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Indefinite quadratic with linear costs optimal control of Markov jump with multiplicative noise systems ☆

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Abstract

In this paper we consider the stochastic optimal control problem of discrete-time Markov jump with multiplicative noise linear systems. The performance criterion is assumed to be formed by a linear combination of a quadratic part and a linear part in the state and control variables. The weighting matrices of the state and control for the quadratic part are allowed to be indefinite. We present a necessary and sufficient condition under which the problem is well posed and a state feedback solution can be derived from a set of coupled generalized Riccati difference equations interconnected with a set of coupled linear recursive equations. For the case in which the quadratic-term matrices are non-negative, this necessary and sufficient condition can be written in a more explicit way. The results are applied to a problem of portfolio optimization.

Keywords: Indefinite stochastic linear quadratic control; Well-posed; Multiplicative noise; Markov process; Discrete-time; Coupled generalized Riccati difference equation

1. Introduction

Lately a great deal of attention has been given to systems with stochastic multiplicative noise and/or abrupt changes in the parameters, due to the fact that the modelling of uncertainties using this kind of formulation has found many applications in engineering and finance. Some examples of such systems can be found in nuclear fission and heat transfer, population models and immunology, portfolio optimization, etc. Solutions to various control and estimation problems have been derived in the literature (see, for instance, Basin, Perez, & Skliar, 2006a,b; Costa, Fragoso, & Marques, 2005; Costa & Kubrusly, 1996; Dombrovskii & Lyashenko, 2003; Dombrovskii, Dombrovskii, & Lyashenko, 2005; Dragan & Morozan, 2002; Gershon & Shaked, 2006 and references therein for H_2 and H_∞ control problems, optimal filtering, robust stability and stabilizability conditions, predictive model-based control, etc.).

In particular, the stochastic quadratic optimal control problem of linear systems with multiplicative noise in both the state and control, and indefinite weighting matrices of the state and control, has been intensively studied lately. Some of the recent works on this subject can be found, for instance, in Ran and Trentelman (1993), Lim and Zhou (1999), Chen, Li, and Zhou (1998), Beghi and D'Alessandro (1998), Moore, Zhou, and Lim (1999), Wu and Zhou (2002), Ait Rami, Moore, and Zhou (2001), Zhu (2005), Luo and Feng (2004), Zhou and Li (2000), Ait Rami, Chen, Moore, and Zhou (2001), Ait Rami and Zhou (2000), Ait Rami, Chen, and Zhou (2002). In Li and Zhou (2002), Li, Zhou, and AitRami (2003) and Liu, Yin, and Zhou (2005) the authors consider continuous-time linear systems with Markov jumps as well as multiplicative noise acting on the parameters of the system. The same problem is treated within a time-varying framework and infinite time horizon in Dragan and Morozan (2004). In Zhou and Yin (2003) and Yin and Zhou (2004) it is presented a Markowitz's mean variance portfolio selection with regime switching problem, which is

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solved through an auxiliary indefinite linear quadratic control problem of a Markov jump with multiplicative noise linear system. Instead of mean variance, other criterions could be considered, for instance, one could consider a portfolio optimization problem in which the performance criterion is composed by a linear combination of a quadratic part, representing the quadratic error between the portfolio value and a benchmark, and a linear part, representing an expected error between the portfolio value and an index which is desired to overcome. This kind of problem motivated us to consider in this paper performance criterions formed by a indefinite quadratic part as well as a linear part in the state and control variables.

The main novelty of this paper is to analyze the stochastic optimal control problem of discrete-time Markov jump with multiplicative noise linear systems, with the performance criterion formed by a quadratic part and a linear part in the state and control variables. The weighting matrices of the state and control for the quadratic part are allowed to be indefinite. To the best of our knowledge there is no other work handling this kind of problem in the literature. Indeed the linear cost, which leads to some interesting conditions in our formulation, is not presented in the previous papers. Even excluding this linear term, the closest formulation to ours would be that in Ait Rami et al. (2002) which, however, does not consider the Markovian jumps. Our first main result consists of deriving a necessary and sufficient condition for this stochastic optimal control problem to be well posed, and obtaining the optimal control law, whenever it exists. This condition and the optimal control law are written in terms of a set of coupled generalized Riccati difference equations interconnected with a set of coupled linear recursive equations. To some extent, this result extends some of those presented in Ait Rami et al. (2002) to the Markovian jump case with linear costs. The second main result is to show that, for the case in which the weighting matrices of the state and control are positive semi-definite, this necessary and sufficient condition can be written is a more explicit way. These conditions are illustrated through an example of a portfolio optimization problem with regime switching. The performance criterion for this example is composed by a linear combination of a quadratic error between the portfolio value and the benchmark, and a linear part representing an expected error between the portfolio value and an index.

This paper is organized in the following way. Section 2 presents the notation that will be used throughout the work and some auxiliary results. Section 3 deals with the formulation of the problem, and presents some operators and a proposition that will be required for the developing of the results. Section 4 presents the necessary and sufficient condition for the optimal control problem to be well posed. Under this condition we derive an optimal control law, based on a set of coupled generalized Riccati difference equations interconnected with a set of coupled linear recursive equations. The case in which the weighting matrices of the state and control are positive semi-definite is considered in Section 5, yielding to a more explicit result. In Section 6 we illustrate the obtained results through an example of a portfolio optimization problem with regime switching. The paper is concluded in Section 7 with some final remarks.

2. Preliminaries

We denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space and by $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ the normed bounded linear space of all $m \times n$ real matrices, with $\mathbb{B}(\mathbb{R}^n) := \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$. For a matrix $A \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\mathrm{N}(A)$ represents the null space of A, $\mathrm{R}(A)$ the range of A, and A' the transpose of A. As usual, for $A \in \mathbb{B}(\mathbb{R}^n)$, $A \geqslant 0$ ($A > 0$, respectively) will denote that the matrix A will be positive semi-definite (positive definite), and $E(.)$ will represent the operator expected value. Set $\mathbb{H}^{n,m}$ the linear space made up of all N-sequences of real matrices $V = (V_1, \ldots, V_N)$ with $V_i \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $i = 1, \ldots, N$ and, for simplicity, set $\mathbb{H}^n := \mathbb{H}^{n,n}$. We say that $V = (V_1, \ldots, V_N) \in \widetilde{\mathbb{H}}^n$ if $V \in \mathbb{H}^n$ and for each $i = 1, \ldots, N$, V_i is a symmetric matrix. For $V = (V_1, \ldots, V_N) \in \widetilde{\mathbb{H}}^n$, $R = (R_1, \ldots, R_N) \in \widetilde{\mathbb{H}}^n$, we write that $V \geqslant R$ if $V_i - R_i \geqslant 0$ for each $i = 1, \ldots, N$. We represent by $\mathbb{B}(\mathbb{H}^n, \mathbb{H}^m)$ the bounded linear space of all operators from $\mathbb{H}^n$ to $\mathbb{H}^m$ and, in particular, $\mathbb{B}(\mathbb{H}^n) := \mathbb{B}(\mathbb{H}^n, \mathbb{H}^n)$. We say that $\mathscr{T} \in \mathbb{B}(\widetilde{\mathbb{H}}^n, \widetilde{\mathbb{H}}^m)$ if $\mathscr{T} \in \mathbb{B}(\mathbb{H}^n, \mathbb{H}^m)$ and it is such that $\mathscr{T}(V) \in \widetilde{\mathbb{H}}^m$ whenever $V \in \widetilde{\mathbb{H}}^n$.

We need the following definition (see Saberi, Sannuti, & Chen, 1995, pp. 12–13).

Definition 1. For a matrix $A \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, the generalized inverse of A (or Moore–Penrose inverse of A) is defined to be the unique matrix $A^\dagger \in \mathbb{B}(\mathbb{R}^m, \mathbb{R}^n)$ such that

(i) $AA^\dagger A = A$,
(ii) $A^\dagger A A^\dagger = A^\dagger$,
(iii) $(AA^\dagger)' = AA^\dagger$,
(iv) $(A^\dagger A)' = A^\dagger A$.

We recall the following result (see Saberi et al., 1995, pp. 12–13):

Proposition 1 (*Schur's complement*). *The following affirmatives are equivalent*:

(a) $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{pmatrix} \geqslant 0.$

(b) $Q_{22} \geqslant 0$, $Q_{12} = Q_{12} Q_{22}^\dagger Q_{22}$, *and*

$$Q_{11} - Q_{12} Q_{22}^\dagger Q_{12}' \geqslant 0.$$

(c) $Q_{11} \geqslant 0$, $Q_{12} = Q_{11} Q_{11}^\dagger Q_{12}$, *and*

$$Q_{22} - Q_{12}' Q_{11}^\dagger Q_{12} \geqslant 0.$$

3. Problem formulation

On a probabilistic space $(\Omega, \mathscr{P}, \mathscr{F})$ we consider the following Markov jump linear system with multiplicative noise:

$$x(k+1) = \left(\bar{A}_{\theta(k)}(k) + \sum_{s=1}^{v^x} \widetilde{A}_{\theta(k),s}(k) w_s^x(k)\right) x(k) + \left(\bar{B}_{\theta(k)}(k) + \sum_{s=1}^{v^u} \widetilde{B}_{\theta(k),s}(k) w_s^u(k)\right) u(k) + C_{\theta(k)}(k) w(k),$$

$$x(0) = x_0, \quad \theta(0) = \theta_0, \tag{1}$$

where $\theta(k)$ denotes a time-varying Markov chain taking values in $\{1, \ldots, N\}$ with transition probability matrix $\mathbb{P}(k)=[p_{ij}(k)]$, $\{w(k); k=0, 1, \ldots, T-1\}$ are mutually independent zero-mean random vectors with covariance matrix equal to the identity, $\{w_s^x(k); s=1, \ldots v^x, k=0, 1, \ldots, T-1\}$ are zero-mean random variables with variance equal to 1 and $E(w_i^x(k) w_j^x(k))=0$, for all k and $i \neq j$, and independent of the Markov chain $\{\theta(k)\}$ and $\{w(k)\}$. Similarly $\{w_s^u(k); s=1, \ldots v^u, k=0, 1, \ldots, T-1\}$ are zero-mean random variables with variance equal to 1 and $E(w_i^u(k) w_j^u(k)) = 0$, for all k and $i \neq j$, and independent of the Markov chain $\{\theta(k)\}$ and $\{w(k)\}$. The initial conditions θ_0 and x_0 are assumed to be independent of $\{w(k)\}$, $\{w_s^x(k)\}$ and $\{w_s^u(k)\}$, with x_0 an n-dimensional random vector with finite second moments. The mutual correlation between $w_{s_1}^x(k)$ and $w_{s_2}^u(k)$ is denoted by

$$E(w_{s_1}^x(k) w_{s_2}^u(k)) = \rho_{s_1,s_2}(k).$$

Without loss of generality, we assume that $v = v^x = v^u$. We also have for each $k = 0, 1, \ldots, T-1$,

$$\bar{A}(k) = (\bar{A}_1(k), \ldots, \bar{A}_N(k)) \in \mathbb{H}^n,$$

$$\widetilde{A}_s(k) = (\widetilde{A}_{1,s}(k), \ldots, \widetilde{A}_{N,s}(k)) \in \mathbb{H}^n, \quad s = 1, \ldots, v,$$

$$\bar{B}(k) = (\bar{B}_1(k), \ldots, \bar{B}_N(k)) \in \mathbb{H}^{m,n},$$

$$\widetilde{B}_s(k) = (\widetilde{B}_{1,s}(k), \ldots, \widetilde{B}_{N,s}(k)) \in \mathbb{H}^{m,n}, \quad s = 1, \ldots, v.$$

We set $\mathscr{F}_\tau$ the σ-field generated by $\{(\theta(t), x(t)); t=0, \ldots, \tau\}$, and write

$$\mathbb{U}(\tau) = \{u_\tau = (u(\tau), \ldots, u(T-1)); u(k) \text{ is an } m\text{-dimensional random vector with finite second moments and } \mathscr{F}_k\text{-measurable for each } k = \tau, \ldots, T-1\}.$$

We consider the following functional for $u_0 \in \mathbb{U}(0)$

$$\mathrm{J}(x_0, \theta_0, u_0) = \sum_{k=0}^{T} E(x(k)' Q_{\theta(k)}(k) x(k) + L_{\theta(k)}(k) x(k)) + \sum_{k=0}^{T-1} E(u(k)' M_{\theta(k)}(k) u(k) + H_{\theta(k)}(k) u(k)), \tag{2}$$

where

$$Q(k) = (Q_1(k), \ldots, Q_N(k)) \in \widetilde{\mathbb{H}}^n,$$

$$L(k) = (L_1(k), \ldots, L_N(k)) \in \mathbb{H}^{n,1},$$

$$M(k) = (M_1(k), \ldots, M_N(k)) \in \widetilde{\mathbb{H}}^m,$$

$$H(k) = (H_1(k), \ldots, H_N(k)) \in \mathbb{H}^{m,1}.$$

Notice that the quadratic cost matrices $Q_i(k)$ and $M_i(k)$ are just assumed to be symmetric, and that there are linear costs on the state $(L_{\theta(k)}(k)x(k))$ and on the control $(H_{\theta(k)}(k)u(k))$ variables. We want to solve the following problem:

$$J(x_0, \theta_0) = \inf_{u_0 \in \mathbb{U}(0)} \mathrm{J}(x_0, \theta_0, u_0). \tag{3}$$

Let us consider the following intermediate problems

$$J(x(\tau), \theta(\tau), \tau) = \inf_{u_\tau \in \mathbb{U}(\tau)} \mathrm{J}(x(\tau), \theta(\tau), \tau, u_\tau), \tag{4}$$

where

$$\mathrm{J}(x(\tau), \theta(\tau), \tau, u_\tau) = \sum_{k=\tau}^{T} E\{(x(k)' Q_{\theta(k)}(k) x(k) + L_{\theta(k)}(k) x(k)) | \mathscr{F}_\tau\} + \sum_{k=\tau}^{T-1} E\{(u(k)' M_{\theta(k)}(k) u(k) + H_{\theta(k)}(k) u(k)) | \mathscr{F}_\tau\}.$$

We make the following definition.

Definition 2. Problem (4) is well posed if

$$J(x(\tau), \theta(\tau), \tau) > -\infty$$

for any random variables $x(\tau)$, $\theta(\tau)$, independent of $\{w(k); k = \tau, \ldots, T-1\}$, $\{w_s^x(k); k = \tau, \ldots, T-1\}$ and $\{w_s^u(k); k = \tau, \ldots, T-1\}$, with $x(\tau)$ an n-dimensional random vector with finite second moments.

We define next for $k = 0, \ldots, T-1$ the following operators $\mathscr{E}(k,.) \in \mathbb{B}(\mathbb{H}^n)$, $\mathscr{A}(k,.) \in \mathbb{B}(\mathbb{H}^n)$, $\mathscr{G}(k,.) \in \mathbb{B}(\mathbb{H}^n, \mathbb{H}^{n,m})$, $\mathscr{R}(k,.) \in \mathbb{B}(\mathbb{H}^n, \mathbb{H}^m)$, $\mathscr{P}(k,.) \in \mathbb{B}(\mathbb{H}^n)$, $\mathscr{T}(k,.) \in \mathbb{B}(\mathbb{H}^{n,1}, \mathbb{H}^{1,n})$, $\mathscr{V}(k,.,.) \in \mathbb{B}(\mathbb{H}^n \times \mathbb{H}^{n,1}, \mathbb{H}^{n,1})$, $\mathscr{H}(k,.) \in \mathbb{B}(\mathbb{H}^{n,1}, \mathbb{H}^{1,m})$, $\mathscr{D}(k,.,.,.) \in \mathbb{B}(\mathbb{H}^n \times \mathbb{H}^{n,1} \times \mathbb{H}^1, \mathbb{H}^1)$. For $X \in \mathbb{H}^n$, $V \in \mathbb{H}^{n,1}$, $\gamma \in \mathbb{H}^1$, and $i = 1, \ldots, N$,

$$\mathscr{E}_i(k, X) = \sum\nolimits_{j=1}^{N} p_{ij}(k) X_j,$$

$$\mathscr{A}_i(k, X) = Q_i(k) + \bar{A}_i(k)' \mathscr{E}_i(k, X) \bar{A}_i(k) + \sum_{s=1}^{v} \widetilde{A}_{i,s}(k)' \mathscr{E}_i(k, X) \widetilde{A}_{i,s}(k),$$

$$\mathscr{G}_i(k,X) = \left(\bar{A}_i(k)'\mathscr{E}_i(k,X)\bar{B}_i(k) + \sum_{s_1=1}^{v}\sum_{s_2=1}^{v}\rho_{s_1,s_2}(k)\widetilde{A}_{i,s_1}(k)'\mathscr{E}_i(k,X)\widetilde{B}_{i,s_2}(k)\right)',$$

$$\mathscr{R}_i(k,X) = \bar{B}_i(k)'\mathscr{E}_i(k,X)\bar{B}_i(k) + \sum_{s=1}^{v}\widetilde{B}_{i,s}(k)'\mathscr{E}_i(k,X)\widetilde{B}_{i,s}(k) + M_i(k),$$

and

$$\mathscr{P}_i(k,X) = \mathscr{A}_i(k,X) - \mathscr{G}_i(k,X)'\mathscr{R}_i(k,X)^{\dagger}\mathscr{G}_i(k,X),$$

$$\mathscr{T}_i(k,V) = (L_i(k) + \mathscr{E}_i(k,V)\bar{A}_i(k))',$$

$$\mathscr{H}_i(k,V) = (H_i(k) + \mathscr{E}_i(k,V)\bar{B}_i(k))',$$

$$\mathscr{V}_i(k,X,V) = \mathscr{T}_i(k,V)' - \mathscr{H}_i(k,V)'\mathscr{R}_i(k,X)^{\dagger}\mathscr{G}_i(k,X),$$

$$\mathscr{D}_i(k,X,V,\gamma) = \mathscr{E}_i(k,\gamma) + tr(C_i(k)'C_i(k)\mathscr{E}_i(k,X)) - \tfrac{1}{4}\mathscr{H}_i(k,V)'\mathscr{R}_i(k,X)^{\dagger}\mathscr{H}_i(k,V).$$

It is easy to see that $\mathscr{E}(k,.) \in \mathbb{B}(\widetilde{\mathbb{H}}^n)$, $\mathscr{A}(k,.) \in \mathbb{B}(\widetilde{\mathbb{H}}^n)$ and $\mathscr{R}(k,.) \in \mathbb{B}(\widetilde{\mathbb{H}}^n, \widetilde{\mathbb{H}}^m)$.

The following proposition will be useful in the sequel, and justifies the definition of the operators above. Notice that this result is closely related to the optimality Bellman equation.

Proposition 2. *Let* $P=(P_1,\ldots,P_N) \in \widetilde{\mathbb{H}}^n$, $V=(V_1,\ldots,V_N) \in \mathbb{H}^{n,1}$ *and* $\gamma \in \mathbb{H}^1$. *For any* $u_k \in \mathbb{U}(k)$, $u(k)=u$, $x(k)=x$ *and* $\theta(k)=i$, *we have that*

$$\begin{aligned} & x'Q_i(k)x + L_i(k)x + u'M_i(k)u + H_i(k)u \\ &\quad + E(x(k+1)'P_{\theta(k+1)}x(k+1) \\ &\quad + V_{\theta(k+1)}x(k+1) + \gamma_{\theta(k+1)}|\mathscr{F}_k) \\ &\quad\quad = x'\mathscr{A}_i(k,P)x + 2x'\mathscr{G}_i(k,P)'u \\ &\quad\quad\quad + u'\mathscr{R}_i(k,P)u + \mathscr{T}_i(k,V)'x + \mathscr{H}_i(k,V)'u \\ &\quad\quad\quad + \mathrm{tr}(C_i(k)'C_i(k)\mathscr{E}_i(k,P)) + \mathscr{E}_i(k,\gamma). \end{aligned} \tag{5}$$

If

$$\mathscr{G}_i(k,P)' = \mathscr{G}_i(k,P)'\mathscr{R}_i(k,P)^{\dagger}\mathscr{R}_i(k,P), \tag{6}$$

$$\mathscr{H}_i(k,V)' = \mathscr{H}_i(k,V)'\mathscr{R}_i(k,P)^{\dagger}\mathscr{R}_i(k,P), \tag{7}$$

then (5) *can be re-written as*

$$\begin{aligned} & x'\mathscr{A}_i(k,P)x + 2x'\mathscr{G}_i(k,P)'u \\ &\quad + u'\mathscr{R}_i(k,P)u + \mathscr{T}_i(k,V)'x + \mathscr{H}_i(k,V)'u \\ &\quad + \mathrm{tr}(C_i(k)'C_i(k)\mathscr{E}_i(k,P)) + \mathscr{E}_i(k,\gamma) \\ &\quad\quad = x'\mathscr{P}_i(k,P)x + (u+a(x))'\mathscr{R}_i(k,P)(u+a(x)) \\ &\quad\quad\quad + \mathscr{V}_i(k,P,V)x + \mathscr{D}_i(k,P,V,\gamma), \end{aligned} \tag{8}$$

where

$$a(x) = \mathscr{R}_i(k,P)^{\dagger}(\mathscr{G}_i(k,P)x + \tfrac{1}{2}\mathscr{H}_i(k,V)). \tag{9}$$

Proof. We have that

$$\begin{aligned} & E(x(k+1)'P_{\theta(k+1)}x(k+1)|\mathscr{F}_k) \\ &\quad = x'\left(\bar{A}_i(k)'\mathscr{E}_i(k,P)\bar{A}_i(k) + \sum_{s=1}^{v}\widetilde{A}_{i,s}(k)'\mathscr{E}_i(k,P)\widetilde{A}_{i,s}(k)\right)x \\ &\quad\quad + 2x'\left(\bar{A}_i(k)'\mathscr{E}_i(k,P)\bar{B}_i(k) + \sum_{s_1=1}^{v}\sum_{s_2=1}^{v}\rho_{s_1,s_2}(k)\widetilde{A}_{i,s_1}(k)'\mathscr{E}_i(k,P)\widetilde{B}_{i,s_2}(k)\right)u \\ &\quad\quad + u'\left(\bar{B}_i(k)'\mathscr{E}_i(k,P)\bar{B}_i(k) + \sum_{s=1}^{v}\widetilde{B}_{i,s}(k)'\mathscr{E}_i(k,P)\widetilde{B}_{i,s}(k)\right)u \\ &\quad\quad + \mathrm{tr}(C_i(k)'C_i(k)\mathscr{E}_i(k,P)), \end{aligned} \tag{10}$$

and

$$E(V_{\theta(k+1)}x(k+1)|\mathscr{F}_k) = \mathscr{E}_i(k,V)(\bar{A}_i(k)x + \bar{B}_i(k)u), \tag{11}$$

$$E(\gamma_{\theta(k+1)}|\mathscr{F}_k) = \mathscr{E}_i(k,\gamma). \tag{12}$$

Adding (10)–(12) to $x'Q_i(k)x + L_i(k)x + u'M_i(k)u + H_i(k)u$ yields (5). Suppose now that (6) and (7) hold. Considering now on the right-hand side of (5) only the terms dependent on u, and calling it $f(u)$, we have

$$f(u) = u'\mathscr{R}_i(k,P)u + 2(x'\mathscr{G}_i(k,P)' + \tfrac{1}{2}\mathscr{H}_i(k,V)')u. \tag{13}$$

From (6) and (7) it follows that (13) can be written as

$$f(u) = u'\mathscr{R}_i(k,P)u + 2(x'\mathscr{G}_i(k,P)' + \tfrac{1}{2}\mathscr{H}_i(k,V)')\mathscr{R}_i(k,P)^{\dagger}\mathscr{R}_i(k,P)u. \tag{14}$$

Writing $a(x)$ as in (9) we have that (14) can be re-written as

$$\begin{aligned} f(u) &= u'\mathscr{R}_i(k,P)u + 2a(x)'\mathscr{R}_i(k,P)u \\ &= (u+a(x))'\mathscr{R}_i(k,P)(u+a(x)) \\ &\quad - a(x)'\mathscr{R}_i(k,P)a(x). \end{aligned}$$

Notice now that

$$\begin{aligned} & -a(x)'\mathscr{R}_i(k,P)a(x) \\ &\quad = -(x'\mathscr{G}_i(k,P)'\mathscr{R}_i(k,P)^{\dagger}\mathscr{G}_i(k,P)x \\ &\quad\quad + \mathscr{H}_i(k,V)'\mathscr{R}_i(k,P)^{\dagger}\mathscr{G}_i(k,P)x \\ &\quad\quad + \tfrac{1}{4}\mathscr{H}_i(k,V)'\mathscr{R}_i(k,P)^{\dagger}\mathscr{H}_i(k,V)), \end{aligned}$$

where we have used the fact that

$$\mathscr{R}_i(k,P)^{\dagger}\mathscr{R}_i(k,P)\mathscr{R}_i(k,P)^{\dagger} = \mathscr{R}_i(k,P)^{\dagger}.$$

Thus we have that

$$
\begin{aligned}
&x'\mathscr{A}_i(k,P)x + 2x'\mathscr{G}_i(k,P)'u + u'\mathscr{R}_i(k,P)u + \mathscr{E}_i(k,\gamma)\\
&\quad + \mathscr{T}_i(k,V)'x + \mathscr{H}_i(k,V)'u + \operatorname{tr}(C_i(k)'C_i(k)\mathscr{E}_i(k,P))\\
&\qquad = x'\mathscr{A}_i(k,P)x + \mathscr{T}_i(k,V)'x\\
&\qquad\quad + \operatorname{tr}(C_i(k)'C_i(k)\mathscr{E}_i(k,P)) + \mathscr{E}_i(k,\gamma) + f(u)\\
&\qquad = x'(\mathscr{A}_i(k,P) - \mathscr{G}_i(k,P)'\mathscr{R}_i(k,P)^{\dagger}\mathscr{G}_i(k,P))x\\
&\qquad\quad + (\mathscr{T}_i(k,V)' - \mathscr{H}_i(k,V)'\mathscr{R}_i(k,P)^{\dagger}\mathscr{G}_i(k,P))x\\
&\qquad\quad + \operatorname{tr}(C_i(k)'C_i(k)\mathscr{E}_i(k,P))\\
&\qquad\quad + (u+a(x))'\mathscr{R}_i(k,P)(u+a(x))\\
&\qquad\quad + \mathscr{E}_i(k,\gamma) - \tfrac{1}{4}\mathscr{H}_i(k,V)'\mathscr{R}_i(k,P)^{\dagger}\mathscr{H}_i(k,V)\\
&\qquad = x'\mathscr{P}_i(k,P)x + \mathscr{V}_i(k,P,V)x + \mathscr{D}_i(k,P,V,\gamma)\\
&\qquad\quad + (u+a(x))'\mathscr{R}_i(k,P)(u+a(x)),
\end{aligned}
$$

showing (8) and completing the proof of the proposition. □

4. The indefinite case

The next theorem presents a necessary and sufficient condition for problems (4) to be well posed for each $k = T, T-1, \ldots, 0$, as well as an optimal control strategy for Problem (3), posed in Section 3. The quadratic weighting matrices of the state and control are allowed to be indefinite. This condition and the optimal control law are written in terms of a set of coupled generalized Riccati difference equations interconnected with a set of coupled linear recursive equations. Define the following sequences for $k = T-1, \ldots, 0$,

$$P(k) = \mathscr{P}(k, P(k+1)), \tag{15}$$

$$V(k) = \mathscr{V}(k, P(k+1), V(k+1)), \tag{16}$$

$$\gamma(k) = \mathscr{D}(k, P(k+1), V(k+1), \gamma(k+1)), \tag{17}$$

with $P(T) = Q(T)$, $V(T) = L(T)$ and $\gamma(T) = 0$.

Theorem 1. *The following assertions are equivalent*:

(a) *For each* $k = T, T-1, \ldots, 0$, *problem* (4) *is well posed.*
(b) *For each* $k = T-1, \ldots, 0$, *we have that*

$$\mathscr{R}_i(k, P(k+1)) \geqslant 0, \tag{18}$$

$$\mathscr{G}_i(k, P(k+1))' = \mathscr{G}_i(k, P(k+1))'\mathscr{R}_i(k, P(k+1))^{\dagger}\mathscr{R}_i(k, P(k+1)), \tag{19}$$

$$\mathscr{H}_i(k, V(k+1))' = \mathscr{H}_i(k, V(k+1))'\mathscr{R}_i(k, P(k+1))^{\dagger}\mathscr{R}_i(k, P(k+1)), \tag{20}$$

with the value functions $J(x(k), \theta(k), k)$ *given by*

$$J(x(k),\theta(k),k) = E((x(k)'P_{\theta(k)}(k)x(k) + V_{\theta(k)}(k)x(k)) + \gamma_{\theta(k)}(k), \tag{21}$$

and an optimal control law achieved by

$$u(k) = -\mathscr{R}_{\theta(k)}(k, P(k+1))^{\dagger}\left(\mathscr{G}_{\theta(k)}(k, P(k+1))x(k) + \frac{1}{2}\mathscr{H}_{\theta(k)}(k, V(k+1))\right). \tag{22}$$

Proof. Clearly (b) implies (a). Let us show that (a) implies (b) by induction on k. For $k = T$ there is no control to take, and it follows that $J(x(T),\theta(T),T) = E(x(T)'Q_{\theta(T)}(T)x(T) + L_{\theta(T)}(T)x(T))$, showing (21) from the definition of $P(T) = Q(T)$, $V(T) = L(T)$ and $\gamma(T) = 0$. Suppose from the induction hypothesis that (21)–(22) hold for $k+1$. From the Bellman equation and (5) we have that for $x(k) = x$, $\theta(k) = i$,

$$
\begin{aligned}
J(x,i,k) &= \inf_{u\in\mathbb{R}^m}\{x'Q_i(k)x + L_i(k)x + u'M_i(k)u\\
&\quad + H_i(k)u + E(J(x(k+1),\theta(k+1),k+1)|\mathscr{F}_k)\}\\
&= \inf_{u\in\mathbb{R}^m}\{x'Q_i(k)x + L_i(k)x + u'M_i(k)u\\
&\quad + H_i(k)u + E(x(k+1)'P_{\theta(k+1)}(k+1)x(k+1)\\
&\quad + V_{\theta(k+1)}(k+1)x(k+1) + \gamma_{\theta(k+1)}(k+1)|\mathscr{F}_k)\}\\
&= \inf_{u\in\mathbb{R}^m}\{x'\mathscr{A}_i(k, P(k+1))x\\
&\quad + 2x'\mathscr{G}_i(k, P(k+1))'u + u'\mathscr{R}_i(k, P(k+1))u\\
&\quad + \mathscr{T}_i(k, V(k+1))'x + \mathscr{H}_i(k, V(k+1))'u\\
&\quad + \operatorname{tr}(C_i(k)'C_i(k)\mathscr{E}_i(k, P(k+1)))\\
&\quad + \mathscr{E}_i(k, \gamma(k+1))\}.
\end{aligned}
\tag{23}
$$

Suppose by contradiction that (a) holds for k, that is, for any $x \in \mathbb{R}^n$ and i,

$$J(x,i,k) > -\infty, \tag{24}$$

but (18), (19) or (20) do not hold for k. Suppose first that there exists $\bar{u} \in \mathbb{R}^m$, $\bar{u} \neq 0$, such that $\bar{u}$ is an eigenvector of $\mathscr{R}_i(k, P(k+1))$ with negative eigenvalue λ. From (23) with $u = \alpha\bar{u}$, α a positive real number, we have that for some positive real numbers a_0, a_1, a_2, a_3, and a_4,

$$\alpha^2\lambda\|\bar{u}\|^2 + a_0\|x\|^2 + \alpha a_1\|x\|\|\bar{u}\| + a_2\|x\| + \alpha a_3\|\bar{u}\| + a_4 \geqslant J(x,i,k). \tag{25}$$

By taking the limit as $\alpha \to \infty$ and recalling that $\lambda < 0$ we have from (25) that $J(x,i,k) \to -\infty$, in contradiction with (24), showing that (18) must hold. As shown in Ait Rami et al. (2002, Lemma 4.2), (19) is equivalent to

$\mathrm{N}(\mathscr{R}_i(k, P(k+1))) \subseteq \mathrm{N}(\mathscr{G}_i(k, P(k+1))')$ and similarly, (20) is equivalent to $\mathrm{N}(\mathscr{R}_i(k, P(k+1))) \subseteq \mathrm{N}(\mathscr{H}_i(k, V(k+1))')$. Suppose by contradiction that there exists $\bar{u} \in \mathbb{R}^m$ such that $\bar{u} \in \mathrm{N}(\mathscr{R}_i(k, P(k+1)))$ but $\bar{u} \notin \mathrm{N}(\mathscr{G}_i(k, P(k+1))')$. Without loss of generality, consider $\bar{x} \in \mathbb{R}^n$ such that $\bar{x}'\mathscr{G}_i(k, P(k+1))'\bar{u}=\psi<0$ (this is possible since by hypothesis (24) holds for any $x \in \mathbb{R}^n$). From (23) with $x=\beta\bar{x}$, $u=\alpha\bar{u}$, β and α positive real numbers, we have again that for positive real numbers a_0, a_2, a_3, and a_4,

$$\beta^2 a_0\|\bar{x}\|^2+\alpha(2\beta\psi+a_3\|\bar{u}\|) + \beta a_2\|\bar{x}\|+a_4 \geqslant J(x,i,k). \tag{26}$$

By choosing β such that $2\beta\psi+a_3\|\bar{u}\|<0$ (which is possible since $\psi<0$), and by taking the limit as $\alpha \to \infty$, we have from (26) that $J(x,i,k) \to -\infty$, in contradiction with (24), showing that (19) must hold. Suppose now by contradiction that there exists $\bar{u} \in \mathbb{R}^m$ such that $\bar{u} \in \mathrm{N}(\mathscr{R}_i(k, P(k+1)))$ but $\bar{u} \notin \mathrm{N}(\mathscr{H}_i(k, V(k+1))')$. Fix any $\bar{x} \in \mathbb{R}^n$. Without loss of generality, suppose that $\mathscr{H}_i(k, V(k+1))'\bar{u}=\psi<0$. From (23) with $x=\beta\bar{x}$, $u=\alpha\bar{u}$, β and α positive real numbers, we have from the same reasoning as before that for positive real numbers a_0, a_1, a_2, and a_4,

$$\beta^2 a_0\|\bar{x}\|^2+\alpha(\beta a_1\|\bar{x}\|\|\bar{u}\| + \psi)+\beta a_2\|\bar{x}\|+a_4 \geqslant J(x,i,k). \tag{27}$$

By choosing β such that $\beta a_1\|\bar{x}\|\|\bar{u}\|+\psi<0$ (which is possible since $\psi<0$), and by taking the limit as $\alpha\to\infty$, we have from (27) that $J(x,i,k) \to -\infty$, in contradiction with (24), and showing that (20) must hold. From (18)–(20), (8), (15)–(17), we have that (23) can be re-written as

$$\begin{aligned} J(x,i,k) &= \inf_{u\in\mathbb{R}^m}\{x'\mathscr{P}_i(k, P(k+1))x \\ &\quad + (u+a(x))'\mathscr{R}_i(k, P(k+1))(u+a(x)) \\ &\quad + \mathscr{V}_i(k, P(k+1), V(k+1))x \\ &\quad + \mathscr{D}_i(k, P(k+1), V(k+1), \gamma(k+1))\} \\ &= \inf_{u\in\mathbb{R}^m}\{(u+a(x))'\mathscr{R}_i(k, P(k+1))(u+a(x))\} \\ &\quad + x'P_i(k)x+V_i(k)x+\gamma_i(k), \end{aligned} \tag{28}$$

where

$$a(x)=\mathscr{R}_i(k, P(k+1))^\dagger(\mathscr{G}_i(k, P(k+1))x + \tfrac{1}{2}\mathscr{H}_i(k, V(k+1))).$$

From (18) it is clear that the right-hand side of (28) reaches its minimum value at $u=-a(x)$, and (21) is satisfied, completing the proof of the theorem. $\square$

Remark 1. It would be possible to have some k and a set $\Gamma(k) \subset \mathbb{R}^n$ such that $J(x,i,k)=-\infty$ for any $x\in\Gamma(k)$, but $J(x,i)=J(x,i,0)>-\infty$ (that is, the problem is well posed for $k=0$ but it is not well posed for some other $k>0$). Indeed, consider the following deterministic example, $N=1$, $T=2$, $n=2$, $m=1$ and the following models (we omit the subscript i since there is no Markov chain):

$$\begin{aligned} \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}+\begin{pmatrix} 1 \\ 0 \end{pmatrix}u(0), \\ \begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix}u(1). \end{aligned} \tag{29}$$

The functional to be minimized is

$$\begin{aligned} \mathrm{J}(x_0,\theta_0,u_0) &= u(1)^2-x_2(2)^2 \\ &= u(1)^2-(x_2(1)+u(1))^2 \\ &= -x_2(1)(x_2(1)+2u(1)). \end{aligned}$$

It is clear that $J(x,1)=-\infty$ whenever $x_2(1)\neq 0$. But from (29), $x_2(1)=0$ and thus $J(x(0))=0$ for any $x(0)\in\mathbb{R}^2$. We can easily see that (19) is not valid for $k=1$. Indeed, we obtain that $\mathscr{G}(1, P(2))'=(0\ \ -1)$ and $\mathscr{R}(1, P(2))=0$.

5. The positive semi-definite case

In this section we consider the special case in which the weighting matrices of the state and control are positive semi-definite, that is, $Q(k)\geqslant 0$ and $M(k)\geqslant 0$ for all $k=0,\ldots,T$. The following proposition will be useful in the sequel.

Proposition 3. *Consider $Y\in\mathbb{B}(\mathbb{R}^n)$ and $M\in\mathbb{B}(\mathbb{R}^m)$ with $Y\geqslant 0$, $M\geqslant 0$. Let A and B be stochastic matrices (that is, each element of the matrix is a random variable) in $\mathbb{B}(\mathbb{R}^n)$ and $\mathbb{B}(\mathbb{R}^m,\mathbb{R}^n)$, respectively. Then*

$$E(A'YA)-E(A'YB)(E(B'YB)+M)^\dagger E(B'YA)\geqslant 0 \tag{30}$$

and

$$E(A'YB)=E(A'YB)(E(B'YB)+M)^\dagger(E(B'YB)+M). \tag{31}$$

Proof. Consider the stochastic matrix

$$Q=\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{pmatrix}=\begin{pmatrix} Y & YB \\ B'Y & R \end{pmatrix},$$

where

$$R=B'YB+M.$$

Clearly we have $Q_{11}=Y\geqslant 0$, and $Q_{12}=YB=Q_{11}Q_{11}^\dagger Q_{12}=YY^\dagger YB$ since from Definition 1, $YY^\dagger Y=Y$. Furthermore, using again that $YY^\dagger Y=Y$, we have

$$\begin{aligned} Q_{22}-Q_{12}'Q_{11}^\dagger Q_{12} &= R-B'YY^\dagger YB \\ &= R-B'YB=M\geqslant 0. \end{aligned}$$

Thus from Schur's complement (Proposition 1), $Q\geqslant 0$, and hence

$$\begin{pmatrix} A'YA & A'YB \\ B'YA & R \end{pmatrix}=\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}'\begin{pmatrix} Y & YB \\ B'Y & R \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\geqslant 0.$$

Taking the expected value of the above equation we get that

$$\begin{pmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{12}' & \widetilde{Q}_{22} \end{pmatrix} = \begin{pmatrix} E(A'YA) & E(A'YB) \\ E(B'YA) & E(R) \end{pmatrix} \geqslant 0.$$

From Schur's complement again (Proposition 1), we have that

$$0 \leqslant \widetilde{Q}_{11} - \widetilde{Q}_{12}\widetilde{Q}_{22}^{\dagger}\widetilde{Q}_{12}' = E(A'YA) - E(A'YB)E(R)^{\dagger}E(B'YA)$$

and that

$$\widetilde{Q}_{12} = E(A'YB) = \widetilde{Q}_{12}\widetilde{Q}_{22}^{\dagger}\widetilde{Q}_{22} = E(A'YB)E(R)^{\dagger}E(R),$$

showing the desired result. □

The following result establishes more explicitly a necessary and sufficient condition for the particular case of non-negative definite quadratic-term matrices.

Theorem 2. *Suppose that $Q(k) \geqslant 0$ and $M(k) \geqslant 0$ for all $k = 0, \ldots, T$. Let $P(k)$, $V(k)$ and $\gamma(k)$ be as in* (15), (16) and (17), *respectively. Then* (18) and (19) *are always satisfied. Moreover the following assertions are equivalent*:

(a) *For each $k = T, T-1, \ldots, 0$, problem* (4) *is well posed.*
(b) *For each $k = T-1, \ldots, 0$, we have that*

$$\mathscr{H}_i(k, V(k+1)) \in \mathrm{R}(\mathscr{R}_i(k, P(k+1))), \tag{32}$$

with the value functions $J(x(k), \theta(k), k)$ given by (21) *and an optimal control law achieved by* (22).

Proof. Let us show first that for each $k = 0, 1, \ldots, T$, $P(k) \geqslant 0$ (and thus that (18) holds). By induction on k, for $k = T$ we have, recalling that $Q(T) \geqslant 0$, that $P(T) = (P_1(T), \ldots, P_N(T)) \geqslant 0$. Setting in Proposition 3

$$A = \bar{A}_i(k) + \sum_{s=1}^{v} \widetilde{A}_{i,s}(k) w_s^x(k),$$

$$B = \bar{B}_i(k) + \sum_{s=1}^{v} \widetilde{B}_{i,s}(k) w_s^u(k),$$

$$Y = \mathscr{E}_i(P(k+1)),$$

$$M = M_i(k),$$

we have from Eq. (30), and the hypothesis made for $\{w_s^x(k)\}$ and $\{w_s^u(k)\}$, that

$$\bar{A}_i(k)'\mathscr{E}_i(P(k+1))\bar{A}_i(k) + \sum_{s=1}^{v} \widetilde{A}_{i,s}(k)'\mathscr{E}_i(P(k+1))\widetilde{A}_{i,s}(k) - \mathscr{G}_i(k, P(k+1))'\mathscr{R}_i(k, P(k+1))^{\dagger}\mathscr{G}_i(k, P(k+1)) \geqslant 0$$

and thus $P_i(k) \geqslant 0$. It also follows from Proposition 3, Eq. (31), that (19) holds. From Ait Rami et al. (2002, Lemma 4.2), (20) is equivalent to

$$\mathrm{N}(\mathscr{R}_i(k, P(k+1))) \subseteq \mathrm{N}(\mathscr{H}_i(k, V(k+1))'),$$

or, in other words, that (20) is equivalent to (32) for each $k = 0, 1, \ldots, T-1$. Therefore by applying Theorem 1 we obtain the desired result. □

Example 1. Let us illustrate the use of (32) through a simple example. Consider a deterministic system given by the equation

$$x(k+1) = ax(k) + (b_1\ b_2)\begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix},$$

with $a \neq 0$, $b_1 \neq 0$, $b_2 \neq 0$. The functional to be minimized is given by

$$\mathrm{J}(x_0, u_0) = \sum_{k=0}^{T}(qx(k)^2 + \ell x(k)) + \sum_{k=0}^{T-1}(h_1\ h_2)\begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix},$$

with $q > 0$, $\ell \neq 0$, $h_1 \neq 0$, $h_2 \neq 0$. We have that (eliminating the dependence on the Markov chain and time)

$$\mathscr{A}(X) = q + a^2X, \quad \mathscr{G}(X) = aX\begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

$$\mathscr{R}(X) = X\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}(b_1\ b_2),$$

$$\mathscr{R}(X)^{\dagger} = \frac{1}{(b_1^2 + b_2^2)^2X}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}(b_1\ b_2),$$

$$\mathscr{P}(X) = q, \quad \mathscr{T}(V) = \ell + aV, \quad \mathscr{H}(V) = \begin{pmatrix} h_1 + b_1V \\ h_2 + b_2V \end{pmatrix},$$

$$\mathscr{V}(X, V) = \ell - \frac{a}{b_1^2 + b_2^2}(b_1h_1 + b_2h_2),$$

and

$$\mathscr{D}(X, V, \gamma) = \gamma - \frac{1}{4X}\left(\frac{1}{b_1^2 + b_2^2}(b_1h_1 + b_2h_2) + V\right)^2.$$

Then we have that (32) is satisfied if and only if $h_1/b_1 = h_2/b_2$, that is, the problem is well posed if and only if $h_1/b_1 = h_2/b_2$. In this case, setting $\alpha = h_1/b_1$, we have that $P(k) = q$, $V(k) = \ell - \alpha a$, $k = 0, \ldots, T-1$, $V(T) = \ell$, $\delta(k) = \delta(k+1) - (1/4q)(\alpha + V(k+1))^2$, and an optimal control law would be

$$u(k) = -\frac{1}{b_1^2 + b_2^2}\left(ax(k) + \frac{1}{2q}(\alpha + V(k+1))\right)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

For the further special case in which $Q(k)\geqslant 0$ and $M(k)>0$ for all $k=0,\ldots,T$ we have the following corollary.

Corolary 1. *Suppose that $Q(k)\geqslant 0$ and $M(k)>0$ for all $k=0,\ldots,T$. Let $P(k)$, $V(k)$ and $\gamma(k)$ be as in* (15), (16) and (17), *respectively. Then* (18), (19) and (32) *are always satisfied, and for each $k=T,T-1,\ldots,0$, problem* (4) *is well posed. Moreover for each $k=0,1,\ldots,T-1$ the value functions $J(x(k),\theta(k),k)$ are given by* (21) *and an optimal control law is achieved by* (22).

Proof. From the proof of Theorem 2 we have that $P_i(k)\geqslant 0$, and from the hypothesis that $M(k)>0$, it follows that $\mathscr{R}(k,P(k+1))>0$, showing that (32) is always satisfied. □

6. Portfolio optimization problem

In this section we present a portfolio optimization problem with regime switching. We assume that the market parameters will depend on the market mode that switches according to a Markov chain among a finite number of states. The portfolio optimization problem will be written as a Markov jump with multiplicative noise LQ optimal control problem with linear and quadratic costs, so that the results presented in Section 5 can be applied to solve the problem.

Consider a financial market model in which there are n assets represented by the random price vector $S(t)\in\mathbb{R}^n$. The components of $S(t)$ will be represented by $S_\ell(t)$, $\ell=1,\ldots,n$, that is, $S(t)'=(S_1(t)\ \cdots\ S_n(t))$. We consider that the investor can allocate his wealth among the assets $\ell=2,\ldots,n$, being the first asset assigned as the benchmark and, as such, ineligible for investment.

Let $\{\theta(t);\,t=0,1,\ldots,T-1\}$ be a Markov chain taking values in a finite set $\{1,\ldots,N\}$ with transition probability matrix $\mathbb{P}=[p_{ij}]$. The Markov parameter $\theta(t)$ will characterize the market mode at the instant t and it will set how the prices are expected to vary from time t to time $t+1$. Indeed, we make the following assumptions:

Condition 1. We assume that for $\ell=1,\ldots,n$,

$$S_\ell(t+1)=(1+R_{\theta(t),\ell}(t))S_\ell(t), \tag{33}$$

where the vector

$$R_{\theta(t)}(t)=\begin{pmatrix} R_{\theta(t),1}(t)\\ \vdots\\ R_{\theta(t),n}(t)\end{pmatrix} \tag{34}$$

can be decomposed as

$$R_{\theta(t)}(t)=\eta_{\theta(t)}(t)+\Sigma_{\theta(t)}(t)w^S(t), \tag{35}$$

with $\Sigma(t)=(\Sigma_1(t),\ldots,\Sigma_N(t))\in\mathbb{H}^{n+}$ for each $t=0,\ldots,T-1$ and $\{w^S(t);\,t=0,\ldots,T-1\}$ are independent zero-mean n-dimensional random vectors with identity covariance matrix and independent of the Markov chain $\{\theta(t);\,t=0,1,\ldots,T-1\}$. The vectors $\eta_i(t)\in\mathbb{R}^n$ are deterministic vectors for each $t=0,\ldots,T-1$, $i=1,\ldots,N$.

We write for $i=1,\ldots,N$,

$$\Sigma_i(t)=\begin{pmatrix}\sigma_{i,1,1}(t) & \cdots & \sigma_{i,1,n}(t)\\ \vdots & \ddots & \vdots\\ \sigma_{i,n,1}(t) & \cdots & \sigma_{i,n,n}(t)\end{pmatrix},$$

$$\eta_i(t)=\begin{pmatrix}\eta_{i,1}(t)\\ \vdots\\ \eta_{i,n}(t)\end{pmatrix},\quad w^S(t)=\begin{pmatrix}w_1^S(t)\\ \vdots\\ w_n^S(t)\end{pmatrix}. \tag{36}$$

Remark 2. From (33) we have that the rate of return of the assets from time period t to $t+1$ depends on the market mode $\theta(t)$ at the instant t. When the market mode at time t is $\theta(t)=i$ we can see from (34) and (35) that the vector of expected rate of returns is given by $\eta_i(t)$, and the covariance matrix is given by $\Sigma_i^2(t)$.

Let $U_\ell(t)$ represent the amount of money invested in asset ℓ, with $\ell=2,\ldots,n$. We write

$$U(t)=\begin{pmatrix}U_2(t)\\ \mathrm{U}(t)\end{pmatrix},\quad \mathrm{U}(t)=\begin{pmatrix}U_3(t)\\ \vdots\\ U_n(t)\end{pmatrix}. \tag{37}$$

Let $X_U(t)$ represent the value of the portfolio associated to the investment strategy U (for simplicity we omit the subscript U whenever no confusion can arise), with an initial wealth $X(0)=X_0$. Let e represent the $(n-2)$-dimensional vector with 1 in all its components. We write

$$R_i(t)=\begin{pmatrix}R_{i,1}(t)\\ R_{i,2}(t)\\ \mathrm{R}_i(t)\end{pmatrix},\quad \mathrm{R}_i(t)=\begin{pmatrix}R_{i,3}(t)\\ \vdots\\ R_{i,n}(t)\end{pmatrix}. \tag{38}$$

Then we have that

$$X(t)=U_2(t)+\mathrm{U}(t)'e \tag{39}$$

and

$$X(t+1)=(1+R_{\theta(t),2}(t))U_2(t)+(e+\mathrm{R}_{\theta(t)}(t))'\mathrm{U}(t). \tag{40}$$

From (39) and (40) we have that

$$X(t+1)=(1+R_{\theta(t),2}(t))X(t)+\mathrm{P}_{\theta(t)}(t)'\mathrm{U}(t), \tag{41}$$

where for each $i=1,\ldots,N$,

$$\mathrm{P}_i(t)=\mathrm{R}_i(t)-R_{i,2}(t)e. \tag{42}$$

Let $Y(t)$ represent the value of the reference portfolio (benchmark), satisfying the following recursive equation:

$$Y(t+1)=(1+R_{i,1}(t))Y(t), \tag{43}$$

with $Y(0)=X(0)$. Notice that

$$R_{i,1}=\eta_{i,1}(t)+\sum_{s=1}^{n}\sigma_{i,1,s}(t)w_s^S(t). \tag{44}$$

We write $\mathscr{F}_t$ as the σ-field generated by the random variables $\{(\theta(t),X(t),Y(t));\ t=0,\ldots,T\}$. We want to minimize the following functional:

$$\mathrm{J}(X(0),\theta(0),\mathrm{U})=\sum_{t=0}^{T}E(\varrho_{\theta(t)}(t)\|X(t)-Y(t)\|^2 - \xi_{\theta(t)}(t)(X(t)-Y(t))), \tag{45}$$

where $\varrho_{\theta(t)}(t)$ and $\xi_{\theta(t)}(t)$ are positive numbers. The quadratic term indicates that we want the portfolio value, $X(t)$, to remain close to the benchmark value, $Y(t)$, while the linear term indicates that we want the portfolio value to be greater than the benchmark value. The trade-off between these two terms is balanced by the weights $\varrho_{\theta(t)}(t)$ and $\xi_{\theta(t)}(t)$. For each t the control variables $\mathrm{U}(t)$ are assumed to be $\mathscr{F}_t$-measurable.

The relevance of the proposed target functional (45) would be, for instance, in multi-period tracking error optimization of funds (see, for instance, Roll, 1992; Rudolf, Wolter, & Zimmermann, 1999 for the uni-period case). In this case the professional money manager is judged by the total return performance relative to a prespecified benchmark portfolio, usually a broadly diversified index of assets. The allocation decision problem is based at each time on the difference between the manager's return and the benchmark return, the so-called tracking error. This means that tracking error optimization problems could be posed as follows: to find a multi-period portfolio with optimal trade-off between the tracking error and expected performance relative to the benchmark along all the periods of time. Notice that the proposed target function in Zhou and Yin (2003) and Yin and Zhou (2004) would not reach this goal, since it looks only at the final variance of the portfolio, so that it does not consider the tracking error performance of the fund at the intermediate periods of time $t=1,\ldots,T-1$.

Set for $i=1,\ldots,N$ and $\ell=1,\ldots,n$,

$$\Lambda_{i,\ell}(t)=\begin{pmatrix}\sigma_{i,3,\ell}(t)-\sigma_{i,2,\ell}(t)\\ \vdots\\ \sigma_{i,n,\ell}(t)-\sigma_{i,2,\ell}(t)\end{pmatrix}, \tag{46}$$

$$\mathscr{J}=(\mathbf{0}\ \ -e\ I), \tag{47}$$

where $\mathbf{0}$ represents an $(n-2)$-dimensional zero vector and I an identity matrix. Defining

$$x(t)=\begin{pmatrix}X(t)\\ Y(t)\end{pmatrix},\quad u(t)=\begin{pmatrix}U_3(t)\\ \vdots\\ U_n(t)\end{pmatrix} \tag{48}$$

and for $i=1,\ldots,N$,

$$\bar{A}_i(t)=\begin{pmatrix}1+\eta_{i,2}(t) & 0\\ 0 & 1+\eta_{i,1}(t)\end{pmatrix},$$

$$\bar{B}_i(t)=\begin{pmatrix}(\mathscr{J}\eta_i(t))'\\ 0\end{pmatrix} \tag{49}$$

and for $s=1,\ldots,n$,

$$\widetilde{A}_{i,s}(t)=\begin{pmatrix}\sigma_{i,2,s}(t) & 0\\ 0 & \sigma_{i,1,s}(t)\end{pmatrix},\quad w_s^x(t)=w_s^S(t),$$

$$\widetilde{B}_{i,s}(t)=\begin{pmatrix}\Lambda_{i,s}(t)'\\ 0\end{pmatrix},\quad w_s^u(t)=w_s^S(t), \tag{50}$$

we have from (41), (43), (48)–(50) that

$$x(t+1)=\left(\bar{A}_{\theta(t)}(t)+\sum_{s=1}^{n}\widetilde{A}_{\theta(t),s}(t)w_s^x(t)\right)x(t) + \left(\bar{B}_{\theta(t)}(t)+\sum_{s=1}^{n}\widetilde{B}_{\theta(t),s}(t)w_s^u(t)\right)u(t). \tag{51}$$

Defining for $i=1,\ldots,N$

$$Q_i(t)=\varrho_i(t)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},$$

$$L_i(t)=\xi_i(t)(-1\ 1),$$

$$M_i(t)=(\mathbf{0}_{n-2\times n-2}),$$

$$H_i(t)=(\mathbf{0}_{1\times n-2}),$$

where $\mathbf{0}_{i\times j}$ is an $i\times j$ dimensional zero matrix, we have that (45) can be re-written as in (2). Moreover if we have that (32) is satisfied for each $i=1,\ldots,N$, and $k=0,\ldots,T-1$, then we can apply Theorem 2 to solve the problem, since that $Q_i(t)\geqslant 0$ and $M_i(t)=0$.

We also compared our results with an uni-period strategy based on the usual Markowitz model, applied successively in time. For each market mode $i=1,2,3$ let $P_i(t)$ denote the uni-period return of the portfolio, $P_{i,b}(t)$ the return of the benchmark, and $P_{i,e}(t)=P_i(t)-P_{i,b}(t)$. Let also $\omega_i(t)$ denote the vector with the percentual allocation of the wealth among the assets $\ell=2,\ldots,n$. It follows that $P_{i,e}(t)=(-1\ \omega_i(t)')R_i(t)$, so that the expected value of $P_{i,e}(t)$ is given by $E(P_{i,e}(t))=(-1\ \omega_i(t)')\eta_i(t)$ and the variance given by $Var(P_{i,e}(t))=(-1\ \omega_i(t)')\Sigma_i^2(t)\begin{pmatrix}-1\\ \omega_i(t)\end{pmatrix}$. Therefore for each i and t we can solve the following problem: $\min_{\omega_i}\varrho_i(t)Var(P_{i,e}(t))-\xi_i(t)E(P_{i,e}(t))$, subject to $\omega_{i,2}+\cdots+\omega_{i,n}=1$. By doing this we obtain a set of uni-period investment strategies $\{\omega_i(t)\}$ that can be applied according to the observed market mode $\theta(t)$ at time t.

The numerical simulations we present next are based on the examples presented in Li and Ng (2000) and Blair and Sworder (1975). Set $i=1$, 2 and 3, for the possible market modes, defined by

Regime	Description
$i=1$	Market in mid-range
$i=2$	Market in low range
$i=3$	Market in high range

where the transition matrices for the modes at each time t, with $t=1,\ldots,20$, are given by

$$\mathbb{P}(t)=\begin{pmatrix}0.5 & 0.3 & 0.2\\ 0.3 & 0.5 & 0.2\\ 0.4 & 0.4 & 0.2\end{pmatrix},$$

and the correlation coefficient between the noises $w^u_{s_1}(t)$ and $w^x_{s_2}(t)$ is given by $\rho_{s_1,s_2}(t)=E(w^u_{s_1}(t)w^x_{s_2}(t))=E(w^S_{s_1}(t)w^S_{s_2}(t))=1$ if $s_1=s_2$, 0 otherwise.

Consider a capital market with four assets, $\ell=1,2,3,4$. Asset $\ell=1$ is defined as benchmark, so that the investor can allocate his wealth only among the assets $\ell=2,3,4$. The vector of expected rate of returns, $\eta_i(t)$, and the matrices, $\Sigma_i(t)$, associated to each market mode, are given by

$$\eta_1(t)=\begin{pmatrix}0.210\\0.162\\0.246\\0.228\end{pmatrix},\quad \eta_2(t)=\begin{pmatrix}0.190\\0.147\\0.223\\0.207\end{pmatrix},\quad \eta_3(t)=\begin{pmatrix}0.231\\0.178\\0.270\\0.250\end{pmatrix},$$

$$\Sigma_1(t)=\begin{pmatrix}0.1679 & 0.1242 & 0.0802 & 0.0947\\ 0.1242 & 0.1930 & 0.0959 & 0.1056\\ 0.0802 & 0.0959 & 0.6405 & 0.0346\\ 0.0947 & 0.1056 & 0.0346 & 0.3510\end{pmatrix},$$

$$\Sigma_2(t)=\begin{pmatrix}0.1591 & 0.1196 & 0.0761 & 0.0899\\ 0.1196 & 0.1836 & 0.0914 & 0.1007\\ 0.0761 & 0.0914 & 0.6105 & 0.0333\\ 0.0899 & 0.1007 & 0.0333 & 0.3349\end{pmatrix},$$

$$\Sigma_3(t)=\begin{pmatrix}0.1760 & 0.1310 & 0.0841 & 0.0991\\ 0.1310 & 0.2021 & 0.1007 & 0.1112\\ 0.0841 & 0.1007 & 0.6716 & 0.0360\\ 0.0991 & 0.1112 & 0.0360 & 0.3681\end{pmatrix}.$$

The problem is to find the best portfolio allocation at each moment t that minimizes the function cost (45). Since that (32) is satisfied for each $i=1,\ldots,N$, and $t=1,\ldots,T-1$, we can apply Theorem 2 to solve this problem. Considering $\varrho_i(t)=0.3$, $\xi_i(t)=0.1$, $\theta(1)=1$ and $X(1)=Y(1)=1$ we present in Fig. 1 a Monte Carlo simulation for the benchmark values, the portfolio values using the multi-period strategy derived in this paper,

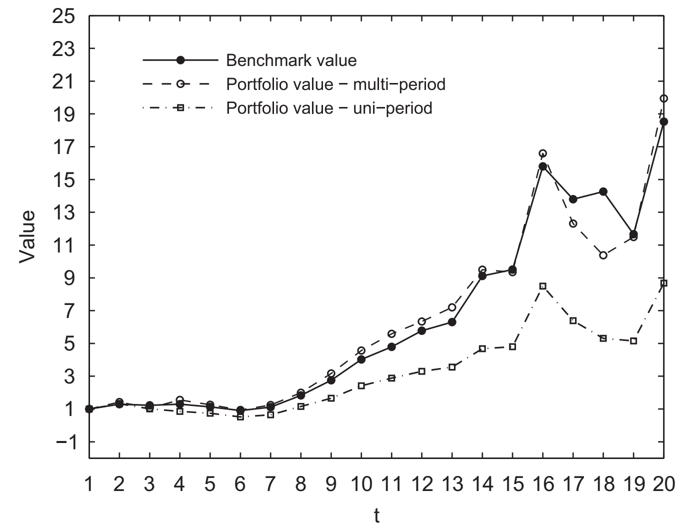

Fig. 1. Monte Carlo simulation of the portfolio values $X(t)$ using the multi-period strategy, the portfolio values $Xm(t)$ using the uni-period strategy, and the benchmark values $Y(t)$.

and the portfolio values using the uni-period strategy based on the Markowitz model described before. Note that the investment goal is to follow the benchmark, having a positive rate of return with respect to it. We can see from Fig. 1 that this goal is indeed achieved, that is, the value of the portfolio follows the value of the benchmark, beating it most of the times. On the other hand, we can see from Fig. 1 that the uni-period strategy cannot reach this goal, due to the fact that in the optimization problem it looks ahead just one step of time.

7. Final remarks

In this paper we have considered the stochastic optimal control problem of discrete-time Markov jump with multiplicative noise linear systems. The performance criterion is assumed to be composed by a linear combination of a quadratic part and a linear part in the state and control variables. The weighting matrices of the state and control for the quadratic part are allowed to be indefinite. Problems of this kind would arise, for instance, in portfolio optimization in which the performance criterion is composed by a linear combination of two terms. The first one is quadratic and represents how closely the portfolio tracks the benchmark. The second one is linear, representing an excess return relative to the benchmark.

We presented a necessary and sufficient condition under which the problem is well posed (Theorem 1) and the optimal control law is written as a state feedback added with a deterministic sequence (Eq. (22)). This solution is derived from a set of coupled generalized Riccati difference equations (15) interconnected with a set of coupled linear recursive equations (16). For the case in which the weighting matrices of the state and control are positive semi-definite, it suffices to verify (32). An example of a portfolio optimization problem with regime switching is presented.

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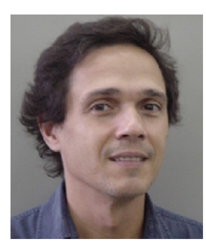

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