V. Conclusion

In this note, *direct* certainty equivalence results for output feedback controllers have been proposed. Theorem 2 allows to deal with observers which do not guarantee uniform convergence of the estimates when a priory boundedness of the system state is not assumed. The certainty equivalence result of Theorem 1 and its *indirect* version have been specialized to a class of nonlinear systems combined with the reduced-order observer design of [2], considering systems which may include several (possibly nonmonotonic) nonlinearities. As an application, we have designed a computed torque plus PD-like certainty-equivalent output feedback controller for a class of 2-DOF mechanical systems with position-only feedback (see Fig. 2).

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Finite Horizon H_2/H_∞ Control for Discrete-Time Stochastic Systems With Markovian Jumps and Multiplicative Noise

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***Abstract*—In this note, we consider the finite horizon mixed H_2/H_∞ control problem for discrete-time stochastic linear systems subject to Markov jump parameters and multiplicative noise. Firstly, we derive a stochastic bounded real lemma (SBRL), which is used to establish a necessary and sufficient condition for the existence of the mixed H_2/H_∞ control via the solvability of four coupled difference matrix-valued recursions (CDMRs). Moreover, a state feedback H_2/H_∞ controller is designed by means of the solutions of CDMRs.**

***Index Terms*—Coupled difference matrix-valued recursions, discrete-time systems, H_2/H_∞ control, Markov jump.**

I. Introduction

In the past two decades, the H_∞ control has become one of the most important research issues in the fields of control theory and engineering applications; see, e.g., [11], [13], [14], [17], and [26] for the treatment of deterministic systems. In recent years, stochastic H_∞ theory has made great progress. For instance, due to successful applications of stochastic Itô systems in radar signal processing [9], portfolio selection [22] and mathematical finance [21], linear and nonlinear stochastic Itô-type H_∞ control problems with state-multiplicative noise have been studied extensively, we refer the reader to [4], [12], [18], [23] and the references therein. [7] developed a discrete-time version of [12] for the system with multiplicative noise, [10] further studied the discrete-time output feedback H_2 and H_∞ control problems. Recently, [24] and [25] extended the results of [4] to discrete-time systems, where it is shown that, despite of finite horizon [24] or infinite horizon case [25], the existence of discrete-time mixed H_2/H_∞ control is equivalent to the solvability of some four CDMRs.

On the other hand, as one of the most basic dynamics models, Markov jump linear systems can be used to represent random failure processes in manufacturing industry [1], and some investment portfolio models [3], [5], [22], which have been researched extensively [15]. For example, the stability analysis and H_∞ control of Markovian jumping parameter systems were dealt with in [6], [8], [16], respectively. In [2], a mixed H_2/H_∞ control problem for discrete-time Markov jump linear systems with additive noise is well treated. The filtering problem for the system with Markovian jumping parameters has also gained much attention [19]. [3] and [5] discussed the linear quadratic (LQ) optimal control problem and its applications to investment portfolio optimization for discrete-time Markov jump linear systems with multiplicative noise.

Manuscript received March 14, 2009; revised September 07, 2009 and September 09, 2009. First published February 02, 2010; current version published May 12, 2010. This work was supported by the Key Project of Natural Science Foundation of Shandong Province under Grant ZR2009GZ001 and by the National Natural Science Foundation of China under Grant 60874032. Recommended by Associate Editor C. Szepesvari.

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Digital Object Identifier 10.1109/TAC.2010.2041987

In [20], the infinite horizon H_∞ control problem has been researched for a type of time-invariant discrete-time Markov jump systems. However, in many engineering cases (target maneuver, missile guidance, etc.), a control strategy of limited time length is needed. Therefore, the objective of this note is to handle the finite horizon mixed H_2/H_∞ control problem for discrete-time Markov jump linear systems with multiplicative noise, which might be more realistic in some practical applications. Our first main result provides a stochastic bounded real lemma, which plays a central role in stochastic H_∞ theory. It should be pointed out that, in the proof of SBRL, we adopt a thorough new simple approach, which is different from [14], [24]. The second main result gives a necessary and sufficient condition for the mixed H_2/H_∞ control via the solvability of four CDMRs. For the single mode state $\theta(\cdot)$, the results of this note specialize to those of [24].

The organization of this note is as follows. In Section II, we formulate some preliminaries and useful lemmas, and then prove a SBRL for Markov jump linear systems with multiplicative noise. Based on the obtained SBRL, a necessary and sufficient condition for the finite horizon mixed H_2/H_∞ control problem is established in Section III. Section IV supplies a recursive procedure to solve the CDMRs. Section V ends this note with some concluding remarks.

For convenience, we adopt the following notations in this note. R^n: n-dimensional Euclidean space with the usual 2-norm $\|\cdot\|$; $R^{n\times m}$: the space of all $n\times m$ real matrices; S^n: the set of all $n\times n$ symmetric matrices; M': the transpose of a matrix M; $M>0(\geq 0)$: M is positive definite (positive semi-definite) symmetric matrix; I: the identity matrix with appropriate dimensions; $N:=\{0,1,\cdots\}$, $N_t:=\{0,\cdots,t\}$, and $\overline{N}:=\{1,\cdots,N\}$.

II. A Stochastic Bounded Real Lemma

In this section, we aim at developing a stochastic bounded real lemma which by itself has theoretical importance in the study of stochastic H_∞ control and estimation. For this purpose, let us present some preliminaries.

Given a probability space $(\Omega,\mathcal{F},\mathcal{P})$, consider the following Markov jump linear system with multiplicative noise:

$$\begin{cases} x(k+1)=A_{\theta(k)}(k)x(k)+\overline{A}_{\theta(k)}(k)x(k)w(k) \\ \quad +B_{\theta(k)}(k)v(k)+\overline{B}_{\theta(k)}(k)v(k)w(k) \\ z(k)=C_{\theta(k)}(k)x(k),\ x(0)=x_0\in R^n, k\in N_T \end{cases} \tag{1}$$

where $x(k)\in R^n$, $v(k)\in R^l$, and $z(k)\in R^m$ represent the state, disturbance signal, and controlled output of the system, respectively. $\theta(k)$ is a time-varying Markov chain taking values in $\overline{N}$ with the transition probability matrix $\mathcal{P}(k)=[p_{ij}(k)]$, $p_{ij}(k)=P(\theta(k+1)=j|\theta(k)=i)$. $A_{\theta(k)}(k)$, $\overline{A}_{\theta(k)}(k)$, $B_{\theta(k)}(k)$, $\overline{B}_{\theta(k)}(k)$ are matrix-valued functions of suitable dimensions. $w(k)(k\in N_T)$ is a sequence of real random variables with $E(w(k))=0$ and $E(w(k)w(s))=\delta_{ks}$(Kronecker function). The random variables $\{w(k),k=0,1,\cdots,T\}$ are independent of the Markov chain $\{\theta(\cdot)\}$ and $w(k)$ is uncorrelated to $\{v(k)\}$ for $k\in N_T$. The initial value $\theta(0)=\theta_0$ is independent of the noise $w(k)(k\in N_T)$. Denote $\mathcal{F}_k$ the σ-field generated by $\{(\theta(t),x(t)),t=0,1,\cdots,k\}$. Let $L^2(\Omega,R^k)$ be the space of R^k-valued square summable random vectors and $l^2(N_T,R^q)$ stand for the space of all finite sequences $y(k)\in L^2(\Omega,R^q)$ that are $\mathcal{F}_k$-measurable for $k\in N_T$. The norm of $l^2(N_T,R^q)$ is defined as $\|y(\cdot)\|_{l^2(N_T,R^q)}=\left(\sum_{k=0}^T E\|y(k)\|^2\right)^{1/2}<+\infty$.

Definition 1: The operator $L_T: l^2(N_T,R^l)\to l^2(N_T,R^m)$ defined by $L_T(v(k))=C_{\theta(k)}(k)x(k)$ for $\forall v(k)\in l^2(N_T,R^l)$, $\theta_0\in\overline{N}$, $x_0=0$, is called the perturbation operator of system (1), and its norm $\|L_T\|$ is defined as the minimal $\delta\geq 0$ such that

$$\|z(\cdot)\|_{l^2(N_T,R^m)}\leq\delta\|v(\cdot)\|_{l^2(N_T,R^l)}.$$

Since the effect of the disturbance v on the controlled output z can be described by $\|L_T\|$, the key of the robust control problem lies in how to design an appropriate controller to attenuate this effect. In the case of $v=0$, i.e., the system is unperturbed, the problem is trivial ($\|L_T\|=0$ when $v=0$ is imposed on system (1)). Therefore, we below only deal with the nonzero disturbance case. When $v\neq 0$, the former definition of $\|L_T\|$ can be equivalently represented as

$$\|L_T\|:=\sup_{v\in l^2(N_T,R^l),v\neq 0,\theta(0)\in\overline{N},\ x_0=0}\frac{\|z(\cdot)\|_{l^2(N_T,R^m)}}{\|v(\cdot)\|_{l^2(N_T,R^l)}}.$$

To make the formulae tighter, we introduce some notations that will be used later. Let P represent a collection of symmetric matrices indexed by the time k and the mode of operation i, namely, $P=\{P_{\theta(k)}(k)\in S^n: k\in N_{T+1},\theta(k)\in\overline{N}\}$, $P(k)=[P_1(k),P_2(k),\cdots,P_N(k)]$. For P and $i\in\overline{N}$, set

$$\begin{aligned}
\Psi_i(k,P(k+1)) &= \sum_{j=1}^N p_{ij}(k)P_j(k+1)\\
R_i(k,P) &:= R_i(k,P(k+1))\\
&= A_i(k)'\Psi_i(k,P(k+1))A_i(k)\\
&\quad+\overline{A}_i(k)'\Psi_i(k,P(k+1))\overline{A}_i(k)\\
&\quad-P_i(k)\\
K_i(k,P) &:= K_i(k,P(k+1))\\
&= A_i(k)'\Psi_i(k,P(k+1))B_i(k)\\
&\quad+\overline{A}_i(k)'\Psi_i(k,P(k+1))\overline{B}_i(k)\\
T_i(k,P) &:= T_i(k,P(k+1))\\
&= B_i(k)'\Psi_i(k,P(k+1))B_i(k)\\
&\quad+\overline{B}_i(k)'\Psi_i(k,P(k+1))\overline{B}_i(k)\\
L_i(k,P) &:= L_i(k,P(k+1))\\
&= A_i(k)'\Psi_i(k,P(k+1))A_i(k)\\
&\quad+\overline{A}_i(k)'\Psi_i(k,P(k+1))\overline{A}_i(k)\\
&\quad-C_i(k)'C_i(k)\\
H_i(k,P) &= \gamma^2 I+T_i(k,P).
\end{aligned} \tag{2}$$

We define the following functional:

$$J^T(x_0,\theta_0,v):=\sum_{k=0}^T E\left[\gamma^2\|v(k)\|^2-\|z(k)\|^2\right]$$

that is closely related with the H_∞ performance. In fact, for any $\theta_0\in\overline{N}$ and $v\in l^2(N_T,R^l)$, we can infer that $J^T(0,\theta_0,v)>0$ from $\|L_T\|<\gamma$. If the signal v^* minimize $J^T(0,\theta_0,v)$, we call it the worst-case disturbance. From the engineering view of point, the worst-case disturbance v^* achieves the maximum possible energy gain from the disturbance v to the output z. Now, we show a result about $J^T(x_0,\theta_0,v)$, which will be used in the sequel.

Lemma 1: Given $v \in l^2(N_T, R^l)$, $x_0 \in R^n$ and $\theta_0 \in \overline{N}$, let $x(k) := x(k; x_0, \theta_0, v)$ be the corresponding solution of (1). Then, for any fixed $T \in N$

$$J^T(x_0,\theta_0,v) = \sum_{k=0}^{T} E\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}' M_{\theta(k)}(k,P) \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + E x_0' P_{\theta_0}(0) x_0 - E\left[x(T+1)' P_{\theta(T+1)}(T+1) x(T+1)\right]$$

where $M_{\theta(k)}(k,P) = M_i(k,P)$ for $\theta(k) = i$ and

$$M_i(k,P) = \begin{bmatrix} L_i(k,P) - P_i(k) & K_i(k,P) \\ K_i(k,P)' & H_i(k,P) \end{bmatrix}. \quad (3)$$

Proof: By the assumption that $w(k)$ is independent of the Markov chain $\{\theta(\cdot)\}$ and uncorrelated to $v(k)$, we have that $A_{\theta(k)}(k)x(k) + B_{\theta(k)}(k)v(k)$ and $\overline{A}_{\theta(k)}(k)x(k) + \overline{B}_{\theta(k)}(k)v(k)$ are $\mathcal{F}_k$- measurable and uncorrelated to $w(k)$, hence

$$E\left\{\left[A_{\theta(k)}(k)x(k) + B_{\theta(k)}(k)v(k)\right]' P_{\theta(k+1)}(k+1) \cdot \left[\overline{A}_{\theta(k)}(k)x(k) + \overline{B}_{\theta(k)}(k)v(k)\right] w(k) | \mathcal{F}_k\right\} = 0.$$

Fixing $\theta(k) = i$, it can be obtained that

$$\begin{aligned} &E\left[x(k+1)' P_{\theta(k+1)}(k+1) x(k+1) - x(k)' P_i(k) x(k) | \mathcal{F}_k, \theta(k) = i\right] \\ &= \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}' \begin{bmatrix} R_i(k,P) & K_i(k,P) \\ K_i(k,P)' & T_i(k,P) \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}. \end{aligned} \quad (4)$$

By taking summation from $k = 0$ to T on both sides of (4), it follows that:

$$E\left[\left(x' P_{\theta(T+1)} x\right)(T+1) - x_0' P_{\theta(0)}(0) x_0\right] = \sum_{k=0}^{T} E\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}' \begin{bmatrix} R_{\theta(k)}(k,P) & K_{\theta(k)}(k,P) \\ K_{\theta(k)}(k,P)' & T_{\theta(k)}(k,P) \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}.$$

This, together with the definition of $J^T(x_0, \theta_0, v)$, shows that

$$\begin{aligned} J^T&(x_0,\theta_0,v) \\ &= \sum_{k=0}^{T} E\left[\gamma^2 v(k)'v(k) - x(k)' C_{\theta(k)}(k)' C_{\theta(k)}(k) x(k)\right] \\ &+ \sum_{k=0}^{T} E\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}' \begin{bmatrix} R_{\theta(k)}(k,P) & K_{\theta(k)}(k,P) \\ K_{\theta(k)}(k,P)' & T_{\theta(k)}(k,P) \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} \\ &+ E\left[x_0' P_{\theta_0}(0) x_0 - x(T+1)' P_{\theta(T+1)}(T+1) x(T+1)\right] \end{aligned}$$

which yields our desired result. ■

We are now equipped to present a stochastic bounded real lemma associated with system (1).

Theorem 1: For system (1) and a given disturbance attenuation level $\gamma > 0$, the following statements are equivalent.

(i) $\|L_T\| < \gamma$.

(ii) For any $(i,k) \in \overline{N} \times N_T$, $P_i(k)$ is solvable in the following recursions:

$$\begin{cases} P_i(k) = L_i(k,P) - K_i(k,P) H_i(k,P)^{-1} K_i(k,P)' \\ P_i(k) \le 0, P(T+1) = 0 \\ H_i(k,P) > 0 \end{cases} \quad (5)$$

where $L_i(k,P)$, $K_i(k,P)$, and $H_i(k,P)$ are defined by (2).

Proof: (ii) ⇒ (i). Suppose that (ii) holds for $k = T, \cdots, 0$. Fix $\theta(k) = i \in \overline{N}$, then for any nonzero $v(k) \in l^2(N_T, R^l)$, $\theta_0 \in \overline{N}$ and $x_0 = 0$, it yields from Lemma 1 that

$$\begin{aligned} J^T(0,\theta_0,v) &= \sum_{k=0}^{T} E\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}' M_{\theta(k)}(k,P) \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} \\ &= \sum_{k=0}^{T} E\left\{E\left\{\begin{bmatrix} x(k) \\ v(k) \end{bmatrix}' M_i(k,P) \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} | \mathcal{F}_{k-1}, \theta(k) = i\right\}\right\}. \end{aligned}$$

By means of the first equality in (5) and a completing squares technique, it can be shown that

$$J^T(0,\theta_0,v) = \sum_{k=0}^{T} E\left\{E\left\{[v(k) - v^*(k)]' H_i(k,P) \cdot [v(k) - v^*(k)] | \mathcal{F}_{k-1}, \theta(k) = i\right\}\right\} \quad (6)$$

where $v^*(k) = -H_{\theta(k)}(k,P)^{-1} K_{\theta(k)}(k,P)' x(k)$. From the positivity of $H_i(k,P)$, (6) implies that $J^T(0,\theta_0,v) \ge 0$ and $J^T(0,\theta_0,v) = 0$ iff $v(k) = v^*(k)$. After substituting $v(k) = v^*(k)$ into the system dynamics (1) under the initial condition $x_0 = 0$, there must be $x(k) \equiv 0, k \in N_T$, which leads to $v^*(k) \equiv 0, k \in N_T$. So we can deduce that $J^T(0,\theta_0,v) = 0$ iff $v = v^* = 0$, which contradicts the assumption of $v \ne 0$. Hence, v cannot achieve v^* in (6), or equivalently, $J^T(0,\theta_0,v) > 0$ for any $v \ne 0$. Furthermore, without loss of generality, we can set $H_i(k,P) > \epsilon I$. Then (6) implies $J^T(0,\theta_0,v) > \epsilon\|v(\cdot) - v^*(\cdot)\|^2_{l^2(N_T,R^l)} > 0$. From the connection between $J^T(0,\theta_0,v)$ and $\|L_T\|$, it is obvious that (i) holds.

(i)⇒(ii). Note that $H_i(T,P) = \gamma^2 I > 0$ for any $i \in \overline{N}$, we can solve $P_i(T)$ from the recursions (5), which is given by

$$P_i(T) = L_i(T,P) - \gamma^{-2} K_i(T,P) K_i(T,P)', \quad i \in \overline{N}.$$

By the same procedure, the recursions (5) can be solved backward iff $H_i(k,P) > 0$ (see later Remark 1) for all $k \in N_T$ and $i \in \overline{N}$.

If (5) fail to proceed for some $k = T_0 \in N_{T-1}$, then there must exist at least one $j \in \overline{N}$ such that $H_j(T_0,P)$ has at least one zero or negative eigenvalue. However, we can show this is infeasible by induction. Set

$$\begin{cases} F_i(k,P) = -H_i(k,P)^{-1} K_i(k,P)', \ T_0 + 1 \le k \le T \\ F_i(k,P) = 0, \ 0 \le k \le T_0, i \in \overline{N}. \end{cases}$$

Consider the following backward matrix recursions:

$$\begin{cases} \overline{P}_i(k) = L_i(k,\overline{P}) + K_i(k,\overline{P}) F_i(k,\overline{P}) \\ \quad + F_i(k,\overline{P})' K_i(k,\overline{P})' + F_i(k,\overline{P})' H_i(k,\overline{P}) F_i(k,\overline{P}) \\ \overline{P}_i(k) \le 0, \overline{P}(T+1) = 0 \\ H_i(k,\overline{P}) > 0, \ i \in \overline{N} \end{cases} \quad (7)$$

which admit solutions $\overline{P}_i(k)$ on $i \in \overline{N}$ and $k \in N_T$. Moreover, $\overline{P}_i(k) = P_i(k)$ for $k = T_0 + 1, \cdots, T$. Now we will show that $H_j(T_0,\overline{P}) > 0$ for all $j \in \overline{N}$. Otherwise, there must exist $u \in R^l$ with $\|u\| = 1$ such that $E[u' H_j(T_0,\overline{P}) u] \le 0$ (i.e., u is a unit eigenvector corresponding to an eigenvalue $\lambda \le 0$ of H_j). Let $v(k) = u$ for $k = T_0$ and $v(k) = 0$ for $k \ne T_0$. Denote $\overline{v}(k) = v(k) + F_{\theta(k)}(k,\overline{P}) x_F(k)$, where $x_F(\cdot)$ is the state of system (1) corresponding to $\overline{v}(k)$, that is

$$\begin{cases} x_F(k+1) = A_{\theta(k)}(k) x_F(k) + \overline{A}_{\theta(k)}(k) x_F(k) w(k) \\ \quad + B_{\theta(k)}(k) \overline{v}(k) + \overline{B}_{\theta(k)}(k) \overline{v}(k) w(k) \\ \theta(0) = \theta_0 \in \overline{N}, \ x_F(0) = x_0 \in R^n, \ k \in N_T. \end{cases} \quad (8)$$

Combining Lemma 1 with the definition of $\overline{v}$, it is therefore concluded that

$$\begin{aligned}
&J^T(0,\theta_0,\overline{v}) \\
&= \sum_{k=0}^{T} E\begin{bmatrix} x_F(k) \\ \overline{v}(k)\end{bmatrix}' M_{\theta(k)}(k,\overline{P})\begin{bmatrix} x_F(k) \\ \overline{v}(k)\end{bmatrix} \\
&= \sum_{k=0}^{T} E\left\{ E\left\{ x_F(k)'\begin{bmatrix} I \\ F_i(k,\overline{P})\end{bmatrix}' M_i(k,\overline{P})\begin{bmatrix} I \\ F_i(k,\overline{P})\end{bmatrix}\right.\right. \\
&\qquad \cdot x_F(k) + v(k)'N_i(k,\overline{P})x_F(k) \\
&\qquad + x_F(k)'N_i(k,\overline{P})'v(k) \\
&\qquad \left.\left. + v(k)'H_i(k,\overline{P})v(k)|\mathcal{F}_{k-1},\theta(k)=i\right\}\right\}
\end{aligned} \tag{9}$$

where we adopt $N_i(k,\overline{P}) = K_i(k,\overline{P})' + H_i(k,\overline{P})F_i(k,\overline{P}), i \in \overline{N}$. The first term in (9) becomes zero due to (7). For the other terms in (9), remembering the definition of $v(k)$ in mind and noting the linearity of (8), it results in $x_F(k) = 0$ for $k \le T_0$, while $v(k) = 0$ for $k > T_0$. So (9) reduces to

$$J^T(0,\theta_0,\overline{v}) = E\left\{E\left\{\left[u'H_{\theta(T_0)}(T_0,\overline{P})u\right]|\mathcal{F}_{k-1}\right\}\right\}. \tag{10}$$

Recalling the preceding definition of u, in the case of $\theta(T_0) = j$, (10) immediately implies $J^T(0,\theta_0,\overline{v}) = \lambda \le 0$, which contradicts the condition (i). Thus, there must be $H_j(T_0,\overline{P}) > 0$ for all $j \in \overline{N}$. Note that $\overline{P}(T_0+1) = P(T_0+1)$, we conclude that $H_j(T_0,P) > 0$ for all $j \in \overline{N}$ and $T_0 \in N_T$. So the recursive procedure can proceed for $k = T_0,\cdots,0$. This shows (5) admit solutions $P_i(k)$ on N_T for $i \in \overline{N}$.

Now, we examine the nonpositivity of $P_i(k)$ for any $(i,k) \in \overline{N}\times N_T$. Given any $k \in N_T$, $\theta(k) = i \in \overline{N}$ and $x(k) = x \in R^n$, from Lemma 1, it follows that

$$\begin{aligned}
J^T(k;x,i,v) &:= \sum_{s=k}^{T} E\left[\gamma^2\|v(s)\|^2 - \|z(s)\|^2|\mathcal{F}_k\right] \\
&= \sum_{s=k}^{T} E\left\{[v(s)-v^*(s)]' H_{\theta(s)}(s,P)\right. \\
&\qquad \left.\cdot [v(s)-v^*(s)]|\mathcal{F}_k\right\} + x'P_i(k)x.
\end{aligned}$$

Note that $H_{\theta(s)}(s,P) > 0$, we have

$$\begin{aligned}
\min_{v\in l^2(N_T,R^l)} J^T(k;x,i,v) &= J^T(k;x,i,v^*) = x'P_i(k)x \\
&\le J^T(k;x,i,0) \\
&= -\sum_{s=k}^{T} E\left[\|z(s)\|^2|\mathcal{F}_k\right].
\end{aligned} \tag{11}$$

Due to the arbitrariness of x, (11) yields $P_i(k) \le 0$ for all $i \in \overline{N}$ and $k \in N_T$. ∎

Remark 1: What should be pointed out is that $H_i(k,P) > 0$ is an inherent part of (5). If in (5), $H_i(k,P) > 0$ is replaced by the existence of $H_i(k,P)^{-1}$, then the first equality of (5) can still be solved recursively. However, in such case, (i) is not equivalent to (ii).

Remark 2: If the state space of the Markov chain $\theta(k)$ consists of only one value, or equivalently, there is no jump in system (1), the result of Theorem 1 reduces to that of Lemma 3 in [24]. Hence, Theorem 1 may be regarded as an extension of [24] for stochastic Markov jump systems. In addition, we remark that a new and simple analytic method is first adopted in the proof of the sufficiency part of Theorem 1, which differs from that adopted in [14] and [24], where an invertible operator was introduced.

III. Mixed H_2/H_∞ Control

In this section, by virtue of the SBRL, we attempt to verify that the existence of a state feedback H_2/H_∞ controller is equivalent to the solvability of four coupled difference matrix-valued recursions. We are interested in the following discrete-time Markov jump linear system with state, disturbance and control dependent noise:

$$\begin{cases}
x(k+1) = A_{\theta(k)}(k)x(k) + \overline{A}_{\theta(k)}(k)x(k)w(k) \\
\quad +B_{\theta(k)}(k)v(k) + \overline{B}_{\theta(k)}(k)v(k)w(k) + B^1_{\theta(k)}(k)u(k) \\
\quad +B^2_{\theta(k)}(k)u(k)w(k) \\
z(k) = \begin{bmatrix} C_{\theta(k)}(k)x(k) \\ D_{\theta(k)}(k)u(k)\end{bmatrix},\ D_{\theta(k)}(k)'D_{\theta(k)}(k) = I \\
x(0) = x_0 \in R^n,\ k\in N_T,\ \theta(0) = \theta_0 \in \overline{N}
\end{cases} \tag{12}$$

where $u(\cdot) \in l^2(N_T,R^q)$ and $B^i_{\theta(k)}(k)(i=1,2)$ are matrix-valued functions with appropriate dimensions. The other terms are the same as those adopted in (1).

Given $T \in N$ and the disturbance attenuation level $\gamma > 0$, our objective is to seek a state feedback controller $u^*(\cdot) \in l^2(N_T,R^q)$ such that:

1) for the closed-loop trajectory of system (12) with $x(0) = 0$, $\|L_T\| < \gamma$ for any nonzero $v(\cdot) \in l^2(N_T,R^l)$ and $\theta_0 \in \overline{N}$;
2) when the worst-case disturbance $v^*(\cdot)$ is imposed on (12), $u^*(\cdot)$ solves

$$\min_{u(\cdot)\in l^2(N_T,R^q)}\left\{\hat{J}^T(x_0,\theta_0,u(k);v^*(k)) := \sum_{k=0}^{T} E\|z(k)\|^2\right\}$$

for any $x_0 \in R^n$ and $\theta_0 \in \overline{N}$. If (u^*,v^*) exist, the mixed H_2/H_∞ control problem is called solvable.

Before presenting the main result of this section, we give the following four coupled matrix recursions on $(i,k) \in \overline{N}\times N_T$:

$$\begin{cases}
P^1_i(k) = \left[A_i(k) + B^1_i(k)K^2_i(k,P^2)\right]'\Psi_i\left(k,P^1(k+1)\right) \\
\quad \cdot\left[A_i(k) + B^1_i(k)K^2_i(k,P^2)\right] \\
\quad + \left[\overline{A}_i(k) + B^2_i(k)\cdot K^2_i(k,P^2)\right]'\Psi_i\left(k,P^1(k+1)\right) \\
\quad \times\left[\overline{A}_i(k) + B^2_i(k)\cdot K^2_i(k,P^2)\right] \\
\quad -C_i(k)'C_i(k) - K^3_i(k,P^1)\cdot H^1_i(k,P^1)^{-1}K^3_i(k,P^1)' \\
\quad -K^2_i(k,P^2)'K^2_i(k,P^2) \\
P^1_i(k) \le 0, P^1(T+1) = 0 \\
H^1_i(k,P^1) > 0
\end{cases} \tag{13}$$

$$K^1_i(k,P^1) = -H^1_i(k,P^1)^{-1}K^3_i(k,P^1)' \tag{14}$$

$$\begin{cases}
P^2_i(k) = \left[A_i(k) + B_i(k)K^1_i(k,P^1)\right]'\Psi_i\left(k,P^2(k+1)\right) \\
\quad \cdot\left[A_i(k) + B_i(k)K^1_i(k,P^1)\right] \\
\quad + \left[\overline{A}_i(k) + \overline{B}_i(k)\cdot K^1_i(k,P^1)\right]'\Psi_i\left(k,P^2(k+1)\right) \\
\quad \times\left[\overline{A}_i(k) + \overline{B}_i(k)\cdot K^1_i(k,P^1)\right] + C_i(k)'C_i(k) \\
\quad -K^4_i(k,P^2)\cdot H^2_i(k,P^2)^{-1}K^4_i(k,P^2)' \\
P^2_i(k) \ge 0, P^2(T+1) = 0 \\
H^2_i(k,P^2) > 0
\end{cases} \tag{15}$$

$$K^2_i(k,P^2) = -H^2_i(k,P^2)^{-1}K^4_i(k,P^2)' \tag{16}$$

where in (13)–(16)

$$\begin{aligned} H_i^1(k,P^1) &= \gamma^2 I + B_i(k)'\Psi_i\left(k,P^1(k+1)\right)B_i(k) \\ &\quad + \overline{B}_i(k)'\Psi_i\left(k,P^1(k+1)\right)\overline{B}_i(k) \\ H_i^2(k,P^2) &= I + B_i^1(k)'\Psi_i\left(k,P^2(k+1)\right)B_i^1(k) \\ &\quad + B_i^2(k)'\Psi_i\left(k,P^2(k+1)\right)B_i^2(k) \\ K_i^3(k,P^1) &= \left[A_i(k)+B_i^1(k)K_i^2(k,P^2)\right]' \\ &\quad \cdot \Psi_i\left(k,P^1(k+1)\right)B_i(k) \\ &\quad + \left[\overline{A}_i(k)+B_i^2(k)K_i^2(k,P^2)\right]' \\ &\quad \cdot \Psi_i\left(k,P^1(k+1)\right)\overline{B}_i(k) \\ K_i^4(k,P^2) &= \left[A_i(k)+B_i(k)K_i^1(k,P^1)\right]' \\ &\quad \cdot \Psi_i\left(k,P^2(k+1)\right)B_i^1(k) \\ &\quad + \left[\overline{A}_i(k)+\overline{B}_i(k)K_i^1(k,P^1)\right]' \\ &\quad \cdot \Psi_i\left(k,P^2(k+1)\right)B_i^2(k). \end{aligned}$$

Theorem 2: A finite horizon H_2/H_∞ control problem is solvable with $u^*(k) = K_{\theta(k)}^2(k,P^2)x(k)$, $v^*(k) = K_{\theta(k)}^1(k,P^1)x(k)$, iff the CDMRs (13)–(16) admit a group of solutions $(P_i^1(k), K_i^1(k,P^1); P_i^2(k), K_i^2(k,P^2))$ for any $(i,k) \in \overline{N} \times N_T$.

Proof: Sufficiency: If the CDMRs (13)–(16) admit a group of solutions $(P_i^1(k), K_i^1(k,P^1); P_i^2(k), K_i^2(k,P^2))$ on $\overline{N} \times N_T$, we can construct $u^*(k) = K_{\theta(k)}^2(k,P^2)x(k)$ and substitute u^* into system (12). By Theorem 1 and recursion (13), it yields that $\|L_T\| < \gamma$ holds for all nonzero $v(\cdot) \in l^2(N_T, R^l)$, $\theta_0 \in \overline{N}$ and $x_0 = 0$. Now, noticing that $P_i^1(k)$ satisfies (13), by Lemma 1 and the technique of completing squares, for any $x_0 \neq 0$, we arrive at

$$\begin{aligned} &J^T(x_0,\theta_0,v;u^*) \\ &= \sum_{k=0}^{T} E\left\{E\left\{[v(k)-v^*(k)]' H_i^1(k,P^1)\right.\right. \\ &\qquad\qquad \left.\left.\cdot[v(k)-v^*(k)]\,|\mathcal{F}_{k-1},\theta(k)=i\right\}\right\} \\ &\quad + x_0' P_{\theta_0}^1(0)x_0 \\ &\geq J^T(x_0,\theta_0,v^*;u^*) = x_0' P_{\theta_0}^1(0)x_0 \end{aligned} \tag{17}$$

where $v^*(k) = K_{\theta(k)}^1(k,P^1)x(k)$ with K_i^1 given by (14). Equation (17) shows that $J^T(x_0,\theta_0,v;u^*)$ is minimized by $v(k) = v^*(k)$ for any $x_0 \in R^n$, so v^* is just the worst-case disturbance. Furthermore, by the recursion (15) and the technique of completing squares, we have

$$\begin{aligned} &\hat{J}^T(x_0,\theta_0,u(k);v^*(k)) \\ &= \sum_{k=0}^{T} E\|z(k)\|^2 \\ &= \sum_{k=0}^{T} E\left\{E\left\{[u(k)-u^*(k)]' H_i^2(k,P^2)\right.\right. \\ &\qquad\qquad \left.\left.\cdot[u(k)-u^*(k)]\,|\mathcal{F}_{k-1},\theta(k)=i\right\}\right\} \\ &\quad + x_0' P_{\theta_0}^2(0)x_0 \\ &\geq \hat{J}^T(x_0,\theta_0,u^*(k);v^*(k)) = x_0' P_{\theta_0}^2(0)x_0 \end{aligned} \tag{18}$$

where $u^*(k) = K_{\theta(k)}^2(k,P^2)x(k)$ with K_i^2 given by (16). Equation (18) implies that $u^*(k)$ minimizes $\hat{J}^T(x_0,\theta_0,u(k);v^*(k))$. Therefore, (u^*,v^*) solve the finite horizon H_2/H_∞ control problem of system (12).

Necessity: Assume that the finite horizon H_2/H_∞ control problem for system (12) is solved by $u^*(k) = K_{\theta(k)}^2(k,P^2)x(k)$, $v^*(k) = K_{\theta(k)}^1(k,P^1)x(k)$. Substituting $u^*(k)$ into (12), we get

$$\begin{cases} x(k+1) = \left[A_{\theta(k)}(k) + B_{\theta(k)}^1(k)K_{\theta(k)}^2(k,P^2)\right]x(k) \\ \quad + \left[\overline{A}_{\theta(k)}(k) + B_{\theta(k)}^2(k)K_{\theta(k)}^2(k,P^2)\right]x(k)w(k) \\ \quad + B_{\theta(k)}(k)v(k) + \overline{B}_{\theta(k)}(k)v(k)w(k) \\ z(k) = \begin{bmatrix} C_{\theta(k)}(k)x(k) \\ D_{\theta(k)}(k)K_{\theta(k)}^2(k,P^2)x(k) \end{bmatrix} \\ D_{\theta(k)}(k)'D_{\theta(k)}(k) = I \\ x(0) = x_0 \in R^n,\ k \in N_T,\ \theta(0) = \theta_0 \in \overline{N}. \end{cases} \tag{19}$$

Applying Theorem 1 to (19), we can derive that $P_i^1(k)$ satisfies recursion (13) on N_T with $P^1 \leq 0$. From the proof of sufficiency, the worst disturbance v^* is $K_{\theta(k)}^1(k,P^1)x(k)$ with K_i^1 given by (14). Imposing v^* on system (12), we have

$$\begin{cases} x(k+1) = \left[A_{\theta(k)}(k) + B_{\theta(k)}(k)K_{\theta(k)}^1(k,P^1)\right]x(k) \\ \quad + \left[\overline{A}_{\theta(k)}(k) + \overline{B}_{\theta(k)}(k)K_{\theta(k)}^1(k,P^1)\right]x(k)w(k) \\ \quad + B_{\theta(k)}^1(k)u(k) + B_{\theta(k)}^2(k)u(k)w(k) \\ z(k) = \begin{bmatrix} C_{\theta(k)}(k)x(k) \\ D_{\theta(k)}(k)u(k) \end{bmatrix} \\ D_{\theta(k)}(k)'D_{\theta(k)}(k) = I \\ x(0) = x_0 \in R^n,\ k \in N_T,\ \theta(0) = \theta_0 \in \overline{N}. \end{cases} \tag{20}$$

By the assumption, $u^*(k)$ is optimal for the following optimization problem:

$$\begin{cases} \min\limits_{u(\cdot)\in l^2(N_T,R^q)} \left\{\hat{J}^T(x_0,\theta_0,u(k);v^*(k)) \right. \\ \quad \left. = \sum\limits_{k=0}^{T} E\left[u(k)'u(k) + x(k)'C_{\theta(k)}(k)'C_{\theta(k)}(k)x(k)\right]\right\}, \\ \text{subject to (20)} \end{cases} \tag{21}$$

which is a standard LQ problem for Markov jump linear system in finite horizon and has been discussed in [3]. By Theorem 1 in [3], it is easy to prove that the recursion (15) is solved by $P_i^2(k) \geq 0$. The proof of this theorem is complete. ■

Remark 3: It should be noted that, by the same procedure, the result of Theorem 2 can be extended to the multiple noises case as discussed in [7] without any essential difficulty.

IV. Algorithm and Numerical Example

In contrast to the continuous-time case, the discrete-time mixed H_2/H_∞ control problem has exact solution if the four CDMRs (13)–(16) is solvable. This section provides a recursive algorithm, by which the four CDMRs (13)–(16) will be solved accurately. The algorithm procedures are as follows.

i) For $k = T$, $\theta(T) = i \in \overline{N}$, $H_i^1(T,P^1)$ and $H_i^2(T,P^2)$ are available by the final conditions $P_{\theta(T+1)}^1(T+1) = 0$ and $P_{\theta(T+1)}^2(T+1) = 0$.
ii) Solve the matrix recursions (14) and (16), then $K_i^1(T,P^1)$ and $K_i^2(T,P^2)$ are obtained.
iii) Substitute the obtained $K_i^1(T,P^1)$ and $K_i^2(T,P^2)$ into the matrix recursions (13) and (15) respectively, then $P_i^1(T) \leq 0$ and $P_i^2(T) \geq 0$ for $i \in \bar{N}$ are available.
iv) Repeat the above procedures, $P_i^1(k)$, $P_i^2(k)$, $K_i^1(k,P^1)$ and $K_i^2(k,P^2)$ can be computed for $k = T-1, T-2, \cdots, 0$ and $i \in \overline{N}$.

TABLE I
PARAMETERS OF SYSTEM (12)

mode	$i=1$			$i=2$		
time	$k=0$	$k=1$	$k=2$	$k=0$	$k=1$	$k=2$
$A_i(k)$	$\begin{bmatrix}0.3 & 0\\0.1 & 0.2\end{bmatrix}$	$\begin{bmatrix}0.5 & 0.1\\0 & 0.4\end{bmatrix}$	$\begin{bmatrix}2.0 & 1.0\\0 & 3.0\end{bmatrix}$	$\begin{bmatrix}0.6 & 0\\0.2 & 0.4\end{bmatrix}$	$\begin{bmatrix}0.4 & 0\\0.2 & 0.5\end{bmatrix}$	$\begin{bmatrix}2.0 & 1.0\\0 & 1.0\end{bmatrix}$
$\overline{A}_i(k)$	$\begin{bmatrix}0.4 & 0.1\\0 & 0.5\end{bmatrix}$	$\begin{bmatrix}0.5 & 0.2\\0.1 & 0.6\end{bmatrix}$	$\begin{bmatrix}1.0 & 2.0\\1.0 & 1.0\end{bmatrix}$	$\begin{bmatrix}0.3 & 0\\0 & 0.4\end{bmatrix}$	$\begin{bmatrix}0.4 & 0.1\\0 & 0.8\end{bmatrix}$	$\begin{bmatrix}2.0 & 0\\0 & 1.0\end{bmatrix}$
$B_i(k)$	$\begin{bmatrix}0.5 & 0\\0 & 0.4\end{bmatrix}$	$\begin{bmatrix}0.8 & 0\\0 & 0.5\end{bmatrix}$	$\begin{bmatrix}3.0 & 1.0\\2.0 & 2.0\end{bmatrix}$	$\begin{bmatrix}0.4 & 0\\0 & 0.3\end{bmatrix}$	$\begin{bmatrix}0.6 & 0\\0 & 0.8\end{bmatrix}$	$\begin{bmatrix}2.0 & 0\\0 & 1.0\end{bmatrix}$
$\overline{B}_i(k)$	$\begin{bmatrix}0.4 & 0\\0 & 0.5\end{bmatrix}$	$\begin{bmatrix}0.5 & 0\\0 & 0.4\end{bmatrix}$	$\begin{bmatrix}2.0 & 0\\0 & 3.0\end{bmatrix}$	$\begin{bmatrix}0.2 & 0\\0 & 0.5\end{bmatrix}$	$\begin{bmatrix}0.6 & 0\\0 & 0.5\end{bmatrix}$	$\begin{bmatrix}3.0 & 0\\1.0 & 1.0\end{bmatrix}$
$B_i^1(k)$	$\begin{bmatrix}0.6 & 0\\0 & 0.5\end{bmatrix}$	$\begin{bmatrix}1.0 & 0\\0 & 0\end{bmatrix}$	$\begin{bmatrix}1.0 & 0\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0.4 & 0\\0.1 & 0.8\end{bmatrix}$	$\begin{bmatrix}0 & 0\\0 & 1.0\end{bmatrix}$	$\begin{bmatrix}2.0 & 0\\0 & 0\end{bmatrix}$
$B_i^2(k)$	$\begin{bmatrix}0.5 & 0\\0 & 0.2\end{bmatrix}$	$\begin{bmatrix}1.0 & 0\\0 & 1.0\end{bmatrix}$	$\begin{bmatrix}1.0 & 0\\1.0 & 1.0\end{bmatrix}$	$\begin{bmatrix}0.2 & 0\\0 & 0.6\end{bmatrix}$	$\begin{bmatrix}0.5 & 0\\0 & 0.8\end{bmatrix}$	$\begin{bmatrix}1.0 & 0\\0 & 1.0\end{bmatrix}$
$C_i(k)$	$\begin{bmatrix}0.5 & 0\\0 & 0.5\end{bmatrix}$	$\begin{bmatrix}1.0 & 0\\0 & 0.5\end{bmatrix}$	$\begin{bmatrix}1.0 & 0\\0 & 2.0\end{bmatrix}$	$\begin{bmatrix}0.3 & 0\\0 & 0.7\end{bmatrix}$	$\begin{bmatrix}0.8 & 0\\0 & 0.6\end{bmatrix}$	$\begin{bmatrix}0.2 & 0.3\\0 & 0.1\end{bmatrix}$
$D_i(k)$	$\begin{bmatrix}0.6 & -0.8\\0.8 & 0.6\end{bmatrix}$	$\begin{bmatrix}-0.6 & 0.8\\0.8 & 0.6\end{bmatrix}$	$\begin{bmatrix}1.0 & 0\\0 & 1.0\end{bmatrix}$	$\begin{bmatrix}0.8 & 0.6\\0.6 & -0.8\end{bmatrix}$	$\begin{bmatrix}-0.8 & 0.6\\0.6 & 0.8\end{bmatrix}$	$\begin{bmatrix}0.6 & -0.8\\0.8 & 0.6\end{bmatrix}$

TABLE II
SOLUTIONS FOR CDMRS (13)–(16)

mode	$i=1$			$i=2$		
time	$k=0$	$k=1$	$k=2$	$k=0$	$k=1$	$k=2$
$-P_i^1(k)$	$\begin{bmatrix}0.51 & 0.17\\0.17 & 0.99\end{bmatrix}$	$\begin{bmatrix}1.11 & 0.10\\0.10 & 0.82\end{bmatrix}$	$\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}$	$\begin{bmatrix}0.57 & 0.08\\0.08 & 0.75\end{bmatrix}$	$\begin{bmatrix}0.86 & 0.14\\0.14 & 1.16\end{bmatrix}$	$\begin{bmatrix}0.04 & 0.06\\0.06 & 0.10\end{bmatrix}$
$P_i^2(k)$	$\begin{bmatrix}0.69 & 0.42\\0.42 & 2.23\end{bmatrix}$	$\begin{bmatrix}1.15 & 0.17\\0.17 & 1.29\end{bmatrix}$	$\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}$	$\begin{bmatrix}0.77 & 0.09\\0.09 & 0.87\end{bmatrix}$	$\begin{bmatrix}1.02 & 0.41\\0.41 & 2.01\end{bmatrix}$	$\begin{bmatrix}0.04 & 0.06\\0.06 & 0.10\end{bmatrix}$
$K_i^1(k)$	$\begin{bmatrix}0.40 & 0.06\\0.15 & 0.95\end{bmatrix}$	$\begin{bmatrix}0.21 & 0.12\\0.10 & 0.85\end{bmatrix}$	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0.43 & 0.01\\-0.01 & 0.29\end{bmatrix}$	$\begin{bmatrix}0.38 & 0.10\\0.34 & 1.15\end{bmatrix}$	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$
$-K_i^2(k)$	$\begin{bmatrix}0.38 & 0.19\\0.17 & 0.59\end{bmatrix}$	$\begin{bmatrix}0.21 & 0.11\\0.76 & 0.44\end{bmatrix}$	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0.35 & 0.05\\0.17 & 0.47\end{bmatrix}$	$\begin{bmatrix}0.09 & 0.03\\0.26 & 1.04\end{bmatrix}$	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

In order to guarantee this recursive algorithm to proceed backward, the *priori* condition $H_i^1(k,P^1) > 0$ and $H_i^2(k,P^2) > 0$ should be checked first. Otherwise, the recursive procedure has to stop. It is noted that $H_i^1(k,P^1)$ and $H_i^2(k,P^2)$ can be computed, provided that $P_i^1(k+1)$ and $P_i^2(k+1)$ are known. In this case, (14) and (16) constitute a group of coupled linear matrix recursions about $K_i^1(k,P^1)$ and $K_i^2(k,P^2)$. Similarly, after $K_i^1(k,P^1)$ and $K_i^2(k,P^2)$ have been obtained, (13) and (15) are also two coupled linear matrix recursions about $P_i^1(k)$ and $P_i^2(k)$.

Next, to show the efficiency of the above algorithm, we consider the following 2-D example.

Example 1: $T = 2, \gamma = 0.8, \overline{N} = \{1,2\}$. The transition probability matrix $P(k)$ of $\theta(k)$ is

$$P(0) = \begin{bmatrix}0.1 & 0.9\\0.5 & 0.5\end{bmatrix},\ P(1) = \begin{bmatrix}0.2 & 0.8\\0.3 & 0.7\end{bmatrix},\ P(2) = \begin{bmatrix}0.85 & 0.15\\0.6 & 0.4\end{bmatrix}.$$

Table I presents the parameters of system (12). Utilizing the above algorithm procedures, we can check the existence of the solutions of the recursions (13)–(16) and then compute them backward. The solutions are given in Table II.

Remark 4: For the infinite horizon stochastic H_2/H_∞ control problem, we need to search for a stabilizing controller, which requires to compute the stabilizing solution to a pair of cross-coupled algebraic Riccati equations [4], [25], but for the finite horizon case, we only need to simply solve the difference matrix recursions (13)–(16) regardless of stabilization. A numerical procedure for designing infinite horizon H_2/H_∞ controller can be found in [25] for discrete-time multiplicative noise systems; how to generalize the results of [25] to system (12) deserves further study.

V. CONCLUSION

In this note, we have discussed the finite horizon H_2/H_∞ control problem for stochastic discrete-time linear systems with Markov jump in parameters. Based on four CDMRs, we derive a necessary and sufficient condition for the solvability of H_2/H_∞ control problem and the controller can be designed explicitly according to the solutions of CDMRs. Compared with the Nash game approach adopted here, the convex optimization technique is another prevalent method in H_2/H_∞ robust control [2], [13], [16], [19], [20], which usually leads to a suboptimal H_2/H_∞ solution via solving convex optimization problems.

ACKNOWLEDGMENT

The authors would like to thank the Associate Editor and the referees for their constructive comments.

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Perturbation Analysis of a Dynamic Priority Call Center

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***Abstract*—We analyze a call center with multiple customer types and dynamic priority service discipline, in which a low-priority customer becomes high priority when its waiting time exceeds a given deterministic service level threshold. Within each priority queue, the service discipline is first come, first served. Based on a fluid approximation of the system, we apply infinitesimal perturbation analysis (IPA) to derive estimators for the derivatives of the queue lengths with respect to the threshold parameter. Numerical examples illustrate the validity of the fluid model approximation and the accuracy of the IPA estimators.**

***Index Terms*—Call center, fluid model, perturbation analysis.**

I. Introduction

A call center is a network of customer service representatives and the physical infrastructure needed to provide services to customers from remote sources. In traditional call centers, customers access the call center by talking to customer service representatives directly over the phone. However, modern call centers have become more complicated, both in size and in operational complexity, allowing additional communication channels, e.g., automatic answering systems, live Web support, e-mails, faxes. Along with the added complexity, more demanding service requirements have created new challenges in modeling, analysis, and design of call centers. As an alternative to the traditional modeling approach using discrete-event queueing models, fluid models provide a simpler and more computationally efficient approach for performance evaluation using first-order approximations to the corresponding queueing system. A central operational problem in call center management is the dynamic routing of service requests to agents, given the current system state. Since modern call centers have multiple classes of customers, e.g., live calls, e-mail, and fax, for which the quality of service (QoS) requirements are differentiated, strict (static) priority is often used to classify customer classes to meet the more stringent requirements of high-priority customers. However, during periods of heavy traffic, this may lead to difficulties in meeting the QoS requirements for lower priority customers, which may experience unacceptable waiting times, so many call centers have adopted dynamic priorities. However, recent literature reviews (e.g., [5], [6]) indicate that modeling dynamic priorities is a challenging problem that has not been addressed to date. One of the simplest versions is

Manuscript received August 27, 2008; revised March 22, 2009 and July 27, 2009. First published February 02, 2010; current version published May 12, 2010. This note was presented in part at the 43rd IEEE Conference on Decision and Control. This work was supported by the Natural Science Foundation of China (NSFC) under grant NSFC 70832002, by the Shanghai Science and Technology PuJiang Funds under grant 09PJ1401500, by a grant from IBM China Research Laboratory, by the National Science Foundation under Grants DMI-9988867, DMI-0323220, and DMI-0540312, and by the Air Force Office of Scientific Research under Grants FA9550-04-1-0210 and FA9550-07-1-0366. Recommended by Associate Editor C.-H. Chen.

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Digital Object Identifier 10.1109/TAC.2010.2041979