Robust H_∞ Filtering for Discrete-Time Systems with Nonlinear Uncertainties

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Abstract

In this paper, we consider the problem of robust H_∞ filtering for discrete-time systems with norm-bounded nonlinear uncertainties. The class of uncertain systems is described by a state-space model with linear nominal parts and norm-bounded nonlinear uncertainties on both state and output measurements. We proposed a method of converting such type of nonlinear uncertain system to a linear uncertain system, and presented a methodology for designing a robust H_∞ filter.

Keywords: nonlinear norm-bounded uncertainties, robust H_∞ filtering.

1 Introduction

Since some early attempts on the H_∞ filtering problems in the late 80's, the problem of estimator design for systems with the H_∞ performance criterion is by now fairly well understood (see e.g., [1], [2] and [3]). One of the main advantages of H_∞ estimation is that it is insensitive to the exact knowledge of the statistics of the noise signals. But in practice, modeling uncertainties exist not only in the form of a bounded energy noise signal but also in the form of parameter uncertainty in the system. This has motivated the study of robust H_∞ filtering in the last few years. Until now, some results (see e.g., [4], [5], [6] [7] and [8].) have been obtained and they make the robust H_∞ filtering approaches suitable for applications where parameter uncertainty exists and little knowledge about the noise statistics is available.

In this paper, motivated by the results in [9], we consider the problem of robust H_∞ filtering for discrete-time systems with norm-bounded nonlinear uncertainties. The class of uncertain systems is described by a state-space model with linear nominal parts and norm-bounded nonlinear time-varying parameter uncertainties on both state and output measurements. Such type of nonlinear uncertain system can be converted to a norm-bounded linear uncertain system. Then the Riccati equality approach can be used to obtain a robust H_∞ filter.

The remainder of this paper is organized as follows: We describe the problem formulation and give some preliminary results in Section 2. In Section 3, finite horizon robust H_∞ filtering problem is solved. Then a solution to the infinite horizon robust H_∞ filtering problem is given in Section 4. The paper ends with conclusion in Section 5.

Notations: Most of the notations used in this note are fairly standard. $\mathcal{K}$ denotes the set of integer numbers. $\Re^n$ denotes the n dimensional Euclidean space and $||\cdot||$ refers to Euclidean vector norm. The superscript 'T' denotes matrix transposition. For a matrix $M \in \Re^{n\times m}$, $\rho(M) = \lambda_{max}(M^T M)$. $l_2[0, N]$ stands for the space of square summable vector sequence over $[0, N]$. $R^{\frac{1}{2}}$ denotes the symmetric square root of a matrix $R = R^T \geq 0$, i.e. $R^{\frac{1}{2}} \cdot R^{\frac{1}{2}} = R$.

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2 Problem Formulation and Preliminaries

2.1 Problem Formulation

Consider the following class of norm-bounded nonlinear uncertain discrete-time systems

$$\begin{aligned} \Sigma: \quad x_{k+1} &= Ax_k + B\omega_k + \Delta_1(x_k,\omega_k) \quad (2.1)\\ y_k &= Cx_k + D\omega_k + \Delta_2(x_k,\omega_k) \quad (2.2)\\ z_k &= Lx_k \quad (2.3) \end{aligned}$$

where the initial state is x_0, $x_k \in \Re^n$ is the system state, $\omega_k \in \Re^q$ is the exogenous noise signal which is assumed to be an arbitrary signal in $l_2[0,N]$, $z_k \in \Re^p$ is a linear combination of the state variables to be estimated, $y_k \in \Re^m$ is the measurements, and A, B, C, D and L are known real bounded constant matrices with appropriate dimensions.

In addition, $\Delta_1(x_k,\omega_k)$ and $\Delta_2(x_k,\omega_k)$ are real time-varying matrices which represent time-varying norm-bounded nonlinear parameter uncertainties. Such uncertainties are assumed to be norm-bounded, i.e. there exists some $\alpha_i \geq 0$, $\beta_i \geq 0, (i=1,2)$ such that

$$\begin{aligned} ||\Delta_i(x_k,\omega_k)|| \leq \alpha_i||x_k|| + \beta_i||\omega_k||, \quad (i=1,2),\\ \text{for all } x_k \in \Re^n,\ \omega_k \in \Re^q,\ (k \in \mathcal{K}). \end{aligned}$$

The corresponding uncertainty sets can be denoted as follows:

$$\begin{aligned} \Omega_i(x_k,\omega_k) &= \{\Delta_i(x_k,\omega_k) : ||\Delta_i(x_k,\omega_k)||\\ &\quad \leq \alpha_i||x_k|| + \beta_i||\omega_k||\}, \quad (i=1,2). \end{aligned}$$

It is clear that the matrices $\Delta_i(x_k,\omega_k), i=1,2$ contain the uncertain parameters in the state and input matrices of the system Σ. And the scalar α_i and β_i represent the influence of the uncertain parameters in $\Delta_i(x_k,\omega_k), i=1,2,$ on the nominal matrices of the system Σ.

In order to simplify our results, we adopt the following assumption:

Assumption 2.1. *$\alpha_1 = \alpha_2 \triangleq \alpha$ and $\beta_1 = \beta_2 \triangleq \beta$, i.e., $\Delta_1(x_k,\omega_k) = \Delta_2(x_k,\omega_k) = \Delta(x_k,\omega_k)$, $\Omega_1(x_k,\omega_k) = \Omega_2(x_k,\omega_k) = \Omega(x_k,\omega_k)$.*

It should be noted that since we are only interested in the robustness for norm-bounded uncertainty, such assumption is without loss of generality. Please refer to Remark 2 of [9].

In this paper, our objective is to obtain an estimate z_{ek} [i] via a linear filter using the measurements $\{y_i,\ 0 \leq i \leq N\}$.

[i]In order to simplify expression, we denote time-varying variables $x_e(k)$, say, as x_{ek} in this paper.

For the finite horizon case, we consider a linear filter of the following form:

$$\begin{aligned} x_{e(k+1)} &= A_{ek}x_{ek} + K_{ek}y_k\ ,\ x_{e0} = x_0 \quad (2.4)\\ z_{ek} &= L_e x_{ek} \quad (2.5) \end{aligned}$$

where $x_{ek} \in \Re^n$ is the filter state, $z_{ek} \in \Re^p$ is the estimate of z_k, and y_k is the measure outputs of system Σ. A_{ek}, K_{ek} and L_e are what need to be determined. Next, we define the estimation error by $e_k = z_k - z_{ek}$.

Then the finite horizon robust H_∞ filtering problem addressed in this paper can be expressed as follows: *Given a prescribed level of noise attenuation $\gamma > 0$, find a linear filter of the form (2.4)-(2.5) such that the filtering error dynamic satisfies the finite horizon H_∞ performance*

$$||e||_2^2 \ \leq\ \gamma^2\{||\omega||_2^2 + x_0^T R_0 x_0\} \quad (2.6)$$

for any non-zero $\omega_k \in l_2[0,N)$ and norm-bounded nonlinear uncertainties $\Delta(x_k,\omega_k) \in \Omega(x_k,\omega_k)$.

2.2 Some Preliminary Results

At first, we introduce the following linear uncertainty set:

$$\begin{aligned} \Omega_l(x_k,\omega_k) \ &\triangleq\ \{\alpha M_{1k}x_k + \beta M_{2k}\omega_k : M_{1k} \in \Re^{n\times n},\\ &\quad M_{2k} \in \Re^{n\times q}, \rho(M_{1k}) \leq 1, \rho(M_{2k}) \leq 1.\} \end{aligned}$$

Then the relationship between the set $\Omega(x_k,\omega_k)$ and the set $\Omega_l(x_k,\omega_k)$ is established with the following lemma.

Lemma 2.1. $\Omega(x_k,\omega_k) = \Omega_l(x_k,\omega_k)$

Proof: It can be carried out by using the same technique as in [10]. $\bigtriangledown\bigtriangledown\bigtriangledown$

Remark 2.1. In the view of Lemma 2.1, the robust H_∞ filtering problem can be solved by considering all admissible uncertainties $\Delta(x_k,\omega_k) \in \Omega(x_k,\omega_k)$ if it can be solved by considering all admissible uncertainties $\Delta(x_k,\omega_k) \in \Omega_l(x_k,\omega_k)$. That means the norm-bounded nonlinear uncertainties can be considered with a linear form which is easier to deal with.

We next introduce the following structure assumption for the uncertain parameter M_{1k} and M_{2k}.

Assumption 2.2.

$$M_{1k} = F_k E_\alpha,\ M_{2k} = F_k E_\beta \quad (2.7)$$

where $E_\alpha \in \Re^{i\times n}$ and $E_\beta \in \Re^{i\times q}$ are known matrices, and $F_k \in \Re^{n\times i}$ is an unknown matrix satisfying $\rho(F_k) \leq 1$.

It should be denoted that such kind of uncertainty structure is widely used in the problems of robust control and filtering of uncertain systems (see e.g. [4], [6], [7], etc.).

Next, we shall recall a version of the bounded real lemma for linear discrete time-varying systems which will be used in the derivation of a solution to the robust H_∞ filtering problem.

Consider the following linear time-varying system:

$$x_{k+1} = A_k x_k + B_k \omega_k \quad (2.8)$$
$$z_k = C_k x_k \quad (2.9)$$

Lemma 2.2. **([4])** *Given a scalar $\gamma > 0$ and weighting matrix $R_0 = R_0^T > 0$, the system (2.8)-(2.9) possesses the following H_∞ performance*

$$||z||_2^2 \le \gamma^2\{||\omega||_2^2 + x_0^T R_0 x_0\}$$

if there exists a bounded time-varying $Y_k = Y_k^T > 0$ over $[0, N]$ satisfying $I - \gamma^{-2} C_k Y_k C_k^T > 0$ and such that

$$A_k Y_k A_k^T - Y_{k+1} + \gamma^{-2} A_k Y_k C_k^T (I - \gamma^{-2} C_k Y_k C_k^T)^{-1}$$
$$\cdot C_k Y_k A_k^T + B_k B_k^T \le 0, \quad Y_0 = R_0^{-1}$$

We end this section by giving a linear matrix inequality result which will be needed in the proof of our main result.

Lemma 2.3. *Let $A \in \Re^{n\times n}$, $H_1 \in \Re^{n\times i}$, $E_1 \in \Re^{j\times n}$, $B \in \Re^{n\times m}$, $H_2 \in \Re^{n\times i}$, $E_2 \in \Re^{j\times m}$ and $Q = Q^T \in \Re^{n\times n}$ be given matrices. Then there exists a real matrix $X = X^T \ge 0$ such that*

$$(A + H_1 F E_1) X (A + H_1 F E_1)^T$$
$$+ (B + H_2 F E_2)(B + H_2 F E_2)^T - Q \le 0$$

for all F satisfying $F^T F \le I$, if there exists a scalar $\epsilon > 0$ such that $\epsilon I - E_1 X E_1^T \ge 0$, $\epsilon I - E_2 E_2^T \ge 0$ and

$$A X A^T + A X E_1^T (\epsilon I - E_1 X E_1^T)^{-1} E_1 X A^T$$
$$+ B(I - \frac{1}{\epsilon} E_2 E_2^T)^{-1} B^T + \epsilon (H_1 H_1^T + H_2 H_2^T) - Q = 0$$

Proof: It can be carried out by using the same technique as in [11]. $\bigtriangledown\bigtriangledown\bigtriangledown$

3 Finite Horizon Robust H_∞ Filter

The solution to the finite horizon robust H_∞ filtering problem formulated in the previous section will be derived in this section.

Firstly, from Lemma 2.1 and Remark 2.1, the problem of robust H_∞ filtering problem for system Σ is solvable if the robust H_∞ filtering problem for the following system

$$\Sigma_l :$$
$$x_{k+1} = (A + \alpha F_k E_\alpha) x_k + (B + \beta F_k E_\beta) \omega_k \quad (3.1)$$
$$y_k = (C + \alpha F_k E_\alpha) x_k + (D + \beta F_k E_\beta) \omega_k \quad (3.2)$$
$$z_k = L x_k \quad (3.3)$$

is solvable, where the initial state is x_0, $x_k \in \Re^n$ is the system state, $\omega_k \in \Re^q$ is the exogenous noise signal which belongs to $l_2[0, N]$, $z_k \in \Re^p$ is a linear combination of the state variables to be estimated, $y_k \in \Re^m$ is the measurements, and A, B, C, D, L, E_α and E_β are the same as in (2.1)-(2.3) and (2.7), where F_k is as in Assumption 2.2.

Motivated by the results on robust H_∞ filtering for uncertain systems (see, e.g. [7] and [8]), the filter (2.4)-(2.5) can be re-expressed in the form of a modified observer as follows:

$$x_{e(k+1)} = (A + \Delta A_{ek}) x_{ek} + K_{ek}\{y_k$$
$$- (C + \Delta C_{ek} x_{ek})\} \quad (3.4)$$
$$z_{ek} = L x_{ek} \quad (3.5)$$

where A, C and L are the same as in the system Σ_l and ΔA_{ek}, $\Delta C_{ek} x$ and K_{ek} are matrices to be chosen such that the filter (3.4)-(3.5) will solve the robust filtering problem for the system Σ_l. The motivation for (3.4)-(3.5) is that the filter structure should take into account the parameter uncertainties in the system Σ_l.

Before presenting the results of this section, some shorthand notations are introduced as follows:

$$R_\beta = (I - \epsilon^{-1} E_\beta^T E_\beta)^{-1} \quad , \quad \bar{L} = \begin{bmatrix} L \\ \gamma \epsilon^{-\frac{1}{2}} E_\alpha \end{bmatrix},$$
$$\bar{B} = \begin{bmatrix} B R_\beta^{\frac{1}{2}} & \epsilon \alpha I & \epsilon \beta I \end{bmatrix},$$
$$\bar{D} = \begin{bmatrix} D R_\beta^{\frac{1}{2}} & \epsilon \alpha I & \epsilon \beta I \end{bmatrix}. \quad (3.6)$$

The next theorem provides a solution to the finite horizon robust H_∞ filtering problem for uncertain system Σ_l.

Theorem 3.1. *Given a scalar $\gamma > 0$ and weighting matrix $R_0 > 0$, the finite horizon robust H_∞ filtering problem associated with the uncertain system Σ_l is solvable if for some scalar $\epsilon > 0$ satisfying $\epsilon I - E_\beta E_\beta^T \ge 0$, the following conditions hold:*
1) There exists a solution $X_k = X_k^T > 0$ to the DRE:

$$X_{k+1} = A X_k A^T - (A X_k C_1^T + \bar{B} D_1^T)$$
$$\cdot (C_1 X_k C_1^T + R_1)^{-1} (A X_k C_1^T + \bar{B} D_1^T)^T$$
$$+ \bar{B} \bar{B}^T \quad (3.7)$$

with $X_0 = R_0^{-1}$, and such that $X_k^{-1} - \gamma^{-2} L^T L > 0$, *where*

$$C_1 = \begin{bmatrix} C \\ \gamma^{-1}\bar{L} \end{bmatrix}, D_1 = \begin{bmatrix} \bar{D} \\ 0 \end{bmatrix}, R_1 = \begin{bmatrix} \bar{D}\bar{D}^T & 0 \\ 0 & -I \end{bmatrix}.$$

2) There exists a solution $P_k = P_k^T \geq S_k = S_k^T > 0$ to the DRE:

$$P_{k+1} - S_{k+1} + X_{k+1} = AP_kA^T + AP_kE_\alpha^T(\epsilon I - E_\alpha P_k E_\alpha^T)^{-1}E_\alpha P_k A^T + \bar{B}\bar{B}^T, \quad P_0 = R_0^{-1} \tag{3.8}$$

and such that $\epsilon I - E_\alpha P_k E_\alpha^T > 0$, where $S_k^{-1} = X_k^{-1} - \gamma^{-2}L^TL$.

Moreover, the filter parameters of (3.4) are given as follows:

$$\Delta A_{ek} = AS_kE_\alpha^T(\epsilon I - E_\alpha S_k E_\alpha^T)^{-1}E_\alpha \tag{3.9}$$

$$\Delta C_{ek} = CS_kE_\alpha^T(\epsilon I - E_\alpha S_k E_\alpha^T)^{-1}E_\alpha \tag{3.10}$$

$$K_{ek} = (AQ_kC^T + \bar{B}\bar{D}^T)(CQ_kC^T + \bar{D}\bar{D}^T)^{-1} \tag{3.11}$$

where $Q_k = (S_k^{-1} - \epsilon^{-1}E_\alpha^TE_\alpha)^{-1}$.

Proof: Define $\tilde{x}_k = x_k - x_{ek}$. Then from (3.1), (3.2), (2.4) and (3.4), we have

$$\begin{aligned} \tilde{x}_{k+1} &= [A + \alpha F_k E_\alpha - K_{ek}(C + \alpha F_k E_\alpha)]\tilde{x}_k \\ &\quad + [A - A_{ek} + \alpha F_k E_\alpha - K_{ek}(C + \alpha F_k E_\alpha)]x_{ek} \\ &\quad + [B + \beta F_k E_\beta - K_{ek}(D + \beta F_k E_\beta)]\omega_k \end{aligned} \tag{3.12}$$

So the estimation error dynamics e_k associated with (2.4) and (3.12) is of the form

$$\eta_{k+1} = (A_1 + H_1F_kE_1)\eta_k + (B_1 + H_2F_kE_2)\omega_k \tag{3.13}$$

$$e_k = L_1\eta_k \tag{3.14}$$

where $\eta_k = [\tilde{x}_k^T \;\; x_{ek}^T]^T$ and

$$\begin{aligned} A_1 &= \begin{bmatrix} A - K_{ek}C & A - A_{ek} - K_{ek}C \\ K_{ek}C & A + K_{ek}C \end{bmatrix}, \\ B_1 &= \begin{bmatrix} B - K_{ek}D \\ K_{ek}D \end{bmatrix}, \quad L_1 = \begin{bmatrix} L & 0 \end{bmatrix}, \\ H_1 &= \begin{bmatrix} \alpha - \alpha K_{ek} \\ \alpha K_{ek} \end{bmatrix}, \quad H_2 = \begin{bmatrix} \beta - \beta K_{ek} \\ \beta K_{ek} \end{bmatrix}, \\ E_1 &= \begin{bmatrix} E_\alpha & E_\alpha \end{bmatrix}, \quad E_2 = E_\beta. \end{aligned} \tag{3.15}$$

Using Lemma 2.2 and the matrix inversion lemma, we conclude that the uncertain system (3.13)-(3.14) satisfies the H_∞ performance constrain (2.6) if there exists a matrix $Y_k = Y_k^T > 0$ such that

$$(A_1 + H_1F_kE_1)\Sigma_k(A_1 + H_1F_kE_1)^T + (B_1 + H_2F_kE_2)(B_1 + H_2F_kE_2)^T - (\Sigma_{k+1}^{-1} + \gamma^{-2}L_1^TL_1)^{-1} \leq 0 \tag{3.16}$$

where $\Sigma_k^{-1} = Y_k^{-1} - L_1^TL_1 > 0$ and $Y_0 = R_0^{-1}$.

Next, we let

$$\begin{aligned} U &= \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} \\ &= A_1\Sigma_kA_1^T + A_1\Sigma_kE_1^T(\epsilon I - E_1\Sigma_kE_1^T)^{-1}E_1\Sigma_kA_1^T \\ &\quad + B_1(I - \frac{1}{\epsilon}E_2E_2^T)^{-1}B_1^T + \epsilon(H_1H_1^T + H_2H_2^T) \\ &\quad - (\Sigma_{k+1}^{-1} + \gamma^{-2}L_1^TL_1)^{-1} \end{aligned} \tag{3.17}$$

where $\epsilon > 0$ is a scalar and $\epsilon I - E_1\Sigma_kE_1^T \geq 0$, $\epsilon I - E_2E_2^T \geq 0$.

Then we show that $U = 0$ when we choose the filter parameters as in (3.9)-(3.11). We denote

$$\Sigma_k = \begin{bmatrix} S_k & 0 \\ 0 & P_k - S_k \end{bmatrix} \tag{3.18}$$

Then consider (3.15) and (3.18), after some tedious algebraic manipulations, we can derive from (3.17) that

$$\begin{aligned} U_1 &= AQ_kA^T - (S_{k+1}^{-1} + \gamma^{-2}L^TL)^{-1} \\ &\quad + \{K_{ek} - \big(AQ_kC^T + BR_\beta D^T + \epsilon(\alpha^2 + \beta^2)I\big) \\ &\quad \cdot \big(CQ_kC^T + DR_\beta D^T + \epsilon(\alpha^2 + \beta^2)I\big)^{-1}\} \\ &\quad \cdot \big(CQ_kC^T + DR_\beta D^T + \epsilon(\alpha^2 + \beta^2)I\big) \\ &\quad \cdot \{K_{ek} - \big(AQ_kC^T + BR_\beta D^T + \epsilon(\alpha^2 + \beta^2)I\big) \\ &\quad \cdot \big(CQ_kC^T + DR_\beta D^T + \epsilon(\alpha^2 + \beta^2)I\big)^{-1}\}^T \\ &\quad - \big(AQ_kC^T + BR_\beta D^T + \epsilon(\alpha^2 + \beta^2)I\big) \\ &\quad \cdot \big(CQ_kC^T + DR_\beta D^T + \epsilon(\alpha^2 + \beta^2)I\big)^{-1} \\ &\quad \cdot \big(AQ_kC^T + BR_\beta D^T + \epsilon(\alpha^2 + \beta^2)I\big)^T \\ &\quad + BR_\beta B^T + \epsilon(\alpha^2 + \beta^2)I \end{aligned} \tag{3.19}$$

where $Q_k = (S_k^{-1} - \epsilon^{-1}E_\alpha^TE_\alpha)^{-1}$ and $R_\beta = (I - \epsilon^{-1}E_\beta^TE_\beta)^{-1}$.

Consider the DRE (3.7) and (3.11), it can be derived from (3.19) that $U_1 = 0$.

Next, we expand (3.17) and obtain U_2 as follows:

$$\begin{aligned} U_2 &= (A - K_{ek}C)S_kC^TK_{ek}^T - (\Delta A_{ek} - K_{ek}\Delta C_{ek}) \\ &\quad \cdot (P_k - S_k)(A + \Delta A_{ek} - K_{ek}\Delta C_{ek})^T \\ &\quad + \{(A - K_{ek}C)S_k - (\Delta A_{ek} - K_{ek}\Delta C_{ek}) \\ &\quad \cdot (P_k - S_k)\}E_\alpha^T(\epsilon I - E_\alpha P_kE_\alpha^T)^{-1}E_\alpha \\ &\quad \cdot \{K_{ek}CS_k + (A + \Delta A_{ek} - K_{ek}\Delta C_{ek}) \\ &\quad (P_k - S_k)\}^T + (B - K_{ek}D)R_\beta D^TK_{ek}^T \\ &\quad + \epsilon(\alpha^2 + \beta^2)(I - K_{ek})K_{ek}^T \end{aligned} \tag{3.20}$$

By applying (3.7) and (3.9)-(3.11), (3.20) can be simplified after some tedious but straightforward algebraic manipulations,

$$\begin{aligned} U_2 &= (A - K_{ek}C)Q_kC^TK_{ek}^T + (B - K_{ek}D)R_\beta D^TK_{ek}^T \\ &\quad +\epsilon(\alpha^2+\beta^2)(I - K_{ek})K_{ek}^T \\ &= 0 \end{aligned} \tag{3.21}$$

Similarly, it follows from (3.17) that

$$\begin{aligned} U_3 &= K_{ek}CS_kC^TK_{ek}^T + (A + \Delta A_{ek} - K_{ek}\Delta C_{ek}) \\ &\quad \cdot(P_k - S_k)(A + \Delta A_{ek} - K_{ek}\Delta C_{ek})^T \\ &\quad +\{K_{ek}CS_k + (A + \Delta A_{ek} - K_{ek}\Delta C_{ek})(P_k - S_k)\} \\ &\quad \cdot E_\alpha^T(\epsilon I - E_\alpha P_k E_\alpha^T)^{-1}E_\alpha \\ &\quad \cdot\{K_{ek}CS_k + (A + \Delta A_{ek} - K_{ek}\Delta C_{ek})(P_k - S_k)\}^T \\ &\quad +K_{ek}DR_\beta D^TK_{ek}^T \\ &\quad +\epsilon(\alpha^2+\beta^2)K_{ek}K_{ek}^T - (P_{k+1} - S_{k+1}) \end{aligned} \tag{3.22}$$

Consider (3.7), (3.8) and (3.9)-(3.11), it follows from (3.22) that $U_3 = 0$.

In conclusion, $U = 0$.

Now, we shall show that $\Sigma_k > 0$. Firstly, it is worth noting that $S_k > 0$ is ensured by Condition 1 of Theorem 3.1. Next, it follows from (3.7) and (3.8) that

$$\begin{aligned} &P_{k+1} - S_{k+1} \\ = \; & A\{(P_k^{-1} - \epsilon^{-1}E_\alpha^TE_\alpha)^{-1} - (S_k^{-1} \\ & -\epsilon^{-1}E_\alpha^TE_\alpha)^{-1}\}A^T \\ & +\{AQ_kC^T + BR_\beta D^T + \epsilon(\alpha^2+\beta^2)I\} \\ & \cdot\{CQ_kC^T + DR_\beta D^T + \epsilon(\alpha^2+\beta^2)I\}^{-1} \\ & \cdot\{AQ_kC^T + BR_\beta D^T + \epsilon(\alpha^2+\beta^2)I\}^T \end{aligned} \tag{3.23}$$

Observe that $P_0 = S_0$ and $P_1 \geq S_1$. On the other hand, if $P_k - S_k \geq 0$, $(P_k^{-1} - \epsilon^{-1}E_\alpha^TE_\alpha)^{-1} - (S_k^{-1} - \epsilon^{-1}E_\alpha^TE_\alpha)^{-1} \geq 0$, it means that $P_{k+1} - S_{k+1} \geq 0$. To sum up, it is obvious that $\Sigma_k > 0$.

Lastly, using Lemma 2.3, (3.17) equals to 0 implies (3.16), i.e. the filter (3.4)-(3.5) solves the robust H_∞ filtering problem for the uncertain system Σ_l. $\bigtriangledown\bigtriangledown\bigtriangledown$

Remark 3.1. Theorem 3.1 provides a solution to the finite horizon robust H_∞ filtering problem for uncertain system Σ_l. The design of a finite horizon robust H_∞ filter can be carried out by computing two DREs (3.7) and (3.8) recursively. Equation (3.8) of Theorem 4.1, which depends on X_{k+1}, do not impose a special difficult since equations (3.7) and (3.8) can be solved in sequence at every step.

Remark 3.2. Note that when there is no norm-bounded nonlinear uncertainties in the system Σ, i.e. $\Delta_1(x_k,\omega_k) = 0$ and $\Delta_2(x_k,\omega_k) = 0$, then the parameter uncertainties in the system Σ_l also disappear. In such case, the DRE (3.7) will recover the the finite horizon H_∞ filtering solution of nominal system without parameter uncertainty ([4]), at the same time, the DRE (3.8) will be redundant.

Now we are in the position to give the main result of this paper. Note that, by Lemma 2.1 and Remark 2.1, the solvability of the robust H_∞ filtering problem for the system Σ_l ensures the problem of robust H_∞ filtering for the system Σ has a solution. To summarize, we have the following main result.

Theorem 3.2. *If for the system Σ_l, there exists a linear filter (3.4)-(3.5), whose parameters are given by (3.9)-(3.11), such that the filtering error dynamic satisfies the finite horizon H_∞ performance (2.6), then for the uncertain system Σ, with the same filter, the filtering error dynamic also satisfies the finite horizon H_∞ performance (2.6).*

Remark 3.3. Theorem 3.2 shows that, instead of designing the robust filter for the system Σ which involves norm-bounded nonlinear uncertainties, we may design a filter based on the system Σ_l which does not involve nonlinear uncertainties.

4 Infinite Horizon H_∞ Filtering

In this section, we design a robust H_∞ filter which solves the infinite-horizon robust H_∞ filtering problem. Consider again the system Σ, Σ_l and the linear filter (2.4)-(2.5) and (3.4)-(3.5) , but now with $x_0 = 0$.

Then the infinite horizon robust H_∞ filtering problem can be expressed as follows: *Given a prescribed level of noise attenuation $\gamma > 0$, find a linear filter of the form (2.4)-(2.5) such that the filtering error dynamic is asymptotically stable and satisfies the infinite horizon H_∞ performance*

$$||e||_2^2 \; < \; \gamma^2|||\omega||_2^2 \tag{4.1}$$

for any non-zero $\omega_k \in l_2[0,\infty)$ and norm-bounded nonlinear uncertainties $\Delta(x_k,\omega_k) \in \Omega(x_k,\omega_k)$.

The following theorem is the stationary counterpart of Theorem 3.1 and provides a solution to the infinite-horizon robust H_∞ filtering problem of the system Σ_l .

Theorem 4.1. *Given a scalar $\gamma > 0$, the infinite horizon robust H_∞ filtering problem associated with the uncertain system Σ_l is solvable if for some scalar $\epsilon > 0$ satisfying $\epsilon I - E_\beta E_\beta^T \geq 0$, the following conditions hold:*
1) There exists a solution $X = X^T > 0$ to the ARE:

$$\begin{aligned} X &= AXA^T - (AXC_1^T + \bar{B}D_1^T)(C_1XC_1^T + R_1)^{-1} \\ &\quad (AXC_1^T + \bar{B}D_1^T)^T + \bar{B}\bar{B}^T \end{aligned} \tag{4.2}$$

and such that $X^{-1} - \gamma^{-2}L^TL > 0$, where

$$C_1 = \begin{bmatrix} C \\ \gamma^{-1}\bar{L} \end{bmatrix}, D_1 = \begin{bmatrix} \bar{D} \\ 0 \end{bmatrix}, R_1 = \begin{bmatrix} \bar{D}\bar{D}^T & 0 \\ 0 & -I \end{bmatrix}.$$

2) There exists a solution $P = P^T \geq S = S^T > 0$ to the ARE:

$$\begin{aligned} P - S + X &= APA^T \\ &\quad + APE_\alpha^T(\epsilon I - E_\alpha P E_\alpha^T)^{-1} E_\alpha P A^T \\ &\quad + \bar{B}\bar{B}^T \end{aligned} \tag{4.3}$$

and such that $\epsilon I - E_\alpha P E_\alpha^T > 0$, where $S^{-1} = X^{-1} - \gamma^{-2} L^T L$.

Moreover, the filter parameters of (3.4) are given by:

$$\Delta A_e = ASE_\alpha^T(\epsilon I - E_\alpha S E_\alpha^T)^{-1} E_\alpha \tag{4.4}$$

$$\Delta C_e = CSE_\alpha^T(\epsilon I - E_\alpha S E_\alpha^T)^{-1} E_\alpha \tag{4.5}$$

$$K_e = (AQC^T + \bar{B}\bar{D}^T)(CQC^T + \bar{D}\bar{D}^T)^{-1} \tag{4.6}$$

where $Q = (S^{-1} - \epsilon^{-1} E_\alpha^T E_\alpha)^{-1}$.

The following theorem provides a solution to the problem of infinite horizon robust H_∞ filtering for the system Σ.

Theorem 4.2. *If for the system Σ_l, there exists a linear filter (3.4)-(3.5), whose parameters are given by (4.4)-(4.6), such that the filtering error dynamic satisfies the infinite horizon H_∞ performance (4.1), then for the uncertain system Σ, with the same filter, the filtering error dynamic also satisfies the infinite horizon H_∞ performance (4.1).*

5 Conclusion

In this paper, the problem of robust H_∞ filtering for discrete-time systems with norm-bounded nonlinear uncertainties has been investigated. A sufficient condition is given. It has been shown that a solution to this estimation problem can be obtained from the solution of robust H_∞ filtering problem involving no nonlinear uncertainties but only one scaling parameter. The desired solution is derived from two Riccati equations. Both the finite horizon case and the infinite horizon case are discussed.

References

[1] J. Doyle, K. Glover, P. Khargonekar, and B. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Trans. on Automatic Control*, vol. 34, pp. 831–846, 1989.

[2] K. Nagpal and P. Khargonekar, "Filtering and smoothing in an H_∞ setting," *IEEE Trans. on Automatic Control*, vol. 36, pp. 152–166, 1991.

[3] U. Shaked and Y. Theodor, "H_∞-optimal estimation: a tutorial," in *Proc. 31st IEEE Conf. on Decision and Control*, (Tucson, TX.), pp. 2278–2286, 1992.

[4] C. E. de Souza and L. Xie, "Robust H_∞ filtering," *Control and Dynamic System*, vol. 65, pp. 323–377, 1994.

[5] L. Xie, C. E. de Souza, and M. Fu, "H_∞ estimation for discrete-time linear uncertain systems," *Int. J. Robust & Nolinear Control*, vol. 1, pp. 111–123, 1991.

[6] L. Xie, Y. C. Soh, and C. E. de Souza, "Robust Kalman filtering for uncertain discrete-time systems," *IEEE Trans. on Automatic Control*, vol. 39, pp. 1310–1314, 1994.

[7] C. Du, L. Xie, and Y. Soh, "Robust H_∞ filtering for uncertain 2-D discrete systems," in *Proc. 1999 American Control Conf.*, 1999.

[8] X. Zhu, Y. C. Soh and L. Xie, "Design & Analysis of Discrete-Time Robust Kalman Filters", *Automatica*, vol. 38, pp. 1069-1077, 2002.

[9] P. Shi and S. Shue, "Robust H_∞ control for linear discrete-time systems with norm-bounded nolinear uncertainties," *IEEE Trans. on Automatic Control*, vol. 44, pp. 108–111, 1999.

[10] L. Wang and W. Zhan, "Robust disturbance attenuation with stability for linear systems with norm-bounded nonlinear uncertainties," *IEEE Trans. on Automatic Control*, vol. 41, pp. 886–888, 1996.

[11] L. Xie and Y. C. Soh, "Guaranteed cost control of uncertain discrete-time systems," *Control Theory and Advanced Technology*, vol. 10, no. 4, pp. 1235–1251, 1995.