

Stochastic Stability of Ito Differential Equations With Semi-Markovian Jump Parameters

Zhenting Hou, Jiaowan Luo, Peng Shi, and Sing Kiong Nguang

Abstract—In this note, the problem of stochastic stability for linear systems with jump parameters being semi-Markovian rather than full Markovian is further investigated. In particular, the system under consideration is described by Itô type nonlinear stochastic differential equations with phase type semi-Markovian jump parameters. Stochastic stability conditions are presented.

Index Terms—Itô stochastic differential equations, semi-Markovian jump parameters, stochastic stability.

I. INTRODUCTION

In engineering applications, frequently occurring dynamical systems which can be represented by different forms depending on the value of an associated Markov chain process are termed Markovian jump systems. Research into this class of systems and their applications span several decades. For some representative prior work on this general topic, we refer the reader to [1]–[4], [7], [9], [10], [13]–[18], and the references therein.

However, because the jump time of a Markov chain is, in general, exponentially distributed, Markovian jump systems have many limitations in applications, and the results obtained on the systems are conservative in some sense. In our recent paper [5], the problem of stochastic stability for linear systems with semi-Markovian jump parameters was first considered and some similar results as in the full-Markovian jump systems have been obtained, which are less conservative, thus it has wider application domains.

On the other hand, it is worth to recall that PH-distribution is the distribution of a hitting time in a finite-state, time-homogenous Markov chain. In 1954, Jensen [6] first introduced this distribution in an economics model, but no feasible solution was provided. The key that makes a PH-distribution to be a powerful tool is the matrix-analysis method developed by Neuts [11] in 1975. Since the 1960s, the PH-distribution has been a very effective method to analyze stochastic models in the queueing theory, the storage theory, the reliable theory, etc. It also replaces the special status of negative exponential distribution. However, a PH-distribution is still new, if not strange, to a lot of scientists and engineers in their research fields. Very recently, it is found out that in virtue of the phase type semi-Markov processes (defined below), the

PH-distribution has important applications in control theory, Markov decision theory, Markov game and stochastic differential equations, etc. It is believed that there are more and more applications of the PH-distribution in many areas in the near future.

In this note, the so called semi-Markovian jump systems are further studied. Our focus is on Itô nonlinear stochastic systems, described by differential equations with phase type semi-Markovian jump parameters. First the comparison principle for the nonlinear stochastic differential equations with semi-Markovian jump parameters is established. Then, using this comparison principle, we present some stochastic stability conditions/criteria, including stability in probability, asymptotic stability in probability, stability in the p th mean, asymptotic stability in the p th mean and the p th-order moment exponential stability on such class of systems.

II. PHASE TYPE SEMI-MARKOV PROCESSES AND MARKOVIZATION

Consider a Markov process $r(t)$ on the state-space $\{1, 2, \dots, m+1\}$, where the states $1, 2, \dots, m$, are transient and the state $m+1$ is absorbing. The infinitesimal generator is

$$Q = \begin{pmatrix} T & T^0 \\ 0 & 0 \end{pmatrix}$$

where T is a square matrix of order m , with $T_{ii} < 0$, $T_{ij} \geq 0$, for $i \neq j$, and such that T^{-1} exists. The n -vector T has nonnegative entries, and is equal to $-Te$. The vector e has all entries equal to one. The vector of initial probabilities is denoted by (a, a_{m+1}) , and satisfies $ae + a_{m+1} = 1$, $0 \leq a_{m+1} < 1$. First, we recall the following result.

Proposition 2.1: [18] The probability distribution $F(\cdot)$ of the time until absorption in the state $m+1$, corresponding to the initial probability vector (a, a_{m+1}) , is given by

$$F(t) = 1 - ae^{Tt}e, t \geq 0. \quad (1)$$

Throughout this note, we say the state of $r(\cdot)$ at time t is the phase of $F(\cdot)$ at time t . To begin with our study in this note, we also recall the following definitions.

Definition 2.2: [12] A probability distribution $F(\cdot)$ on $[0, \infty)$ is a continuous distribution of phase type (PH-distribution) if and only if it is the distribution of the time until absorption in a finite Markov process of the type defined above. The pair (a, T) is called a representation of order m of $F(\cdot)$.

It should be noted that there are many distributions are PH-distribution, for example, a negative exponential distribution is a continuous PH-distribution, and a k -order Erlang distribution E_k is also a continuous PH-distribution, etc.

Similar to the study on continuous-time Markov chain, we consider the discrete-time Markov chain on the state space $\{1, 2, \dots, m+1\}$, also let the state $m+1$ be absorbing, and denote the transition matrix by

$$P = \begin{pmatrix} T & T^0 \\ 0 & 1 \end{pmatrix}$$

where $T = (T_{ij})$ is a sub-stochastic matrix, $T_{ij} \geq 0$, $Te \leq e$, $T^0 = (I - T)e$ is a column vector, and $I - T$ is nonsingular. Define $a = (a_1, a_2, \dots, a_m) \geq 0$, $a_{m+1} \geq 0$, $ae + a_{m+1} = 1$.

Definition 2.3: [12] The discrete distribution taking values on the nonnegative integers space P_h is called a discrete phase type distribution ((PH-distribution), if and only if it is a distribution of the transition steps when the Markov chain, which has the transition matrix P and

Manuscript received July 15, 2003; revised November 13, 2004 and March 15, 2006. Recommended by Associate Editor S. Dey. This work was supported in part by the NNSF of China under Grant 10301036; by the Research Fund for Ph.D. Programs of MOE of China under Grant 20010533001; and by the Hunan Provincial Natural Science Foundation for Outstanding Young Scientists under Grant 04JJ1001. The work of P. Shi was supported by the Harbin Institute of Technology, Nanjing University of Aeronautics and Astronautics, and LCSIS, Institute of Automation, Chinese Academy of Sciences.

Z. Hou is with the School of Mathematics, Central South University, Hunan 410075, China (e-mail: zhou@mail.csu.edu.cn).

J. Luo is with the School of Mathematics and Information Science, Guangzhou University, Guangdong 510006, China (e-mail: jwluo@mail.csu.edu.cn).

P. Shi is with the School of Technology, University of Glamorgan, Pontypridd CF37 1DL, U.K. (e-mail: pshi@glam.ac.uk).

S. K. Nguang is with the Department of Electrical and Electronic Engineering, University of Auckland, 92019 Auckland, New Zealand (e-mail: sk.nguang@auckland.ac.nz).

Digital Object Identifier 10.1109/TAC.2006.878746

the initial distribution (\mathbf{a}, a_{m+1}) , reaches the absorbing state $m + 1$. (\mathbf{a}, T) is called its representation of order m .

Notice that a geometric distribution can be seen as a discrete PH-distribution.

In general, a continuous PH-distribution and a discrete PH-distribution are all called PH-distribution.

Proposition 2.4: [12] The PH-distributions are dense in the set of all probability distributions on $[0, +\infty)$.

Sketch of Proof: It is obvious that any probability distribution on $[0, \infty]$ may be arbitrarily closely and uniformly approximated by a discrete distribution with finite support. Such a distribution is clearly a finite mixture of degenerate distributions. Any degenerate distribution at $x = a > 0$, is the uniform limit of a sequence of Erlang distributions with mean a and increasing orders. Any probability distribution $F(\cdot)$ on $[0, \infty]$ may, therefore, be obtained as the uniform limit of a sequence of probability distributions, each of which consists of a finite mixture of Erlang distributions and possibly a jump at 0. Such distributions are clearly of phase type.

Remark 2.5: The importance of Proposition 2.4 is that, for every probability distribution on $[0, +\infty)$, we may choose a PH-distribution to approximate the original distribution in any accuracy.

Definition 2.6: Let E be a finite or countable set. A stochastic process $\hat{r}(t)$ on the state-space E is called a phase semi-Markov process or a denumerable phase semi-Markov process (when E is finite, $\hat{r}(t)$ is also called a finite phase semi-Markov process), if the following hold.

- 1) The sample paths of $(\hat{r}(t), t < +\infty)$ are right-continuous step functions and have left-hand limits with probability one.
- 2) Denote the n th jump point of the process $\hat{r}(t)$ by τ_n ($n = 0, 1, 2, \dots$), where $\tau_0 \equiv 0 < \tau_1 < \tau_2 < \dots < \tau_n < \dots$, $\tau_n \uparrow +\infty$, all τ_n ($n = 0, 1, 2, \dots$) are Markovian of the process $\hat{r}(t)$.
- 3) $F_{ij}(t) := P(\tau_{n+1} - \tau_n \leq t | r(\tau_n) = i, r(\tau_{n+1}) = j) = F_i(t)$ ($i, j \in E, t \geq 0$) do not depend on j and n .
- 4) $F_i(t)$ ($i \in E$) is a phase type distribution.

Obviously, in the case when $F_i(t)$ ($i \in E$) is a negative exponential distribution, the denumerable phase semi-Markov process is a Markov chain. A denumerable phase semi-Markov process can successfully overcome the restriction of the negative exponential distribution of the time when a Markov chain sojourns at a state. However, in the previous work on stochastic systems, in particular, systems involved Markov chain, the key problem whether a denumerable phase semi-Markov process can replace a Markov chain, is equivalent to whether the denumerable phase semi-Markov process can be transformed to a Markov chain. In fact, in the following, we will show that if the process is a finite phase semi-Markov process, then it could be transformed to a finite Markov chain.

Let E be a finite or countable nonempty set, $\hat{r}(t)$ be a denumerable phase semi-Markov process on the state-space E . Denote the n th jump point of the process $\hat{r}(t)$ by τ_n ($n = 0, 1, 2, \dots$), where $\tau_0 \equiv 0 < \tau_1 < \tau_2 < \dots < \tau_n < \dots$. Let $(\mathbf{a}^{(i)}, T^{(i)})$ ($i \in E$) denote the $m^{(i)}$ order representation of $F_i(t)$, and $E^{(i)}$ be the corresponding all transient states set (obviously, the number of the elements in $E^{(i)}$ is $m^{(i)}$), where

$$F_i(t) = P(\tau_{n+1} - \tau_n \leq t | \hat{r}(\tau_n) = i) \quad (i \in E) \quad (2)$$

$$\mathbf{a}^{(i)} = (a_1^{(i)}, a_2^{(i)}, \dots, a_{m^{(i)}}^{(i)}) \quad (3)$$

$$T^{(i)} = (T_{jk}^{(i)}, j, k \in E^{(i)}) \quad (4)$$

Let

$$p_{ij} = P(\hat{r}(\tau_{n+1}) = j | \hat{r}(\tau_n) = i) \quad (i, j \in E) \quad (5)$$

$$P = (p_{ij}, i, j \in E) \quad (6)$$

$$(\mathbf{a}, T) = \left\{ \left(\mathbf{a}^{(i)}, T^{(i)} \right), \quad i \in E \right\}. \quad (7)$$

It is ready to see that the probability distribution of $\hat{r}(t)$ can be determined only by $\{P, (\mathbf{a}, T)\}$.

To continue with our study, we introduce the following definitions.

Definition 2.7: $\{P, (\mathbf{a}, T)\}$ is called the pair of a denumerable phase semi-Markov process $\hat{r}(t)$.

For every n ($n = 0, 1, \dots$), $\tau_n \leq t < \tau_{n+1}$, define

$$J(t) = \text{the phase of } F_{\hat{r}(t)}(\cdot) \text{ at time } t - \tau_n. \quad (8)$$

Definition 2.8: $J(t)$, defined in (2.8), is called the phase of $\hat{r}(\cdot)$ at time t .

For any $i \in E$, we define

$$T_j^{(i,0)} = - \sum_{k=1}^{m^{(i)}} T_{jk}^{(i)} \quad (j = 1, 2, \dots, m^{(i)}) \quad (9)$$

$$G = \left\{ \left(i, k^{(i)} \right) \middle| i \in E, k^{(i)} = 1, 2, \dots, m^{(i)} \right\}. \quad (10)$$

From the previous analysis, we can easily get the following result.

Theorem 2.9: $Z(t) = (\hat{r}(t), J(t))$ is a Markov chain with state-space G (G is finite if and only if E is finite). The infinitesimal generator of $Z(t)$ given by $Q = (q_{\mu\nu}, \mu, \nu \in G)$ is determined only by the pair of $\hat{r}(t)$ given by $\{P, (\mathbf{a}, T)\}$ as follows:

$$\begin{cases} q_{(i,k^{(i)})((i,k^{(i)}))} = T_{k^{(i)}k^{(i)}}^{(i)}, & (i, k^{(i)}) \in G \\ q_{(i,k^{(i)})((i,\bar{k}^{(i)}))} = T_{k^{(i)}\bar{k}^{(i)}}^{(i)}, & k^{(i)} \neq \bar{k}^{(i)}, (i, k^{(i)}) \in G \\ & \text{and } (i, \bar{k}^{(i)}) \in G \\ q_{(i,k^{(i)})((j,k^{(j)}))} = p_{ij} T_{k^{(i)}k^{(j)}}^{(i,0)} a_{k^{(j)}}^{(j)}, & i \neq j, (i, k^{(i)}) \in G \\ & \text{and } (j, k^{(j)}) \in G. \end{cases} \quad (11)$$

Proof: For any $\Delta > 0$.

- 1) For any $(i, k^{(i)}) \in G$, we have

$$\begin{aligned} P \left\{ (\hat{r}(t + \Delta), J(t + \Delta)) = (i, k^{(i)}) \middle| (\hat{r}(t), J(t)) = (i, k^{(i)}) \right\} \\ = 1 + T_{k^{(i)}k^{(i)}}^{(i)} \Delta + o(\Delta). \end{aligned}$$

- 2) For any $(i, k^{(i)}) \in G, (i, \bar{k}^{(i)}) \in G, k^{(i)} \neq \bar{k}^{(i)}$, we have

$$\begin{aligned} P \left\{ (\hat{r}(t + \Delta), J(t + \Delta)) = (i, \bar{k}^{(i)}) \middle| (\hat{r}(t), J(t)) = (i, k^{(i)}) \right\} \\ = 1 + T_{k^{(i)}\bar{k}^{(i)}}^{(i)} \Delta + o(\Delta). \end{aligned}$$

- 3) For any $(i, k^{(i)}) \in G, (j, k^{(j)}) \in G, i \neq j$, we have

$$\begin{aligned} P \left\{ (\hat{r}(t + \Delta), J(t + \Delta)) = (j, k^{(j)}) \middle| (\hat{r}(t), J(t)) = (i, k^{(i)}) \right\} \\ = p_{ij} T_{k^{(i)}k^{(j)}}^{(i,0)} a_{k^{(j)}}^{(j)} \Delta + o(\Delta). \end{aligned}$$

The proof is complete.

Assume that G has $s = \sum_{i \in E} m^{(i)}$ elements, so the state space of $Z(t)$ has s elements. We number the s elements according to the following method: Denote the number of (i, k) by $\sum_{r=1}^{i-1} m^{(r)} + k$ ($1 \leq k \leq m^{(i)}$), also denote this transformation by φ , hence one has

$$\varphi((i, k)) = \sum_{r=1}^{i-1} m^{(r)} + k \quad (i \in E, 1 \leq k \leq m^{(i)}). \quad (12)$$

Moreover, we define

$$\alpha_{\varphi((i,k))\varphi((i',k'))} = q_{(i,k)(i',k')} \quad (13)$$

$$r(t) = \varphi(Z(t)). \quad (14)$$

Therefore, $r(t)$ is a Markov chain with the state space $S = \{1, 2, \dots, s\}$ and the infinitesimal generator $Q = (\alpha_{im}, 1 \leq i, m \leq s)$.

We end this section by giving the following definition.

Definition 2.10: $r(t)$ is called an associated Markov chain of $\hat{r}(t)$.

III. PROBLEM STATEMENT AND MAIN RESULTS

Let (Ω, \mathcal{F}, P) be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. Let $w(t) = (w_1(t), w_2(t), \dots, w_m(t))^T$ be an m -dimensional Brownian motion defined on the probability space. Let $\{\hat{r}(t), t \geq 0\}$ be a finite phase semi-Markovian chain on the probability space (Ω, \mathcal{F}, P) taking values in a finite state-space E , the probability distribution of $\hat{r}(t)$ can be determined by $\{P, (a, T)\}$ defined in (2.6) and (2.7).

Consider a class of stochastic differential equations with phase type semi-Markovian jump parameters in the probability space (Ω, \mathcal{F}, P) for $t > 0$

$$\begin{cases} dx(t) = \hat{f}(x(t), t, \hat{r}(t)) dt + \hat{g}(x(t), t, \hat{r}(t)) dw(t) \\ x(0) = x_0 \end{cases} \quad (15)$$

where we assume that the semi-Markov chain $\hat{r}(\cdot)$ is independent of the Brownian motion $w(\cdot)$. The initial state $x_0 \in R^n$ is a fixed nonrandom constant vector, $r(0) = r_0$. $f : R^n \times R \times E \rightarrow R^n$, $g : R^n \times R \times E \rightarrow R^{n \times m}$. Moreover, both \hat{f} and \hat{g} satisfy the local Lipschitz condition and the linear growth condition. Then, system (3.1) has a unique continuous solution. In the sequel, without loss of generality, we assume that $f(0, t, \cdot) \equiv 0$ and $g(0, t, \cdot) \equiv 0$. Thus system (3.1) has a trivial solution $x(t) \equiv 0$.

For any $\alpha \in R^n$, $\beta \in R$, $i \in E$, $k = 1, 2, \dots, m^{(i)}$, we define function h as follows:

$$h(\alpha, \beta, \varphi(i, k)) \triangleq \hat{h}(\alpha, \beta, i) \quad (16)$$

where $h = f$ or g , and φ is defined in (2.12)

It is easy to show that for any (ω, t)

$$\begin{aligned} f(x(t), t, r(\omega, t)) &\equiv \hat{f}(x(t), t, \hat{r}(\omega, t)) \\ g(x(t), t, r(\omega, t)) &\equiv \hat{g}(x(t), t, \hat{r}(\omega, t)). \end{aligned} \quad (17)$$

Subsequently, we have the following theorem.

Theorem 3.1: System (3.1) is equivalent to the following system for $t > 0$

$$\begin{cases} dx(t) = f(x(t), t, r(t)) dt + g(x(t), t, r(t)) dw(t) \\ x(0) = x_0 \end{cases} \quad (18)$$

where $r(t)$ is the associated Markov chain of phase type semi-Markovian chain $\hat{r}(t)$.

System (18) has been studied in [8] and [9]. In [8], a comparison method is used and the results in [9] have been improved. By use of Theorem 3.1 and the results of [8], we are ready to present our main result in this note.

Let $C^{2,1}(R^n \times [0, +\infty) \times S; R_+)$ denote the family of all non-negative function $V(x, t, i)$ on $R^n \times [0, +\infty) \times S$ which are continuously twice differentiable in x and once differentiable in t . For any $(x, t, i) \in R^n \times [0, +\infty) \times S$, define an operator L by

$$\begin{aligned} LV(x, t, i) &= V_t(x, t, i) + V_x(x, t, i)f(x, t, i) \\ &\quad + \frac{1}{2} \text{trace} \left[g^T(x, t, i) V_{xx}(x, t, i) g(x, t, i) \right] \\ &\quad + \sum_{j=1}^N \alpha_{ij} V(x, t, j) \end{aligned} \quad (19)$$

where α_{ij} is defined in (2.13) and

$$\begin{aligned} V_t(x, t, i) &= \frac{\partial V(x, t, i)}{\partial t} \\ V_x(x, t, i) &= \left(\frac{\partial V(x, t, i)}{\partial x_1}, \dots, \frac{\partial V(x, t, i)}{\partial x_n} \right) \\ V_{xx}(x, t, i) &= \left(\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j} \right)_{n \times n}. \end{aligned}$$

Definition 3.2: A function $\varphi(u)$ is said to belong to the class \mathcal{K} if $\varphi \in C(R^+, R^+)$, $\varphi(0) = 0$ and $\varphi(u)$ is strictly increasing in u . A function $\varphi(u)$ is said to belong to the class \mathcal{VK} if φ belongs to \mathcal{K} and φ is convex. A function $\varphi(t, u)$ is said to belong to the class \mathcal{CK} if $\varphi \in C(R^+ \times R^+, R^+)$, $\varphi(t, 0) \equiv 0$ and $\varphi(t, u)$ is concave and strictly increasing in u for each $t \in R^+$.

For convenience, we restate some results of [8] as Theorem A.

Theorem A: [8] Consider the stochastic differential equation (3.4) and the following ordinary differential equation:

$$\frac{dx(t)}{dt} = h(t, x(t)), \quad t \geq 0. \quad (20)$$

where $h : R \times R^+ \rightarrow R$ is a continuous mapping, $h(t, 0) \equiv 0$.

a) Assume that there exists a nonnegative function V such that

$$LV(x(t), t, i) \leq h(t, V(x(t), t, i)), \quad t \geq 0.$$

b) $Eh(t, \xi) \leq h(t, E\xi)$ for any n -dimensional stochastic vector ξ on the probability space (Ω, \mathcal{F}, P) .

c) There exist a function $b(x) \in \mathcal{K}$ and a function $a(t, x) \in \mathcal{CK}$ such that

$$b(\|x(t)\|) \leq V(x(t), t, i) \leq a(t, \|x(t)\|). \quad (21)$$

Then, the following hold.

- 1) The trivial solution of (20) is stable (asymptotic stable) implies that the trivial solution of (18) is stable in probability (asymptotic stable in probability). In particular, If $a(t, x) \equiv a(x)$ in (21), then the trivial solution of (20) is uniformly stable (uniformly asymptotic stable) implies that the trivial solution of (18) is uniformly stable in probability (uniformly asymptotic stable in probability).
- 2) If condition (21) is replaced by

$$b(\|x(t)\|^p) \leq V(x(t), t, i) \leq a(t, \|x(t)\|^p) \quad (22)$$

then the trivial solution of (20) is stable (asymptotic stable, exponentially stable) implies that the trivial solution of (18) is the p th moment stable (p th moment asymptotic stable, p th moment exponentially stable). In particular, If $a(t, x) \equiv a(x)$ in (22), then

the trivial solution of (20) is uniformly stable (uniformly asymptotic stable, uniformly exponentially stable) implies that the trivial solution of (18) is the p th moment uniformly stable (p th moment uniformly asymptotic stable, p th moment uniformly exponentially stable).

Now, we state our main results in this note. The proof can be completed by use of Theorem 3.1 and Theorem A and is omitted.

Theorem 3.3: Assume the following.

- a) $LV(x(t), t, i) \leq h(t, V(x(t), t, i))$, where $h : R \times R^+ \rightarrow R$ is a continuous mapping, $h(t, 0) \equiv 0$.
- b) $Eh(t, \xi) \leq h(t, E\xi)$ for any n -dimensional stochastic vector ξ on the probability space (Ω, \mathcal{F}, P) .
- c) There exist a function $b \in \mathcal{K}$ and a function $a \in \mathcal{CK}$ such that

$$b(\|x(t)\|) \leq V(x(t), t, i) \leq a(t, \|x(t)\|). \quad (23)$$

Then, the trivial solution of the following ordinary equation:

$$\frac{dx(t)}{dt} = h(t, x(t)) \quad (24)$$

is stable (asymptotic stable) implies that the trivial solution of (3.1) is stable in probability (asymptotic stable in probability).

Theorem 3.4: If $a(t, x) = a(x)$ in Theorem 3.3, then the trivial solution of (24) is uniformly stable (uniformly asymptotic stable) implies that the trivial solution of (3.1) is uniformly stable in probability (uniformly asymptotic stable in probability).

Theorem 3.5: Assume conditions a) and b) in Theorem 3.3 are satisfied. Moreover: c') there exist a function $b \in \mathcal{K}$ and a function $a \in \mathcal{CK}$ such that

$$b(\|x(t)\|^p) \leq V(x(t), t, i) \leq a(t, \|x(t)\|^p). \quad (25)$$

Then, the trivial solution of (24) is stochastic stable implies that the trivial solution of (3.1) is the p th moment stable (p th moment p th moment asymptotic stable, the p th moment exponentially stable).

Our last result in this note is stated as follows.

Theorem 3.6: If $a(t, x) = a(x)$ in Theorem 3.5, then the trivial solution of (3.10) is uniformly stable (uniformly asymptotic stable, uniformly exponentially stable) implies that the trivial solution of (3.1) is the p th moment uniformly stable (p th moment uniformly asymptotic stable, p th moment uniformly exponentially stable).

Remark 3.7: It is worth to mention that the advantages of Theorems 3.3–3.6 are that when we study stochastic stability problem, we can replace Markovian jump systems with semi-Markovian jump systems, and achieve the same results, while semi-Markovian jump systems are much less restrictive and it can be widely found and used in many real system applications. Furthermore, more importantly, almost all the nice results obtained so far on Markovian jump systems, for example, [7] and [13]–[16] are valid on semi-Markovian jump systems.

IV. NUMERICAL EXAMPLE

Let us consider the one dimension case, i.e., $x(t)$ takes values in $R = (-\infty, +\infty)$. We assume that the phase semi-Markovian process $\hat{r}(t)$ has two states denoted by 1 and 2. The sojourn time in the first state 1 is a random variable with negative exponential distribution with parameter λ_1 . The sojourn in the second state 2 is divided into two parts, the sojourn times in the two parts are two random variables which are negative exponentially distributed with parameters λ_2 and λ_3 , respectively. More specially, if the process $\hat{r}(t)$ enters the state 2, it must first stay at the first part of the state 2 for some time and then stay

at the second part, at last it will enter the state 1. We assume that $p_{12} = p_{21} = 1$. Obviously

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (26)$$

$$\mathbf{a}^{(1)} = \begin{pmatrix} a_1^{(1)} \end{pmatrix} = (1) \quad (27)$$

$$T^{(1)} = \begin{pmatrix} T_{11}^{(1)} \end{pmatrix} = (-\lambda_1) \quad (28)$$

$$\mathbf{a}^{(2)} = \begin{pmatrix} a_1^{(2)} & a_2^{(2)} \end{pmatrix} = (1, 0) \quad (29)$$

$$T^{(2)} = \begin{pmatrix} T_{11}^{(2)} & T_{12}^{(2)} \\ T_{21}^{(2)} & T_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} -\lambda_2 & \lambda_2 \\ 0 & -\lambda_3 \end{pmatrix}. \quad (30)$$

It is easy to know that the state space of $Z(t) = (\hat{r}(t), J(t))$ is $G = ((1, 1), (2, 1), (2, 2))$. We number all the elements of G as follows:

$$\left. \begin{aligned} \varphi((1, 1)) &= 1 \\ \varphi((2, 1)) &= 2 \\ \varphi((2, 2)) &= 3. \end{aligned} \right\} \quad (31)$$

Hence, the infinitesimal generator of $\varphi(Z(t))$ is

$$Q = (\alpha_{ij}) = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_3 \end{pmatrix}. \quad (32)$$

We assume that both \hat{f} and \hat{g} satisfy the local Lipschitz condition, i.e., for each $k = 1, 2, \dots$, there exists an h_k such that

$$\left| \hat{f}(x, t, i) - \hat{f}(\bar{x}, t, i) \right| + |\hat{g}(x, t, i) - \hat{g}(\bar{x}, t, i)| \leq h_k |x - \bar{x}| \quad (33)$$

for all $t \geq 0$, $i = 1, 2$ and those $x, \bar{x} \in R$ with $|x| \vee |\bar{x}| \leq k$.

Furthermore, for each $i = 1, 2, 3$, there exist constants $\beta_i \in R$ and $\sigma_i \geq 0$ such that

$$2x^T \hat{f}(x, t, i) \leq \beta_i |x|^2 \quad |\hat{g}(x, t, i)|^2 \leq \sigma_i |x|^2 \quad (34)$$

where $x \in R$, $t \geq 0$.

Thus, for f and g , we have

$$\begin{aligned} f(x, t, 1) &= \hat{f}(x, t, 1) & f(x, t, 2) &= \hat{f}(x, t, 2) \\ f(x, t, 3) &= \hat{f}(x, t, 2). \end{aligned}$$

Denote the Lyapunov function by $V(x(t), t, i) = q_i |x(t)|^2$ where q_i ($i = 1, 2, 3$) is a positive constant. So we get

$$LV(x, t, i) \leq \frac{1}{q_i} \left\{ q_i(\beta_i + \sigma_i) + \sum_{j=1}^3 \alpha_{ij} q_j \right\} V(x(t), t, i). \quad (35)$$

Hence, if there exist three positive constants q_i , $i = 1, 2, 3$, such that

$$\frac{1}{q_i} \left\{ q_i(\beta_i + \sigma_i) + \sum_{j=1}^3 \alpha_{ij} q_j \right\} \leq -1 \quad (36)$$

then

$$LV(x, t, i) \leq -V(x(t), t, i). \quad (37)$$

Because of the trivial solution of the following equation:

$$\frac{dx(t)}{dt} = -x(t) \quad (38)$$

is exponentially stable, by Theorems 3.3 and 3.5, we have that the trivial solution of (3.1) is asymptotic stable in probability, mean square asymptotic stable, and mean square exponentially stable.

V. CONCLUSION

In this note, a new approach has been established to study the problem of stochastic stability for a class of nonlinear stochastic systems with semi-Markovian jump parameters. It has been shown that the existing results on stochastic stability for Markovian jump systems also hold for semi-Markovian jump systems. The semi-Markovian jump systems are less conservative and more applicable in real practices. A numerical example is given to illustrate the feasibility and effectiveness of the theoretic results obtained.

ACKNOWLEDGMENT

The authors would like to thank the Associate Editor, Prof. S. Dey, and the referees, for their very helpful comments and suggestions which have greatly improved the presentation of this note.

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On the Observability of Linear Differential-Algebraic Systems With Delays

V. M. Marchenko, O. N. Poddubnaya, and Z. Zaczekiewicz

Abstract—The problem of \mathbb{R}^n -observability is considered for the simplest linear time-delay differential-algebraic system consisting of differential and difference equations. A determining equation system is introduced and a number of algebraic properties of the determining equation solutions is established, in particular, the well-known Hamilton–Cayley matrix theorem is generalized to the solutions of determining equation. As a result, an effective parametric rank criterion for the \mathbb{R}^n -observability is given. A dual controllability result is also formulated.

Index Terms—Determining equations, differential-algebraic systems, duality, observability, time-delay.

I. INTRODUCTION

The note deals with linear stationary differential-algebraic systems with delays (DAD systems), with some equations being differential, the other—difference, with some variables being continuous the other—piecewise continuous (see also [1]–[5]). Observe that some kinds of neutral type time-delay and discrete-continuous hybrid systems can be regarded as examples of DAD systems.

Example 1: Consider a linear neutral type time-delay system

$$\frac{d}{dt}(y(t) - A_{22}y(t-h)) = A_{11}y(t) + A_{12}y(t-h). \quad (1)$$

If we denote $x(t) = y(t) - A_{22}y(t-h)$, we obtain the following DAD system:

$$\begin{aligned} \dot{x}(t) &= A_{11}x(t) + (A_{11}A_{22} + A_{12})y(t-h) \\ y(t) &= x(t) + A_{22}y(t-h). \end{aligned}$$

Example 2: Consider the following linear discrete-continuous system:

$$\dot{x}(t) = A_{11}x(t) + A_{12}y[k], \quad t \in [kh, (k+1)h) \quad (2a)$$

$$y[k] = A_{21}x(kh) + A_{22}y[k-1], \quad k = 0, 1, \dots \quad (2b)$$

with initial conditions

$$x(0) = x(0+) = x_0 \quad y[-1] = y_0,$$

Manuscript received May 20, 2004; revised February 17, 2005 and September 9, 2005. Recommended by Associate Editor E. Jonckheere. This work was supported by the KBN under Bialystok Technical University Grant W/IMF/3/05 and by the Belarusian State Technological University "Math Structure" Grant.

V. M. Marchenko is with the Institute of Mathematics and Physics, Bialystok Technical University, 15-351 Bialystok, Poland, and also with the Department of Higher Mathematics, Belarusian State Technological University, 220630 Minsk, Belarus (e-mail: vmar@bstu.unibel.by).

O. N. Poddubnaya is with the Department of Higher Mathematics, Belarusian State Technological University, 220630 Minsk, Belarus (e-mail: olesya@bstu.unibel.by).

Z. Zaczekiewicz is with the Institute of Mathematics and Physics, Bialystok Technical University, 15-351 Bialystok, Poland (e-mail: pbzaczki@pb.bialystok.pl).

Digital Object Identifier 10.1109/TAC.2006.876803