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$\mathscr{H}_\infty$ filtering for discrete-time linear systems with Markovian jumping parameters†

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SUMMARY

This paper investigates the problem of $\mathscr{H}_\infty$ filtering for discrete-time linear systems with Markovian jumping parameters. It is assumed that the jumping parameter is available. This paper develops necessary and sufficient conditions for designing a discrete-time Markovian jump linear filter which ensures a prescribed bound on the ℓ_2-induced gain from the noise signals to the estimation error. The proposed filter design is given in terms of linear matrix inequalities. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: $\mathscr{H}_\infty$ filtering; Markovian jump linear systems; discrete-time systems; state estimation; linear matrix inequalities

1. INTRODUCTION

The $\mathscr{H}_\infty$ approach has crop out as a very powerful tool within the context of filtering theory in recent years. This is particularly true if we are dealing with the very common situation in which we do not know *precisely* the *statistics* of the additive noise actuating in the system. The $\mathscr{H}_\infty$ approach hinges on the use of a nowadays very popular measure of performance, the $\mathscr{H}_\infty$-norm, or equivalently, the ℓ_2-induced gain (in the discrete-time case). In this context, the filtering problem is known in the literature as the $\mathscr{H}_\infty$ *filtering problem*, and has attracted considerable attention (see, e.g. References [1–7] and the references therein). Roughly speaking, in the $\mathscr{H}_\infty$ filtering approach the noise sources one considers are arbitrary signals with bounded energy, or bounded average power, and the estimator is designed to guarantee that the ℓ_2-induced gain, from the noise signals to the estimation error, be less than a prescribed bound. The potential of $\mathscr{H}_\infty$ filtering lies far beyond its insensitivity to the noise statistics. It has been recognized in

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Reference [4] that the $\mathscr{H}_\infty$ filtering scheme is also less sensitive than its $\mathscr{H}_2$ counterpart to uncertainty in the system parameters.

The main purpose of this paper is to study the problem of $\mathscr{H}_\infty$ filtering for a class of linear discrete-time systems whose structures are subject to abrupt parameters changes (jumps), modelled here via a discrete-time finite-state Markov chain (it is also known in the literature as the class of Markovian jump linear systems). These changes may be a consequence of random component failures or repairs, abrupt environmental disturbances, changes in the operating point of a nonlinear plant, etc. This can be found, for instance, in control of solar thermal central receivers, robotic manipulator systems, aircraft control systems, large flexible structures for space stations (such as antenna, solar arrays), etc. Several authors have analysed different aspects of such a class of systems and some successful applications have, in part, spurred a considerable interest on it (see, e.g. References [8–37] and the references therein). In particular with regard to the filtering problem, minimum variance filtering schemes have been studied in, for instance, References [9,10,13,17,18,22,26,35,37], whereas $\mathscr{H}_\infty$ filtering for the continuous-time case has been addressed in [20,21]. To the best of the authors' knowledge, to date the problem of $\mathscr{H}_\infty$ filtering for discrete-time Markovian jump linear systems has not yet been fully resolved.

The problem considered in this paper is the design of a Markovian jump linear filter for discrete-time Markovian jump linear systems which provides a mean square stable error dynamics and a prescribed bound on the ℓ_2-induced gain from the noise signals to the estimation error. Necessary and sufficient conditions in terms of linear matrix inequalities (LMIs) are proposed for solving this $\mathscr{H}_\infty$ filtering problem.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\Re^n$ denotes the n-dimensional Euclidean space, $\Re^{n\times m}$ is the set of $n \times m$ real matrices, I_n is the $n \times n$ identity matrix, and ℓ_2 stands for the space of squared summable vector sequences over the non-negative integers. The notation $P > 0$ (respectively, $P < 0$) for $P \in \Re^{n\times n}$, means that P is symmetric and positive definite (respectively, negative definite).

2. PROBLEM FORMULATION

Fix an underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and consider the stochastic system (Σ):

$$(\Sigma) \quad x(k+1) = A(\theta_k)x(k) + B(\theta_k)w(k), \; x(0) = x_0 \tag{1}$$

$$y(k) = C(\theta_k)x(k) + D(\theta_k)w(k) \tag{2}$$

$$z(k) = L(\theta_k)x(k) \tag{3}$$

where $x(k) \in \Re^n$ is the state, $x_0 \in \Re^n$ is an unknown initial state, $w(k) \in \Re^q$ is the noise signal which is assumed to be an arbitrary signal in ℓ_2, $y(k) \in \Re^m$ is the measurement, and $z(k) \in \Re^p$ is the signal to be estimated. $\{\theta_k\}$ is a discrete-time homogeneous Markov chain with finite state-space $\mathscr{S} = \{1, \ldots, N\}$ and stationary transition probability matrix $\Lambda = [p_{ij}]$, where

$$p_{ij} := \mathbb{P}\{\theta_{n+1} = j \mid \theta_0, \ldots, \theta_n = i\} = \mathbb{P}\{\theta_{n+1} = j \mid \theta_n = i\}$$

The set $\mathscr{S}$ comprises the various operation modes of system (Σ) and for each possible value of $\theta_k = i, \; i \in \mathscr{S}$, we will denote the matrices associated with the 'ith mode' by

$$A_i := A(\theta_k), \quad B_i := B(\theta_k), \quad C_i := C(\theta_k), \quad D_i := D(\theta_k), \quad L_i := L(\theta_k) \quad \text{for } \theta_k = i$$

where A_i, B_i, C_i, D_i and L_i are known real constant matrices for any $i \in \mathcal{S}$. Finally, the jumping process $\{\theta_k\}$ is assumed to be accessible, i.e. the operation mode of system (Σ) is known for every $k \geqslant 0$.

The filtering problem to be addressed is to obtain an estimate, $\hat{z}(k)$, of $z(k)$ via a causal Markovian jump linear filter which provides a uniformly small estimation error, $z(k) - \hat{z}(k)$, for all $w \in \ell_2$ and $x_0 \in \Re^n$.

In order to put the $\mathcal{H}_\infty$ filtering problem for system (Σ) in a stochastic setting, we introduce the space $\ell_2[(\Omega, \mathcal{F}, \mathbb{P})]$ of $\mathcal{F}$-measurable sequences, $\{z(k) - \hat{z}(k)\}$, for which

$$\|z - \hat{z}\|_2 := \left\{ E\left[\sum_{k=0}^{\infty} [z(k) - \hat{z}(k)]^{\mathrm{T}} [z(k) - \hat{z}(k)] \right] \right\}^{1/2} < \infty$$

where $E[\cdot]$ stands for the mathematical expectation. For the sake of simplifying the notation, $\|\cdot\|_2$ will be used to denote the norm either in $\ell_2[(\Omega, \mathcal{F}, \mathbb{P})]$ or in ℓ_2, the later defined by

$$\|w\|_2 := \left[\sum_{k=0}^{\infty} w^{\mathrm{T}}(k) w(k) \right]^{1/2} \quad \text{for } w \in \ell_2$$

We also define

$$\|v\|_P = v^{\mathrm{T}} P v \quad \text{for } v \in \Re^n$$

where P is a given symmetric positive definite matrix.

Before formulating the $\mathcal{H}_\infty$ filtering problem, we recall the notions of *internal mean square stability* and *mean square detectability*.

Definition 2.1
System (1) is said to be internally mean square stable (IMSS), if the solution to the stochastic difference equation

$$x(k+1) = A(\theta_k) x(k)$$

is such that $E[\|x(k)\|^2] \to 0$, as $k \to \infty$ for any finite initial condition $x_0 \in \Re^n$ and $\theta_0 \in \mathcal{S}$.

Definition 2.2
$(C(\theta_k), A(\theta_k), \Lambda)$ is said to be mean square detectable if there exist matrices $M_i \in \Re^{n \times m}$, $i = 1, \ldots, N$, such that with $M(\theta_k) = M_i$ for $\theta(k) = i,\ i \in \mathcal{S}$, the system

$$x(k+1) = [A(\theta) + M(\theta_k) C(\theta)] x(k)$$

is internally mean square stable.

Remark 2.1
It is noteworthy here that *mean square detectability*, as defined above, is not equivalent to the detectability of each mode $(C_i, A_i),\ i \in \mathcal{S}$. In fact, it can be shown that an operation mode which is a 'rare event' (is not a persistent mode) may not be detectable and yet the system be mean square detectable.

This paper is concerned with the following $\mathcal{H}_\infty$ filtering problem for system (Σ):

Given an *a-priori* estimate, $\hat{x}_0$, of the initial state, x_0, design a Markovian jump linear filter that provides an estimate, $\hat{z}(k)$, of $z(k)$ based on $\{y(j), 0 \leqslant j \leqslant k-1\}$ and $\{\theta(j),\ 0 \leqslant j \leqslant k-1\}$

such that the estimation error system is internally mean square stable and

$$\mathscr{J}(\hat{x}_0, R) := \sup_{x_0 \in \Re^n, w \in \ell_2} \left\{ \left[\frac{\|z - \hat{z}\|_2^2}{\|w\|_2^2 + \|x_0 - \hat{x}_0\|_R^2} \right]^{1/2} : \|w\|_2^2 + \|x_0 - \hat{x}_0\|_R^2 \neq 0 \right\} < \gamma \tag{4}$$

where $R > 0$ is a given weighting matrix for the initial state estimation error and $\gamma > 0$ is a given scalar which specifies the level of 'noise' attenuation in the estimation error.

The weighting matrix R is a measure of the confidence in the estimate $\hat{x}_0$ relative to the uncertainty in w. A 'large' value of R indicates that $\hat{x}_0$ is very close to x_0.

In filtering problems where the effect of the initial state is ignored, without loss of generality, x_0 and $\hat{x}_0$ can be set to zero and (4) is replaced by

$$\mathscr{J}_0 := \sup_{w \in \ell_2} \left\{ \frac{\|z - \hat{z}\|_2}{\|w\|_2} : x_0 = 0, \;\; \hat{x}_0 = 0, \;\; \|w\|_2 \neq 0 \right\} < \gamma \tag{5}$$

Remark 2.2

Observe that no 'non-singularity' assumption, namely

$$D^{\mathrm{T}}(\theta_k) D(\theta_k) > 0 \quad \forall \theta_k \in \mathscr{S} \text{ and } \forall k \geqslant 0$$

is imposed to the filtering problem treated in this paper. This is in contrast with the $\mathscr{H}_\infty$ filtering approaches in References [6,7] for discrete-time linear systems without jumps.

We conclude this section by stating some stability results required in the next section. We recall that (Theorem 2 in Reference [16] or Theorem 2.2.1 in Reference [31]):

Lemma 2.1

The system (1) is IMSS if and only if there exist matrices $X_i > 0, \; i = 1, \ldots, N$, satisfying the following inequalities:

$$A_i^{\mathrm{T}} \left(\sum_{j=1}^{N} p_{ij} X_j \right) A_i - X_i < 0, \quad i = 1, \ldots, N$$

Finally,

Lemma 2.2

System (1) is IMSS if and only if $x(k) \in \ell_2[(\Omega, \mathscr{F}, \mathbb{P})]$ for every $w(k) \in \ell_2$, i.e.,

$$\sum_{k=0}^{\infty} E[\|x(k)\|^2] = c < \infty \quad \text{for any } w \in \ell_2 \tag{6}$$

where c is a positive number.

Proof

See the appendix.

3. MAIN RESULTS

Attention will be focused on the design of an nth order filter. Since the matrices of system (Σ) are known at time k (as θ_k is available) and it is required that in the absence of w,

$$E[\|x(k)-\hat{x}(k)\|^2] \to 0 \quad \text{and} \quad E[\|z(k)-\hat{z}(k)\|^2] \to 0 \quad \text{as } k \to \infty$$

where $\hat{x}(k)$ is the state of the filter, irrespective of the internal mean square stability of (1), without loss of generality, the following structure for the Markovian jump linear filter will be adopted:

$$(\Sigma_f) \quad \hat{x}(k+1) = A(\theta_k)\hat{x}(k) + K(\theta_k)[y(k) - C(\theta_k)\hat{x}(k)], \quad \hat{x}(0) = \hat{x}_0 \tag{7}$$

$$\hat{z}(k) = L(\theta_k)\hat{x}(k) \tag{8}$$

where the filter gain matrix $K(\theta_k)$ is to be determined.

We shall make the following assumption for system (Σ).

Assumption 3.1
$(C(\theta_k), A(\theta_k), \Lambda)$ is mean square detectable.

Before presenting our filter design, we first establish an LMI based $\mathcal{H}_\infty$ performance result for discrete-time Markovian jump linear systems. To this end, consider the Markovian jump linear system as follows:

$$(\Sigma_1) \quad x(k+1) = A(\theta_k)x(k) + B(\theta_k)w(k), \quad x(0) = x_0 \tag{9}$$

$$\xi(k) = C(\theta_k)x(k) \tag{10}$$

where $x(k) \in \Re^n$, $w(k) \in \Re^q$, $\xi(k) \in \Re^m$, and consider the $\mathcal{H}_\infty$-type performance index

$$\mathcal{J}(\Sigma_1, R) := \sup_{x_0 \in \Re^n, w \in \ell_2} \left\{ \left[\frac{\|\xi\|_2^2}{\|w\|_2^2 + \|x_0\|_R^2} \right]^{1/2} : \|w\|_2^2 + \|x_0\|_R^2 \neq 0 \right\} \tag{11}$$

where $R > 0$ is a given weighting matrix for the initial state of system (Σ_1).

Lemma 3.1
Consider system (Σ_1) and let $\gamma > 0$ be a given scalar and $R > 0$ a given initial state weighting matrix. Then the following conditions are equivalent:

(a) The system (Σ_1) is IMSS and $\mathcal{J}(\Sigma_1, R) < \gamma$.

(b) There exist matrices $Y_i > 0$, $i = 1, \ldots, N$, satisfying the LMIs

$$\begin{bmatrix} A_i^{\mathrm{T}} \\ B_i^{\mathrm{T}} \end{bmatrix} \left(\sum_{j=1}^{N} p_{ij} Y_j \right) [A_i \quad B_i] + \begin{bmatrix} C_i^{\mathrm{T}} C_i & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} Y_i & 0 \\ 0 & \gamma^2 I_q \end{bmatrix} < 0, \quad i = 1, \ldots, N \tag{12}$$

$$Y_i - \gamma^2 R \leqslant 0, \quad i = 1, \ldots, N \tag{13}$$

Proof

(b) $\Rightarrow$ (a) First note that by using Schur's complements, (12) is equivalent to the following coupled Riccati inequalities:

$$A_i^{\mathrm{T}} \bar{Y}_i A_i - Y_i + A_i^{\mathrm{T}} \bar{Y}_i B_i (\gamma^2 I - B_i^{\mathrm{T}} \bar{Y}_i B_i)^{-1} B_i^{\mathrm{T}} \bar{Y}_i A_i + C_i^{\mathrm{T}} C_i < 0, \quad i = 1, \ldots, N \tag{14}$$

$$\gamma^2 I - B_i^{\mathrm{T}} \bar{Y}_i B_i > 0, \quad i = 1, \ldots, N \tag{15}$$

where

$$\bar{Y}_i = \sum_{j=1}^{N} p_{ij} Y_j, \quad i = 1, \ldots, N$$

By Lemma 2.1, it follows from (14) that system (9) is internally mean square stable.

In order to establish the $\mathscr{H}_\infty$ performance of (11), introduce the cost functional

$$J(k) := E\left\{ \sum_{j=0}^{k-1} [\xi^{\mathrm{T}}(j)\xi(j) - \gamma^2 w^{\mathrm{T}}(j) w(j)] \right\} - \gamma^2 x_0^{\mathrm{T}} R x_0 \tag{16}$$

Next, define

$$\Omega(k) := x^{\mathrm{T}}(k+1) Y(\theta_{k+1}) x(k+1) - x^{\mathrm{T}}(k) Y(\theta_k) x(k)$$

where $Y(\theta_k) = Y_i$ when $\theta_k = i$, $i \in \mathscr{S}$, and with Y_i satisfying (12) and (13), or equivalently, (13)–(15). Also, let $\bar{Y}(\theta_k)$ be such that $\bar{Y}(\theta_k) = \bar{Y}_i$ when $\theta_k = i$, $i \in \mathscr{S}$.

By considering (9) and (10), it can be easily established that $\Omega(k)$ calculated along the trajectory of system (Σ_1) implies that

$$E[\Omega(k) \mid x(k), \theta_k] = x^{\mathrm{T}}(k) Q(\theta_k) x(k) + \gamma^2 w^{\mathrm{T}}(k) w(k) - h^{\mathrm{T}}(k) V(\theta_k) h(k) \tag{17}$$

where

$$\begin{aligned} Q(\theta_k) = {} & A^{\mathrm{T}}(\theta_k) \bar{Y}(\theta_k) A(\theta_k) - Y(\theta_k) \\ & + A^{\mathrm{T}}(\theta_k) \bar{Y}(\theta_k) B(\theta_k) V^{-1}(\theta_k) B^{\mathrm{T}}(\theta_k) \bar{Y}(\theta_k) A(\theta_k) \end{aligned} \tag{18}$$

$$V(\theta_k) = \gamma^2 I - B^{\mathrm{T}}(\theta_k) \bar{Y}(\theta_k) B(\theta_k) \tag{19}$$

$$h(k) = w(k) - V^{-1}(\theta_k) B^{\mathrm{T}}(\theta_k) \bar{Y}(\theta_k) A(\theta_k) x(k) \tag{20}$$

In view of the definition of $\Omega(k)$, we can rewrite (16) as

$$\begin{aligned} J(k) = {} & E\left\{ \sum_{j=0}^{k-1} [\xi^{\mathrm{T}}(j)\xi(j) - \gamma^2 w^{\mathrm{T}}(j) w(j)] \right\} + E\left\{ \sum_{j=0}^{k-1} E[\Omega(j) \mid x(j), \theta_j] \right\} \\ & - E[x^{\mathrm{T}}(k) Y(\theta_k) x(k)] + x_0^{\mathrm{T}} [Y(\theta_0) - \gamma^2 R] x_0 \\ = {} & E\left\{ \sum_{j=0}^{k-1} [x^{\mathrm{T}}(j)(Q(\theta_j) + C^{\mathrm{T}}(\theta_j) C(\theta_j)) x(j) - h^{\mathrm{T}}(j) V(\theta_j) h(j)] \right\} \\ & - E[x^{\mathrm{T}}(k) Y(\theta_k) x(k)] + x_0^{\mathrm{T}} [Y(\theta_0) - \gamma^2 R] x_0 \end{aligned} \tag{21}$$

where the last equality follows by taking into account (10) and (17).

Note that as system (9) is IMSS and $w \in \ell_2$, it follows from Lemma 2.2 that

$$\lim_{k\to\infty} E[x^{\mathrm{T}}(k)Y(\theta_k)x(k)] = 0$$

as well as that $J(k)$ remains bounded as $k \to \infty$. Hence, it results from (21) that

$$\begin{aligned} &\|\xi\|_2^2 - \gamma^2[\|w\|_2^2 + x_0^{\mathrm{T}} R x_0] \\ &\quad = E\left\{\sum_{j=0}^{\infty} [x^{\mathrm{T}}(j)(Q(\theta_j) + C^{\mathrm{T}}(\theta_j)C(\theta_j))x(j) - h^{\mathrm{T}}(j)V(\theta_j)h(j)]\right\} \\ &\quad + x_0^{\mathrm{T}}[Y(\theta_0) - \gamma^2 R]x_0 \end{aligned} \tag{22}$$

Finally, considering (13)–(15), (18) and (19), the result follows from (22).

(a) $\Rightarrow$ (b) Suppose that system (Σ_1) given by (9) and (10) is IMSS and $\mathscr{J}(\Sigma_1, R) < \gamma$. Consider the Markovian jump system obtained from (9) and (10) by adding the output $\varepsilon x(t)$, where ε is a positive number, i.e. let the system

$$(\Sigma_{1a}) \quad x(k+1) = A(\theta_k)x(k) + B(\theta_k)w(k), \quad x(0) = x_0 \tag{23}$$

$$\xi_a(k) = \begin{bmatrix} C(\theta_k) \\ \varepsilon I \end{bmatrix} x(k) \tag{24}$$

Noticing that system (Σ_{1a}) is internally mean square stable and considering that $\mathscr{J}(\Sigma_1, R) < \gamma$ and $\|\xi_a\|_2^2 = \|\xi\|_2^2 + \varepsilon^2\|x\|_2^2$, where ξ is the output of system (Σ_1), it follows that there exists a sufficiently small number $\varepsilon > 0$ such that

$$\mathscr{J}(\Sigma_{1a}, R) := \sup_{x_0 \in \Re^n, w \in \ell_2} \left\{ \left[\frac{\|\xi_a\|_2^2}{\|w\|_2^2 + \|x_0\|_R^2}\right]^{1/2} : \|w\|_2^2 + \|x_0\|_R^2 \neq 0 \right\} < \gamma$$

This implies that for system (Σ_{1a}),

$$\sup_{w\in\ell_2} \left\{ \frac{\|\xi_a\|_2}{\|w\|_2} : x(0) = 0, \|w\|_2 \neq 0 \right\} < \gamma$$

Hence, it follows from the main Theorem in Reference [14] that there exist matrices $X_i > 0$ $\forall i \in \mathscr{S}$, satisfying the following coupled algebraic Riccati equations:

$$A_i^{\mathrm{T}}\bar{X}_iA_i - X_i + A_i^{\mathrm{T}}\bar{X}_iB_i(\gamma^2 I - B_i^{\mathrm{T}}\bar{X}_iB_i)^{-1}B_i^{\mathrm{T}}\bar{X}_iA_i + C_i^{\mathrm{T}}C_i + \varepsilon^2 I = 0, \quad i = 1, \ldots, N \tag{25}$$

and such that

$$\gamma^2 I - B_i^{\mathrm{T}}\bar{X}_iB_i > 0, \quad i = 1, \ldots, N \tag{26}$$

where

$$\bar{X}_i = \sum_{j=1}^{N} p_{ij}X_j, \quad i = 1, \ldots, N$$

Moreover, the Markovian jump linear system

$$x(k+1) = [A(\theta_k) + B(\theta_k)U^{-1}(\theta_k)B^{\mathrm{T}}(\theta_k)\bar{X}(\theta_k)A(\theta_k)]x(k) \tag{27}$$

is IMSS where $\bar{X}(\theta_k)$ is such that $\bar{X}(\theta_k) = \bar{X}_i$ when $\theta_k = i,\ i \in \mathcal{S}$, and

$$U(\theta_k) := \gamma^2 I - B^{\mathrm{T}}(\theta_k)\bar{X}(\theta_k)B(\theta_k) \tag{28}$$

Also, let $X(\theta_k)$ be such that $X(\theta_k) = X_i$ when $\theta_k = i,\ i \in \mathcal{S}$. Note that $X(\theta_k)$ satisfies

$$\begin{aligned} &A^{\mathrm{T}}(\theta_k)\bar{X}(\theta_k)A(\theta_k) - X(\theta_k) + A^{\mathrm{T}}(\theta_k)\bar{X}(\theta_k)B(\theta_k)U^{-1}(\theta_k)B^{\mathrm{T}}(\theta_k)\bar{X}(\theta_k)A(\theta_k) \\ &\quad + C^{\mathrm{T}}(\theta_k)C(\theta_k) + \varepsilon^2 I = 0 \end{aligned} \tag{29}$$

Next, it follows from (25) that

$$A_i^{\mathrm{T}}\bar{X}_iA_i - X_i + A_i^{\mathrm{T}}\bar{X}_iB_i(\gamma^2 I - B_i^{\mathrm{T}}\bar{X}_iB_i)^{-1}B_i^{\mathrm{T}}\bar{X}_iA_i + C_i^{\mathrm{T}}C_i < 0, \quad i = 1, \dots, N$$

which together with (26) are equivalent to (12).

It remains to be shown that $X_i - \gamma^2 R \leqslant 0,\ i = 1, \dots, N$. To this end, suppose that this is not true, namely, there exist $i_0 \in \mathcal{S}$ and a non-zero $\eta \in \Re^n$ such that $\eta^{\mathrm{T}}(X_{i_0} - \gamma^2 R)\eta > 0$. Further, define

$$\hat{\Omega}(k) := x^{\mathrm{T}}(k+1)X(\theta_{k+1})x(k+1) - x^{\mathrm{T}}(k)X(\theta_k)x(k)$$

where $X(\theta_k)$ satisfies (29). Similarly to the proof of (b) $\Rightarrow$ (a), it can be shown that $\hat{\Omega}(k)$ calculated along the trajectory of system (Σ_{1a}) implies that

$$\begin{aligned} &E[\hat{\Omega}(k) \mid x(k), \theta_k] \\ &\quad = x^{\mathrm{T}}(k)\hat{Q}(\theta_k)x(k) + \gamma^2 w^{\mathrm{T}}(k)w(k) - [w(k) - \hat{w}(k)]^{\mathrm{T}}U(\theta_k)[w(k) - \hat{w}(k)] \\ &\quad = \gamma^2 w^{\mathrm{T}}(k)w(k) - \xi_a^{\mathrm{T}}(k)\xi_a(k) - [w(k) - \hat{w}(k)]^{\mathrm{T}}U(\theta_k)[w(k) - \hat{w}(k)] \end{aligned} \tag{30}$$

where $U(\theta_k)$ is as in (28) and

$$\begin{aligned} \hat{Q}(\theta_k) = &A^{\mathrm{T}}(\theta_k)\bar{X}(\theta_k)A(\theta_k) - X(\theta_k) \\ &+ A^{\mathrm{T}}(\theta_k)\bar{X}(\theta_k)B(\theta_k)U^{-1}(\theta_k)B^{\mathrm{T}}(\theta_k)\bar{X}(\theta_k)A(\theta_k) \end{aligned}$$

$$\hat{w}(k) = U^{-1}(\theta_k)B^{\mathrm{T}}(\theta_k)\bar{X}(\theta_k)A(\theta_k)x(k)$$

and where the second equality of (30) has been obtained by considering (29) and the definition of $\xi_a(k)$ given by (24).

Now choose $x_0 = \eta$, $\theta_0 = i_0$ and $w(k) = \hat{w}(k),\ \forall k \geqslant 0$, for system (Σ_{1a}). Under these conditions, and since system (27) is IMSS, it follows that for system (Σ_{1a}):

$$\lim_{k \to \infty} E[x^{\mathrm{T}}(k)X(\theta_k)x(k)] = 0$$

Moreover, summing (30) from 0 to ∞ and taking expectation, one obtains that

$$-\eta^{\mathrm{T}}X_{i_0}\eta = \gamma^2\|w\|_2^2 - \|\xi_a\|_2^2$$

or yet

$$\gamma^2[\|w\|_2^2 + \eta^{\mathrm{T}}R\eta] - \|\xi_a\|_2^2 = \eta^{\mathrm{T}}(\gamma^2 R - X_{i_0})\eta < 0$$

which is a contradiction as $\mathcal{J}(\Sigma_{1a}, R) < \gamma$.

We now present a solution to the $\mathscr{H}_\infty$ filtering problem for the Markovian jumping linear system (Σ). Associated with system (Σ), define the following matrices:

$$F_i := \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad G_i := \begin{bmatrix} L_i^{\mathrm{T}} L_i & 0 \\ 0 & 0 \end{bmatrix}, \quad J := \begin{bmatrix} I_n \\ 0 \end{bmatrix}^{\mathrm{T}}, \quad H := \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix} \tag{31}$$

Then, we have the following result:

Theorem 3.1

Consider system (Σ) satisfying Assumption 3.1 and let $\gamma > 0$ be a given scalar. Let $\hat{x}_0$ be an *a-priori* estimate of the initial state and $R > 0$ a given initial state error weighting matrix. Then there exists a Markovian jump filter of the form of (7) and (8) such that the estimation error system is IMSS and $\mathscr{J}(\hat{x}_0, R) < \gamma$ if and only if for all $i \in \mathscr{S}$ there exist positive definite matrices $X_{1i} \in \Re^{n\times n}$ and matrices $X_{2i} \in \Re^{n\times m}$ and $X_{3i} \in \Re^{m\times m}$ satisfying the following LMIs:

$$\begin{bmatrix} \bar{X}_{1i} & X_{2i} \\ X_{2i}^{\mathrm{T}} & X_{3i} \end{bmatrix} \geqslant 0, \quad i = 1, \ldots, N \tag{32}$$

$$F_i^{\mathrm{T}} X_i F_i - J^{\mathrm{T}} X_{1i} J + G_i - \gamma^2 H < 0, \quad i = 1, \ldots, N \tag{33}$$

$$X_{1i} - \gamma^2 R \leqslant 0, \quad i = 1, \ldots, N \tag{34}$$

where

$$\bar{X}_{1i} = \sum_{j=1}^{N} p_{ij} X_{1i} \quad \text{and} \quad X_i = \begin{bmatrix} \bar{X}_{1i} & X_{2i} \\ X_{2i}^{\mathrm{T}} & X_{3i} \end{bmatrix}, \quad i = 1, \ldots, N \tag{35}$$

Moreover, a suitable filter is given by

$$\hat{x}(k+1) = A_i \hat{x}(k) + K_i[y(k) - C_i \hat{x}(k)], \quad \hat{x}(0) = \hat{x}_0 \tag{36}$$

$$\hat{z}(k) = L_i \hat{x}(k) \tag{37}$$

when $\theta_k = i, \; i \in \mathscr{S}$, where

$$K_i = K(\theta_k = i) = -\bar{X}_{1i}^{-1} X_{2i} \tag{38}$$

Proof

Sufficiency: First note that by defining $\tilde{x} := x - \hat{x}$ and considering (1)–(3), (7) and (8), it follows that a state-space model for the estimation error, $z - \hat{z}$, is given by

$$\tilde{x}(k+1) = \tilde{A}(\theta_k)\tilde{x}(k) + \tilde{B}(\theta_k) w(k), \quad \tilde{x}(0) = x_0 - \hat{x}_0 \tag{39}$$

$$z(k) - \hat{z}(k) = L(\theta_k)\tilde{x}(k) \tag{40}$$

where

$$\tilde{A}(\theta_k) = A(\theta_k) - K(\theta_k) C(\theta_k), \quad \tilde{B}(\theta_k) = B(\theta_k) - K(\theta_k) D(\theta_k)$$

In the sequel, for each possible value of $\theta_k = i, \; i \in \mathscr{S}$, we denote

$$\tilde{A}_i := \tilde{A}(\theta_k = i) = A_i - K_i C_i, \quad \tilde{B}_i := \tilde{B}(\theta_k = i) = B_i - K_i D_i \tag{41}$$

With the definitions in (35) it can be easily obtained that

$$F_i^{\mathrm{T}} X_i F_i = \begin{bmatrix} (A_i + \bar{X}_{1i}^{-1} X_{2i} C_i)^{\mathrm{T}} \\ (B_i + \bar{X}_{1i}^{-1} X_{2i} D_i)^{\mathrm{T}} \end{bmatrix} \bar{X}_{1i} [\, A_i + \bar{X}_{1i}^{-1} X_{2i} C_i \quad B_i + \bar{X}_{1i}^{-1} X_{2i} D_i \,] + \begin{bmatrix} C_i^{\mathrm{T}} \\ D_i^{\mathrm{T}} \end{bmatrix} (X_{3i} - X_{2i}^{\mathrm{T}} \bar{X}_{1i}^{-1} X_{2i}) [\, C_i \quad D_i \,] \tag{42}$$

Taking into account (38), (41) and (42), it follows that (33) can be rewritten as

$$\begin{bmatrix} \tilde{A}_i^{\mathrm{T}} \\ \tilde{B}_i^{\mathrm{T}} \end{bmatrix} \bar{X}_{1i} [\, \tilde{A}_i \quad \tilde{B}_i \,] + \begin{bmatrix} C_i^{\mathrm{T}} \\ D_i^{\mathrm{T}} \end{bmatrix} (X_{3i} - X_{2i}^{\mathrm{T}} \bar{X}_{1i}^{-1} X_{2i}) [\, C_i \quad D_i \,] + \begin{bmatrix} L_i^{\mathrm{T}} L_i & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} X_{1i} & 0 \\ 0 & \gamma^2 I_q \end{bmatrix} < 0, \quad i = 1, \ldots, N \tag{43}$$

In view of (32) and since $X_{1i} > 0$ for $i = 1, \ldots, N$, then $\bar{X}_{1i} > 0$ and $X_{3i} - X_{2i}^{\mathrm{T}} \bar{X}_{1i}^{-1} X_{2i} \geqslant 0$ for $i = 1, \ldots, N$. Thus, (43) implies that

$$\begin{bmatrix} \tilde{A}_i^{\mathrm{T}} \\ \tilde{B}_i^{\mathrm{T}} \end{bmatrix} \left(\sum_{j=1}^{N} p_{ij} X_{1j} \right) [\, \tilde{A}_i \quad \tilde{B}_i \,] + \begin{bmatrix} L_i^{\mathrm{T}} L_i & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} X_{1i} & 0 \\ 0 & \gamma^2 I_q \end{bmatrix} < 0, \quad i = 1, \ldots, N \tag{44}$$

Finally, considering (44) and (34), the result follows from Lemma 3.1.

Necessity: Suppose that there exists a filter of the form of (7) and (8) such that the estimation error system given by (39) and (40) is IMSS and $\mathcal{J}(\hat{x}_0, R) < \gamma$. Hence, it follows from Lemma 3.1 that there exist matrices $X_{1i} > 0$, $i = 1, \ldots, N$, satisfying the following inequalities for $i = 1, \ldots, N$:

$$\begin{bmatrix} (A_i - K_i C_i)^{\mathrm{T}} \\ (B_i - K_i D_i)^{\mathrm{T}} \end{bmatrix} \bar{X}_{1i} [\, A_i - K_i C_i \quad B_i - K_i D_i \,] + \begin{bmatrix} L_i^{\mathrm{T}} L_i & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} X_{1i} & 0 \\ 0 & \gamma^2 I_q \end{bmatrix} < 0 \tag{45}$$

$$X_{1i} - \gamma^2 R \leqslant 0 \tag{46}$$

where

$$\bar{X}_{1i} = \sum_{j=1}^{N} p_{ij} X_{1j} \tag{47}$$

Define the matrices $X_{2i} := -\bar{X}_{1i} K_i$ and $X_{3i} := X_{2i}^{\mathrm{T}} \bar{X}_{1i}^{-1} X_{2i}$. Note that

$$\begin{bmatrix} \bar{X}_{1i} & X_{2i} \\ X_{2i}^{\mathrm{T}} & X_{3i} \end{bmatrix} \geqslant 0, \quad i = 1, \ldots, N \tag{48}$$

With the above definitions, (45) can be rewritten as

$$\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{X}_{1i} & X_{2i} \\ X_{2i}^{\mathrm{T}} & X_{3i} \end{bmatrix} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} - \begin{bmatrix} C_i^{\mathrm{T}} \\ D_i^{\mathrm{T}} \end{bmatrix} (X_{3i} - X_{2i}^{\mathrm{T}} \bar{X}_{1i}^{-1} X_{2i}) [\, C_i \quad D_i \,]$$
$$+ \begin{bmatrix} L_i^{\mathrm{T}} L_i & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} X_{1i} & 0 \\ 0 & \gamma^2 I_q \end{bmatrix} < 0, \quad i = 1, \ldots, N$$

which is equivalent to (33). Moreover, taking into account (46) and (48), one concludes that the matrices X_{1i}, X_{2i} and X_{3i} as above satisfy the LMIs (32)–(34).

Remark 3.1
Theorem 3.1 provides a method for designing $\mathscr{H}_\infty$ Markovian jump linear filters for linear discrete-time systems subject to Markovian jumping parameters. The proposed design is given in terms of linear matrix inequalities, which has the advantage that the filter gain can be determined numerically very efficiently using recently developed algorithms for solving LMIs.

Note that the problem of designing an optimal $\mathscr{H}_\infty$ filter, namely for the smallest possible $\gamma > 0$, can be easily solved via the following convex optimization problem:

$$\begin{aligned} &\text{minimize} \quad \kappa \\ &\text{subject to} \quad \kappa > 0, \quad X_{1i} > 0, \; i = 1, \ldots, N, \; \text{and} \quad (32){-}(34) \text{ with } \gamma^2 = \kappa \end{aligned}$$

Remark 3.2
It should be remarked that although Assumption 3.1 has not been explicitly used in the proof of Theorem 3.1, it is a necessary condition for the inequalities of (33) to hold. Indeed, this can be easily verified from the inequalities of (43) which are equivalent to (33).

When the effect of the initial state of system (Σ) is ignored, without loss of generality, x_0 and $\hat{x}_0$ can be set to zero. Thus, the inequalities of (34) will no longer be required as this case corresponds to choosing a sufficient large R (in the sense that its smallest eigenvalue approaches infinity). In such situation, Theorem 3.1 specializes as follows:

Corollary 3.1
Consider system (Σ) with $x(0) = 0$ and satisfying Assumption 3.1. Given a scalar $\gamma > 0$, there exists a Markovian jump filter of the form of (7) and (8) such that the estimation error dynamics is IMSS and $\mathscr{J}_0 < \gamma$ if and only if for all $i \in \mathscr{S}$ there exist positive definite matrices $X_{1i} \in \Re^{n \times n}$ and matrices $X_{2i} \in \Re^{n \times m}$ and $X_{3i} \in \Re^{m \times m}$ satisfying the LMIs (32) and (33). Moreover, a suitable filter is given by (36)–(38) with $\hat{x}_0 = 0$.

In the case of one mode operation, i.e. there are no jumps in system (Σ), we have $N = 1$, $\mathscr{S} = \{1\}$ and $p_{11} = 1$. Denoting the matrices of system (Σ) by $A,\ B,\ C,\ D$ and L, Theorem 3.1 reduces to the following result.

Corollary 3.2
Consider system (Σ) with no jumps and let $\gamma > 0$ be a given scalar. Let $\hat{x}_0$ be an a-priori estimate of the initial state and $R > 0$ a given initial state error weighting matrix. Then there exists an nth order strictly proper linear filter such that the estimation error dynamics is asymptotically stable and $\mathscr{J}(\hat{x}_0, R) < \gamma$ if and only if there exist a positive definite matrix $X_1 \in \Re^{n\times n}$ and matrices $X_2 \in \Re^{n\times m}$ and $X_3 \in \Re^{m\times m}$ satisfying the LMIs:

$$\begin{bmatrix} X_1 & X_2 \\ X_2^{\mathrm{T}} & X_3 \end{bmatrix} \geqslant 0 \tag{49}$$

$$F^{\mathrm{T}}XF - J^{\mathrm{T}}X_1J + G - \gamma^2 H < 0 \tag{50}$$

$$X_1 - \gamma^2 R \leqslant 0 \tag{51}$$

where the matrices J and H are as in (31) and

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^{\mathrm{T}} & X_3 \end{bmatrix}, \quad F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad G = \begin{bmatrix} L^{\mathrm{T}}L & 0 \\ 0 & 0 \end{bmatrix}$$

Moreover, a suitable filter is given by

$$\hat{x}(k+1) = A\hat{x}(k) + K[y(k) - C\hat{x}(k)], \quad \hat{x}(0) = \hat{x}_0$$

$$\hat{z}(k) = L\hat{x}(k)$$

where $K = -X_1^{-1}X_2$.

Remark 3.3
Corollary 3.2 provides LMI-based necessary and sufficient conditions for designing an $\mathscr{H}_\infty$ filter for linear discrete-time systems with non-zero initial conditions. This is in contrast with the approach developed in References [6,7] which are based on algebraic Riccati equations and are also restricted to 'non-singular' $\mathscr{H}_\infty$ filtering problems.

4. AN EXAMPLE

Consider the following Markovian jump linear system, adapted from the closed-loop system of an example in Reference [19]. The system is of form (Σ) with two operating modes described by

$$A_1 = \begin{bmatrix} 1 & 5.2529\times 10^{-2} \\ 1.5146\times 10^{-3} & 1.1022 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.9955 & 4.9660\times 10^{-2} \\ -0.2669 & 0.8075 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \quad C_1 = C_2 = [-1 \quad 1], \quad D_1 = D_2 = [0 \quad 1],$$

$$L_1 = L_2 = [0 \quad 1], \quad \Lambda = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$$

$\mathscr{H}_\infty$ filters have been designed for the above system using the methods developed in this paper and two cases have been treated. First, considering an initial state weighting matrix $R = 5I_2$, the minimum upper-bound γ on the $\mathscr{H}_\infty$ performance index $\mathscr{J}(\hat{x}_0, R)$ obtained from Theorem 3.1 was $\gamma = 1.4606$ and the corresponding filter gains are:

$$K_1 = [\,-0.8913 \quad 0.1125\,], \quad K_2 = [\,-1.48067 \quad 1.5954 \times 10^{-2}\,]$$

The second case deals with the design of an $\mathscr{H}_\infty$ filter without considering initial conditions. The minimum upper-bound γ on $\mathscr{J}_0$ obtained from Corollary 3.1 was $\gamma = 0.8770$ which, as expected, is smaller than in the previous case because the effect of the uncertainty in the initial state of the system is ignored. The filter gains for the latter case are:

$$K_1 = [\,-0.9202 \quad 8.2259 \times 10^{-2}\,], \quad K_2 = [\,-1.5511 \quad -2.0452 \times 10^{-2}\,]$$

To illustrate the filter performance in the time-domain, Monte Carlo simulations have been performed for the first case with the values of $\theta(k)$ generated randomly. We have performed 5000 simulations for the estimation error system, with the same x_0, $\hat{x}_0$ and $w(k)$ given by

$$x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \hat{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad w(k) = 0.9^k \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix}$$

where $w_1(k)$ and $w_2(k)$ are independent zero-mean white sequences with Gaussian distribution and unitary variance. Note that $w(k) \in \ell_2$.

Figure 1 displays the square root of the mean square (RMS) value of the estimation error at time $k = 0, \ldots, 60$, obtained from the 5000 realizations.

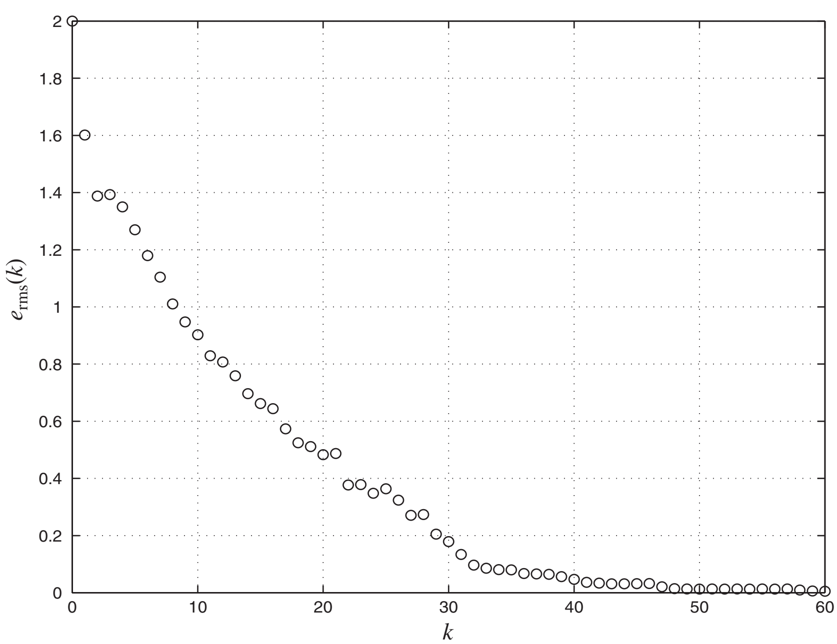

Figure 1. RMS value of the estimation error.

5. CONCLUSIONS

This paper has addressed the problem of $\mathscr{H}_\infty$ filtering for a class of discrete-time Markovian jump linear systems under the assumption that the jumping parameter is accessible. Necessary and sufficient conditions have been derived for designing discrete-time $\mathscr{H}_\infty$ Markovian jump linear filters. The filter design is given in terms of linear matrix inequalities.

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APPENDIX A: PROOF OF LEMMA 2.2

Throughout this appendix, we shall denote by $\mathbb{C}^n$ the n-dimensional complex Euclidean spaces and by $\mathbb{B}(\mathbb{C}^n, \mathbb{C}^m)$ the normed bounded linear space of all $m \times n$ complex matrices, with $\mathbb{B}(\mathbb{C}^n) := \mathbb{B}(\mathbb{C}^n, \mathbb{C}^n)$ and $\mathbb{B}(\mathbb{C}^n)^+ := \{L \in \mathbb{B}(\mathbb{C}^n) : L = L^* \geqslant 0\}$. The superscript * will denote complex conjugate transpose and $\mathrm{tr}(\cdot)$ the trace of a matrix. Either the uniform induced norm in $\mathbb{B}(\mathbb{C}^n)$ or the standard Euclidean norm in $\mathbb{C}^n$ is represented by $\|\cdot\|$.

Set $H^{n,m}$ the linear space made up of all N-sequences of complex matrices $V = (V_1, \ldots, V_N)$ with $V_i \in \mathbb{B}(\mathbb{C}^n, \mathbb{C}^m)$, $i = 1, \ldots, N$ and, for simplicity, set $\mathbb{H}^n := \mathbb{H}^{n,n}$. For $V = (V_1, \ldots, V_N) \in \mathbb{H}^{n,m}$, we consider the following norm in $\mathbb{H}^{n,m}$:

$$\|V\|_1 := \sum_{i=1}^{N} \|V_i\|$$

It is easy to verify that $\mathbb{H}^{n,m}$ equipped with the above norm is a Banach space.

Since we shall derive the results here for a more general set-up, namely, for the complex scenario, without any additional difficult, let us repeat here Equation (1)

$$(\Sigma_a) \quad x(k+1) = A(\theta_k)x(k) + B(\theta_k)w(k), \quad x(0) = x_0$$

where now $A(\cdot)$ and $B(\cdot)$ are such that $A(\theta_k) = A_j$ and $B(\theta_k) = B_j$ for $\theta_k = j$, $j \in \mathscr{S}$, with A_j and B_j, $j = 1, \ldots, N$, being constant matrices in $\mathbb{B}(\mathbb{C}^n)$ and $\mathbb{B}(\mathbb{C}^m, \mathbb{C}^n)$, respectively.

Define also:

(D.1) $\mathscr{L}(H) = (\mathscr{L}_1(H), \ldots, \mathscr{L}_N(H))$ with $H = (H_1, \ldots, H_N) \in \mathbb{H}^n$ and $\mathscr{L}_j(H) := \sum_{i=1}^N p_{ij} A_i H_i A_i^*$.

(D.2) $P(k) = (P_1(k), \ldots, P_N(k))$ with $P_i(k) = E\{x(k)x^*(k)\mathbb{1}_{\{\theta_k = i\}}\} \in \mathbb{B}(\mathbb{C}^n)$, where $\mathbb{1}_{\{.\}}$ stands for the Dirac measure.

(D.3) $F(q) = (F_1(q), \ldots, F_N(q))$ with $q = (q_1, \ldots, q_N)$ and $F_j(q) := \sum_{i=1}^N p_{ij} A_i q_i$.

(D.4) $q(k) = (q_1(k), \ldots, q_N(k))$ with $q_i(k) := E\{x(k)\mathbb{1}_{\{\theta_k = i\}}\}$.

(D.5) $v(k) = (v_1(k), \ldots, v_N(k))$ with $v_j(k) := \sum_{i=1}^N p_{ij} B_i w(k) \mathbb{P}(\theta_k = i)$.

(D.6) $V(k) = (V_1(k), \ldots, V_N(k))$ with

$$V_j(k) := \sum_{i=1}^{N} p_{ij}\{B_i w(k) w^*(k) B_i^* \mathbb{P}(\theta_k = i) + A_i q_i(k) w^*(k) B_i^* + B_i w(k) q_i^*(k) A_i^*\}$$

Lemma A.1
For system (Σ_a), we have

$$q(k+1) = Fq(k) + v(k) \tag{A1}$$

$$P(k+1) = \mathscr{L}(P(k)) + V(k) \tag{A2}$$

where $q(k)$ and $P(k)$ are defined as in (D.4) and (D.2), respectively.

Proof
Use (Σ_a) and the definitions above.

Lemma A.2
System (Σ_a) is IMSS if and only if $r_\sigma(\mathscr{L}) < 1$, where $\mathscr{L}$ is defined as in (D.1), and $r_\sigma(.)$ denotes the usual spectral radius of an operator.

Proof
Follows from Lemma 1 in Reference [15] and the fact that stochastic and mean square stability are equivalent in our set up.

Proof of Lemma 2.2
($\Rightarrow$) First note that, if (Σ_a) is IMSS, then by Lemma A.2 $r_\sigma(\mathscr{L}) < 1$. It follows that there exist $0 < \xi < 1$ and $\beta \geqslant 0$ such that

$$\|\mathscr{L}^k\|_1 \leqslant \beta \xi^k, \quad k \in \mathbb{N}^0 \tag{A3}$$

Now, from Lemma A.1 we have that

$$P(k) = \mathscr{L}^k(P(0)) + \sum_{l=0}^{k-1} \mathscr{L}^{k-1-l} V(l) \tag{A4}$$

and therefore

$$P(k) \leqslant \mathscr{L}^k(P(0)) + \sum_{l=0}^{k-1} \mathscr{L}^{k-1-l} V^w(l) + \sum_{l=0}^{k-1} \mathscr{L}^{k-1-l} \bar{V}(l) \tag{A5}$$

where

$$V_j^w(l) = \sum_{i=1}^{N} p_{ij} B_i w(l) w^*(l) B_i^* [\mathbb{P}(\theta_k = i) + 1] \tag{A6}$$

$$\bar{V}_j(l) = \sum_{i=1}^{N} p_{ij} A_i q_i(l) q_i^*(l) A_i^* = \mathscr{L}_j(qq^*(l)) \tag{A7}$$

with $qq^*(l) := (q_1(l)q_1^*(l), \ldots, q_N(l)q_N^*(l))$. Thus, it follows that

$$
\begin{aligned}
\|V^w(l)\|_1 &= \sum_{j=1}^{N} \|V_j^w(l)\| \\
&= \sum_{j=1}^{N} \left\| \sum_{i=1}^{N} p_{ij} B_i w(l) w^*(l) B_i^* [\mathbb{P}(\theta_k = i) + 1] \right\| \\
&\leqslant \sum_{i=1}^{N} \sum_{j=1}^{N} 2\, p_{ij} \|B_i w(l) w^*(l) B_i^*\| \\
&= 2 \sum_{i=1}^{N} \|B_i w(l) w^*(l) B_i^*\| \\
&\leqslant 2\|w(l)\|^2 \sum_{i=1}^{N} \|B_i B_i^*\| = 2\|w(l)\|^2 \|BB^*\|_1
\end{aligned} \tag{A8}
$$

where $BB^* := (B_1 B_1^*, \ldots, B_N B_N^*)$.

Now, from (A5) and bearing in mind (A7) and (A8), we have

$$
\|P(k)\|_1 \leqslant \|\mathscr{L}^k\|_1 \|P(0)\|_1 + \sum_{l=0}^{k-1} \|\mathscr{L}^{k-1-l}\|_1 \|V^w(l)\|_1 + \sum_{l=0}^{k-1} \|\mathscr{L}^{k-l}\|_1 \|qq(l)\|_1 \tag{A9}
$$

From (A3), it follows that

$$
\begin{aligned}
\|P(k)\|_1 &\leqslant \beta\xi^k \|P(0)\|_1 + 2 \sum_{l=0}^{k-1} \beta\xi^{k-1-l} \|BB^*\|_1 \|w(l)\|^2 + \sum_{l=0}^{k-1} \beta\xi^{k-l} \|qq(l)\|_1 \\
&= \sum_{l=0}^{k} \zeta_{k-l}\beta_l + \sum_{l=0}^{k-1} \zeta_{k-l}\gamma_l
\end{aligned} \tag{A10}
$$

where $\zeta_{k-l} := \xi^{k-l}$, $\beta_0 := \beta\|P(0)\|_1$, $\beta_l := 2\beta\|BB^*\|_1 \|w(l-1)\|^2$ and $\gamma_l := \|qq(l)\|_1$. Now, since $\sum_{l=0}^{k-1} \zeta_{k-l}\gamma_l = \sum_{l=0}^{k} \gamma_{k-l}\eta_l$, with $\eta_0 := 0$ and $\eta_l := \xi^l$, for $l \geqslant 1$, we have

$$
\|P(k)\|_1 \leqslant \sum_{l=0}^{k} \zeta_{k-l}\beta_l + \sum_{l=0}^{k} \gamma_{k-l}\eta_l \tag{A11}
$$

Furthermore, let $\zeta = \{\zeta_k\}_{k=0}^{\infty}$, $\beta = \{\beta_k\}_{k=0}^{\infty}$, $\gamma = \{\gamma_k\}_{k=0}^{\infty}$, $\eta = \{\eta_k\}_{k=0}^{\infty}$ and define the convolutions

$$
\begin{aligned}
(\zeta * \beta)(k) &:= \sum_{l=0}^{k} \zeta_{k-l}\beta_l \\
(\gamma * \eta)(k) &:= \sum_{l=0}^{k} \gamma_{k-l}\eta_l
\end{aligned} \tag{A12}
$$

where, from (A11) and (A12), we have

$$||P(k)||_1 \leqslant (\zeta * \beta)(k) + (\gamma * \eta)(k) \tag{A13}$$

Since ζ, β, γ and $\eta \in \ell_1$, we have that $||P(k)||_1 \in \ell_1$.

On the other hand, we have that

$$\begin{aligned} E\left[||x(k)||^2\right] &= \operatorname{tr}\left(\sum_{i=1}^{N} P_i(k)\right) \leqslant n\left|\left|\sum_{i=1}^{N} P_i(k)\right|\right| \\ &\leqslant n \sum_{i=1}^{N} ||P_i(k)|| = n||P(k)||_1 \end{aligned} \tag{A14}$$

where n is the dimension of $x(k)$.

Now, since $E\left[||x(k)||^2\right] \leqslant n||P(k)||_1$ and $||P(k)||_1 \in \ell_1$, we have from (A13) that

$$\sum_{k=0}^{\infty} E\left[||x(k)||^2\right] \leqslant n \sum_{k=0}^{\infty} (\zeta * \beta)(k) + n \sum_{k=0}^{\infty} (\gamma * \eta)(k) \leqslant \infty \tag{A15}$$

($\Leftarrow$): Just make $w(t) = 0$ for all $t \in \mathbb{R}^+$ and the result follows.

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