the so-called multilevel schemes existing today involve iterations of one form or another.

In the conclusion, Pearson states that the suggested approximations work quite well. It would be most helpful if he could provide some evidence or references to verify this statement, as well as being more explicit about the approximations themselves.

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Author's Reply[5]

Masak is quite correct in his conclusions that my approximations are not exact. As far as decomposition is concerned, a further "published attempt" with Reich shows that for a cascaded structure of 15 third-order systems, i.e., $n=45$, decomposition produces the optimal open-loop control with less work than the Riccati solution.[6] Unfortunately, here a feedback solution is desired, and the approximation methods are the attempts to obtain it.

Assuming μ and ν constant in (11) and (12) with[7] $l_1=l_2=0$ is a standard trick[8] equivalent to assuming the reference signal fixed for the purpose of designing the feedforward. Thus $l_1(\pi_{k-1}, \pi_k)$ and $l_2(\pi_{k-1}, \pi_k)$ in (8) and (9) essentially define $\delta x^0(\pi_{k-1}, \pi_k)$, which can be used to satisfy in some manner $\delta x_k = \delta y_{k+1}$, with δy_{k+1} defined by observation of the $(k+1)$th vehicle. Approximation 2 is merely integral action, based on the error between vehicle spacing, which is not too critical. The equation $d\pi_i/d\sigma = \delta y_{i+1} - \delta x_i^0$ is included only for comparison with a proposed iteration which turns out to be fairly useless compared with the conjugate gradient method.[6] An example has appeared elsewhere.[9] Neither approximation suggests iteration.

Finally, at the very least, these techniques reduce to proportional integral control of a stable system a problem that is not well known for its intransigence.

Although these tricks amount to very little, they represent an *attempt* (possibly unsuccessful) to solve the large-scale structured feedback problem as it should be solved. I would be delighted if somebody would produce an alternative method, which was the point of my first correspondence.

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[5] Manuscript received September 15, 1967.
[6] J. D. Pearson and S. Reich, "Decomposition of large optimal control problems," *Proc. IEE* (London), vol. 114, June 1967.
[7] Equation numbers of the original correspondence,[1] in which l_1 and l_2 should read $\bar{l}_1$ and $\bar{l}_2$.
[8] I believe it dates back to Kalman and Englar, "Fundamental study of adaptive control systems, I," Wright-Patterson AFB, Dayton, Ohio, Tech. Rept. ASD-TR-61-27, 1961.
[9] J. D. Pearson, "Multilevel control systems," presented at the IFAC Symp. on Adaptive Control, London, 1965.

On an Iterative Technique for Riccati Equation Computations

In this correspondence an iterative technique for solving the linear regulator problem with infinite terminal time is presented. The scheme is based on the method of successive substitutions, which has been applied elsewhere to solve linear regulator problems,[1]–[3] albeit under the assumption of a finite terminal time. The results presented below are based on research reported by this author,[3],[5] in which successive substitution methods were applied to solve the infinite time problem. By using the concept of a *cost matrix*, it is directly shown that the iterations are monotonically convergent. Recently, Puri and Gruver[4] have obtained similar results by applying concepts of the Hamilton–Jacobi theory. This latter technique gives an interesting and different insight to the nature of the iterative scheme.

The linear time-invariant, completely controllable system is described by the state equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0. \tag{1}$$

As is well known,[3],[6] the control $u^*(\cdot)$ that minimizes the quadratic cost functional

$$J(x_0; u(\cdot)) = \int_0^\infty [x'(t)C'Cx(t) + u'(t)\overset{D^*D}{R}u(t)]dt \tag{2}$$

where $R>0$[1] and the pair $[A, C]$ is completely observable, is given by the linear feedback law

$$u^*(x(t)) = -R^{-1}B'Kx(t) = -L^*x(t) \tag{3}$$

where K is the unique positive definite solution of the algebraic equation

$$0 = KA + A'K + C'C - KBR^{-1}B'K. \tag{4}$$

In addition, K has the property

$$J(x_0; u^*(\cdot)) = \min_{u(\cdot)} J(x_0; u(\cdot)) = x_0'Kx_0. \tag{5}$$

Before presenting the iterative scheme for determining K, the concept of a cost matrix is introduced. Suppose that $u_L(x(t)) = -Lx(t)$ is an arbitrary feedback law. If $u_L(\cdot)$ is applied to the system (1), the resulting cost (2) may be written as

$$J(x_0; u_L(\cdot)) = x_0'V_Lx_0 \tag{6}$$

where V_L is defined as the cost matrix associated with the feedback gains L and is given by[2] $\dot{x}=(A-BL)x \Rightarrow x=x_0e^{(A-BL)t}$ 代入(2)式

$$\boxed{V_L = \int_0^\infty e^{(A-BL)'t}(C'C + L'RL) \cdot e^{(A-BL)t}dt.} \tag{7}$$

V_L is finite if and only if the closed-loop system matrix $A-BL$ has eigenvalues with negative real parts. In this case, V_L is the unique (positive definite) solution of the linear equation

$$\boxed{0 = (A-BL)'V + V(A-BL) + C'C + L'RL.} \tag{8}$$

Lastly, if L_1 and L_2 are gain matrices with associated cost matrices V_1 and V_2, it can be shown[5] that

$$V_1 - V_2 = \int_0^\infty e^{(A-BL_2)'\tau}[(L_1-L_2)'R(L_1-L_2) - (L_1-L_2)'(B'V_1-RL_2) - (B'V_1-RL_2)'(L_1-L_2)] \cdot e^{(A-BL_2)\tau}d\tau \tag{9}$$

or, alternatively,

$$V_1 - V_2 = \int_0^\infty e^{(A-BL_1)'\tau}[(L_1-L_2)'R(L_1-L_2) - (L_1-L_2)'(B'V_2-RL_2) - (B'V_2-RL_2)'(L_1-L_2)] \cdot e^{(A-BL_1)\tau}d\tau. \tag{10}$$

Care must be exercised in using the above formula if V_1 and/or V_2 are unbounded.

We now state and prove the main result.

Theorem

Let V_k, $k=0, 1, \cdots$, be the (unique) positive definite solution of the linear algebraic equation

$$0 = A_k'V_k + V_kA_k + C'C + L_k'RL_k \tag{11}$$

where, recursively,

$$L_k = R^{-1}B'V_{k-1}, \qquad k = 1, 2, \cdots$$
$$A_k = A - BL_k$$

and where L_0 is chosen such that the matrix $A_0 = A - BL_0$ has eigenvalues with negative real parts. Then

1) $K \le V_{k+1} \le V_k \le \cdots$, $k=0, 1, \cdots$
2) $\lim_{k\to\infty} V_k = K$.

Proof

1) Let V_0 be the cost matrix associated with L_0. V_0 then satisfies (11) with $k=0$. Now set $L_1 = R^{-1}B'V_0$ and let V_1 be its associated cost matrix. Using (9) we obtain

$$V_0 - V_1 = \int_0^\infty e^{A_1'\tau}(L_0-L_1)'R(L_0-L_1)e^{A_1\tau}d\tau \ge 0$$

so that $V_1 \le V_0$. In addition, we have by (10)

$$V_1 - K = \int_0^\infty e^{A_1'\tau}(L_1-L^*)R(L_1-L^*)e^{A_1\tau}d\tau \ge 0.$$

Hence V_1 is also bounded below and therefore has finite norm. Thus A_1 has eigenvalues with negative real parts, and so V_1 satisfies (11) with $k=1$. Repeating the above argument for $k=2, 3, \cdots$ yields the desired result 1).

2) $\lim_{k\to\infty} V_k = V_\infty$ exists by a theorem on monotonic convergence of positive operators (Kantorovich and Akilov,[7] p. 189). Thus by taking the limit of (11) as $k\to\infty$ we obtain

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[1] If X and Y are positive semidefinite, the notation $X>Y[X\ge Y]$ means that the matrix $X-Y$ is positive [semi] definite.
[2] Note that the cost matrix associated with $L^* = R^{-1}BK$ is simply K.

$$0 = A'V_\infty + V_\infty A + C'C - V_\infty BR^{-1}B'V_\infty. \quad (12)$$

Since K is the unique positive definite solution of (12), $V_\infty = K$ and the proof is completed. Q.E.D.

REMARKS

1) Since the system (1) is completely controllable, it is always possible to choose an L, such that Re $\{\lambda_i(A_0)\} < 0$.[2],[8] It is necessary for Re $\{\lambda_i(A_0)\} < 0$ to insure the boundedness of the cost matrix V_0. Otherwise the iterations may converge to an indefinite solution of (4), if they converge at all.

2) It can be shown that the iterative scheme embodied in (11) is precisely that which is obtained by applying Newton's method (in function spaces) to solve (4) (Kleinman,[5] Appendix E). However, Newton's method alone will not provide conditions that will insure monotonic convergence.

3) In addition to being monotonically convergent, the iterates V_k are also quadratically convergent. This is expected from Newton's method and can be shown directly by taking the matrix norm of (10) with $L_1 = R^{-1}B'V_k = L_k$, $L_2 = R^{-1}B'K$. It can be shown that, uniformly in k,

$$\int_0^\infty \|e^{A_k\tau}\|^2 d\tau \leq c_1$$

so that

$$\boxed{\|V_{k+1} - K\| \leq c_2 \|V_k - K\|^2.} \quad (13)$$

Therefore, convergence of V_k to K is rapid in the vicinity of K. This is not the case in most other iterative schemes (ASP program, Runge–Kutta, etc.), which display only linear convergence to K.

4) The theorem provides a useful iterative scheme for the numerical solution of (4) and the associated regulator problem. If the number of state variables n is small, V_k may be obtained by solving an $n(n+1)/2$-dimensional linear vector equation. A computer program for finding K in this manner has been written for up to 10 state variables. Convergence to K is rapid with approximately 10 iterations of (11) needed in most cases.

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REFERENCES

[1] R. Bellman and R. Kalaba, "Quasi-linearization and nonlinear boundary value problems," The Rand Corp., Santa Monica, Calif., Rept. R-438, 1965.
[2] W. M. Wonham, "Lecture notes on stochastic control," Center for Dynamical Systems, Brown University, Providence, R. I., Lecture Notes 67-2, 1967.
[3] D. L. Kleinman, "On the linear regulator problem and the matrix Riccati equation," M.I.T. Electronic Systems Lab., Cambridge, Mass., Rept. 271, 1966.
[4] N. N. Puri and W. A. Gruver, "Optimal control design via successive approximations," Preprints, Joint Automatic Control Conf., Philadelphia, Pa., June 1967, pp. 335–344.
[5] D. L. Kleinman, "Suboptimal design of linear regulator systems subject to computer storage limitations," M.I.T. Electronic Systems Lab., Cambridge, Mass., Rept. 297, 1967.
[6] M. Athans and P. F. Falb, *Optimal Control.* New York: McGraw-Hill, 1966.
[7] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces.* New York: Macmillan, 1964.
[8] W. A. Wolovich, "On the stabilization of controllable systems" (to be published).

Simultaneous Generation of First- and Higher-Order Gradient Functions of a Cost Function

Some promising applications of the first-, second-, and higher-order gradient functions of a cost function for a linear system may be made in the gradient method of hill-climbing processes, where an efficient computer optimization may be achieved if these gradient functions are made available simultaneously. Sinusoidal perturbation signals[1] and binary m sequences[2] have been used extensively to derive the first-order gradient function. Recently, the use of three-level m sequences[3] has been described to generate the first-order and second-order gradient functions simultaneously, but the results for the higher-order gradient functions are subject to the complications inherent in the generation of higher-order m sequences.

A method is presented here for the simultaneous generation of all the necessary gradient functions using the low-frequency sinusoidal perturbation signals.

DESCRIPTION OF METHOD

The gradient function of order m of a given cost function is defined as

$$g_m(K) = \frac{\partial^m P(K)}{\partial K^m} \quad (1)$$

where $P(K)$ is a cost function for a given linear system with stationary inputs, and $g_m(K)$ represents the mth-order gradient function with respect to the variable parameter K.

If the variable parameter is perturbed by a sinusoidal signal $\Delta K \cos \omega_p t$, the perturbed parameter is given by

$$K(t) = K + \Delta K \cos \omega_p t. \quad (2)$$

The dynamic response in the cost function may be evaluated knowing the dynamics of the process and the statistics of the input signal. It has been shown[4] that if

$$\omega_p T \leq 0.1$$

where T is the system time constant, the dynamic effect of the parameter perturbation signal transmission path, from parameter variation to response in the cost function, is negligible.

Thus, for low perturbation frequencies, the response in the cost function may be evaluated simply by Taylor expansion which yields

$$P(K + \Delta K \cos \omega_p t) = P(K) + \sum_{m=1,2,3\cdots} (\Delta K \cos \omega_p t)^m \frac{1}{m!} g_m(K). \quad (3)$$

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[1] P. H. Hammond and M. J. Duckenfield, "Automatic optimization by continuous perturbation of parameters," *Automatica*, vol. 1, pp. 147–175, 1963.

[2] J. L. Douce and K. C. Ng, "The use of pseudo-random binary test signals in process optimization," *Proc. IFAC Symp. on the Theory of Self-Adaptive Control Systems.* New York: Plenum Press (for the Instrument Society of America), 1966.

[3] D. W. Clarke and K. B. Godfrey, "Simultaneous estimation of first and second derivatives of a cost function," *Electronics Lett.* vol., 2, p. 338, 1966.

[4] J. L. Douce, K. C. Ng, and M. M. Gupta, "Dynamics of the parameter perturbation process," *Proc. IEE* (London), vol. 113, no. 6, pp. 1077–1083, 1966.

Define l_n as the average of the cross product of the perturbed cost function $P(K(t))$ and the nth harmonic of the perturbation signal,

$$l_n = \langle P(K + \Delta K \cos \omega_p t) \cos n\omega_p t \rangle_{\text{avg}}. \quad (4)$$

The first-order and other successive higher-order gradient functions may be obtained from (4). For example, for the first-order gradient function, putting $n = 1$,

$$l_1 = \langle P(K + \Delta K \cos \omega_p t) \cdot \cos \omega_p t \rangle_{\text{avg}}. \quad (5a)$$

Substituting (3) in (4) it can be shown that

$$l_1 = \tfrac{1}{2}\Delta K g_1(K) + \sum_{m=3,5\cdots} \frac{1}{m!} (\Delta K)^m g_m(K) \cdot \langle (\cos \omega_p t)^{m+1} \rangle_{\text{avg}}. \quad (5b)$$

For $\Delta K/K \ll 1$, assuming that

$$\frac{2}{m!} (\cos \omega_p t)^{m+1} (\Delta K)^{m-1} \frac{g_m(K)}{g_1(K)} \ll 1, \quad m = 3, 5, 7 \cdots$$

it yields

$$l_1 \simeq \tfrac{1}{2}\Delta K g_1(K) = c_1 g_1(K)$$

or

$$g_1(K) = l_1/c_1 \quad (6)$$

where $c_1 = \frac{1}{2}\Delta K$. In (5b) the convergence of the series is guaranteed assuming the convergences of the Taylor expansion in (3).

The second-order gradient function may be generated by putting $n = 2$ in (4). Thus, again for $\Delta K/K \ll 1$, neglecting the higher-order terms, $g_2(K)$ is approximated to

$$g_2(K) \simeq \frac{8 l_2}{\Delta K^2} = l_2/c_2 \quad (7)$$

where $c_2 = \Delta K^2/8$.

The process, as shown in Fig. 1, may be continued for the generation of the higher-order gradient functions. The technique may be extended for multivariable parameter cost function, using uncorrelated low-frequency sinusoidal perturbation signals. In practice, the usefulness of the method, for more than a couple of the parameters, will be limited by the cross-coupling effect due to the generation of the low-frequency cross-modulating terms during the process of the perturbation.

The estimates of the gradient functions depend upon the amplitude of the perturbation signal ΔK. A large amplitude of the perturbation signal will cause large errors in the estimate of the gradients due to the significant coupling effect of the higher-order gradients, and at the same time large ΔK will create large disturbances in the system. For small ΔK, there is a possibility of large experimental errors in the measurement of these gradients. However, the method seems to have promising applications over other techniques. First, such signals are easy to generate and second, it does not introduce unwanted disturbances in the system as introduced by the multilevel sequences. The gradient estimation under