poles are nonnegative; the rest of the transfer function is required to be a strictly positive real function. We then proved that the closed-loop system is asymptotically stable provided that $\omega_1 \neq m\pi$ for some natural number $m \in \boldsymbol{N}$. We also studied the case where the output of the controller is corrupted by a disturbance. We showed that if the frequency spectrum of the disturbance is known, then by choosing the controller appropriately we can obtain better disturbance rejection. To support this idea, we presented some numerical simulation results.

We note that the ideas presented here can also be applied to other flexible structures (e.g., flexible beams). The work on this subject is still in progress and the results will be presented elsewhere.

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Mixed H_2/H_∞-Control of Discrete-Time Markovian Jump Linear Systems

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***Abstract*—In this paper we consider the mixed H_2/H_∞-control problem for the class of discrete-time linear systems with parameters subject to Markovian jump linear systems (MJLS's). It is assumed that both the state variable and the jump variable are available to the controller. The transition probability matrix may not be exactly known, but belongs to an appropriate convex set. For this controlled discrete-time Markovian jump linear system, the problem of interest can be stated in the following way. Find a robust (with respect to the uncertainty on the transition Markov probability matrix) mean-square stabilizing state and jump feedback controller that minimizes an upper bound for the H_2-norm, under the restriction that the H_∞-norm is less than a prespecified value δ. The problem of the determination of the smallest H_∞-norm is also addressed. We present an approximate version of these problems via linear matrix inequality optimization.**

***Index Terms*—Coupled Lyapunov equations, LMI optimization, Markovian jump systems, mixed H_2/H_∞-control.**

I. Introduction

A great deal of attention has been given nowadays to a class of stochastic linear systems subject to abrupt variations, namely, Markovian jump linear systems (MJLS's). This family of systems is modeled by a set of linear systems with the transitions between the models determined by a Markov chain taking values in a finite set. Due to a large number of applications in control engineering, several results on this field can be found in the current literature, regarding applications, stability conditions, and optimal control problems (see, for instance, [1]–[11], [13]–[18], and [21]–[28]).

The mixed H_2/H_∞ and H_∞ control problems for time-invariant discrete-time linear systems has been studied in the current literature, usually using a state-space approach, leading to nonstandard algebraic Riccati equations and Lyapunov-like equations (see, for instance, [12], [19], and [20]). The H_2- and H_∞-control problems for MJLS's have recently been analyzed in [5], [6], and [11]. For the H_2-control problem, a convex programming approach was applied in [5] and numerical algorithms developed. In this paper we study the mixed H_2/H_∞-control and H_∞-control problems of a discrete-time MJLS's. We will assume that the transition probability matrix for the Markov chain is not exactly known, but belongs to an appropriate convex set. In this case a robust mean-square (state and jump feedback) stabilizing controller is defined as a state-feedback controller, which also depends on the jump Markov variable, that stabilizes in the mean-square sense the MJLS for every appropriate Markov transition probability matrix. This kind of concept was first introduced by Rami and El Ghaoui in [27] for continuous-time MJLS's. Under these conditions, the mixed H_2/H_∞-control problem of an MJLS's can be formulated as follows: we are interested in finding a robust mean-square stabilizing controller that minimizes an upper bound for the H_2-norm, under the restriction that the H_∞-norm is less than a prespecified value δ. The problem of minimizing

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the H_∞-norm is also addressed. We trace a parallel with the discrete-time linear system theory of H_2/H_∞ and H_∞ control to derive our results. When restricted to the case with no jumps, the equations presented here can be seen as dual to the ones derived in [12]. As in [12], we present an approximate version of the mixed H_2/H_∞- and H_∞-control problems of MJLS's based on linear matrix inequalities (LMI) optimization.

The paper is organized in the following way. Section II presents the notation that will be used throughout the work. Section III deals with previous results derived for stability, H_2- and H_∞-control of MJLS's as well as some other auxiliary results. Section IV presents a sufficient condition for the existence of a mean-square stabilizing controller that makes the H_∞-norm of the MJLS's less than a prespecified value δ. The condition is written in terms of the existence of a solution $P = (P_1, \cdots, P_N)$ and $K = (K_1, \cdots, K_N)$ for a set of coupled Lyapunov-like equations. This solution P leads to an upper bound for the H_2-norm of the MJLS's so that an approximation for the mixed H_2/H_∞-control problem for the MJLS's can be determined by minimizing this functional over the set of solutions P and K. The H_∞-control problem can also be addressed through this Lyapunov-like equation. In Section V we consider the case in which the transition probability matrix belongs to an appropriate convex set and, using the results of Section IV, derive an LMI optimization problem that leads to an approximation for the mixed H_2/H_∞ and H_∞-control problems. Numerical examples are presented in Section VI, and the paper is concluded in Section VII with some final comments.

II. Notation

We shall write $\mathbb{C}^n$ and $\mathbb{R}^n$ to denote the n-dimensional complex and real spaces, respectively, and $\mathbb{M}(\mathbb{C}^n, \mathbb{C}^m)$ the normed linear space of all m by n complex matrices. For simplicity we set $\mathbb{M}(\mathbb{C}^n, \mathbb{C}^n) = \mathbb{M}(\mathbb{C}^n)$. We write * to indicate the adjoint operator and, for real matrices, $'$ will indicate transpose. L≥ 0 and L> 0 will be used if a self-adjoint matrix is positive semidefinite or positive definite, respectively, and we write $\mathbb{M}(\mathbb{C}^n)^+ = \{L \in \mathbb{M}(\mathbb{C}^n); L = L^* \geq 0\}$. We denote by $\|.\|$ the standard norm in $\mathbb{C}^n$.

Let $\mathcal{H}^{m,n}$ be the linear space made up of all N-sequence of matrices $V = (V_1, \cdots, V_N)$, $V_i \in \mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$). For $V \in \mathcal{H}^{m,n}$ we define the following norm $\|.\|_2$:

$$\|V\|_2 = \left(\sum_{i=1}^{N} \mathrm{tr}(V_i^* V_i)\right)^{1/2}$$

(where tr(.) denotes the trace operator).

It is easy to verify that $\mathcal{H}^{m,n}$ equipped with the norm $\|.\|_2$ is a complex Hilbert space with inner product given by

$$\langle V; H\rangle = \sum_{i=1}^{N} \mathrm{tr}((V_i^* H_i)).$$

We set $\mathcal{H}^{n,n} = \mathcal{H}^n$ and $\mathcal{H}^{n+} = \{V = (V_1, \cdots, V_N) \in \mathcal{H}^n; V_i \in \mathbb{M}(\mathbb{C}^n)^+, i = 1, \cdots, N\}$. For $H = (H_1, \cdots, H_N)$ and $V = (V_1, \cdots, V_N)$ in $\mathcal{H}^{n+}$ the notation $H \leq L (H < L)$ indicates that $H_i \leq L_i$ $(H_i < L_i)$ for each $i = 1, \cdots, N$.

For an increasing filtration $\{\mathcal{F}_k\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, we set $\ell_2^r(\mathcal{F}_k)$ as the Hilbert space formed by the sequence of second-order random variables $z = (z(0), z(1), \cdots)$ with $z(k) \in \mathbb{R}^r$ and $\mathcal{F}_k$-adapted for each $k = 0, 1, \cdots$, and such that

$$\|z\|_2^2 := \sum_{k=0}^{\infty} \|z(k)\|_2^2 < \infty \text{ where } \|z(k)\|_2^2 := E(\|z(k)\|^2).$$

For any complex Banach space $\mathbb{Z}$ we denote by $\mathbb{B}(\mathbb{Z})$ the Banach space of all bounded linear operators of $\mathbb{Z}$ into $\mathbb{Z}$ with the uniform-induced norm represented by $\|.\|$, and for $L \in \mathbb{B}(\mathbb{Z})$ we denote by $r_\sigma(L)$ the spectral radius of L.

Finally, we conclude this section with the following well-known result used in LMI's, which will be useful in the sequel.

Remark 1: If $R > 0$, then $W = \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$ if and only if $Q \geq SR^{-1}S'$.

III. Auxiliary Results

A. Stability Results

Consider the following stochastic system on an appropriate probability space $(\Omega, \{\mathcal{F}_k\}, \mathcal{F}, P)$:

$$x(k+1) = \tilde{A}_{\theta(k)} x(k) \tag{1a}$$

$$x(0) = x_0, \qquad \theta(0) = \theta_0 \tag{1b}$$

where $\{\theta(k);\ k = 0, 1, \cdots\}$ is a discrete-time Markov chain with finite state space $\{1, \cdots, N\}$ with transition probability matrix $\mathbb{P} = [p_{ij}]$. We consider $\tilde{A} = (\tilde{A}_1, \cdots, \tilde{A}_N) \in \mathcal{H}^n$ real, and x_0 a second-order random variable in $\mathbb{R}^n$. We set $Q(k) = (Q_1(k), \cdots, Q_N(k))$, where

$$Q_j(k) := E(x(k)x(k)' 1_{\{\theta(k)=j\}}) \in \mathbb{M}(\mathbb{C}^n)^+ \tag{2}$$

and $1_{\{.\}}$ stands for the Dirac measure.

For $S = (S_1, \cdots, S_N) \in \mathcal{H}^n$ we define the operator $\mathcal{T} \in \mathbb{B}(\mathcal{H}^n)$ as: $\mathcal{T}(S) = (\mathcal{T}_1(S), \cdots, \mathcal{T}_N(S))$, where

$$\mathcal{T}_j(S) = \sum_{i=1}^{N} p_{ij} \tilde{A}_i S_i \tilde{A}_i'. \tag{3}$$

It is easy to verify that with the inner product as defined above we have $\mathcal{L} := \mathcal{T}^*$ given by

$$\mathcal{L}_i(S) = \tilde{A}_i' \left(\sum_{j=1}^{N} p_{ij} S_j\right) \tilde{A}_i.$$

In particular, $r_\sigma(\mathcal{L}) = r_\sigma(\mathcal{T})$. The following result, shown in [7, Proposition 3] provides a connection between (2) and (3).

Proposition 1: For every $k = 0, 1, 2, \cdots, Q(k+1) = \mathcal{T}(Q(k))$.

We make the following definitions.

Definition 1: Model (1) is mean-square stable (MSS) if $\|Q(k)\|_2^2 \to 0$ as $k \to \infty$ for any initial condition x_0 and initial distribution for θ_0.

Remark 2: It can be shown that stability of each model (that is, $r_\sigma(\tilde{A}_i) < 1$ for $i = 1, \cdots, N$) is neither necessary nor sufficient for MSS (see [16]). Moreover, if (1) is MSS, then $\|x(k)\| \to 0$ as $k \to \infty$ with probability one (see [7]).

The next result has been proved in [7, Ths. 1 and 2].

Proposition 2: The following assertions are equivalent.

1) Model (1) is MSS.
2) $r_\sigma(\mathcal{T}) < 1$.
3) $r_\sigma(\mathcal{L}) < 1$.
4) There exists $\alpha \in$(0,1) and $a \in \mathbb{R}$, $a > 0$, such that for each $k = 0, 1, \cdots$,
$$E(\|x(k)\|^2) \leq a\alpha^k.$$
5) (Coupled Lyapunov Equations) Given any $S = (S_1, \cdots, S_N) > 0$ in $\mathcal{H}^{n+}$ there exists $P = (P_1, \cdots, P_N) > 0$ in $\mathcal{H}^{n+}$ satisfying $P - \mathcal{T}(P) = S$ with $P = \sum_{k=0}^{\infty} \mathcal{T}^k(S)$.
6) (Adjoint Coupled Lyapunov Equations) Given any $S = (S_1, \cdots, S_N) > 0$ in $\mathcal{H}^{n+}$ there exists $P = (P_1, \cdots, P_N) > 0$ in $\mathcal{H}^{n+}$ satisfying $P - \mathcal{L}(P) = S$ with $P = \sum_{k=0}^{\infty} \mathcal{L}^k(S)$.

Moreover, if $r_\sigma(\mathcal{T}) < 1$ then for any $S \in \mathcal{H}^n$ there exists a unique $P \in \mathcal{H}^n$ such that $P - \mathcal{T}(P) = S$. If $S \geq T \geq 0$ (>0, respectively) and $P - \mathcal{T}(P) = S, L - \mathcal{T}(L) = T$ then $P \geq L \geq 0$ (>0). These results also hold replacing $\mathcal{T}$ by $\mathcal{L}$.

We present now the definition of mean-square stabilizability and detectability. Consider $A = (A_1, \cdots, A_N) \in \mathcal{H}^n$, $B = (B_1, \cdots, B_N) \in \mathcal{H}^{m,n}$ and $C = (C_1, \cdots, C_N) \in \mathcal{H}^{n,p}$ real.

Definition 2: We say that (A, B) is mean-square stabilizable if there exists $K = (K_1, \cdots, K_N) \in \mathcal{H}^{n,m}$ such that model (1) is MSS with $\tilde{A}_i = A_i - B_iK_i$. In this case we say that K stabilizes (A, B) in the mean-square sense and set $\mathbb{K} = \{K \in \mathcal{H}^{n,m};\ K$ stabilizes (A, B) in the mean-square sense$\}$. Similarly, we say that (C, A) is mean-square detectable if there exists $H = (H_1, \cdots, H_N) \in \mathcal{H}^{p,n}$ such that model (1) is MSS with $\tilde{A}_i = A_i - H_iC_i$, and we say that H stabilizes (C, A).

The next proposition follows from [9, Proposition 6]. Consider $D = (D_1, \cdots, D_N) \in \mathcal{H}^{m,p}$ such that $D_i'D_i > 0$ and $C_i'D_i = 0$ and set $\mathcal{E}_i(L) = \sum_{j=1}^N p_{ij}L_j$, $i = 1, \cdots, N$, for $L = (L_1, \cdots, L_N)$.

Proposition 3: Suppose (C, A) is mean-square detectable and $P = (P_1, \cdots, P_N) \geq 0, K = (K_1, \cdots, K_N) \in \mathcal{H}^{n,m}$ satisfy

$$\begin{aligned} &- P_i + (A_i - B_iK_i)'\mathcal{E}_i(P)(A_i - B_iK_i) \\ &\quad + (C_i - D_iK_i)'(C_i - D_iK_i) \leq 0. \end{aligned} \tag{4}$$

Then $K = (K_1, \cdots, K_N) \in \mathbb{K}$.

B. The H_2-Norm

Consider again on $(\Omega, \{\mathcal{F}_k\}, \mathcal{F}, P)$ the following system $\mathcal{G}$:

$$\mathcal{G} = \begin{cases} x(k+1) = \tilde{A}_{\theta(k)}x(k) + Jw(k) & \text{(5a)} \\ x(0) = 0, \qquad \theta(0) = \theta_0 & \text{(5b)} \\ z(k) = \tilde{C}_{\theta(k)}x(k) & \text{(5c)} \end{cases}$$

where $\tilde{A} = (\tilde{A}_1, \cdots, \tilde{A}_N) \in \mathcal{H}^n$, $\tilde{C} = (\tilde{C}_1, \cdots, \tilde{C}_N) \in \mathcal{H}^{n,p}$, and $J \in \mathbb{M}(\mathbb{C}^r, \mathbb{C}^n)$, with $\tilde{A}, \tilde{C}, J$ real and $JJ' > 0$.

Suppose that $r_\sigma(\mathcal{T}) < 1$ (that is, model (1) is MSS) and $w = (w(0), \cdots)$ is an impulse input. From Proposition 2-4) we have $z = (z(0), z(1), \cdots) \in \ell_2^p(\mathcal{F}_k)$. The next definition is a generalization of the H_2-norm from discrete-time deterministic systems to the stochastic Markovian jump case.

Definition 3: We define the H_2-norm of system $\mathcal{G}$ as

$$\|\mathcal{G}\|_2^2 = \sum_{s=1}^r \sum_{j=1}^N \|z_{s,j}\|_2^2$$

where $z_{s,j}$ represents the output sequence $(z(0), z(1), \cdots)$ given by (5c) when

1) the input sequence is given by $w = (w(0), w(1), \cdots)$, $w(0) = e_s$, $w(k) = 0$, $k > 0$, $e_s \in \mathbb{R}^r$ the unitary vector formed by one at the sth position and zero elsewhere;
2) $\theta(0) = \theta(1) = j$.

For the deterministic case ($N = 1$ and $p_{11} = 1$) the above definition reduces to the usual H_2-norm. As in the deterministic case, we have that the H_2-norm as defined above can be calculated as the solution of the discrete-time coupled gramian of observability and controllability. For this, define $\mathcal{C} = (\tilde{C}_1'\tilde{C}_1, \cdots, \tilde{C}_N'\tilde{C}_N) \in \mathcal{H}^{n+}$, $\mathcal{J} = (JJ', \cdots, JJ') \in \mathcal{H}^{n+}$, and $L = (L_1, \cdots, L_N) \in \mathcal{H}^{n+}$, $P = (P_1, \cdots, P_N) \in \mathcal{H}^{n+}$, the unique solution of the equations (recall that $r_\sigma(\mathcal{T}) = r_\sigma(\mathcal{L}) < 1$ and see Proposition 2)

$$L = \mathcal{L}(L) + \mathcal{C} \text{ (observability gramian)} \tag{6}$$

$$P = \mathcal{T}(P) + \mathcal{J} \text{ (controllability gramian).} \tag{7}$$

The next result was proved in [5] and represents a characterization of the H_2-norm in terms of the solution of the observability and controllability gramians.

Proposition 4: $\|\mathcal{G}\|_2^2 = \sum_{j=1}^N \operatorname{tr}(J'L_jJ) = \sum_{j=1}^N \operatorname{tr}(\tilde{C}_jP_j\tilde{C}_j')$.

C. The H_∞-Norm

Consider again system $\mathcal{G}$ as in (5) above with $w = (w(0), \cdots) \in \ell_2^r(\mathcal{F}_k)$. The following result was proved in [6, Proposition 2].

Proposition 5: $r_\sigma(\mathcal{T}) < 1$ if and only if $x = (0, x(1), \cdots) \in \ell_2^n(\mathcal{F}_k)$ for every $w = (w(0), w(1), \cdots) \in \ell_2^r(\mathcal{F}_k)$.

Suppose that $r_\sigma(\mathcal{T}) < 1$. From the above proposition, $x = (0, x(1), \cdots) \in \ell_2^n(\mathcal{F}_k)$, and thus $z = (0, z(1), \cdots) \in \ell_2^p(\mathcal{F}_k)$. The H_∞-norm of system $\mathcal{G}$ is defined as follows.

Definition 4: $\|\mathcal{G}\|_\infty := \sup_{\theta_0} \sup_{w \in \ell_2^r(\mathcal{F}_k)} (\|z\|_2/\|w\|_2)$.

Again, for the deterministic case ($N = 1$ and $p_{11} = 1$), the above definition reduces to the usual H_∞-norm.

IV. Mixed H_2/H_∞-Control Problem

Consider now a controlled version of system $\mathcal{G}$

$$\mathcal{G} = \begin{cases} x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k) + Jw(k) & \text{(8a)} \\ x(0) = 0, \qquad \theta(0) = \theta_0 & \text{(8b)} \\ z(k) = C_{\theta(k)}x(k) + D_{\theta(k)}u(k) & \text{(8c)} \end{cases}$$

where $A = (A_1, \cdots, A_N) \in \mathcal{H}^n$, $B = (B_1, \cdots, B_N) \in \mathcal{H}^{m,n}$, $C = (C_1, \cdots, C_N) \in \mathcal{H}^{n,p}$, $D = (D_1, \cdots, D_N) \in \mathcal{H}^{m,p}$, $J \in \mathbb{M}(\mathbb{C}^r, \mathbb{C}^n)$ are real, and $D_i'D_i > 0$, $C_i'D_i = 0$ for each $i = 1, \cdots, N$.

For $K = (K_1, \cdots, K_N)$ set $\mathcal{G}_K$ as (8) with $u(k) = -K_{\theta(k)}x(k)$. We have the following result.

Theorem 1: Suppose (C, A) is mean-square detectable and $\delta > 0$ a fixed real number. If there exists $P = (P_1, \cdots, P_N) \geq 0$ and $K = (K_1, \cdots, K_N) \in \mathcal{H}^{n,m}$ such that for each $i = 1, \cdots, N$

$$\begin{aligned} &-P_i + (A_i - B_iK_i)'\mathcal{E}_i(P)(A_i - B_iK_i) \\ &\quad + (C_i - D_iK_i)'(C_i - D_iK_i) + \frac{1}{\delta^2} P_iJJ'P_i \leq 0 \end{aligned} \tag{9}$$

then $K = (K_1, \cdots, K_N) \in \mathbb{K}$ and

$$\|\mathcal{G}_K\|_\infty^2 \leq \delta^2(1 - \nu) \leq \delta^2$$

where $\nu \in (0, (1/\delta^2) \sum_{i=1}^N \operatorname{tr}(J'P_iJ))$. Moreover

$$\|\mathcal{G}_K\|_2^2 \leq \sum_{i=1}^N \operatorname{tr}(J'P_iJ).$$

Proof: Comparing (4) and (9), it is immediate from Proposition 3 that $K \in \mathbb{K}$. Set $\tilde{A}_i = A_i - B_iK_i$ and $\tilde{C}_i = C_i - D_iK_i$. Recalling from Proposition 5 that for any $w = (w(0), \cdots) \in \ell_2^r(\mathcal{F}_k)$ we have $x = (0, x(1), \cdots) \in \ell_2^n(\mathcal{F}_k)$, and that $x(k)$, $\theta(k)$, and $w(k)$ are $\mathcal{F}_k$-measurable, we get from (9) that

$$\begin{aligned} &E(x(k+1)'P_{\theta(k+1)}x(k+1)) \\ &\quad = E(E(x(k+1)'P_{\theta(k+1)}x(k+1)|\mathcal{F}_k)) \\ &\quad = E(x(k+1)'E(P_{\theta(k+1)} \mid \mathcal{F}_k)x(k+1)) \\ &\quad = E((\tilde{A}_{\theta(k)}x(k) + Jw(k))'\mathcal{E}_{\theta(k)}(P) \\ &\qquad \cdot (\tilde{A}_{\theta(k)}x(k) + Jw(k))) \\ &\quad \leq E\Bigg(x(k)'\bigg(P_{\theta(k)} - \tilde{C}_{\theta(k)}'\tilde{C}_{\theta(k)} - \frac{1}{\delta^2} P_{\theta(k)}JJ'P_{\theta(k)}\bigg) \\ &\qquad \cdot x(k) + w(k)'J'\mathcal{E}_{\theta(k)}(P)\tilde{A}_{\theta(k)}x(k) + x(k)'\tilde{A}_{\theta(k)} \\ &\qquad \cdot \mathcal{E}_{\theta(k)}(P)Jw(k) + w(k)'J'\mathcal{E}_{\theta(k)}Jw(k)\Bigg) \end{aligned}$$

so that

$$\begin{aligned}
&\|P^{1/2}_{\theta(k+1)}x(k+1)\|_2^2 - \|P^{1/2}_{\theta(k)}x(k)\|_2^2 + \|z(k)\|_2^2 \\
&\quad \le -\frac{1}{\delta^2}\|J'P_{\theta(k)}x(k)\|_2^2 + E(w(k)'J'\mathcal{E}_{\theta(k)}(P)\tilde{A}_{\theta(k)}x(k)) \\
&\qquad + E(x(k)'\tilde{A}'_{\theta(k)}\mathcal{E}_{\theta(k)}(P)Jw(k)) + \|\mathcal{E}^{1/2}_{\theta(k)}(P)Jw(k)\|_2^2 \\
&\quad = -\frac{1}{\delta^2}\|J'P_{\theta(k)}x(k)\|_2^2 + \frac{1}{\delta^2}\|J'P_{\theta(k+1)}x(k+1)\|_2^2 \\
&\qquad - \frac{1}{\delta^2}\|J'P_{\theta(k+1)}x(k+1)\|_2^2 \\
&\qquad + 2E(w(k)'J'\mathcal{E}_{\theta(k)}(P)(\tilde{A}_{\theta(k)}x(k) + Jw(k))) \\
&\qquad - E(w(k)'J'\mathcal{E}_{\theta(k)}(P)Jx(k)).
\end{aligned}$$

Thus

$$\begin{aligned}
&\|P^{1/2}_{\theta(k+1)}x(k+1)\|_2^2 - \|P^{1/2}_{\theta(k)}x(k)\|_2^2 \\
&\quad - \frac{1}{\delta^2}\|J'P_{\theta(k+1)}x(k+1)\|_2^2 \\
&\quad + \frac{1}{\delta^2}\|J'P_{\theta(k)}x(k)\|_2^2 + \|z(k)\|_2^2 \\
&\le -\frac{1}{\delta^2}\|J'P_{\theta(k+1)}x(k+1)\|_2^2 + 2E(w(k)' \\
&\quad \cdot J'P_{\theta(k+1)}x(k+1)) - \delta^2\|w(k)\|_2^2 \\
&\quad + E(w(k)'(\delta^2 I - J'\mathcal{E}_{\theta(k)}(P)J)w(k)) \\
&= -\left\|\frac{1}{\delta}J'P_{\theta(k+1)}x(k+1) - \delta w(k)\right\|_2^2 \\
&\quad + E(w(k)'(\delta^2 I - J'\mathcal{E}_{\theta(k)}(P)J)w(k)) \\
&\le E(w(k)'(\delta^2 I - J'\mathcal{E}_{\theta(k)}(P)J)w(k)).
\end{aligned}$$

Taking the sum for $k = 0$ to ∞, and recalling that $x(0) = 0$, $\|x(k)\|_2 \to 0$ as $k \to \infty$, we get

$$\begin{aligned}
\|z\|_2^2 &\le \delta^2 \sum_{k=0}^{\infty} E(w(k)'\left(I - \frac{1}{\delta^2}J'P_{\theta(k+1)}J)w(k)\right) \\
&\le \delta^2(1-\nu)\|w\|_2^2
\end{aligned}$$

where $\nu \in (0, (1/\delta^2)\sum_{i=1}^N \mathrm{tr}(J'P_iJ))$. Thus

$$\|\mathcal{G}_K\|_\infty = \sup_{\theta_0} \sup_{w\in\ell_2^r(\mathcal{F}_k)} \frac{\|z\|_2}{\|w\|_2} \le \delta(1-\nu)^{1/2} < \delta.$$

Finally, notice from Proposition 4, $\|\mathcal{G}_K\|_2^2 = \sum_{i=1}^N \mathrm{tr}(J'S_iJ)$, where

$$S_i = \tilde{A}'_i\mathcal{E}_i(S)\tilde{A}_i + \tilde{C}'_i\tilde{C}_i.$$

From (9) and some $V_i \ge 0$, $i = 1, \cdots, N$

$$P_i = \tilde{A}'_i\mathcal{E}_i(P)\tilde{A}_i + \tilde{C}'_i\tilde{C}_i + \frac{1}{\delta^2}P_iJJ'P_i + V'_iV_i$$

so that, from Proposition 2, $P_i \ge S_i$ for all $i = 1, \cdots, N$. This implies that

$$\|\mathcal{G}_K\|_2^2 = \sum_{i=1}^N \mathrm{tr}(J'S_iJ) \le \sum_{i=1}^N \mathrm{tr}(J'P_iJ)$$

completing the proof of the theorem. □

The above theorem suggests the following approximation for the mixed H_2/H_∞-control problem: for $\delta > 0$ fixed, find $P = (P_1, \cdots, P_N) \ge 0$ and $K = (K_1, \cdots, K_N)$ such that

$$\min \mathrm{tr}\left(\sum_{i=1}^N J'P_iJ\right)$$

subject to (9). If we are interested in minimizing the H_∞-norm, then δ becomes a variable of our problem, and we just have to replace $(\mathrm{tr}\sum_{i=1}^N J'P_iJ)$ above by δ^2. For the case in which $N = 1, p_{11} = 1$, (9) can be seen as dual to the one obtained in [12, Lemma 3.1].

V. Convex Approach

We will assume now that the transition probability matrix $\mathbb{P}$ is not exactly known but belongs to a convex set $\mathbb{D} := \{\mathbb{P};\ \mathbb{P} = \sum_{t=1}^q \alpha_t\mathbb{P}^t,\ \alpha_t \ge 0,\ \sum_{t=1}^q \alpha_t = 1\}$, where $\mathbb{P}^t$, $t = 1, \cdots, q$ are known transition probability matrices. We make the following definition.

Definition 5: We say that $K = (K_1, \cdots, K_N) \in \mathcal{H}^{n,m}$ robustly stabilizes (A, B) in the mean-square sense if (1) with $\tilde{A}_i = A_i - B_iK_i$ is MSS for every $\mathbb{P} \in \mathbb{D}$, and we set $\mathbb{K}_r := \{K \in \mathcal{H}^{n,m};\ K$ robustly stabilizes (A, B) in the mean-square sense$\}$.

We want to solve the following mixed H_2/H_∞ control problem: given $\delta > 0$, find $K \in \mathbb{K}_r$ which minimizes ζ subject to $\|\mathcal{G}_K\|_2 \le \zeta$, $\|\mathcal{G}_K\|_\infty \le \delta$, for every $\mathbb{P} \in \mathbb{D}$. Let us show now that an approximation for this problem can be obtained via an LMI optimization problem. Set $\Gamma_i^t = [\sqrt{p_{i1}^t}I \cdots \sqrt{p_{iN}^t}I] \in \mathbb{M}(\mathbb{C}^{Nn}, \mathbb{C}^n)$ for $i = 1, \cdots, N$, $t = 1, \cdots, q$, and define the following problem.

Problem I: Set $\mu = \delta^2$. Find $P = (P_1, \cdots, P_N) > 0$, $Q = (Q_1, \cdots, Q_N) > 0$, $L = (L_1, \cdots, L_N) > 0$, $Y = (Y_1, \cdots, Y_N)$ such that

$$\xi = \min \mathrm{tr}\left(\sum_{i=1}^N J'P_iJ\right)$$

subject to

$$\begin{bmatrix} Q_i & Q_iA'_i + Y'_iB'_i & Q_iC'_i & Y'_iD'_i & J \\ A_iQ_i + B_iY_i & L_i & 0 & 0 & 0 \\ C_iQ_i & 0 & I & 0 & 0 \\ D_iY_i & 0 & 0 & I & 0 \\ J' & 0 & 0 & 0 & \mu I \end{bmatrix} \ge 0, \quad i = 1, \cdots, N \tag{10}$$

$$\begin{bmatrix} L_i & L_i\Gamma_i^t \\ \Gamma_i^{t\prime}L_i & \mathrm{diag}\{Q_\kappa\} \end{bmatrix} \ge 0, \qquad i = 1, \cdots, N,\ t = 1, \cdots, q \tag{11}$$

$$\begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} \ge 0, \qquad i = 1, \cdots, N \tag{12}$$

where $\mathrm{diag}\{Q_\kappa\}$ is the matrix in $\mathbb{M}(\mathbb{C}^{Nn})$ formed by $Q_1, \cdots, Q_N$ in the diagonal and zero elsewhere.

Theorem 2: Suppose Problem I has a solution P, Q, L, and Y. Set $K = (K_1, \cdots, K_N)$ as $K_i = -Y_iQ_i^{-1}$, $i = 1, \cdots, N$ and $\xi = \sum_{i=1}^N \mathrm{tr}(J'P_iJ)$. Then $K \in \mathbb{K}_r$ and $\|\mathcal{G}_K\|_2 \le \xi^{1/2}$, $\|\mathcal{G}_K\|_\infty \le \delta$, for every $\mathbb{P} \in \mathbb{D}$.

Proof: First of all notice that (10)–(12) are equivalent to (see Remark 1)

$$\begin{aligned}
Q_i \ge\ & Q_i(A_i - B_iK_i)'L_i^{-1}(A_i - B_iK_i)Q_i \\
&+ Q_i(C_i - D_iK_i)'(C_i - D_iK_i)Q_i \\
&+ \mu^{-1}Q_i(Q_i^{-1})JJ'(Q_i^{-1})Q_i
\end{aligned} \tag{13}$$

$$L_i \ge L_i(\sum_{j=1}^N p_{ij}^tQ_j^{-1})L_i,\ t = 1, \cdots, q \tag{14}$$

$$P_i \ge Q_i^{-1}. \tag{15}$$

Since we are minimizing $\mathrm{tr}(\sum_{i=1}^N J'P_iJ)$ and $JJ' > 0$ by hypothesis, we must have from (15) that $P_i = Q_i^{-1}$. Consider any $\mathbb{P} \in \mathbb{D}$. Then by definition we have $p_{ij} = \sum_{t=1}^q \alpha_tp_{ij}^t$ for some $\alpha_t \ge 0$, $\sum_{t=1}^q \alpha_t = 1$. Thus from (14) we get

$$L_i^{-1} \ge \left(\sum_{j=1}^N p_{ij}Q_j^{-1}\right) = \mathcal{E}_i(P)$$

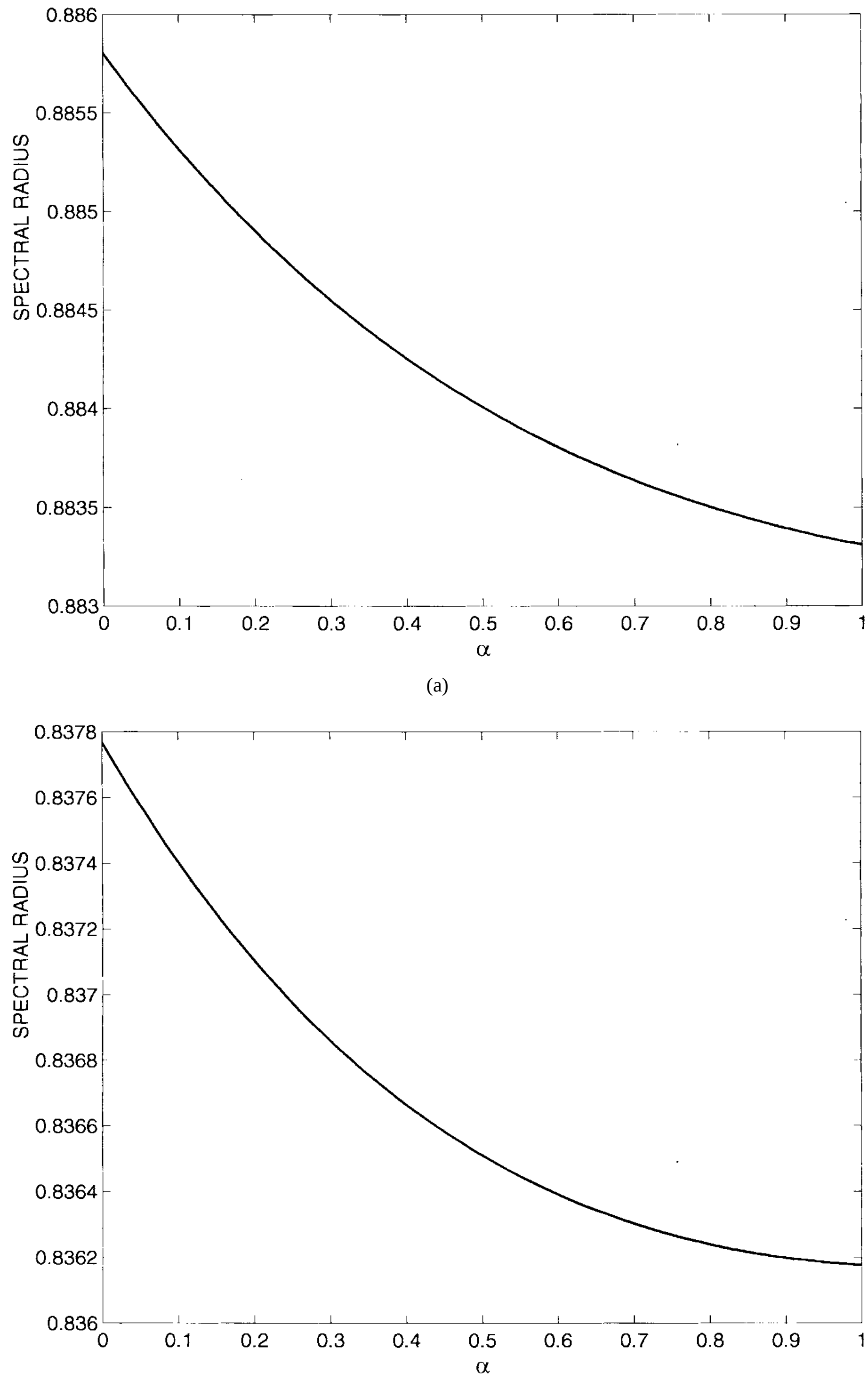

Fig. 1. Spectral radii for $\mathbb{P} \in \mathbb{D}$ (parameterized by α).

and from (13)

$$\begin{aligned} P_i = Q_i^{-1} &\geq (A_i - B_iK_i)'L_i^{-1}(A_i - B_iK_i) \\ &\quad + (C_i - D_iK_i)'(C_i - D_iK_i) + \mu^{-1}P_iJJ'P_i \\ &\geq (A_i - B_iK_i)'\mathcal{E}_i(P)(A_i - B_iK_i) \\ &\quad + (C_i - D_iK_i)'(C_i - D_iK_i) + \mu^{-1}P_iJJ'P_i. \qquad (16) \end{aligned}$$

The desired result follows from (16), Proposition 2-6), and Theorem 1. ■

Remark 3: If we desire to minimize the H_∞-norm, then μ becomes a variable in Problem I above and we just have to replace the value function $\mathrm{tr}(\sum_{i=1}^{N} J'P_iJ)$ by μ. The inequalities in (12) can be eliminated.

VI. Numerical Examples

This example is adapted from [12] for the case in which we have two modes of operation with transition probability matrix between the models given by $\mathbb{P}$. The matrices are

$$A_1 = A_2 = \begin{bmatrix} 0.9974 & 0.0539 \\ -0.1078 & 1.1591 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.0013 \\ 0.0539 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0.0013 \\ 0.1078 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 1 & 0.1 \end{bmatrix}$$
$$C_1 = C_2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_1 = D_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We consider the following cases.

a1) H_2/H_∞-control problem with $\delta = 80$, and transition probability matrix exactly known, given by
$$\mathbb{P} = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}.$$
For this case the obtained solution is $K_1 = [1.36 \quad 4.43]$, $K_2 = [1.6 \quad 4.64]$, and the optimal value function is $\xi = 728$. The closed-loop system is MSS, with $r_\sigma(\mathcal{T}) = 0.8624$.

a2) The same as above but with $\mathbb{P}$ belonging to $\mathbb{D}$, where $\mathbb{D}$ is defined through the transition probability matrices $\mathbb{P}^1$ and $\mathbb{P}^2$ defined below
$$\mathbb{P}^1 = \begin{bmatrix} 0.65 & 0.35 \\ 0.25 & 0.75 \end{bmatrix}, \qquad \mathbb{P}^2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.15 & 0.85 \end{bmatrix}.$$
For this case the obtained solution is $K_1 = [2.13 \quad 4.95]$, $K_2 = [2.4 \quad 6]$, and the optimal value function is $\xi = 983$. Fig. 1(a) shows $r_\sigma(\mathcal{T})$ for all elements of the convex set $\mathbb{D}$. This set is parameterized by α, where $\mathbb{P}(\alpha) = \alpha\mathbb{P}^1 + (1 - \alpha)\mathbb{P}^2$, $\alpha \in [0, 1]$. Notice by the curve that the system is MSS for all elements of the convex set $\mathbb{D}$.

b1) H_∞-control problem with the same data as in a1) above. The minimal value obtained for $\mu(=\delta^2)$ is 4369, with the controllers given by $K_1 = [5.47 \quad 7.02]$, $K_2 = [4.89 \quad 5.97]$. For this case, $r_\sigma(\mathcal{T}) = 0.8126$.

b2) The same as above but with $\mathbb{P} \in \mathbb{D}$, where $\mathbb{D}$ is defined as in a2). The minimal value obtained for μ is 5042, with the controllers given by $K_1 = [5.86 \quad 6.96]$, $K_2 = [5.97 \quad 7.32]$. Fig. 1(b) shows the spectral radius of $\mathcal{T}(.)$ as a function of α, as in a2). It can be seen from the curve that the closed-loop system is MSS for all elements of the convex set $\mathbb{D}$.

VII. Conclusions

In this paper, we have considered the problem of mixed H_2/H_∞-control of discrete-time MJLS's. It has been assumed that both the state variable and the jump variable are available to the controller. The transition probability matrix may belong to an appropriate convex set. We are interested in finding a state and jump feedback controller that robustly stabilizes an MJLS in the mean-square sense and minimizes an upper bound for the H_2 norm, under the restriction that the H_∞-norm is less than a prespecified value δ. This kind of problem has been studied in the current literature for discrete-time deterministic linear systems, usually using a state-space approach, leading to nonstandard algebraic Riccati and Lyapunov-like equations. We have traced a parallel with the discrete-time linear system theory of H_2/H_∞ and H_∞ control to derive our results. An approximation for the problem has been proposed by minimizing a linear functional over the positive semidefinite solutions of a set of coupled Lyapunov-like equations. Furthermore, it has been shown that this problem can be written in a convex programming formulation, leading to numerical algorithms. The H_∞-control problem has also been addressed.

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