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Controllability, observability and discrete-time markovian jump linear quadratic control

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Controllability, observability and discrete-time markovian jump linear quadratic control

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This paper is concerned with the controllability and observability of discrete-time linear systems that possess randomly jumping parameters described by finite-state Markov processes, and the relationship between these properties and the solution of the infinite time jump linear quadratic (JLQ) optimal control problem. The solution of the markovian JLQ problem with finite or infinite time horizons is known. Necessary and sufficient conditions for the existence of optimal constant control laws that lead to finite optimal expected costs as the time horizon becomes infinite are also known. Sufficient conditions for these steady-state control laws to stabilize the controlled system are also available (Chizeck *et al.* 1986). These conditions are not easy to test, however. Various definitions of controllability and observability for stochastic systems exist in the literature. These definitions are unfortunately not related to the steady-state JLQ control problem in a manner that is analogous to the role of deterministic controllability and observability in the linear quadratic optimal control problem. In this paper, new and refined definitions of the controllability and observability of jump linear systems are developed. These conditions have relatively simple algebraic tests. More importantly, these controllability and observability conditions can be used to determine the existence of finite steady-state JLQ solutions.

1. Problem formulation and basic definitions

We consider jump linear systems described by

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k \tag{1}$$

$$y_k = C(r_k)x_k \tag{2}$$

where $x \in R^n$ is the x-process state, $u \in R^m$ is the x-process input, $y \in R^p$ is the x-process output, and $A(r_k)$, $B(r_k)$, $C(r_k)$ are appropriately dimensioned matrices. Here $k \in Z^+$ is the time index and r_k is the *form process* which takes values in the finite set $\mathbf{M} = \{1, 2, \ldots, M\}$. It is a finite state discrete-time Markov chain, with transition probability matrix

$$\Pr\{r_{k+1} = j \mid r_k = i\} = \{p_{ij}\} \in R^{M \times M} \tag{3}$$

where $i, j = 1, 2, \ldots, M$.

The optimal control of this system with a quadratic cost was first studied by Blair and Sworder (1975). Chizeck (1982) considered the steady-state solution of this problem. But these results have not been related to properties of controllability and observability of this system. This will be done here.

Since this system is stochastic (due to the random form process), definitions of controllability and observability for deterministic systems do not directly apply.

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Appropriate concepts of stochastic controllability and observability are needed for discrete-time jump linear systems. These are derived below.

First consider the set of admissible control laws

$$U:[k_0, T]\times R^n \times \mathbf{M} \to R^m$$

Here admissibility means the control law must be causal; that is

$$u_k = u_k(x_s, r_s, s \leqslant k)$$

and we assume that for r_k can be perfectly observed at time k.

Let $x(k, r_0, x_0, u)$ denote the x-state at time k, starting from the initial state (x_0, r_0) at initial time k_0, under the admissible control law $u_k = u_k(x_s, r_s, s \leqslant k)$. We introduce the following definitions.

Definition 1: *Weak controllability and absolute controllability*

Consider the jump linear system (1), (3). For every initial state (x_0, r_0) and any choice of target value $\mathbf{x} \in R^n$, if there exists an admissible control law u such that the random time

$$T_{cw} = \inf_{u\in U}\{k > 0: \Pr(x(k, x_0, r_0, u) = \mathbf{x}) > 0\}$$

has finite expectation, $E\{T_{cw}\} < \infty$, then the system is *weakly controllable*. If time

$$T_{ca} = \inf_{u\in U}\{k > 0: x(k, x_0, r_0, u) = \mathbf{x}\}$$

is finite then the system is *absolutely controllable*.

This definition of weak controllability is in the spirit of that given in Zabczyk (1981) and Ehrhardt and Kliemann (1982) for continuous-time linear systems with stochastic disturbances.

We can define concepts of observability for discrete-time jump linear systems in a similar manner. Let $y_k(x_0 = x_{\#^1})$ denote the output obtained at time k from system (1)–(3) having $x_0 = x_{\#^1}$.

Definition 2: *Weak observability and absolute observability*

Consider the jump linear system (1)–(3). For any initial form r_0, and any two initial x-states, $x_{\#^1}$ and $x_{\#^2}$, let T_{ow} be the minimum time such that if the outputs $y_k(x_0 = x_{\#^1}) = y_k(x_0 = x_{\#^2})$ for all k between k_0 and T_{ow}, and the inputs u_k are known in this time interval, then $\Pr(x_{\#^1} = x_{\#^2}) > 0$. We say that this system is *weakly observable* if $E\{T_{ow}\} < \infty$.

Similarly, let T_{oa} be the minimum time such that equivalent outputs $y_k(x_0 = x_{\#^1}) = y_k(x_0 = x_{\#^2})$ and known inputs in the interval $k_0 \leqslant k \leqslant T_{oa}$ imply that $x_{\#^1} = x_{\#^2}$ with certainty. We say that this system is *absolutely observable* if this time T_{oa} is finite.

Necessary and sufficient conditions for these controllability and observability conditions to hold for a discrete-time jump linear system with markovian forms are given by algebraic tests, which are presented in § 2. In § 3, we discuss the relationships among controllability, controllability to the origin, observability and constructibility properties using these definitions. We show that the controllability properties are invariant under state feedback, and the observability properties are invariant under linear observers. In § 4, these controllability and observability properties are related to the existence of steady-state finite-cost solutions to the markovian JLQ problem which stabilize the controlled system as the time horizon becomes infinite. Absolute controllability and observability are shown to play a role in the optimal control of jump linear systems similiar to that of deterministic controllability and observability in linear quadratic control problems.

For notational simplicity, in the following discussion we will indicate the form value by subscripts. The countably infinite space of all the possible sample paths is indicated by $\Omega(w)$. The terminology used below regarding discrete-time Markov processes is listed in Appendix A.

3. Algebraic tests of controllability and observability of jump linear systems

In this section we develop algebraic tests for weak and absolute controllability and observability. The first theorem states that the weak controllability of a jump linear system depends only on the closed communicating classes.

Theorem 1

A discrete-time jump linear system (1), (3) is weakly controllable if and only if the system is weakly controllable for every form in any of the closed communicating classes of **M**.

Proof

Necessity is obvious. To prove sufficiency, note that after a random time T_1, the form r will enter $C^{\#}$, one of the closed communicating classes. Let $r_{T_1} \in C^{\#}$ denote the first form in $C^{\#}$ that is entered. Let $x_{T_1} \in R^n$ be the corresponding x-state. Since we assume that the system is weakly controllable when the form is in any communicating class, there exists a random time $T \geqslant T_1$ with $E\{T - T_1\} < \infty$ and an admissible control law $u(k, r_k)$ for $k = T_1, T_1 + 1, \ldots, T$ such that

$$\Pr(x(T, x_0, r_0, u) = \mathbf{x}) > 0 \tag{4}$$

Since (4) is valid for arbitrary x_{T_1}, we can select the values of u before time T_1 arbitrarily to obtain a control law $u(k, r_k)$ for $k = k_0, k_0 + 1, \ldots, T_1, T_1 + 1, \ldots, T - 1$, such that

$$\Pr(x(T, x_0, r_0, u) = \mathbf{x}) > 0$$

Now $E\{T\} = E\{T_1 + (T - T_1)\} = E\{T_1\} + E\{T - T_1\}$ and the second term is known to be finite. Lemma 1 below (established by direct computation), is used to show that $E\{T_1\}$ is finite.

Lemma 1

Consider a Markov chain with $M = 2$ forms and $p_{12} \neq 0$. Then the expected time until form 2 is entered is finite.

To establish that $E\{T_1\} < \infty$, divide **M** into two sets; set $1'$ is all the transient states, and set $2'$ consists of all states in the closed communicating classes. We can then aggregate the Markov chain **M** into a two-state process, by combining the probabilities of staying in set $1'$ into $p_{1'1'}$, with $p_{2'1'} = 0$ and $p_{2'2'} = 1$. After making the necessary time-scale transformation, this two-state form process is markovian. By Lemma 1, we know that r' will enter form $2'$ in finite expected time. Thus $E\{T_1\} < \infty$, which completes the proof of sufficiency of Theorem 1.

The next theorem provides an algebraic test for weak controllability of closed communicating classes.

Theorem 2

Closed communicating class $C^\#$ of a discrete-time jump linear system (1), (3) is weakly controllable if and only if for some $r_{i0} \in C^\#$, there exists a possible transition sequence $i_0, i_1, \ldots, i_{T-1}$, with $T < \infty$, such that the *jump controllability matrix* C_J has

$$\operatorname{rank} C_J = \operatorname{rank}\left[B_{iT-1} : A_{iT-1}B_{iT-2} : \ldots : \prod_{j=1}^{T-1} A_{ij}B_0\right] = n \tag{5}$$

This result is proved in Appendix B.

If there is only one state in a closed communicating class (an absorbing state), then the condition of the above theorem is equivalent to the deterministic controllability. Theorems 1 and 2 can be used to establish the weak controllability of a discrete-time jump linear system as shown in the following example.

Example 1

Consider a system as in (1) with form transition matrix

$$P = (p_{ij}) = \begin{bmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 \\ 0 & p_{22} & 0 & 0 & 0 \\ 0 & 0 & p_{33} & p_{34} & p_{35} \\ 0 & 0 & p_{43} & p_{44} & p_{45} \\ 0 & 0 & 0 & p_{54} & p_{55} \end{bmatrix}$$

with

$$A_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad A_1 = A_3 = I, \quad A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_1 = B_3 = 0, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

There are two closed communicating classes, $\{2\}$ and $\{3, 4, 5\}$. Form 2 is an absorbing state. Thus if $r = 2$, the condition for weak controllability is the same as in the deterministic case. This system is weakly controllable when $r_0 \in \{2\}$, since

$$\operatorname{rank}\,[B_2 : A_2 B_2 : A_2^2 B_2] = \operatorname{rank} \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & 7 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 5 \end{bmatrix} = 3$$

Now for $r_0 \in \{3, 4, 5\}$, we have

$$\operatorname{rank}\,[B_5 : A_5 B_4] = \operatorname{rank} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 3$$

Thus for any r_0 in this closed communicating class $\{3, 4, 5\}$, we will have a finite length transition sequence $\{r_0, \ldots, 4, 5\}$, beginning in r_0, with the last two stages at (4, 5), such that

$$\operatorname{rank}\,[B_5 : A_5 B_4 : \ldots : A_5 A_4 \ldots B_{r0}] = 3$$

According to Theorem 2, the closed communicating class $\{3, 4, 5\}$ is weakly controllable. Since both closed communicating classes $\{2\}$ and $\{3, 4, 5\}$ are weakly controllable, by Theorem 1 this system is weakly controllable.

The previous example illustrates that controllability in each form is *not necessary* for weak controllability of a closed communicating class. The next example shows that deterministic controllability of each form of a closed communicating class is *not sufficient* for weak controllability.

Example 2

Consider system (1) with the two-state form structure. Let $p_{11} = p_{22} = 0$ and $p_{21} = p_{12} = 1$, and

$$A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In each form ($r = 1, 2$) the system is deterministically controllable. But for this flip-flop system, there exist only two kinds of transition sequences: $\{\ldots, 2, 1, 2, 1, 2, 1\}$ or $\{\ldots, 1, 2, 1, 2, 1, 2\}$. For any finite transition sequences ending in $r = 1$

$$\operatorname{rank}\,[B_1 : A_1 B_2 : A_1 A_2 B_1 : \ldots] = 1 < 2$$

If the transition sequence ends at $r = 2$, then

$$\operatorname{rank}\,[B_2 : A_2 B_1 : A_2 A_1 B_2 : \ldots] = 1 < 2$$

Thus by Theorems 1 and 2, this system is not weakly controllable.

Unfortunately, Theorem 2 is usually difficult to use in deciding whether a system is not weakly controllable, because the rank of C_J for all possible transition sequences must be checked. A sufficient condition that is easier to use follows immediately from Theorems 1 and 2.

Corollary 1

A sufficient condition for a discrete-time Markov jump linear system (1), (3) to be weakly controllable is that in every closed communicating class of the form process, there exists at least one form i with $p_{ii} > 0$ that is (deterministically) controllable.

Next we establish necessary and sufficient conditions for absolute controllability.

Theorem 3

System (1), (3) is absolutely controllable if and only if:

(i) for every sample path of form transitions ($\forall w \in \Omega$), there exists a finite time $T_{ca}(w) < \infty$ such that the first $T = T_{ca}(w)$ forms $\{i_0, i_1, \ldots, i_{T-1}\}$ in the sample path yield a controllability matrix C_J with

$$\operatorname{rank} C_J = \operatorname{rank}\left[B_{iT-1} : A_{iT-1} B_{iT-2} : \ldots : \prod_{j=1}^{T-1} A_{ij} B_{i0} \right] = n$$

(ii) For each x_0, r_0 and $\mathbf{x}$, the control sequence

$$U = [C_J' C_J]^{-1} C_J' \left\{ \mathbf{x} - \prod_{j=0}^{T-1} A_{ij} x_0 \right\}$$

where $U = [u_0' \quad \ldots \quad u_{T-2}' \quad u_{T-1}']'$ which accomplishes this transition is causal. That is, u_k is determined by $\{(r_t, x_t)$ for $t = k_0, \ldots, k\}$.

Proof

Necessity can be proved by contradiction. First, suppose that there is a sample path $i_0, \ldots, i_{T-1}$ on which (i) fails. That is, for every finite T we have

$$\operatorname{rank} C_J = \operatorname{rank}\left[B_{iT-1} : A_{iT-1} B_{iT-2} : \ldots : \prod_{j=1}^{T-1} A_{ij} B_{i0} \right] < n$$

By the proof of Theorem 2, on this sample path there exists no control law that can drive x-state to $\mathbf{x}$ in a finite time. This contradicts absolute controllability. Next suppose that (i) holds, but only for a noncausal control law. Then admissibility is violated, which contradicts the definition of absolute controllability.

Sufficiency. Given (i), for any $\mathbf{x}$ and initial state (x_0, r_0), we have

$$\mathbf{x} = x_T = \prod_{j=0}^{T-1} A_{ij} x_0 + \prod_{j=1}^{T-1} A_{ij} B_{i0} u_0 + \ldots + A_{iT-1} B_{iT-2} u_{T-2} + B_{iT-1}$$

Thus

$$C_J U = \mathbf{x} - \prod_{j=0}^{T-1} A_{ij} x_0$$

where

$$C_J = \left[\prod_{j=1}^{T-1} A_{ij} B_0 : \ldots : A_{iT-1} B_{iT-2} : B_{iT-1} \right]$$

and

$$U = [u_0' \ldots u_{T-2}' \quad u_{T-1}']'$$

Since rank $C_J = n$, we can choose $u_0, \ldots, u_{T-1}$ as

$$U = [C_J' C_J]^{-1} C_J' \left\{ \mathbf{x} - \prod_{j=0}^{T-1} A_{ij} x_0 \right\}$$

With (ii), the causal condition ensures admissibility even though the values $\{r_{k+s}, s > 0\}$ are unknown at time k. So the system is absolutely controllable. □

The following example demonstrates the importance of the second condition in this theorem.

Example 3

Consider system (1) with $M = 2$ and non-zero transition probabilities. Let

$$A_1 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then condition (i) is satisfied since

$$\operatorname{rank}[B_1 : A_1 B_1] = \operatorname{rank}[B_2 : A_2 B_1] = \operatorname{rank}[B_2 : A_2 B_2] = \operatorname{rank}[B_1 : A_1 B_2] = 2$$

However, this does not mean we can find an admissible control law to drive any (x_0, r_0) to any given $\mathbf{x}$. For example, if $r_0 = 1$ and $\mathbf{x} = 0$, then for $r_1 = 1$, we have

$$\begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = -[B_1 : A_1 B_1]^{-1} A_1^2 x_0 \quad \text{hence } u_0 = -[1 \quad 1] x_0$$

But if $r_1 = 2$, we have

$$\begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = -[B_2 : A_2 B_1]^{-1} A_2 A_1 x_0 \quad \text{hence } u_0 = -[0 \quad 2] x_0$$

Note that the control law is non-causal—we have to choose u_0 based on the value of r_1, which is not available at time $k = k_0$. Thus this control law is not admissible and the system is not absolutely controllable.

A simpler necessary condition that is sometimes useful in disproving absolute controllability is given by the following corollary. The proof (by contradiction) is immediate.

Corollary 2

If the discrete-time jump linear system (1), (3) is absolutely controllable, then the pair (A_i, B_i) must be deterministically controllable for each form i with $p_{ii} > 0$.

Condition (i) of Theorem 3 is not sufficient for the absolute controllability except in certain special cases.

Corollary 3

(i) A scalar discrete-time Markov jump linear system (1), (3) is absolutely controllable if and only if both

(a) $b_i \neq 0$ if $p_{ii} > 0$, and

(b) within every communicating class, there is at least one form i with $b_i \neq 0$.

(ii) In general, system (1), (3) is absolutely controllable if Theorem 3 (i) holds, and on every infinite sample path, r_{k+s} is completely determined from r_k, for all $s \geqslant 0$. That is, after some finite time the form process is deterministic.

Note that condition (i) (b) of Corollary 3 refers to all the communicating classes, not only the closed communicating classes.

The following example is an absolutely controllable system that does not satisfy Corollary 3 (ii).

Example 4

Consider system (1), (3) that satisfies Theorem 3 (i) with the form structure of Fig. 1.

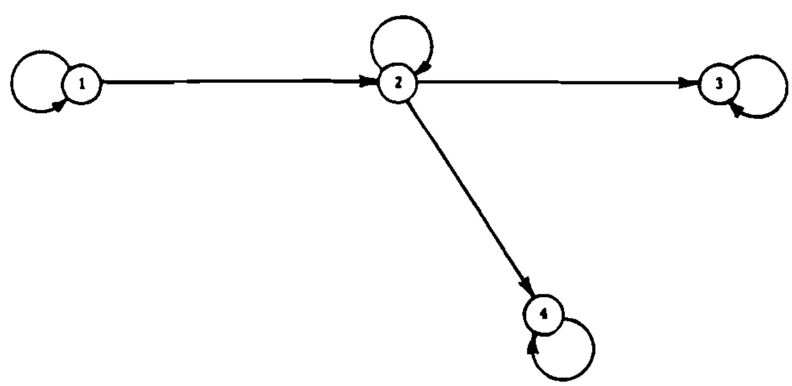

Figure 1. Form structure of Example 4.

We first demonstrate absolute controllability. This is done by showing that there exists a control strategy which can drive x-state to $\mathbf{x}$ from any (x_0, r_0) in at most $3n - 1$ steps. When $r_0 = 3$ or 4, use the deterministic control law. For $r_0 = 2$, use the control law that results from assuming that $r_k = 2$ for $k > 0$. If $r_k = 2$ for $k \leqslant n - 1$, the x-state will be driven to $\mathbf{x}$; if not, then r will enter form 3 or 4, where the control law can drive the x-state to $\mathbf{x}$ in at most n steps. If $r_0 = 1$, we use the control law that assumes $r_k = 1$ for $0 < k < n - 1$. If the system does enter 2 before the $(n - 1)$th step, the control law will achieve $\mathbf{x}$ in $2n$ time steps. That is, there exists a control strategy which can drive the x-state to $\mathbf{x}$ from any (x_0, r_0) in at most $3n - 1$ steps. Thus the system is absolutely controllable although Corollary 3(ii) does not hold.

The next example illustrates the use of Theorem 3 and Corollary 3 for a more complex example.

Example 5

Consider system (1), (3) with form structure as shown in Fig. 2, where

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To check the absolute controllability of this system, first consider form transition sequences ending in the closed communicating class $\{2, 4\}$. If the final form is $r = 2$, we have

$$\operatorname{rank}\,[B_2 : A_2 B_2] = 2$$

If the final form is 4, we have

$$\operatorname{rank}\,[B_4 : A_4 B_4] = 2$$

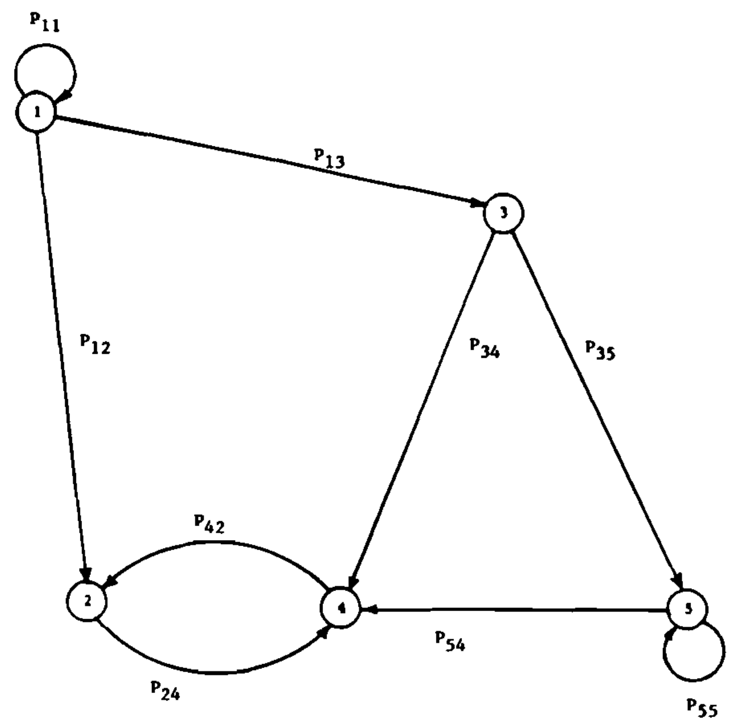

Figure 2. Form structure of Example 5.

The form process is deterministic inside this class. So by Corollary 3(ii), we can use a causal control law to achieve **x** in it.

For form transition sequences ending in form 1 or 5, the same arguments as in Example 4 will give a causal control law that achieves **x** in finite time. Since $p_{33} = 0$, there is no infinite form sequence ending in 3.

Thus by Theorem 3, the system is absolutely controllable.

Note that demonstrating absolute controllability is harder than disproving it. This was not the case for weak controllability.

The above results for controllability of Markov jump linear systems have analogous theorems and corollaries regarding observability. We state them here without proof.

Theorem 4

Discrete-time jump linear system (1)–(3) is weakly observable if and only if the system is weakly observable in every closed communicating class of $\mathbf{M}$.

Theorem 5

Discrete-time jump linear system (1)–(3) with the initial form $r_0 \in C^{\#}$ (one of the closed communicating classes of $\mathbf{M}$), is weakly observable if and only if there exists a sample path $\{i_0, i_1, \dots, i_{T-1}\}$ within $C^{\#}$ with $T < \infty$, such that the *jump observability matrix* 0_J has

$$\operatorname{rank} 0_J = \operatorname{rank} \begin{bmatrix} C_{i0} \\ C_{i1} A_{i0} \\ \vdots \\ C_{iT-1} \prod_{j=0}^{T-2} A_{ij} \end{bmatrix} = n$$

Corollary 4

A sufficient condition for discrete-time Markov jump linear system (1)–(3) to be weakly observable is that for each closed communicating class of the form process $\mathbf{M}$, there is at least one form i with $p_{ii} > 0$ and the pair (C_i, A_i) is observable.

Theorem 6

System (1), (3) is absolutely observable if and only if:

(i) for every sample path of form process, there exists a finite time $T_{oa}(w) < \infty$ such that the first $T = T_{oa}(w)$ forms $\{i_0, i_1, \dots, i_{T-1}\}$, yield a jump observability matrix 0_J with:

$$\operatorname{rank} 0_J = \operatorname{rank} \begin{bmatrix} C_{i0} \\ C_{i1} A_{i0} \\ \vdots \\ C_{iT-1} \prod_{j=0}^{T-2} A_{ij} \end{bmatrix} = n$$

(ii) For each r_0 and any given output sequence $\{y_0, y_1, \dots, y_{T-1}\}$ and corresponding input sequence $\{u_0, u_1, \dots, u_{T-1}\}$, the initial x-state x_0 can be determined uniquely (without the knowledge of the values of $r_1, r_2, \dots, r_{T-1}$).

Corollary 5

If the discrete-time Markov jump linear system (1)–(3) is absolutely observable, then the pair (C_i, A_i) must be deterministically observable for each form i with $p_{ii} > 0$.

Corollary 6

(i) A scalar system (1)–(3) is absolutely observable if and only if both

(*a*) $C_i \neq 0$ for each form i with $p_{ii} > 0$, and

(*b*) within every communicating class, there is at least one form i with $C_i \neq 0$.

(ii) In general, system (1)–(3) is absolutely observable if on every infinite sample path, Theorem 6 (i) holds and r_{k+s} is completely determined from r_k, for all $s \geqslant 0$. That is, after some finite time the form process is deterministic.

3. Duality and invariance properties

The definitions of weak and absolute controllability correspond in the deterministic case to controllability from the origin (reachability). For controllability to the origin (i.e. $\mathbf{x} = 0$ in Definition 1), the rank condition on the jump controllability matrix C_J is replaced by

$$\prod_{j=0}^{T-1} A_{ij} x_0 \subset R\left[B_{iT-1} : A_{iT-1} B_{iT-2} : \ldots : \prod_{j=1}^{T-1} A_{ij} B_{i0} \right]$$

in Theorems 2, 3 and Corollaries 1–3, where $R[\cdot]$ denotes the range space of the matrix. Similarly, the weak and absolute observability definitions correspond to observability of the state from future outputs in the deterministic case.

Consequently, weak controllability implies weak controllability to the origin, but weak controllability to the origin need not imply weak controllability. The previous sentence is also true if 'weak' is replaced by 'absolute'. Weak and absolute observability (from the future outputs) and constructibility (i.e. observability from the past outputs) have the similar relationships.

To set up the duality between the controllability and observability, consider the dual system of (1)–(2)

$$x^*_{k+1} = -A'(r_k)x^*_k + C'(r_k)u^*_k \tag{1)*}$$

$$y^*_k = B'(r_k)x^*_k \tag{2)*}$$

where $x^* \in R^n$ is the state vector, $u^* \in R^p$ is the input and $y^* \in R^m$ is the output vector. Here A', B' and C' denote the transpose of A, B and C respectively. In addition, we interchange the input and output constraints of (1), (2), and use inverse time in (1)*, (2)*.

In the following, we give two theorems about the basic properties of controllability and observability defined in this paper.

Theorem 7

Absolute controllability and weak controllability of a jump linear system are invariant under (known) form-dependent x-state feedback.

Proof

Suppose the known feedback law is $f(r_k, x_k)$. Consider the controllability for system (1) with the input $u_k = f_k(r_k, x_k) + v_k$:

$$x_{k+1} = A(r_k) + B(r_k)[f_k(r_k, x_k) + v_k]$$

This can be written as

$$\mathbf{x}_{k+1} = A(r_k)x_k + B(r_k)v_k \tag{*1}$$

$$x_{k+1} = \mathbf{x}_{k+1} + B(r_k)f_k(r_k, x_k) \tag{*2}$$

Since $f_k(r_k, x_k)$ is known at time k, so as long as we have $\mathbf{x}_{k+1}$, then x_{k+1} is known from (*2). Thus the controllability of system (1) is equivalent to the controllability of (*1). □

This proof is in the spirit of Chen (1984).

Under linear feedback this theorem means that if a jump linear system is absolutely controllable, then by using a linear form-dependent state feedback $u_k = K_k(i)x_k + v_k(i)$ we can change the dynamic behaviour of the closed-loop system. Here $K_k(r_k = i)$ is an $m \times n$ form-dependent (and possibly time-dependent) gain matrix. This is in the spirit of pole placement, but because jump systems are time varying, the concept of pole assignment is not directly applicable. We have an analogous result for observers.

Theorem 8

Weak observability and absolute observability of a jump linear system are invariant under the action of form-dependent linear x-state observers.

4. Steady-state markovian JLQ optimal controller

We now consider the time-invariant jump linear quadratic (JLQ) control problem. As in Chizeck *et al.* (1986), for system (1)–(3) with the assumption that complete form and x-state information is available within one time step, we seek to minimize

$$J_k(x_0, r_0) = E\left\{\sum_{k=k_0}^{N-1} [u_k' R(r_k)u_k + x_{k+1}' Q(r_{k+1})x_{k+1}] + x_N' K_T(r_N)x_N\right\} \tag{6}$$

over choice of $u_0, \ldots, u_{N-1}$. The matrices $R(j)$, $Q(j)$ and $K_T(j)$ are symmetric and positive-semidefinite for each j. In addition, we assume that

$$R(j) + B_j'\left[\sum_{j=1}^{M} P_{ij}Q(i)\right]B_j > 0 \quad \text{for each } j \in \mathbf{M} \tag{7}$$

Note that (7) is satisfied if $R(j) > 0$ and $Q(j) \geqslant 0$ for all $j \in \mathbf{M}$. For this discrete-time noiseless markovian-form jump linear quadratic optimal control problem, the optimal control law is given by:

$$u_k = -L_k(j)x_k \quad \text{for } r_k = j \in \mathbf{M}, \quad k = 0, 1, \ldots, N-1 \tag{8}$$

where for each possible form j the optimal gain is given by

$$L_k(j) = [R(j) + B_j'Q_{k+1}^*(j)B_j]^{-1} B_j'Q_{k+1}^*(j)A_j \tag{9}$$

$$Q_{k+1}^*(j) = \sum_{i=1}^{M} p_{ji}[Q(i) + K_{k+1}(i)] \tag{10}$$

Hence the sequence of sets of positive-semidefinite symmetric matrices $\{K_k(j) : j \in \mathbf{M}\}$ satisfies the set of M coupled matrix difference equations

$$K_k(j) = A_j'Q_{k+1}^*(j)[A_j - B_jL_k(j)] \tag{11}$$

with terminal conditions

$$K_N(j) = K_T(j)$$

The value of the optimal expected cost that is achieved with this control law is given by

$$x_0' K_{k0}(r_0) x_0$$

Note that the $\{K_k(j) : j \in \mathbf{M}\}$ and optimal gains $\{L_k(j) : j \in \mathbf{M}\}$ can be recursively computed off-line, using the M coupled difference equations (9)–(11).

For the steady-state problem, we wish to determine the feedback control law to minimize

$$\lim_{N-k_0 \to \infty} E\left\{\sum_{k=k_0}^{N-1} [u_k' R(r_k) u_k + x_{k+1}' Q(r_{k+1}) x_{k+1}] + x_N' K_T(r_N) x_N \mid x_0, r_0\right\} \tag{12}$$

In Chizeck *et al.* (1986), necessary and sufficient conditions for the existence of constant steady-state control laws yielding finite expected cost were obtained. In the following we relate this problem to absolute controllability and observability.

Theorem 9

For the discrete time-invariant markovian JLQ problem:

(i) If the system is absolutely controllable (or absolutely controllable to the origin), the solution of the set of coupled matrix difference equations (9)–(11) converges to a constant steady-state set

$$\{K(j) > 0 : j \in \mathbf{M}\}$$

as $N - k_0 \to \infty$. These $K(j)$ are given by the M coupled equations

$$K(j) = A_j' Q^*(j) D(j) \tag{13}$$

where

$$D(j) = \{I - B_j[R(j) + B_j' Q^*(j) B_j]^{-1} B_j' Q^*(j)\} A_j \tag{14}$$

and $Q^*(j)$ is defined in (10) with $K_k(i)$ replaced by $K(i)$:

$$Q^*(j) = \sum_{i=1}^{M} p_{ji}[Q(i) + K(i)] \tag{15}$$

The optimal closed-loop dynamics in each form are given by

$$x_{k+1} = D_k(r_k) x_k$$

Furthermore, the steady-state gains $L(j)$ in the optimal control law

$$u(r_k, x_k) = -L(r_k) x_k \tag{16}$$

are given by

$$L_j = [R(j) + B_j' Q^*(j) B_j]^{-1} B_j' Q^*(j) A_j \tag{17}$$

and the optimal infinite-time horizon expected cost is

$$V(x_0, r_0) = x_0' K(r_0) x_0$$

In addition, the expected cost-to-go from time k remains finite as $N - k \to \infty$.

(ii) If there exists a steady-state controller and the expected cost is finite, then absolute observability of the system with

$$C_j'C_j = Q(j)$$

implies the stability of the closed-loop system, in the sense that $E\{x_k'x_k\} \to 0$ as $k \to \infty$.

Proof

(i) Absolute controllability means that there exists a control law $\{u_0, \ldots, u_{T-1}\}$ (not necessarily constant for each form or optimal) which can drive the x-state to the origin in a finite time with a finite expected cost-to-go, although the future value of r is not known at time k. The optimal control law minimizes the expected cost-to-go, so the resulting optimal expected cost-to-go must be less than that obtained by using $\{u_0, \ldots, u_{T-1}\}$. Thus it must be finite. This establishes an upper bound on the optimal cost matrices $K_k(j)$ for the infinite time horizon problem. Since $K_k(j)$ are monotone-increasing as $N-k$ increases, they therefore converge. It follows from (11) that the limits

$$\lim_{N-k_0\to\infty} K_{k0}(j) = K(j)$$

satisfy (12). It is straightforward (as in the deterministic LQ case) to show that there is a unique set of positive-definite solutions of (12). Thus optimal control law is constant (in k) for each form.

(ii) Finiteness of the optimal expected cost does not imply that the optimal closed-loop system is stable, however. For example, an undetectable (i.e. unstable and unobservable) mode could result in $E\{x_k'x_k\} \to \infty$ but with finite expected cost. Absolute observability of (1)–(3) requires that for all the possible form sample paths we will have observability. That is, all the x modes will enter into the expected cost. Then finiteness of the expected optimal cost means $E\{x_N'x_N\} \to 0$ as $N-k \to \infty$, and the controlled system is stable. □

This theorem avoids the awkward need of finding constant (non-optimal) control laws in order to check for steady-state convergence, as in Proposition 2 of Chizeck *et al.* (1986). However, this theorem involves only sufficient conditions.

In the deterministic linear quadratic control problem, controllability of (A, B) and the observability of (C, A) ensure that K_k converges to a unique, constant, symmetric, positive semi-definite solution with asymptotically stable closed-loop dynamics. In the JLQ problem, absolute controllability and absolute observability play the same role. Note that in Theorem 9 the conclusions still hold if we replace absolute controllability by absolute controllability to the origin and absolute observability by absolute contructibility.

The following example illustrates the role of absolute controllability and observability in the JLQ problem.

Example 6

Consider the flip-flop system ($p_{12} = p_{21} = 1$, $p_{11} = p_{22} = 0$) with

$$A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R(1) = 1$$

$$A_2 = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R(2) = 1$$

Let $C_1'C_1 = Q(1)$ and $C_2'C_2 = Q(2)$, hence $C_1 = [Q(1)]^{1/2}$ and $C_2 = [Q(2)]^{1/2}$. Clearly this system is absolutely controllable and absolutely observable. However for each form this system is not controllable. By Theorem 9, the optimal cost-to-go will be finite and $E\{x'x\}$ tends to zero as $(N - k_0) \to \infty$. Then the finite-time solution to (9)–(11):

$$u_k(x_k, i) = -L_k(i)x_k \tag{18}$$

$$V_k(x_k, i) = x_k' K_k(i) x_k \tag{19}$$

converges as $(N - k) \to \infty$, to

$$L_\infty(1) = [0 \quad 1{\cdot}7913], \quad L_\infty(2) = [1{\cdot}7913 \quad 0]$$

$$K_\infty(1) = \begin{bmatrix} 0 & 0 \\ 0 & 7{\cdot}5823 \end{bmatrix}, \quad K_\infty(2) = \begin{bmatrix} 7{\cdot}5823 & 0 \\ 0 & 0 \end{bmatrix}$$

This example illustrates that deterministic controllability in each form is not necessary for finite JLQ solutions in infinite time. The next example demonstrates the difference between the role of weak and absolute controllability.

Example 7

Consider the system of Example 6, except that the form structure is given by $p_{11} = p_{12} = p_{21} = p_{22} = 0{\cdot}5$. This system is weakly controllable but not absolutely controllable. Here the expected cost increases without bound, because the constant form sample paths are uncontrollable.

In the next example we demonstrate the role of absolute observability in JLQ problem.

Example 8

Consider a jump linear system with the same form structure as Example 6 and dynamics

$$A_1 = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0{\cdot}5 \\ 1 \end{bmatrix}, \quad Q(1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad R(1) = 1$$

$$A_2 = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0{\cdot}5 \end{bmatrix}, \quad Q(2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R(2) = 1$$

Let $C_1'C_1 = Q(1)$ and $C_2'C_2 = Q(2)$, hence $C_1 = [0 \quad 1]$ and $C_2 = [1 \quad 0]$ in the JLQ control problem. This system is absolutely controllable to the origin but not absolutely observable. On the other hand, it is deterministically observable in each form. We can use this example to show that deterministic observability of every form is not sufficient for the steady-state Markov JLQ controller to stabilize the controlled system (in the sense that $E\{x_N'x_N\} = 0$ as $(N - k_0) \to \infty$).

Since this system is absolutely controllable to the origin, by Theorem 9 the optimal steady-state Markov JLQ controller exists and the resulting optimal expected cost-to-go will be finite as $(N-k_0)\to\infty$. Suppose we have $K_T(1)=K_T(2)=0$. The optimal constant JLQ controllers obtained by using (12)–(16), are

$$L_\infty(1)=L_\infty(2)=[0 \quad 0]$$

The infinite-time costs are

$$K_\infty(1)=K_\infty(2)=43{\cdot}4658\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

By (13) the closed-loop system matrices are:

$$D(1)=\begin{bmatrix}0 & 0\\ 3 & 3\end{bmatrix},\quad D(2)=\begin{bmatrix}3 & 3\\ 0 & 0\end{bmatrix}$$

To check the stability of the closed-loop system,

$$x_{k+1}=D(1)x_k,\quad \text{if } r_k=1$$

$$x_{k+1}=D(2)x_k,\quad \text{if } r_k=2$$

We can consider the expected cost with $Q(1)=Q(2)=I$, the identity matrix, and $K_T(1)=K_T(2)=0$ in (5). Direct computation gives

$$K_k(1)=3^{N-k}\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix},\quad K_k(2)=3^{N-k}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

which increases without bound as $N-k\to\infty$. Thus $E\{x'_N x_N\}=E\{x'_N I x_N\}$ will not tend to zero as $N-k_0\to\infty$; that is, the closed-loop system is not stable.

5. Conclusions

In this paper the definitions of weak and absolute controllability and observability for discrete-time jump linear system are introduced. Algebraic tests for these concepts are also given and the properties of duality and invariance are discussed. Finally, the results are related to the sufficient conditions of the existence of the optimal steady-state JLQ controller. It is shown that the concepts of absolute controllability and observability play the same role in JLQ problems as controllability and observability in deterministic linear quadratic problems.

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Appendix A

Terminology for finite-state Markov chains

A state is *transient* if a return to it is not guaranteed.

A state i is *recurrent* if an eventual return to i is guaranteed.

A state i is *accessible* from state j if it is possible to begin in j and arrive in i in some finite number of steps.

States i and j are said to *communicate* if each is accessible from the other; a *communicating class* is a class in which all the states are accessible from each other.

A communicating class is *closed* if there are no possible transitions from inside the class to any state outside it.

A *closed communicating class* containing only one member j is an *absorbing* state, that is $p_{jj} = 1$.

A Markov-chain state set can be divided into disjoint sets $\mathbf{T}$, $C^1, \ldots, C^s$, where all the states in $\mathbf{T}$ are transient and each C^i is a closed communicating class of recurrent states.

The cover C_j^* of a form $j \in \mathbf{M}$ is the set of all the forms accessible from j in one time step, i.e.

$$C_j^* = \{i \in \mathbf{M} : p(j, i) \neq 0\}$$

Appendix B

Proof of Theorem 2

Assume that (5) holds. For the corresponding form transition sequence $i_0, \ldots, i_{T-1}$ (inside $C^{\#}$), we have

$$\mathbf{x} = x_T = \prod_{j=0}^{T-1} A_{ij} x_0 + \prod_{j=1}^{T-1} A_{ij} B_{i0} u_0 + \ldots + A_{iT-1} B_{iT-2} u_{T-2} + B_{iT-1}$$

which implies that

$$\underbrace{\left[\prod_{j=1}^{T-1} A_{ij} B_{i0} : \ldots : A_{iT-1} B_{iT-2} : B_{iT-1}\right]}_{C_J} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_{T-2} \\ u_{T-1} \end{bmatrix}}_{U} = \mathbf{x} - \prod_{j=0}^{T-1} A_{ij} x_0$$

Since rank $C_J = n$, we can choose $u_0, \ldots, u_{T-1}$ as

$$U = [C'_J \quad C_J]^{-1} C'_J \left\{\mathbf{x} - \prod_{j=0}^{T-1} A_{ij} x_0\right\}$$

This control law drives the system to $x_T = \mathbf{x}$ with a probability greater than zero from x_0, since the form process will take this sample path with a non-zero probability. That is, we have weak controllability if we can show that this transition sequence can occur within a finite expected time $E\{T_{cw}\} < \infty$. Let $s = |C^{\#}|$ denote the number of states in closed communicating class $C^{\#}$. Take all the possible length T transition sequences as the form set of a new, larger jump process. This new form process r' takes values in a finite set. With an appropriate time-scale transformation, it is a Markov process, each stage of r' corresponds to T of the r process time steps. We can aggregate this into a two-state Markov process r'', with the states

- a'' is the sequence $i_0, i_1, \ldots, i_{T-1}$ of Theorem 2
- b'' is all other length T transition sequences of $\{r\}$ (i.e. all other elements in the process r')

This Markov process r'' is now like that in Lemma 1. That is, the expected time until entry into form a'', which we call $E\{T''\}$, is finite. In the original time scale of $\{r\}$, we have $E\{TT''\} < \infty$. Note that the actual time until the specified length T sequence corresponding to a'' is reached (in the r form) may be less than TT'', since the a'' sequence does not have to begin at a time that is a multiple of T. We know that a'' will occur within $T_{cw} \triangleq TT''$, with $E\{T_{cw}\} < \infty$. Thus sufficiency is proved.

Necessity follows by contradiction. Assume that the system is weakly controllable but there no transition sequence exists with the property (4). Then we cannot find a control law which can drive the x-state to $\mathbf{x}$ from any initial state in finite expected time with a probability greater than zero. This contradicts with the assumption that the system is weakly controllable.

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