A FREQUENCY THEOREM FOR THE CASE IN WHICH THE STATE AND CONTROL SPACES ARE HILBERT SPACES, WITH AN APPLICATION TO SOME PROBLEMS OF SYNTHESIS OF OPTIMAL CONTROLS, II

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INTRODUCTION

This paper is a continuation of [1]. As in [1], a frequency theorem is defined by us as a set of assertions about the solution of certain special matrix (the finite-dimensional case), or operator (the infinite case) inequalities. (The first results were obtained in [2-5] for the finite-dimensional case.) These inequalities are encountered in various problems of control theory (see [1] and the references given in [1]). The aim of this paper, just as in the case of [1], is to extend the results of [2-5] to the infinite-dimensional case.

In this paper the frequency theorem is established for the case that the control system satisfies a condition that is weaker than the condition given in [1]. This weaker assumption about the original system makes it necessary to considerably modify various aspects of the proof. The basic idea of the proof is the same as in [1], i.e., it is based on the study of an auxiliary variational problem. (The adequacy of this procedure follows, for example, from [5], where we examined in detail the connection between the frequency theorem and a variational problem.) In contrast to [1], we shall consider below also the degenerate case (i.e., the case of nonstrict inequalities).

The second principal result obtained below is a theorem on linear stabilization that asserts that L_2 controllability implies exponential stabilizability. It is well known that this theorem which is important in finite-dimensional control theory is quite laborious to prove in the finite-dimensional case (if the control is a vector in R^n with $n > 1$); this proof is essentially "finite-dimensional," i.e., it involves the reduction of matrices to Jordan form. The proof presented below is based on other considerations that make it possible to carry it out also in the infinite-dimensional case, i.e., below it is fairly easy to derive the theorem on linear stabilization from the frequency theorem.†

The principal part of this paper (including the formulation of the results) can be read independently of [1]. Only in some sections of the proofs will the results of [1] be mentioned.

§ 1. Formulation of Results

Let $\mathbf{X} = \{x\}$ and $\mathbf{U} = \{u\}$ be Hilbert spaces that are either both real, or both complex, and whose elements are called a state (x) and a control (u). The scalar product and the norm in $\mathbf{X}$ are written in the form $(x_1, x_2) = x_2^* x_1$ and $|x| = \sqrt{x^*x}$. (The function $x_2^* x_1$ is linear in x_1 and antilinear in x_2 if $\mathbf{X}$ is a complex

†Let us note that a number of results similar to those obtained in this paper (and in [1]) were obtained in interesting articles (still in print) by V. A. Brusin, A. A. Nudel'man, and P. A. Shvartsman. In contrast to this paper, it is assumed by them (and also in [1]) that the original system is either exponentially stable, or exponentially stabilizable (or that other similar conditions hold). Nudel'man and Shvartsman have moreover considered the case of a control space R^1, and the idea of their proofs differs considerably from [1] and from the present paper. Brusin has considered also the case of a control system with an unbounded operator.

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space.) Similarly we can write the scalar product and the norm in $\mathbf{U}$, i.e., $u_2^* u_1$ and $|u| = \sqrt{u^*u}$. The null elements in X and $\mathbf{U}$ will be denoted by 0_X and 0_U, whereas the unit operators are denoted by I_X and I_U.

Let $\mathbf{Z} = \{z\}$ be a Hilbert space (real or complex) with a norm $|z|$, and $\Delta = (T_1, T_2)$ a finite or infinite interval of the real axis. We shall denote by $L_2(Z, \Delta)$ a Hilbert space of functions $z(\cdot): \Delta \to Z$ such that $|z(t)| \in L_2[\Delta]$ with a scalar product $(z_1(\cdot), z_2(\cdot)) = \int_\Delta z_1(t)^* z_1(t)dt$ and a norm $\|z(\cdot)\| = \sqrt{(z(\cdot), z(\cdot))}$.

We are given linear bounded operators

$$A: \mathbf{X} \to \mathbf{X}, \; b: \mathbf{U} \to \mathbf{X}.$$

Let us consider the system of equations†

$$\frac{dx}{dt} = Ax + bu. \tag{1.1}$$

Definition 1. a) A system (1.1) (or pair $\{A, b\}$) is said to be completely controllable if for a $T > 0$ and any $a \in \mathbf{X}$ there exists a function $u_a(\cdot) \in L_2[\mathbf{U}, (0, 1)]$ such that the solution of Eq. (1.1) with an initial condition $x(0) = a$ and a control $u = u_a(t)$ satisfies the relation $x(T) = 0_X$, and the function $u_a(t)$ can be obtained by the formula

$$u_a(t) = Ra, \tag{1.2}$$

where $R: X \to L_2[\mathbf{U}, (0, T)]$ is a linear bounded operator.

b) A system (1.1) (or pair $\{A, b\}$) is said to be controllable if for a $T > 0$, any $a \in \mathbf{X}$, and any $\delta > 0$, there exists a function $u_a(\cdot) \in L_2[\mathbf{U}, (0, T)]$ such that the solution of system (1.1) with an initial condition $x(0) = a$ satisfies the relation $|x(T)| < \delta$.

c) A system (1.1) (or pair $\{A, b\}$) is said to be L_2 controllable if for any $a \in \mathbf{X}$ there exists a function $u_a(\cdot) \in L_2[\mathbf{U}, (0, \infty)]$ such that the solution of system (1.1) with an initial condition $x(0) = a$ satisfies the relation $x(t) \in L_2[\mathbf{X}, (0, \infty)]$, and $u_a(\cdot) = S_1 a$, $x(\cdot) = S_2 a$, where

$$S_1: \mathbf{X} \to L_2[\mathbf{U}, (0, \infty)], \; S_2: \mathbf{X} \to L_2[\mathbf{X}, (0, \infty)]$$

are bounded linear operators.‡

It is evident that any completely controllable system is controllable and L_2 controllable. [Indeed, the pair $x(t)$, $u_a(t)$ $(0 \le t \le \infty)$ in Definition 1c can be obtained from the pair $x(t)$, $u_a(t)$ $(0 \le t \le T)$ in Definition 1a by setting $x(t) = 0_X$ and $u_a(t) = 0_U$ for $t \ge T$.] The system (1.1) may be L_2 controllable, but not controllable and, all the more so, not completely controllable. The definition of a completely controllable system differs from Kalman's definition for the finite-dimensional case (when X and $\mathbf{U}$ are finite-dimensional spaces) by the condition (1.2). It is easy to show that in the finite-dimensional case this condition is superfluous, since it follows from the previous conditions. For the infinite-dimensional case this is not so. In the finite-dimensional case, complete controllability is equivalent to controllability.

The following simple proposition whose proof is given in § 3 below establishes a connection between the properties of controllability of the system (1.1) with the properties of the operator

$$\mathcal{K} = \int_0^T e^{At} bb^* e^{A^*t} dt. \tag{1.3}$$

LEMMA 1.1. For complete controllability of the system (1.1) it is sufficient that for a $T > 0$ the operator $\mathcal{K}$ be uniformly positive†† ($\mathcal{K} \gg 0$), and necessary that for a $T > 0$ the operator $\mathcal{K}$ be positive

†Let us note that although we are considering here the case of bounded operators, almost all the results of this paper can be extended to the case that A in (1.1) is an unbounded operator which is a generating operator of a strongly continuous semigroup.

‡A. L. Likhtarnikov has shown that in Definition 1c it is possible to drop the last condition (before the word "where"), since it is automatically satisfied.

††An operator $\mathcal{K} = \mathcal{K}^*$, acting in a Hilbert space $Z = \{z\}$ is said to be uniformly positive if $\exists \delta > 0: z^* \mathcal{K} z \ge \delta |z|^2 \; \forall z \in Z$. An operator $\mathcal{K} = \mathcal{K}^*$ is said to be positive (nonnegative) if $z^* \mathcal{K} z > 0$ for $z \ne 0_Z$ (if $z^* \mathcal{K} z \ge 0 \; \forall z \in Z$). We shall write $\mathcal{K} \gg 0$, $\mathcal{K} > 0$, $\mathcal{K} \geqslant 0$, depending on whether $\mathcal{K}$ is a uniformly positive, positive, or nonnegative operator.

(> 0). For controllability of the system (1.1) it is necessary and sufficient that $\mathcal{K} > 0$ for a $T > 0$. Let

$$\mathcal{K} = \int_0^\Lambda \lambda dE_\lambda$$

be the spectral decomposition of the operator $\mathcal{K}$ and let $Q_\varepsilon = \int_0 dE_\lambda$. For L_2 controllability of the system (1.1) it is sufficient that for a positive ε the spectrum of the operator $Q_\varepsilon A Q_\varepsilon$ considered in the subspace $Q_\varepsilon X$ be located in the open left half-plane (or that for a positive T the spectrum of the operator $Q_\varepsilon e^{AT} Q_\varepsilon$ in the subspace $Q_\varepsilon X$ be located in the unit circle).

Let us note that these conditions of L_2 controllability are only sufficient, but not necessary conditions. It is possible to formulate more exact sufficient conditions of L_2 controllability, but they are cumbersome.

We shall present without proof some criteria of controllability of the system (1.1). (The proofs are almost an exact repetition of the well-known proofs of similar assertions for the finite-dimensional case.)

LEMMA 1.2. The following assertions are equivalent, and each of them is equivalent to controllability of the system (1.1).

(I). For any $a \in \mathbf{X}$, $a' \in \mathbf{X}$, $\delta > 0$, $T_1 > 0$, $T_2 > T_1$ there exists a function $u(\cdot) \in L_2[\mathbf{U}, (T_1, T_2)]$ such that the solution of system (1.1) with an initial condition $x(T_1) = a$ satisfies the relation $|x(T_2) - a'| < \delta$.

(II). For any $a \in \mathbf{X}$, $a' \in \mathbf{X}$, $\delta > 0$, $T_1 > 0$, $T_2 > T_1$ there exists a function $u(\cdot) \in L_2(\mathbf{U}, (T_1, T_2)]$ and a solution $x(t)$ of system (1.1) such that $|x(T_1) - a| < \delta$, $x(T_2) = a'$.

(I'), (II'). Assertion (I) [or (II)], with the condition "a function $u(\cdot) \in L_2[\mathbf{U}, (T_1, T_2)]$" being replaced by the condition "a function $u(t)$ defined on (T_1, T_2) and analytic for any $t \in (T_1, T_2)$."

(III). There coincides with X a linear closed hull of vectors $\{A^n bu\}$, where $n = 0, 1, \ldots$; $u \in U$.

(IV). There coincides with X a linear closed hull of vectors $\{e^{At}bu\}$, where $u \in \mathbf{U}$, $t \in \Delta$, with Δ being a fixed interval of the number axis.

(V). There coincides with X a linear closed hull of vectors $\{\lambda I - A)^{-1}bu\}$, where $u \in \mathbf{U}$, $|\lambda - \lambda_0| < \delta_0$, λ_0 being a fixed point such that the circle $|\lambda - \lambda_0| < \delta_0$ does not belong to the spectrum of the operator A.

(VI). If $\mathfrak{N}$ is a (closed) subspace in X that is invariant with respect to the operator A and $bu \in \mathfrak{N}$ $\forall u \in U$, then $\mathfrak{N} = X$.

(VII). If $\mathfrak{M}$ is a (closed) subspace in X that is invariant with respect to the operator A and for any $u \in \mathbf{U}$ the vector $bu (\in X)$ is orthogonal to $\mathfrak{M}$, then $\mathfrak{M} = 0_X$.

Definition 2. A system (1.1) (or pair $\{A, b\}$) is said to be L_2 stabilizable (exponentially stabilizable) if there exists a linear bounded operator $c : \mathbf{U} \to \mathbf{X}$ such that the operator $C = A + bc^*$ has the property

$$\int_0^\infty |e^{Ct}a|^2 \, dt < \infty \quad (\forall a \in \mathbf{X}) \tag{1.4}$$

(the operator C is Hurwitz †).

The properties mentioned in Definition 2 signify that it is possible to assign a "feedback"

$$u = c^* x, \tag{1.5}$$

such that the closed-loop system (1.1), (1.5) is L_2 stable [i.e., $|x(t)| \in L_2(0, \infty)$], or exponentially stable [i.e., for any solution $x(t)$ we have $|x(t)| \le \gamma e^{-\varepsilon t} |x(0)|$, where $\gamma > 0$ and $\varepsilon > 0$ do not depend on the solution].

Remarks. 1°. Any exponentially stabilizable system is evidently also L_2 stabilizable. It is not trivial to note that in fact any L_2-stabilizable system is also exponentially stabilizable (with the same operators

†A linear bounded operator C is said to be Hurwitz if its spectrum lies in the open left half-plane. A Hurwitz operator has a bound $|e^{Ct}| \le \gamma e^{-\varepsilon t}$ $\forall t \ge 0$ with $\gamma > 0$, $\varepsilon > 0$. (In [1], an exponentially stabilizable system was called by us a stabilizable system).

c and C); below we shall show (Theorem 3a) that from (1.4) there follows the bound $|e^{Ct}| \leq \text{const} \cdot e^{-\alpha t}$ with an $\alpha > 0$. Thus, in fact, exponential stabilizability and L_2 stabilizability are equivalent properties. However, until this is proved, we shall have to distinguish between these concepts.

2°. It is evident that an exponentially stabilizable system is L_2 controllable [the corresponding $u(\cdot) = S_1 a$ and $x(\cdot) = S_2 a$ have the form $u(t) = c^* e^{Ct} a$ and $x(t) = e^{Ct} a$]. It is easy to show (see § 2 below, Lemma 2.1) that an L_2-stabilizable system is also L_2 controllable.

3°. It follows directly from the definition that in the case of L_2 stabilizability the operator C has in addition to the property (1.4) also the property

$$|e^{Ct}a| \leqslant \text{const} \quad \text{for} \quad t \geqslant 0 \quad \forall a \in \mathbf{X}. \tag{1.6}$$

Indeed, from (1.1) and (1.5) we obtain for the solution $x(t) = e^{Ct}a$:

$$\frac{dx}{dt} = Cx \in L_2[\mathbf{X}, (0, \infty)],$$

$$|x(t)|^2 = |a^2| + \int_0^t 2 \operatorname{Re} x^* \frac{dx}{dt} dt \leqslant |a|^2 + \|x\|^2_{(0,\infty)} + \|Cx\|^2_{(0,\infty)}.$$

As a rule, we shall assume below that one of the following two conditions is satisfied:

(I) The system (1.1) is L_2 controllable.

(II) The system (1.1) is L_2 stabilizable.

We have noted above that (II) yields (I). It is important that in fact the properties (I) and (II) are equivalent (Theorem 3 on linear stabilization). However the proof of this proposition is not easy; therefore we shall have to distinguish for the time being between these properties.

Now let us formulate the frequency theorem. Depending on whether the spaces X and **U** are real or complex, we shall refer to a real or complex case. We shall consider a quadratic (in the real case) or Hermitian (in the complex case) form

$$F(x, u) = \begin{pmatrix} x \\ u \end{pmatrix}^* \hat{F} \begin{pmatrix} x \\ u \end{pmatrix}$$

on $X \times \mathbf{U}$. Here $\hat{F} = \hat{F}^*$ is a linear bounded self-adjoint operator on $X \times \mathbf{U}$ with a scalar product $x_1^* x_2 + u_1^* u_2$. We are interested in the existence of an operator $H = H^*: X \to X$ that has the following property: The form $F(x, u) + \operatorname{Re} x^* H(Ax + bu)$ is uniformly positive, i.e.,

$$\exists \delta > 0: F(x, u) + \operatorname{Re} x^* H(Ax + bu) \geqslant \delta[|x|^2 + |u|^2] \quad (\forall x, u). \tag{1.7}$$

(in the real case the sign of Re in (1.7) can evidently be omitted.) In the real case we shall denote by $X_C = X + (iX)$, $U_C = \mathbf{U} + (i\mathbf{U})$ the corresponding complex extensions, and extend with preservation of hermitivity the form $F(x, u)$ to $X_C \times U_C$ by setting $F(x_1 + ix_2, u_1 + iu_2) = F(x_1, u_1) + F(x_2, u_2)$ (for details, see [1]). The operator $K: X_C \to X_C$ is said to be real if $KX \subset X$. In the same way we can define real operators acting from U_C to X_C, from X_C to U_C, and from U_C to U_C. (In the finite-dimensional case there corresponds to a real operator a real matrix in a basis belonging to X).

THEOREM 1 (Frequency Theorem for Nondegenerate Case). We shall assume that the system (1.1) is either L_2 controllable, or L_2 stabilizable. For the existence of a linear bounded self-adjoint operator $H: X \to X$ that satisfies (1.7), it is necessary and sufficient that for a positive δ_0 we have

$$F(x, u) \geqslant \delta_0 |u|^2 \quad (\forall x \in \mathbf{X}_c, \ u \in \mathbf{U}_c, \ \omega \in \mathbf{R}^1: \ i\omega x = Ax + bu). \tag{1.8}$$

If (1.8) is satisfied, there exist linear bounded operators

$$H = H^*: \mathbf{X} \to \mathbf{X}, \quad h: \mathbf{U} \to \mathbf{X}, \quad \varkappa: \mathbf{U} \to \mathbf{U}, \tag{1.9}$$

such that

$$F(x, u) + \operatorname{Re} x^* H(Ax + bu) = |\varkappa(u - h^* x)|^2 \quad (\forall x \in \mathbf{X}, \ u \in \mathbf{U}) \tag{1.10}$$

and the operator $B = A + bh^*$ has the property $|e^{Bt}a| \in L_2(0, \infty)$ for any $a \in X_0$. The operator $\varkappa$ can be any operator (real in the real case) such that $F(0, u) = |\varkappa u|^2$. After selection of the operator $\varkappa$, the operators h and H with the above properties will be uniquely determined.

Remarks. 1°. In the real case the vectors x and u occurring in condition (1.8) are complex. Condition (1.8) signifies that for any $x \in X_C$, $u \in U_C$, and $\omega \in R^1$ connected by the relation $i\omega x = Ax + bu$ we have the bound $F(x, u) \geq \delta_0 |u|^2$. Let us note that in the real case Theorem 1 asserts the existence of real operators $H = H^*$, h, and $\varkappa$.

2°. If the operator A does not have spectral points on the imaginary axis, we can rewrite (1.8) in the form

$$F[(i\omega I-A)^{-1}bu, u] \geqslant \delta_0 |u|^2, \quad (\forall\, u \in U_c,\ \omega \in R^1). \tag{1.11}$$

3°. The operators H in (1.7) and (1.10) are distinct.

4°. It is easy to show that the condition $|e^{Bt}a| \in L_2(0, \infty)$ for any $a \in X$ implies the property that the spectrum of the operator B lies in the half-plane $\operatorname{Re}\lambda \leq 0$.† However, as we noted above, this condition signifies in fact that B is a Hurwitz operator, i.e., that its spectrum lies in the half-plane $\operatorname{Re}\lambda < 0$ (see Theorem 3a below). Thus $B = A + bh^*$ in Theorem 1 will be a Hurwitz operator.

5°. The necessity of condition (1.8) is obvious, i.e., from (1.7) we directly obtain (1.8). Therefore we shall prove below only the sufficiency of condition (1.8).

6°. For the validity of a representation (1.10) such that the operator $B = A + bh^*$ will have the property mentioned in the theorem, it is necessary that the system (1.1) be L_2 controllable. Indeed, this property of the operator B signifies that (1.1) is an L_2 stabilizable, and hence also L_2-controllable system.

THEOREM 2 (Frequency Theorem for Degenerate Case). We shall assume that the pairs $\{A, b\}$ and $\{-A, -b\}$ are L_2 controllable. For the existence of a linear bounded self-adjoint operator $H: X \to X$ that satisfies the relation

$$F(x, u) + \operatorname{Re} x^* H(Ax + bu) \geqslant 0 \quad (\forall\, x \in X,\ u \in U), \tag{1.12}$$

it is necessary and sufficient that

$$F(x, u) \geqslant 0 \quad (\forall\, x \in X_c,\ u \in U_c,\ \omega \in R^1 : i\omega x = Ax + bu). \tag{1.13}$$

If (1.13) holds, there exist linear bounded operators (1.9) such that the following relation holds‡:

$$F(x, u) + \operatorname{Re} x^* H(Ax + bu) = |\varkappa u - h^* x|^2. \tag{1.14}$$

The assumption of L_2 controllability of the pair $\{-A, -b\}$ can be replaced by the assumption of boundedness from below (for any fixed $a \in X$) of the functional

$$J[x(\cdot), u(\cdot)] = \int_0^\infty F[x(t), u(t)]\, dt$$

on a set of pairs $x(\cdot)$, $u(\cdot)$, where $x(\cdot) \in L_2[X, (0, \infty)]$, $u(\cdot) \in L_2[U, (0, \infty)]$ that satisfy almost everywhere the equation (1.1) and the condition $x(0) = a$. Boundedness from below of the functional $J[x(\cdot), u(\cdot)]$ is necessary for the existence of an operator $H = H^*$ satisfying the relation (1.12).

Let us note that the condition (1.13) signifies that for any $x \in X_C$, $u \in U_C$, and $\omega \in R^1$ connected by the relation $i\omega x = Ax + bu$, we have the bound $F(x, u) \geq 0$. If the operator A does not have spectral points on the imaginary axis, it is possible to rewrite condition (1.13) in the form

$$F[(A - i\omega I)^{-1}bu, u] \geqslant 0 \quad (\forall\, u \in U,\ \omega \in R^1). \tag{1.15}$$

Let us write the form F(x, u) explicitly:

$$F(x, u) = x^* G x + 2\operatorname{Re} x^* g u + u^* \Gamma u, \tag{1.16}$$

where $G = G^*$ and $\Gamma = \Gamma^*$. Formula (1.14) is equivalent to the operator relations

$$HA + A^*H + 2G = 2hh^*, \quad Hb + 2g = 2h\varkappa, \quad \Gamma = \varkappa^*\varkappa. \tag{1.17}$$

These relations, interpreted as equations for the operators $H = H^*$, h, and $\varkappa$, will be called Lur'e's equations (just as in the finite-dimensional case).

†The author is grateful to M. Z. Solomyak who showed that this assumption follows directly from the results of [6], p. 45.

‡Let us point out the difference between the right-hand sides of (1.14) and (1.10). Formula (1.14) can be written in the form (1.10) if F(0, u) is a positive-definite form.

In the corollaries that follow it will be assumed that the conditions of Theorem 2 are satisfied.

COROLLARY 1. For Lur'e's equation (1.17) to have a solution, it is necessary and sufficient that the "frequency condition" (1.13) be satisfied.

COROLLARY 2. (On the existence of a solution of Lyapunov's operator inequality in the presence of a linear constraint.) Let $A: \mathbf{X} \to \mathbf{X}$, $b: \mathbf{U} \to \mathbf{U}$ be assigned linear bounded operators, and suppose that the operator A does not have spectral points on the imaginary axis. For the existence of an operator $H = H^*: \mathbf{X} \to \mathbf{X}$ satisfying the relations

$$HA + A^*H \geqslant 0,\ Hb + g = 0, \tag{1.18}$$

it is necessary and sufficient that†

$$\operatorname{Re} g^*(i\omega I - A)^{-1}b \geqslant 0 \quad (\forall \omega \in R^1). \tag{1.19}$$

If the operator A has spectral points on the imaginary axis, we have a similar assertion, with the condition (1.19) being replaced by the condition

$$\operatorname{Re} u^*g^*x \geqslant 0 \ \ (\forall\ x \in \mathbf{X}_c,\ u \in \mathbf{U}_c,\ \omega \in R^1 : Ax + bu = i\omega x).$$

The proof follows directly from Theorem 2 for

$$F(x, u) = \operatorname{Re} u^*g^*x.$$

COROLLARY 3. Let

$$A: \mathbf{X} \to \mathbf{X},\ b: \mathbf{U} \to \mathbf{U},\ g: \mathbf{U} \to \mathbf{X},\ \Gamma = \Gamma^*: \mathbf{U} \to \mathbf{U}$$

be assigned bounded operators, and suppose that the operator A does not have spectral points on the imaginary axis. Let us write

$$F = A^*H + HA,\ f = Hb + g,$$

where $H = H^*: \mathbf{X} \to \mathbf{X}$ is an operator. For the existence of an operator $H = H^*$ satisfying the operator inequality

$$\begin{pmatrix} F & f \\ f^* & \Gamma \end{pmatrix} \geqslant 0, \tag{1.20}$$

it is necessary and sufficient that

$$\Gamma + 2\operatorname{Re} g^*(i\omega I_X - A)^{-1}b \geqslant 0 \quad (\forall \omega \in \mathbf{R}^1). \tag{1.21}$$

Let $\Gamma = \Gamma^* \gg 0$. For the existence of an operator $H = H^*$ satisfying the inequality

$$\begin{pmatrix} F & f \\ f^* & \Gamma \end{pmatrix} \gg 0 \tag{1.22}$$

or the equivalent quadratic inequality

$$A^*H + HA - (Hb + g)^*\Gamma^{-1}(Hb + g) \gg 0, \tag{1.23}$$

it is necessary and sufficient that for a positive δ we have

$$\Gamma + 2\operatorname{Re}[g^*(i\omega I_X - A)^{-1}b] \geqslant \delta I_U \ (\forall \omega \in \mathbf{R}^1). \tag{1.24}$$

For the proof it is sufficient to use Theorems 1 and 2, by taking the form.

$$F(x, u) = u^*\Gamma u + 2\operatorname{Re} x^*gu.$$

The assertions of Corollaries 2 and 3 for $\mathbf{X} = \mathbf{R}^n$ and $\mathbf{U} = \mathbf{R}^1$ were obtained for the first time in [2], where we considered the inequalities (1.18) (with the sign > instead of ≥) and (1.22); the problems (1.18) and (1.20) reduce to the problem using the inequality (1.22). For the case that $\mathbf{X}$ is a Hilbert space and $\mathbf{U} = \mathbf{R}^1$, the Corollaries 2 and 3 were proved by A. A. Nudel'man and P. A. Shvartsman. Their proof differs essentially from the proof given below, and it relies strongly on the finite-dimensionality of $\mathbf{U}$.

From the frequency theorem it is fairly easy to obtain the following assertion.

†In (1.18) and below, where Q is a bounded operator in Hilbert space, we denoted as usual $\operatorname{Re} Q = (Q + Q^*)/2$.

THEOREM 3 (On Linear Stabilization). Suppose that the system (1.1) is L_2 controllable. Then it will be exponentially stabilizable, i.e., there exists a linear bounded operator $c: \mathbf{U} \to \mathbf{X}$ such that the spectrum of the operator $B = A + bc^*$ [i.e., the operator of the system (1.1) and (1.5)] lies in the half-plane $\mathrm{Re}\,\lambda < 0$, and hence for a positive ε and any $t \geq 0$ we have the bound

$$|e^{Bt}| \leqslant \mathrm{const} \cdot e^{-\varepsilon t}.$$

Theorem 3 shows that any L_2-controllable system (1.1) can be exponentially stabilized by introducing an appropriate feedback (1.5).

COROLLARY. Suppose that the pair $\{-A, -b\}$ is L_2 controllable. Then there exists a linear bounded operator $c: \mathbf{U} \to \mathbf{X}$ such that the spectrum of the operator $B = A + bc^*$ lies in the half-plane

$$\mathrm{Re}\,\lambda > 0.$$

For the proof is suffices to apply Theorem 3 to the system

$$\frac{dx}{dt} = -Ax - bu.$$

On the other hand, Theorem 3 yields directly the following (above-mentioned) assertion which is of intrinsic interest.

THEOREM 3a. Let A be a linear bounded operator in a Hilbert space X that has the property $|e^{At}a| \in L_2(0, \infty)$ for any $a \in \mathbf{X}$. Then A will be a Hurwitz operator, i.e.,

$$\exists\, C > 0,\ \varepsilon > 0: |e^{At}| \leqslant Ce^{-\varepsilon t} \quad (\forall t \geqslant 0).$$

Indeed, a system (1.1) with an L_2-stable operator A i.e., $|e^{At}a| \in L_2(0, \infty)$ for $\forall a \in \mathbf{X}]$ and with $b = 0$ will evidently be L_2 controllable ($u_a(t) \equiv 0$).

As in [1], we shall prove Theorem 1 by solving an auxiliary variational problem. The corresponding result is of intrinsic interest; let us consider it.

A function $u(t) \in L_2[\mathbf{U}, (0, \infty)]$ is called an admissible control if the solution $x(t)$ of Eq. (1.1) with a fixed initial condition $x(0) = a \in \mathbf{X}$ has the property $x(t) \in L_2[\mathbf{X}, (0, \infty)]$. Let $F(x, u)$ be a form of type (1.16). On the set $\mathfrak{A}_a$ of admissible controls let us consider a functional

$$J[x(\cdot), u(\cdot)] = \int_0^\infty F[x(t), u(t)]\, dt, \tag{1.25}$$

where $x(t)$ is the corresponding solution. A pair $x(t)$, $u(t)$ that satisfies Eq. (1.1) is called a process. A control $u^0(t) \in \mathfrak{A}_a$ [a process $x^0(t)$, $u^0(t)$ for $x^0(0) = a$] is said to be optimal if

$$J[x^0(\cdot), u^0(\cdot)] \leqslant J[x(\cdot), u(\cdot)] \tag{1.26}$$

for any $u(t) \in \mathfrak{A}_a$ [for all the processes with $x(0) = a$].

THEOREM 4. Suppose that the system (1.1) is L_2 controllable. For the existence of an optimal control (process) for any $a \in \mathbf{X}$, it is sufficient that the "frequency condition" (1.8) be satisfied for a positive δ_0 [or the condition (1.11) if the operator A does not have spectral points on the imaginary axis], and necessary that the following "relaxed frequency condition" be satisfied:

$$F(x, u) \geqslant 0 \quad (\forall\, x \in \mathbf{X}_c,\ u \in \mathbf{U}_c,\ \omega \in \dot{\mathbf{R}}^1 : i\omega x = Ax + bu). \tag{1.27}$$

Suppose that (1.8) holds [or (1.11) under the corresponding assumption]. Then for any $a \in \mathbf{X}$ the optimal process will be unique and can be constructed in the form of a "feedback"

$$u^0(t) = h^* x^0(t), \tag{1.28}$$

where $h: \mathbf{U} \to \mathbf{X}$ is a linear bounded operator (defined in Theorem 1) that does not depend on $a \in \mathbf{X}$. Thus, $\inf\limits_{\mathfrak{A}_a} J[x(\cdot), u(\cdot)] = a^* H a$, where $H = H^*$ is the operator defined in Theorem 1. If (1.27) is violated, we have

$$\inf_{\mathfrak{A}_a} J[x(\cdot), u(\cdot)] = -\infty.$$

In applied problems it is often necessary to ensure not only stabilization of the system, but also a so-called "prescribed transient performance." This requirement can be formalized as follows. Let $\alpha \in \mathbf{R}^1$ be a given number, usually positive. (For the sake of generality, this assumption is not made here.)

A control $u(t):[0, \infty] \to \mathbf{U}$ is said to be α-admissible if

$$e^{\alpha t}u(t) \in L_2[\mathbf{U}, (0, \infty)], \; e^{\alpha t}x(t) \in L_2[\mathbf{X}, (0, \infty)], \; x(0) = a, \tag{1.29}$$

where x(t) is a solution of Eq. (1.1). The set of α-admissible controls will be denoted by $\mathfrak{A}_a^{(\alpha)}$. On the set $\mathfrak{A}_a^{(\alpha)}$ let us consider the functional

$$J[x(\cdot), u(\cdot)] = \int_0^\infty e^{2\alpha t} F[x(t), u(t)]\, dt, \tag{1.30}$$

where x(t) is the corresponding solution. A control $u^0(t)$ [process $x^0(t)$, $u^0(t)$] is said to be α-optimal [with respect to the functional (1.30)] if (1.26) holds for all controls $u(t) \in \mathfrak{U}_a^{(\alpha)}$ [all processes with $x(0) = a$].

THEOREM 5. Let the pair $\{A + \alpha I, b\}$ be L_2 controllable. For the existence of an α-optimal control (process) with respect to the functional (1.30), it is sufficient if the condition

$$\left.\begin{array}{c} F[x, u] \geqslant \delta |u|^2 \\ (\forall\, x \in \mathbf{X}_c,\; u \in \mathbf{U}_c,\; \omega \in \mathbf{R}^1 : (i\omega - \alpha)x = Ax + bu) \end{array}\right\} \tag{1.31}$$

holds for a positive δ, and necessary that the following condition hold:

$$\left.\begin{array}{c} F[x, u] \geqslant 0 \\ (\forall\, x \in \mathbf{X}_c,\; u \in \mathbf{U}_c,\; \omega \in \mathbf{R}^1 : (i\omega + \alpha)x = Ax + bu) \end{array}\right\}. \tag{1.32}$$

Suppose that (1.31) holds. Then an α-optimal process will be unique for any $a \in \mathbf{X}$, and it can be realized in the form of a "feedback"

$$u^0(t) = h_\alpha^* x^0(t), \tag{1.33}$$

where $h_\alpha : \mathbf{U} \to \mathbf{X}$ is a linear bounded operator that does not depend on a. This operator (together with the operator $H = H^*$) is uniquely determined by the representation

$$F(x, u) + \operatorname{Re} x^* H[(A + \alpha I)x + bu] = (u - h_\alpha^* x)^* \Gamma (u - h_\alpha^* x) \quad (\forall x \in \mathbf{X},\; u \in U) \tag{1.34}$$

and the Hurwitz condition of the operator $B + \alpha I$, where $B = A + bh_\alpha^*$ is the operator of the "closed-loop system" (1.1), (1.33). Thus $\inf J = a^* H a$. If the condition (1.32) is violated, then $\inf J = -\infty$. Here J is the functional (1.30), and the infimum is taken over all the α-admissible controls.

Theorem 5 follows from Theorem 4 if we effect a change of variables $x_1 = e^{\alpha t}x$, $u_1 = e^{\alpha t}u$ in (1.1). Convenient algorithms of determination of the operator h_α for the finite-dimensional case are given in [5].

In [1, 5] it is shown that a frequency theorem can be used for solving various problems of optimal control synthesis involving minimization of quadratic functionals. Since we have proved above a frequency theorem under assumptions that are weaker than in [1], it is possible to establish the corresponding results under assumptions that are weaker than in [1]. As an example, let us present the following assertion.

THEOREM 6. Suppose that the system (1.1) is L_2 controllable. Let the frequency condition (1.8) be satisfied [or, if the operator A does not have spectral points on the imaginary axis, the condition (1.11)]. Let h and H be the operators occurring in (1.10), and suppose that the operator $B = A + bh^*$ has the properties mentioned in the theorem. Suppose there exists a bounded inverse operator H^{-1}, and that the operator (1.24) in [1] has a bounded inverse for any $0 \le t \le T$. Then the assertions of Theorem 5 of [1] with regard to the existence and the form of an optimal control that minimizes the functional $J_T(u)$ in [1] will be valid.

In the same way it is possible to solve many other problems of optimal control synthesis involving minimization of quadratic funtionals. For example, the results of [8, 9, 10] can be extended to the infinite-dimensional case.

§ 2. Proof of Theorems 1 and 4

The general scheme of the proof of Theorem 1 will be almost the same as in [1]. We shall consider an optimization problem whose solution is given by Theorem 4, and at the same time we shall prove Theorems 1 and 4.

For various reasons (see the next footnote) we shall, however, in contrast to [1], reduce this optimization problem to a problem of minimization of a quadratic functional on a plane in a Hilbert space $\mathcal{H}$. More precisely, we shall consider the Hilbert space

$$\mathcal{H}=L_2[\mathbf{X},\ (0,\ \infty)]\times L_2[\mathbf{U},\ (0,\ \infty)]. \tag{2.1}$$

Let F(x, u) be a quadratic (in the real case), or Hermitian (in the complex case) form (1.16) and let $J[x(\cdot),\ u(\cdot)]$ be a functional (1.25) defined on $\mathcal{H}$. Let $a \in \mathbf{X}$ be a fixed vector. By $\mathfrak{M}_a$ we denote a set of pairs $\{x(\cdot),\ u(\cdot)\} \in \mathcal{H}$, for which (1.1) holds almost everywhere, and let

$$x(0)=a. \tag{2.2}$$

[It follows from (1.1) that x(t) is an absolutely continuous function.] The problem of interest to us is the minimization of the functional $J[x(\cdot),\ u(\cdot)]$ on the set $\mathfrak{M}_a$.

Let us note that the set $\mathfrak{M}_a$ can be defined also as a set of pairs $\{x(\cdot),\ u(\cdot)\} \in \mathcal{H}$, such that

$$x(t)=e^{At}a+\int\limits_0^t e^{A(t-s)}bu(s)\,ds. \tag{2.3}$$

In general the set $\mathfrak{M}_a$ can be empty. [This will be the case, for example, if (–A) is a Hurwitz operator and b = 0]. In this section, just as in § 1, we shall assume that one of the following conditions holds.

(I) The system (1.1) is L_2 controllable.

(II) The system (1.1) is L_2 stabilizable.

It is quite obvious that under either of these conditions, $\mathfrak{M}_a$ will be a nonempty closed set for any $a \in \mathbf{X}$ (this will be shown below). At first we shall prove the following assertion.

LEMMA 2.1. Suppose that the bounded operator $C : \mathbf{X} \to \mathbf{X}$ has the property

$$\varphi(x)=\int\limits_0^\infty |e^{Ct}x|^2\,dt<\infty \quad (\forall x \in \mathbf{X}).$$

Then there exists a positive constant γ such that for any $x \in \mathbf{X}$ we have $\varphi(x) \leq \gamma |x|^2$, i.e., the operator R which maps X into $L_2[X,\ (0,\ \infty)]$ and is defined by the formula $Rx = e^{Ct}x$ will be linear and bounded. In particular, an L_2-stabilizable system (1.1) will be L_2 controllable.

Proof. Let us write

$$H_T=\int\limits_0^T e^{C^*t}e^{Ct}dt,\quad \varphi_T(x)=x^*H_Tx=\int\limits_0^T |e^{Ct}x|^2\,dt.$$

It is evident that $H_T = H_T^* > 0$. Hence there exists an $H_T^{1/2}$ and $\varphi_T(x) = |H_T^{1/2}x|^2$. By virtue of the condition of the lemma we have $\sup\limits_{T>0} |H^{1/2}x| < +\infty$ $\forall x \in \mathbf{X}$. By virtue of the Banach – Steinhaus theorem ([6], p. 21) we have $\sup\limits_{T>0} |H^{1/2}| = \gamma^{1/2} < \infty$. Therefore $|H_T| = |H_T^{1/2}|^2 \leq \gamma$, and hence also $\varphi(x) \leq \gamma |x|^2$. Let (1.1) be an L_2-stabilizable system, and let C and c be the operators occurring in Definition 2. The pair $x(\cdot) = Ra$, $u_a(\cdot) = c^*Ra$ evidently belongs to $\mathfrak{M}_a$. Therefore (1.1) will be an L_2-controllable system and $S_1 = R$, $S_2 = c^*R$. This completes the proof of Lemma 2.1.

LEMMA 2.2. Suppose that one of the conditions (I) and (II) holds. Then $\mathfrak{M}_a$ will be a nonempty closed set for any $a \in \mathbf{X}$, and there exists a linear bounded operator $S : \mathbf{X} \to \mathcal{H}$, such that $Sa \in \mathfrak{M}_a$ for any $a \in \mathbf{X}$.

Proof. If (I) holds [and hence by virtue of Lemma 2.1, also if (II) holds], $\mathfrak{M}_a$ will be a nonempty set. Let us show that $\mathfrak{M}_a$ is closed. Let $\{x_n,\ u_n\} \xrightarrow{\mathcal{H}} \{x,\ u\}$, $\{x_n,\ u_n\} \in \mathfrak{M}_a$. Let us take a positive T. For the corresponding sections of functions considered in $L_2[\mathbf{X},\ (0,\ T)]$ and $L_2[\mathbf{U},\ (0,\ T)]$ we have $x_n \to x$ and $u_n \to u$ in $L_2[\mathbf{U},\ (0,\ T)]$. From the relation (2.3) for the functions x_n and u_n we find that (2.3) holds for limiting functions on [0, T], and hence almost everywhere on [0, ∞]. Thus $\mathfrak{M}_a$ is closed. The existence of the operator S follows from Definition 1c.

Now let us return to the optimization problem considered above. As we can see, this problem involves minimization of the form (1.25) defined on the plane $\mathfrak{M}_a$.† We shall show that under the condition (I) or (II), there exists for any $a \in X$ a uniquely determined optimal process $\{x^0(t, a),\ u^0(t, a)\} \in \mathfrak{M}_a$, if the frequency condition (1.8) is satisfied. [It follows from Lemma 2.1 that it suffices to consider the condition (I)].

Next we shall show that

$$\inf_{\mathfrak{M}_a} J[x(\cdot),\ u(\cdot)] = a^*Ha, \quad u^0(t,\ a) \equiv h^*x^0(t,\ a),$$

where $H = H^*$, h are operators that have the properties mentioned in Theorem 1. Thus we have proved the "sufficiency" in Theorems 1 and 4. Incidentally we shall prove an assertion that yields the necessity of the frequency condition (1.27) in Theorem 4. [As we noted in Remark 5° about Theorem 1, the necessity of condition (1.8) is obvious].

LEMMA 2.3.‡ Let $Z = \{z\}$ be a Hilbert space, $J(z) = z^*Gz + 2\operatorname{Re} g^*z$ a quadratic functional on Z ($G = G^* : Z \to Z$ being a bounded operator, $g \in Z$), $\mathfrak{M}_0$ a subspace in Z, $m \in Z$, and $\mathfrak{M} = \mathfrak{M}_0 + m$ a closed plane in Z. Let

$$\alpha = \inf_{y \in \mathfrak{M}_0} \frac{y^*Gy}{|y|^2}. \tag{2.4}$$

For the existence of an element $z_0 \in \mathfrak{M}$, such that

$$J(z) \geqslant J(z_0) \quad \forall z \in \mathfrak{M} \tag{2.5}$$

(and called optimal), it is necessary and sufficient that $\alpha \geq 0$ and that the following relations be solvable in z:

$$z \in \mathfrak{M},\ Gz + g \perp \mathfrak{M}_0. \tag{2.6}$$

Any element z_0 satisfying these relations is called an optimal element. The relations (2.6) are uniquely solvable for $\alpha > 0$. If $\alpha < 0$, then $\inf_{\mathfrak{M}} J(z) = -\infty$.

Proof. Necessity. Suppose that (2.5) holds, but $\alpha < 0$, i.e., $\exists y_0 \in \mathfrak{M}_0$, $y_0^*Gy_0 < 0$. For $z_\lambda = \lambda y_0 + m \in \mathfrak{M}$ we have $J(z_\lambda) \to -\infty$ for $\lambda \to +\infty$. Thus $\alpha \geq 0$. Let $h \in \mathfrak{M}_0$, and hence $z_0 + h \in \mathfrak{M}$. From (2.5) we obtain

$$J(z_0 + h) = J(z_0) + \operatorname{Re} h^*(Gz_0 + g) + O(|h|^2) \geqslant J(z_0) \quad \forall h \in \mathfrak{M}_0.$$

Hence $\operatorname{Re} h^*(Gz_0 + g) \geq 0$ $\forall h \in \mathfrak{M}_0$, i.e., $h^*(Gz_0 + g) = 0$ $\forall h \in \mathfrak{M}_0$. Therefore (2.6) will hold for $z = z_0$.

Sufficiency. Let $\alpha \geq 0$ and let (2.6) be satisfied for $z = z_0$, i.e., $z_0 \in \mathfrak{M}$, $Gz_0 + g = n \perp \mathfrak{M}_0$. We have $J(z) - J(z_0) = [z^*Gz + 2\operatorname{Re} z^*(n - Gz_0)] - [z_0^*Gz_0 + 2\operatorname{Re} z^*(n - Gz_0)] = y^*Gy + 2\operatorname{Re} n^*y$, where $y = z - z_0$. For $z \in \mathfrak{M}$ we have $y \in \mathfrak{M}_0$, $J(z) - J(z_0) = y^*Gy \geq 0$, since $\alpha \geq 0$. Thus we have obtained (2.5). For $z = m + y$ the relations (2.6) are written in the form $y \in \mathfrak{M}_0$, $PGP_y + P(Gm + g) = 0z$, where P is an orthogonal projection operator in $\mathfrak{M}_0$. For $\alpha > 0$ the restriction of the operator PGP to $\mathfrak{M}_0$ is a bounded invertible operator, i.e., the equation for $y \in \mathfrak{M}_0$ can be solved uniquely. This completes the proof of the lemma.

Now let us determine the sign of the number (2.4) for the case under consideration, when $Z = \mathscr{H}$ and the functional has the form (1.25). Let us write

$$\left.\begin{array}{l} \alpha' = \inf \dfrac{F(x,\ u)}{|x|^2 + |u|^2} \\ (\omega \in \mathbf{R}^1,\ x \in \mathbf{X}_c,\ u \in \mathbf{U}_c,\ i\omega x = Ax + bu) \end{array}\right\}. \tag{2.7}$$

†Let us show how this method differs from [1]. In [1] we have considered the case of a Hurwitz operator A. Then formula (2.3) will define a bounded operator from $L_2[\mathbf{U},\ (0,\ \infty)]$ to $L_2[\mathbf{X},\ (0,\ \infty)]$, and the functional $J[x(\cdot),\ u(\cdot)]$ will be defined on the entire space $L_2[\mathbf{U},\ (0,\ \infty)]$. In our case this is not so. Moreover, the set of all $u(\cdot) \in L_2[\mathbf{U},\ (0,\ \infty)]$ such that $x(t) \in L_2[\mathbf{X},\ (0,\ \infty)]$ [where x(t) is determined from (2.3)] may not be a subspace in $L_2[\mathbf{U},\ (0,\ \infty)]$. This makes it necessary to modify the scheme of the proof given in [1]. Let us note that similar difficulties arise by using the method of Lions [11].

‡The assertion of this lemma is not new, but the author cannot find the source in which it is formulated in such a way. Similar assertions can be found in [7]. The symbol $\perp$ in (2.6) denotes orthogonality to the subspace $\mathfrak{M}_0$.

(The infimum is taken over all $\omega \in R^1$, $x \in X_C$, $u \in U_C$ connected by the relation $i\omega x = Ax + bu$). If the operator A does not have spectral points on the imaginary axis, we shall write

$$\alpha'' = \inf_{\omega \in R^1,\, u \in U_c} \frac{F[(i\omega I - A)^{-1} bu, u]}{|u|^2}. \tag{2.8}$$

LEMMA 2.4. Suppose that one of the conditions (I) and (II) is satisfied. If $\alpha' > 0$, or (in the absence of spectral points of the operator A on the imaginary axis) $\alpha'' > 0$, then $\alpha > 0$, and hence any $a \in X$ there exists a unique optimal process $\{x^0(\cdot), u^0(\cdot)\} \in \mathfrak{M}_a$, i.e., a process such that

$$J[x^0(\cdot), u^0(\cdot)] \leqslant J[x(\cdot), u(\cdot)] \quad (\forall \{x(\cdot), u(\cdot)\} \in \mathfrak{M}_a). \tag{2.9}$$

Proof. Let us show that the inequality $\alpha'' > 0$ (in the assertion of the lemma) implies $\alpha' > 0$. For $x = [A - i\omega I]^{-1} bu$ we have $|x|^2 + |u|^2 \le \varkappa |u|^2$, where $\varkappa = 1 + \sup_\omega |(A - i\omega I)^{-1} b|^2$. Let $\alpha'' > 0$. In (2.7) we then have $F > 0$, and from (2.7)-(2.8) we obtain $\alpha' \ge \varkappa^{-1}\alpha'' > 0$.

Let $\alpha' > 0$. Let us show that $\alpha > 0$. In our case,

$$\alpha = \inf \frac{1}{\|x\|^2 + \|u\|^2} \int_0^\infty F[x(t), u(t)]\, dt, \tag{2.10}$$

where the infimum is taken over all $\{x(t), u(t)\} \in \mathfrak{M}_0$, i.e., over all the processes satisfying (2.3) for $t \ge 0$ with $a = 0_X$. Let us assign x(t) and u(t) for $t \le 0$ by setting $x(t) = 0_X$ and $u(t) = 0_U$ for $t < 0$. Then x(t) will remain an absolutely continuous function, and the relation $dx/dt = Ax + bu$ will be satisfied almost everywhere on $(-\infty, +\infty)$. Let us write

$$\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} x(t)\, dt, \quad \tilde{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} u(t) dt.$$

(The integrals are convergent in the mean-square in the sense of the principal value). We have $i\omega\tilde{x}(\omega) = A\tilde{x}(\omega) + b\tilde{u}(\omega)$. By using Parseval's equation and the boundedness of the operator of the form F(x, u), we obtain

$$J = \int_{-\infty}^{+\infty} F[x(t), u(t)]\, dt = \int_{-\infty}^{+\infty} F[\tilde{x}(\omega), \tilde{u}(\omega)]\, d\omega. \tag{2.11}$$

Similarly we obtain for the norms in $L_2[X, (0, \infty)]$ and $L_2[U, (0, \infty)]$ the formulas

$$\|x\|^2 + \|u\|^2 = \int_{-\infty}^{+\infty} (|x(t)|^2 + |u(t)|^2)\, dt = \int_{-\infty}^{+\infty} (|\tilde{x}(\omega)|^2 + |\tilde{u}(\omega)|^2)\, d\omega. \tag{2.12}$$

From (2.7) we obtain

$$F[\tilde{x}(\omega), \tilde{u}(\omega)] \geqslant \alpha' [|\tilde{x}(\omega)|^2 + |\tilde{u}(\omega)|^2].$$

From (2.10)-(2.12) it follows that $\alpha \ge \alpha'$, i.e., $\alpha > 0$, which completes the proof.

LEMMA 2.5.* Suppose that either the system (1.1) is L_2 stabilizable, or that both pairs $\{A, b\}$ and $\{-A, -b\}$ are completely controllable. Let $\alpha' < 0$, or (in the absence of spectral points of the operator A on the imaginary axis) $\alpha'' < 0$. Then $\alpha < 0$, and hence for any $a \in X$ the infimum of the functional $J[x(\cdot), u(\cdot)]$ on the set $\mathfrak{M}_a$ will be equal to $(-\infty)$.

Proof. Let (in the assertion of the lemma) $\alpha'' < 0$. Then $F < 0$ in (2.8) for some u and ω. It follows from (2.7) that

$$\alpha' < 0.$$

Let us assume that $\alpha' < 0$, i.e., there exist $x_0 \in X_C$, $u_0 \in U_C$, and $\omega_0 \in R^1$ such that $Ax_0 + bu_0 = i\omega_0 x_0$, $F(x_0, u_0) < 0$. Let us show that $J[x(t), u(t)] < 0$ for a pair $\{x(t), u(t)\} \in \mathfrak{M}_0$, i.e., that $\alpha < 0$. Since the pair $x'(t) = e^{i\omega_0 t} x_0$, $u'(t) = e^{i\omega_0 t} u_0$ satisfies the equation $dx/dt = Ax + bu$ and the integral $\int_0^t F[x'(t), u'(t)]dt = tF(x_0, u)$ is an arbitrarily large (in absolute value) negative number, it is natural to take the process $\{x'(t), u'(t)\}$ as the "principal part" of the sought process $\{x(t), u(t)\}$. The process $\{x'(t), u'(t)\}$ must be "adjusted," since it does not belong to $\mathfrak{M}_0$.

*This lemma is not used in the proof of Theorems 1 and 2.

Suppose that the pairs $\{A, b\}$ and $\{-A, -b\}$ are completely controllable. It follows from the complete controllability of the pair $\{-A, -b\}$ that for the system (1.1) there exists a control $u_1(t)|_0^T$ that carries in the interval [0, T] the state $x(0) = 0_X$ into the state $x(T) = x_0$; let $x_1(t)$ be the corresponding solution of the system (1.1). [Indeed, it follows from Definition 1a that there exists a pair $x_1'(t)$, $u_1'(t)$ $(0 \le t \le T)$ such that $dx_1'(t)/dt = -Ax_1'(t) - bu_1'(t)$, $x_1'(0) = x_0$, $x_2'(T) = 0_X$; then $x_1(t) = x_1'(T-t)$, $u_1(t) = u_1(T-t)$). It follows from the complete controllability of the pair $\{A, b\}$ that there exists for the system (1.1) a control $u_2(t)|_0^T$ that carries in [0, T] the state $x(0) = x_0$ into the state $x(T) = 0_X$; let $x_2(t)$ be the corresponding solution of the system (1.1). Then for any $s > 0$ the control $u_2(t-s)$ will carry in the interval $[s, T+s]$ the state $x(s) = x_0$ into the state $x(T+s) = 0_X$, and $x_2(t+s)$ is the corresponding solution. Let us write $T_0 = 2\pi/\omega_0$, and

$$\begin{aligned}
u(t) &= u_1(t) \text{ on } [0, T],\\
u(t) &= e^{i\omega_0(t-T)}u_0 \text{ on } [T, T+nT_0],\\
u(t) &= u_2(t) \text{ on } [T+nT_0,\ 2T+nT_0],\\
u(t) &= 0_U \text{ on } [2T+nT_0,\ \infty].
\end{aligned}$$

Here n is an integer. The corresponding solution of Eq. (1.1) will be (see Fig. 1):

$$\begin{aligned}
x(t) &= x_1(t) \text{ on } [0, T],\\
x(t) &= e^{i\omega_0(t-T)} \text{ on } [T, T+nT_0],\\
x(t) &= x_2(t-T-nT_0) \text{ on } [T+nT_0,\ 2T+nT_0],\\
x(t) &= 0_X \text{ on } [2T+nT_0,\ \infty].
\end{aligned}$$

In this case,

$$\int_0^\infty F\,[x(t),\ u(t)]\,dt = \int_0^T + \int_T^{T+nT_0} + \int_{T+nT_0}^{2T+nT_0} = \int_0^T F[x_1(t), u_1(t)]\,dt + n\,T_0 F\,[x_0,\ u_0] + \int_0^T F\,[x_2(t),\ u_2(t)]\,dt < 0$$

for sufficiently large n.

Suppose that we have the complex case, i.e., $X_C = X$ and $U_C = U$. Then $\{x(t), u(t)\} \in \mathfrak{M}_0$, i.e., $\alpha < 0$. Now suppose that we have the real case. Then $\mathfrak{M}_0 \in X$ and since the pair $\{x(t), u(t)\}$ constructed by us is "complex" (i.e., $\in X_C \times U_C$), it will not belong to $\mathfrak{M}_0$, since it satisfies (2.3) with $a = 0_X$. Let $x(t) = x'(t) + ix''(t)$ and $u(t) = u'(t) + iu''(t)$, where $x'(t) \in X$, $x''(t) \in X$, $u'(t) \in U$, and $u''(t) \in U$ are "real." We have $F[x, u] = F[x', u'] + F[x'', u'']$, $J[x, u] = J[x', u'] + J[x'', u'']$, and since $J[x, u] < 0$, it follows that either $J[x', u'] < 0$, or $J[x'', u''] < 0$, with $\{x', u'\} \in \mathfrak{M}_0$ and $\{x'', u''\} \in \mathfrak{M}_0$. Thus at least one of the pairs $\{x', u'\}$ and $\{x'', u''\}$ satisfies the required condition, i.e., $\alpha < 0$.

Let (II) be satisfied; $C = A + b^*c$ is an L_2-stable operator (see Definition 2). In the system $dx/dt = Ax + bu$ which specifies the subspace $\mathfrak{M}_0$ let us effect a change of variables $u = v + c^*x$. We obtain the system $dx/dt = Cx + bv$, $x(0) = 0_X$. Let $F(x, u) = F(x_1, v + c^*x) = F_1(x, v)$, $v_0 = u_0 + c^*x_0$. Then $i\omega_0 x_0 = Cx_0 + bv_0$. Let us write $v(t) = v_0 e^{i\omega_0 t}$ on [0, T] and $v(t) = 0_U$ on $[T, \infty)$, i.e., $u(t) = u_0 e^{i\omega_0 t}$ on [0, T] and $u(t) = c^*x(t)$ on $[T, \infty)$. The corresponding solution will be $x(t) = e^{i\omega_0 t}x_0 - e^{Ct}x_0$ on [0, T], and $x(t) = e^{C(t-T)}x_T$ on $[T, \infty)$, where $x_T = x(T) = e^{i\omega_0 T}x_0 - e^{Ct}x_0$. In the complex case we have $\{x(t), u(t)\} \in \mathfrak{M}_0$. For $J = J[x(t), u(t)]$ we have

$$J = \int_0^\infty F_1\,[x(t),\ v(t)]\,dt = \int_0^T + \int_T^\infty. \tag{2.13}$$

Let us evaluate these two integrals. Since $v(t) = 0_U$ on $[T, \infty)$, we have

$$\left|\int_T^\infty\right| = \left|\int_0^T F_1\,[x(t),\ O_U]\,dt\right| = \text{const}\int_T^\infty |x(t)|^2\,dt.$$

Here and in the following, "const" is a constant not dependent on T. On (T, ∞) we have

$$|x(t)|^2 \leqslant [\,|e^{C(t-T)}x_0| + |e^{Ct}x_0|\,]^2 \leqslant 2|e^{C(t-T)}x_0|^2 + 2|e^{Ct}x_0|^2.$$

Hence

$$\left|\int_T^\infty F_1(x,\ v)\,dt\right| \leqslant \text{const}. \tag{2.14}$$

On [0, T] we have $F_1(x, v) = F_1(x_0, v_0) + \mathrm{Re}\,[k^* e^{Ct} x_0 e^{i\omega_0 t}] + F_1[e^{\beta t} x_0, 0_U]$, where $k \in X_C$.

The integral of the absolute value of the third term does not exceed

$$\text{const} \int_0^T |e^{Ct} x_0|\, dt,$$

a quantity which is bounded by virtue of property (II). The integral of the absolute value of the second term does not exceed

$$\text{const} \int_0^T |e^{Ct} x_0|\, dt \leqslant \text{const} \left[T \int_0^T |e^{Ct} x_0|^2\, dt \right]^{1/2} = O(\sqrt{T}).$$

Since $F_1(x_0, v_0) = F(x_0, u_0)$, it follows that

$$\int_0^T F_1(x, v)\, dt = F(x_0, v_0)\, T + O(\sqrt{T}). \tag{2.15}$$

From (2.13)-(2.15) it follows that $J < 0$ for sufficiently large T. We have found that $\alpha < 0$ in the complex case. By the same reasoning as above, we find that $\alpha < 0$ also in the real case, which completes the proof of Lemma 2.5.

Let us note that from Lemma 2.5 and Theorem 3 there follows the necessity of the frequency condition (1.27) in Theorem 4, as well as the last assertion of Theorem 4.

Let us assume that $\alpha > 0$ and that one of the conditions (I)-(II) holds. Let us denote by

$$x^0(t, a),\ u^0(t, a) \tag{2.16}$$

the optimal process existing by virtue of Lemma 2.4, and by

$$V(a) = \int_0^\infty F[x^0(t, a),\ u^0(t, a)]\, dt \tag{2.17}$$

the minimum value of the functional (2.1) on the set $\mathfrak{M}_a$. Let us ascertain the dependence of the functions (2.16)-(2.17) on $a \in X$.

For this purpose we return to the abstract situation of Lemma 2.3, taking into account that in our case there exists by virtue of Lemma 2.2 an operator $S: X \to \mathscr{H}$, such that $Sa \in \mathfrak{M}_a$ for any $a \in X$.

LEMMA 2.6. Let $Z = \{z\}$ and X be Hilbert spaces, let $J(z) = z^*Gz + 2\,\mathrm{Re}\, g^*z$ be a quadratic functional on Z, $\mathfrak{M}_0$ a subspace of Z, $S: X \to Z$ a linear bounded operator, $\mathfrak{M}_a = \mathfrak{M}_0 + Sa$, and $\{\mathfrak{M}_a\}$ a family of planes that correspond to all sorts of $a \in X$. Let $\alpha > 0$, where α is defined by (2.4), and hence there exists for any $a \in X$ an element $z(a)$ (said to be optimal) such that

$$J(z) \geqslant J[z(a)]\ \ (\forall z \in \mathfrak{M}_a). \tag{2.18}$$

Then there exist linear bounded operators $M: X \to Z$ and $N: Z \to Z$ that do not depend on g and a, and such that $z(a) = Ma + Ng$ $(\forall a \in X)$.

Proof. Let P be an orthogonal projection operator of the subspace $\mathfrak{M}_0$ that acts in Z. By virtue of Lemma 2.3 the element $z(a)$ is uniquely determined by the relations $z(a) \in \mathfrak{M}_a$, and $Gz(a) + g \perp \mathfrak{M}_0$, i.e., $z(a) = y + Sa$, $y \in \mathfrak{M}_0$, $P(Gz(a) + g) = 0_Z$. We shall rewrite the last relation in the form $G_0 y + P(GSa + g) = 0_Z$, where $G_0 = PGP$. Since $a > 0$, the operator $G_0 = G_0^*$ will be uniformly positive on $\mathfrak{M}_0$, and hence bounded invertible on $\mathfrak{M}_0$. Therefore $y = -G_0^{-1}P(GSa + g)$ and $z(a) = Ma + Ng$, where $M = S - G_0^{-1}PGS$ and $N = -G_0^{-1}P$ are linear bounded operators.

LEMMA 2.7. Suppose that one of the conditions (I)-(II) holds, and that $\alpha' > 0$, or if the operator A does not have spectral points on the imaginary axis, then $\alpha'' > 0$. Hence

$$x^0(t, a) = N_1 a,\ u^0(t, a) = N_2 a,\ V(a) = a^* H a, \tag{2.18}$$

where $N_1: X \to L_2[X, (0, \infty)]$, $N_2: X \to L_2[U, (0, \infty)]$, and $H = H^*: X \to X$ are linear bounded operators.

Proof. Let us apply Lemma 2.6. In our case $Z = \mathscr{H}$, $z = \{x(\cdot), u(\cdot)\} \in \mathscr{H}$, the functional $J(z)$ is specified by formula (1.25), and hence $g = 0_Z$. It follows from Lemma 2.6 that $z(a) = \{x^0(\cdot, a), u^0(\cdot, a)\} = Ma$, where $M: X \to \mathscr{H}$ is a linear bounded operator, i.e., the first two formulas (2.18) are valid. Since

$$V(a) = z(a)^* G z(a) = a^* M^* G M a,$$

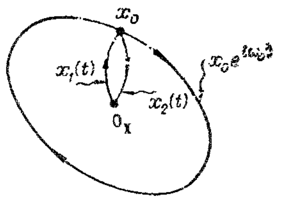

Fig. 1

it follows that $V(a) = a^*Ha$, where $H = M^*GM = H^*$ is a linear bounded operator from X to X. This completes the proof of Lemma 2.7.

The subsequent proof of Theorems 1 and 4 is exactly the same as in [1]. The functions $x^0(t, a)$ and $u^0(t, a)$ have the semigroup property:

$$u^0(t+s, a) = u^0[t, x^0(s, a)], \; x^0(t+s, a) = x^0[t, x^0(s, a)] \tag{2.19}$$

(see [1], Lemma 6; here $s \geq 0$, the first equation is understood as an equality between elements of $L_2[\mathbf{U}, (0, \infty)]$ for any fixed $s \geq 0$, whereas the second equation holds almost everywhere in the interval $(0 \leq t < \infty)$). Let

$$G(x, u) = F(x, u) + \mathrm{Re}\, x^*H(Ax+bu). \tag{2.20}$$

Then we have almost everywhere in the interval $0 \leq t < \infty$ the relations

$$G[x^0(t, a), v] \geqslant 0, \; G[x^0(t, a), u^{(0)}(t, a)] = 0 \tag{2.21}$$

(see [1], Lemma 7; here $v \in \mathbf{U}$ is arbitrary). Let us write

$$h = -(Hb+g)\Gamma^{-1}, \tag{2.22}$$

where g and Γ are operators in the representation (1.16). It follows from (2.21) (by virtue of Lemmas 8 and 9 of [1]) that the form (2.20) can be represented as follows:

$$G(x, u) = |\varkappa(u-h^*x)|^2 \tag{2.23}$$

and almost everywhere in the interval $0 \leq t < \infty$ we have

$$u^0(t, a) = h^*x^0(t, a). \tag{2.24}$$

It is evidently possible to assume that (2.24) holds for any $t \in [0, \infty)$. From (2.24) we obtain $x^0(t, a) = e^{Bt}a$, where $B = A + bh^*$, and hence the operator has the property mentioned in Theorem 1. The uniqueness of the operators h and H with the properties mentioned above can be established in exactly the same way as in [1].

We have proved all the assertions of Theorems 1 and 4, apart from the necessity of the frequency condition (1.8) and of Theorem 1. The necessity of condition (1.8) is obvious, i.e., (1.8) follows directly from (1.7).

Let us note that the necessity of the frequency condition (1.27) in Theorem 4 has been proved for the time being only conditionally, since we used Theorem 3 in the proof. [From Theorem 3 there follows L_2 stabilizability of system (1.1), which made it possible to apply Lemma 2.5.] Below we shall prove Theorem 3 independently of Theorem 4, and thus the proof of Theorem 4 will be complete.

§ 3. Proof of Lemma 1.1 and of Theorem 3

At first we shall prove Lemma 1.1. From (1.3) it follows that $\mathscr{K} \geq 0$. Let us show that controllability, and (all the more so) complete controllability of system (1.1) implies $\mathscr{K} > 0$. Let us assume the contrary: $\exists r \in X$, $r \neq 0_X$, $r^*Kr = 0$. From (1.3) it follows what $r^*\mathscr{K}r = \int_0^T |r^*e^{At}b|^2 dt$, i.e., $r^*e^{At}b = 0$ for $0 \leq t \leq T$. Hence the solution of (1.1) with initial condition $x(0) = a$ for any positive T and any control u(t) will satisfy the relation $r^*[x(T) - e^{AT}a] = 0$. Hence it follows that x(T) cannot be an arbitrary vector, i.e., (1.1) is not a completely controllable system. If the sequence $u_n(t)$ is such that $x_n(T) \to 0$, then $r^*e^{AT}a = 0$. This equation cannot be satisfied for any a. Therefore (1.1) is not a controllable system.

Let $\mathscr{K} \gg 0$ for a positive T. Let us show that the system (1.1) is completely controllable. In the interval [0, T] let us take a control u(t)

$$u(t) = b^* e^{A^*(T-t)} c, \tag{3.1}$$

$c \in \mathbf{X}$, and a vector c such that $x(0) = a$ and $x(T) = 0_{\mathbf{X}}$. Since there exists a bounded invertible operator $\mathscr{K}^{-1}$, it follows from (1.1) and (1.3) that $c = -K^{-1}e^{At}a$. Hence the required control $u(\cdot)$ and operator R exist; formula (1.2) has the form $u(t) = b^* e^{A(T-t)} K^{-1} a$.

Let $\mathscr{K} > 0$ for a positive T. Let us show that the system (1.1) is controllable. Let

$$\mathscr{K} = \int_0^{\Lambda} \lambda dE_\lambda$$

be a spectral decomposition of the operator $\mathscr{K}$. Let us write $P_\varepsilon = \int_\varepsilon^\Lambda dE_\lambda$, $Q_\varepsilon = \int_0^\varepsilon dE_\lambda$, $K_\varepsilon^{(-1)} = \int_\varepsilon^\Lambda \lambda^{-1} dE_\lambda$,

and define on [0, T] a control by the formula (3.1) in which $c = c_\varepsilon = -\mathscr{K}_\varepsilon^{(-1)} e^{AT} a$. From (1.1) and (1.3) we have $x(T) = e^{AT}a + \mathscr{K} c_\varepsilon$. Since $\mathscr{K}\mathscr{K}_\varepsilon^{(-1)} = \mathscr{K}_\varepsilon^{(-1)}\mathscr{K} = P_\varepsilon$, $P_\varepsilon + Q_\varepsilon = I$, it follows that $x(T) = Q_\varepsilon e^{AT} a$. Since $\lambda = 0$ is not an eigenvalue of the operator, it follows that $Q_{+0} = 0$, $|x(T)| \to 0$ for $\varepsilon \to 0$, i.e., the system (1.1) is controllable.

Let us assume that for a positive ε the spectrum of the operator $Q_\varepsilon A Q_\varepsilon$ in the subspace $Q_\varepsilon \mathbf{X}$ is located in the open left half-plane. Let us show that the system (1.1) is L_2 controllable. In accordance with (3.1) we shall define on (nT, (n + 1)T) the control u(t) by the formula $u(t) = b^* e^{A[(n+1)T-t]} c_n$, where

$$c_n = -K_\varepsilon^{(-1)} e^{AT} a_n. \tag{3.2}$$

Assuming that $a_n = x(nT)$, $a_0 = a$, we obtain for a_n the formula

$$a_{n+1} = Q_\varepsilon e^{AT} a_n, \quad n = 0, 1, \ldots.$$

Hence follows that

$$Q_\varepsilon a_{n+1} = a_{n+1}, \text{ т. е. } a_{n+1} \in Q_\varepsilon \mathbf{X} \quad \text{for} \quad n = 0, 1, \ldots,$$

and therefore

$$a_{n+1} = Q_\varepsilon e^{AT} Q_\varepsilon a_n, \quad a_{n+1} = (Q_\varepsilon e^{AT} Q_\varepsilon)^n a_1, \quad n = 1, 2, \ldots.$$

From the condition imposed on the spectrum of the operator $Q_\varepsilon A Q_\varepsilon$ it follows that the spectrum of the operator $Q_\varepsilon e^{AT} Q_\varepsilon$ considered in the space $Q_\varepsilon \mathbf{X}$ will lie inside the circle $|\lambda| < 1$. Hence

$$\exists\ \gamma_0 > 0, \ \rho \in (0, 1); \ |a_n| \leqslant \gamma_0 \rho^n, \ n = 0, 1, 2, \ldots.$$

From this formula and (3.2) we find that $u(\cdot) \in L_2[\mathbf{U}, (0, \infty)]$, and, moreover, $\|u(\cdot)\|^2 \le \gamma_1 |a|^2$. Therefore the formula (3.2) can be written in the form $u(\cdot) = S_1 a$, where $S_1 : \mathbf{X} \to L_2[\mathbf{U}, (0, \infty)]$ is a linear bounded operator. Since on [nT, (n + 1)T] we have

$$x(t) = e^{A(t-nT)} a_n + \int_{nT}^{t} e^{A(t-\tau)} b b^* e^{A[(n+1)T-\tau]} d\tau \cdot a_n,$$

it follows that

$$x(\cdot) \in L_2[\mathbf{X}, (0, \infty)], \ \|x(\cdot)\|^2 \leqslant \gamma^2 |a|^2.$$

Hence $x(\cdot) = S_2 a$, where $S_2 : \mathbf{X} \to L_2[\mathbf{U}, (0, x)]$ is a linear bounded operator. We have found that the system (1.1) is L_2 stabilizable. This completes the proof of Lemma 1.1.

Now let us prove Theorem 3. At first we shall prove the following auxiliary proposition which is also of intrinsic interest.

LEMMA 3.1. Let us assume that the system (1.1) is L_2 controllable, and that F(x, u) is a positive-definite form, i.e., $F(x, u) \ge \varepsilon_0(|x|^2 + |u|^2)$ $\forall x \in \mathbf{X}$, $u \in \mathbf{U}$, where $\varepsilon_0 > 0$; hence there exist operators H and h with the properties mentioned in Theorem 1. Then $B = A + bh^*$ will be a Hurwitz operator and $H = H^* \gg 0$.

Proof. In our case, $\Gamma = \Gamma^* \gg 0$. [Let us recall that Γ is an operator of the form F(0, u).] It follows from Theorem 1 that for $\varkappa = \Gamma^{1/2}$ the formula (1.10) holds. By setting $u = h^* x$ in (1.10), we obtain

$$\operatorname{Re} x^*HBx + F(x, h^*x) = 0 \quad (\forall x \in X). \tag{3.3}$$

By virtue of (2.17), $V(a) = a^*Ha = J[x^0(\cdot), u^0(\cdot)]$. Since in our case $J \geq 0$, it follows that $a^*Ha \geq 0$ for any $a \in X$, i.e., $H \geq 0$. Let us take a positive ε that is so small that $\varepsilon \operatorname{Re} x^*Bx \leq (\varepsilon_0/2)|x|^2$ and then write $H_0 = H + \varepsilon I$. By virtue of (3.21) we have for any $x \in X$:

$$\operatorname{Re} x^*H_0Bx = -F(x, h^*x) - \varepsilon \operatorname{Re} x^*Bx \leqslant -\frac{\varepsilon_0}{2}|x|^2$$

or, in other notations:

$$H_0B + B^*H_0 \leqslant -\varepsilon_0 I.$$

We have obtained Lyapunov's inequality with an operator $H_0 \gg 0$. Hence (see, for example, [6], p. 51) B will be a Hurwitz operator. From (3.3) we also obtain Lyapunov's inequality

$$HB + B^*H \leqslant -2\varepsilon_0 I$$

with a Hurwitz operator B. Hence (see, for example, [6], p. 51, or Lemma 4.1 below) $H \gg 0$. This completes the proof of Lemma 3.1.

From Lemma 3.1 and Theorem 1 we obtain the assertion of Theorem 3. Indeed, let us take a positive-definite form $F(x, u)$, for example, $F(x, u) = |x|^2 + |u|^2$. Since the conditions of Lemma 3.1 are satisfied, there exists an operator $h: \mathbf{U} \to X$ such that $B = A + bh^*$ is a Hurwitz operator.

§ 4. Proof of Theorem 2

Let us consider the Lyapunov equation

$$A^*H + HA = G \tag{4.1}$$

for the operator $H = H^*$. Let us present the following well-known lemma.

LEMMA 4.1. Let us assume that A is either a Hurwitz operator, or that A is an anti-Hurwitz operator i.e., $(-A)$ is a Hurwitz operator]. Then the solution of Eq. (4.1) will be expressed by the formulas

$$H = -\int_0^\infty e^{A^*t}Ge^{At}dt \quad \text{(if A is a Hurwitz operator)}, \tag{4.2}$$

$$H = \int_{-\infty}^0 e^{A^*t}Ge^{At}dt \quad \text{(if A is an anti-Hurwitz operator)}. \tag{4.3}$$

The proof is obvious, since the integrals (4.2) and (4.3) are convergent and satisfy Eq. (4.1), which can be verified by direct substitution into (4.1). Let us note that under the conditions of the lemma, the solution of Eq. (4.1) is unique [6]; but this assertion is not needed by us.

LEMMA 4.2. Suppose that the operator $H = H^*$ satisfies the inequality†

$$A^*H + HA \geqslant G. \tag{4.4}$$

If A is a Hurwitz operator and $H_1 = H_1^*$ a solution of Eq. (4.1), then $H \leq H_1$. If A is an anti-Hurwitz operator and $H_2 = H_2^*$ a solution of Eq. (4.1), then $H \geq H_2$.

Proof. Let us write $\Delta H = H - H_j$, $j = 1, 2$. Then $A^*\Delta H + A\Delta H = G \geq 0$. From Lemma 4.1 and formulas (4.2)-(4.3) it follows that $\Delta H \leq 0$ if A is a Hurwitz operator, and $\Delta H \geq 0$ if A is an anti-Hurwitz operator.

LEMMA 4.3. Suppose that (1.10) is satisfied. a) Let us assume that the pair $\{A, b\}$ is L_2 controllable and that $c_1: \mathbf{U} \to X$ is a bounded operator that exists by virtue of Theorem 3 and such that $A_1 = A + bc_1^*$ is a Hurwitz operator. Let $F(x, c_1^*x) = -1/2x^*F_1x$ and let the operator $H_1 = H_1^*$ be defined by the relation $A_1^*H_1 + H_1A_1 = F_1$. Then $H \leq H_1$.

b) Let us assume that the pair $\{-A, -b\}$ is L_2 controllable and that $c_2: \mathbf{U} \to X$ is an operator that exists by virtue of the corollary of Theorem 3 and such that $A_2 = A + bc_2^*$ is an anti-Hurwitz operator. Let $F(x, c_2^*x) = -1/2x^*F_2x$ and let the operator $H_2 = H_2^*$ be defined by the relation $A_2^*H_2 + H_2A_2 = F_2$. Then $H \geq H_2$.

†Here and in the following the notation $\mathscr{H}_1 \geqslant \mathscr{H}_2$ (where $\mathscr{H}_j = \mathscr{H}_j^*$) signifies that $\mathscr{H}_1 - \mathscr{H}_2 \geq 0$, i.e., $\forall x \in X : x^*\mathscr{H}_1 x \geq x^*\mathscr{H}_2 x$.

Proof. For $u = c_1^* x$ and $u = c_2^* x$ we obtain from (1.10) the relation $\mathrm{Re}\, x^* H A_j x + F(x, c_j^* x) \geq 0$, i.e., $HA_j + A_j^* H \geq F_j$; $j = 1, 2$. From Lemma 4.2 it follows that $H_1 \geq H \geq H_2$.

Now let us proceed directly to the proof of Theorem 2. The necessity of condition (1.13) is obvious. Let us prove the sufficiency of this condition. Suppose that (1.13) is satisfied. From (1.13) it follows for $\omega \to \infty$ that $F(0, u) = u^* \Gamma u \geq 0$ ($\forall u \in \mathbf{U}$). Let us take a positive δ and $\varkappa_\delta = (\Gamma + \delta I_U)^{1/2}$, $F_\delta(x, u) = F(x, u) + \delta |u|^2$. Then form $F_\delta(x, u)$ satisfies (1.8), and hence by virtue of Theorem 1 there exist operators H_δ and h_δ such that

$$F(x, u) + \delta |u|^2 + \mathrm{Re}\, x^* H_\delta (Ax + bu) = |\varkappa_\delta u - h_\delta^* x|^2. \tag{4.5}$$

Let us show that on a sequence $\delta_n \to 0$ we have weak convergence

$$H_{\delta_n} \to H, \quad h_{\delta_n} \to h, \quad \varkappa_{\delta_n} \to \varkappa_0, \tag{4.6}$$

where $H = H^*$, h, and $\varkappa_0$ are bounded operators.

It follows from Lemma 4.3 that

$$H_1(\delta) \geqslant H_\delta \geqslant H_2(\delta) \quad (\forall \delta : 0 < \delta \leqslant 1), \tag{4.7}$$

where $H_j(\delta) = H_j(\delta)^*$ are operators corresponding to the forms $F_j(x) = F(x, c_j^* x) + \delta |c_j^* x|^2 : \mathrm{Re}\, x^* H_j(\delta) A_j x = -F_j(x)$. For $0 \leq \delta \leq 1$ we have $F_j(x) \leq F_j^0(x) = F(x, c_j^*, x) + |c_j^* x|^2$. Denoting by H_1^0 and H_2^0 operators corresponding to the forms $F_1^0(x)$ and $F_2^0(x)$ [i.e., $\mathrm{Re}\, x^* H_j^0 A_j x = -F_j^0(x)$], we find by virtue of Lemma 4.2 that $H_1(\delta) \leq H_1^0$, $H_2(\delta) \geq H_2^0$, and from (4.7) we obtain

$$H_1^0 \geqslant H_\delta \geqslant H_2^0 \quad \text{for} \quad 0 < \delta \leqslant 1. \tag{4.8}$$

Since any operator $H = H^*$ satisfies the relation $|H| = \sup |x^* H x|$ for $|x| = 1$, it follows from (4.8) that

$$|H_\delta| \leqslant \max(|H_1^0|, |H_2^0|) \quad \text{for} \quad 0 < \delta \leqslant 1. \tag{4.9}$$

From (4.5) we obtain [see also (1.16)] $2G + H_\delta A + A^* H_\delta = 2h_\delta h_\delta^*$. Since $|h_\delta h_\delta^*| = |h_\delta|^2$, it follows from this formula and (4.9) that $|h_\delta| \leq \mathrm{const}$ for $0 < \delta \leq 1$. From the definition of $\varkappa_\delta$ it follows that $|\varkappa_\delta| \leq \mathrm{const}$ for $0 < \delta \leq 1$. By using the weak compactness of the unit sphere in Hilbert space, we find that for a sequence $\delta_n \to 0$ and any $a \in \mathbf{X}$ and $u \in \mathbf{U}$ we have weak convergence

$$H_{\delta_n} a \to Ha, \quad h_{\delta_n} u \to hu, \quad \varkappa_{\delta_n} u \to \varkappa u.$$

The right-hand sides define by virtue of the Banach–Steinhaus theorem ([6], p. 21) the linear bounded operators

$$H = H^* : \mathbf{X} \to \mathbf{X}, \quad h : \mathbf{U} \to \mathbf{X}, \quad \varkappa : \mathbf{U} \to \mathbf{U}.$$

Thus we have the weak convergence (4.6). By going over to the limit in (4.5) for $\delta = \delta_n \to 0$, we obtain (1.14).

Suppose that the assumption of L_2 controllability of the pair $\{-A, -b\}$ has been replaced by the assumption of boundedness from below, for any $a \in \mathbf{X}$, of the functional mentioned in Theorem 2, i.e., by the assumption

$$J[x(\cdot), u(\cdot)] = \int_0^\infty F[x(t), u(t)]\, dt \geqslant -\gamma_a \quad \text{for} \quad \{x(\cdot), u(\cdot)\} \in \mathfrak{M}_a, \tag{4.10}$$

where $\gamma_a > 0$. By repeating the proof, we find by virtue of Lemma 4.3 that $H_\delta \leq H_1(\delta) \leq H_1^0$. [But the inequalities $H_\delta \geq H_2(\delta) \geq H_1^0$ are no longer valid, and, moreover, the operators $H_2(\delta)$, H_2^0, and c_2 are not defined here.] Since $F_\delta(x, u) \geq F(x, u)$, it follows from (4.10) and (2.17) that

$$a^* H_\delta a = \inf_{\mathfrak{M}_a} \int_0^\infty F_\delta[x(t), u(t)]\, dt \geqslant -\gamma_a. \tag{4.11}$$

By virtue of (4.11) there exists a constant $\gamma > 0$ such that $H_\delta \geq -\gamma I$ for $0 < \delta \leq 1$. Indeed, let $a^* H_\delta a = a^* H_\delta^{(+)} a - a^* H_\delta^{(-)} a$, where $H_\delta^{(+)} \geq 0$, $H_\delta^{(-)} \geq 0$. From (4.11) it follows that $a^* H_\delta^{(-)} a = |\sqrt{H^{(-)}} a|^2 \leq \gamma_a$. By virtue of the Banach–Steinhaus theorem ([6], p. 21), $\gamma > 0 : |\sqrt{H_\delta^{(-)}} a| \leq \gamma |a|^2$ ($\forall a \in \mathbf{X}$), i.e.,

$$a^* H_\delta a \geqslant -a^* H_\delta^{(-)} a \geqslant -\gamma |a|^2 \quad (\forall a \in \mathbf{X}),$$

as asserted above. From the inequalities $-\gamma I \le H_\delta \le H_1^0$, $\forall\delta : 0 < \delta \le 1$ it follows (as before) that $|H_\delta|$ $\le$ const for $0 < \delta \le 1$. Hence, in the same way as above, we obtain (4.6) (in the sense of weak convergence), and (1.14).

It remains to prove that formula (4.10) is necessary for the existence of an operator $H = H^*$ that satisfies the inequality (1.12). Suppose that (1.12) is satisfied. By substituting into (1.12) the pair $\{x(t), u(t)\} \in \mathfrak{M}_a$ and integrating the obtained inequality from $t = 0$ to $t = \infty$, we obtain

$$J[x(\cdot),\ u(\cdot)] \geqslant {}^1/_2\, a^* Ha,$$

whence follows (4.10). [We used the limit relation $|x(t)| \to 0$ for $t \to \infty$, which holds by virtue of the fact that $|x(t)| \in L_2(0, \infty)$ and $|dx/dt| = |Ax + bu| \in L_2(0, \infty)$.] This completes the proof of Theorem 2.

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