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Infinite horizon H_2/H_∞ control for stochastic systems with Markovian jumps☆

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Abstract

This paper studies robust H_2/H_∞ control problem for systems subjected to multiplicative noise and Markovian parameter jumps. A necessary/sufficient condition for the existence of H_2/H_∞ control is presented by means of two coupled algebraic Riccati equations, respectively. Finally, a suboptimal H_2/H_∞ controller design algorithm is also obtained by solving a convex optimization problem.

Keywords: Stochastic H_2/H_∞ control; Stochastic systems; Markovian jumps; Coupled algebraic Riccati equations

1. Introduction

H_2/H_∞ control is an important robust control method, which requires a controller not only to attenuate external disturbances efficiently, but also to minimize the H_2 performance simultaneously. Compared with sole H_∞ control or H_∞ filter, H_2/H_∞ control and filtering design are more attractive in engineering practice, see Gao, Lam, Xie, and Wang (2005), Limebeer, Anderson, and Hendel (1994) and Wang and Huang (2000). Before 1998, most works are focused on deterministic H_2/H_∞ control for the systems governed by an ordinary differential equation. In recent years, stochastic H_∞ control for the systems governed by stochastic Itô equations has become a popular research field, and attracted many researchers' attentions. For example, in Hinrichsen and Pritchard (1998), H_∞ control for general linear stochastic Itô systems was first discussed very extensively, and more importantly, a very useful lemma called "stochastic bounded real lemma (SBRL)" was given therein in terms of linear matrix inequalities, which is very useful in H_∞ filtering design, see Gershon, Shaked, and Yaesh (2005) and Xu and Chen (2002). Recently, the authors in Zhang and Chen (2006) investigated nonlinear H_∞ control for stochastic affine systems, where a nonlinear SBRL and a nonlinear Lur'e equation were obtained. The authors in Chen and Zhang (2004) and Zhang, Zhang, and Chen (2006) studied the mixed H_2/H_∞ control with state and (x, u, v)-dependent noise, respectively, which generalized the results of Limebeer et al. (1994) to stochastic case. For other relevant developments in this regard, we refer readers to the monographs (Gershon et al., 2005; Petersen, Ugrinovskii, & Savkin, 2000).

This paper will deal with stochastic H_2/H_∞ state feedback control for Itô systems with Markovian jumping, where the state is subjected to both Gaussian white noise and Markovian parameter jumping. Such class of systems has important applications in engineering practice since they can be used to represent random failure processes in manufacturing systems, electric power systems and so on, see Boukas, Zhang, and Yin (1995), Bjork (1980), Dragan and Morozan (2002), Dragan and Morozan (2004), Fragoso and Rocha (2006), Li, Zhou, and Ait Rami (2003), Mao, Yin, and Yuan (2007) and Willsky (1976). In particular, stability and robust stabilization for such perturbed

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systems were investigated extensively in Fragoso and Rocha (2006), Dragan and Morozan (2002) and Mao et al. (2007). A bounded real lemma for Markovian jumping stochastic systems was derived in Dragan and Morozan (2002). Bjork (1980) studied the optimal filtering problem for such systems, while Dragan and Morozan (2004) and Li et al. (2003) addressed the issue of linear quadratic regulator. The goal of this paper is to develop the mixed H_2/H_∞ control theory for stochastic Itô systems with Markovian jumps, a necessary/sufficient condition is, respectively, developed for the existence of a static state feedback H_2/H_∞ control in terms of two coupled algebraic Riccati equations (AREs), which can be viewed as a generalized version of the results in Chen and Zhang (2004). Because it is not easy to solve the two coupled AREs, a suboptimal H_2/H_∞ controller design method is further developed by solving a convex optimization problem.

For convenience, we make use of the following notations:

A': transpose of a matrix or vector A; $A\geqslant 0$ ($A>0$): positive semi-definite (positive definite) symmetric matrix A; χ_A: indicator function of a set A; $\mathrm{tr}(A)$: trace of square matrix A; $M^l_{n,m}$: space of all $A=(A(1),A(2),\ldots,A(l))$ with $A(i)$ being $n\times m$ matrix, $i=1,2,\ldots,l$; $M^l_n := M^l_{n,n}$; S^l_n: space of all $A=(A(1),A(2),\ldots,A(l))$ with $A(i)$ being $n\times n$ symmetric matrix, $i=1,2,\ldots,l$; $A=(A(1),A(2),\ldots,A(l))>0$ ($\geqslant 0$) means $A(i)>0$ ($\geqslant 0$) for $i=1,2,\ldots,l$; R^n: space of all n-dimensional real vectors with usual 2-norm $|\cdot|$; I: identity matrix. $col(A_1,A_2,\ldots,A_n) := [A_1',A_2',\ldots,A_n']'$.

2. Definitions and preliminaries

Throughout this paper, let $(\Omega, F, \{F_t\}_{t\geqslant 0}, P)$ be a given filtered probability space where there exists a standard one-dimensional Wiener process $w(t)$, $t\geqslant 0$, and a right continuous homogeneous Markov chain $r_t, t\geqslant 0$ with state space $D=\{1,2,\ldots,l\}$. We assume that r_t is independent of $w(t)$ and has the following transition probability:

$$P\{r_{t+\Delta t}=j/r_t=i\}=\begin{cases} q_{ij}\Delta t+o(\Delta t), & i\neq j,\\ 1+q_{ii}\Delta t+o(\Delta t), & i=j,\end{cases}$$

where $q_{ij}\geqslant 0$ for $i\neq j$ and $q_{ii}=-\sum_{i\neq j} q_{ij}$. F_t stands for the smallest σ-algebra generated by processes $w(s), r(s), 0\leqslant s\leqslant t$, i.e. $F_t=\sigma\{w(s), r(s)|0\leqslant s\leqslant t\}$.

By $L^2_F([0,\infty), R^k)$ we denote the space of all measurable functions $u(t,\omega):[0,\infty)\times\Omega\mapsto R^k$, which is F_t-measurable for every $t\geqslant 0$, and $E[\int_0^\infty |u(t)|^2\,\mathrm{d}t|r_0=i]<\infty$, $i\in D$. Obviously, $L^2_F([0,\infty), R^k)$ is a Hilbert space with the inner product

$$\langle u,v\rangle=\sum_{i=1}^{l} E\left[\int_0^\infty u'(t)v(t)\,\mathrm{d}t|r_0=i\right].$$

In a similar way, $L^2_F([0,T], R^k)$, $T>0$ can be defined. Consider the following linear stochastic controlled system with Markovian jumps

$$\mathrm{d}x(t)=[A(r_t)x(t)+B(r_t)u(t)]\,\mathrm{d}t+A_1(r_t)x(t)\,\mathrm{d}w(t), \quad (1)$$

where $x(t)\in R^n$ and $u(t)\in R^{n_u}$ are the state and control input, respectively. The coefficients $A, A_1\in M^l_n$ and $B\in M^l_{n,n_u}$ with $A(i), B(i), A_1(i), i\in D$, being constant matrices of compatible dimensions.

It is well known that for any $u(t)\in L^2_F([0,T], R^{n_u})$ and $(x_0,i)\in R^n\times D$, there exists a unique solution $x(t)\in L^2_F([0,T], R^n)$ of (1) with initial condition $x(0)=x_0, r_0=i$. Next, we first introduce the definitions of stochastic stabilizability and stochastic detectability, which are essential assumptions in this paper.

Definition 2.1. System (1) or (A,B,A_1) is called stochastic stabilizable (in mean-square sense), if there exists a feedback control $u(t)=\sum_{i=1}^{l} K(i)x(t)\chi_{r_t=i}(t)$ with $K(1), K(2),\ldots,K(l)$ being constant matrices, such that for any initial state $x(0)=x_0, r_0=i$, the closed-loop system

$$\mathrm{d}x(t)=[A(r_t)+B(r_t)K(r_t)]x(t)\,\mathrm{d}t+A_1(r_t)x(t)\,\mathrm{d}w(t)$$

is asymptotically mean-square stable, i.e.

$$\lim_{t\to\infty} E[|x(t)|^2|r_0=i]=0.$$

Definition 2.2. The following state-measurement system:

$$\begin{cases} \mathrm{d}x(t)=A(r_t)x(t)\,\mathrm{d}t+A_1(r_t)x(t)\,\mathrm{d}w(t),\\ y(t)=C(r_t)x(t)\end{cases} \quad (2)$$

or $(A,A_1|C)$ is called stochastically detectable, if there exists a constant matrix $H=(H(1),H(2),\ldots,H(l))$ such that $(A+HC, A_1)$ is mean-square stable, where the measurement $y(t)\in R^{n_y}$, $C\in M^l_{n_y,n}$, the ith-component of $(A+HC)$ is defined as

$$(A+HC)(i)=A(i)+H(i)C(i), \quad i\in D.$$

Obviously, when C is of full row-rank, $(A,A_1|C)$ is stochastically detectable. Now we give some lemmas which are important in our subsequent analysis. For system (1), we apply Itô's formula Bjork (1980) to $x'P(i)x$, we immediately obtain the following result.

Lemma 2.1. *Suppose $P=(P(1),P(2),\ldots,P(l))\in S^l_n$ is given, then for system (1) with initial state $(x_0,i)\in R^n\times D$, we have for any $T>0$,*

$$\begin{aligned} E\Bigg\{\int_0^T &[x'(t)(P(r_t)A(r_t)+A'(r_t)P(r_t)\\ &+\underline{A_1'(r_t)P(r_t)A_1(r_t)}+\underline{\sum_{j=1}^{l} q_{r_t j}P(j))x(t)}\\ &\underline{+2u'(t)B'(r_t)P(r_t)x(t)]\,\mathrm{d}t}|r_0=i\Bigg\}\\ &=E[x'(T)P(r_T)x(T)|r_0=i]-x_0'P(i)x_0. \end{aligned} \quad (3)$$

Lemma 2.2. *If* $(A, A_1|C)$ *is stochastically detectable, then* (A, A_1) *is mean-square stable iff the following Lyapunov-type equation*:

$$L(P) := P(i)A(i) + A'(i)P(i) + A_1'(i)P(i)A_1(i) + \sum_{j=1}^{l} q_{ij}P(j) + C'(i)C(i) = 0, \quad i \in D \tag{4}$$

has a unique positive semi-definite solution $P = (P(1), P(2), \ldots, P(l)) \in S_n^l$ *with* $P(i) \geqslant 0, i \in D$.

Proof. The sufficiency is a corollary of Dragan and Morozan (2002, Theorem 6.5) in the stationary case. To prove the necessity part, let $Q_\varepsilon(i) = C'(i)C(i) + \varepsilon I > 0, i = 1, 2, \ldots, l, \varepsilon > 0$. Since (A, A_1) is mean-square stable, by Proposition 4.11 of Dragan and Morozan (2002), we deduce that

$$P(i)A(i) + A'(i)P(i) + A_1'(i)P(i)A_1(i) + \sum_{j=1}^{l} q_{ij}P(j) + Q_\varepsilon'(i)Q_\varepsilon(i) = 0, \quad i = 1, 2, \ldots, l$$

admit a unique solution $P_\varepsilon = (P_\varepsilon(1), P_\varepsilon(2), \ldots, P_\varepsilon(l)) \in S_n^l$ with $P_\varepsilon(i) > 0, i \in D$. Moreover, it is easy to show that P_ε is monotonically decreasing as ε decreases and is bounded from below. Hence, $P =: \lim_{\varepsilon \to 0} P_\varepsilon$ exists and solves Eq. (4). Obviously, $P(i) \geqslant 0, i = 1, 2, \ldots, l$.

Next, we show that P is a unique solution to (4). For system (1) with $u(t) \equiv 0$, for any $(x_0, i) \in R^n \times D$ and $T > 0$, by Lemma 2.1 we have

$$E\left[\int_0^T x'(t)C'(r_t)C(r_t)x(t)\,\mathrm{d}t | r_0 = i\right] = x_0'P(i)x_0 - E[x'(T)P(r_T)x(T)|r_0 = i].$$

Set $T \to \infty$, $\lim_{T\to\infty} E[x'(T)P(r_T)x(T)|r_0 = i] = 0$ due to the stability of (A, A_1). So

$$E\left[\int_0^\infty x'(t)C'(r_t)C(r_t)x(t)\,\mathrm{d}t | r_0 = i\right] = x_0'P(i)x_0.$$

Assume P_1 is another solution of Eq. (4), by the same procedure as above, we obtain

$$E\left[\int_0^\infty x'(t)C'(r_t)C(r_t)x(t)\,\mathrm{d}t | r_0 = i\right] = x_0'P(i)x_0 = x_0'P_1(i)x_0.$$

Therefore $P_1 = P$ due to the arbitrariness of $x_0 \in R^n$, $i \in D$, the uniqueness is proved. □

Remark 2.1. In fact, it is easy to show that if the Lyapunov-type inequality $L(P) < 0$ admits a solution $P \geqslant 0$, then we must have $P > 0$. In addition, it is interesting to compare Lemma 2.2 with Theorem 3.2 of Zhang, Zhang, and Chen (2005), where a similar theorem for stochastic stability without Markovian jumps was obtained under a weaker assumption called "exact detectability".

The following lemma can be verified by following the line of Lemma 4 of Chen and Zhang (2004).

Lemma 2.3. *Suppose* $A, A_1 \in M_n^l$, $B_1 \in M_{n,m_1}^l$, $B_2 \in M_{n,m_2}^l$, $C \in M_{m,n}^l$, $P_1, P_2 \in S_n^l$, $\gamma \neq 0$, *let*

$$\tilde{A}_2 = col(C, \gamma^{-1}B_1'P_1, B_2'P_2), \quad \tilde{A}_3 = col(C, B_2'P_2). \tag{5}$$

(i) *If* $(A, A_1|C)$ *is stochastically detectable, then so is* $(A - B_2B_2'P_2, A_1|\tilde{A}_2)$.
(ii) *If* $(A + \gamma^{-2}B_1B_1'P_1, A_1|C)$ *is stochastically detectable, then so is* $(A + \gamma^{-2}B_1B_1'P_1 - B_2B_2'P_2, A_1|\tilde{A}_3)$.

Consider the following control system with Markovian jumps:

$$\begin{cases} \mathrm{d}x(t) = [A(r_t)x(t) + B(r_t)u(t)]\,\mathrm{d}t + A_1(r_t)x(t)\,\mathrm{d}w(t), \\ z(t) = C(r_t)x(t, u, x_0, i), \end{cases} \tag{6}$$

where the initial value $x(0) = x_0, r_0 = i$, the control input $u(t) \in L_F^2([0, \infty), R^{n_u})$ and the controlled output $z(t) \in L_F^2([0, \infty), R^{n_z})$.

Lemma 2.4. *For system* (6), *if* (A, B, A_1) *is stabilizable*, $(A, A_1|C)$ *is stochastically detectable, then the following ARE has a unique solution* $P = (P(1), P(2), \ldots, P(l))$ *with* $P(i) \geqslant 0, i \in D$.

$$P(i)A(i) + A'(i)P(i) + A_1'(i)P(i)A_1(i) + C'(i)C(i) + \sum_{j=1}^{l} q_{ij}P(j) - P(i)B(i)B'(i)P(i) = 0. \tag{7}$$

In addition,

$$\min_{u \in L_F^2([0,\infty), R^{n_u})} E\left[\int_0^\infty (|z(t)|^2 + |u(t)|^2)\,\mathrm{d}t | r_0 = i\right] = x_0'P(i)x_0 \tag{8}$$

and the optimal control law u^* *is given by*

$$u^*(t) = -B'(r_t)P(r_t)x(t).$$

Proof. From Li et al. (2003, Theorem 4.6), the ARE (7) admits a maximal solution

$$P = (P(1), P(2), \ldots, P(l)) \in S_n^l, \quad P(i) \geqslant 0, \quad i \in D.$$

Note that (7) can be written as $(i \in D)$

$$\begin{aligned} &P(i)(A(i) - B(i)B'(i)P(i)) \\ &\quad + (A(i) - B(i)B'(i)P(i))'P(i) \\ &\quad + A_1'(i)P(i)A_1(i) + \sum_{j=1}^{l} q_{ij}P(j) \\ &\quad + [col(C(i), B'(i)P(i))]'col(C(i), B'(i)P(i)) = 0. \end{aligned} \tag{9}$$

By Lemma 2.3, $(A - BB'P, A_1/[C' \ PB]')$ is stochastically detectable due to the stochastic detectability of $(A, A_1|C)$, so

$(A - BB'P, A_1)$ is stable according to Lemma 2.2, which implies that $u^*(t)$ is a stabilizing control law of system (6). Then (8) is a consequence of Li et al. (2003, Theorems 5.1–5.2).

3. Infinite horizon stochastic H_2/H_∞ control

Consider the following stochastic linear system with Markovian jumps:

$$\begin{cases} \mathrm{d}x(t) = [A(r_t)x(t) + B_2(r_t)u(t) + B_1(r_t)v(t)]\,\mathrm{d}t \\ \qquad\qquad + A_1(r_t)x(t)\,\mathrm{d}w(t), \\ z(t) = col(C(r_t)x(t), M(r_t)u(t)), \end{cases} \tag{10}$$

where $M'(i)M(i) = I, i \in D$, $u(t), v(t), z(t)$ are, respectively, the control input, external disturbance and controlled output. Define two associated performances as follows:

$$J_1^\infty(u, v; x_0, i) = E\left[\int_0^\infty (|z(t)|^2 - \gamma^2|v(t)|^2)\,\mathrm{d}t|r_0 = i\right]$$

and

$$J_2^\infty(u, v; x_0, i) = E\left[\int_0^\infty |z(t)|^2\,\mathrm{d}t|r_0 = i\right], \quad i \in D.$$

The infinite horizon H_2/H_∞ stochastic control problem of system (10) is described as follows.

Definition 3.1. For given disturbance attenuation level $\gamma > 0$, if we can find $u^*(t) \times v^*(t) \in L_F^2([0,\infty), R^{n_u}) \times L_F^2([0,\infty), R^{n_v})$, such that

(i) $u^*(t)$ stabilizes system (10) internally, i.e. when $v(t) = 0, u = u^*$, the state trajectory of (10) with any initial value $(x_0, i) \in R^n \times D$ satisfies

$$\lim_{t\to\infty} E[|x(t)|^2|r_0 = i] = 0.$$

(ii) $\|L_{u^*}\|_\infty < \gamma$ with

$$\|L_{u^*}\|_\infty = \sup_{\substack{v\in L_F^2([0,\infty),R^{n_v}),\\ v\neq 0, u=u^*, x_0=0}} \frac{\{\sum_{i=1}^l E[\int_0^\infty |z(t)|^2\,\mathrm{d}t|r_0 = i]\}^{1/2}}{\{\sum_{i=1}^l E[\int_0^\infty |v(t)|^2\,\mathrm{d}t|r_0 = i]\}^{1/2}}.$$

(iii) When the worst case disturbance $v^*(t) \in L_F^2([0,\infty), R^{n_v})$, if existing, is applied to (10), $u^*(t)$ minimizes the output energy

$$J_2^\infty(u, v^*; x_0, i) = E\left[\int_0^\infty |z(t)|^2\,\mathrm{d}t|r_0 = i\right], \quad i \in D.$$

Then we say that the infinite horizon stochastic H_2/H_∞ control problem has a pair of solutions. Obviously, (u^*, v^*) is the Nash equilibrium strategies such that $J_1^\infty(u^*, v^*; x_0, i) \geqslant J_1^\infty(u^*, v; x_0, i)$, $i \in D$, and $J_2^\infty(u^*, v^*; x_0, i) \leqslant J_2^\infty(u, v^*; x_0, i)$, $i \in D$.

The following theorem presents a sufficient condition for the existence of (u^*, v^*).

Theorem 3.1. *For system* (10), *suppose the coupled AREs* $(i, j \in D)$

$$\begin{aligned} \Upsilon_{11} := {} & P_1(i)A(i) + A'(i)P_1(i) + A_1'(i)P_1(i)A_1(i) \\ & + C'(i)C(i) + \sum_{j=1}^l q_{ij}P_1(j) + \begin{bmatrix} P_1(i) \\ P_2(i) \end{bmatrix}' \\ & \times \begin{bmatrix} \gamma^{-2}B_1(i)B_1'(i) & -B_2(i)B_2'(i) \\ -B_2(i)B_2'(i) & B_2(i)B_2'(i) \end{bmatrix} \begin{bmatrix} P_1(i) \\ P_2(i) \end{bmatrix} = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} \Upsilon_{22} := {} & P_2(j)A(j) + A'(j)P_2(j) + A_1'(j)P_2(j)A_1(j) \\ & + C'(j)C(j) + \sum_{k=1}^l q_{jk}P_2(k) + \begin{bmatrix} P_1(j) \\ P_2(j) \end{bmatrix}' \\ & \times \begin{bmatrix} 0 & \gamma^{-2}B_1(j)B_1'(j) \\ \gamma^{-2}B_1(j)B_1'(j) & -B_2(j)B_2'(j) \end{bmatrix} \begin{bmatrix} P_1(j) \\ P_2(j) \end{bmatrix} = 0 \end{aligned} \tag{12}$$

have solutions $P_1 = (P_1(1), P_1(2), \ldots, P_1(l)) \geqslant 0 \in S_n^l$, $P_2 = (P_2(1), P_2(2), \ldots, P_2(l)) \geqslant 0 \in S_n^l$. *If* $(A, A_1|C)$ *and* $(A + \gamma^{-2}B_1B_1'P_1, A_1|C)$ *are stochastically detectable, then the* H_2/H_∞ *control problem has a pair of solutions*

$$u^*(t) = -\sum_{i=1}^l B_2'(i)P_2(i)\chi_{r_t=i}(t)x(t),$$

$$v^*(t) = \gamma^{-2}\sum_{i=1}^l B_1'(i)P_1(i)\chi_{r_t=i}(t)x(t).$$

Proof. By Definition 3.1, we will prove the theorem by three steps as follows:

Step 1: $(u^*, v^*) \in L_F^2([0,\infty), R^{n_u}) \times L_F^2([0,\infty), R^{n_v})$, and $(A - B_2B_2'P_2, A_1)$ is stable.

We note that (11) and (12) can be rewritten as

$$\begin{aligned} & P_1(i)[A(i) - B_2(i)B_2(i)'P_2(i)] \\ & + [A(i) - B_2(i)B_2'(i)P_2(i)]'P_1(i) + A_1(i)'P_1(i)A_1(i) \\ & + \sum_{j=1}^l q_{ij}P_1(j) + \tilde{A}_2'(i)\tilde{A}_2(i) = 0, \quad i \in D \end{aligned} \tag{13}$$

and

$$\begin{aligned} & P_2(j)[A(j) - B_2(j)B_2'(j)P_2(j) \\ & + \gamma^{-2}B_1(j)B_1'(j)P_1(j)] + [A(j) - B_2(j)B_2'(j)P_2(j) \\ & + \gamma^{-2}B_1(j)B_1'(j)P_1(j)]'P_2(j) + A_1'(j)P_2(j)A_1(j) \\ & + \sum_{k=1}^l q_{jk}P_2(k) + \tilde{A}_3'(j)\tilde{A}_3(j) = 0, \quad j \in D, \end{aligned} \tag{14}$$

respectively, where $\tilde{A}_2, \tilde{A}_3$ are defined in Lemma 2.3. From Lemma 2.3, $(A - B_2B_2'P_2 + \gamma^{-2}B_1B_1'P_1, A_1/\tilde{A}_3)$ is also stochastically detectable for the reason of $(A + \gamma^{-2}B_1B_1'P_1, A_1/C)$ being stochastically detectable, which yields $(A-B_2B_2'P_2+\gamma^{-2}B_1B_1'P_1, A_1)$ being stable by applying Lemma 2.2. Hence,

$$u^*(t)=-\sum_{i=1}^{l}B_2'(i)P_2(i)\chi_{r_t=i}(t)x(t)\in L_F^2([0,\infty), R^{n_u})$$

and

$$v^*(t)=\gamma^{-2}\sum_{i=1}^{l}B_1'(i)P_1(i)\chi_{r_t=i}(t)x(t)\in L_F^2([0,\infty), R^{n_v}).$$

From Lemmas 2.2–2.3, (13) yields the stability of $(A-B_2B_2'P_2, A_1)$, i.e. system (10) is internally stabilized by $u^*(t)=-\sum_{i=1}^{l}B_2'(i)P_2(i)\chi_{r_t=i}(t)x(t)$.

Step 2: $\|L_{u^*}\|_\infty<\gamma$.

Substituting $u^*(t)=-\sum_{i=1}^{l}B_2'(i)P_2(i)\chi_{r_t=i}(t)x(t)$ into system (10) yields

$$\begin{cases} dx(t)=[(A(r_t)-B_2(r_t)B_2'(r_t)P_2(r_t))x(t) \\ \qquad +B_1(r_t)v(t)]\,dt+A_1(r_t)x(t)\,dw(t), \\ z(t)=col(C(r_t)x(t), -M(r_t)B_2'(r_t)P_2(r_t)x(t)). \end{cases} \tag{15}$$

Note that (11) can be rearranged as

$$\begin{aligned} &P_1(i)(A(i)-B_2(i)B_2'(i)P_2(i)) \\ &\quad+(A(i)-B_2(i)B_2'(i)P_2(i))'P_1(i) \\ &\quad+A_1'(i)P_1(i)A_1(i)+\sum_{j=1}^{l}q_{ij}P_1(j) \\ &\quad+\gamma^{-2}P_1(i)B_1(i)B_1'(i)P_1(i) \\ &\quad+C'(i)C(i)+P_2(i)B_2(i)B_2'(i)P_2(i)=0, \quad i\in D. \end{aligned} \tag{16}$$

Furthermore, $(A-B_2B_2'P_2+\gamma^{-2}B_1B_1'P_1, A_1)$ is stable. By Dragan and Morozan (2002, Corollary 7.11), system (15) is internally stable and the norm of the perturbation operator is less than γ, i.e. $\|L_{u^*}\|_\infty<\gamma$.

Step 3: u^* minimizes the output energy when v^* is applied in system (10).

Since $(A-B_2B_2'P_2, A_1)$ is stable, then for any $v(t)\in L_F^2([0,\infty), R^{n_v})$, we have $x(t)\in L_F^2([0,\infty), R^n)$ by Dragan and Morozan (2002, Theorem 5.1), where $x(t)$ is the trajectory of (15). Considering (15), (16) and applying Lemma 2.1 for $T\to\infty$, we have

$$\begin{aligned} J_1^\infty(u^*, v; x_0, i) &= x_0'P_1(i)x_0 \\ &\quad-\gamma^2E\left[\int_0^\infty |v(t)-v^*(t)|^2\,dt|r_0=i\right] \\ &\leqslant J_1^\infty(u^*, v^*; x_0, i)=x_0'P_1(i)x_0, \end{aligned}$$

from which we can see that $v^*(t)=\gamma^{-2}B_1'(r_t)P_1(r_t)x(t)\in L_F^2([0,\infty), R^{n_v})$ is the worst case disturbance corresponding to u^*. Finally, when $v=v^*$ is applied to system (10), we obtain

$$\begin{cases} dx(t)=[(A(r_t)+\gamma^{-2}B_1(r_t)B_1'(r_t)P_1(r_t))x(t) \\ \qquad +B_2(r_t)u(t)]\,dt+A_1(r_t)x(t)\,dw(t), \\ z(t)=col(C(r_t)x(t), M(r_t)u(t)). \end{cases} \tag{17}$$

Under the constraint of (17), the optimization problem $\min_{u\in L_F^2([0,\infty), R^{n_u})} J_2^\infty(u, v^*; x_0, i)$ is a standard linear quadratic optimal control problem. By Lemma 2.4, we immediately have

$$\begin{aligned} \min_{u\in L_F^2([0,\infty), R^{n_u})} J_2^\infty(u, v^*; x_0, i) &= J_2^\infty(u^*, v^*; x_0, i) \\ &= x_0'P_2(i)x_0 \end{aligned}$$

which completes the proof. □

In using Theorem 3.1, one needs to test the stochastic detectability of $(A+\gamma^{-2}B_1B_1'P_1, A_1|C)$ and $(A, A_1|C)$, but if $C(i)C'(i)>0$, $i\in D$, the assumption of stochastic detectability can be removed from Theorem 3.1.

The following theorem presents a necessary condition for the existence of stochastic H_2/H_∞ control, which can be shown following the line of Chen and Zhang (2004, Theorem 2).

Theorem 3.2. *If the stochastic H_2/H_∞ control problem admits a pair of solutions (u^*, v^*) such that*

$$u^*(t)=K_2(r_t)x(t), \quad v^*=K_1(r_t)x(t)$$

with $K_2\in M_{n_u,n}^l$, $K_1\in M_{n_v,n}^l$ being constant matrices, and suppose $(A+B_1K_1, A_1/C)$ is stochastically detectable, then the coupled AREs (11) *and* (12) *have a unique pair of solutions* $(P_1\geqslant 0, P_2\geqslant 0)\in S_n^l\times S_n^l$. *Furthermore,*

$$u^*(t)=-B_2'(r_t)P_2(r_t)x(t),\ v^*(t)=\gamma^{-2}B_1'(r_t)P_1(r_t)x(t).$$

It is not easy to solve the coupled AREs (11) and (12) though they may be solved numerically as shown in Limebeer et al. (1994). Now, we present a suboptimal H_2/H_∞ controller design algorithm as follows. It is easy to see that if the coupled AREs (11) and (12) are, respectively replaced by the coupled algebraic Riccati inequalities (ARIs):

$$\Upsilon_{11}(P_1, P_2)<0, \tag{18}$$

$$\Upsilon_{22}(P_1, P_2)<0 \tag{19}$$

and $(P_1>0, P_2>0)$ solve (18) and (19), then the conditions (i) and (ii) of Definition 3.1 are satisfied, but

$$\begin{aligned} &\inf_{u\in L_F^2([0,\infty), R^{n_u})}\left\{J_2^\infty(u, v^*; x_0, i)=E\left[\int_0^\infty |z(t)|^2\,dt|r_0=i\right]\right\} \\ &\quad\leqslant |x_0|^2\mathrm{tr}(P_2(i)), \quad i\in D. \end{aligned}$$

If we let $P(i):=P_1(i)=P_2(i)$ in (18) and (19), then a sufficient condition for (18) and (19) to hold is

$$\begin{aligned} &P(j)A(j)+A'(j)P(j)+A_1'(j)P(j)A_1(j)+C'(j)C(j) \\ &\quad+\sum_{k=1}^{l}q_{jk}P(k)+2\gamma^{-2}P(j)B_1(j)B_1'(j)P(j) \\ &\quad-P(j)B_2(j)B_2'(j)P(j)<0, \quad j\in D. \end{aligned} \tag{20}$$

Because $P(j)B_2(j)B_2'(j)P(j)\geqslant 0$, (20) holds if

$$P(j)A(j)+A'(j)P(j)+A_1'(j)P(j)A_1(j)+C'(j)C(j)+\sum_{k=1}^{l}q_{jk}P(k)+2\gamma^{-2}P(j)B_1(j)B_1'(j)P(j)<0,\quad j\in D. \tag{21}$$

Applying Schur's complement, (21) is equivalent to

$$\begin{bmatrix} \Delta_{11} & P(j)B_1(j) \\ B_1'(j)P(j) & -\frac{1}{2}\gamma^2 I \end{bmatrix}<0,\quad j\in D, \tag{22}$$

where

$$\Delta_{11}=P(j)A(j)+A'(j)P(j)+A_1'(j)P(j)A_1(j)+C'(j)C(j)+\sum_{k=1}^{l}q_{jk}P(k).$$

Summarizing the above discussion, we have the following result.

Theorem 3.3. *A suboptimal H_2/H_∞ controller may be obtained by solving the following convex optimization problem*

$$\inf_{s.t.(22),P(1)>0,\ldots,P(l)>0}\operatorname{tr}(P(i)),\quad i\in D \tag{23}$$

with $u^*(t)=-B_2'(r_t)P(r_t)x(t)$.

The convex optimization problem (23) may be easily solved by using Matlab control toolbox (Gahinet, Nemirovski, Laub, & Chilali, 1995)

Example 3.1. Consider system (10) with the coefficients as follows:

$$D=\{1,2\},\quad Q=\begin{bmatrix}-7 & 7\\ 1 & -1\end{bmatrix},\quad A(1)=\begin{bmatrix}-20 & 1.5\\ 0.3 & -50\end{bmatrix},$$

$$A(2)=\begin{bmatrix}-30 & 1.2\\ 3.6 & -10\end{bmatrix},\quad A_1(1)=\begin{bmatrix}0.5 & 0.8\\ 2 & 1\end{bmatrix},$$

$$A_1(2)=\begin{bmatrix}2 & 3\\ 1 & 0.5\end{bmatrix},\quad B_1(1)=\begin{bmatrix}3\\ 0.8\end{bmatrix},\quad B_1(2)=\begin{bmatrix}1\\ 1.2\end{bmatrix},$$

$$B_2(1)=\begin{bmatrix}10\\ 2\end{bmatrix},\quad B_2(2)=\begin{bmatrix}0.5\\ 8\end{bmatrix},$$

$$C(1)=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\quad C(2)=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix},$$

$$M(1)=M(2)=I_{2\times 2}.$$

Set $\gamma=0.9$, by solving (23), we have

$$P(1)=\begin{bmatrix}0.0548 & 0.0230\\ 0.0230 & 0.0313\end{bmatrix},\quad P(2)=\begin{bmatrix}0.0389 & 0.0720\\ 0.0720 & 0.3004\end{bmatrix}.$$

Therefore, the suboptimal H_2/H_∞ controller is given by $u(t)=-0.5940x_1(t)-0.2926x_2(t)$ while $r_t=1$, and $u(t)=-0.59545x_1(t)-2.4392x_2(t)$ while $r_t=2$.

4. Conclusions

This paper has studied the infinite horizon mixed H_2/H_∞ control of stochastic Itô systems with Markovian jumps and state-dependent noise, a couple of sufficient and necessary conditions for the existence of such mixed H_2/H_∞ control have been obtained, which are expressed in a pair of coupled AREs. The results obtained in this paper generalize the results of Chen and Zhang (2004).

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