

# Robust $\mathcal{H}_\infty$ Filtering for Uncertain Markovian Jump Linear Systems\*

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## ABSTRACT

This paper investigates the problem of  $\mathcal{H}_\infty$  filtering for a class of uncertain Markovian jump linear systems. The uncertainty is assumed to be norm-bounded and appears in all the matrices of the system state-space model, including the coefficient matrices of the noise signals. It is also assumed that the jumping parameter is available. We develop a methodology for designing a Markovian jump linear filter that ensures a prescribed bound on the  $\mathcal{L}_2$ -induced gain from the noise signals to the estimation error, irrespective of the uncertainty. The proposed design is given in terms of linear matrix inequalities.

## 1. INTRODUCTION

The  $\mathcal{H}_\infty$  filtering approach is concerned with the design of an estimation procedure which ensures that the  $\mathcal{L}_2$ -induced gain from the noise signals to the estimation error will be less than a prescribed level. In  $\mathcal{H}_\infty$  filtering, the noise sources are arbitrary signals with bounded energy, or bounded average power, which renders this approach insensitive to the noise statistics, and thus very appropriate to applications where the statistics of the noise signals are not exactly known; see, e.g. [1], [2], [12], [16]-[18], [23] and the references therein. The potential of  $\mathcal{H}_\infty$  filtering lies far beyond its insensitivity to the noise statistics. It has been recognized in [18] that the  $\mathcal{H}_\infty$  filtering scheme is also less sensitive than its  $\mathcal{H}_2$  counterpart to uncertainty in the system parameters.

Recently, there have been a lot of interest on the problem of  $\mathcal{H}_\infty$  filtering for systems with parameter uncertainty. It has been established that the  $\mathcal{H}_\infty$  approach can guarantee a prescribed bound to the  $\mathcal{L}_2$ -induced gain from the noise signals to the estimation error in the presence of norm-bounded parameter uncertainty ([11]). Filtering schemes for uncertain linear systems have been developed in [7], [11] and [22].

Over the past few years, Markovian jump linear systems have been attracting an increasing attention in the literature. This class of systems is very appropriate to model plants whose structure is subject to random abrupt changes due to, for instance, component and/or interconnections failures, sudden environment changes, modification of the operating point of a linearized model of a nonlinear system, etc. A number of estimation and control problems

related to these systems has been analysed by several authors (see, for example, [3]-[6], [9], [14], [15], [19]-[21] and the references therein). In particular with regard to estimation, minimum variance filtering schemes for discrete-time systems have been studied in, for instance, [3], [4] and [20], whereas the  $\mathcal{H}_\infty$  filtering is addressed in [6].

In this paper we consider the problem of  $\mathcal{H}_\infty$  filtering for a class of uncertain Markovian jump linear systems. This class of systems is described by a nominal Markovian jump linear system subject to real norm-bounded parameter uncertainty, which appears in all the matrices of the system state-space model, including the coefficient matrices of the noise signals. It is assumed that the jumping parameter is available. The problem we address is the design of a Markovian jump linear filter with a mean square stable error dynamics and which provides a prescribed bound on the  $\mathcal{L}_2$ -induced gain from the noise signals to the estimation error, irrespective of the uncertainty. This problem will be referred to as *robust Markovian jump  $\mathcal{H}_\infty$  filtering*. A linear matrix inequality (LMI) approach is developed for solving this robust  $\mathcal{H}_\infty$  filtering problem. The results obtained in this paper extend those in [6] to deal with norm-bounded uncertainty in the state-space model of Markovian jump linear systems.

**Notation.** Throughout the paper the superscript " $T$ " stands for matrix transposition,  $\mathbb{R}^n$  denotes the  $n$  dimensional Euclidean space,  $\mathbb{R}^{n \times m}$  is the set of all  $n \times m$  real matrices,  $I_n$  is the  $n \times n$  identity matrix,  $\|\cdot\|$  denotes the induced matrix 2-norm, and  $\mathcal{L}_2$  stands for the space of square integrable vector functions over  $[0, \infty)$ . For a real matrix  $P$ ,  $P > 0$  (respectively,  $P < 0$ ), means that  $P$  is symmetric and positive definite (respectively, negative definite).

## 2. PROBLEM FORMULATION

Fix a complete probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  and consider the following class of dynamic systems:

$$\begin{aligned} \dot{x}(t) &= [A(\theta_t) + \Delta A(\theta_t)]x(t) + [B(\theta_t) + \Delta B(\theta_t)]w(t); \\ x(0) &= x_0, \quad \theta_0 = i_0 \end{aligned} \quad (1)$$

$$y(t) = [C(\theta_t) + \Delta C(\theta_t)]x(t) + [D(\theta_t) + \Delta D(\theta_t)]w(t) \quad (2)$$

$$z(t) = L(\theta_t)x(t) \quad (3)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $x_0 \in \mathbb{R}^n$  is an unknown initial state,  $w(t) \in \mathbb{R}^q$  is the noise signal which is assumed to be an arbitrary signal in  $\mathcal{L}_2$ ,  $y(t) \in \mathbb{R}^m$  is the measure-

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ment, and  $z(t) \in \mathbb{R}^p$  is the signal to be estimated.  $\{\theta_t\}$  is a continuous-time Markov process with right continuous trajectories and taking values on a finite set  $\phi = \{1, 2, \dots, N\}$  with stationary transition probabilities:

$$\text{Prob}\{\theta_{t+h} = j | \theta_t = i\} = \begin{cases} \lambda_{ij}h + o(h), & i \neq j \\ 1 + \lambda_{ii}h + o(h), & i = j \end{cases}$$

where  $h > 0$  and  $\lambda_{ij} \geq 0$  is the transition rate from the state  $i$  to  $j$ ,  $i \neq j$ , and

$$\lambda_{ii} = - \sum_{\substack{j=1 \\ j \neq i}}^N \lambda_{ij}. \quad (4)$$

$A(\theta_t), B(\theta_t), C(\theta_t), D(\theta_t)$  and  $L(\theta_t)$  are known real matrices for all  $\theta_t \in \phi$ , and  $\Delta A(\theta_t), \Delta B(\theta_t), \Delta C(\theta_t)$  and  $\Delta D(\theta_t)$  are unknown matrices representing parameter uncertainties. The admissible uncertainties are assumed to be of the form:

$$\begin{aligned} \Delta A(\theta_t) &= H_1(\theta_t)F_1(\theta_t)E_1(\theta_t), \quad \Delta C(\theta_t) = H_2(\theta_t)F_2(\theta_t)E_2(\theta_t) \\ \Delta B(\theta_t) &= H_3(\theta_t)F_3(\theta_t)E_3(\theta_t), \quad \Delta D(\theta_t) = H_4(\theta_t)F_4(\theta_t)E_4(\theta_t) \end{aligned}$$

where  $F_i(\theta_t) \in \mathbb{R}^{m_i \times n_i}$ ,  $i = 1, \dots, 4$ , are unknown real matrices satisfying

$$\|F_i(\theta_t)\| \leq 1, \quad \forall t \geq 0, \quad i = 1, \dots, 4 \quad (5)$$

and  $H_i(\theta_t)$  and  $E_i(\theta_t)$ ,  $i = 1, \dots, 4$ ,  $\forall \theta_t \in \phi$ , are known real matrices of appropriate dimensions.

The set  $\phi$  comprises the various operation modes of system  $(\Sigma)$  and for each possible value of  $\theta_t = i$ ,  $i \in \phi$ , we will denote the matrices associated with the “ $i$ -th mode” by:

$$\begin{aligned} A_i &\triangleq A(\theta_t = i), \quad B_i \triangleq B(\theta_t = i), \quad C_i \triangleq C(\theta_t = i), \\ D_i &\triangleq D(\theta_t = i), \quad L_i \triangleq L(\theta_t = i), \\ E_{ji} &\triangleq E_j(\theta_t = i), \quad H_{ji} \triangleq H_j(\theta_t = i); \quad j = 1, \dots, 4 \end{aligned}$$

where  $A_i, B_i, C_i, D_i, L_i$  and  $E_{ji}, H_{ji}$ ,  $j = 1, \dots, 4$ , are constant matrices for any  $i \in \phi$ .

It is assumed that no *a priori* estimate of the initial state,  $x_0$ , is available and the jumping process,  $\{\theta_t\}$ , is accessible, i.e. the operation mode of system  $(\Sigma)$  is known for every  $t \geq 0$ .

The filtering problem we will address is to obtain an estimate,  $\hat{z}(t)$ , of  $z(t)$  via a causal Markovian jump linear filter which provides a uniformly small estimation error,  $z(t) - \hat{z}(t)$ , for all  $w \in \mathcal{L}_2$ , irrespective of the uncertainty.

In order to put the robust Markovian jump  $\mathcal{H}_\infty$  filtering problem in a stochastic setting, we introduce the space  $\mathcal{L}_2[\Omega, \mathcal{F}, \mathcal{P}]$  of  $\mathcal{F}$ -measurable processes,  $z(t) - \hat{z}(t)$ , for which

$$\|z - \hat{z}\|_2 = \left\{ E \left[ \int_0^\infty (z - \hat{z})^T (z - \hat{z}) dt \right] \right\}^{\frac{1}{2}} \leq \infty \quad (6)$$

where  $E[\cdot]$  denotes mathematical expectation. For the sake of notation simplification, we shall use indistinctly  $\|\cdot\|_2$  to

denote the norm either in  $\mathcal{L}_2[\Omega, \mathcal{F}, \mathcal{P}]$  or in  $\mathcal{L}_2$ , defined by

$$\|w\|_2 \triangleq \left[ \int_0^\infty w^T w dt \right]^{\frac{1}{2}}, \quad \text{for } w \in \mathcal{L}_2.$$

Attention will be focused on the design of an  $n$ -order filter. Since the jumping process  $\{\theta_t\}$  is available for  $t \geq 0$  and in view of the structure of system  $(\Sigma)$ , without loss of generality, the following structure for the Markovian jump linear filter will be adopted:

$$(\Sigma_f): \quad \dot{\hat{x}}(t) = G(\theta_t)\hat{x}(t) + K(\theta_t)y(t); \quad \hat{x}(0) = 0 \quad (7)$$

$$\hat{z}(t) = L(\theta_t)\hat{x}(t) \quad (8)$$

where the matrix functions  $G(\theta_t)$  and  $K(\theta_t)$  are to be determined.

This paper is concerned with the following robust filtering problem:

*Design a Markovian jump filter of the form of (7)-(8) such that the estimation error system is robustly mean square stable and*

$$\|z - \hat{z}\|_2 \leq \gamma (\|w\|_2^2 + x_0^T R x_0)^{\frac{1}{2}} \quad (9)$$

for all  $w \in \mathcal{L}_2$ ,  $x_0 \in \mathbb{R}^n$ , and for all possible uncertainties, where  $R > 0$  is a given weighting matrix for the initial state and  $\gamma \geq 0$  is a given scalar which specifies the level of “noise” attenuation in the estimation error.

In the above, *robust mean square stability* is defined in the following sense:

**Definition 2.1** Consider the following Markovian jump system

$$\begin{aligned} \dot{x}(t) &= [A(\theta_t) + \Delta A(\theta_t)]x(t) + [B(\theta_t) + \Delta B(\theta_t)]w(t); \\ x(0) &= x_0, \quad \theta_0 = i_0 \end{aligned}$$

where  $\Delta A(\theta_t)$  and  $\Delta B(\theta_t)$  are norm-bounded uncertain matrices for all possible  $\theta_t$ . This system is said to be *robustly mean square stable*, if it is *internally mean square stable* for all admissible  $\Delta A(\theta_t)$  and  $\Delta B(\theta_t)$ , i.e. if the solution of

$$\dot{x}(t) = [A(\theta_t) + \Delta A(\theta_t)]x(t) \quad (10)$$

is such that  $E(\|x(t)\|^2) \rightarrow 0$ , as  $t \rightarrow \infty$  for arbitrary initial condition  $(x_0, \theta_0)$  and for all admissible uncertainties.  $\square$

The weighting matrix  $R$  in (9) is a measure of the uncertainty in the initial state,  $x_0$ , relative to the uncertainty in  $w$ . A “large” value of  $R$  indicates that the initial state is very close to zero. In the case of filtering problems where the effect of the initial state is ignored,  $x_0$  is set to zero and the performance measure of (9) is replaced by

$$\|z - \hat{z}\|_2 \leq \gamma \|w\|_2. \quad (11)$$

Note that (11) can be viewed as the limit of (9) as the smallest eigenvalue of  $R$  approaches infinity.

**Remark 2.1** We observe that no “non-singularity assumption” is imposed to the filtering problem treated in this paper, as is the case with the robust  $\mathcal{H}_\infty$  filtering approaches in [7], [11] and [22] for uncertain linear systems without jumps.  $\square$

### 3. THE ROBUST $\mathcal{H}_\infty$ MARKOVIAN JUMP FILTER

First, note that by defining  $\tilde{x} \triangleq x - \hat{x}$ , it follows from (1), (2) and (7) that

$$\begin{aligned} \dot{\tilde{x}}(t) = & G(\theta_t)\tilde{x} + \{A(\theta_t) + \Delta A(\theta_t) - K(\theta_t)[C(\theta_t) + \Delta C(\theta_t)] \\ & - G(\theta_t)\}x(t) + \{B(\theta_t) + \Delta B(\theta_t) \\ & - K(\theta_t)[D(\theta_t) + \Delta D(\theta_t)]\}w(t), \quad \tilde{x}(0) = x_0. \end{aligned} \quad (12)$$

Hence, it follows that a state-space model for the estimation error,  $z - \hat{z}$ , in terms of  $\tilde{x}$  and  $x$  is given by

$$\begin{aligned} \dot{\xi}(t) = & [\hat{A}(\theta_t) + \hat{H}_1(\theta_t)F_1(\theta_t)\hat{E}_1(\theta_t) + \hat{H}_2(\theta_t)F_2(\theta_t)\hat{E}_2(\theta_t)]\xi(t) \\ & + [\hat{B}(\theta_t) + \hat{H}_3(\theta_t)F_3(\theta_t)\hat{E}_3(\theta_t) + \hat{H}_4(\theta_t)F_4(\theta_t)\hat{E}_4(\theta_t)]w(t) \end{aligned} \quad (13)$$

$$z(t) - \hat{z}(t) = \hat{L}(\theta_t)\xi(t) \quad (14)$$

where  $\xi = [\tilde{x}^T \ x^T]^T$  and

$$\hat{A}(\theta_t) = \begin{bmatrix} G(\theta_t) & A(\theta_t) - K(\theta_t)C(\theta_t) - G(\theta_t) \\ 0 & A(\theta_t) \end{bmatrix} \quad (15)$$

$$\hat{B}(\theta_t) = \begin{bmatrix} B(\theta_t) - K(\theta_t)D(\theta_t) \\ B(\theta_t) \end{bmatrix}, \quad \hat{E}_1(\theta_t) = \begin{bmatrix} 0 \\ E_1^T(\theta_t) \end{bmatrix}^T \quad (16)$$

$$\hat{E}_2(\theta_t) = [0 \ E_2(\theta_t)], \quad \hat{E}_3(\theta_t) = E_3(\theta_t) \quad (17)$$

$$\hat{E}_4(\theta_t) = E_4(\theta_t), \quad \hat{L}(\theta_t) = [L(\theta_t) \ 0] \quad (18)$$

$$\hat{H}_1(\theta_t) = \begin{bmatrix} H_1(\theta_t) \\ H_1(\theta_t) \end{bmatrix}, \quad \hat{H}_2(\theta_t) = \begin{bmatrix} -K(\theta_t)H_2(\theta_t) \\ 0 \end{bmatrix} \quad (19)$$

$$\hat{H}_3(\theta_t) = \begin{bmatrix} H_3(\theta_t) \\ H_3(\theta_t) \end{bmatrix}, \quad \hat{H}_4(\theta_t) = \begin{bmatrix} -K(\theta_t)H_4(\theta_t) \\ 0 \end{bmatrix}. \quad (20)$$

In the sequel, for each possible value of  $\theta_t = i$ ,  $i \in \phi$ , we denote

$$\hat{A}_i \triangleq \hat{A}(\theta_t = i), \quad \hat{B}_i \triangleq \hat{B}(\theta_t = i), \quad \hat{L}_i \triangleq \hat{L}(\theta_t = i) \quad (21)$$

$$\hat{H}_{ji} \triangleq \hat{H}_j(\theta_t = i), \quad \hat{E}_{ji} \triangleq \hat{E}_j(\theta_t = i); \quad j = 1, \dots, 4. \quad (22)$$

Our first result deals with the problem of analysing if a given Markovian jump filter as in (7)-(8) provides a robustly mean square stable error dynamics and satisfies the performance constraint of (9) for all admissible uncertainties.

**Theorem 3.1** Consider the system  $(\Sigma)$  and let  $R > 0$  be a given initial state weighting matrix and  $\gamma > 0$  a given scalar. Given a Markovian jump linear filter of the form of (7)-(8), then the corresponding estimation error system is robustly mean square stable and

$$\|z - \hat{z}\|_2 \leq \gamma (\|w\|_2^2 + x_0^T R x_0)^{\frac{1}{2}}$$

for all  $w \in \mathcal{L}_2$ ,  $x_0 \in \mathbb{R}^n$ , and for all admissible uncertainty, if for all  $i \in \phi$  there exist matrices  $P_i > 0$  and positive scalars  $\alpha_i$ ,  $\beta_i$ ,  $\delta_i$  and  $\varepsilon_i$  satisfying the following inequalities:

$$\begin{bmatrix} Q_i + \sum_{j=1}^N \lambda_{ij} P_j & P_i \hat{B}_i \\ \hat{B}_i^T P_i & -S_i \end{bmatrix} < 0, \quad \forall i \in \phi \quad (23)$$

$$[I_n \ I_n] P_{i_0} [I_n \ I_n]^T - \gamma^2 R \leq 0 \quad (24)$$

where  $i_0$  is the state assumed by the jumping process  $\{\theta_t\}$  at  $t = 0$ , and for all  $i \in \phi$ :

$$\begin{aligned} Q_i = & P_i \hat{A}_i + \hat{A}_i^T P_i + \alpha_i \hat{E}_{1i}^T \hat{E}_{1i} + \beta_i \hat{E}_{2i}^T \hat{E}_{2i} + \hat{L}_i^T \hat{L}_i \\ & + P_i \left( \frac{1}{\alpha_i} \hat{H}_{1i} \hat{H}_{1i}^T + \frac{1}{\beta_i} \hat{H}_{2i} \hat{H}_{2i}^T + \frac{1}{\delta_i} \hat{H}_{3i} \hat{H}_{3i}^T + \frac{1}{\varepsilon_i} \hat{H}_{4i} \hat{H}_{4i}^T \right) P_i \end{aligned} \quad (25)$$

$$S_i = \gamma^2 I - \delta_i \hat{E}_{3i}^T \hat{E}_{3i} - \varepsilon_i \hat{E}_{4i}^T \hat{E}_{4i}. \quad (26)$$

Before proceeding to the proof of Theorem 3.1, we recall the following well known inequality:

**Proposition 3.1** Let  $F, X$  and  $Y$  be real matrices of appropriate dimensions with  $\|F\| \leq 1$ . Then for any scalar  $\varepsilon > 0$ ,

$$X F Y + Y^T F^T X^T \leq \frac{1}{\varepsilon} X X^T + \varepsilon Y^T Y.$$

**Proof of Theorem 3.1.** First note that in view of (23)

$$\begin{aligned} P_i \hat{A}_i + \hat{A}_i^T P_i + \alpha_i \hat{E}_{1i}^T \hat{E}_{1i} + \beta_i \hat{E}_{2i}^T \hat{E}_{2i} + P_i \left( \frac{1}{\alpha_i} \hat{H}_{1i} \hat{H}_{1i}^T \right. \\ \left. + \frac{1}{\beta_i} \hat{H}_{2i} \hat{H}_{2i}^T \right) P_i + \sum_{j=1}^N \lambda_{ij} P_j < 0, \quad \forall i \in \phi. \end{aligned} \quad (27)$$

Considering Proposition 3.1, it follows from (27) that for any matrices  $F_{1i}$ , and  $F_{2i}$ ,  $\forall i \in \phi$ , of appropriate dimensions and satisfying  $\|F_{1i}\| \leq 1$  and  $\|F_{2i}\| \leq 1$ :

$$\begin{aligned} P_i (\hat{A}_i + \hat{H}_{1i} F_{1i} \hat{E}_{1i} + \hat{H}_{2i} F_{2i} \hat{E}_{2i}) + (\hat{A}_i + \hat{H}_{1i} F_{1i} \hat{E}_{1i} \\ + \hat{H}_{2i} F_{2i} \hat{E}_{2i})^T P_i + \sum_{j=1}^N \lambda_{ij} P_j < 0, \quad \forall i \in \phi. \end{aligned}$$

By Theorem 3.1 and Proposition 3.5 in [8], this implies that the error system of (13) is internally mean square stable for all  $F_1(\theta_t)$  and  $F_2(\theta_t)$ , satisfying (5).

Now for  $T > 0$ , introduce the cost function:

$$J(T) = E \left\{ \int_0^T [(z - \hat{z})^T (z - \hat{z}) - \gamma^2 w^T w] dt \right\}. \quad (28)$$

Note that by Proposition A.1 in [5],  $\{\xi(t), \theta_t\}$  is a Markov process with infinitesimal operator given by

$$\begin{aligned} \tilde{V} \cdot g(\xi(t), \theta_t) = & \{ \xi^T(t) [\hat{A}(\theta_t) + \hat{H}_1(\theta_t)F_1(\theta_t)\hat{E}_1(\theta_t) \\ & + \hat{H}_2(\theta_t)F_2(\theta_t)\hat{E}_2(\theta_t)]^T + w^T(t) [\hat{B}(\theta_t) + \hat{H}_3(\theta_t)F_3(\theta_t)\hat{E}_3(\theta_t) \\ & + \hat{H}_4(\theta_t)F_4(\theta_t)\hat{E}_4(\theta_t)]^T \} g_\xi(\xi(t), \theta_t) + \sum_{j=1}^N \lambda_{\theta_t j} g(\xi(t), j) \\ & + g_\theta(\xi(t), \theta_t) \end{aligned} \quad (29)$$

where  $g(\xi(t), \theta_t)$  is a real continuous, bounded, functional with partial derivatives:

$$g_t \triangleq \frac{\partial g}{\partial t} \quad \text{and} \quad g_\xi \triangleq \left[ \frac{\partial g}{\partial \xi_1}, \frac{\partial g}{\partial \xi_2}, \dots, \frac{\partial g}{\partial \xi_n} \right]^T$$

where  $\xi_j$  denotes the  $j$ -th component of  $\xi$ .

Adding and subtracting  $\tilde{V} \cdot [\xi^T(t)P(\theta_t)\xi(t)]$  to (28), where  $P(\theta_t) = P_i$  when  $\theta_t = i$ ,  $i \in \phi$ , and with  $P_i$  satisfying (23)-(24), and considering (14) and (29) we have:

$$\begin{aligned} J(T) = E \left\{ \int_0^T \left\{ \xi^T(t) [(\hat{A}(\theta_t) + \Delta \hat{A}(\theta_t))^T P(\theta_t) \right. \right. \\ + P(\theta_t)(\hat{A}(\theta_t) + \Delta \hat{A}(\theta_t)) + \sum_{j=1}^N \lambda_{\theta_t j} P_j + \hat{L}^T(\theta_t) \hat{L}(\theta_t)] \xi(t) \\ - \gamma^2 w^T(t) w(t) + \xi^T(t) P(\theta_t) [\hat{B}(\theta_t) + \Delta \hat{B}(\theta_t)] w(t) \\ \left. \left. w^T(t) [\hat{B}(\theta_t) + \Delta \hat{B}(\theta_t)]^T P(\theta_t) \xi(t) - \tilde{V} \cdot [\xi^T(t) P(\theta_t) \xi(t)] \right\} dt \right\} \end{aligned} \quad (30)$$

where

$$\begin{aligned} \Delta \hat{A}(\theta_t) &\triangleq \hat{H}_1(\theta_t) F_1(\theta_t) \hat{E}_1(\theta_t) + \hat{H}_2(\theta_t) F_2(\theta_t) \hat{E}_2(\theta_t) \\ \Delta \hat{B}(\theta_t) &\triangleq \hat{H}_3(\theta_t) F_3(\theta_t) \hat{E}_3(\theta_t) + \hat{H}_4(\theta_t) F_4(\theta_t) \hat{E}_4(\theta_t). \end{aligned}$$

Using Dynkin's formula (see, e.g., [13]) and considering Proposition 3.1, it follows from (30) that

$$\begin{aligned} J(T) \leq E \left\{ \int_0^T \left\{ \xi^T(t) [Q(\theta_t) + \sum_{j=1}^N \lambda_{\theta_t j} P_j] \xi(t) \right. \right. \\ + \xi^T(t) P(\theta_t) \hat{B}(\theta_t) w(t) + w^T(t) \hat{B}^T(\theta_t) P(\theta_t) \xi(t) \\ \left. \left. - w^T(t) S(\theta_t) w(t) \right\} dt \right\} - E \left\{ \xi^T(T) P(\theta_T) \xi(T) \right\} \\ + \xi^T(0) P(\theta_0) \xi(0) \end{aligned} \quad (31)$$

where

$$\begin{aligned} Q(\theta_t) &\triangleq P(\theta_t) \hat{A}(\theta_t) + \hat{A}^T(\theta_t) P(\theta_t) + \alpha(\theta_t) \hat{E}_1^T(\theta_t) \hat{E}_1(\theta_t) \\ &+ \beta(\theta_t) \hat{E}_2^T(\theta_t) \hat{E}_2(\theta_t) + \hat{L}^T(\theta_t) \hat{L}(\theta_t) \\ &+ P(\theta_t) [\alpha^{-1}(\theta_t) \hat{H}_1(\theta_t) \hat{H}_1^T(\theta_t) + \beta^{-1}(\theta_t) \hat{H}_2(\theta_t) \hat{H}_2^T(\theta_t) \\ &+ \delta^{-1}(\theta_t) \hat{H}_3(\theta_t) \hat{H}_3^T(\theta_t) + \varepsilon_4^{-1}(\theta_t) \hat{H}_4(\theta_t) \hat{H}_4^T(\theta_t)] P(\theta_t) \end{aligned}$$

$$S(\theta_t) \triangleq \gamma^2 I - \delta(\theta_t) \hat{E}_3^T(\theta_t) \hat{E}_3(\theta_t) - \varepsilon(\theta_t) \hat{E}_4^T(\theta_t) \hat{E}_4(\theta_t)$$

where  $\alpha(\theta_t)$ ,  $\beta(\theta_t)$ ,  $\delta(\theta_t)$  and  $\varepsilon(\theta_t)$  assume positive values  $\alpha_i$ ,  $\beta_i$ ,  $\delta_i$  and  $\varepsilon_i$ , respectively, when  $\theta_t = i$ ,  $i \in \phi$ .

Observe that since  $\hat{x}(0) = 0$  and  $\theta_0 = i_0$ ,

$$\begin{aligned} \xi^T(0) P(\theta_0) \xi(0) &= \begin{bmatrix} x_0^T & x_0^T \end{bmatrix} P_{i_0} \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} \\ &= x_0^T \begin{bmatrix} I_n & I_n \end{bmatrix} P_{i_0} \begin{bmatrix} I_n & I_n \end{bmatrix}^T x_0. \end{aligned} \quad (32)$$

It was shown in [10] that, since the system of (13) is robustly mean square stable and  $w \in \mathcal{L}_2$ , it follows that

$$\lim_{T \rightarrow \infty} E [\xi^T(T) P(\theta_T) \xi(T)] = 0. \quad (33)$$

This implies that  $J(T)$  of (28) is well defined as  $T \rightarrow \infty$ .

Now, considering (32) and (33), it results from (31) that

$$\begin{aligned} \|z - \hat{z}\|_2^2 - \gamma^2 (\|w\|_2^2 + x_0^T R x_0) &\leq E \left[ \int_0^\infty \eta^T \Xi(\theta_t) \eta dt \right] \\ &+ x_0^T \{ \begin{bmatrix} I_n & I_n \end{bmatrix} P_{i_0} \begin{bmatrix} I_n & I_n \end{bmatrix}^T - \gamma^2 R \} x_0 \end{aligned} \quad (34)$$

where

$$\begin{aligned} \eta &= [\xi^T \quad w^T]^T, \\ \Xi(\theta_t) &= \begin{bmatrix} Q(\theta_t) + \sum_{j=1}^N \lambda_{\theta_t j} P_j & P(\theta_t) \hat{B}(\theta_t) \\ \hat{B}^T(\theta_t) P(\theta_t) & -S(\theta_t) \end{bmatrix}. \end{aligned}$$

Finally, denoting  $Q_i(P_i, \alpha_i, \beta_i, \delta_i, \varepsilon_i) \triangleq Q(\theta_t = i)$  and  $S_i(\gamma, \delta_i, \varepsilon_i) \triangleq S(\theta_t = i)$ ,  $i \in \phi$ , and considering the inequalities (23) and (24), the results follows from (34).  $\nabla \nabla \nabla$

The next result presents a solution to the robust  $\mathcal{H}_\infty$  filtering problem for Markovian jump linear systems in terms of linear matrix inequalities.

**Theorem 3.2** Consider the system  $(\Sigma)$  and let  $R > 0$  be a given initial state weighting matrix and  $\gamma > 0$  a given scalar. Then there exists a Markovian jump filter of the form of (7)-(8) such that the estimation error system is robustly mean square stable and

$$\|z - \hat{z}\|_2 \leq \gamma (\|w\|_2 + x_0^T R x_0)^{\frac{1}{2}}$$

for all  $w \in \mathcal{L}_2$ ,  $x_0 \in \mathbb{R}^n$ , and for all admissible uncertainties, if for all  $i \in \phi$  there exist matrices  $X_i > 0$ ,  $Z_i > 0$ ,  $Y_i$  and  $W_i$ , and positive scalars  $\alpha_i$ ,  $\beta_i$ ,  $\delta_i$  and  $\varepsilon_i$  satisfying the following LMIs:

$$\begin{bmatrix} M_i & T_i & \hat{T}_i & U_i \\ T_i^T & N_i & Z_i B_i & V_i \\ \hat{T}_i^T & B_i^T Z_i & -S_i & 0 \\ U_i^T & V_i^T & 0 & -J_i \end{bmatrix} < 0, \quad \forall i \in \phi \quad (35)$$

$$X_{i_0} + Z_{i_0} - \gamma^2 R \leq 0 \quad (36)$$

where  $i_0$  is the state assumed by  $\{\theta_i\}$  at  $t = 0$ , and for all  $i \in \phi$ :

$$M_i = W_i + W_i^T + \sum_{j=1}^N \lambda_{ij} X_j + L_i^T L_i \quad (37)$$

$$T_i = X_i A_i - Y_i C_i - W_i, \quad \hat{T}_i = X_i B_i - Y_i D_i \quad (38)$$

$$N_i = Z_i A_i + A_i^T Z_i + \sum_{j=1}^N \lambda_{ij} Z_j + \alpha_i E_{1i}^T E_{1i} + \beta_i E_{2i}^T E_{2i} \quad (39)$$

$$S_i = \gamma^2 I - \delta_i E_{3i}^T E_{3i} - \varepsilon_i E_{4i}^T E_{4i} \quad (40)$$

$$U_i = [X_i H_{1i} \quad -Y_i H_{2i} \quad X_i H_{3i} \quad -Y_i H_{4i}] \quad (41)$$

$$V_i = [Z_i H_{1i} \quad 0 \quad Z_i H_{3i} \quad 0] \quad (42)$$

$$J_i = \text{diag} \{ \alpha_i I, \beta_i I, \delta_i I, \varepsilon_i I \}. \quad (43)$$

Moreover, a suitable filter is given by

$$\hat{x}(t) = G_i \hat{x}(t) + K_i y(t); \quad \hat{x}(0) = 0 \quad (44)$$

$$\hat{z}(t) = L_i \hat{x}(t) \quad (45)$$

for  $\theta_t = i$ ,  $i \in \phi$ , where

$$G_i = X_i^{-1} W_i, \quad K_i = X_i^{-1} Y_i, \quad \forall i \in \phi. \quad (46)$$

*Proof.* First, note that by using Schur complements the inequality (23) of Theorem 3.1 is equivalent to:

$$\begin{bmatrix} \hat{Q}_i + \sum_{j=1}^N \lambda_{ij} P_j & P_i \hat{B}_i & P_i \hat{H}_i \\ \hat{B}_i^T P_i & -S_i & 0 \\ -\hat{H}_i^T P_i & 0 & -J_i \end{bmatrix} < 0, \quad \forall i \in \phi \quad (47)$$

where  $\hat{B}_i$ ,  $S_i$  and  $J_i$  are as in (21), (40) and (43), respectively, and

$$\begin{aligned} \hat{Q}_i &= P_i \hat{A}_i + \hat{A}_i^T P_i + \alpha_i \hat{E}_{1i}^T \hat{E}_{1i} + \beta_i \hat{E}_{2i}^T \hat{E}_{2i} + \hat{L}_i^T \hat{L}_i \\ \hat{H}_i &= [\hat{H}_{1i} \quad \hat{H}_{2i} \quad \hat{H}_{3i} \quad \hat{H}_{4i}] \end{aligned}$$

where  $\hat{A}_i$ ,  $\hat{L}_i$ ,  $\hat{E}_{1i}$ ,  $\hat{E}_{2i}$  and  $\hat{H}_{ji}$ ,  $j = 1, \dots, 4$ , are given in (21) and (22).

Let  $P_i \triangleq \text{diag}\{X_i, Z_i\}$ ,  $\forall i \in \phi$ , where  $X_i$  and  $Z_i$  are symmetric positive definite matrices to be found. Denoting

$$Y_i = X_i K_i, \quad W_i = X_i G_i, \quad \forall i \in \phi \quad (48)$$

where  $K_i \triangleq K(\theta_t = i)$  and  $G_i \triangleq G(\theta_t = i)$ ,  $\forall i \in \phi$ , and considering (15)-(20) it can be easily shown that (47) is equivalent to (35). Moreover, we also have that (36) ensures the condition (24) of Theorem 3.1. Further, by (48),

$$G_i = X_i^{-1} W_i, \quad K_i = X_i^{-1} Y_i, \quad \forall i \in \phi.$$

Hence, by Theorem 3.1 we conclude that the filter of (44)-(46) solves the robust  $\mathcal{H}_\infty$  filtering problem.  $\nabla\nabla\nabla$

**Remark 3.1** Theorem 3.2 provides a method for designing a robust  $\mathcal{H}_\infty$  Markovian jump linear filter for Markovian jump linear systems subject to norm-bounded parameter uncertainty. The proposed design is based on LMIs which has the advantage that can be solved numerically very efficiently using recently developed algorithms for solving LMIs.

We observe that the problem of finding the robust  $\mathcal{H}_\infty$  filter of Theorem 3.2 for the smallest possible  $\gamma \geq 0$  can be easily solved in terms of the following linear programming problem in  $\gamma$ ,  $X_i$ ,  $Z_i$ ,  $Y_i$ ,  $W_i$ ,  $\alpha_i$ ,  $\beta_i$ ,  $\delta_i$  and  $\varepsilon_i$ ,  $\forall i \in \phi$ :

$$\begin{aligned} &\text{minimize} \quad \gamma \\ &\text{subject to} \quad \gamma \geq 0, \quad X_i > 0, \quad Z_i > 0, \quad \alpha_i > 0, \quad \beta_i > 0, \\ &\quad \delta_i > 0, \quad \varepsilon_i > 0, \quad (35) \quad \text{and} \quad (36). \quad \square \end{aligned}$$

When the effect of the initial state of system  $(\Sigma)$  is not taken into account, which can be viewed as having  $x(0)$  that is certain to be zero, the inequality of (36) will no longer be required as this case corresponds to choosing a sufficient large  $R$  (in the sense that its smallest eigenvalue approaches infinity). In such situation, Theorem 3.2 specializes as follows:

**Corollary 3.1** Consider the system  $(\Sigma)$  with  $x(0) = 0$  and let  $\gamma > 0$  a given scalar. Then there exists a filter of the form of (7)-(8) such that the estimation error system is robustly mean square stable and

$$\|z - \hat{z}\|_2 \leq \gamma \|w\|_2$$

for all  $w \in \mathcal{L}_2$  and for all admissible uncertainties, if for all  $i \in \phi$  there exist matrices  $X_i > 0$ ,  $Z_i > 0$ ,  $Y_i$  and  $W_i$ , and

positive scalars  $\alpha_i$ ,  $\beta_i$ ,  $\delta_i$  and  $\varepsilon_i$  satisfying the LMI (35). Moreover, a suitable filter is given by (44)-(46).  $\square$

In the case of one mode operation, i.e. there no jumps in system  $(\Sigma)$ , we have  $N = 1$ ,  $\phi = \{1\}$  and  $\lambda_{11} = 0$ . Denoting the matrices of system  $(\Sigma)$  by  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $L$ ,  $E_j$ ,  $H_j$ ,  $j = 1, \dots, 4$ , Theorem 3.2 reduces to the following result:

**Corollary 3.2** Consider the system  $(\Sigma)$  with no jumps and let  $R > 0$  be a given initial state weighting matrix and  $\gamma > 0$  a given scalar. Then there exists a causal linear filter such that the estimation error system is robustly stable and

$$\|z - \hat{z}\|_2 \leq \gamma (\|w\|_2 + x_0^T R x_0)^{\frac{1}{2}}$$

for all  $w \in \mathcal{L}_2$ ,  $x_0 \in \mathcal{R}^n$ , and for all admissible uncertainties, if there exist matrices  $X > 0$ ,  $Z > 0$ ,  $Y$  and  $W$ , and positive scalars  $\alpha$ ,  $\beta$ ,  $\delta$  and  $\varepsilon$  satisfying the following LMIs:

$$\begin{bmatrix} \Omega & \Lambda & XB - YD & U \\ \Lambda^T & \Psi & ZB & V \\ B^T X - D^T Y^T & B^T Z & -\Lambda & 0 \\ U^T & V^T & 0 & -J \end{bmatrix} < 0 \quad (49)$$

$$X + Z - \gamma^2 R \leq 0 \quad (50)$$

where

$$\Omega = W + W^T + L^T L, \quad \Lambda = XA - YC - W$$

$$\Psi = ZA + A^T Z + \alpha E_1^T E_1 + \beta E_2^T E_2$$

$$\Gamma = \gamma^2 I - \delta E_3^T E_3 - \varepsilon E_4^T E_4$$

$$U = [XH_1 \quad -YH_2 \quad XH_3 \quad -YH_4],$$

$$V = [ZH_1 \quad 0 \quad ZH_3 \quad 0]$$

$$J = \text{diag}\{\alpha I, \beta I, \delta I, \varepsilon I\}.$$

Moreover, a suitable filter is given by

$$\dot{\hat{x}}(t) = G\hat{x}(t) + Ky(t); \quad \hat{x}(0) = 0$$

$$\hat{z}(t) = L\hat{x}(t)$$

where

$$G = X^{-1}W, \quad K = X^{-1}Y.$$

**Remark 3.2** Corollary 3.2 provides a methodology for designing robust  $\mathcal{H}_\infty$  filters for a class of uncertain linear systems in terms of linear matrix inequalities. This is in contrast with the approach developed in [22] which is based on algebraic Riccati equations and is also restricted to "non-singular" robust  $\mathcal{H}_\infty$  filtering problems.  $\square$

## 4. CONCLUSIONS

In this paper we have investigated the problem of robust  $\mathcal{H}_\infty$  filtering for a class of Markovian jump linear system with norm-bounded parameter uncertainty and unknown initial state. The  $\mathcal{H}_\infty$  filter derived here ensures robust mean square stability for the estimation error and a prescribed bound on the  $\mathcal{L}_2$ -induced gain from the noise signals to the estimation error irrespective of the uncertainty, while preserves the system feature of having Markovian jumps, i.e. the filter is a Markovian jump linear system as well. The main result is tailored via linear matrix inequalities.

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