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Stability Analysis of Semi-Markov Jump Stochastic Nonlinear Systems

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***Abstract*—The problem of exponential stability for semi-Markov jump stochastic nonlinear systems is studied in this paper. Semi-Markov chain is well known as an extension of the Markov chain, whose sojourn time distribution depends on the current and next states, and is no longer limited to the exponential distribution. However, in the existing works, the independence and the distribution function limitation are imposed on the sojourn time of semi-Markov jump systems. In this paper, without additional constraints for the sojourn time, the problem of almost surely exponential stability is investigated for semi-Markov jump stochastic nonlinear systems by developing a new stochastic analysis method. In addition, mode-dependent linear comparable relationships are assumed among Lyapunov like functions, which can effectively reduce the conservatism caused by mode-independent case. To validate the developed theoretic results, two examples are provided in this paper.**

***Index Terms*—Semi-Markov chain, exponential stability, nonlinear systems, stochastic systems.**

I. Introduction

Stochastic switching systems are usually described as a family of dynamic systems exhibiting stochastic variations among a finite number of dynamic systems, where the stochastic switching is usually induced by some internal and external disturbances, such as random faults, configuration changes, and unexpected events [1, 2]. Some typical applications of stochastic switching systems include electric circuit systems [3], aircrafts [4], vehicles [5], and communication networks [6], etc. Over the past decades, as an important class of stochastic switching systems, Markov jump systems have been extensively considered for linear systems [1, 2] and nonlinear systems [7–11].

For the Markov chain, there are always some restrictions on the sojourn time distribution because of its memoryless characteristic. In discrete-time and continuous-time cases [12], the distribution regarding sojourn time should be geometrically distributed and exponentially distributed, respectively. Due to these restrictions, some practical systems, such as maneuvering target tracking systems [13], biological systems [14] and reward systems [15], are unsuitable to be considered as Markov jump systems. A naturally generalized research frontier is semi-Markov jump systems, whose distribution of sojourn time is allowed to satisfy arbitrary continuous-time distribution, and depends on the current and next states [16]. The stability and control problems have gained considerable attention for semi-Markov jump systems [8, 17–23]. It is pointed out in [24] that the almost sure stability for Markov jump systems is usually the more desirable property in practical applications, because only one path is observed in each time and the mean stability criteria sometimes may be too conservative to be practically useful.

The multiple Lyapunov-like function method (MLLFM), which is widely used to consider the stability of switching systems under constrained switching signals [25], is firstly introduced to analyze the asymptotical stability of nonlinear systems, when considering the stochastic switching [7]. This method can effectively present the differences among each subsystem, and has been extensively utilized in Markov/semi-Markov jump systems to analyze their stability [8, 18, 26–28]. For example, the asymptotical stability is presented for semi-Markov jump nonlinear systems in [8], and the stochastic stability is investigated for semi-Markov jump stochastic nonlinear systems in [27], respectively. For semi-Markov jump nonlinear systems, the generalized moment-stability is tackled in [18] and is further derived for semi-Markov jump stochastic nonlinear systems in [19]. In MLLFMs, linear comparable relationships are assumed to establish the connection among different Lyapunov-like functions [25]. Recently, these assumptions are further generalized to the model-dependent case, which can efficiently reduce the conservatism of the MLLFM, see e. g., Proposition 4 in [29] and Remark 2 in [30].

When analyzing the stability for semi-Markov jump *linear* systems, there is no additional constraint considered for the sojourn time in the existing works [21, 22]. However, some additional constraints are imposed on the sojourn time for semi-Markov jump *nonlinear* systems in some existing works [8, 17, 19, 20, 27]. For example, the distribution of sojourn time only depends on the current state in [8, 17, 20, 27]; the sojourn time sequence is independent of the embedded chain in [8, 27]; the distribution of sojourn time of unstable subsystems should follow some special distribution functions in [19, 27]. On the other hand, although it has been indicated in [29, 30] that the mode-dependent linear comparable assumption can reduce the conservatism, due to the difficulty in estimating the number of mode-dependent state transitions, the mode-independent linear comparable assumption is made for Lyapunov-like functions in [8, 18, 26–28]. Hence, there is still some room to analyze the stability of semi-Markov jump stochastic nonlinear systems when removing some assump-

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tions and reducing the conservativeness simultaneously.

Based on the above discussions, for semi-Markov jump stochastic nonlinear systems, the almost surely exponential stability (ASES) is investigated in this paper. The contributions of this paper are summarized as follows:

1) Under the mode-dependent linear comparable assumption, the exponential stability is considered for semi-Markov jump stochastic nonlinear systems without additional assumptions for the sojourn time as in [8, 17, 19, 20, 27];

2) For the ASES of semi-Markov jump stochastic nonlinear systems, we develop a new analysis technique in this paper. The system state in semi-Markov jump stochastic nonlinear systems is estimated with the technique of stochastic analysis and the law of large numbers.

Notations: Denote $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty)$, $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\mathbb{N}^+ = \{1, 2, \cdots\}$ for sets of real numbers, positive real numbers, nonnegative integers and positive integers, respectively. Let $\Gamma = \{1, 2, \cdots, \upsilon\}$, where $\upsilon > 0$ is a finite integer. The Euclidean norm of x is denoted by $|x|$, for $x \in \mathbb{R}^d$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Denote by $\omega(t) = (\omega_1(t), \cdots, \omega_m(t))^T$ a m-dimensional Brownian motion defined on the probability space. Let $r(t) \in \Gamma$ be a stochastic switching signal on the same probability space, which is independent of the Brownian motion $\omega(\cdot)$. For $(x, i) \in \mathbb{R}^d \times \Gamma$, $\mathcal{C}^2$ is a family of nonnegative functions $V(x, i)$. Moreover, functions $V(x, i)$ in $\mathcal{C}^2$ are assumed to be continuously twice differentiable in x. $\mathbb{E}$ denotes the expectation operator. For real numbers a_1 and a_2, $a_1 \wedge a_2$ denotes $\min\{a_1, a_2\}$. The symbols $\mathcal{E}(b_1)$ and $U(b_2, b_3)$ denote, respectively, the exponential distribution with parameter b_1 and the uniform distribution with parameters b_2, b_3.

II. Preliminaries

First, we concentrate on stochastic nonlinear systems as follows,

$$dx(t) = f_i(x(t))dt + g_i(x(t))d\omega(t), t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^d$ and $i \in \Gamma$. For each $i \in \Gamma$, it is assumed that functions $f_i : \mathbb{R}^d \to \mathbb{R}^d$ and $g_i : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ satisfy some sufficient conditions to guarantee the existence and uniqueness of solution for each system (1). Let $r(t) \in \Gamma$ be a stochastic switching signal, and τ_k be the kth switching time instant with $r(t) = r(\tau_k) \in \Gamma$ for $t \in [\tau_k, \tau_{k+1})$, where $k \in \mathbb{N}$ and $\tau_0 = 0$. Based on the stochastic nonlinear systems defined as (1), special cases with stochastic switching are considered and provided as follows,

$$dx(t) = f_{r(t)}(x(t))dt + g_{r(t)}(x(t))d\omega(t), t \geq 0, \tag{2}$$

where the initial value $x(0) = x_0$ and initial state $r(0) = r_0$ of system (2) are non-random. Let $f_i(0) \equiv 0$ and $g_i(0) \equiv 0$ for $\forall i \in \Gamma$, which yield that system (2) has a trivial solution $x(t) \equiv 0$.

For convenience, some notations will be given for the stochastic switching $r(t)$. Over the interval $[0, t]$, we denote $N_j(t)$, $N_{ji}(t)$ and $N(t)$, for the occurrence number for the state j, the number of transitions from the state j to state i and the number for total switching, respectively. Assume that $T_j(t)$ and $T_{ji}(t)$ are the total sojourn time for state j, and for state j before transferring from state j to state i over the interval $[0, t]$, respectively. Set $s(k) = \tau_{k+1} - \tau_k, k \in \mathbb{N}$. For $k \in \mathbb{N}^+$, $s_j(k)$ and $s_{ji}(k)$ are the sojourn time of state j for the kth visiting, and the sojourn time of the state j at the kth transition from state j to state i, respectively.

Some definitions and assumptions are introduced in the following part before presenting our main results.

Definition 1: [16] A stochastic switching $r(t)$ is said to be a continuous-time semi-Markov chain, if let $r_k = r(\tau_k)$ for $k \in \mathbb{N}$ be the embedded chain, and the following two conditions satisfy:

i) the discrete-time process $\{r_k, s(k)\}$ is a Markov renewal process and

$$\begin{aligned} &P(r_{k+1} = j, s(k) \leq t | r_k, s(k-1), \cdots, r_1, s(0)) \\ =&P(r_{k+1} = j, s(k) \leq t | r_k), j \in \Gamma, t \geq 0, k \in \mathbb{N}, \end{aligned}$$

ii) for $\forall i, j \in \Gamma$, the distribution function of sojourn time $s(k)$ is denoted by

$$F_{ij}(t) = P(s(k) \leq t | r_{k+1} = j, r_k = i), t \geq 0, k \in \mathbb{N},$$

which depends on r_k and r_{k+1}, and is independent of k.

For $\forall V \in \mathcal{C}^2$, let

$$\mathcal{K}V(x, i) = V_x(x, i)g_i(x),$$

and $\mathcal{L}V$ be a mapping defined as in [27],

$$\mathcal{L}V(x, i) = V_x(x, i)f_i(x) + \frac{1}{2}\text{trace}[g_i^T(x)V_{xx}(x, i)g_i(x)],$$

where $i \in \Gamma$, $V_x(x, i) = (\frac{\partial V(x,i)}{\partial x_1}, \cdots, \frac{\partial V(x,i)}{\partial x_d})$, $V_{xx}(x, i) = (\frac{\partial^2 V(x,i)}{\partial x_{d_1} \partial x_{d_2}})_{d\times d}$ and $1 \leq d_1, d_2 \leq d$.

Assumption 1: Suppose that there are constants $\lambda_i \in \mathbb{R}$, $p > 0, c > 0$, $\eta_i \geq 0$ and $\mu_{ji} > 0$ for $\forall i, j \in \Gamma$ such that

(H_1) $c|x|^p \leq V(x, i)$,

(H_2) for $V \in \mathcal{C}^2$,

$$\mathcal{L}V(x, i) \leq \lambda_i V(x, i), \tag{3}$$

and

$$\eta_i |V(x, i)|^2 \leq |\mathcal{K}V(x, i)|^2, \tag{4}$$

(H_3) $V(x, i) \leq \mu_{ji} V(x, j)$, with $\mu_{jj} = 1$.

Remark 1: (H_1) implies that these Lyapunov-like functions are radially unbounded. In (H_2), assumption (3) implies that stochastic subsystems may be stable or unstable. While Assumption (4) in (H_2) is widely used to study the ASES for stochastic differential systems driven by Brownian motion [11, 31, 32], it is introduced to deal with the stochastic integral in the following. The mode-dependent linear comparable relationship is assumed in (H_3) among the Lyapunov-like functions, which can effectively establish the connection between different Lyapunov-like functions. In [8, 18, 26, 27], a mode-independent relationship is assumed that $V(x, i) \leq \mu V(x, j)$ with a constant $\mu \geq 1$. In this paper, $\mu \geq 1$ is generalized to $\mu_{ij} > 0$, which can effectively reduce the conservativeness [29], [30]. It is worth pointing out that, by using (H_1), (3) and the local Lipschitz condition, the existence and uniqueness of solution can be guaranteed for each system (1) by means of Theorem 3.19 in [9].

Assumption 2: For the stochastic switching $r(t)$, assume that the expectation of sojourn time $\mathbb{E}s_j(k) < +\infty$ for $\forall j \in \Gamma$ and $k \in \mathbb{N}^+$.

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Definition 2: [9] Let $x(t)$ be the state trajectory of system (2) with stochastic switching. Then system (2) has the almost surely exponential stability if

$$\limsup_{t\to\infty} \frac{1}{t} \ln |x(t)| < 0, \quad \text{a. s.,}$$

for any initial condition (x_0, r_0).

In the following, we always assume that $x_0 \neq 0$. In fact, if $x_0 = 0$, then the solution $x(t) \equiv 0$. It yields that system (2) is almost surely exponentially stable. In addition, for any initial state $x_0 \neq 0$, it can be got from Lemma 2.1 in [31] that $x(t)$ in system (2) will never reach to zero in probability 1.

III. Main results

This section will be divided into two parts. The ASES of semi-Markov jump stochastic nonlinear systems and Markov jump systems will be separately studied with the limit theory and stochastic analysis method in these two parts.

A. Stability of semi-Markov jump stochastic nonlinear systems

Let the stochastic switching $r(t)$ in system (2) be a semi-Markov chain, which implies that $\{r_k, k \in \mathbb{N}\}$ is a Markov chain. Assume that this Markov chain $\{r_k, k \in \mathbb{N}\}$ is irreducible, which has a transition probability matrix $P = [p_{ij}]_{\upsilon\times\upsilon}$ and a stationary distribution $\bar{\pi} = \{\bar{\pi}_1, \bar{\pi}_2, \cdots, \bar{\pi}_\upsilon\}$, where $p_{ii} = 0$ for $\forall i \in \Gamma$.

By utilizing the strong law of large numbers, a lemma will be shown in the following to estimate the switching times between states of semi-Markov chain.

Lemma 1: Assume that $r(t)$ is a semi-Markov chain, and its embedded chain $\{r_k, k \in \mathbb{N}\}$ has a transition matrix P and a stationary distribution $\bar{\pi}$. Then for $\forall i, j \in \Gamma$

$$\lim_{t\to\infty} \frac{N_{ji}(t)}{t} = \frac{\pi_j p_{ji}}{m_j}, \quad \text{a. s.,}$$

where $m_j = \mathbb{E}s_j(1)$ and $\pi_j = \frac{\bar{\pi}_j m_j}{\sum_{l\in\Gamma} \bar{\pi}_l m_l}$.

Proof: For the embedded chain $\{r_k, k \in \mathbb{N}\}$, let $n_j(k)$ be the time instant for the kth visiting to the state j, then for $\forall k \in \mathbb{N}^+$,

$$\begin{aligned}
&P(s_j(k) \le t | s_j(k-1), \cdots, s_j(1)) \\
=&P(\tau_{n_j(k)+1} - \tau_{n_j(k)} \le t | r_{n_j(k)} = j, s_j(k-1), \cdots, s_j(1)) \\
=&P(\tau_{n_j(k)+1} - \tau_{n_j(k)} \le t | r_{n_j(k)} = j),
\end{aligned} \tag{5}$$

which implies that $\{s_j(k), k \in \mathbb{N}^+\}$ is an independent stochastic sequence. It can be verified from Definition 1 that

$$\begin{aligned}
&P(\tau_{n_j(k)+1} - \tau_{n_j(k)} \le t | r_{n_j(k)} = j) \\
=&\sum_{l\in\Gamma} P(\tau_{n_j(k)+1} - \tau_{n_j(k)} \le t, r_{n_j(k)+1} = l | r_{n_j(k)} = j) \\
=&\sum_{l\in\Gamma} F_{jl}(t) p_{jl},
\end{aligned} \tag{6}$$

which is independent of the switching instant and next mode. By using (5) and (6), we derive that $s_j(k)$ are i. i. d. random variables and $\mathbb{E}s_j(k) = m_j$, for $\forall j \in \Gamma$ and $k \in \mathbb{N}^+$. Assume that for the embedded chain, $\bar{n}_{ji}(k)$ is the time instant for the kth visiting to the state j and $r_{\bar{n}_{ji}(k)+1} = i$, then for $\forall k \in \mathbb{N}^+$,

$$\begin{aligned}
&P(s_{ji}(k) \le t) \\
=&P(\tau_{\bar{n}_{ji}(k)+1} - \tau_{\bar{n}_{ji}(k)} \le t | r_{\bar{n}_{ji}(k)+1} = i, r_{\bar{n}_{ji}(k)} = j) \\
=&F_{ji}(t),
\end{aligned}$$

which is independent of the switching instant. Similarly, it can be checked that $s_{ji}(k)$ are i. i. d. for $\forall j, i \in \Gamma$ and $k \in \mathbb{N}^+$. If $\mathbb{E}s_{ji}(1) = m_{ji}$, it follows that $\mathbb{E}s_{ji}(k) = m_{ji}$ for $\forall k \in \mathbb{N}^+$. Thus, for $\forall i, j \in \Gamma$, we can get from the strong law of large numbers that

$$\begin{aligned}
&\lim_{t\to\infty} \frac{T_j(t)}{N_j(t)} = \lim_{t\to\infty} \frac{\sum_{k=1}^{N_j(t)} s_j(k)}{N_j(t)} \\
=&\mathbb{E}s_j(k) = m_j, \quad \text{a. s.,}
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&\lim_{t\to\infty} \frac{T_{ji}(t)}{N_{ji}(t)} = \lim_{t\to\infty} \frac{\sum_{k=1}^{N_{ji}(t)} s_{ji}(k)}{N_{ji}(t)} \\
=&\mathbb{E}s_{ji}(k) = m_{ji}, \quad \text{a. s.}
\end{aligned} \tag{8}$$

Since for $\forall k \in \mathbb{N}$,

$$P(r_{k+1} = i | r_k = j) = p_{ji},$$

by using the strong law of large numbers, we obtain that

$$\lim_{t\to\infty} \frac{N_{ji}(t)}{N_j(t)} = p_{ji}, \quad \text{a. s.} \tag{9}$$

Combining (7) and (8) with (9) implies that

$$\lim_{t\to\infty} \frac{T_{ji}(t)}{T_j(t)} = \lim_{t\to\infty} \frac{\frac{T_{ji}(t)}{N_{ji}(t)}}{\frac{T_j(t)}{N_j(t)}} \frac{N_{ji}(t)}{N_j(t)} = \frac{m_{ji} p_{ji}}{m_j}, \quad \text{a. s.} \tag{10}$$

Noticing that the embedded Markov chain $\{r_k, k \in \mathbb{N}\}$ is an irreducible Markov chain, we can get from Ergodic theorem [12] that

$$\lim_{t\to\infty} \frac{N_i(t)}{N(t)} = \bar{\pi}_i, \quad \text{a. s.} \tag{11}$$

Obviously,

$$\begin{aligned}
\frac{T_j(t)}{t} =& \frac{\sum_{k=1}^{N_j(t)} s_j(k)}{\sum_{l\in\Gamma} \sum_{k=1}^{N_l(t)} s_l(k)} \\
=& \frac{\frac{N_j(t)}{N(t)} \frac{1}{N_j(t)} \sum_{k=1}^{N_j(t)} s_j(k)}{\sum_{l\in\Gamma} \frac{N_l(t)}{N(t)} \frac{1}{N_l(t)} \sum_{k=1}^{N_l(t)} s_l(k)}.
\end{aligned} \tag{12}$$

Substituting (7) and (11) into (12) yields that

$$\lim_{t\to\infty} \frac{T_j(t)}{t} = \frac{\bar{\pi}_j m_j}{\sum_{l\in\Gamma} \bar{\pi}_l m_l} = \pi_j, \quad \text{a. s.} \tag{13}$$

It can be checked from (10) and (13) that

$$\lim_{t\to\infty} \frac{T_{ji}(t)}{t} = \lim_{t\to\infty} \frac{T_j(t)}{t} \frac{T_{ji}(t)}{T_j(t)} = \pi_j \frac{m_{ji} p_{ji}}{m_j}, \quad \text{a. s.} \tag{14}$$

By using (8) and (14), we have

$$\lim_{t\to\infty} \frac{N_{ji}(t)}{t} = \lim_{t\to\infty} \frac{T_{ji}(t)}{t} \frac{N_{ji}(t)}{T_{ji}(t)} = \frac{\pi_j p_{ji}}{m_j}, \quad \text{a. s.}$$

The proof is completed. □

To obtain an upper bound for the Lyapunov-like function, the following lemma is provided, which is vital to investigate the exponential stability of system (2).

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Lemma 2: Let Assumption 1 hold, and constants $\lambda_i \in \mathbb{R}, \mu_{ji} > 0$ for $\forall i, j \in \Gamma$, then,

$$\begin{aligned} &\ln V(x(t), r(t)) \\ \leq &\ln V(x_0, r_0) + \sum_{i,j\in\Gamma} N_{ji}(t) \ln \mu_{ji} \\ &+ \int_0^t [\lambda_{r(s)} - 0.5\Xi(s)]ds + \Lambda(t, 0), \end{aligned} \tag{15}$$

where

$$\Xi(t) = \frac{|\mathcal{K}V(x(t), r(t))|^2}{V^2(x(t), r(t))},$$

and

$$\Lambda(t, 0) = \int_0^t \frac{\mathcal{K}V(x(s), r(s))}{V(x(s), r(s))} d\omega(s).$$

Proof: With the Itô formula in [9], it can be obtained that

$$\begin{aligned} &d[\ln V(x(t), r(t))] \\ =&\frac{1}{V(x(t), r(t))}[\mathcal{L}V(x(t), r(t))dt + \mathcal{K}V(x(t), r(t))d\omega(t)] \\ &- \frac{1}{2V^2(x(t), r(t))}|\mathcal{K}V(x(t), r(t))|^2 dt, \end{aligned}$$

where $\forall t \in [\tau_k, \tau_{k+1})$ and $k \in \mathbb{N}^+$. We have that

$$\begin{aligned} &\ln V(x(t), r(t)) \\ \leq &\ln V(x(\tau_k), r(\tau_k)) + \Lambda(t, \tau_k) \\ &+ \int_{\tau_k}^t \big[\frac{\mathcal{L}V(x(s), r(s))}{V(x(s), r(s))} - 0.5\Xi(s)\big]ds. \end{aligned} \tag{16}$$

Combining (16) with (H_2) in Assumption 1, it implies that

$$\begin{aligned} &\ln V(x(t), r(t)) \leq \ln V(x(\tau_k), r(\tau_k)) \\ &+ \int_{\tau_k}^t (\lambda_{r(s)} - 0.5\Xi(s))ds + \Lambda(t, \tau_k), \end{aligned} \tag{17}$$

where $\forall t \in [\tau_k, \tau_{k+1})$ and $k \in \mathbb{N}^+$. It can be verified from (H_3) in Assumption 1 that

$$\begin{aligned} &\ln V(x(\tau_k), r(\tau_k)) \\ \leq &\ln V(x(\tau_k), r(\tau_{k-1})) + \ln \mu_{r(\tau_{k-1})r(\tau_k)}. \end{aligned} \tag{18}$$

By (17) and (18), we have

$$\begin{aligned} &\ln V(x(t), r(t)) \\ \leq &\ln V(x(\tau_{k-1}), r(\tau_{k-1})) + \ln \mu_{r(\tau_{k-1})r(\tau_k)} \\ &+ \int_{\tau_{k-1}}^t (\lambda_{r(s)} - 0.5\Xi(s))ds + \Lambda(t, \tau_{k-1}). \end{aligned}$$

By repeating this procedure, one sees that

$$\begin{aligned} &\ln V(x(t), r(t)) \\ \leq &\ln V(x_0, r_0) + \sum_{k=1}^{N(t)} \ln \mu_{r(\tau_{k-1})r(\tau_k)} \\ &+ \int_0^t [\lambda_{r(s)} - 0.5\Xi(s)]ds + \Lambda(t, 0). \end{aligned} \tag{19}$$

Set $\mathcal{V}(j) = \{i : p_{ji} > 0\}$. Since $\mu_{jj} = 1$ for $\forall j \in \Gamma$, we get that

$$\prod_{k=1}^{N(t)} \mu_{r(t_{k-1})r(t_k)} = \prod_{\substack{\forall i\in\Gamma,\\ i\in\mathcal{V}(j)}} \mu_{ji}^{N_{ji}(t)} = \prod_{i,j\in\Gamma} \mu_{ji}^{N_{ji}(t)},$$

where $N_{ji}(t) = 0$, for $i \notin \mathcal{V}(j)$. This implies that

$$\sum_{k=1}^{N(t)} \ln \mu_{r(\tau_{k-1})r(\tau_k)} = \sum_{i,j\in\Gamma} N_{ji}(t) \ln \mu_{ji}. \tag{20}$$

Thus, (15) is obtained by utilizing (19) and (20). □

For the semi-Markov jump stochastic nonlinear system in (2), the ASES will be investigated with the help of Lemmas 1 and 2.

Theorem 1: Let constants $\lambda_i \in \mathbb{R}$, $\eta_i \geq 0, \mu_{ij} > 0$, π_i and m_i be defined the same as in Assumption 1 and Lemma 1 for $i, j \in \Gamma$. If Assumption 1 holds and

(H_4) $\sum_{i\in\Gamma} \pi_i\big(\lambda_i - 0.5\eta_i + \sum_{j\in\Gamma} \frac{p_{ij} \ln \mu_{ij}}{m_i}\big) < 0$,

then the semi-Markov jump stochastic nonlinear system in (2) is almost surely exponentially stable.

Proof: It can be deduced from (H_4) that

$$\sum_{i\in\Gamma} \pi_i(\lambda_i - 0.5\eta_i) + \sum_{i,j\in\Gamma} \frac{\pi_j p_{ji} \ln \mu_{ji}}{m_j} < 0. \tag{21}$$

Thus, there should exist two sufficiently small constants $\varrho > 0$ and $0 < \varepsilon < 1$ such that $\lambda < 0$, where

$$\lambda = \sum_{i\in\Gamma} \pi_i(\lambda_i - 0.5(1-\varepsilon)\eta_i) + \sum_{i,j\in\Gamma} \big(\frac{\pi_j p_{ji} \ln \mu_{ji}}{m_j} + \varrho|\ln \mu_{ji}|\big). \tag{22}$$

According to Lemma 1, there should exist a constant $T(\varrho) > 0$ such that if $t \geq T(\varrho)$, for $\forall i, j \in \Gamma$,

$$(\frac{\pi_j p_{ji}}{m_j} - \varrho)t \leq N_{ji}(t) \leq (\frac{\pi_j p_{ji}}{m_j} + \varrho)t, \quad \text{a. s.} \tag{23}$$

It leads to that for any $t \geq T(\varrho)$,

$$\begin{aligned} &\sum_{i,j\in\Gamma} N_{ji}(t) \ln \mu_{ji} \\ \leq &\sum_{i,j\in\Gamma} (\frac{\pi_j p_{ji} \ln \mu_{ji}}{m_j} + \varrho|\ln \mu_{ji}|)t, \quad \text{a. s.} \end{aligned} \tag{24}$$

In fact, if $0 < \mu_{ji} \leq 1$,

$$N_{ji}(t) \ln \mu_{ji} \leq (\frac{\pi_j p_{ji} \ln \mu_{ji}}{m_j} - \varrho \ln \mu_{ji})t, \quad \text{a. s.} \tag{25}$$

On the other hand, if $\mu_{ji} > 1$,

$$N_{ji}(t) \ln \mu_{ji} \leq (\frac{\pi_j p_{ji} \ln \mu_{ji}}{m_j} + \varrho \ln \mu_{ji})t, \quad \text{a. s.} \tag{26}$$

Thus, inequality (24) is proved by using (25) and (26). Let

$$\begin{aligned} \Theta(t) = &\int_0^t (\lambda_{r(s)} - 0.5(1-\varepsilon)\eta_{r(s)})ds \\ &+ \big(\sum_{i,j\in\Gamma} (\frac{\pi_j p_{ji} \ln \mu_{ji}}{m_j} + \varrho|\ln \mu_{ji}|)\big)t. \end{aligned}$$

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We can deduce from (13) that

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\int_0^t(\lambda_{r(s)}-0.5(1-\varepsilon)\eta_{r(s)})ds\\ =&\lim_{t\to\infty}\frac{1}{t}\sum_{i\in\Gamma}\int_0^t(\lambda_i-0.5(1-\varepsilon)\eta_i)I_{(r(s)=i)}(s)ds\\ =&\sum_{i\in\Gamma}(\lambda_i-0.5(1-\varepsilon)\eta_i)\lim_{t\to\infty}\frac{T_i(t)}{t}\\ =&\sum_{i\in\Gamma}(\lambda_i-0.5(1-\varepsilon)\eta_i)\pi_i,\quad \text{a. s.}\end{aligned}\tag{27}$$

According to (22) and (27), one sees that

$$\lim_{t\to\infty}\frac{\Theta(t)}{t}=\lambda<0,\quad \text{a. s.}\tag{28}$$

On the other side, by using Theorem 1.24 in [9], we have $\Lambda(0,0)=0$ and

$$\Lambda(t,0)=\int_0^t\frac{\mathcal{K}V(x(s),r(s))}{V(x(s),r(s))}d\omega(s),$$

is a continuous martingale. Its quadratic variation is given as follows

$$\langle\Lambda(t,0),\Lambda(t,0)\rangle=\int_0^t\Xi(s)ds.$$

It can be shown from the exponential martingale inequality in [9] that for ε defined in (22) and $\nu\in\mathbb{N}^+$,

$$P\{\sup_{1\le t\le\nu}[\Lambda(t,0)-\frac{\varepsilon}{2}\langle\Lambda(t,0),\Lambda(t,0)\rangle]\ge\frac{2\ln\nu}{\varepsilon}\}\le\frac{1}{\nu^2}.$$

According to the Borel-Cantelli lemma, there is an integer $\bar{k}>T(\varrho)+1$ such that if $\nu>\bar{k}$,

$$\Lambda(t,0)\le\frac{2\ln\nu}{\varepsilon}+\frac{\varepsilon}{2}\langle\Lambda(t,0),\Lambda(t,0)\rangle,\quad \text{a. s.},\tag{29}$$

where $0\le t\le\nu$. Noticing that the positive constant $\varepsilon<1$ is assumed in (22), by using (4) and (29), we have for $\nu>\bar{k}$,

$$\begin{aligned}&\int_0^t[\lambda_{r(s)}-0.5\Xi(s)]ds+\Lambda(t,0)\\ \le&\frac{2\ln\nu}{\varepsilon}+\int_0^t[\lambda_{r(s)}-0.5\Xi(s)]ds+\frac{\varepsilon}{2}\int_0^t\Xi(s)ds\\ =&\frac{2\ln\nu}{\varepsilon}+\int_0^t[\lambda_{r(s)}-0.5(1-\varepsilon)\frac{|\mathcal{K}V(x(s),r(s))|^2}{V^2(x(s),r(s))}]ds\\ \le&\frac{2\ln\nu}{\varepsilon}+\int_0^t[\lambda_{r(s)}-0.5(1-\varepsilon)\eta_{r(s)}]ds,\quad \text{a. s.}\end{aligned}\tag{30}$$

Substituting (24) and (30) into (15), we can find that for any $\bar{k}<t\le\nu$,

$$\ln V(x(t),r(t))\le\ln V(x_0,r_0)+\frac{2\ln\nu}{\varepsilon}+\Theta(t),\quad \text{a. s.},$$

which implies that for $\bar{k}\le\nu-1<t\le\nu$,

$$\begin{aligned}&\frac{1}{t}\ln V(x(t),r(t))\\ \le&\frac{1}{\nu-1}(\ln V(x_0,r_0)+\frac{2\ln\nu}{\varepsilon})+\frac{\Theta(t)}{t},\quad \text{a. s.}\end{aligned}\tag{31}$$

Combining (28) with (31) leads to

$$\limsup_{t\to\infty}\frac{1}{t}\ln V(x(t),r(t))=\lambda<0,\quad \text{a. s.}$$

In view of (H_1) in Assumption 1, we have

$$\limsup_{t\to\infty}\frac{1}{t}\ln|x(t)|<0,\quad \text{a. s.}$$

Then, the almost surely exponential stability is obtained for the semi-Markov jump system in (2). □

Remark 2: For a semi-Markov chain, the distribution of its sojourn time is allowed to satisfy arbitrary continuous-time distribution, and depends on the current and next states. For the semi-Markov chain in [8, 17, 19, 20, 27], in order to facilitate the stability analysis, some special assumptions are imposed on its sojourn time. For example, the distribution of sojourn time only depends on the current state in [8, 17, 20, 27]; the sojourn time sequence is independent of the embedded chain in [8, 27]; the distribution of sojourn time of unstable subsystems should follow some special distribution functions in [19, 27]. In Theorem 1, the ASES is investigated for semi-Markov jump stochastic nonlinear systems, where the above assumptions imposed on the sojourn time in [8, 17, 19, 20, 27] are removed.

Remark 3: It is pointed out in [29] and [30] that the mode-dependent linear comparable assumption for Lyapunov-like functions can effectively reduce the conservatism in the M-LLFM. The *mode-independent* linear comparable assumption is provided in [8, 18, 26, 27] for Lyapunov-like functions, which is generalized to *mode-dependent* case in (H_3) of Theorem 1. In Theorem 1, under this mode-dependent assumption, the exponential stability is considered for semi-Markov jump stochastic nonlinear systems.

The ASES will be considered in the following for semi-Markov jump stochastic nonlinear systems with a partial mode-dependent linear comparable assumption, where the coefficients only depend on the current state.

Corollary 1: Let constants $\mu_i>1$, $\lambda_i\in\mathbb{R},\eta_i\ge 0$, π_i and m_i be defined the same as in Assumption 1 and Lemma 1 for $i,j\in\Gamma$. Assume that (H_1) and (H_2) in Assumption 1 hold and

(H_3') $V(x,j)\le\mu_iV(x,i)$, for $\forall i,j\in\Gamma$,

(H_5) $\sum_{i\in\Gamma}\pi_i\big(\lambda_i-0.5\eta_i+\frac{\ln\mu_i}{m_i}\big)<0$,

then the semi-Markov jump stochastic nonlinear system in (2) is almost surely exponentially stable.

Proof: By letting $\mu_{ij}\equiv\mu_i$ in (H_4) of Theorem 1, we get that

$$\begin{aligned}&\sum_{i\in\Gamma}\pi_i\big(\lambda_i-0.5\eta_i+\sum_{j\in\Gamma}\frac{p_{ij}\ln\mu_{ij}}{m_i}\big)\\ =&\sum_{i\in\Gamma}\pi_i\big(\lambda_i-0.5\eta_i+\frac{\ln\mu_i}{m_i}\big),\end{aligned}$$

where $\sum_{j\in\Gamma}p_{ij}=1$ is used. Thus, Corollary 1 follows from Theorem 1. □

If let $g_i(x)\equiv 0$ in system (2), then the ASES can be derived for system (2) without stochastic disturbances.

Corollary 2: Let constants $\mu_i>1$, λ_i, π_i and m_i be defined the same as in Corollary 1 for $i\in\Gamma$. Assume that (H_1) and (H_2) in Assumption 1 hold and

(H_3') $V(x,j)\le\mu_iV(x,i)$, for $\forall i,j\in\Gamma$,

(H_6) $\sum_{i\in\Gamma}\pi_i\big(\lambda_i+\frac{\ln\mu_i}{m_i}\big)<0$, then the semi-Markov jump nonlinear system in (2) without stochastic disturbances is almost surely exponentially stable.

Proof: Noticing that $g_i(x) \equiv 0$, we get that $\eta_i \equiv 0$ in Corollary 1 for any $i \in \Gamma$. Therefore, Corollary 2 follows from Corollary 1. $\square$

Remark 4: Compared with [28], the contributions of this paper lie in the following two aspects. 1) The model is more general. This paper has extended the semi-Markov jump deterministic systems in [28] to stochastic systems; 2) The assumptions are more general. In this paper, the partial mode-dependent assumption (H_3') in [28] has been generalized to mode-dependent assumption (H_3), with the purpose of reducing the conservatism as shown in Case II of Example 2.

B. Stability of Markov jump stochastic nonlinear systems

In this subsection, the stochastic switching $r(t)$ in system (2) is a continuous-time Markov chain, and the ASES will be considered for the Markov jump stochastic nonlinear system in (2).

Let $r(t)$ be a Markov chain with generator $Q = [q_{ij}]_{\upsilon\times\upsilon}$ denoted by

$$P\{r(t+\vartheta) = j|r(t) = i\} = \begin{cases} q_{ij}\vartheta + o(\vartheta), & i \neq j, \\ 1 + q_{ij}\vartheta + o(\vartheta), & i = j, \end{cases}$$

where $\lim_{\vartheta\to 0} \frac{o(\vartheta)}{\vartheta} = 0$ with $i, j \in \Gamma, \vartheta > 0$. For $i \neq j$, $q_{ij} \geq 0$ is the transition rate from i to j, and $q_{ii} = -\sum_{j\neq i} q_{ij}$. Set $q_i = |q_{ii}| > 0$, for $\forall i \in \Gamma$, i. e., there is no absorbing state for the Markov chain $r(t)$. Assume that the embedded chain $r_k = r(\tau_k)$ for $k \in \mathbb{N}$, which is an irreducible discrete-time Markov chain. It has a transition matrix $P = [p_{ij}]_{\upsilon\times\upsilon}$ and a stationary distribution $\bar{\pi} = (\bar{\pi}_1, \bar{\pi}_2, \cdots, \bar{\pi}_\upsilon)$, where

$$p_{ij} = \begin{cases} \frac{q_{ij}}{q_i}, & i \neq j, \\ 0, & i = j. \end{cases}$$

Thus, it can be obtained from the properties of Markov chain [12] that $\mathbb{E}s_i(1) = \frac{1}{q_i}$, and there exits a unique stationary distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_\upsilon)$ for Markov chain $r(t)$, where $\pi_j = \frac{\frac{\bar{\pi}_j}{q_j}}{\sum_{l\in\Gamma}\frac{\bar{\pi}_l}{q_l}}, j \in \Gamma$.

Firstly, an estimation for the number of switching will be derived for states of Markov chain.

Lemma 3: Assume that Markov chain $r(t)$ has a generator Q and a stationary distribution π. Then

$$\lim_{t\to\infty} \frac{N_{ji}(t)}{t} = \pi_j q_{ji}, \quad \text{a. s.},$$

with $i \neq j$ and $i, j \in \Gamma$.

Proof: It can be checked that $r(t)$ is a semi-Markov chain with $m_i = \frac{1}{q_i}$ and $p_{ji} = \frac{q_{ji}}{q_i}$. Thus, according to Lemma 1, we have

$$\lim_{t\to\infty} \frac{N_{ji}(t)}{t} = \frac{\pi_j p_{ji}}{m_j} = \pi_j q_{ji}, \quad \text{a. s.},$$

where $i \neq j$. The proof is completed. $\square$

The ASES will be presented in the following for the Markov jump stochastic nonlinear system in (2).

Theorem 2: Let constants $\lambda_i \in \mathbb{R}$, $\mu_{ij} > 0, \eta_i \geq 0$, π_i and q_{ij} be defined the same as in Assumption 1 and Lemma 3 for $i, j \in \Gamma$. If Assumption 1 holds and
(H_7) $\sum_{i\in\Gamma} \pi_i\big(\lambda_i - 0.5\eta_i + \sum_{j\in\Gamma} q_{ij}\ln\mu_{ij}\big) < 0$,
then the Markov jump stochastic nonlinear system in (2) is almost surely exponentially stable.

Proof: Since $\mu_{ii} = 1$ for $\forall i \in \Gamma$, we have

$$\begin{aligned}&\sum_{i\in\Gamma} \pi_i\big(\lambda_i - 0.5\eta_i + \sum_{\substack{j\in\Gamma,\\ j\neq i}} q_{ij}\ln\mu_{ij}\big) \\ =&\sum_{i\in\Gamma} \pi_i\big(\lambda_i - 0.5\eta_i + \sum_{j\in\Gamma} q_{ij}\ln\mu_{ij}\big).\end{aligned}$$

Thus, Theorem 2 can be similarly proved by the proof presented in Theorem 1, and it is omitted here. $\square$

By using Theorem 2, we can derive the following corollary for Markov jump nonlinear systems in (2) without stochastic disturbances.

Corollary 3: Let constants $\mu_i > 1$, $\lambda_i \in \mathbb{R}, \eta_i \geq 0$, π_i and q_{ij} be defined the same as in Assumption 1 and Lemma 3 for $i, j \in \Gamma$. Assume that (H_1) and (H_2) in Assumption 1 hold and
(H_3') $V(x, j) \leq \mu_i V(x, i)$, for $\forall i, j \in \Gamma$,
(H_8) $\sum_{i\in\Gamma} \pi_i\big(\lambda_i + q_i \ln\mu_i\big) < 0$,
then the Markov jump stochastic nonlinear system in (2) is almost surely exponentially stable.

Proof: This corollary can be similarly proved by Corollary 2, here we omit it. $\square$

Remark 5: With the help of the stationary distribution regarding semi-Markov/Markov chain, the results in this paper show that the ASES can be obtained for semi-Markov/Markov jump systems. However, it should be pointed out that the stationary distribution is insufficient to have the mean square stability. The following example illustrates this point.

Example 1: Consider the following Markov jump linear scalar stochastic systems,

$$dx(t) = a_{r(t)}x(t)dt + x(t)d\omega(t), \tag{32}$$

where $r(t) \in \{1, 2\}$, $a_1 = -2$, $a_2 = 2$, and $r(t)$ is a Markov chain with the generator

$$Q = \begin{bmatrix} -\alpha & \alpha \\ 2\alpha & -2\alpha \end{bmatrix},$$

with $\alpha > 0$ and $\omega(t)$ is a standard scalar Brownian motion. Assume that $r(t)$ is independent of $\omega(t)$. Obviously,

$$x(t) = x_0 \exp\{\int_0^t (a_{r(s)} - 0.5)ds + \int_0^t d\omega(s)\},$$

and $r(t)$ exits a stationary distribution $\pi = (\frac{2}{3}, \frac{1}{3})$. In view of $\sum_{i=1}^2 \pi_i(a_i - 0.5) < -1$, we get that system (32) is almost surely exponentially stable. However, system (32) is mean square unstable with $\alpha < 2.5$. Let the initial distribution of $r(t)$ be $\pi_0 = (0, 1)$. Since

$$P(r(u) = 2, 0 \leq u \leq t|r(0) = 2) = e^{-2\alpha t},$$

and $\mathbb{E}e^{2\omega(t)} = e^{2t}$,

$$\begin{aligned}&\mathbb{E}|x(t)|^2 \\ \geq&|x_0|^2 e^{\int_0^t (2a_2-1)ds}\mathbb{E}e^{2\omega(t)} P(r(u) = 2, 0 \leq u \leq t|r(0) = 2) \\ =&|x_0|^2 e^{(2a_2-1)t} e^{2t} P(r(u) = 2, 0 \leq u \leq t|r(0) = 2) \\ =&|x_0|^2 e^{(5-2\alpha)t}.\end{aligned}$$

Thus, if $\alpha < 2.5$, then

$$\liminf_{t\to\infty} \mathbb{E}|x(t)|^2 = +\infty,$$

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which implies that (32) is mean square unstable. This example shows that for semi-Markov/Markov jump systems, the stationary distribution of $r(t)$ is insufficient to have the mean square stability.

IV. Examples

To validate the proposed results, two examples are provided.

Example 2: A stochastic nonlinear system with stochastic switching is considered as follows:

$$dx(t) = f_{r(t)}(x(t))dt + g_{r(t)}(x(t))d\omega(t), \tag{33}$$

where

$$f_1(x) = \begin{bmatrix} -3x_1 + x_2 \\ (x_1 + x_2)\sin x_1 - 4x_2 \end{bmatrix}, g_1(x) = \begin{bmatrix} x_1 \cos x_2 \\ x_2 \end{bmatrix},$$
$$f_2(x) = \begin{bmatrix} -1.5x_1 - x_1^3 \\ x_1 - 1.5x_2 \end{bmatrix}, g_2(x) = \begin{bmatrix} x_1 \\ x_2 \sin x_1 \end{bmatrix},$$
$$f_3(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, g_3(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Let

$$V_1(x) = x_1^2 + x_2^2, V_2(x) = \frac{1}{2}(x_1^2 + x_2^2), V_3(x) = \frac{1}{2}x_1^2 + x_2^2,$$

be the candidate Lyapunov-like functions, which yield that $\lambda_1 = -3$, $\lambda_2 = -1$ and $\lambda_3 = 3$. We can select $\mu_{21} = \mu_{31} = 2, \mu_{12} = 0.5, \mu_{32} = 1, \mu_{13} = 1$, $\mu_{23} = 2$, $\eta_1 = \eta_2 = 0$ and $\eta_3 = 1$. It can be verified that the 1st and 2nd subsystems are stable, and the 3rd subsystem is unstable. In the following, the stochastic nonlinear system in (33) is shown to be exponentially stable with semi-Markov jump and Markov jump, respectively.

Case I: The switching signal $r(t)$ is a semi-Markov chain, which has an embedded chain $\{r_k, k \in \mathbb{N}\}$ with a transition probability matrix

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.8 & 0 & 0.2 \\ 0.7 & 0.3 & 0 \end{bmatrix}.$$

It follows that $\bar{\pi} = [0.4292, 0.2968, 0.2740]$ is the unique stationary distribution for $\{r_k, k \in \mathbb{N}\}$. Suppose that $F_{12}(t) = F_{13}(t) \sim \mathcal{E}(1)$, $F_{21}(t) \sim \mathcal{E}(0.5)$, $F_{23}(t) \sim U(0,3)$ and $F_{31}(t) = F_{32}(t) \sim U(0,2)$. Noticing that

$$m_i = \sum_l \mathbb{E}[s_i(k), r_{n(k)+1} = l | r_{n(k)} = i] = \sum_l \mathbb{E}s_{il}p_{il},$$

we can obtain that $m_1 = 1, m_2 = 1.9$ and $m_3 = 1$. According to Lemma 1, we have $\pi = [0.3387, 0.4450, 0.2163]$. It can be checked that

$$\sum_{i\in\Gamma} \pi_i \big(\lambda_i - 0.5\eta_i + \sum_{j\in\Gamma} \frac{p_{ij}\ln\mu_{ij}}{m_i}\big) = -0.7691 < 0.$$

The semi-Markov jump system in (33) is almost surely exponentially stable by utilizing Theorem 1. Fig. 1 (a) indicates that system (33) is stable with the semi-Markov switching.

It should be pointed out that the results in [8, 17, 19, 27] are unable to obtain the stability of system (33) in Case I. The reasons are listed as follows:

1) The distribution $F_{ij}(t)$ of the sojourn time depends on the current state i and the next state j in Case I. However, the distribution $F_{ij}(t)$ is supposed to only depend on the current state i in [8, 17, 27]. Thus, the results in [8, 17, 27] cannot be used for this example.

2) Because the sojourn time $s_3(k)$ of the 3rd unstable subsystem follows the uniform distribution, the results in [19, 27] are unable to consider the stability of system (33).

Case II: The switching signal $r(t)$ is a Markov chain with a generator

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & -4 \end{bmatrix},$$

which yields that it exists a stationary distribution $\pi = (0.4, 0.4, 0.2)$. It can be verified that

$$\sum_{i\in\Gamma} \pi_i \big(\lambda_i - 0.5\eta_i + \sum_{j\in\Gamma} q_{ij}\ln\mu_{ij}\big) = -0.01296 < 0.$$

According to Theorem 2, we have that the Markov jump system in (33) is almost surely exponentially stable. Fig. 1 (b) indicates system (33) is stable with the Markov switching.

In addition, it can be verified that $\sum_{i\in\Gamma}\pi_i\big(\lambda_i - 0.5\eta_i + q_i\ln\mu_i\big) = 0.0090 > 0$, which implies that Corollary 2 with the partial mode-dependent assumption (H_3') cannot be used. Thus, the mode-dependent linear comparable assumption (H_3) in this paper can effectively reduce the conservatism.

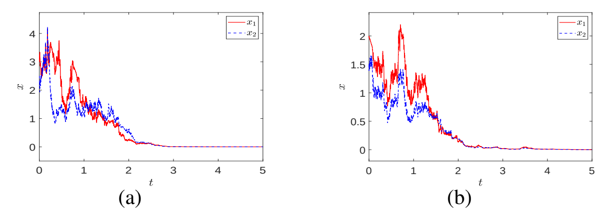

Fig. 1. (a) State trajectories for the semi-Markov jump system in (33). (b) State trajectories for the Markov jump system in (33).

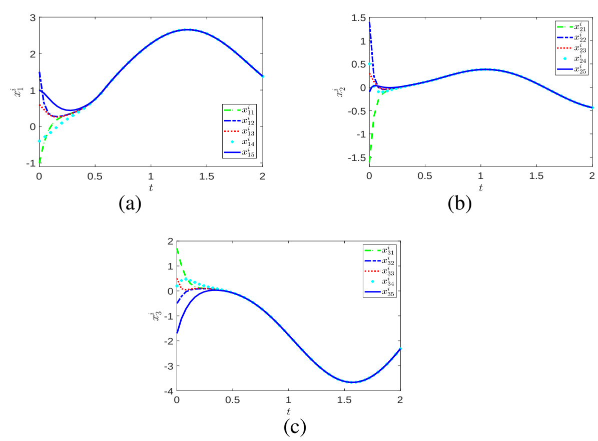

Fig. 2. State trajectories for the multi-agent system in (34) with semi-Markov switching.

Example 3: Consider a multi-agent systems with stochastic switching as in [33],

$$\frac{dx^i(t)}{dt} = f(x^i(t), t) + \epsilon \sum_{j=1}^{m} a_{ij}^{r(t)} x^j(t), t \in [\tau_{k-1}, \tau_k), \tag{34}$$

where $i = 1, \cdots, 5$ and $k \in \mathbb{N}^+$; ϵ is the coupling strength; $A_{r(t)} = [a_{ij}^{r(t)}] \in S_G = \{G_1, G_2, G_3\}$ is the coupling matrix and $r(t) \in \Gamma = \{1, 2, 3\}$ is a semi-Markov chain. Let $f(x,t)$ be a Chua's circuit and G_1, G_2 and G_3 be coupling matrixes, which are defined the same as in [33]. Let $V(x(t),1) = \widetilde{V}(t)$, $V(x(t),2) = 0.5\widetilde{V}(t)$ and $V(x(t),3) = 2\widetilde{V}(t)$, where $\widetilde{V}(t) = x_{\bar{k}(t)}^{\bar{i}(t)}(t) - x_{\bar{k}(t)}^{\bar{j}(t)}(t)$ is defined the same as in equation (3) of [33]. Thus, we can set $\mu_{21} = 2, \mu_{31} = 0.5, \mu_{12} = 0.5, \mu_{32} = 0.25, \mu_{13} = 2$, $\mu_{23} = 4$ and $\eta_i \equiv 0$ for $\forall i \in \Gamma$. According to Theorem 1 in [33], if we can prove $V(x(t), r(t)) = 0$, then the multi-agent system in (34) with semi-Markov switching achieves consensus. By using the same notations in [33], we can derive that $\lambda_i = L - \epsilon RV(G_i)$, where $L = 12.87$, $\epsilon = 10.5$, $RV(G_1) = 1.7835, RV(G_2) = 1.2884, RV(G_3) = 0.6402$ are provided in [33]. It follows that $\lambda_1 = -5.8568$, $\lambda_2 = -0.6582$ and $\lambda_3 = 6.1479$.

Assume that the embedded chain of semi-Markov chain $r(t)$ has the following transition probability matrix

$$P = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0.6 & 0 & 0.4 \\ 0.7 & 0.3 & 0 \end{bmatrix},$$

with a stationary distribution $\bar{\pi} = [0.3929, 0.3214, 0.2857]$. Let $F_{12}(t) = F_{13}(t) \sim \mathcal{E}(0.5)$, $F_{21}(t) = F_{23}(t) \sim U(0,2)$ and $F_{31}(t) = F_{32}(t) \sim U(0,4)$, which follow that $m_1 = 2, m_2 = 1$ and $m_3 = 2$. It can be derived from Lemma 1 that $\pi = [0.4681, 0.1915, 0.3404]$. Thus, we have

$$\sum_{i \in \Gamma} \pi_i \big(\lambda_i - 0.5\eta_i + \sum_{j \in \Gamma} \frac{p_{ij} \ln \mu_{ij}}{m_i}\big) = -0.7748 < 0.$$

Therefore, Theorem 1 implies that the multi-agent system in (34) reaches the consensus with semi-Markov switching. It is indicated in Fig. 2 that the multi-agent system in (34) with semi-Markov switching reaches the consensus.

V. Conclusions

The stability problem has been considered in this paper for semi-Markov jump stochastic nonlinear systems. For the exponential stability of semi-Markov stochastic nonlinear systems, we have developed a new stochastic analysis method under the mode-dependent linear comparable assumption, and sufficient conditions are established. Examples are given to illustrate the validity and usefulness of the proposed new analysis approach.

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