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Brief Paper

On the Solution of Discrete-time Markovian Jump Linear Quadratic Control Problems*

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Key Words—Discrete-time systems; linear systems; optimal control; Riccati equations; Markovian jumps.

Abstract—A necessary and sufficient condition for the existence of a positive-semidefinite solution of the coupled algebraic discrete-time Riccati-like equation occurring in Markovian jump control problems is derived. By verifying a simple matrix inequality, it is shown that such a solution exists and can be obtained as a limit of a monotonic sequence. This leads to a straightforward numerical algorithm for the computation of the solution. An example is given to illustrate the proposed method.

1. *Introduction*

We consider the jump-linear system described by

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k, \quad 0 \le k \le N,$$
$$y_k = C(r_k)x_k,$$

with initial state $x(0) = x_0$, $r(0) = r_0$, where $x_k \in \mathbf{R}^n$, is the plant state, $u_k \in \mathbf{R}^m$ is the control vector and $y_k \in \mathbf{R}^q$ is the process output. Here k is the time index; r_k is the form process taking values in the set $\mathbf{M} = \{1, 2, \ldots, M\}$ and is a finite-state discrete-time Markov chain with transition probabilities

$$\Pr\{r_{k+1} = j \mid r_k = i\} = p_{ij}, \quad 1 \le i, j \le M, \quad \text{with } p_{ii} > 0.$$

The cost criterion to be minimized is given by

$$J(x_0, r_0) = \lim_{N\to\infty} \mathbf{E}\left[\sum_{k=0}^{N-1} [x_k^T Q(r_k)x_k + u_k^T R(r_k)u_k] + x_N^T K_T(r_N)x_N\right], \tag{1}$$

where $Q(r_k) \ge 0$, $R(r_k) > 0$ and $K_T(r_N) \ge 0$ for all k.

Discrete-time Markovian jump-linear quadratic optimal control problems have been studied in several papers. Blair and Sworder (1975) solved the finite-time-horizon case. Chizeck *et al.* (1986) derived necessary and sufficient conditions for the existence of a steady-state solution; however, these conditions are not easy to test. Ji and Chizeck (1988) introduced new and refined definitions of the controllability and observability of jump-linear systems, leading to relatively simple algebraic tests to determine the existence of a steady-state solution.

In the aforementioned references, the feedback law to minimize (1) is obtained by solving the system of coupled algebraic Riccati-like equations

$$K(r_k) = A^T(r_k)G_kA(r_k) + Q(r_k) - A^T(r_k)G_kB(r_k)[R(r_k) + B^T(r_k)G_kB(r_k)]^{-1}B^T(r_k)G_kA(r_K), \tag{2}$$

where

$$G_k = \sum_{i=1}^{M} p_{ki}K(r_i), \quad 1 \le k \le M.$$

According to Chizeck *et al.* (1986), it is not possible to write (2) as a higher-dimensional single Riccati equation. Therefore it is clear that known results and algorithms for solving the linear-quadratic optimal control problem cannot be applied directly to jump-linear systems.

The purpose of this paper is to give necessary and sufficient conditions for the existence of a positive-semidefinite solution of the set of coupled Riccati equations (2). Moreover, a simple algorithm to compute the sought solution is given.

The paper is structured as follows. In Section 2 notation and preliminary results are introduced. The main theorems are given in Section 3, while Section 4 is dedicated to a numerical example.

2. *Notation and preliminaries*

With $r_k = j$, we shall use the following notation:

$$A_j := \sqrt{p_{jj}}\,A(r_k), \quad B_j := \sqrt{p_{jj}}\,B(r_k)R(r_k)^{-1/2},$$
$$K_j := K(r_k), \quad Q_j := Q(r_k), \quad \pi_{ij} = \frac{p_{ij}}{p_{jj}},$$

$1 \le i, j \le M$. Using these abbreviations, we can write (2) as

$$K_j = A_j^TF_jA_j + Q_j - A_j^TF_jB_j(I + B_j^TF_jB_j)^{-1}B_j^TF_jA_j =: \varphi_j(F_j, Q_j) \quad 1 \le j \le M, \tag{3}$$

where

$$F_j = \sum_{i=1}^{M} \pi_{ji}K_i = K_j + \sum_{i\ne j} \pi_{ji}K_i$$

and where φ_j is defined by (3). To (3) there correspond M decoupled standard Riccati difference equations

$$P_j\,(=\varphi_j(P_j, Q_j)) = A_j^TP_jA_j + Q_j - A_j^TP_jB_j(I + B_j^TP_jB_j)^{-1}B_j^TP_jA_j, \quad 1 \le j \le M. \tag{4}$$

The following properties of the strong solution of (4) are known from the literature (Kučera, 1972; Payne and Silverman, 1973; Chan *et al.*, 1984; De Souza *et al.*, 1986):

- the strong solution P_j^+ of (4) exists and is unique if and only if (A_j, B_j) is stabilizable;
- the strong solution exists and is the only positive-semidefinite solution of (4) if and only if (A_j, B_j) is stabilizable and there are no $(\sqrt{Q_j}, A_j)$-unobservable modes outside the unit circle;

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- P_j^+ exists and is stabilizing if and only if (A_j, B_j) is stabilizable and there are no $(\sqrt{Q_j}, A_j)$-unobservable modes on the unit circle;
- P_j^+ exists and is positive-definite if and only if (A_j, B_j) is stabilizable and there are no $(\sqrt{Q_j}, A_j)$-unobservable modes inside or on the unit circle;
- P_j^+ depends monotonically on the coefficients of (4), i.e. it depends monotonically on the matrix

$$\begin{pmatrix} Q_j & A_j^T \\ A_j & -B_jB_j^T \end{pmatrix}$$

(see Wimmer, 1992).

3. *Main results*

For convenience we make the following assumptions:
(H.1) B_j, $1 \le j \le M$, has full rank, i.e. $B_j^TB_j$ is regular;
(H.2) (A_j, B_j), $1 \le j \le M$, is stabilizable.
Let $K_{j0} \ge 0$, $1 \le j \le M$; then we define the sequences $(K_j(v))_{v\in \mathbf{N}_0}$ and $(P_j(v))_{v\in\mathbf{N}_0}$, $1 \le j \le M$, by

$$K_j(0) = P_j(0) = K_{j0},$$

$$K_j(v+1) = \varphi_j\Big(K_j(v) + \sum_{i\neq j} \pi_{ji}K_i(v), Q_j\Big), \tag{5}$$

$$P_j(v+1) = \varphi_j(P_j(v), Q_j),$$

$1 \le j \le M$, $v \in \mathbf{N}_0$. In order to get further information on the behaviour of these sequences, we use the following result (see Bitmead *et al.*, 1985, (3.1), (3.2); de Souza, 1989, Lemma 3.1):

Lemma 1. Let $\hat{Q}$ and $\tilde{Q}$ be symmetric matrices.
(a) For $1 \le j \le M$ let $0 \le \tilde{P}_j(v) < \hat{P}_j(v)$, and

$$\hat{P}_j(v+1) = \varphi_j(\hat{P}_j(v), \hat{Q}), \quad \tilde{P}_j(v+1) = \varphi_j(\tilde{P}_j(v), \tilde{Q}).$$

Then $\hat{P}_j(v) - \tilde{P}_j(v)$ satisfies the Riccati difference equation

$$\begin{aligned}\hat{P}_j(v+1) - \tilde{P}_j(v+1) &= \hat{Q} - \tilde{Q}\\ &+ A_j^T\{I - \tilde{P}_j(v)B_j[I + B_j^T\tilde{P}_j(v)B_j]^{-1}\}\\ &\times g(\hat{P}_j(v) - \tilde{P}_j(v))\{I - \tilde{P}_j(v)B_j[I + B_j^T\tilde{P}_j(v)B_j]^{-1}\}^TA,\end{aligned} \tag{6}$$

where

$$\begin{aligned}g(\hat{P}_j(v) - \tilde{P}_j(v)) &= [\hat{P}_j(v) - \tilde{P}_j(v)]B_j\{B_j^T[\hat{P}_j(v) - \tilde{P}_j(v)]B_j\\ &+ B_j^T[\hat{P}_j(v) - \tilde{P}_j(v)]B_j[B_j\tilde{P}_j(v)B_j + I]^{-1}\\ &\times B_j^T[\hat{P}_j(v) - \tilde{P}_j(v)]B_j\}^{-1}B_j^T[\hat{P}_j(v) - \tilde{P}_j(v)]\\ &+ \{I - [\hat{P}_j(v) - \tilde{P}_j(v)]B_j\\ &\times \{B_j^T[\hat{P}_j(v) - \tilde{P}_j(v)]B_j\}^{-1}B_j^T\}[\hat{P}_j(v) - \tilde{P}_j(v)]\\ &\times \{I - [\hat{P}_j(v) - \tilde{P}_j(v)]B_j\{B_j^T[\hat{P}_j(v) - \tilde{P}_j(v)]B_j\}^{-1}B_j^T\}^T.\end{aligned}$$

(b) If $0 \le \tilde{P}_j(v) \le \hat{P}_j(v)$ and $\hat{Q} - \tilde{Q} \ge 0$ then

$$\hat{P}_j(v+1) - \tilde{P}_j(v+1) \ge 0.$$

The inverses in (a) exist, since B_j has full rank. Hence in this case $\hat{P}_j(v+1) - \tilde{P}_j(v+1)$ is the sum of nonnegative-definite matrices if $\hat{P}_j(v) - \tilde{P}_j(v) \ge 0$ and $\hat{Q} - \tilde{Q} \ge 0$. The assertion in (b) is obtained from (a) by a continuity argument (see Bitmead *et al.*, 1985).

The next lemma shows that the sequences $(K_j(v))$ and $(P_j(v))$ defined in (5) are bounded from below for $K_{j0} \ge P_j^+$.

Lemma 2. Let $K_{j0} \ge P_j^+$. Then $K_j(v) \ge P_j(v) \ge P_j^+ \ge 0$ for $1 \le j \le M$, $v \in \mathbf{N}_0$.

Proof. From

$$P_j^+ \le K_j(0) = P_j(0) \le K_j(0) + \sum_{i\neq j} \pi_{ji}K_i(0),$$

and

$$P_j^+ = \varphi_i(P_j^+, Q_j),$$

we infer, using Lemma 1, that $K_j(1) \ge P_j(1) \ge P_j^+$, $1 \le j \le M$. A simple induction argument completes the proof.

The following hypothesis ensures that the sequences $(K_j(v))$ and $(P_j(v))$ are decreasing (see also Bitmead *et al.*, 1985, Lemma 2).
(H.3) There exists matrices $K_{j0} \ge 0$, $1 \le j \le M$ such that $K_j(1) \le K_{j0}$, $1 \le j \le M$.

Theorem 1. Under Hypothesis (H.3), for $1 \le j \le M$ the limits

$$K_j^\infty := \lim_{v\to\infty} K_j(v), \quad P_j^\infty := \lim_{v\to\infty} P_j(v)$$

exist; furthermore, the following monotonicity properties hold:

$$0 \le P_j^\infty \le P_j(v+1) \le P_j(v) \le K_j(v), \; v \in \mathbf{N}_0, \tag{7}$$

$$P_j^\infty \le K_j^\infty \le K_j(v+1) \le K_j(v), \; v \in \mathbf{N}_0. \tag{8}$$

Proof. The monotonicity of the sequences $(K_j(v))$ and $(P_j(v))$ is obtained from (H.3) and Lemma 1 by induction. Since $P_j(0) \ge 0$ implies $P_j(v) \ge 0$ for $v \in \mathbf{N}_0$ (see Bitmead *et al.*, 1985, Section 2), the limit $P_j^\infty \ge 0$ exists. The estimates $P_j(v) \le K_j(v)$, $1 \le j \le M$, $v \in \mathbf{N}_0$, are also obtained by induction from $P_j(0) = K_j(0) \ge 0$ and Lemma 1. Hence K_j^∞ exists with $K_j^\infty \ge P_j^\infty$.

Note that $P_j^\infty = P_j^+$, $1 \le j \le M$, if there are no $(\sqrt{Q_j}, A_j)$ unobservable modes outside the unit circle, since in this case the decoupled algebraic Riccati equations (4) have unique positive-semidefinite solutions.

Corollary 1. The coupled system (3) has a set of positive-semidefinite solutions K_j^∞, $1 \le j \le M$, if and only if (H.3) is satisfied.

Proof. It is obvious that (H.3) is necessary for the existence of solutions K_j^∞, $1 \le j \le M$, of (3). The sufficiency of (H.3) follows from Theorem 1.

The next lemma generalizes a well-known fact for standard difference equations (see e.g. Bitmead and Gevers, 1991).

Lemma 3. The sequences $(K_j^0(v))_{v\in\mathbf{N}_0}$ defined by

$$K_j^0(0) = 0, \quad K_j^0(v+1) = \varphi_j\Big(K_j^0(v) + \sum_{i\neq j} \pi_{ji}K_i^0(v), Q_j\Big)$$

for $v \in \mathbf{N}_0$ are nondecreasing. These sequences are bounded if and only if (H.3) is valid. In this case,

$$K_j^{0,\infty} := \lim_{v\to\infty} K_j^0(v)$$

exists and defines a solution of the coupled system (3).

Proof. Obviously $K_j(1) \ge 0$, $1 \le j \le M$; hence the monotonicity of the sequences $(K_j^0(v))$ is obtained by induction from Lemma 1; moreover it follows from Corollary 1 and Lemma 1 that these sequences are bounded and convergent if and only if (H.3) is satisfied.

Remarks.
(i) Hypothesis (H.3) is important from a theoretical point of view, since it ensures the existence of positive-semidefinite solutions of (3). However, in practical applications, one can always start with the zero matrix, since (H.3) is automatically fulfilled if the sequences $(K_j^0(v))$ turn out to be bounded.
(ii) Let $K_j^{0,\infty}$ exist and $0 \le \tilde{K}_{j0} \le K_{j0}$, $1 \le j \le M$. If $K_j^{0,\infty} = K_j^\infty$, $1 \le j \le M$, where K_j^∞ is defined as in Theorem 1, then

$$\lim_{v\to\infty} \tilde{K}_j(v) = K_j^{0,\infty}, \quad 1 \le j \le M,$$

for

$$\tilde{K}_j(v+1) = \varphi_j\Big(\tilde{K}_j(v) + \sum_{i \neq j} \pi_{ji}\tilde{K}_i(v), Q_j\Big),$$
$$1 \le j \ge M, \quad v \in \mathbf{N}_0, \quad \tilde{K}_j(0) = \tilde{K}_{j0}.$$

Note that (H.3) does not imply that (3) has a unique positive-semidefinite solution.

(iii) Let $K_j^\infty \ge 0$, $1 \le j \le M$, be a solution of (3) and let

$$F_j^\infty = \sum_{i=1}^{M} \pi_{ji}K_i^\infty = K_j^\infty + \sum_{i \neq j} \pi_{ji}K_i^\infty, \quad 1 \le j \le M.$$

Then

$$F_j^\infty = A_j^{\mathrm{T}}F_j^\infty A_j + Q_j^\infty - A_j^{\mathrm{T}}F_j^\infty B_j(I + B_j^{\mathrm{T}}F_j^\infty B_j)^{-1}B_j^{\mathrm{T}}F_j^\infty A_j, \quad 1 \le j \le M, \tag{9}$$

where $Q_j^\infty := Q_j + \sum_{i \neq j} \pi_{ji}K_i^\infty$, $1 \le j \le M$. Since (9) can be written as

$$F_j^\infty = (A_j + B_jH_j^\infty)^{\mathrm{T}}F_j^\infty(A_j + B_jH_j^\infty) + H_j^{\infty\mathrm{T}}H_j^\infty + Q_j^\infty,$$

with $H_j^\infty = -(I + B_j^{\mathrm{T}}F_j^\infty B_j)^{-1}B_j^{\mathrm{T}}F_j^\infty A_j$, a direct application of Theorem 10.14 of Bitmead and Gevers (1991) yields that the closed-loop matrices

$$A_j^\infty := A_j + B_jH_j^\infty, \quad 1 \le j \le M,$$

are stable if $(\sqrt{Q_j^\infty}, A_j)$ is detectable; obviously this implicit detectability condition is fulfilled if $(\sqrt{Q_j}, A_j)$ is detectable.

(iv) The stability of the matrices $A_j + B_jH_j^\infty$, $1 \le j \le M$, does not imply the stability of the closed-loop matrices

$$A_j^{\mathrm{cl}} := A(j) - B(j)\Big[I + B(j)^{\mathrm{T}}\Big(\sum_{i=1}^{M} p_{ji}K_i^\infty\Big)B(j)\Big]^{-1} \times B(j)^{\mathrm{T}}\Big(\sum_{i=1}^{M} p_{ji}K_i^\infty\Big)A(j)$$

of the original system, since $A_j + B_jH_j^\infty = \sqrt{p_{jj}}\,A_j^{\mathrm{cl}}$ and $A_j = \sqrt{p_{jj}}\,A(j)$. To ensure the stability of the matrices A_j^{cl}, the cost criteria (1) should be modified as follows (see Bourlès *et al.*, 1990):

$$J(x_0, r_0) = \lim_{N\to\infty} \mathbf{E}\Big[\sum_{k=0}^{N-1} \rho^{2k}[x_k^{\mathrm{T}}Q(r_k)x_k + u_k^{\mathrm{T}}R(r_k)u_k] + x_N^{\mathrm{T}}K_T(r_N)x_N\Big],$$

where $\rho^{-1} := \min\{\sqrt{p_{jj}} \mid 1 \le j \le M\}$. This criteria modification will shift all eigenvalues of the matrices A_j^{cl} to inside the unit disk. All results in this paper remain valid for this modified problem.

(v) The monotonicity results of Wimmer (1992) (see Section 1) can be generalized to coupled systems of the form (3); details are given by Freiling *et al.* (1994).

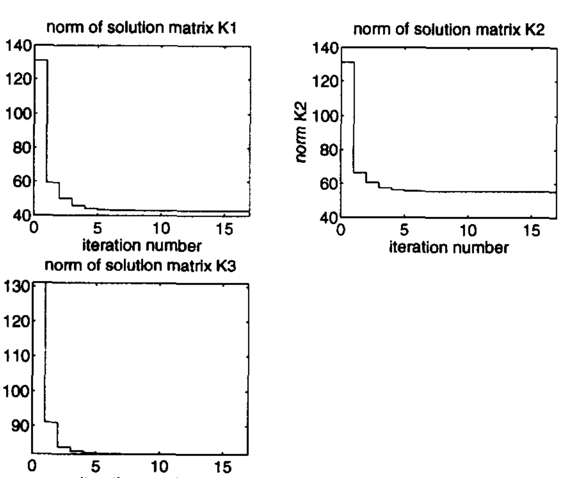

Fig. 1.

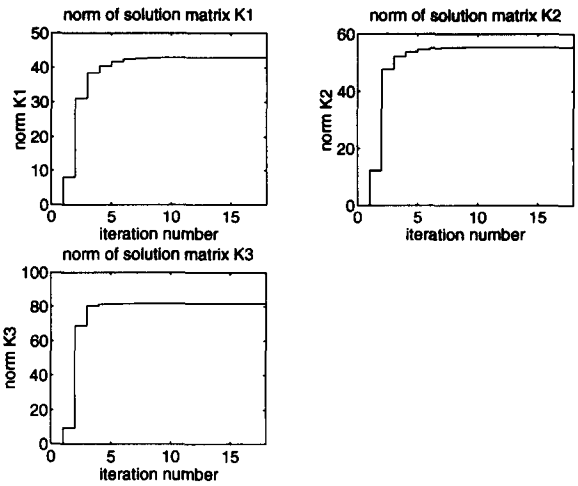

Fig. 2.

4. *Example*

We consider the following three-mode discrete-time jump-linear system (Blair and Sworder, 1975):

mode 1:

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2.5 & 3.2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$Q_1 = \begin{pmatrix} 3.6 & -3.8 \\ -3.8 & 4.87 \end{pmatrix}, \quad R_1 = 2.6;$$

mode 2:

$$A_2 = \begin{pmatrix} 0 & 1 \\ -4.3 & 4.5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$Q_2 = \begin{pmatrix} 10 & -3 \\ -3 & 8 \end{pmatrix}, \quad R_2 = 1.165;$$

mode 3:

$$A_3 = \begin{pmatrix} 0 & 1 \\ 5.3 & -5.2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$Q_3 = \begin{pmatrix} 5 & -4.5 \\ -4.5 & 4.5 \end{pmatrix}, \quad R_3 = 1.111.$$

The discrete state transition probability matrix is

$$\begin{pmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{pmatrix}.$$

The set of coupled Riccati-like equations is integrated as follows:

$$K_j(v+1) = [A_j + B_jH_j(v)]^{\mathrm{T}}F_j(v)[A_j + B_jH_j(v)] + H_j(v)^{\mathrm{T}}H_j(v) + Q_j =: \Phi_j\Big(K_j(v) + \sum_{i \neq j} \pi_{ji}K_i(v), Q_j\Big),$$

with

$$H_j(v) = -[I + B_j^{\mathrm{T}}F_j(v)B_j],$$
$$F_j(v) = K_j(v) + \sum_{i \neq j} \pi_{ji}K_i(v), \quad 1 \le j \le 3.$$

The iterative procedure is stopped if $\max \|E_j(v)\| \le \epsilon$, $1 \le j \le 3$, where $\|\cdot\|$ is the spectral norm and

$$E_j(v) = K_j(v+1) - \Phi_j\Big(K_j(v) + \sum_{i \neq j} \pi_{ji}K_i(v), Q_j\Big).$$

For $\epsilon = 10^{-6}$ and the initial values

$$K_j(0) = K_{j0} = \begin{pmatrix} 50 & -50 \\ -50 & 100 \end{pmatrix}, \quad j = 1, 2, 3,$$

Hypothesis (H.3) is satisfied, and, as expected, decreasing sequences are obtained as depicted in Fig. 1. For the initial values $K_j(0) = K_{j0} = 0$, the monotonic evolution of the norm of the solution matrices is shown in Fig. 2. In both cases, the algorithm converges to the following solution:

$$K_1^\infty = \begin{pmatrix} 18.6616 & -18.9560 \\ -18.9560 & 28.1086 \end{pmatrix},$$

$$K_2^\infty = \begin{pmatrix} 30.8818 & -21.6010 \\ -21.6010 & 36.2739 \end{pmatrix},$$

$$K_3^\infty = \begin{pmatrix} 35.4175 & -38.6192 \\ -38.6129 & 49.7079 \end{pmatrix}.$$

5. *Conclusions*

The solution of coupled discrete-time Riccati-like equations appearing in Markovian jump-linear control problems has been discussed. A necessary and sufficient condition for the existence of a set of positive semidefinite solutions to these equations has been obtained, and it has been shown that this condition ensures the existence of monotonic sequences converging to the solution of the Riccati-like equations. This leads to a straightforward numerical algorithm.

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