Technical Notes and Correspondence

Asynchronous Output Feedback Control of Hidden Semi-Markov Jump Systems With Random Mode-Dependent Delays

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***Abstract*—This article is concerned with the problem of output feedback control for a class of continuous-time hidden semi-Markov jump systems with time delays. Due to the limitations of the actual environment, system modes are usually undetectable, which are called hidden modes. The controller modes are described as observable modes. Emission probabilities are used to establish the relationship between abovementioned two concepts. The jump parameters are governed by the hidden semi-Markov process, which can better describe the asynchronous information between the controller modes and the system modes. Besides, time delays are considered to be time-varying and dependent on the hidden modes. By employing some mathematical transformation and constructing a novel Lyapunov–Krasovskii functional, some new parameter-dependent sufficient stabilization conditions can be obtained by designing an observed-mode-dependent static output-feedback controller. Finally, a practical example is provided to illustrate the effectiveness and merits of the proposed methods.**

***Index Terms*—Asynchronous, emission probabilities, hidden semi-Markov jump systems, mode-dependent delay, output feedback control.**

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I. Introduction

Random system faults, random communication failures, and so on are commonly seen in realistic industrial systems [1], [2]. Stochastic jumping systems (SJSs), which can jump among a finite number of system modes governed by a stochastic process, can be used to model above phenomenon and has been widely concerned and studied by researchers in recent years [3]. As a special type of SJSs, semi-Markov jump systems (SMJSs) have been extensively used in many practical systems such as communication networks [4], transportation system [5], and dependability analysis [6]. Essentially, the greatest advantage of SMJSs is that it relaxes the random distribution condition of mode sojourn time, so that its statistical characteristics are no longer limited to the memoryless random distribution (geometric or exponential distribution). This not only brings more challenges to the theoretical analysis of SMJSs, but also attracts more attentions to its application. Until now, a great many of results have been reported, such as [7]–[14], and the references therein.

It is a common case that system modes information and controller modes information are not necessarily synchronized. For example, due to communication delays and packet loss are inevitable in networked control systems, which leads to the asynchronization phenomenon between controller modes and system modes. Therefore, it is necessary to study the issue of asynchronous for SJSs. From another perspective, system modes may be considered to be unobservable and are called hidden, and controller modes are observable. A double layer stochastic process with unknown parameters is introduced to described this situation, in which the lower layer and the upper layer are called the hidden layer and the observable layer, respectively. The hidden layer is a homogeneous semi-Markov process, in which the real system modes are hidden to controllers, and the observable layer process is an observable mode sequence, which is detected by virtue of emission probabilities (EPs) of real system modes. The aforesaid model is called the hidden semi-Markov (HSM) model, and it is an extension of the semi-Markov model. The HSM model has been widely used in many engineering applications, such as reliability and DNA analysis [15], mobility tracking [16], and speech synthesis system [17]. Nevertheless, most existing results on control and filtering of SJSs are based on synchronous modes information. Even when the asynchronous mode switching dynamics are considered, the HSM model is rarely used to described them. So far, there is very little results have been launched on the stability analysis and controller synthesis problems for the underlying hidden SMJSs [18], [19].

On the other hand, time-delay is immanent phenomenon in engineering systems, which involved in challenging areas of communication and information technologies fields. The stability analysis and robust control of time-delay systems are, therefore, of theoretical and practical importance. In light of the nonnegligible of system state time-delay in practical systems, recently, the control or filtering problems have been investigated for SJSs under influence of system state time-delay, see, e.g., [20]–[24]. Some early works on mode-dependent delays for a class of SJSs can be found in [20], [21], and [25]. In [20] and [21], the issue of robust H_∞ control and filtering has been investigated for a class of linear Markov jump system (MJSs) with time delay. The guaranteed cost control problem for continuous-time time-delay MJSs can be found in [25].

In addition, it is well known that output-based feedback information should be easier to implement than state-based feedback. Using output-based feedback information to synthesize controllers can be found in many literatures [26]–[31]. They cover a lot of scenarios, such as output-feedback control, state estimation, tracking control, output regulation, observed-based control, and so on, static output-feedback control is one of them. In [28], observer-based stabilization of nonhomogeneous linear SMJSs has been addressed, in which the delayed switchings exist in the observer-based output feedback controllers. Unfortunately, the asynchronous output feedback control of hidden SMJSs with random mode-dependent delay still remains unsettled nowadays, which motivates us to conduct this study for such dynamic systems.

Inspired by the observations mentioned earlier, in this article, the main intention is to establish the stabilization criteria for hidden SMJSs with random mode-dependent delays via output-feedback control. The main contributions of this article can be summarized as follows.

1) A novel framework for asynchronous output-feedback control problem of hidden SMJSs is established for the first time, in which the highlight of systems include hidden modes and random mode-dependent delays. This is a challenge to design asynchronous controller for the resulting systems.
2) A new type of Lyapunov–Krasovskii functions are constructed, and some improved transformation means are employed to handle the problem of slack variables introduced by dimension expansion and time-varying delay. These are more generalized methods and also achieve less conservative stabilization results.
3) Some parameter-dependent sufficient criteria are proposed to guarantee hidden SMJSs stabilization via output feedback control, where the system mode is hidden. Then, the output feedback controller can be obtained by the algorithm of solvable stabilization. Meanwhile, an F-404 aircraft engine system as a practical example is supplied to demonstrate the merits of the proposed control strategy.

Notations: The subscripts "$\top$" and "-1" signify matrix transposition and inverse, respectively. $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{Z}_{\geq z}$ denote n dimensional Euclidean space, set of all $n\times m$ real matrices, the sets of nonnegative real numbers and integers greater than z, respectively. $\mathbb{E}\{\cdot\}$ means the expectation. diag$\{\cdot\}$ stands for a block diagonal matrix, I and $\mathbf{0}$ denote identity and zero matrices of appropriate dimensions, respectively. $\mathbb{P}\{x=1\}$ denotes the probability of $x=1$. $\mathbf{He}(A)$ is used to present $A+A^\top$; $\|\cdot\|$ refers to the Euclidean norm of a vector or its induced norm of a matrix. $\otimes$ means the Kronecker product of matrices.

II. Problem Statement

A. System Description

Consider a stochastic process $\{h_t\}_{t\in\mathbb{R}_{\geq 0}}$ taking values in a finite set $\mathcal{N}=\{1,2,\ldots,N\}$, and defined on a complete probability space $\aleph=\{\Omega,\mathcal{F},\mathbb{P}\}$, where Ω is a sample space, $\mathcal{F}$ is the Borel σ-algebra and $\mathbb{P}$ is the corresponding probability measure on $\mathcal{F}$.

Describe the following plant over the space $\aleph$:

$$(\mathfrak{T})\begin{cases}\dot{x}(t)=A_{h_t}x(t)+\gamma(t)C_{h_t}x(t-\tau^t_{h_t})+B_{h_t}u(t) & \text{(1a)}\\ y(t)=D_{h_t}x(t) & \text{(1b)}\\ x(t)=\hat{x}(t)\quad \forall t\in[-\tau_2,-\tau_1] & \text{(1c)}\end{cases}$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input, $y(t)\in\mathbb{R}^r$ is the control output, and $A_{h_t}, B_{h_t}, C_{h_t}$ and D_{h_t} are given real matrices, $\hat{x}(t)$ is initial condition.

The variables $\gamma(t)$ is Bernoulli-distributed white sequences taking values on 0 and 1 with the following probabilities:

$$\begin{cases}\mathbb{P}\{\gamma(t)=1\}=\mathbb{E}\{\gamma(t)\}=\bar{\gamma}\\ \mathbb{P}\{\gamma(t)=0\}=1-\mathbb{E}\{\gamma(t)\}=1-\bar{\gamma}\end{cases}$$

where $\bar{\gamma}\in[0,1]$ is known constant.

τ^t_i stands for the time-varying mode-dependent delay, and it satisfies, for $h_t=i\in\mathcal{N}$

$$\tau^t_i\in[\underline{\tau}_i,\bar{\tau}_i],\quad \dot{\tau}^t_i\leq\wp_i<1 \tag{2}$$

where $\bar{\tau}_i$ and $\underline{\tau}_i$ are positive scalars. Besides, we denote $\tau_1=\min\{\underline{\tau}_i, i\in\mathcal{N}\}$, $\tau_2=\max\{\bar{\tau}_i, i\in\mathcal{N}\}$, $\tau_{12}=\tau_2-\tau_1$.

In this article, the system jumping parameters $\{h_t\}_{t\in\mathbb{R}_{\geq 0}}$ are assumed to be nonaccessible or hidden and described by a hidden semi-Markov process (SMP). To introduce the hidden SMP formally, we first recall some important definitions about it.

B. Hidden Semi-Markov Process

1) Semi-Markov Process: We present only SMP $\{h_t\}_{t\in\mathbb{R}_{\geq 0}}$ with a discrete state space, the more details can be found in [7] and [15].

Definition 1: A bivariate random variables sequence $\{H_k,t_k\}_{k\in\mathbb{Z}_{\geq 0}}$ is said to be a homogenous Markov renewal process (MRP), if for $\forall i,j\in\mathcal{N}$

$$\begin{cases}\mathbb{P}(\mathcal{S}_{k+1}\leq\sigma+\Delta_t, H_{k+1}=j\mid\mathcal{S}_{k+1}>\sigma, H_k=i)\\ =\mathbb{P}(\mathcal{S}_1\leq\sigma+\Delta_t, H_1=j\mid\mathcal{S}_1>\sigma, H_0=i)\\ =\delta_{ij}(\sigma)\Delta_t+o(\Delta_t),\ i\neq j\\ \mathbb{P}(\mathcal{S}_{k+1}>\sigma+\Delta_t, H_{k+1}=j\mid\mathcal{S}_{k+1}>\sigma, H_k=i)\\ =\mathbb{P}(\mathcal{S}_1>\sigma+\Delta_t, H_1=j\mid\mathcal{S}_1>\sigma, H_0=i)\\ =1+\delta_{ii}(\sigma)\Delta_t+o(\Delta_t),\ i=j\end{cases} \tag{3}$$

where $\{\mathcal{S}_{k+1}\in\mathbb{R}_{\geq 0}|\mathcal{S}_{k+1}=t_{k+1}-t_k\}_{k\in\mathbb{Z}_{\geq 0}}$ be the set of sojourn time indicate the system between the kth and the $(k+1)$th jumps, $\{t_k\}_{k\in\mathbb{Z}_{\geq 0}}$ be the set of the kth jumping instant of the process, $\{H_k\}_{k\in\mathbb{Z}_{\geq 0}}$ be a discrete stochastic process taking values on $\mathcal{N}$, $\delta_{ij}(\sigma)$ is transition rate, σ represents the sojourn time. Δ_t is time increment, and $\lim_{\Delta_t\to 0}\frac{o(\Delta_t)}{\Delta_t}=0$.

Definition 2: Consider a homogenous MRP $\{H_k,t_k\}_{k\in\mathbb{Z}_{\geq 0}}$, then $\{H_k\}_{k\in\mathbb{Z}_{\geq 0}}$ is called the embedded Markov process (EMP) of MRP, which transition probabilities $q_{ij}=\mathbb{P}\{H_{k+1}=j|H_k=i\}$ for $i=j$ and $q_{ii}=0$.

Definition 3: Stochastic process $\{h_t\}_{t\in\mathbb{R}_{\geq 0}}$ is called an SMP associated with the MRP $\{H_k,t_k\}_{k\in\mathbb{Z}_{\geq 0}}$, if $h_t=H_{N(t)}$, where $N(t)=\sup\{k: t_k\leq t\}$.

Cumulative distribution function $G_i(\sigma)$ of the random sojourn time $\{\mathcal{S}_k\in\mathbb{R}_{\geq 0}\}_{k\in\mathbb{Z}_{\geq 1}}$ is described by $G_i(\sigma)=\mathbb{P}\{\mathcal{S}_k<\sigma|h_{t_k}=i\}$ when the system remains in mode i. Correspondingly, the transition rate matrix is defined by $\Omega(\sigma)=[\delta_{ij}(\sigma)]_{N\times N}$. Then, following [13], by simple calculations, one has

$$\begin{aligned}\delta_{ij}(\sigma)&=\lim_{\Delta_t\to 0}\frac{\mathbb{P}(h_{t+\Delta_t}=j\mid h_t=i)}{\Delta_t\to 0}\\ &=q_{ij}\frac{\chi_i(\sigma)}{1-G_i(\sigma)}, i\neq j\end{aligned} \tag{4}$$

and $\delta_{ii}(\sigma)=-\sum_{j=1,\,j\neq i}^N\delta_{ij}(\sigma)<0$, where $\chi_i(\sigma)$ is probability density function (PDF) of mode sojourn time.

2) Hidden Semi-Markov Process: Compared with the SMP, the most important feature of hidden SMP is that its states are hidden (i.e., unobservable), however, these hidden states can emit some observations (observable signals). Hence, in describing the hidden SMP, the following two concepts are necessary.

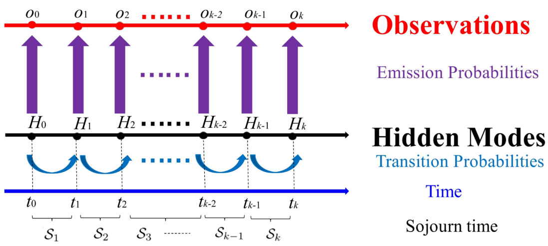

Fig. 1. Hidden semi-Markov model.

1) Transition probabilities: to describe the transition rule between two hidden states.
2) Emission probabilities: to describe the law of hidden state emit the observable signals.

Fig. 1 can illustrate HSM process more clearly.

For observable signals $\{o_t\}_{t\in\mathbb{R}_{\geq 0}}$, taking values in another finite index set $\mathcal{N}' = \{1, 2, \ldots, N'\}$. The EPs matrix $\Pi(t) = [\rho^t_{im}]_{N\times N'}$ is defined by

$$\mathbb{P}\{o_t = m | h_t = i\} = \rho^t_{im} \quad \forall i \in \mathcal{N}, m \in \mathcal{N}' \tag{5}$$

where $\rho^t_{im} \in [0, 1]$ and $\sum_{m\in\mathcal{N}'} \rho^t_{im} = 1$.

Remark 1: The EP ρ^t_{im} is assumed to be time-varying, which depends on the current moment. Specially, $\Pi(t)$ has the following expression:

$$\Pi(t) = \sum_{\upsilon=1}^{\upsilon_0} \hbar_\upsilon(t)\Pi^\upsilon \tag{6}$$

where $\hbar_\upsilon(t) \geq 0, \sum_{\upsilon=1}^{\upsilon_0} \hbar_\upsilon(t) = 1$, and $\Pi^\upsilon = [\rho^\upsilon_{im}]_{N\times N'}$.

In this article, a static output feedback controller dependent on the observable signals for system $(\mathfrak{T})$ is given as

$$u(t) = K_m y(t) \tag{7}$$

when controller (7) is applied to system $(\mathfrak{T})$, then, the closed-loop system is governed by

$$(\mathfrak{T}_{c_1}) \begin{cases} \dot{x}(t) = (A_i + B_i K_m D_i)x(t) + \gamma(t)C_i x(t - \tau^t_i) & \text{(8a)}\\ y(t) = D_i x(t) & \text{(8b)}\\ x(t) = \hat{x}(t) \quad \forall t \in [-\tau_2, -\tau_1]. & \text{(8c)} \end{cases}$$

Adding $n - p$ columns to B_i, such that $\hat{B}_i = [B_i \quad \bar{B}_i]$ is square and full rank. In general, control matrix B_i is belong to space $\mathbb{R}^{n\times p}$, where $p < n$. To ensure that the augmented matrix $\hat{B}_i$ is invertible, the following two statements are important: matrix B_i is row full rank, the column rank is p; matrix $\bar{B}_i$ is row full rank, the column rank is $n - p$. It is easy to see that there are an infinite number of matrices $\bar{B}_i$ satisfying this condition.

Introduce the new state variable $\psi(t) = \hat{B}^{-1}_i x(t)$, then system $(\mathfrak{T}_{c_1})$ becomes

$$(\mathfrak{T}_{c_2}) \begin{cases} \dot{\psi}(t) = \mathcal{A}_{im}\psi(t) + \gamma(t)\bar{C}_i\psi(t - \tau^t_i) & \text{(9a)}\\ y(t) = D_i\hat{B}_i\psi(t) & \text{(9b)}\\ \psi(t) = \hat{\psi}(t) = \hat{B}^{-1}_i\hat{x}(t) \quad \forall t \in [-\tau_2, -\tau_1] & \text{(9c)} \end{cases}$$

where $\mathcal{A}_{im} = \bar{A}_i + \bar{K}_m\bar{D}_i$, with $\bar{A}_i = \hat{B}^{-1}_i A_i\hat{B}_i, \bar{D}_i = D_i\hat{B}_i, \bar{C}_i = \hat{B}^{-1}_i C_i\hat{B}_i$, and $\bar{K}_m = [K^\top_m \quad \mathbf{0}]^\top$.

Remark 2: It is should be pointed out that linear transformation $\hat{B}^{-1}_i$ is invertible. Thus, dynamics of the closed-loop system $(\mathfrak{T}_{c_1})$ is the same as that of the closed-loop system $(\mathfrak{T}_{c_2})$. The main purpose of this transformation is to make the static output-feedback controller easier to synthesize. In the following, all definitions, lemmas, and theorems are subject to system $(\mathfrak{T}_{c_2})$.

Definition 4: The closed-loop system $(\mathfrak{T}_{c_2})$ is said to be stochastic stabilizable (SS), if for all continuous functions $\hat{\psi}(t)$ defined on $[-\tau_2, -\tau_1]$, the following inequality holds:

$$\mathbb{E}\left\{\int_0^\infty \|\psi(t, \hat{\psi}(t), h_0)\|^2 dt | (\hat{\psi}(t), h_0)\right\} < \infty$$

for any initial conditions $\hat{\psi}(t) \in \mathbb{R}^n$, and $h_0 \in \mathcal{N}$.

Lemma 1 [33]: Let $\psi(t) \in [\alpha, \beta] \to \mathbb{R}^n$ be a differentiable function M is a positive definite matrix and satisfy the following inequalities:

$$(\beta - \alpha)\int_\alpha^\beta \dot{\psi}(s)M\dot{\psi}(s)\mathrm{d}s \geq \varrho^\top_1\mathrm{diag}\{M, 3\,M, 5M\}\varrho_1 \tag{10}$$

$$\int_\alpha^\beta (s - \alpha)\dot{\psi}(s)M\dot{\psi}(s)\mathrm{d}s \geq \varrho^\top_2\mathrm{diag}\{2\,M, 4M\}\varrho_2 \tag{11}$$

$$\int_\alpha^\beta (\beta - s)\dot{\psi}(s)M\dot{\psi}(s)\mathrm{d}s \geq \varrho^\top_3\mathrm{diag}\{2\,M, 4M\}\varrho_3 \tag{12}$$

where

$$\varrho_1 = [\psi^\top(\beta) - \psi^\top(\alpha), \psi^\top(\beta) + \psi^\top(\alpha) - 2\varpi^\top_0(\alpha, \beta),$$
$$\psi^\top(\beta) - \psi^\top(\alpha) - 6\varpi^\top_1(\alpha, \beta)]^\top$$
$$\varrho_2 = [\psi^\top(\beta) - \varpi^\top_0(\alpha, \beta), \psi^\top(\beta) - \varpi^\top_0(\alpha, \beta) - 3\varpi^\top_1(\alpha, \beta)]^\top$$
$$\varrho_3 = [\psi^\top(\alpha) - \varpi^\top_0(\alpha, \beta), \psi^\top(\alpha) - \varpi^\top_0(\alpha, \beta) + 3\varpi^\top_1(\alpha, \beta)]^\top$$
$$\varpi_0(\alpha, \beta) = \int_\alpha^\beta \frac{\psi(s)}{\beta - \alpha}\mathrm{d}s$$
$$\varpi_1(\alpha, \beta) = \int_\alpha^\beta \frac{(2s - \beta - \alpha)\psi(s)}{(\beta - \alpha)^2}\mathrm{d}s.$$

Lemma 2 [33]: For positive matrices $M_l (l = 1, 2, 3, 4)$, and any matrices T_1, T_2, there exist positive scalars a_1, a_2 such that

$$\begin{bmatrix} \frac{M_1 + a_2M_3}{a_1} & \mathbf{0}\\ * & \frac{M_2 + a_1M_4}{a_2} \end{bmatrix} \geq \begin{bmatrix} M_1 + a_2\Phi_1 & a_2T_1 + a_1T_2\\ * & M_2 + a_1\Phi_2 \end{bmatrix} \tag{13}$$

where $\Phi_1 = M_1 + M_3 - T_2(M_2 + M_4)^{-1}T^\top_2$, $\Phi_2 = M_2 + M_4 - T_1(M_1 + M_3)^{-1}T^\top_1$, and $a_1 + a_2 = 1$.

III. Main Results

In this section, the static output-feedback control problem for system $(\mathfrak{T}_{c_2})$ will be investigated. At the beginning, a sufficient stabilization condition via static output-feedback control will be shown. For convenience presentation, define

$$\kappa_i = \tau^t_i, \quad \dot{\kappa}_i = \dot{\tau}^t_i, \quad \bar{\kappa}_i = 1 - \wp_i$$
$$\xi(t) = \mathrm{col}[\psi(t), \psi(t - \tau_1), \psi(t - \kappa_i), \psi(t - \tau_2), \dot{\psi}(t), \dot{\psi}(t - \tau_1)$$
$$\dot{\psi}(t - \tau_2), \varpi_0(t - \tau_1, t), \varpi_1(t - \tau_1, t)$$
$$\varpi_0(t - \kappa_i, t - \tau_1), \varpi_1(t - \kappa_i, t - \tau_1), \varpi_0(t - \tau_2, t - \kappa_i)$$
$$\varpi_1(t - \tau_2, t - \kappa_i)]$$
$$e_\iota = [0_{n\times(\iota-1)n}, I_n, 0_{n\times(13-\iota)n}] (\iota = 1, 2, \ldots, 13)$$
$$\mathcal{R}_1 = R_1 + R_3, \quad \mathcal{R}_2 = R_1 + R_2$$
$$\mathcal{R}_3 = \mathrm{diag}\{2R_2, 4R_2\}, \quad \mathcal{R}_4 = \mathrm{diag}\{2R_3, 4R_3\}$$
$$\mathcal{R}_5 = \mathrm{diag}\{Q_1, 3Q_1, 5Q_1, 2Q_2, 4Q_2, 2Q_3, 4Q_3\}$$
$$\varsigma_{11} = e_2 - e_3, \varsigma_{12} = e_2 + e_3 - 2e_{10}, \varsigma_{13} = e_2 - e_3 - 6e_{11}$$
$$\varsigma_{21} = e_3 - e_4, \varsigma_{22} = e_3 + e_4 - 2e_{12}, \varsigma_{23} = e_3 - e_4 - 6e_{13}$$
$$\vartheta^{\kappa_i}_1 = \mathrm{col}\{e_1, \tau_1e_8, (\tau_2 - \kappa_i)e_{12} + (\kappa_i - \tau_1)e_{10}\}$$
$$\vartheta_2 = \mathrm{col}\{e_5, e_1 - e_2, e_2 - e_4\},\ L_1 = [I, \mathbf{0}],\ L_2 = [\mathbf{0}, I]$$

$$\phi_1 = \mathrm{col}\{e_2 - e_{10}, e_2 - e_{10} - 3e_{11}, e_3 - e_{12}, e_3 - e_{12} - 3e_{13}$$
$$e_3 - e_{10}, e_3 - e_{10} + 3e_{11}, e_4 - e_{12}, e_4 - e_{12} + 3e_{13}\}$$
$$\phi_2 = \mathrm{col}\{e_1 - e_2, e_1 + e_2 - 2e_8, e_1 - e_2 - 6e_9, e_1 - e_8$$
$$e_1 - e_8 - 3e_9, e_2 - e_8, e_2 - e_8 + 3e_9\}$$
$$\mathcal{O}_{im} = \left[\sum_{m\in\mathcal{N}'} \rho_{im}^t \mathcal{A}_{im}, \mathbf{0}, \gamma(t)\bar{C}_i, \mathbf{0}, -I, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{8}\right]$$
$$W_1 = [I, \mathbf{0}, \mathbf{0}],\ W_2 = [\mathbf{0}, I, \mathbf{0}],\ W_3 = [\mathbf{0}, \mathbf{0}, I].$$

Theorem 1: Given finite constants $\tau_1 < \tau_2$ and $\wp_i$. For the time-varying EPs matrix $\Pi(t)$, the closed-loop system $(\mathfrak{T}_{c_2})$ is SS if, for any $i \in \mathcal{N}$ and $m \in \mathcal{N}'$, there exist a set of positive definite matrices P_i, $S_\imath$, $R_\imath$, $Q_\imath$, $\imath = 1, 2, 3$, J_ϵ, $\epsilon = 1, 2$, appropriate dimension matrices $\mathcal{Z}_i$, T_1, T_2, and K_m, satisfying

$$\Phi_i(\tau_1) = \Upsilon_i(\tau_1) + \Lambda_{\tau_1} + \Omega_1(\tau_1) + \mathbf{He}\{\mathcal{Z}_i\mathcal{O}_{im}\} < 0 \quad (14)$$
$$\Phi_i(\tau_2) = \Upsilon_i(\tau_2) + \Lambda_{\tau_2} + \Omega_1(\tau_2) + \mathbf{He}\{\mathcal{Z}_i\mathcal{O}_{im}\} < 0 \quad (15)$$

where

$$\Upsilon_i(\kappa_i) = \Upsilon_{1i}(\kappa_i) + \Upsilon_2 + \Omega_2,\ \kappa_i \in [\tau_1, \tau_2]$$
$$\Lambda_{\tau_\zeta} = (\vartheta_1^{\tau_\zeta})^\top \sum_{j=1}^{N} \bar{\delta}_{ij} P_j \vartheta_1^{\tau_\zeta}, \zeta = 1, 2$$

with

$$\Upsilon_{1i}(\kappa_i) = \mathbf{He}\{e_5^\top W_1 P_i (W_1^\top e_1 + \tau_1 W_2^\top e_8 + (\kappa_i - \tau_1) W_3^\top e_{10}$$
$$+ (\tau_2 - \kappa_i) W_3^\top e_{12})\} + \mathbf{He}\{(e_1 - e_2)^\top W_2 P_i W_1^\top e_1$$
$$+ \tau_1 (e_1 - e_2)^\top W_2 P_i W_2^\top e_8 + (e_2 - e_4)^\top W_3 P_i W_1^\top e_1$$
$$+ \tau_1 (e_2 - e_4)^\top W_3 P_i W_2^\top e_8$$
$$+ (e_1 - e_2)^\top W_2 P_i W_3^\top ((\kappa_i - \tau_1) e_{10} + (\tau_2 - \kappa_i) e_{12})$$
$$+ (e_2 - e_4)^\top W_3 P_i W_3^\top ((\kappa_i - \tau_1) e_{10} + (\tau_2 - \kappa_i) e_{12})\}$$
$$\Omega_1(\kappa_i) = \sum_{\epsilon=1}^{3} (2 - 4\epsilon) \Bigg((\varsigma_{1\epsilon}^\top R_1 \varsigma_{1\epsilon} + \varsigma_{2\epsilon}^\top R_1 \varsigma_{2\epsilon})$$
$$+ \begin{bmatrix} \varsigma_{1\epsilon} \\ \varsigma_{2\epsilon} \end{bmatrix}^\top \left(\frac{\kappa_i - \tau_1}{\tau_{12}} \begin{bmatrix} \mathbf{0} & T_2 \\ * & \mathcal{R}_1 - T_1^\top \mathcal{R}_2^{-1} T_1 \end{bmatrix} \right.$$
$$\left. + \frac{\tau_2 - \kappa_i}{\tau_{12}} \begin{bmatrix} \mathcal{R}_2 - T_2^\top \mathcal{R}_1^{-1} T_2 & T_1 \\ * & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \varsigma_{1\epsilon} \\ \varsigma_{2\epsilon} \end{bmatrix} \Bigg)$$
$$\Upsilon_2 = e_1^\top (\mu \tau_{12} S_1 + S_2 + S_3) e_1 + e_2^\top (S_1 - S_2) e_2 - e_4^\top S_3 e_4$$
$$- \bar{\kappa}_i e_3^\top S_1 e_3 + \tau_1^2 e_5^\top (2Q_1 + Q_2 + Q_3) e_5$$
$$+ e_6^\top (2\tau_{12}^2 R_1 + \tau_{12}^2 R_2 + \tau_2^2 R_3) e_6$$
$$- e_7^\top (\tau_1 (2\tau_2 - \tau_1) R_3) e_7 + e_1^\top L_1 J_1 L_1^\top e_1$$
$$+ e_5^\top L_2 J_1 L_2^\top e_5 - e_4^\top L_1 J_2 L_1^\top e_4$$
$$- e_7^\top L_2 J_2 L_2^\top e_7 + e_2^\top L_1 (J_2 - J_1) L_1^\top e_2$$
$$+ e_6^\top L_2 (J_2 - J_1) L_2^\top e_6$$
$$+ \mathbf{He}\{e_1^\top L_1 J_1 L_2^\top e_5 - e_4^\top L_1 J_2 L_2^\top e_7$$
$$+ e_2^\top L_1 (J_2 - J_1) L_2^\top e_6\}$$
$$\Omega_2 = -2\phi_1^\top \mathrm{diag}\{\mathcal{R}_3, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_4\}\phi_1 - 2\phi_2^\top \mathcal{R}_5 \phi_2$$
$$\bar{\delta}_{ij} = \mathbb{E}\{\delta_{ij}(\sigma)\} = \int_0^\infty \delta_{ij}(\sigma)\chi_i(\sigma)\mathrm{d}\sigma,\ \mu = \max\{|\ \bar{\delta}_{ii}\ |, i \in \mathcal{T}\}$$

and $\chi_i(\sigma)$ is the PDF of sojourn time σ staying at mode i.

Proof: Let $\mathbb{C}_{[-\tau_2, -\tau_1]}$ be a continuous functions space. Define $\psi_s(t) = \psi(t + s)$, $\psi_s(t) \in \mathbb{C}_{[-\tau_2, -\tau_1]}, s \in [-\tau_2, -\tau_1]$. Then, $\{\psi_s(t), h_t\}_{t \in \mathbb{R}_{\geq 0}}$ is a hidden SMP with initial state $(\hat{B}^{-1}\hat{x}(t), h_0)$. Construct the following Lyapunov–Krasovskii functional (LKF) candidate

$$V(\psi_s(t), i, t) = \sum_{\iota=1}^{5} V_\iota(\psi_s(t), i, t) \quad (16)$$

where

$$V_1(\psi_s(t), i, t) = \eta^\top(t) P_i \eta(t)$$
$$V_2(\psi_s(t), i, t) = \int_{t-\tau_i^t}^{t-\tau_1} \psi^\top(s) S_1 \psi(s)\mathrm{d}s$$
$$+ \sum_{\ell=2}^{3} \int_{t-\tau_{\ell-1}}^{t} \psi^\top(s) S_\ell \psi(s)\mathrm{d}s$$
$$+ \mu \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} \psi^\top(s) S_1 \psi(s)\mathrm{d}s\mathrm{d}\theta$$
$$V_3(\psi_s(t), i, t) = \int_{t-\tau_1}^{t} \begin{bmatrix} \psi(s) \\ \dot{\psi}(s) \end{bmatrix}^\top J_1 \begin{bmatrix} \psi(s) \\ \dot{\psi}(s) \end{bmatrix} \mathrm{d}s$$
$$+ \int_{t-\tau_2}^{t-\tau_1} \begin{bmatrix} \psi(s) \\ \dot{\psi}(s) \end{bmatrix}^\top J_2 \begin{bmatrix} \psi(s) \\ \dot{\psi}(s) \end{bmatrix} \mathrm{d}s$$
$$V_4(\psi_s(t), i, t) = \int_{t-\tau_2}^{t-\tau_1} \dot{\psi}^\top(s)(2(\tau_2 - t + s)\tau_{12} R_1$$
$$+ (t - \tau_2 - s)^2 R_2 + (\tau_2 - t + s$$
$$+ \tau_1)(\tau_2 + t - s - \tau_1) R_3)\dot{\psi}(s)\mathrm{d}s$$
$$V_5(\psi_s(t), i, t) = \int_{t-\tau_1}^{t} (\tau_1 - t + s)\dot{\psi}^\top(s)(2\tau_1 Q_1 + (\tau_1 - t + s) Q_2$$
$$+ (\tau_1 + t - s) Q_3)\dot{\psi}(s)\mathrm{d}s$$

with

$$\eta(t) = \left[\psi^\top(t) \quad \int_{t-\tau_1}^{t} \psi^\top(s)\mathrm{d}s \quad \int_{t-\tau_2}^{t-\tau_1} \psi^\top(s)\mathrm{d}s\right]^\top.$$

For the SMJSs $(\mathfrak{T}_{c_2})$, an infinitesimal operator $\mathcal{L}$ [3] of the Lyapunov functional is defined as

$$\mathcal{L}V(\psi_s(t), i, t) = \lim_{\Delta_t \to 0} \frac{1}{\Delta_t} \Bigg[\mathbb{E}\Bigg\{ V(\psi_s(t + \Delta_t), h_{t+\Delta_t} = j, t + \Delta_t)$$
$$|\psi_s(t), h_t = i, t \Bigg\} - V(\psi_s(t), i, t) \Bigg].$$

Then, one has

$$\mathcal{L}V_1(\psi_s(t), i, t) = \lim_{\Delta_t \to 0} \frac{1}{\Delta_t} \Bigg[\mathbb{E}\Bigg\{ \sum_{j=1, j\neq i}^{N} \frac{\mathbb{P}\{H_{k+1} = j, H_k = i\}}{\mathbb{P}\{H_k = i\}}$$
$$\times \frac{\mathbb{P}\{\sigma < \mathcal{S}_{k+1} \leq \sigma + \Delta_t | H_{k+1} = j, H_k = i\}}{\mathbb{P}\{\mathcal{S}_{k+1} > \sigma | H_k = i\}}$$
$$\times \eta^\top(t + \Delta_t) P_j \eta(t + \Delta_t)$$
$$+ \frac{\mathbb{P}\{\mathcal{S}_{k+1} > \sigma + \Delta_t | H_k = i\}}{\mathbb{P}\{\mathcal{S}_{k+1} > \sigma | H_k = i\}}$$

$$\times \eta^\top(t+\Delta_t)P_i\eta(t+\Delta_t)\Bigg\} - \eta^\top(t)P_i\eta(t)\Bigg]$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Bigg[\mathbb{E}\Bigg\{\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(G_i(\sigma+\Delta_t)-G_i(\sigma))}{1-G_i(\sigma)}$$
$$\times \eta^\top(t+\Delta_t)P_j\eta(t+\Delta_t) + \frac{1-G_i(\sigma+\Delta_t)}{1-G_i(\sigma)}$$
$$\times \eta^\top(t+\Delta_t)P_i\eta(t+\Delta_t)\Bigg\} - \eta^\top(t)P_i\eta(t)\Bigg].$$

By PDF, it yields that

$$\mathcal{L}V_1(\psi_s(t),i,t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Bigg[\mathbb{E}\Bigg\{\sum_{j=1,j\neq i}^{N}\frac{q_{ij}(G_i(\sigma+\Delta_t)-G_i(\sigma))}{1-G_i(\sigma)}$$
$$\times \eta^\top(t+\Delta_t)P_j\eta(t+\Delta_t) + \frac{1-G_i(\sigma+\Delta_t)}{1-G_i(\sigma)}$$
$$\times [(\eta(t+\Delta_t)-\eta(t))^\top P_i\eta(t+\Delta_t)$$
$$+ \eta(t+\Delta_t)^\top P_i(\eta(t+\Delta_t)-\eta(t))]$$
$$- \frac{G_i(\sigma+\Delta_t)-G_i(\sigma)}{1-G_i(\sigma)}\eta^\top(t)P_i\eta(t)\Bigg\}\Bigg]$$
$$= \mathbb{E}\Bigg\{\mathbf{He}\{\eta^\top(t)P_i\dot{\eta}(t)\} + \eta^\top(t)\left(\sum_{j=1}^{N}\delta_{ij}(\sigma)P_j\right)\eta(t)\Bigg\}$$
$$= \xi^\top(t)[\mathbf{He}(\vartheta_2^\top P_i\vartheta_1^{\kappa_i}) + (\vartheta_1^{\kappa_i})^\top\sum_{j=1}^{N}\bar{\delta}_{ij}P_j\vartheta_1^{\kappa_i}]\xi(t).$$

The derivatives of $V_i(\psi_s(t),i,t), i=2,3,4$ via infinitesimal operator $\mathcal{L}$, which can be derived that

$$\mathcal{L}V_2(\psi_s(t),i,t) = \sum_{j=1}^{N}\bar{\delta}_{ij}\int_{t-\tau_j^t}^{t-\tau_1}\psi^\top(s)S_1x(s)ds$$
$$+ \psi^\top(t-\tau_1)S_1\psi(t-\tau_1) - (1-\dot{\tau}_i^t)\psi^\top(t-\tau_i^t)$$
$$\times S_1\psi(t-\tau_i^t)$$
$$+ \mu\tau_{12}\psi^\top(t)S_1\psi(t) - \mu\int_{t-\tau_2}^{t-\tau_1}\psi^\top(s)S_1\psi(s)ds$$
$$+ \psi^\top(t)S_2\psi(t) - \psi^\top(t-\tau_1)S_2\psi(t-\tau_1)$$
$$+ \psi^\top(t)S_3\psi(t) - \psi^\top(t-\tau_2)S_3\psi(t-\tau_2)$$
$$\leq \xi^\top(t)(e_1^\top(\mu\tau_{12}S_1+S_2+S_3)e_1$$
$$+ e_2^\top(S_1-S_2)e_2 - e_4^\top S_3e_4 - \bar{\kappa}_ie_3^\top S_1e_3)\xi(t)$$

and

$$\mathcal{L}V_3(\psi_s(t),i,t) = \xi^\top(t)\left(\begin{bmatrix}e_1\\e_5\end{bmatrix}J_1\begin{bmatrix}e_1\\e_5\end{bmatrix} - \begin{bmatrix}e_4\\e_7\end{bmatrix}J_2\begin{bmatrix}e_4\\e_7\end{bmatrix}\right.$$
$$\left. + \begin{bmatrix}e_2\\e_6\end{bmatrix}(J_2-J_1)\begin{bmatrix}e_2\\e_6\end{bmatrix}\right)\xi(t)$$

and

$$\mathcal{L}V_4(\psi_s(t),i,t)$$
$$= \dot{\psi}^\top(t-\tau_1)(2\tau_{12}^2R_1+\tau_{12}^2R_2+\tau_2^2R_3)\dot{\psi}(t-\tau_1)$$
$$- \dot{\psi}^\top(t-\tau_2)(\tau_1(2\tau_2-\tau_1)R_3)\dot{\psi}(t-\tau_2) + 2\Xi \tag{17}$$

where $\Xi = -\int_{t-\tau_2}^{t-\tau_1}\dot{\psi}^\top(s)(\tau_{12}R_1+(\tau_2-t+s)R_2+(t-s-\tau_1)R_3)\dot{\psi}(s)\mathrm{d}s$.

In order to eliminate the integral terms of Ξ, the following expression is needed

$$\begin{aligned}\Xi = &-\int_{t-\kappa_i}^{t-\tau_1}\dot{\psi}^\top(s)(\tau_{12}R_1+(\tau_2-\kappa_i)R_2)\dot{\psi}(s)\mathrm{d}s\\&-\int_{t-\kappa_i}^{t-\tau_1}\dot{\psi}^\top(s)((s-t+\kappa_i)R_2+(t-s-\tau_1)R_3)\dot{\psi}(s)\mathrm{d}s\\&-\int_{t-\tau_2}^{t-\kappa_i}\dot{\psi}^\top(s)(\tau_{12}R_1+(\kappa_i-\tau_1)R_3)\dot{\psi}(s)\mathrm{d}s\\&-\int_{t-\tau_2}^{t-\kappa_i}\dot{\psi}^\top(s)((s-t+\tau_2)R_2+(t-s-\kappa_i)R_3)\dot{\psi}(s)\mathrm{d}s.\end{aligned} \tag{18}$$

By (17), (18) and Lemmas 1 and 2, one has

$$\mathcal{L}V_4(\psi_s(t),i,t)$$
$$\leq \xi^\top(t)(e_6^\top(2\tau_{12}^2R_1+\tau_{12}^2R_2+\tau_2^2R_3)e_6 - e_7^\top(\tau_1(2\tau_2-\tau_1)$$
$$\times R_3)e_7 + \Omega_1(\kappa_i) - 2\phi_1^\top\mathrm{diag}\{\mathcal{R}_3,\mathcal{R}_3,\mathcal{R}_4,\mathcal{R}_4\}\phi_1)\xi(t)$$

where

$$\Omega_1(\kappa_i) = \sum_{\epsilon=1}^{3}(2-4\epsilon)\Bigg((\varsigma_{1\epsilon}^\top R_1\varsigma_{1\epsilon}+\varsigma_{2\epsilon}^\top R_1\varsigma_{2\epsilon})$$
$$+ \begin{bmatrix}\varsigma_{1\epsilon}\\\varsigma_{2\epsilon}\end{bmatrix}^\top\Bigg(\frac{\kappa_i-\tau_1}{\tau_{12}}\begin{bmatrix}\mathbf{0} & T_2\\ * & \mathcal{R}_1-T_1^\top\mathcal{R}_2^{-1}T_1\end{bmatrix}$$
$$+ \frac{\tau_2-\kappa_i}{\tau_{12}}\begin{bmatrix}\mathcal{R}_2-T_2\mathcal{R}_1^{-1}T_2^\top & T_1\\ * & \mathbf{0}\end{bmatrix}\Bigg)\begin{bmatrix}\varsigma_{1\epsilon}\\\varsigma_{2\epsilon}\end{bmatrix}\Bigg).$$

Similar to abovementioned treatment, it is can be found that

$$\mathcal{L}V_5(\psi_s(t),i,t)$$
$$= \tau_1^2\dot{\psi}^\top(t)(2Q_1+1Q_2+1Q_3)\dot{\psi}(t) - 2\int_{t-\tau_1}^{t}\dot{\psi}^\top(s)(\tau Q_1$$
$$+ (\tau_1-t+s)Q_2+(t-s)Q_3)\dot{\psi}(s)\mathrm{d}s$$
$$\leq \xi(t)\Big\{\tau_1^2e_5^\top(2Q_1+Q_2+Q_3)e_5 - 2\phi_2^\top\mathcal{R}_5\phi_2\Big\}\xi(t).$$

Furthermore, considering (9a), for any appropriate dimension matrix $\mathcal{Z}_i$, we have

$$2\sum_{m\in\mathcal{N}'}\rho_{im}^t\xi^\top(t)\mathcal{Z}_i[-\dot{\psi}(t)+\mathcal{A}_{im}\psi(t)+\gamma(t)\bar{C}_i\psi(t-\tau_i^t)] = 0. \tag{19}$$

Above all, it can be derived that

$$\sum_{\iota=1}^{5}\mathcal{L}V_\iota(\psi_s(t),i,t) \leq \xi^\top(t)\Phi_i(\kappa_i)\xi(t). \tag{20}$$

It is easy to find that negative definite of $\Phi_i(\kappa_i)$ can guarantee system $(\mathfrak{T}_{c_2})$ is SS. Notice that $\Phi_i(\kappa_i)$ is affine with respect to the time-varying delay κ_i. Since $\kappa_i\to\tau_1$ and $\kappa_i\to\tau_2$ lead to conditions (14) and (15), respectively. Combined with conditions (14) and (15), the closed-loop system $(\mathfrak{T}_{c_2})$ is SS on the basis of Lyapunov theory.

Remark 3: It is worth mentioning that the positive definiteness of LKF $V(\psi_s(t),t)$: i) Positive definiteness of $V_\iota,(\iota=1,2,3)$ are obviously; ii) It is easy to find that the proof of V_ζ's $(\zeta=4,5)$ positive definiteness in [33, Lemma 4]. Besides, it is noted that $\bar{\delta}_{ij} \geq 0$, for $j \neq i$ and $\bar{\delta}_{ii} \leq 0$, we have

$$\begin{aligned}\sum_{j=1}^{N} \bar{\delta}_{ij} \int_{t-\tau_j^t}^{t-\tau_1} \psi^\top(s) S_1 \psi(s) ds &\leq \sum_{j\neq i}^{N} \bar{\delta}_{ij} \int_{t-\tau_2}^{t-\tau_1} \psi^\top(s) S_1 \psi(s) ds \\ &= |\bar{\delta}_{ii}| \int_{t-\tau_2}^{t-\tau_1} \psi^\top(s) S_1 \psi(s) ds \\ &\leq \mu \int_{t-\tau_2}^{t-\tau_1} \psi^\top(s) S_1 \psi(s) ds.\end{aligned}$$

It is crucial for handling mode-dependent delay.

Remark 4: It is obvious that the processing of the delay term depends on the construction of the LKF. In this article, the LKF with augmented vectors $\eta(t)$ and $[\psi^\top(s)\ \dot{\psi}^\top(s)]$ is constructed, which increases the cross terms among no internal vectors and integral vectors. Therefore, compared with the existing results, the conditions derived in this article are more relaxed and less conservative.

Theorem 2: Given finite constants $\tau_1 < \tau_2$, $\varepsilon_i > 0$, $\wp_i$, and $\bar{\gamma} > 0$. For the time-varying EPs matrix $\Pi(t)$, the closed-loop system $(\mathfrak{T}_{c_2})$ is SS if, for any $i \in \mathcal{N}$ and $m \in \mathcal{N}'$, there exist a set of positive definite matrices $P_i, S_\imath, R_\imath, Q_\imath, \imath = 1, 2, 3, J_\epsilon, \epsilon = 1, 2$, appropriate dimension matrices $Z, T_1, T_2, Y, \mathcal{G}_i$, and $\mathcal{K}_m$, such that the following LMIs are satisfied:

$$\widetilde{\digamma}_i(\tau_1) = \begin{bmatrix} \widetilde{\Phi}_i(\tau_1) & -\overline{\mathcal{G}}_i^\top & \hat{P} \\ * & \bar{\delta}_{ii} P_i & \mathbf{0} \\ * & * & -\mathcal{P} \end{bmatrix} < 0 \tag{21}$$

$$\widetilde{\digamma}_i(\tau_2) = \begin{bmatrix} \widetilde{\Phi}_i(\tau_2) & -\overline{\mathcal{G}}_i^\top & \hat{P} \\ * & \bar{\delta}_{ii} P_i & \mathbf{0} \\ * & * & -\mathcal{P} \end{bmatrix} < 0 \tag{22}$$

where

$$\widetilde{\Phi}_i(\tau_1) = \begin{bmatrix} \Upsilon_i(\tau_1) + \Theta_1 + \mathbf{H}e\{\mathcal{Z}_i \widetilde{\mathcal{O}}_{im}\} + \mathbf{H}e\{\mathcal{G}_i^\top \vartheta_1^{\tau_1}\} & \mathcal{J}_1 \\ * & -\nu \otimes \mathcal{R}_1 \end{bmatrix}$$

$$\widetilde{\Phi}_i(\tau_2) = \begin{bmatrix} \Upsilon_i(\tau_2) + \Theta_2 + \mathbf{He}\{\mathcal{Z}_i \widetilde{\mathcal{O}}_{im}\} + \mathbf{He}\{\mathcal{G}_i^\top \vartheta_1^{\tau_2}\} & \mathcal{J}_2 \\ * & -\nu \otimes \mathcal{R}_2 \end{bmatrix}$$

$$\Theta_1 = \sum_{\epsilon=1}^{3} (2 - 4\epsilon) \begin{bmatrix} \varsigma_{1\epsilon} \\ \varsigma_{2\epsilon} \end{bmatrix}^\top \begin{bmatrix} \mathcal{R}_2 + R_1 & T_1 \\ * & R_1 \end{bmatrix} \begin{bmatrix} \varsigma_{1\epsilon} \\ \varsigma_{2\epsilon} \end{bmatrix}$$

$$\Theta_2 = \sum_{\epsilon=1}^{3} (2 - 4\epsilon) \begin{bmatrix} \varsigma_{1\epsilon} \\ \varsigma_{2\epsilon} \end{bmatrix}^\top \begin{bmatrix} R_1 & T_2 \\ * & \mathcal{R}_1 + R_1 \end{bmatrix} \begin{bmatrix} \varsigma_{1\epsilon} \\ \varsigma_{2\epsilon} \end{bmatrix}$$

$$\mathcal{J}_1 = [\varsigma_{11}^\top T_2\ \varsigma_{12}^\top T_2\ \varsigma_{13}^\top T_2],\ \mathcal{J}_2 = [\varsigma_{21}^\top T_1\ \varsigma_{22}^\top T_1\ \varsigma_{23}^\top T_1]$$

with

$$\mathcal{T}_i = [\sqrt{\bar{\delta}_{i1}} I, \ldots, \sqrt{\bar{\delta}_{i(i-1)}} I, \sqrt{\bar{\delta}_{i(i+1)}} I, \ldots, \sqrt{\bar{\delta}_{iN}} I]$$

$$\mathcal{P} = \mathrm{diag}\{P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, \ldots, \ldots, P_N\}$$

$$\nu = \mathrm{diag}\{1/2, 1/6, 1/10\}, Y = \mathrm{diag}\{I_p, \mathbf{0}\}$$

$$\widetilde{\mathcal{O}}_{im} = \left[\sum_{\upsilon=1}^{\upsilon_0} \sum_{m\in\mathcal{N}'} \rho_{im}^{\upsilon} \mathcal{A}_{im}, \mathbf{0}, \gamma(t)\bar{C}_i, 0, -I, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{8}\right]$$

$$\hat{P}^\top = \begin{bmatrix} \mathcal{P}\mathcal{T}_i^\top \vartheta_1^{\tau_1} & \mathbf{0} \end{bmatrix}.$$

Then, the gains K_m of the observed-mode-dependent controller (7) are given by $\bar{K}_m = Z^{-1} Y \mathcal{K}_m$.

Proof: Rewrite (14) as

$$\Gamma^\top \check{\Phi}_i(\tau_1) \Gamma < 0 \tag{23}$$

where

$$\Gamma = \begin{bmatrix} I_{13n} \\ \vartheta_1^{\tau_1} \end{bmatrix},\ \check{\Phi}_i(\tau_1) = \mathrm{diag}\{\Phi_i(\tau_1) - (\vartheta_1^{\tau_1})^\top \bar{\delta}_{ii} P_i \vartheta_1^{\tau_1}, \bar{\delta}_{ii} P_i\}.$$

By employing Projection Lemma to (23), it holds that

$$\check{\Phi}_i(\tau_1) + \mathbf{He}\{\overline{\mathcal{G}}_i^\top \Gamma_\perp\} < 0 \tag{24}$$

where $\overline{\mathcal{G}}_i = \begin{bmatrix} \mathcal{G}_i & 0_{3n\times 3n} \end{bmatrix}$, $\Gamma_\perp$ is the right null matrix of Γ, $\mathcal{G}_i \in \mathbb{R}^{3n\times 13n}$.

By Schur complement lemma, (24) is equivalent to the following

$$\digamma_i(\tau_1) = \begin{bmatrix} \overline{\Phi}_i(\tau_1) & -\mathcal{G}_i^\top & (\vartheta_1^{\tau_1})^\top \mathcal{T}_i \mathcal{P} \\ * & \bar{\delta}_{ii} P_i & \mathbf{0} \\ * & * & -\mathcal{P} \end{bmatrix} < 0 \tag{25}$$

where

$$\overline{\Phi}_i(\tau_1) = \Upsilon_i(\tau_1) + \Omega_1(\tau_1) + \mathbf{He}\{\mathcal{Z}_i \mathcal{O}_{im}\} + \mathbf{He}\{\mathcal{G}_i^\top \vartheta_1^{\tau_1}\}.$$

We prescribe the slack matrices $\mathcal{Z}_i$ implicitly in (19) as

$$\mathcal{Z}_i = [Z^\top, \mathbf{0}, \mathbf{0}, \mathbf{0}, (\varepsilon_i Z)^\top, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{8}]^\top. \tag{26}$$

Letting $Y\mathcal{K}_m = Z\bar{K}_m$ in (19), which purpose is to ensure the structure of the matrix $\bar{K}_m$. In detail

$$\begin{aligned}\bar{K}_m &= Z^{-1} Y \mathcal{K}_m \\ &= \begin{bmatrix} Z_1^{-1} & \mathbf{0} \\ \mathbf{0} & Z_2^{-1} \end{bmatrix} \begin{bmatrix} I_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathcal{K}_m \\ &= \begin{bmatrix} Z_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathcal{K}_m = \begin{bmatrix} K_m \\ \mathbf{0} \end{bmatrix}\end{aligned}$$

and the structure for Z is diagonal.

Besides, owing to there exist time-varying EPs $\Pi(t)$, (25) is not an LMI. Considering condition (6), it means

$$\overline{\digamma}_i(\tau_1) = \begin{bmatrix} \hat{\Phi}_i(\tau_1) & -\mathcal{G}_i^\top & (\vartheta_1^{\tau_1})^\top \mathcal{T}_i \mathcal{P} \\ * & \bar{\delta}_{ii} P_i & \mathbf{0} \\ * & * & -\mathcal{P} \end{bmatrix} < \mathbf{0} \tag{27}$$

where

$$\hat{\Phi}_i(\tau_1) = \Upsilon_i(\tau_1) + \Omega_1(\tau_1) + \mathbf{He}\{\mathcal{Z}_i \hat{\mathcal{O}}_{im}\} + \mathbf{He}\{\mathcal{G}_i^\top \vartheta_1^{\tau_1}\}$$

with

$$\hat{\mathcal{O}}_{im} = \left[\sum_{\upsilon=1}^{\upsilon_0} \sum_{m\in\mathcal{N}'} \hbar_\upsilon(t) \rho_{im}^{\upsilon} \mathcal{A}_{im}, 0, \gamma(t)\bar{C}_i, \mathbf{0}, -I, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{8}\right].$$

Because $\hbar_\upsilon(t)$ is semipositive definite, and by Schur complement lemma, (21) can be obtained. Similar to the abovementioned treatment, condition (22) can also be obtained. Thus, it can be confirmed that conditions (21) and (22) can guarantee the SS of the closed-loop system $(\mathfrak{T}_{c_2})$.

Remark 5: It is not hard to see conditions (14) and (15) are nonlinear with respect to the quadratic term $(\vartheta_1^{\tau_\zeta})^\top \sum_{j=1}^{N} \bar{\delta}_{ij} P_j \vartheta_1^{\tau_\zeta}$. Due to $\bar{\delta}_{ii} < 0$, such that Schur complement cannot be directly applied to perform this decoupling. This will bring difficulties in the numerical testable of the stability analysis problem. Fortunately, the powerful projection lemma [34] will be utilized and introducing a free matrix variable. This decoupling technique makes it easier to obtain solvable stability and stabilization conditions.

Remark 6: It is worth pointing out that the augmented matrix $\hat{B}_i$ is crucial. The significance of its existence lies in that the known matrices among unknown matrices can be eliminated by the inverse of matrix $\hat{B}_i$, so as to achieve the purpose of decoupling, which is also the highlight of this article.

TABLE I
PARAMETERS OF AN F-404 AIRCRAFT ENGINE SYSTEM

Parameters	$i=1$	$i=2$	$i=3$
θ_i	-0.5	1.5	2
b_{11}^i	0.15	0.15	0.15
b_{12}^i	0.12	0.2	0.2
b_{21}^i	0.15	0.13	0.14
b_{22}^i	-1.5	-1.3	-1.4
b_{31}^i	0.2	0.3	0.4
b_{32}^i	0.2	0.3	0.4

IV. SIMULATION RESULTS

In this section, the proposed controller design method will be applied to an F-404 aircraft engine system [32] to demonstrate the applicability of the developed theoretical results.

Using an F-404 aircraft engine system to describe system $(\mathfrak{T})$ with the following state-space model:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -1.46 & 0 & 2.428 \\ 0.1643 + 0.5\theta_i & -0.4 + \theta_i & -0.3788 \\ 0.3107 & 0 & -2.23 \end{bmatrix}}_{A_{h_t}} x(t)$$

$$+ \gamma(t) C_{h_t} x(t - \tau_{h_t}^t) + \underbrace{\begin{bmatrix} b_{11}^i & b_{12}^i \\ b_{21}^i & b_{22}^i \\ b_{31}^i & b_{32}^i \end{bmatrix}}_{B_{h_t}} u(t)$$

where $x(t) = [x_1^\top(t) \quad x_2^\top(t) \quad x_3^\top(t)]^\top$ in which $x_1(t)$ represents the sideslip angle, $x_2(t)$ represents the roll rate, and $x_3(t)$ is the yaw rate of the aircraft, respectively. The parameters can be found in Table 1, and the other parameters are presented as follows:

$$C_{h_t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D_{h_t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.3 \end{bmatrix}, \bar{B}_{h_t} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

The system hidden jumping modes $\{h_t\}_{t\in\mathbb{R}_{\geq 0}}$ are described by a hidden SMP, and $h_t = i \in \{1, 2, 3\}$. It assumed that the mode sojourn-time obeys Weibull distribution, when $i = 1, 2$. Let $\chi_i(\sigma) = \frac{d}{c^d}\sigma^{d-1}\exp[-(\frac{\sigma}{c})^d]$ be Weibull distribution PDF from mode i, where c, d are the scale parameter and the shape parameter, respectively. Here, when $i = 1$, we set $c = 1$ and $d = 2$. For $i = 2$, it is chosen that $c = 1$ and $d = 3$. The last mode sojourn-time obeys exponential distribution with parameter 0.5. Then, the transition probabilities matrix of EMP q_{ij} is chosen as follows

$$[q_{ij}] = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0.8 \\ 0.8 & 0.2 & 0 \end{bmatrix},$$

the time-varying EPs matrix $\Pi(t)$ is characterized by a polytope with three vertices

$$\Pi^1 = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}, \Pi^2 = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$\Pi^3 = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.5 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}.$$

The time-varying mode-dependent delays are given as $\tau_1^t = 1 + 0.4\sin(t)$, $\tau_2^t = 1 + 0.8\sin(t)$, and $\tau_3^t = 1 + 0.2\cos(t)$. This gives

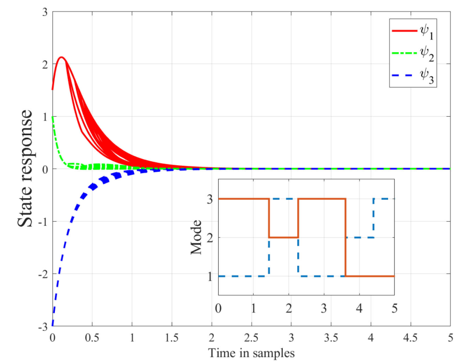

Fig. 2. State response of the closed-loop system $(\mathfrak{T}_{c_2})$ (50 realizations).

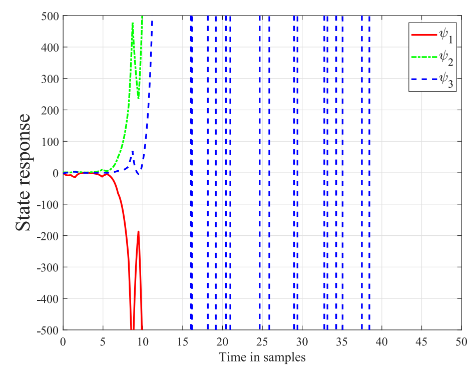

Fig. 3. State response of the open-loop system $(\mathfrak{T}_{c_2})$.

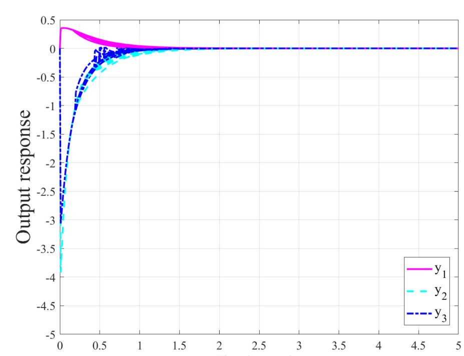

Fig. 4. Output response of the closed-loop system $(\mathfrak{T}_{c_2})$ (50 realizations).

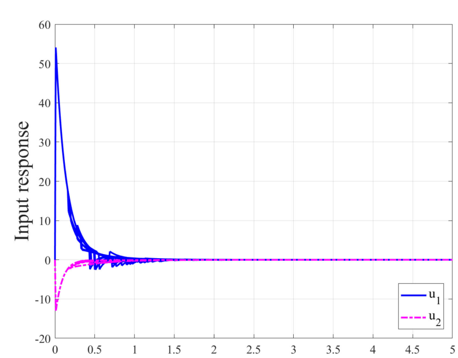

Fig. 5. Input response of the closed-loop system $(\mathfrak{T}_{c_2})$ (50 realizations).

$\tau_1 = 0.2, \tau_2 = 1.8$, and $\wp_1 = 0.4, \wp_2 = 0.8, \wp_3 = 0.2$. The probability is assumed to be $\bar{\gamma} = 0.5$.

Furthermore, its mathematical expectation is derived as

$$\mathbb{E}\{\Omega(\sigma)\} = \begin{bmatrix} -3.545 & 1.7725 & 1.7725 \\ 1.0833 & -5.4164 & 4.3331 \\ 3.2 & 1.6 & -4.8 \end{bmatrix}.$$

For system $(\mathfrak{T}_{c_2})$, the free parameters values are taken as $\varepsilon_1 = 0.5,\ \varepsilon_2 = 1,\ \varepsilon_3 = 0.5$, and the initial conditions are given as $\psi(0) = [1.5\ \ 1\ \ -3]^\top$. Then, Fig. 2 exhibits the closed-loop system state response with randomly generating 50 realizations of jumping sequences. It can be clearly observed from Fig. 2 that all the curves converges to zero. Fig. 3 shows one of 50 realizations for the open-loop system state trajectories and it is unstable. It is easy to find that the designed controller is effective.

By the inspection of Figs. 4 and 5, it is easily observed that the system output and input response with randomly generating 50 realizations of jumping sequences, respectively, converge to zero. Besides, subfigure of Fig. 2 gives the possible time sequences evolution of the values of system modes and controller modes.

V. Conclusion

In this article, a novel asynchronous stabilization approach has been proposed to solve the static output-feedback control problem for a class of hidden SMJSs with random mode-dependent delays. In the proposed approach, through carefully utilizing the information of the emission probability (EP) and choosing an appropriate LKF depend on HSM jumping parameters, a parameter-dependent sufficient and solvable criteria have been derived by improved techniques, which guarantees the stabilization for the underlying systems. The designed strategy has been verified by an F-404 aircraft engine systems. The future works will focus on the analysis and synthesis problems of two-dimension SMJSs with hidden modes.

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