{\theta(k)}x(k) \leq \rho_\iota^2\end{aligned} \tag{34}$$

since $x(k)'P_{\theta(k)}x(k) \leq 1$, completing the proof. □

Remark 4: From the proof of Theorem 2, items i) and ii), we notice that to get the stochastically stabilizing controller $K = \{K_\ell; \ell \in \mathbb{M}\} \in \mathcal{K}$ such that the quadratic cost $J(K)$ is upper-bounded by δ we only needed the LMIs (14) and (15). Therefore for the case without the constraints (13) the results of Theorem 2 show that the solution of Problem 1 without the LMIs (16), (17) provides a guaranteed upper bound for the quadratic control problem, based on the observability gramian associated to the operator $\mathcal{L}$. A similar result was obtained in Theorem 3 of [20], but considering the controllability gramian equations. In this sense the problem of minimizing δ under the LMIs (14) and (15) (related to the observability gramian equations) derived in this paper can be seen as the dual of the LMI optimization problem considered in Theorem 3 of [20] (associated to the controllability gramian equations). Furthermore for the full-observation case (and without constraints), that is, $M = N$ and $\alpha_{ii} = 1$, for $i \in \mathbb{N}$ (case a) in Section III which corresponds to the situation in which $\widehat{\theta}(k) = \theta(k)$, that is, $\theta(k)$ is known and $\mathcal{I}_i = \{i\}$, $\mathbb{M} = \mathbb{N}$) we could take in (14) $U_i = R_{ii}$ so that from Remark 1 the LMI (14) would be equivalent to $P_i > (A_i + B_iK_i)'(\sum_{i\in\mathbb{N}} p_{ij}P_j)(A_i + B_iK_i) + (C_i + D_iK_i)'(C_i + D_iK_i)$, with $P_i = R_{ii}^{-1}$, $K_i = Y_iR_{ii}^{-1}$, which corresponds to the observability gramian of the closed loop system. As seen in Theorem 4.10 in [9] in this case the result of Theorem 2 items i) and ii) retrieves the H_2 optimal control solution associated to the coupled algebraic Riccati equations associated to the problem (see Chapter 4 in [9]).

We conclude this section with 2 alternative problems associated to Problem 1, referred to as Problem 2 and Problem 3. First notice that in Theorem 2 the initial condition x_0 and the initial probability for θ_0 are not fixed and it is only assumed that $(x_0, \theta_0) \in L_P(1)$. Suppose now that the initial condition x_0 and the initial probability for θ_0 are fixed, with $\mu_i = \mathcal{P}(\theta_0 = i)$ for $i \in \mathbb{N}$. In this case Problem 2 can be defined, taking into account these specific initial conditions, as follows:

Problem 2: Find $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, such that

$$\min \delta$$

subject to,

$$\begin{bmatrix} \delta & x_0'\begin{bmatrix}\mu_1^{1/2}\mathbf{I}_1 & \cdots & \mu_N^{1/2}\mathbf{I}_N\end{bmatrix} \\ \bullet & \mathrm{diag}\{R_{s\zeta}\} \end{bmatrix} \geq 0, \tag{35}$$

$$\begin{bmatrix} 1 & x_0' \\ \bullet & Q_i \end{bmatrix} \geq 0, \text{for } i \in \mathbb{N} \text{ with } \mu_i > 0, \tag{36}$$

and the LMIs (15), (16), (17). We have the following corollary:

Corollary 1: If there is a solution $\delta > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, for Problem 2 then, by defining $K_\ell = Y_l U_\ell^{-1}$ for $\ell \in \mathbb{M}$ and $P(x, i) = x'P_ix$, $P_i = Q_i^{-1}$, $i \in \mathbb{N}$ we have that i), ii), iii), iv) of Theorem 2 are satisfied.

Proof: Using the same notation as in the proof of Theorem 2 we have from (35) and Remark 1 that

$$\begin{aligned}\delta &\geq x_0'\Big(\sum_{i=1}^{N}\mu_i(\sum_{\zeta\in\mathcal{I}_\rangle} R_{i\zeta}^{-1})\Big)x_0\\ &= x_0'\Big(\sum_{i=1}^{N}\mu_iV_i\Big)x_0 = E(x_0'V_{\theta_0}x_0)\end{aligned} \tag{37}$$

so that, from (24), we obtain that $J(K) \leq \delta$. From (36) it follows that $1 \geq x_0'Q_i^{-1}x_0 = x_0'P_ix_0$ for any $i \in \mathbb{N}$ with $\mu_i > 0$, and thus $(x_0, \theta_0) \in L_P(1)$. From (30) and the fact that $(x_0, \theta_0) \in L_P(1)$ we get that $(x(k), \theta(k)) \in L_P(1)$ for every $k = 0, 1, \ldots$, and the reaming of the proof of Theorem 2 can be applied. □

As seen in Theorem 2, by solving Problem 1 we minimize the upper-bound δ of the quadratic cost $J(K)$ and derive the invariant set $L_P(1)$ such that the constraints (13) will be satisfied whenever the initial conditions (x_0, θ_0) are in $L_P(1)$ (which implies that $(x(k), \theta(k)) \in L_P(1)$ for all $k = 0, 1, \ldots$). Another approach would be, for $\delta > 0$ fixed, consider an objective function that enlarges the invariant set $L_P(1)$ at the same time that ensures that the quadratic cost $J(K)$ is upper-bounded by δ. A possible way to do this would be, for instance, to consider a problem that aims at getting the largest inner ball (with radius $\frac{1}{\upsilon}$) $\mathcal{D}_\upsilon \doteq \{x_0 \in \mathbb{R}^n; \|x_0\|^2 \leq \frac{1}{\upsilon}\}$ included in the set $\widehat{L}_P(1) := \{x \in \mathbb{R}^n; (x, i) \in L_P(1) \text{ for some } i \in \mathbb{N}\}$, in other words, to obtain the minimum $\upsilon > 0$ such that $\mathcal{D}_\upsilon \subseteq \widehat{L}_P(1)$. This problem would be in general too hard to be solved, so that we need to consider a simplified convex version of this problem. Notice that, by restricting our choice for υ as $\upsilon = \max_{i\in\mathbb{N}} \|P_i\|$, where $P_i = Q_i^{-1}$, then we have that $\mathcal{D}_\upsilon \times \mathbb{N} \subset L_P(1)$ since that, if $x_0 \in \mathcal{D}_\upsilon$ then for any $i \in \mathbb{N}$ we have that

$$x_0' P_i x_0 \leq \|x_0\|^2 \|P_i\| \leq \|x_0\|^2 \upsilon \leq 1.$$

Thus, by minimizing $\max_{i\in\mathbb{N}} \|P_i\|$ we get the largest inner ball as defined in $\mathcal{D}_\upsilon$ with $\upsilon = \max_{i\in\mathbb{N}} \|P_i\|$, included in the set $\widehat{L}_P(1)$. Having this in mind, for $\delta > 0$ fixed, we re-write Problem 1 as follows:

Problem 3: Find $\upsilon > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, such that

$$\min\ \upsilon$$

subject to,

$$\begin{bmatrix} \upsilon I & I \\ \bullet & Q_j \end{bmatrix} \geq 0, \tag{38}$$

and the LMIs (14), (15), (16), (17). We have the following corollary:

Corollary 2: If for $\delta > 0$ fixed there is a solution $\upsilon^* > 0$, $Q = (Q_1, \ldots, Q_N) > 0$, $R_{i\zeta} > 0$, $i \in \mathbb{N}$, $\zeta \in \mathcal{I}_i$, Y_ℓ, U_ℓ, $\ell \in \mathbb{M}$, for Problem 3 then, by defining $K_\ell = Y_l U_\ell^{-1}$ for $\ell \in \mathbb{M}$ and $P(x, i) = x' P_i x$, $P_i = Q_i^{-1}$, $i \in \mathbb{N}$ we have that i), ii), iii), iv) of Theorem 2, and $\mathcal{D}_{\upsilon^*} \times \mathbb{N} \subset L_P(1)$ with $\upsilon^* = \max_{i\in\mathbb{N}} \|Q_i^{-1}\|$, are satisfied.

Proof: Using the same notation as in the proof of Theorem 2 we have from (38) and Remark 1 that $\upsilon I \geq Q_j^{-1}$, so that $\upsilon \geq \max_{j\in\mathbb{N}} \|Q_j^{-1}\|$. Since we want to minimize υ the optimal solution υ^* will be such that $\upsilon^* = \max_{j\in\mathbb{N}} \|Q_j^{-1}\|$. The remaining of the proof follows from the proof of Theorem 2. □

V. Numerical Simulations

For the simulations it is considered a simple economic system based on the Samuelson's multiplier–accelerator model (see [34]). According to this model, the national income at time k, denoted by $Y(k)$, may be written as the sum of three components: consumption, $C(k)$, induced private investment, $I(k)$, and governmental expenditure, $G(k)$, so that $Y(k) = C(k) + I(k) + G(k)$. Moreover the terms $C(k)$ and $I(k)$ are related to the national income $Y(k)$ through the equations $C(k) = (1 - s)Y(k - 1)$, and $I(k) = w(Y(k-1) - Y(k-2))$, where s represents the marginal propensity to save and w the accelerator coefficient (see [34], [35], [36]). We end up with the following difference equation:

$$Y(k) = (1-s)Y(k-1) + w(Y(k-1) - Y(k-2)) + G(k). \tag{39}$$

Considering a target governmental expenditure $\bar{G}$ we have that the fixed point of (39) is given by $\bar{Y} = \frac{\bar{G}}{s}$. Setting $\widetilde{Y}(k) = Y(k) - \bar{Y}$, $\widetilde{G}(k) = G(k) - \bar{G}$, we have that (39) can be re-written as

$$\widetilde{Y}(k) = (1-s)\widetilde{Y}(k-1) + w(\widetilde{Y}(k-1) - \widetilde{Y}(k-2)) + \widetilde{G}(k). \tag{40}$$

A state-space version of (40) is

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -w & 1 - s + w \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \tag{41}$$

where $x_2(k) = \widetilde{Y}(k)$, $x_1(k) = \widetilde{Y}(k-1)$, and $u(k) = \widetilde{G}(k+1)$, and it is desirable that the expected value of $E(x_2(k))$ converges to zero as k goes to infinity.

In [35] it was presented a MJLS version of (41) by considering a Markov chain $\theta(k) \in \mathbb{N} = \{1, 2, 3\}$ and three possible values for the parameters s_i and w_i representing the possible states of the economy (see [37] for more details); $i = 1$ for the "Norm" state, $i = 2$ for the "Boom" state, and $i = 3$ for the "Slump" state. Here we adopted the following values for these parameters: $w_1 = 1.5$, $s_1 = 0.5$; $w_2 = 2.5$, $s_2 = 0.4$; and $w_3 = 1$, $s_3 = 0.6$. Since the classification of these states rely on some economical indexes that may take some time to be evaluated, it would be possible to have some mismatch between the estimated value (represented by $\widehat{\theta}(k)$) and the real state of the economy (represented by $\theta(k)$).

Thus from (41) considering the MJLS, the matrices for our example are:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1.5 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -2.5 & 3.1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 1.4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1.5477 & -1.0976 \\ -1.0976 & 1.9145 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1.7025 & -1.2074 \\ -1.2074 & 2.1060 \\ 0 & 0 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1.3929 & -0.9878 \\ -0.9878 & 1.7231 \\ 0 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 \\ 0 \\ 1.6125 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0 \\ 1.7738 \end{bmatrix}, D_3 \begin{bmatrix} 0 \\ 0 \\ 1.4513 \end{bmatrix},$$

where we have adopted the same cost weighting matrices C_i and D_i as in [9]. The transition matrix that relates the system operation modes is given by (as used in [35]):

$$\mathbf{P} = \begin{bmatrix} 0.67 & 0.17 & 0.16 \\ 0.30 & 0.47 & 0.23 \\ 0.26 & 0.10 & 0.64 \end{bmatrix}.$$

The initial condition x_0 and the initial state θ_0 are assumed to be known (Corollary 1) with the following values:

$$x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}', \quad \theta_0 = 1.$$

To limit great variations on the amount invested in the economy, it is considered a constraint related to the private investment (that depends on $x_2(k) - x_1(k)$) and the government expenditure $u(k)$ defined by the following matrices: $F = c_I \begin{bmatrix} -1 & 1 \end{bmatrix}$, $c_I = 2$ $G = 1$ and $\rho = 0.2$, so that the constraint is $\|2(x_2(k) - x_1(k)) + u(k)\| \leq 0.2$. We consider three cases as below:

(a) Algorithm developed in [19], with perfect information on $\theta(k)$.
(b) Ideal detector, i.e., the detector provides perfect information (equivalent to Case (a) above, and also discussed in Section III).
(c) Detector provides information according to the probability matrix

$$[\alpha_{i\ell}] = \begin{bmatrix} 0.8 & 0 & 0.2 \\ 0.15 & 0.85 & 0 \\ 0 & 0.25 & 0.75 \end{bmatrix}.$$

Table I presents the obtained results for the three cases (upper-bounds, total cost and controllers). Notice that Case (a) attained a larger values for the upperbound δ and cost $J(K)$ than the equivalent

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Case (b). Hence, for this example, the algorithm developed in the present work is an improvement with respect to the one introduced in [19]. We can also conclude that, comparing Case (b) and Case (c), a more reliable information (regarding the parameters $\alpha_{i\ell}$) yields to a lower value for δ and total cost $J(K)$.

The mean value of the state (second component), obtained out of 1000 Monte Carlo simulations, is shown in Figure 1. It can be seen that Case (a) obtained the best performance in terms of driving the expected value of the state to zero but with a total cost $J(k)$ larger than the one obtained from Case (b), and Case (c), as expected, attained the slowest stabilization behavior. Figure 2 presents the extreme values of all the realizations of $Fx(k) + Gu(k)$ for the unconstrained (dash line) and constrained (solid line) algorithms. We can observe that the extreme realizations for the constrained algorithm are bounded by the pre-fixed limit $\rho_\iota = 0.2$ (constant star line) in all the cases (in comparison with the unconstrained case) but these values are distant from the limit constraint line, from which we can infer that the obtained controller is conservative.

Case	δ	$J(K)$	K_ℓ
(a)	5.9547	5.0408	$K_1 = \begin{bmatrix}1.4929 & -1.4089\end{bmatrix}$ $K_2 = \begin{bmatrix}1.5958 & -1.5137\end{bmatrix}$ $K_3 = \begin{bmatrix}1.4442 & -1.3845\end{bmatrix}$
(b)	4.6133	4.5526	$K_1 = \begin{bmatrix}1.6537 & -1.5257\end{bmatrix}$ $K_2 = \begin{bmatrix}2.1303 & -1.9797\end{bmatrix}$ $K_3 = \begin{bmatrix}1.5670 & -1.4422\end{bmatrix}$
(c)	5.6768	5.0996	$K_1 = \begin{bmatrix}1.8253 & -1.6355\end{bmatrix}$ $K_2 = \begin{bmatrix}1.7771 & -1.5891\end{bmatrix}$ $K_3 = \begin{bmatrix}1.6274 & -1.4742\end{bmatrix}$

Table I
PERFORMANCE PARAMETERS FOR DIFFERENT CASES.

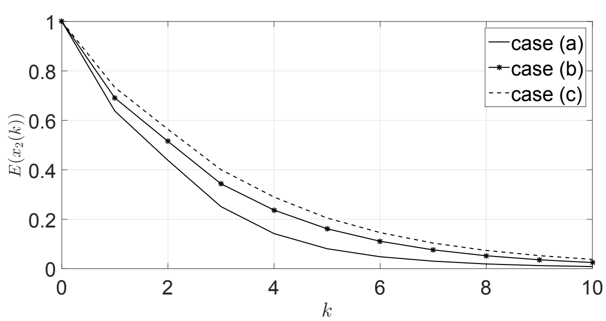

Figure 1. Mean value of the state for different cases.

The problem for maximizing the invariant set derived from Corollary 2 was also implemented using the parameters of Case (c) (set $\rho = 0.4$) and fixing δ attained by computing the algorithm from Theorem 2 ($\delta = 104.4839$). For this problem, it was obtained the optimal value $\upsilon^* = 1.6032$ and the same solution $Q = (Q_1, \ldots, Q_N) > 0$ and K_ℓ as achieved from Theorem 2 ($K_1 = \begin{bmatrix}1.8832 & -1.7640\end{bmatrix}$, $K_2 = \begin{bmatrix}1.7349 & -1.6055\end{bmatrix}$, $K_3 = \begin{bmatrix}1.7146 & -1.6076\end{bmatrix}$). Thus in this case there is no gain in increasing the invariant set $L_P(1)$. But if we increase δ by 10%, that is, by fixing $\delta = 114.9323$, it is obtained a lower value for υ^* ($\upsilon^* = 1.2624$) and, as expected, we get a bigger invariant region $\widehat{L}_P(1)$ for this case (recall that the radius of the inner ball inside $\widehat{L}_P(1)$ is given by $\frac{1}{\upsilon^*}$).

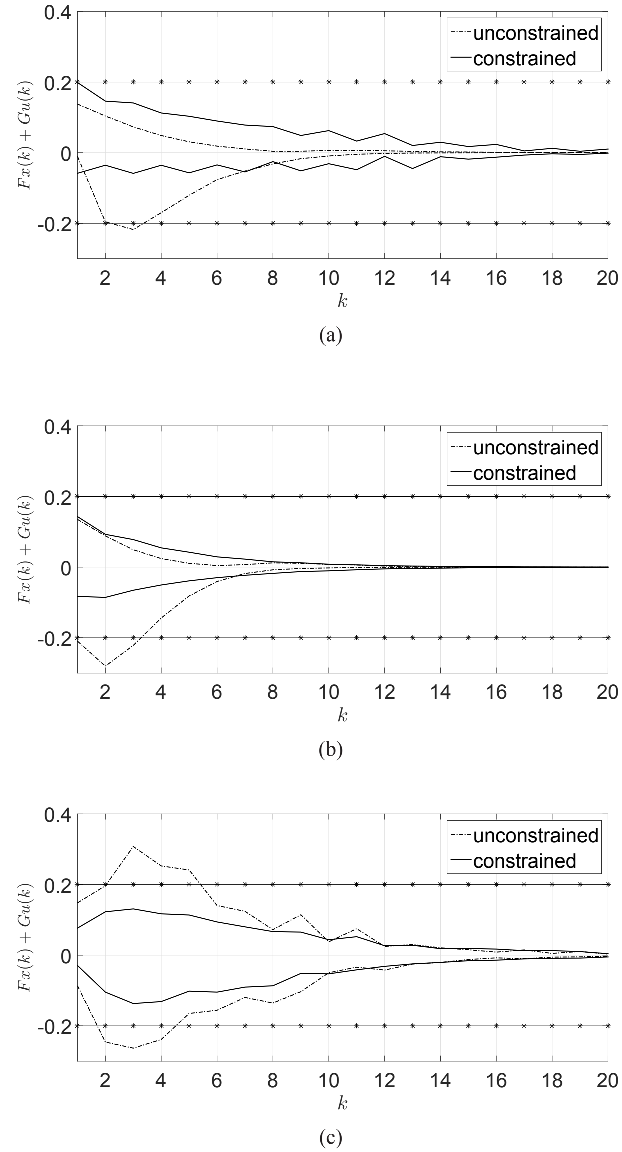

Figure 2. Extreme values of realizations of $Fx(k) + Gu(k)$ for the Cases (a), (b) and (c).

VI. CONCLUSION

In this work it was considered a quadratic control problem of discrete time MJLS with constraints on the norm of the state and control variables. A feedback linear control is derived using the information provided by the detector, so that the closed loop system is stochastically stabilized, an upper-bound δ for the quadratic cost is minimized, and the constraints are satisfied provided that the initial conditions are inside an invariant set. We also show that 2 other problems can be formulated by LMI optimization problems under the proposed framework; one in which δ is fixed and it is desired to maximize the estimate of the domain of an invariant set, and the other to minimize the guaranteed quadratic cost for fixed initial conditions. This work is concluded with numerical simulations in which it was observed that the algorithm developed in this paper, for the considered example, presented better results than the one introduced in [19]. Moreover, as expected, it was noticed that a more reliable detector yields to a lower the value of δ and the respective total cost $J(K)$. A possible continuation of the present research is to consider probabilistic constraints as, for instance, analyzed in [38], for MJLS under the detector-based approach.

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