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A Unified Approach to Linear Estimation for Discrete-Time Systems-Part II: H_∞ Estimation

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Abstract

This paper focuses on developing a unified state-space approach to the H_∞ filtering, multi-step prediction and fixed-lag smoothing. It is shown that the H_∞ estimation is in fact an H_2 estimation for an associated system with current and delayed measurements in Krein space. A necessary and sufficient condition for the existence of an H_∞ estimator is therefore derived using an innovation analysis method together with projection in Krein space. More importantly, the approach leads to a solution for the long standing H_∞ fixed-lag smoothing problem. Our approach does not require augmenting the state of the system and is similar to the forward and backward procedure in the H_2 fixed-lag smoothing. The H_∞ steady state estimation is also solved in terms of a Riccati equation of the same dimension as the system state.

1 Introduction

The problem of estimation includes filtering, prediction and smoothing and has been one of the key research topics of control community since the paper by Wiener [11]. The Kalman filter [3], which addresses the minimization of filtering error covariance, emerged as a major tool of state estimation in the 1960s. It is well known that the state filtering and prediction can be solved in terms of a Riccati difference equation in the finite horizon case and an algebraic Riccati equation in the infinite horizon case. The smoothing problem is classified into three categories by Mendel [4], namely fixed-point smoothing, fixed-interval smoothing and fixed-lag smoothing. The fixed-interval smoothing is to seek optimal estimates at some or all interior points based on a fixed time interval of measurements. The fixed-point smoothing is to find an optimal estimate at a fixed time instant with continuing measurements ahead of the point estimation. The fixed-lag smoothing is concerned with estimating a signal at a fixed length of time back in the past. Of the three different smoothing problems, the fixed-lag one is the most complicated. The optimal fixed-lag smoothing in the H_2 sense can be solved by applying the Kalman filtering to an augmented system or using a forward-backward procedure [4].

On the other hand, the H_∞ estimation has become another important estimation method since the 1980s. An H_∞ estimator is such that the ratio between the energy of estimation error and the energy of input noise is bounded by a prescriped level γ^2; see [5] for the continuous-time case and [8] for the discrete-time one, and is applicable to situations where no information on statistics of input noises is available. Most of the existing work on the H_∞ estimation concentrates on filtering and one step ahead prediction [7, 9]. Very recently, [2] considered the H_∞ multi-step ahead prediction without resorting to system augmentation for the first time, where the estimator is derived by the minimization of certain quadratic function. It is worth noting that different derivation approaches have been adopted for filtering and multi-step prediction.

For the smoothing problem under H_∞ performance, the fixed-interval smoothing and fixed-point smoothing have been shown to be identical to the H_2 smoother; see, for example, [5, 10]. The H_∞ fixed-lag smoothing problem, where an estimate of the current state is sought based on measurements in a finite interval ahead, remains the least investigated in the H_∞ sense. It was first discussed in [1] for scalar systems. In [9], the H_∞ smoothing is addressed through system augmentation and filtering which is computationally expensive especially when the dimension of the state is high and/or the estimation lag is large. Recall that

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the H_2 fixed-lag smoothing can be approached by the forward and backward procedure [4]. To our knowledge, there has been no counterpart in the H_∞ fixed-lag smoothing.

The purpose of the present paper is twofold. First, we shall seek a unified solution to the H_∞ filtering, multi-step prediction and fixed-lag smoothing. Secondly, we aim to present a solution for the H_∞ fixed-lag smoothing similar to the forward and backward approach of the H_2 smoothing. The general tool to be applied is the innovation analysis and projection in Krein space. It is shown that the H_∞ estimation is in fact an H_2 one for systems with current and delayed measurements in Krein space. Our approach offers the advantages that it treats the filtering, prediction and smoothing in a unified framework and thus resolves the H_∞ fixed-lag smoothing problem. It is also worth pointing out that our approach gives a simpler drivation and solution to the multi-step prediction than the existing solution [2]. The steady-state case will also be investigated.

2 Problem Statement

Consider a linear time-variant system described by the following discrete-time model:

$$\begin{aligned} x(t+1) &= \Phi_t x(t) + \Gamma_t u(t) && (1)\\ y(t) &= H_t x(t) + v(t) && (2)\\ z(t) &= L_t x(t) && (3) \end{aligned}$$

where $x(t) \in R^n, u(t) \in R^r, y(t) \in R^m$, $v(t) \in R^m$ and $z(t) \in R^p$ represent the state, input noise, measurement output, measurement noise and the signal to be estimated, respectively. It is assumed that the input and measurement noises are deterministic signals and are from $\ell_2[0, N]$ where N is the time-horizon of the estimation problem under investigation.

The H_∞ estimation problem under investigation is stated as follows: *Given a scalar $\gamma > 0$, an integer l and the observation $\{y(j)\}_{j=0}^{t}$, find an estimate $\check{z}(t-l \mid t)$ of $z(t-l)$, if exists, such that the following inequality is satisfied:*

$$\sup_{(u,v)\neq 0} \frac{\sum_{t=l}^{N} [\check{z}(t-l \mid t) - z(t-l)]^T [\check{z}(t-l \mid t) - z(t-l)]}{x^T(0)P_0^{-1}x(0) + \sum_{t=0}^{N-l_0-1} u^T(t)u(t) + \sum_{t=0}^{N} v^T(t)v(t)} < \gamma^2 \quad (4)$$

where $l_0 = min(0, l)$ and P_0 is a given positive definite matrix which reflects the relative uncertainty of the initial state to the input and measurement noises.

Similar to the H_2 case, the estimator $\check{z}(t-l \mid t)$ will be a filter($l = 0$), a fixed-lag smoother ($l > 0$) and a predictor ($l < 0$).

3 H_∞ Estimation

In this section, the H_∞ smoothing or filtering ($l \geq 0$) and prediction ($l < 0$) estimation are given based on a projection approach in Krein space ([6]). First, the H_∞ estimation problem is equivalent to that $J_{l,N}(x(0), u^N; y_s^N)$ has a minimum over $\{x(0), u^N\}$ and the estimator is such that the minimum is positive [6, 12], where

$$J_{l,N}(x(0), u^N; y_s^N) = x^T(0)P_0^{-1}x(0) + \sum_{t=0}^{N-l_0-1} u^T(t)u(t) + \sum_{t=0}^{N} v^T(t)v(t) - \gamma^{-2} \sum_{t=l}^{N} v_z^T(t)v_z(t) \quad (5)$$

and

$$v_z(t) \triangleq \check{z}(t_l \mid t) - L_{t_l}x(t_l), \quad t_l \triangleq t - l \quad (6)$$

In the case of $l \geq 0$ (smoothing or filtering), (5) can be rewritten as

$$\begin{aligned} & J_{l,N}(x(0), u^N; y_s^N) \\ = & \begin{bmatrix} x(0) \\ u^N \\ v_s^N \end{bmatrix}^T \begin{bmatrix} P_0 & 0 & 0 \\ 0 & R_u^N & 0 \\ 0 & 0 & R_{v_s}^N \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ u^N \\ v_s^N \end{bmatrix} \end{aligned} \quad (7)$$

where

$$\begin{aligned} u^N &= col\{u(0), u(1), \cdots, u(N-1)\} && (8)\\ R_u^N &= \overbrace{Q_u \oplus \cdots \oplus Q_u}^{N \text{ blocks}}, \; Q_u = I_r && (9) \end{aligned}$$

and

$$\begin{aligned} v_s^N &= col\{v_s(0), \cdots v_s(N)\}, \\ v_s(t) &= \begin{cases} v(t), & 0 \leq t < l \\ \begin{bmatrix} v(t) \\ v_z(t_l) \end{bmatrix}, & t \geq l \end{cases} && (10)\\ R_{v_s}^N &= Q_{v_s}(0) \oplus \cdots \oplus Q_{v_s}(N), \\ Q_{v_s}(t) &= \begin{cases} I_m, & 0 \leq t < l \\ \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix}, & t \geq l \end{cases} && (11) \end{aligned}$$

Furthermore, putting together (2), (3) and (6) we have

$$y_s(t) = \begin{cases} H_t x(t) + v_s(t), & 0 \leq t < l \\ \begin{bmatrix} H_t & 0 \\ 0 & L_{t_l} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_l) \end{bmatrix} + v_s(t), & t \geq l \end{cases} \quad (12)$$

where

$$y_s(t) = \begin{cases} y(t), & 0 \le t < l \\ \begin{bmatrix} y(t) \\ \check{z}(t_l \mid t) \end{bmatrix}, & t \ge l \end{cases}$$

It follows that

$$\begin{bmatrix} x(0) \\ u^N \\ y_s^N \end{bmatrix} = \Psi \begin{bmatrix} x(0) \\ u^N \\ v_s^N \end{bmatrix}, \quad \Psi = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{r(N+1)} & 0 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{bmatrix} \quad (13)$$

where

$$y_s^N = col\{y_s(0), \cdots y_s(N)\} \quad (14)$$

and Ψ is invertible.

Thus, it is easy to know that (see also [12])

$$J_{l,N}(x(0), u^N; y_s^N) = \begin{bmatrix} x(0) \\ u^N \\ y_s^N \end{bmatrix}^T \left(\Psi \begin{bmatrix} P_0 & 0 & 0 \\ 0 & R_u^N & 0 \\ 0 & 0 & R_{v_s}^N \end{bmatrix} \Psi^T \right)^{-1} \begin{bmatrix} x(0) \\ u^N \\ y_s^N \end{bmatrix} \quad (15)$$

Similarly, in the case of $l < 0$ (prediction), we denote

$$\begin{aligned} u^N &\triangleq col\{u(0), u(1), \cdots, u(N-l-1)\} \quad (16) \\ R_u^N &\triangleq \overbrace{Q_u \oplus \cdots \oplus Q_u}^{N-l \text{ blocks}}, \; Q_u = I_r \quad (17) \\ v_s^N &\triangleq col\{v_s(l), \cdots v_s(N)\}, \\ v_s(t) &= \begin{cases} v_z(t), & l \le t < 0 \\ \begin{bmatrix} v(t) \\ v_z(t) \end{bmatrix}, & t \ge 0 \end{cases} \quad (18) \\ R_{v_s}^N &\triangleq Q_{v_s}(l) \oplus \cdots \oplus Q_{v_s}(N), \\ Q_{v_s}(t) &= \begin{cases} -\gamma^2 I_p & l \le t < 0 \\ \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix}, & t \ge 0 \end{cases} \quad (19) \\ y_s^N &\triangleq col\{y_s(l), y_s(l+1) \cdots, y_s(N)\}, \\ y_s(t) &= \begin{cases} \check{z}(t_l \mid t), & l \le t < 0 \\ \begin{bmatrix} y(t) \\ \check{z}(t_l \mid t) \end{bmatrix}, & t \ge 0 \end{cases} \quad (20) \end{aligned}$$

where $y_s(t)$ is of the form

$$y_s(t) = \begin{cases} L_{t_l} x(t_l) + v_s(t), & l \le t < 0 \\ \begin{bmatrix} H_t x(t) \\ L_{t_l} x(t_l) \end{bmatrix} + v_s(t), & t \ge 0 \end{cases} \quad (21)$$

Use the above notation, $J_{l,N}(x(0), u^N; y_s^N)$ can be put in the same expression as (15).

On the other hand, based on the state space model (1), (12) and (21), the following stochastic system in Krein spaces are introduced for $l \ge 0$ and $l < 0$ respectively:

$l \ge 0$:

$$\mathbf{x}(t+1) = \Phi_t \mathbf{x}(t) + \Gamma_t \mathbf{u}(t) \quad (22)$$

$$\mathbf{y}_s(t) = \begin{cases} H_t \mathbf{x}(t) + \mathbf{v}_s(t), & 0 \le t < l \\ \begin{bmatrix} H_t \mathbf{x}(t) \\ L_{t_l} \mathbf{x}(t_l) \end{bmatrix} + \mathbf{v}_s(t), & t \ge l \end{cases} \quad (23)$$

$l < 0$:

$$\mathbf{x}(t+1) = \Phi_t \mathbf{x}(t) + \Gamma_t \mathbf{u}(t) \quad (24)$$

$$\mathbf{y}_s(t) = \begin{cases} L_{t_l} \mathbf{x}(t_l) + \mathbf{v}_s(t), & l \le t < 0 \\ \begin{bmatrix} H_t \mathbf{x}(t) \\ L_{t_l} \mathbf{x}(t_l) \end{bmatrix} + \mathbf{v}_s(t), & t \ge 0 \end{cases} \quad (25)$$

where $\mathbf{u}(i)$ and $\mathbf{v}_s(i)$ are uncorrelated white noises, with

$$\begin{aligned} \langle \mathbf{u}(i), \mathbf{u}(j) \rangle &= Q_u(i) \delta_{ij} \\ \langle \mathbf{v}_s(i), \mathbf{v}_s(j) \rangle &= Q_{v_s}(i) \delta_{ij} \end{aligned} \quad (26)$$

while $Q_u(i) = I_r$, and $Q_{v_s}(i)$ is given as in (11) for $l \ge 0$ or (19) for $l < 0$.

Observe that the introduced stochastic system (22)-(23) is of the same form as in the stochastic system (1) and (4) in Part I with $l = d$ where the H_2 estimation for systems with both current and delayed measurements has been considered. The difference is that the former involves indefinite covariance matrices. Therefore, all discussions in this section will be given in Krein space rather than in Hilbert space.

Based on the stochastic system in Krein space, we introduce the innovation sequence:

$$\mathbf{w}_s(t) \triangleq \begin{bmatrix} \mathbf{w}_{s_y}(t) \\ \mathbf{w}_{s_z}(t) \end{bmatrix} = \mathbf{y}_s(t) - \hat{\mathbf{y}}_s(t \mid t-1) \quad (27)$$

where for $l \ge 0$, $\hat{\mathbf{y}}_s(t \mid t-1)$ is obtained from the Krein space projection of $\mathbf{y}_s(t)$ onto the following linear space

$$\mathcal{L}[\{\mathbf{y}_s(i)\}_{i=0}^{t-1}] \equiv \begin{cases} \mathcal{L}[\mathbf{y}(0), \cdots, \mathbf{y}(t-1)], & l \ge t \ge 0 \\ \mathcal{L}[\mathbf{y}_f(0), \cdots, \mathbf{y}_f(t_l - 1), \mathbf{y}(t_l), \cdots, \mathbf{y}(t-1)], & t > l \end{cases} \quad (28)$$

where $\mathbf{y}_f(i) = \begin{bmatrix} \mathbf{y}(i) \\ \check{\mathbf{z}}(i \mid i+l) \end{bmatrix}$. For $l < 0$, note that the observation $\mathbf{y}_s(t)$ starts from $t = l < 0$ and $\hat{\mathbf{y}}_s(t \mid t-1)$ is obtained from the Krein space projection of $\mathbf{y}_s(t)$ onto the following linear space

$$\mathcal{L}[\mathbf{y}_s(i)]_{i=l}^{t-1} \equiv$$

$$\mathcal{L} \begin{cases} [\check{\mathbf{z}}_0(l), \cdots, \check{\mathbf{z}}_{t_l - 1}(t-1)], & 0 \ge t \ge l \\ \mathcal{L}[\mathbf{y}_f(0), \cdots, \mathbf{y}_f(t_l - 1), \check{\mathbf{z}}_t(t+l), \\ \cdots, \check{\mathbf{z}}_{t_l - 1}(t-1)], & t > 0 \end{cases} \quad (29)$$

It is obvious that $\mathbf{w}_s(t)$ is a white noise sequence with the covariance matrix $Q_{w_s}(t)$, i.e.,

$$\langle \mathbf{w}_s(t), \mathbf{w}_s(i)\rangle = Q_{w_s}(t)\delta_{ti} \tag{30}$$

It will be seen that the covariance matrix $Q_{w_s}(t)$ plays an important role in the H_∞ estimation. It is used to check if the H_∞ estimator $\check{\mathbf{z}}(t_l \mid t)$ exists or not. By resorting to the augmented state system for (22)-(23) or (24)-(25), it's easy to know that the covariance matrix $Q_{w_s}(t)$ can be computed with the Kalman filtering formulation. In the following, we shall give a direct method for calculating $Q_{w_s}(t)$ without the need of system augmentation.

Similar to the H_2 counterpart in Part I of this paper, we introduce the following Riccati difference equation (RDE) for the case of $l \geq 0$:

$$\begin{aligned} P(j+1,t) &= \Phi_j P(j,t)\Phi_j^T + \Gamma_j Q_u \Gamma_j^T \\ &\quad -\Phi_j P(j,t) H_j^T Q_w^{-1}(j,t) H_j P(j,t)\Phi_j^T \end{aligned} \tag{31}$$

where $j = t+1, t+2, \cdots$ and $P(t+1,t) = P(t+1,t+1)$ and $P(t+1,t+1)$ is the solution of the following standard RDE:

$$\begin{aligned} P(t+1,t+1) &= \Phi_t P(t,t)\Phi_t^T - \Phi_t P(t,t)\begin{bmatrix} H_t \\ L_t\end{bmatrix}^T \\ &\quad \times Q_w^{-1}(t,t)\begin{bmatrix} H_t \\ L_t\end{bmatrix} P(t,t)\Phi_t^T + \Gamma_t Q_u \Gamma_t^T \end{aligned} \tag{32}$$

with $P(0,0) = P_0$ and

$$Q_w(t+i,t) = \begin{cases} H_{t+i}P(t+i,t)H_{t+i}^T + Q_v(t+i), & i \geq 0 \\ \begin{bmatrix} H_t \\ L_t\end{bmatrix} P(t,t) \begin{bmatrix} H_t \\ L_t\end{bmatrix}^T + Q_{v_f}(t), & i = 0 \end{cases} \tag{33}$$

with $Q_u = I_r$, $Q_v(t) = I_m$, $Q_{v_z}(t) = -\gamma^2 I_p$ and $Q_{v_f}(t) = \begin{bmatrix} Q_v(t) & 0 \\ 0 & Q_{v_z}(t)\end{bmatrix}$.

For the case of $l < 0$, denote

$$\begin{aligned} &P(t,t+i+1) = \Phi_{t+i}P(t,t+i)\Phi_{t+i}^T - \\ &\Phi_{t+i}P(t,t+i)L_{t+i}^T Q_w^{-1}(t,t+i)L_{t+i}^T P(t,t+i)\Phi_{t+i}^T \\ &+\Gamma_{t+i}Q_u(t+i)\Gamma_{t+i}^T, \quad i > 0 \end{aligned} \tag{34}$$

with $P(t,t+1) = P(t,t)$ and

$$Q_w(t,t+i) = L_{t+i}P(t,t+i)L_{t+i}^T + Q_{v_z}(t+i) \tag{35}$$

Theorem 1 For $l \geq 0$ (filtering or smoothing case), the innovation covariance matrix $Q_{w_s}(t)$ is given by

$$Q_{w_s}(t) = \begin{cases} Q_{11}(t), & 0 \leq t < l \\ \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t)\end{bmatrix}, & t \geq l \end{cases} \tag{36}$$

where

$$\begin{aligned} Q_{11}(t) &= H_t P(t,t_l-1)H_t^T + Q_v(t) \\ Q_{12}(t) &= H_t [R_{t,t_l-1}^{t_l}]^T L_{t_l}^T \\ Q_{21}(t) &= L_{t_l} R_{t,t_l-1}^{t_l} H_t^T \\ Q_{22}(t) &= L_{t_l}\mathcal{P}(t_l)L_{t_l}^T + Q_{v_z}(t) \\ \mathcal{P}(t_l) &= P(t_l,t_l-1) - \sum_{i=0}^{l-1} R_{t_l+i,t_l-1}^{t_l} H_{t_l+i}^T \\ &\quad Q_w^{-1}(t_l+i,t_l-1)H_{t_l+i}[R_{t_l+i,t_l-1}^{t_l}]^T, \end{aligned} \tag{37}$$

$Q_w(\cdot,\cdot)$ and $P(\cdot,\cdot)$ are as in (33) and (31), respectively, and $R_{t_l+i,t_l-1}^{t_l}$, $i = 0,1,\cdots,l$ are calculated recursively by Lemma 3 in Part I, with $Q_v(t) = I_m$ and $Q_{v_z}(t) = -\gamma^2 I_p$.

For $l < 0$, the innovation covariance matrix $Q_{w_s}(t)$ is given by

$$Q_{w_s}(t) = \begin{cases} Q_{22}(t), & l \leq t < 0 \\ \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t)\end{bmatrix}, & t \geq 0 \end{cases} \tag{38}$$

where

$$\begin{aligned} Q_{11}(t) &= H_t\mathcal{P}(t)H_t^T + Q_v(t) \\ Q_{12}(t) &= H_t[R_{t-1,t_l}^t]^T L_t^T \\ Q_{21}(t) &= L_t R_{t-1,t_l}^t H_t^T \\ Q_{22}(t) &= L_t P(t-1,t_l)L_t^T + Q_{v_z}(t) \\ \mathcal{P}(t_l) &= P(t-1,t) - \sum_{i=1}^{-l} R_{t-1,t+i-1}^t L_{t+i-1}^T \\ &\quad Q_w^{-1}(t-1,t+i-1)L_{t+i-1}[R_{t-1,t+i-1}^t]^T, \end{aligned} \tag{39}$$

$P(\cdot,\cdot)$ and $Q_w(\cdot,\cdot)$ are as in (34)-(35) and $R_{t-1,t+i-1}^t, i = 0,1,\cdots,-l$ is computed recursively using Lemma 4 of Part I, with $Q_v = I_m$ and $Q_{v_z}(t) = -\gamma^2 I_p$.

Proof. See [13].

Now, we are in the position to state the main results of the H_∞ estimation. The H_∞ filter, predictor and smoother will be given in a unified form in the thorem below.

Theorem 2 Consider the system (1)-(3) and the associated performance criterion (4). Then for a given scalar $\gamma > 0$, an H_∞ estimator $\check{z}(t_l \mid t)$ that achieves (4) exists if and only if, for each $t = l,\cdots,N$, $Q_{w_s}(t)$ and $Q_{v_s}(t)$ have the same inertia, where $Q_{w_s}(t)$ is given in Theorem 1 and $Q_{v_s}(t)$ is as in (11) for $l \geq 0$ or (19) for $l < 0$.

In this case, the possible central estimator $\check{z}(t_l \mid t)$ is given by

$$\check{z}(t_l \mid t) = L_t\hat{x}(t_l \mid t, t_l - 1) \tag{40}$$

where

1) for ≥ 0, $\hat{x}(t_l \mid t, t_l - 1)$, $t > l$ is the projection of $\mathbf{x}(t_l)$ on the linear space generated by

$$\left\{\begin{bmatrix}\mathbf{y}(0)\\ \check{\mathbf{z}}(0 \mid l)\end{bmatrix}, \cdots, \begin{bmatrix}\mathbf{y}(t_l - 1)\\ \check{\mathbf{z}}(t_l - 1 \mid t - 1)\end{bmatrix}, \mathbf{y}(t_l), \cdots \mathbf{y}(t)\right\} \tag{41}$$

and is given in Theorem 1 of Part I with $d = l + 1$ and $\mathbf{y}_f(i) = \begin{bmatrix}\mathbf{y}(i)\\ \check{\mathbf{z}}(i \mid i + l)\end{bmatrix}$ $(i = 0, 1, \cdots t_l - 1)$, and all covariance matrices as given by (26).

2) for $l < 0$ and $t \geq 0$, $\hat{x}(t_l \mid t, t_l - 1)$ is the projection of $\mathbf{x}(t_l)$ on the linear space generated by

$$\begin{gathered}\begin{bmatrix}\mathbf{y}(0)\\ \check{\mathbf{z}}(0 \mid l)\end{bmatrix}, \cdots, \begin{bmatrix}\mathbf{y}(t)\\ \check{\mathbf{z}}(t \mid t + l)\end{bmatrix},\\ \check{\mathbf{z}}(t + 1 \mid t + l + 1), \cdots \check{\mathbf{z}}(t_l - 1 \mid t - 1),\ t > 0\end{gathered} \tag{42}$$

and is given by Theorem 2 of Part I with $d = l + 1$ and $\mathbf{y}_f(i) = \begin{bmatrix}\mathbf{y}(i)\\ \check{\mathbf{z}}(i \mid i + l)\end{bmatrix}$ $(i = 0, 1, \cdots t)$, $\mathbf{z}_{t+i}(t + i + d) = \check{\mathbf{z}}(t + i \mid t + i + l)$, and all the covariance matrices as given in (26).

In the case of $0 > t \geq l$, $\hat{x}(t_l \mid t, t_l - 1)$ is given by

$$\hat{x}(t_l \mid t, t_l - 1) = \Phi_{t_l - 1} \cdots \Phi_0 \hat{x}(0 \mid -1, -1)$$

Proof See [13].

In the remainder of this section, we shall give an equivalent existence condition of an H_∞ estimator.

Theorem 3 1) An H_∞ smoother $\check{z}(t_l \mid t)(l > 0)$ that achieves (4) exists if and only if

$$R_w^l = Q_w(0,0) \oplus \overbrace{Q_w(1,0) \oplus \cdots \oplus Q_w(l,0)}^{(l)\text{ blocks}} \tag{43}$$

and

$$\begin{aligned} R_{v_s}^l &= Q_{v_s}(0) \oplus \cdots \oplus Q_{v_s}(l),\\ Q_{v_s}(t) &= \begin{cases} I_m, & 0 \leq t < l\\ \begin{bmatrix} I_m & 0\\ 0 & -\gamma^2 I_p\end{bmatrix}, & t = l\end{cases}\end{aligned} \tag{44}$$

have the same inertia, and for each $t = 1, \cdots, N - l$,

$$\begin{aligned}&I_{-[Q_w(t,t) \oplus Q_w(t+1,t) \oplus \cdots \oplus Q_w(t+l,t)]}\\ &- I_{-[Q_w(t,t-1) \oplus \cdots \oplus Q_w(t+l-1,t-1)]} = p\end{aligned} \tag{45}$$

where $Q_w(.,t)$ is as (33). $I_{-[A]}$ denotes the negative inertia of matrix A.

2) An H_∞ predictor $\check{z}(t_l \mid t)$ $(l < 0)$ that achieves (4) exists if and only if, for each $t = l, l + 1, \cdots, N$,

$$Q_w(t, t + i) < 0,\ i = 1, \cdots, -l \tag{46}$$

and $Q_w(t,t)$ and $Q_{v_s}(t) = \begin{bmatrix} I_m & 0\\ 0 & -\gamma^2 I_p\end{bmatrix}$ have the same inertia for $t = 0, \cdots, N$, where $Q_w(t, t + i)$, $i > 0$ as in (35).

4 Steady-state H_∞ Estimation

Similar to Section 3 of Part I, we assume that the system to be considered is time-invariant, i.e.

$$\Phi_t = \Phi,\ H_t = H,\ L_t = L,\ \Gamma_t = \Gamma$$

It should be noted that for the H_∞ estimation, $Q_u = I_r$, $Q_v = I_m$ and $Q_{v_z} = -\gamma^2 I_p$.

Assumption Given the initial value P_0, the Riccati equation (32) with $Q_u = I_r$, $Q_v = I_m$ and $Q_{v_z} = -\gamma^2 I_p$ has a time-invariant solution P as $t \longrightarrow \infty$.

Similar to Part I, the matrices $P(i,0)$, $Q_w(i,0)$, $A(i,0)$, $R_{i,0}^j$, $K_{i,0}^j$, $P(0,i)$, $Q_w(0,i)$, $A(0,i)$, $R_{0,i}^j$, $K_{0,i}^j$ can be calculated based on the Riccati equation solution P. The innovation covariance matrix $Q_{w_s}(t)$ can be computed from Theorem 1 as follows:

For $l \geq 0$ (filtering or smoothing case), the innovation covariance matrix $Q_{w_s}(t)$ is given by

$$Q_{w_s}(t) = \begin{cases} HP(l+1,0)H^T + Q_v, & 0 \leq t < l\\ \begin{bmatrix} HP(l+1,0)H^T + Q_v & H[R_{l+1,0}^1]^T L^T\\ LR_{l+1,0}^1 H^T & L\mathcal{P}(l)L^T + Q_{v_z}\end{bmatrix}, & t \geq l\end{cases} \tag{47}$$

where

$$\begin{aligned}\mathcal{P}(l) &= P(1,0) -\\ &\sum_{i=0}^{l-1} R_{i+1,0}^1 H^T Q_w^{-1}(i+1,0) H [R_{i+1,0}^1]^T\end{aligned} \tag{48}$$

For $l < 0$, the innovation covariance matrix Q_{w_s} is given by

$$Q_{w_s}(t) = \begin{cases} LP(0,-l+1)L^T + Q_{v_z}, & l \leq t < 0\\ \begin{bmatrix} H\mathcal{P}(l)H^T + Q_v & H[R_{0,-l+1}^1]^T L^T\\ LR_{0,-l+1}^1 H^T & LP(0,-l+1)L^T + Q_{v_z}\end{bmatrix}, & t \geq 0\end{cases} \tag{49}$$

where

$$\mathcal{P}(l) = P(0,1) - \sum_{i=1}^{-l} R_{0,i}^1 L^T Q_w^{-1}(0,i) L [R_{0,i}^1]^T \quad (50)$$

From Theorem 2, an H_∞ estimator $\check{\mathbf{z}}(t_l \mid t)$ is given by

$$\check{\mathbf{z}}(t_l \mid t) = L\hat{\mathbf{x}}(t_l \mid t, t_l - 1) = L\hat{\mathbf{x}}(t_l \mid t, t_{l+1}) \qquad (51)$$

where $\hat{\mathbf{x}}(t_l \mid t, t_{l+1})$ can be calculated from (31) for $l \geq -1$ or (42) for $l < -1$ of Part I, with $d = l+1$ and $\mathbf{z}_{i_d}(i) = \check{\mathbf{z}}(i_l \mid i)$. We now have the following results for the steady-state H_∞ estimation.

Theorem 4 Suppose $\mathcal{T}_{l,d}(q^{-1})$ and $T_{l,d}(q^{-1})$ are obtained by Theorem 3 in Part I with $Q_u = I_r$, $Q_v = I_m$, $Q_{v_z} = -I_p\gamma^2$ and $d = l+1$, and decompose

$$L \times \mathcal{T}_{l,d}(q^{-1}) = [\,\mathcal{T}_{l,d}^1(q^{-1}) \quad \mathcal{T}_{l,d}^2(q^{-1})\,] \qquad (52)$$

where $\mathcal{T}_{l,d}^1(q^{-1}) \in \mathcal{R}^{p\times m}$ and $\mathcal{T}_{l,d}^2(q^{-1}) \in \mathcal{R}^{p\times p}$. Suppose $Q_{w_s}(t)$ and $Q_{v_s}(t)$ have the same inertia for $t = l, \cdots, N$. Then,

1) for $d = l+1 \geq 0$, if

$$\left[I_p - q^{-1}\mathcal{T}_{l,d}^2(q^{-1})\right]^{-1} \in \mathcal{R}H_\infty$$

the steady-state H_∞ estimator $\check{\mathbf{z}}(t_l \mid t)$ is given as

$$\begin{aligned} & \left[I_p - q^{-1}\mathcal{T}_{l,d}^2(q^{-1})\right]\check{\mathbf{z}}(t_l \mid t) \\ = \; & \left[q^{-d}\mathcal{T}_{l,d}^1(q^{-1}) + L \times T_{l,d}(q^{-1})\right]\mathbf{y}(t) \qquad (53) \end{aligned}$$

2) for $d = l+1 < 0$, if

$$\left[I_p - q^{l}\mathcal{T}_{l,d}^2(q^{-1}) - q^{-1}L \times T_{l,d}(q^{-1})\right]^{-1} \in \mathcal{R}H_\infty$$

the steady state H_∞ estimator $\check{\mathbf{z}}(t_l \mid t)$ is given as

$$\begin{aligned} & \left[I_p - q^{l}\mathcal{T}_{l,d}^2(q^{-1}) - q^{-1}L \times T_{l,d}(q^{-1})\right]\check{\mathbf{z}}(t_l \mid t) \\ = \; & \mathcal{T}_{l,d}^1(q^{-1})\mathbf{y}(t) \qquad (54) \end{aligned}$$

Proof. For $d = l+1 \geq 0$, from (52) and noting that $\mathbf{y}_f(i) = [\,\mathbf{y}^T(i) \quad \check{\mathbf{z}}^T(i \mid i+l)\,]^T$, it follows that

$$\begin{aligned} L\hat{\mathbf{x}}(t_l \mid t, t_d) &= \mathcal{T}_{l,d}^1(q^{-1})\mathbf{y}(t_d) + L \times T_{l,d}(q^{-1})\mathbf{y}(t) \\ &\quad + \mathcal{T}_{l,d}^2(q^{-1})\check{\mathbf{z}}(t_l - 1 \mid t - 1) \qquad (55) \end{aligned}$$

which is (53). Similarly, we can prove (54).

5 Conclusion

In this paper, we have considered the H_∞ estimation problem for discrete-time systems based on the innovation analysis method and projection in Krein space. Our contribution is a unified state-space solution to the H_∞ filtering, multi-step ahead prediction and fixed-lag smoothing. In particular, we solved the long standing H_∞ fixed-lag smoothing problem without resorting to system augmentation which greatly lessens the computational demand. A unified solution to the steady state H_∞ estimation has also been obtained.

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