Stability Analysis of Discrete-Time Semi-Markov Jump Linear Systems With Time Delay

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***Abstract*—The stability and stabilization problems for discrete-time semi-Markov jump linear systems with time delay are discussed. By virtue of a novel Lyapunov–Krasovskill functional and the probability structure of semi-Markov switching signal, the sufficient stability conditions for the considered systems are presented in terms of a set of linear matrix inequalities and a proper semi-Markov switching condition. Based on such conditions, the time delay and mode dependent controller can be designed to stabilize the corresponding closed-loop systems. Finally, two examples are given to illustrate the effectiveness of our results.**

***Index Terms*—Controller design, discrete-time semi-Markov jump linear systems, linear matrix inequality (LMI), semi-Markov switching condition, time delay.**

I. Introduction

Markov jump systems (MJSs) are referred to as a class of stochastic jump systems with Markovian switching signal. Such systems can efficiently model dynamic systems that are subject to random abrupt changes. Over the past decades, many works on the issues of stability analysis and stabilization synthesis for MJSs have been obtained (see [1], [2], [3], [4], [5], [6], [7], [8]). Nevertheless, the Markov switching singal signal in their works requires an ideal assumption that the sojourn time of each mode should be subject to exponential distribution in the continuous-time domain or geometric distribution in the discrete-time domain. It means that the transition probabilities of mode switching in such systems should be memoryless, which cannot be suitable for many practical dynamic systems. In order to relax such restriction, semi-Markov process (SMP) has been introduced to model a more general switching signal, in which the sojourn times are allowed to be arbitrary random variables. Meanwhile, the research on stability and stabilization synthesis problems of semi-Markov jump systems (SMJSs) have been received much attention (see [9], [10]). However, many ideal assumptions are still imposed on their semi-Markov switching signal. For instance, in some published works on continuous-time SMJSs, the sojourn times are also assumed to follow some special probability distributions, such as phase-type distributions (see [11], [12], [13]) and Weibull distributions (see [14], [15]). On discrete-time SMJSs, by virtue of the notion of semi-Markov kernel (SMK) (see [16], [17]), the probability distributions of sojourn times are allowed to be dependent both on the current and the next modes, but in their sufficient stability conditions, expectation values of Lyapunov function on switching instants $\{t_n, n \in \mathbf{Z}_{\geq 0}\}$ must be monotonous decreased. Furthermore, in their numerically testable controller existence conditions, the sojourn times are required to be bounded. These unreasonable restrictions mentioned above manifest a fact that the research on SMJSs are still far away from maturity. On the other hand, the factor of time delay is quite common in many practical application environments, which is frequently the main source of instability and poor performance for dynamical systems. The effect of time delay on the stability of deterministic jump systems (see [18], [19], [20]), MJSs (see [21], [22], [23], [24], [25]), and continuous-time SMJSs (see [26], [27], [28], [29], [30]) has been studied extensively. However, to the best of the authors' knowledge, there has no related work on discrete-time SMJSs with time delay, which motivates our present study. This article is concerned with the problems of stability analysis and stabilizing controller design for a class of discrete-time semi-Markov jump linear systems (SMJLSs) with time delay. The main contributions are summarized as follows.

1) The sufficient stability conditions for discrete-time SMJLSs have been proposed by virtue of a novel semi-Markov switching condition, which removes a strict restriction in [16] and [17] that the expectation of Lyapunov function taking value on each jump instant should be less than the one taking value on the previous jump instant.
2) Based on the obtained conditions, the controller gains can be designed by solving a set of linear matrix inequalities (LMIs) to stabilize the considered systems with unbounded sojourn times, which are different from the ones in [16] and [17] that the sojourn times of their systems are required to be bounded.

Notations: $\mathbf{R}$ and $\mathbf{Z}$ denote the real number set and the integer set, respectively. For any subset C of $\mathbf{R}$, $\mathbf{Z}_C = \mathbf{Z} \cap C$, and $\mathbf{R}_C = \mathbf{R} \cap C$. For $(\Omega, \mathcal{F}, P_r)$, Ω represents the sample space, $\mathcal{F}$ is the $\boldsymbol{\sigma}$-algebra of subsets of sample space, and P_r is the probability measure on $\mathcal{F}$. $\mathbf{E}[\cdot]$ denotes the mathematical expectation with respect to P_r. $I(\cdot)$ denotes the indicator function. $||\cdot||$ denotes the Euclidean norm in $\mathbf{R}^n$. For a real symmetric matrix P, $P > 0$ $(P < 0)$ means that P is positive (negative) definite. $\lambda_{\min}(P)(\lambda_{\max}(P))$ denotes the minimum (maximum) eigenvalue of P. diag$\{\cdots\}$ stands for a block-diagonal matrix. $\mathbf{I}$ and $\mathbf{0}$ represent the identity matrix and zero matrix with appropriate dimensions, respectively. Symbol $\star$ is used as an ellipsis for the terms that are induced by symmetry. The function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}_\infty$, if it is continuous, strictly increasing, unbounded, and $\alpha(0) = 0$.

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II. Preliminaries and Problem Formulation

We first introduce the formal definitions of discrete-time Markov renewal process (MRP) and SMP.

Definition 2.1 (see [16]): For any $n \in \mathbf{Z}_{\geq 0}$, t_n denotes the time instant at the nth switching with $t_0 = 0$, and R_n denotes the mode index at the nth switching with a finite state space $\mathcal{I} = \{1, 2, \ldots, M\}$. If for any $j \in \mathcal{I}, \tau \in \mathbf{Z}_{\geq 1}$ and $n \in \mathbf{Z}_{\geq 0}$

$$\begin{aligned} P_r(R_{n+1} &= j, t_{n+1} - t_n = \tau | R_l, t_l, l \in \mathbf{Z}_{[0,n]}) \\ &= P_r\left(R_{n+1} = j, t_{n+1} - t_n = \tau | R_n\right) \\ &= P_r\left(R_1 = j, t_1 - t_0 = \tau | R_0\right) \end{aligned} \tag{1}$$

$\{(R_n, t_n)\}_{n \in \mathbf{Z}_{\geq 0}}$ is called the homogeneous discrete-time MRP with SMK $\Pi(\tau) = [\pi_{ij}(\tau)]_{i,j \in \mathcal{I}}$, in which $\pi_{ij}(\tau) := P_r(R_{n+1} = j, t_{n+1} - t_n = \tau | R_n = i)$ with $\pi_{ii}(\tau) = 0$ for any $\tau \in \mathbf{Z}_{\geq 1}$ and i, $j \in \mathcal{I}$.

Remark 2.2: From (1), we can obtain

$$\begin{aligned} &P_r(R_{n+1} = j | R_l, l \in \mathbf{Z}_{[0,n]}) \\ &\quad = \sum_{\tau=1}^{+\infty} P_r\left(R_{n+1} = j, t_{n+1} - t_n = \tau | R_l, l \in \mathbf{Z}_{[0,n]}\right) \\ &\quad = P_r\left(R_{n+1} = j | R_n\right) = P_r\left(R_1 = j | R_0\right) \end{aligned} \tag{2}$$

which implies that $\{R_n\}_{n \in \mathbf{Z}_{\geq 0}}$ has homogeneous Markov property. $\{R_n\}_{n \in \mathbf{Z}_{\geq 0}}$ is called the embedded Markov chain (EMC) with respect to MRP $\{(R_n, t_n)\}_{n \in \mathbf{Z}_{\geq 0}}$. Its transition probability matrix is denoted as $\Theta = [\theta_{ij}]_{i,j \in \mathcal{I}}$, in which $\theta_{ij} = P_r(R_{n+1} = j | R_n = i)$ and $\theta_{ii} = 0$.

Definition 2.3 (see [16]): The process $\{r(k)\}_{k \in \mathbf{Z}_{\geq 0}}$ is called the discrete-time SMP with respect to MRP $\{(R_n, t_n)\}_{n \in \mathbf{Z}_{\geq 0}}$, if for any $k \in \mathbf{Z}_{[t_n, t_{n+1})}$ and $n \in \mathbf{Z}_{\geq 0}$

$$r(k) = R_n. \tag{3}$$

Remark 2.4: We denote τ_{ij} be the sojourn time $t_{n+1} - t_n$ with the current mode i and the next mode j, and let $\omega_{ij}(\tau) = P_r(t_{n+1} - t_n = \tau | r(t_n) = i, r(t_{n+1}) = j)$ be its distribution law. Similarly, let τ_i be the sojourn time $t_{n+1} - t_n$ with the current mode i, and let $\omega_i(\tau) = P_r(t_{n+1} - t_n = k | r(t_n) = i)$ be its distribution law. If for each $i \in \mathcal{I}$, τ_i subjects to geometric distributions, $\{r(k)\}_{k \in \mathbf{Z}_{\geq 0}}$ degenerates into discrete-time Markov process.

In this article, we consider the following discrete-time SMJLS with time delay:

$$\begin{aligned} x(k+1) &= A(r(k))x(k) + A_d(r(k))x(k-d) \\ x(l) &= \phi(l), l \in \mathbf{Z}_{[-d,0]} \end{aligned} \tag{4}$$

where $x(k) \in \mathbf{R}^{n_x}$ is the system state, $d \in \mathbf{Z}_{\geq 1}$ is a constant time delay, and $\phi(l) \in \mathbf{R}^{n_x}, l \in \mathbf{Z}_{[-d,0]}$ is a vector-valued initial function on $\mathbf{Z}_{[-d,0]}$. The switching signal $\{r(k)\}_{k \in \mathbf{Z}_{\geq 0}}$ is assumed to be a discrete-time SMP defined above. For simplicity, we denote $A_i = A(r(k))|_{r(k)=i}$ and $A_{di} = A_d(r(k))|_{r(k)=i}, i \in \mathcal{I}$. By denoting $\xi(k) = \mathbf{col}(x(k), x(k-1), \ldots, x(k-d))$, $k \in \mathbf{Z}_{\geq 0}$, system (4) has the following equivalent form:

$$\xi(k+1) = \mathbf{\Gamma}_d(r(k))\xi(k) \tag{5}$$

where

$$\mathbf{\Gamma}_d(r(k)) = \begin{bmatrix} [A(r(k)), \mathbf{0}] & A_d(r(k)) \\ \mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{6}$$

At the last part of this section, we give the definition of mean square stability (MSS) for the considered systems and introduce a lemma that can facilitate our proof in the following section.

Definition 2.5: System (4) is MSS, if

$$\lim_{k \to \infty} \mathbf{E}\left[\|x(k)\|^2\right] = 0 \tag{7}$$

holds for any $\phi(l) \in \mathbf{R}^{n_x}, l \in \mathbf{Z}_{[-d,0]}$ and $i_0 \in \mathcal{I}$.

Lemma 2.6 (see [32]): For any matrix $P > 0$, integers r_1, r_2 with $r_2 > r_1 > 0$ and vector function $\omega : \{r_1, r_1 + 1, \ldots r_2\} \to \mathbf{R}^n$, the following inequality holds:

$$\sum_{j=r_1}^{r_2-1} \omega^T(j) P \omega(j) \geq \frac{1}{r_2 - r_1} \left(\sum_{j=r_1}^{r_2-1} \omega(j)\right)^T P \left(\sum_{j=r_1}^{r_2-1} \omega(j)\right).$$

III. Main Results

A. Stability Analysis

We first give the following sufficient stability conditions for discrete-time SMJLSs with time delay.

Lemma 3.1: For given constants $\lambda_i \in \mathbf{R}_{>0}, i \in \mathcal{I}$, if there exist function $V(\xi, i) : \mathbf{R}^{(d+1) \times n_x} \times \mathcal{I} \to \mathbf{R}_{>0}$, $\mathcal{K}_\infty$ functions α_1, α_2, and constants $u_i \in \mathbf{R}_{>0}, i \in \mathcal{I}$, such that

$$\alpha_1(\|\xi(k)\|) \leq V(\xi(k), r(k)) \leq \alpha_2(\|\xi(k)\|) \tag{8}$$

$$\begin{aligned} &V(\xi(k+1), i) < \lambda_i V(\xi(k), i) \\ &\forall k \in \mathbf{Z}_{[t_n, t_{n+1}-1]}, r(t_n) = i \end{aligned} \tag{9}$$

$$V(\xi(k), j) \leq u_i V(\xi(k), i) \quad \forall i, j \in \mathcal{I} \tag{10}$$

$$\max_{l \in \mathcal{I}} \left\{ \sum_{i \in \mathcal{I}} u_i \left[\max_{j \in \mathcal{I}, j \neq i} \left\{ \mathbf{E}\left(\lambda_i^{\tau_{ij}}\right) \right\} \right] \theta_{li} \right\} < 1. \tag{11}$$

Then, system (4) is MSS.

Proof: See Appendix. ■

Remark 3.2: How to construct the proper stabilizing switching condition is the basic problem in stability analysis of deterministic jump systems (see [34, Problem C]), condition (11) solves this problem for discrete-time SMJLSs, we call it as the semi-Markov switching condition. For each $i \in \mathcal{I}$, λ_i is utilized to measure the stability degree of the ith subsystem mode, if $\lambda_i \in \mathbf{R}_{(0,1]}$, the ith subsystem mode is stable, else if $\lambda_i \in \mathbf{R}_{>1}$, the ith subsystem mode may be unstable. Inequality (11) implies a fact that the proper probability structure of semi-Markov switching signal has an stabilizing effect on the considered systems. More specifically, even if there exist some unstable modes in the whole system, the larger instability degree and the larger sojourn times of these unstable modes still can be compensated for by a smaller probability activating the corresponding modes. Furthermore, our stability conditions do not require the decreasing property of the Lyapunov function and the boundary of the sojourn times, thus, our results are less conservative than the ones in [16] and [17].

The following theorem presents the LMI-based sufficient conditions for system (4).

Theorem 3.3: For given constants $\lambda_i \in \mathbf{R}_{>0}, i \in \mathcal{I}$, if there exist matrices $P_i > 0$, $Q_i > 0$, $R_i > 0$, such that

$$\begin{bmatrix} -\lambda_i P_i - \widetilde{\lambda}_i R_i & \widetilde{\lambda}_i R_i & A_i^T P_i & d(A_i - \mathbf{I})^T R_i & Q_i \\ \star & -\widetilde{\lambda}_i R_i - \lambda_i^d Q_i & A_{di}^T P_i & d A_{di}^T R_i & \mathbf{0} \\ \star & \star & -P_i & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -R_i & \mathbf{0} \\ \star & \star & \star & \star & -Q_i \end{bmatrix} < 0 \tag{12}$$

in which

$$\widetilde{\lambda}_i = \lambda_i I(\lambda_i > 1) + \lambda_i^d I(0 < \lambda_i \le 1). \tag{13}$$

Then, system (4) is MSS under semi-Markov switching condition (11) with

$$u_i = \max_{j\in\mathcal{I}} \left\{ \frac{\lambda_{\max}(\Psi_j)}{\lambda_{\min}(\Psi_i)} \right\}, i \in \mathcal{I} \tag{14}$$

where

$$\Psi_i = \begin{pmatrix} P_i + d^2 R_i & -d^2 R_i & \mathbf{0} & \cdots & \mathbf{0} \\ \star & Q_i + d^2 R_i + d(d-1)\lambda_i R_i & -d(d-1)\lambda_i R_i & \cdots & \mathbf{0} \\ \star & \star & \lambda_i Q_i + d(d-1)\lambda_i R_i + d(d-2)\lambda_i^2 R_i & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & \lambda_i^{d-1} Q_i + d\lambda_i^{d-1} R_i \end{pmatrix}. \tag{15}$$

Proof: See Appendix. ■

If we further assume that there exist matrices P, Q, R, such that $P = P_i, Q = Q_i, R = R_i, i \in \mathcal{I}$, and let $\lambda = \min\{1, \lambda_i, i \in \mathcal{I}\}$, condition (12) can be guaranteed by

$$\begin{bmatrix} -\lambda_i P - \lambda^d R & \lambda^d R & A_i^T P & d(A_i - \mathbf{I})^T R & Q \\ \star & -\lambda^d R - \lambda_i \lambda^{d-1} Q & A_{di}^T P & d A_{di}^T R & \mathbf{0} \\ \star & \star & -P & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -R & \mathbf{0} \\ \star & \star & \star & \star & -Q \end{bmatrix} < 0, \tag{16}$$

Ψ_i reduces to

$$\Psi = \begin{pmatrix} P + d^2 R & -d^2 R & \mathbf{0} & \cdots & \mathbf{0} \\ \star & Q + d^2 R + d(d-1)\lambda R & -d(d-1)\lambda R & \cdots & \mathbf{0} \\ \star & \star & \lambda Q + d(d-1)\lambda R + d(d-2)\lambda^2 R & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \cdots & \lambda^{d-1} Q + d\lambda^{d-1} R \end{pmatrix},$$

which together with (14) implies that $\mu_i \equiv 1, i \in \mathcal{I}$, thus, the switching condition (11) reduces to the following simple form:

$$\max_{l\in\mathcal{I}} \left\{ \sum_{i\in\mathcal{I}} \left[\max_{j\in\mathcal{I}, j\neq i} \left\{ \mathbf{E}\left(\lambda_i^{\tau_{ij}}\right) \right\} \right] \theta_{li} \right\} < 1. \tag{17}$$

Therefore, we can obtain the following theorem.

Theorem 3.4: For given constants $\lambda_i \in \mathbf{R}_{>0}$ $i \in \mathcal{I}$, if there exist matrices $P > 0, Q > 0, R > 0$, such that (16) holds. Then, system (4) is MSS under semi-Markov switching condition (17).

Another basic problem for jump systems is to find the stability conditions for such systems under arbitrary switching signal (see [34, Problem A]). Based on Theorem 3.4, we can present the following sufficient stability conditions for system (4) with arbitrary semi-Markov switching signal.

Corollary 3.5: Let $\lambda \in \mathbf{R}_{(0,1)}$ be a given constant, if there exist matrices $P > 0$, $Q > 0$, $R > 0$, such that

$$\begin{bmatrix} -\lambda P - \lambda^d R & \lambda^d R & A_i^T P & d(A_i - \mathbf{I})^T R & Q \\ \star & -\lambda^d R - \lambda^d Q & A_{di}^T P & d A_{di}^T R & \mathbf{0} \\ \star & \star & -P & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -R & \mathbf{0} \\ \star & \star & \star & \star & -Q \end{bmatrix} < 0. \tag{18}$$

Then, system (4) is MSS under arbitrary semi-Markov switching signal.

Proof: See Appendix. ■

B. Controller Design

The aim of this section is to design the stabilizing controller $u(x(k), x(k-d), r(k))$, such that the resulting controlled system

$$\begin{aligned} x(k+1) &= A(r(k))x(k) + A_d(r(k))x(k-d) \\ &\quad + B(r(k))u(x(k), x(k-d), r(k)) \\ x(l) &= \phi(l), l \in \mathbf{Z}_{[-d,0]} \end{aligned} \tag{19}$$

becomes stable. We construct the following time delay and mode dependent stabilizing controller:

$$\begin{aligned} &u(x(k), x(k-d), r(k)) \\ &= K(r(k))x(k) + J(r(k))x(k-d), \end{aligned} \tag{20}$$

by substituting (20) into (19), we have

$$\begin{aligned} x(k+1) &= [A(r(k)) + B(r(k))K(r(k))]\, x(k) \\ &\quad + [A_d(r(k)) + B(r(k))J(r(k))]\, x(k-d) \\ x(l) &= \phi(l), l \in \mathbf{Z}_{[-d,0]}. \end{aligned} \tag{21}$$

For simplicity, we denote $B_i = B(r(k))|_{r(k)=i}$, $K_i = K(r(k))|_{r(k)=i}$, and $J_i = J(r(k))|_{r(k)=i}$, where K_i and $J_i, i \in \mathcal{I}$ are the controller gains to be determined.

Theorem 3.6: For given constants $\lambda_i \in \mathbf{R}_{>0}$, if there exist matrices $U > 0, V > 0, Z > 0$ and $L_i, W_i, i \in \mathcal{I}$, such that

$$\begin{pmatrix} -\lambda_i V & \mathbf{0} & V A_i^T + W_i B_i^T & d\left(V A_i^T + W_i B_i^T - V\right) & V \\ \star & -\lambda_i^d U_i & U A_{di}^T + L_i B_i^T & d\left(U A_{di}^T + L_i B_i^T\right) & \mathbf{0} \\ \star & \star & -V & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -Z & \mathbf{0} \\ \star & \star & \star & \star & -U \end{pmatrix} < 0. \tag{22}$$

Then, system (21) under switching condition (17) is MSS with respect to the controller gains

$$K_i = \left(V^{-1}W_i\right)^T, J_i = \left(U^{-1}L_i\right)^T, i \in \mathcal{I}. \tag{23}$$

Proof: See Appendix. ■

Similarly, based on Corollary 3.5, the existence conditions of controller (20) under arbitrary switching can be obtained.

Theorem 3.7: Let $\lambda \in \mathbf{R}_{(0,1)}$ be a given constant, if there exist matrices $U > 0, V > 0, Z > 0$, and $L_i, W_i, i \in \mathcal{I}$, such that

$$\begin{pmatrix} -\lambda V & \mathbf{0} & VA_i^T + W_iB_i^T & d\left(VA_i^T + W_iB_i^T - V\right) & V \\ \star & -\lambda^d U & UA_{di}^T + L_iB_i^T & d\left(UA_{di}^T + L_iB_i^T\right) & \mathbf{0} \\ \star & \star & -V & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -Z & \mathbf{0} \\ \star & \star & \star & \star & -U \end{pmatrix} < 0. \tag{24}$$

Then, system (21) under arbitrary switching is MSS with respect to the controller gains (23).

Remark 3.8: Different from the mode dependent controller design method presented by [16] and [17], Theorems 3.6 and 3.7 consider the time delay factor and the sojourn times are allowed us to be unbounded.

IV. Examples

In this section, we present two examples to illustrate the effectiveness of Theorem 3.4, Corollary 3.5, and Theorem 3.7, respectively.

Example 1. Consider system (4) with the following three modes:

$$A_1 = \begin{bmatrix} 0.1 & 0 \\ 0.02 & 0.3 \end{bmatrix}, A_{d1} = \begin{bmatrix} -0.2 & 0.1 \\ -0.2 & 0.1 \end{bmatrix}, \tag{25}$$

$$A_2 = \begin{bmatrix} 0.2 & 0 \\ 0.05 & 0.5 \end{bmatrix}, A_{d2} = \begin{bmatrix} -0.5 & 0.1 \\ -0.5 & 0.2 \end{bmatrix}, \tag{26}$$

$$A_3 = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, A_{d3} = \begin{bmatrix} -0.2 & 0.22 \\ -0.25 & -0.2 \end{bmatrix}. \tag{27}$$

The state trajectories of modes (25), (26), and (27) with $d = 2$ are shown in Fig. 1(a)–(c), respectively. It can be seen that these three modes are all stable. By taking $\lambda = 0.99$, LMI (18) has the feasible solutions. Thus, by Corollary 3.5, the whole system (4) with $d = 2$ remains stable under any semi-Markov switching signal. Fig. 1(d) illustrates the state trajectories of the whole system with $d = 2$ switched by 100 randomly generating SMPs without any restriction. It can be seen that all the state trajectories converge to zero. When time delay d increases from 2 to 5, modes (25) and (26) remain stable, but mode (27) becomes unstable. The state trajectories of three modes with $d = 5$ are shown in Fig. 2(a)–(c), respectively. Next, we show the effect of the semi-Markov switching condition (17) on the stability of the whole system with unstable modes. The proper semi-Markov switching signal can be designed to stabilize the considered system. By taking $\lambda_1 = \lambda_2 = 0.95$ and $\lambda_3 = 1.2$, LMI (16) of Theorem 3.4 has the feasible solutions. Let the sojourn time probability distributions be

$$\omega_{12}(\tau) = \omega_{13}(\tau) = p^{\tau-1}(1-p), \omega_{11}(\tau) = 0,\ \tau \in \mathbf{Z}_{[1,+\infty)}$$

$$\omega_{21}(\tau) = \omega_{23}(\tau) = \frac{2^\tau}{\tau!}e^{-2},\ \omega_{22}(\tau) = 0,\ \tau \in \mathbf{Z}_{[0,+\infty)}$$

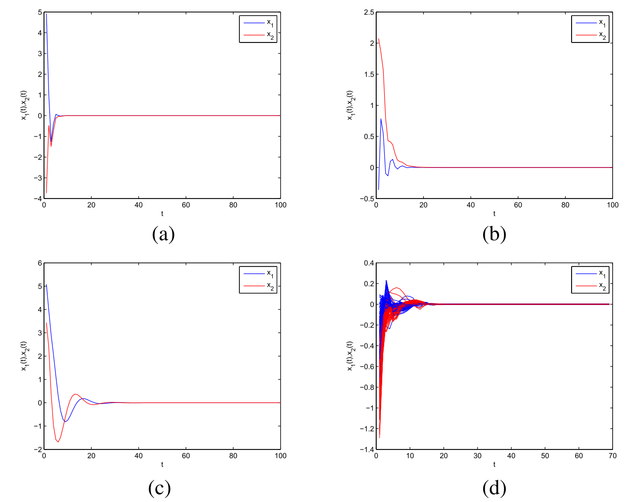

Fig. 1. State responses of three modes and the whole system with $d = 2$. (a) State trajectory of mode (25) with $d = 2$. (b) State trajectory of mode (26) with $d = 2$. (c) State trajectory of mode (27) with $d = 2$. (d) State trajectories of the whole system (4) with $d = 2$ under 100 randomly generating SMPs without any restriction.

$$\omega_{31}(\tau) = \omega_{32}(\tau) = \frac{2}{3} \times \left(\frac{1}{3}\right)^{\tau-1}, \omega_{33}(\tau) = 0, \tau \in \mathbf{Z}_{[1,+\infty)}$$

in which the value of p indicates the probability of staying at the current mode (25) [or (26)] on the next step. Let the transition probability matrix of its EMC be $\Theta = \begin{bmatrix} 0 & 1-\theta & \theta \\ 1-\theta & 0 & \theta \\ 0 & 1 & 0 \end{bmatrix}$, in which the value of θ indicates the transition probability from mode (25) [or (26)] to mode (27). By $\mathbf{E}(\lambda_i^{\tau_{ij}}) = \sum_{k=1}^{\infty} \lambda_i^k \omega_{ij}(k)$, we have $\max_{j=2,3}\left\{\mathbf{E}\left(\lambda_1^{\tau_{1j}}\right)\right\} = \frac{0.95(1-p)}{1-0.95p}, \max_{j=1,3}\left\{\mathbf{E}\left(\lambda_2^{\tau_{2j}}\right)\right\} = e^{-0.1}, \max_{j=1,2}\left\{\mathbf{E}\left(\lambda_3^{\tau_{3j}}\right)\right\} = \frac{4}{3}$, which together with the abovementioned transition probability matrix implies that the condition (17) has the following form:

$$\frac{0.95(1-p)(1-\theta)}{1-0.95p} + e^{-0.1} + \frac{4}{3}\theta < 1, \tag{28}$$

which can be guaranteed by

$$0.3495p - 0.3833\theta < 0.3179. \tag{29}$$

This inequality implies that if the value of p, the probability of staying at the stable modes, is large enough, and the value of θ, the transition probability from stable mode to unstable mode, is small enough, the whole system can be stabilized by the proper semi-Markov signal satisfying inequality (29). The state trajectories of system (4) with $d = 5$ switched by 100 randomly generating SMPs satisfying (29) are shown in Fig. 2(d), it can be seen that all the state trajectories converge to zero.

Remark 4.1: Different from the sufficient stability criterion presented by [16] and [17], our results highlight the stabilizing effect of the switching signal on discrete-time SMJLSs. Based on switching condition (17), the suitable switching signal is designed to stabilize the considered system, which manifests the our design method for stabilizing switching signal is effective.

Example 2. In this example, we aim to design the controller (20) to stabilize system (21) under arbitrary semi-Markov switching signal.

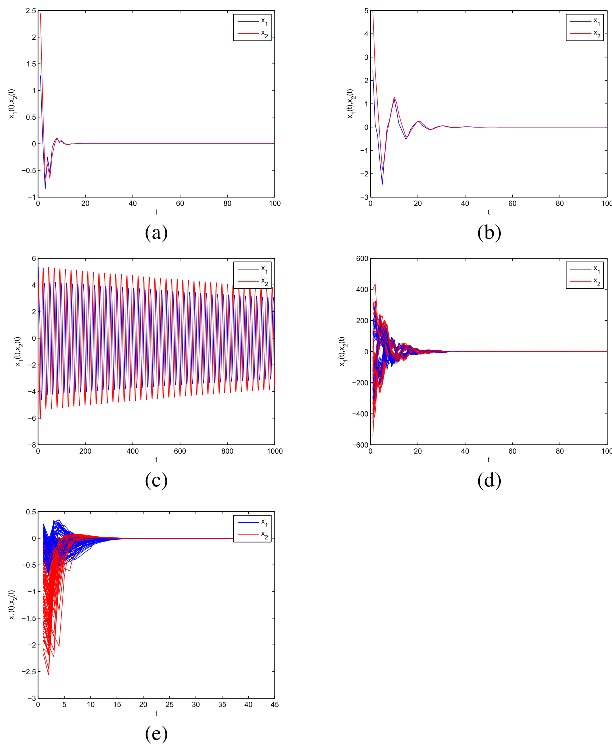

Fig. 2. State trajectories of three modes and the whole system with $d = 5$. (a) State trajectory of mode (25) with $d = 5$. (b) State trajectory of mode (26) with $d = 5$. (c) State trajectory of mode (27) with $d = 5$. (d) State trajectories of the whole system (4) with $d = 5$ under 100 randomly generating SMPs satisfying (29). (e) State trajectories of the whole system (21) with $d = 5$ stabilized by the controller gains in (30) under 100 randomly generating SMPs without any restriction.

Consider system (21) with $d = 5$, which has three modes:

$$A_1 = \begin{bmatrix} 0.1 & 0 \\ 0.02 & 0.3 \end{bmatrix}, A_{d1} = \begin{bmatrix} -0.2 & 0.1 \\ -0.2 & 0.1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.2 & 0 \\ 0.05 & 0.5 \end{bmatrix}, A_{d2} = \begin{bmatrix} -0.5 & 0.1 \\ -0.5 & 0.2 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, A_{d3} = \begin{bmatrix} -0.2 & 0.2 \\ -0.25 & 0.2 \end{bmatrix}, B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By taking $\lambda = 0.9$, LMI (22) has the feasible solutions. By (23), the admissible controller gain matrices can be obtained as follows:

$$\begin{aligned} &K_1 = (0.0628, 0.0185), J_1 = (0.2, -0.1) \\ &K_2 = (0.0156, -0.1323), J_2 = (0.5000, -0.1754) \\ &K_3 = (-0.1323, -0.4362), J_3 = (0.2378, 0.0967). \end{aligned} \tag{30}$$

The state trajectories of system (21) with $d = 5$ stabilized by (30) with 100 randomly generating arbitrary semi-Markov switching processes are shown in Fig. 2(e). It can be seen that all state trajectories converge to zero, which implies that the obtained controller can stabilize the considered system under arbitrary semi-Markovian switching signal.

V. Conclusion

In this article, the stability analysis and stabilizing controller design problems for discrete-time SMJLSs with time delay have been studied. The sufficient stability conditions in terms of a set of LMIs and a novel semi-Markov switching condition have been provided. Based on such conditions, the stabilizing controller design method for the considered systems under arbitrary semi-Markov switching signal can be obtained. Two examples have been given to illustrate the effectiveness of our results.

Appendix

Proof of Lemma 3.1. Similar as [33, proof of Lemma 1], for any $i_0, i_1, i_2, \ldots, i_n \in \mathcal{I}$, by condition (9), we can obtain

$$\begin{aligned} &\mathbf{E}\left[V(\xi(t_n), r(t_n)) \middle| \cap_{m=0}^{n} \{r(t_m) = i_m\}\right] \\ &< \mathbf{E}\left[V(\xi(0), i_0)\right] \prod_{m=1}^{n} u_{i_{m-1}} \left[\mathbf{E}\left(\lambda_{i_{m-1}}^{\tau_{i_{m-1} i_m}}\right)\right] \\ &\leq \mathbf{E}\left[V(\xi(0), i_0)\right] \prod_{m=1}^{n} u_{i_{m-1}} \left[\max_{j \in \mathcal{I}, j \neq i_{m-1}} \left\{\mathbf{E}\left(\lambda_{i_{m-1}}^{\tau_{i_{m-1} j}}\right)\right\}\right]. \end{aligned} \tag{31}$$

By Markov property of $\{r(t_n)\}_{n \in \mathbf{Z}_{\geq 0}}$, we have $P_r\left(\cap_{m=0}^{n}\{r(t_m) = i_m\}\right) = \prod_{m=1}^{n} \theta_{i_{m-1} i_m}$. By the definition of expectation, we see that for any $n \in \mathbf{Z}_{\geq 1}$

$$\begin{aligned} &\mathbf{E}[V(\xi(t_n), r(t_n))] \\ &= \sum_{i_1, \ldots, i_n \in \mathcal{I}} \Big\{\mathbf{E}[V(\xi(t_n), r(t_n)) \big| \cap_{m=0}^{n} \{r(t_m) = i_m\}] \\ &\quad \times P_r\left(\cap_{m=0}^{n}\{r(t_m) = i_m\}\right)\Big\}. \end{aligned} \tag{32}$$

Thus, we know that for any $n \in \mathbf{Z}_{\geq 2}$

$$\begin{aligned} &\mathbf{E}\left[V(\xi(t_n), r(t_n))\right] \\ &< \mathbf{E}\left[V(\xi(0), i_0)\right] \\ &\quad \times \sum_{i_1, \ldots, i_n \in \mathcal{I}} \left\{\prod_{m=1}^{n} u_{i_{m-1}} \left[\max_{j \in \mathcal{I}, j \neq i_{m-1}} \left\{\mathbf{E}\left(\lambda_{i_{m-1}}^{\tau_{i_{m-1} j}}\right)\right\}\right] \theta_{i_{m-1} i_m}\right\} \\ &= u_{i_0} \mathbf{E}[V(\xi(0), i_0)] \left[\max_{j \in \mathcal{I}, j \neq i_0} \left\{\mathbf{E}\left(\lambda_{i_0}^{\tau_{i_0 j}}\right)\right\}\right] \\ &\quad \times \sum_{i_1, \ldots, i_n \in \mathcal{I}} \left\{\prod_{m=2}^{n} u_{i_{m-1}} \left[\max_{j \in \mathcal{I}, j \neq i_{m-1}} \left\{\mathbf{E}\left(\lambda_{i_{m-1}}^{\tau_{i_{m-1} j}}\right)\right\}\right] \theta_{i_{m-2} i_{m-1}} \theta_{i_{n-1} i_n}\right\} \\ &\leq u_{i_0} \mathbf{E}\left[V(\xi(0), i_0)\right] \left[\max_{j \in \mathcal{I}, j \neq i_0} \left\{\mathbf{E}\left(\lambda_{i_0}^{\tau_{i_0 j}}\right)\right\}\right] \\ &\quad \times \sum_{i_1, \ldots, i_n \in \mathcal{I}} \left\{\prod_{m=2}^{n} u_{i_{m-1}} \left[\max_{j \in \mathcal{I}, j \neq i_{m-1}} \left\{\mathbf{E}\left(\lambda_{i_{m-1}}^{\tau_{i_{m-1} j}}\right)\right\}\right] \theta_{i_{m-2} i_{m-1}}\right\} \\ &= u_{i_0} M \mathbf{E}\left[V(\xi(0), i_0)\right] \left[\max_{j \in \mathcal{I}, j \neq i_0} \left\{\mathbf{E}\left(\lambda_{i_0}^{\tau_{i_0 j}}\right)\right\}\right] \\ &\quad \times \sum_{i_1, \ldots, i_{n-1} \in \mathcal{I}} \left\{\prod_{m=2}^{n} u_{i_{m-1}} \left[\max_{j \in \mathcal{I}, j \neq i_{m-1}} \left\{\mathbf{E}\left(\lambda_{i_{m-1}}^{\tau_{i_{m-1} j}}\right)\right\}\right] \theta_{i_{m-2} i_{m-1}}\right\} \\ &\leq u_{i_0} M \mathbf{E}\left[V(\xi(0), i_0)\right] \left[\max_{j \in \mathcal{I}, j \neq i_0} \left\{\mathbf{E}\left(\lambda_{i_0}^{\tau_{i_0 j}}\right)\right\}\right] \\ &\quad \times \sum_{i_1, \ldots, i_{n-2} \in \mathcal{I}} \left\{\prod_{m=2}^{n-1} u_{i_{m-1}} \left[\max_{j \in \mathcal{I}, j \neq i_{m-1}} \left\{\mathbf{E}\left(\lambda_{i_{m-1}}^{\tau_{i_{m-1} j}}\right)\right\}\right] \theta_{i_{m-2} i_{m-1}}\right\} \\ &\quad \times \max_{i_{n-2} \in \mathcal{I}} \left\{\sum_{i_{n-1} \in \mathcal{I}} u_{i_{n-1}} \left[\max_{j \in \mathcal{I}, j \neq i_{n-1}} \left\{\mathbf{E}\left(\lambda_{i_{n-1}}^{\tau_{i_{n-1} j}}\right)\right\}\right] \theta_{i_{n-2} i_{n-1}}\right\} \end{aligned}$$

$$\leq u_{i_0} M\left[\max_{j\in\mathcal{I},j\neq i_0}\left\{\mathbf{E}\left(\lambda_{i_0}^{\tau_{i_0 j}}\right)\right\}\right]\mathbf{E}\left[V(\xi(0),i_0)\right]$$
$$\times\prod_{m=1}^{n-1}\left[\max_{i_{m-1}\in\mathcal{I}}\left\{\sum_{i_m\in\mathcal{I}}u_{i_m}\left[\max_{j\in\mathcal{I},j\neq i_m}\left\{\mathbf{E}\left(\lambda_{i_m}^{\tau_{i_m j}}\right)\right\}\right]\theta_{i_{m-1}i_m}\right\}\right]$$
$$= u_{i_0} M\left[\max_{j\in\mathcal{I},j\neq i_0}\left\{\mathbf{E}\left(\lambda_{i_0}^{\tau_{i_0 j}}\right)\right\}\right]\mathbf{E}\left[V(\xi(0),i_0)\right]$$
$$\times\left[\max_{l\in\mathcal{I}}\left\{\sum_{i\in\mathcal{I}}u_i\left[\max_{j\in\mathcal{I},j\neq i}\left\{\mathbf{E}\left(\lambda_i^{\tau_{ij}}\right)\right]\theta_{li}\right\}\right]^{n-1}$$

which together with condition (11) implies

$$\lim_{n\to\infty}\mathbf{E}\left[V\left(\xi(t_n),r(t_n)\right)\right]=0. \tag{33}$$

Combining with (8) and (9), for any $k\in\mathbf{Z}_{[t_n+1,t_{n+1})}$, $n\in\mathbf{Z}_{\geq 0}$, we have

$$\begin{aligned}&\mathbf{E}\left[\alpha_1(||\xi(k)||)\right]\leq\mathbf{E}\left[V(\xi(k),r(k))\right]\\&=\mathbf{E}\left[V\left(\xi(k),r(t_n)\right)\right]\leq\mathbf{E}\left[\lambda_{r(t_n)}^{k-t_n}V\left(\xi(t_n),r(t_n)\right)\right].\end{aligned} \tag{34}$$

By the assumption that the probability distribution of sojourn time of switching only depends on the current mode and the next mode, we derive that for any $n\in\mathbf{Z}_{\geq 0}$

$$\begin{aligned}&\mathbf{E}\left[\lambda_{r(t_n)}^{k-t_n}V(\xi(t_n),r(t_n))\right]\\&=\mathbf{E}\left[\mathbf{E}\left[\lambda_{r(t_n)}^{k-t_n}V(\xi(t_n),r(t_n))|r(t_{n-1}),\xi(t_n),r(t_n)\right]\right]\\&=\mathbf{E}\left[V(\xi(t_n),r(t_n))\mathbf{E}\left[\lambda_{r(t_n)}^{k-t_n}|r(t_{n-1}),\xi(t_n),r(t_n)\right]\right]\\&\leq\max_{i\in\mathcal{I}}\left\{\max_{j\in\mathcal{I},j\neq i}\left\{\mathbf{E}\left(\lambda_i^{\tau_{ij}}\right)\vee 1\right\}\right\}\mathbf{E}\left[V(\xi(t_n),r(t_n))\right].\end{aligned}$$

Combining with (34) and (33), similar as [16, proof of Lemma 1], we obtain

$$\lim_{k\to\infty}\mathbf{E}\left[\|\xi(k)\|^2\right]=0. \tag{35}$$

By the definition of $\xi(k)$, we have $\|x(k)\|^2\leq\|\xi(k)\|^2$, $k\in\mathbf{Z}_{\geq 0}$, and so

$$\lim_{k\to\infty}\mathbf{E}\left[\|x(k)\|^2\right]=0. \tag{36}$$

Thus, we complete the proof of this lemma. Proof of Theorem 3.3. We construct

$$\begin{aligned}&V(\xi(k),r(k))\\&=V_1(\xi(k),r(k))+V_2(\xi(k),r(k))+V_3(\xi(k),r(k))\end{aligned} \tag{37}$$

in which

$$V_1(\xi(k),r(k))=x^T(k)P_{r(k)}x(k) \tag{38}$$

$$V_2(\xi(k),r(k))=\sum_{s=k-d}^{k-1}\lambda_{r(k)}^{k-1-s}x^T(s)Q_{r(k)}x(s), \tag{39}$$

$$V_3(\xi(k),r(k))=d\sum_{\theta=1}^{d}\sum_{s=k-\theta}^{k-1}\lambda_{r(k)}^{k-1-s}\eta^T(s)R_{r(k)}\eta(s), \tag{40}$$

where $\eta(k)=x(k+1)-x(k)$, $k\in\mathbf{Z}_{\geq 0}$. It is easy to know that (37) has the following form:

$$V(\xi(k),r(k))=\xi^T(k)\Psi_{r(k)}\xi(k). \tag{41}$$

For any $k\in\mathbf{Z}_{\geq 0}$

$$\begin{aligned}&\min_{i\in\mathcal{I}}\{\lambda_{\min}(\Psi_i)\}(\|\xi(k)\|^2)\leq\xi^T(k)\Psi_{r(k)}\xi(k)\\&\leq\max_{i\in\mathcal{I}}\{\lambda_{\max}(\Psi_i)\}(\|\xi(k)\|^2),\end{aligned} \tag{42}$$

which implies that condition (8) of Lemma 3.1 holds. For any $i,j\in\mathcal{I}$, it follows from (14) that

$$\begin{aligned}&V(\xi(k),j)=\xi^T(k)\Psi_j\xi(k)\\&\leq\max_{j\in\mathcal{I}}\left\{\frac{\lambda_{\max}(\Psi_j)}{\lambda_{\min}(\Psi_i)}\right\}\xi^T(k)\Psi_i\xi(k)=u_iV(\xi(k),i)\end{aligned} \tag{43}$$

and so condition (10) of Lemma 3.1 holds. For any $k\in\mathbf{Z}_{[t_n,t_{n+1})}$ and $r(t_n)=i$, $n\in\mathbf{Z}_{\geq 0}$, $i\in\mathcal{I}$, the generalized forward differences of the $V_1(\xi(k),i)$, $V_2(\xi(k),i)$, and $V_3(\xi(k),i)$ along the state trajectory of system (4) can be computed as

$$\begin{aligned}&V_1(\xi(k+1),i)-\lambda_iV_1(\xi(k),i)\\&=\begin{bmatrix}x(k)\\x(k-d)\end{bmatrix}^T\begin{bmatrix}A_i^TP_iA_i-\lambda_iP_i & A_i^TP_iA_{di}\\ \star & A_{di}^TP_iA_{di}\end{bmatrix}\\&\quad\times\begin{bmatrix}x(k)\\x(k-d)\end{bmatrix}\\&V_2(\xi(k+1),i)-\lambda_iV_2(\xi(k),i)\\&=\begin{bmatrix}x(k)\\x(k-d)\end{bmatrix}^T\begin{bmatrix}Q_i & \mathbf{0}\\ \star & -\lambda_i^dQ_i\end{bmatrix}\begin{bmatrix}x(k)\\x(k-d)\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}&V_3(\xi(k+1),i)-\lambda_iV_3(\xi(k),i)\\&=d^2\eta^T(k)R_i\eta(k)-d\sum_{s=1}^{d}\lambda_i^s\eta^T(k-s)R_i\eta(k-s).\\&\leq d^2\eta^T(k)R_i\eta^T(k)-d\widetilde{\lambda}_i\sum_{s=1}^{d}\eta^T(k-s)R_i\eta(k-s)\\&\leq d^2[x(k+1)-x(k)]^TR_i[x(k+1)-x(k)]\\&\quad-\widetilde{\lambda}_i\left[\sum_{s=1}^{d}\eta^T(k-s)\right]R_i\left[\sum_{s=1}^{d}\eta(k-s)\right]\\&=\begin{bmatrix}x(k)\\x(k-d)\end{bmatrix}^T\\&\quad\times\begin{bmatrix}d^2(A_i-\mathbf{I})^TR_i(A_i-\mathbf{I})-\widetilde{\lambda}_iR_i & d^2(A_i-\mathbf{I})^TR_iA_{di}+\widetilde{\lambda}_iR_i\\ \star & d^2A_{di}^TR_iA_{di}-\widetilde{\lambda}_iR_i\end{bmatrix}\\&\quad\times\begin{bmatrix}x(k)\\x(k-d)\end{bmatrix},\end{aligned} \tag{44}$$

in which the second $\leq$ of (44) is based on Lemma 2.6. Thus, for any $k\in\mathbf{Z}_{[t_n,t_{n+1})}$ and $r(t_n)=i\in\mathcal{I}$, $n\in\mathbf{Z}_{[0,+\infty)}$, we get the generalized forward difference of $V(\xi(k),i)$ along the state trajectory of system (4) as

$$V(\xi(k+1),i) - \lambda_i V(\xi(k),i) = \begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix}^T \begin{bmatrix} \Delta_{11i} & \Delta_{12i} \\ \star & \Delta_{22i} \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-d) \end{bmatrix}, \tag{45}$$

in which $\Delta_{11i} = A_i^T P_i A_i + Q_i + d^2(A_i - \mathbf{I})^T R_i (A_i - \mathbf{I}) - \lambda_i P_i - \widetilde{\lambda}_i R_i$, $\Delta_{12i} = A_i^T P_i A_{di} + d^2(A_i - \mathbf{I})^T R_i A_{di} + \widetilde{\lambda}_i R_i$ and $\Delta_{22i} = A_{di}^T P_i A_{di} + d^2 A_{di}^T R_i A_{di} - \widetilde{\lambda}_i R_i - \lambda_i^d Q_i$. By the Schur complement lemma, condition (12) is equivalent to

$$\begin{bmatrix} \Delta_{11i} & \Delta_{12i} \\ \star & \Delta_{22i} \end{bmatrix} < 0, \tag{46}$$

which together with (45) ensures that condition (9) of Lemma 3.1 holds. Thus, all the conditions of Lemma 3.1 are satisfied. It follows from Lemma 3.1 that system (4) is MSS under semi-Markov switching condition (11). Thus, we complete the proof of this theorem. Proof of Corollary 3.5. By letting $\lambda_i = \lambda \in \mathbf{R}_{(0,1)}, i \in \mathcal{I}$, (18) ensures that (16) holds. For any distributions of τ_{ij} and values of θ_{ij}, $i, j \in \mathcal{I}$, we can obtain that for any $l \in \mathcal{I}$

$$\sum_{i\in\mathcal{I}} \left[\max_{j\in\mathcal{I}, j\neq i} \left\{ \mathbf{E}\left(\lambda_i^{\tau_{ij}}\right) \right\} \right] \theta_{li} < \sum_{i\in\mathcal{I}} \theta_{li} = 1. \tag{47}$$

Thus, by Theorem 3.4, system (4) is MSS under arbitrary semi-Markov switching signal. Thus, we complete the proof of this corollary. Proof of Theorem 3.6. By letting $K_i = W_i^T V^{-T}$ and $J_i = L_i^T U^{-T}$, $i \in \mathcal{I}$, we have

$$V A_i^T + W_i B_i^T = V(A_i + B_i K_i)^T, \tag{48}$$

$$U A_{di}^T + L_i B_i^T = U(A_{di} + B_i J_i)^T. \tag{49}$$

By letting $\widetilde{A}_i = A_i + B_i K_i$, $\widetilde{A}_{di} = A_{di} + B_i J_i$, $i \in \mathcal{I}$ and substituting (48) and (49) into (22), we have

$$\begin{bmatrix} -\lambda_i V & \mathbf{0} & V\widetilde{A}_i^T & dV\left(\widetilde{A}_i - \mathbf{I}\right)^T & V \\ \star & -\lambda^d U & U\widetilde{A}_{di}^T & dU\widetilde{A}_{di}^T & \mathbf{0} \\ \star & \star & -V & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -Z & \mathbf{0} \\ \star & \star & \star & \star & -U \end{bmatrix} < 0. \tag{50}$$

By letting

$$Q = U^{-1}, P = V^{-1}, R = Z^{-1} \tag{51}$$

and applying the congruence transformation to (49) by diag$\{P, Q, P, R, Q\}$, we obtain

$$\begin{bmatrix} -\lambda_i P & \mathbf{0} & \widetilde{A}_i^T P & d(\widetilde{A}_i - \mathbf{I})^T R & Q \\ \star & -\lambda^d Q & \widetilde{A}_{di}^T P & d\widetilde{A}_{di}^T R & \mathbf{0} \\ \star & \star & -P & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -R & \mathbf{0} \\ \star & \star & \star & \star & -Q \end{bmatrix} < 0. \tag{52}$$

It is easy to know that

$$\begin{bmatrix} -\lambda^d R & \lambda^d R \\ \star & -\lambda^d R \end{bmatrix} \leq 0,$$

which together with (52) can ensure that condition (16) of Theorem 3.4 holds with respect to system (21). Hence, it follows from Theorem 3.4 that system (21) with semi-Markov switching condition (17) is MSS with the controller gains $K_i = (V^{-1}W_i)^T$ and $J_i = (U^{-1}L_i)^T, i \in \mathcal{I}$. Thus, we complete the proof of this theorem.

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