Stochastic Stability Properties of Jump Linear Systems

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Abstract—**Jump linear systems are defined as a family of linear systems with randomly jumping parameters (usually governed by a Markov jump process) and are used to model systems subject to failures or changes in structure. In this paper, we study stochastic stability properties of jump linear systems and the relationship among various moment and sample path stability properties. It is shown that all second moment stability properties are equivalent and are sufficient for almost sure sample path stability, and a testable necessary and sufficient condition for second moment stability is derived. The Lyapunov exponent method for the study of almost sure sample stability is discussed and a theorem which characterizes the Lyapunov exponents of jump linear systems is presented. Finally, for one-dimensional jump linear systems, we prove that the region for δ-moment stability is monotonically converging to the region for almost sure stability as $\delta \downarrow 0^+$.**

I. Introduction

Jump linear systems with plant and observation noises are usually modeled by a system of stochastic differential equations in the following form:

$$\begin{cases} \dot{x}(t) = A(r(t))x(t) + B(r(t))u(t) + D(r(t))\dot{W}(t), \\ \quad t \geq 0 \\ y(t) = C(r(t))x(t) + F(r(t))\dot{V}(t), \\ \quad x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^k$ is the control, $y(t) \in \mathbb{R}^m$ is the output, and $\dot{W}(t) \in \mathbb{R}^p$ and $\dot{V}(t) \in \mathbb{R}^q$ are independent vector-valued white noise processes. In (1.1), $\{r(t),\ t \geq 0\}$ is a finite state time homogeneous Markov process, which is assumed to be independent of the noise terms $\dot{W}$ and $\dot{V}$ and is defined on the state space $S = \{1, 2, \cdots, s\}$ with the infinitesimal matrix

$$Q = (q_{ij})_{s \times s}, \qquad |q_{ij}| < +\infty. \tag{1.2}$$

Here, $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, and $F(\cdot)$ are matrix-valued functions defined on S with appropriate dimensions.

The model (1.1) has variable structure and can be used as a model for a system subject to ramdom failures and structure changes, as indicated by the dependence of all matrix parameters on the so-called form or indicant process $r(t)$. The terminology *jump linear system* results from the fact that it would be linear, if not for the Markov jump process $r(t)$. Note that the state $\{x(t), r(t)\}$ of the system is jointly Markovian.

The JLQ(G) problem, namely, the optimal control problem associated with (1.1) and the minimization of a quadratic cost functional

$$J = E\left\{ \int_0^T x'(t)Q(r(t))x(t) + u'(t)R(r(t))u(t)\, dt \right\} \tag{1.3}$$

was first studied by Krasovskii and Lidskii [20]. The problem was solved by Sworder [21] for the finite-time horizon ($T < +\infty$) using a stochastic maximum principle. Wonham [22] presented a nice dynamic programming approach to the problem, which was followed in most later work and he provided a solution to the infinite-time horizon case ($T = +\infty$) as well. The output feedback problem was considered by Mariton and Bertrand [31]. A discrete-time version of the problem was attacked by several researchers. Recently, Ji and Chizeck [23]–[28] have systematically investigated the JLQ(G) problem and refined many results concerned with controllability, observability, filtering, the separation principle, and optimal control. A critical assumption in all the above work is the perfect observation of the form process $r(t)$. Imperfect knowledge of $r(t)$ leads to the dual control problem; see [29], [30] for a discussion of the optimal control problem for jump linear system with the dual effect. Since our intention here is to study the stochastic stability of the system, we will not discuss the control problem in detail. A brief summary of the results for the control problem is the following: for the JLQ problem ($D \equiv F \equiv 0$) under a certain controllability (stabilizability) condition [27], an optimal state feedback control exists for the case $T < +\infty$. This state feedback control law tends to a stationary one in the form $u(t) = -K(r(t))x(t)$ when considering the infinite-time horizon problem ($T = +\infty$) and a finite cost $J < +\infty$ is obtained. A testable necessary and sufficient condition for the existence of stabilizing (in a stochastic stability sense) state feedback control and henceforth optimal state feedback control law is available [27]. In the case when $x(t)$ is not directly observable, an optimal output feedback control law exists [31]. When the noise terms are introduced, the JLQG problem has a solution in the sense that, under certain

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conditions, a separation principle holds and thus, the optimal controller consists of the optimal state feedback control law for the JLQ problem together with a Kalman–Bucy type filter [25]. Here, the jump process is observable and if not, some results are still available, see Hijab [38] and Casiello and Loparo [30]. In the JLQG problem, the value of the cost functional tends to infinity when T tends to infinity. In general, a model for the closed-loop system without filtering dynamics (or with perfect state or output observations) can be considered in the form

$$\dot{x}(t) = A_c(r(t))x(t) + D(r(t))\dot{W}(t). \quad (1.4)$$

The stability properties of stochastic systems were systematically investigated by Kushner [2] and Has'minskii [3]. Kozin, in his comprehensive survey [1], presented an excellent summary of the problem some twenty years ago. Kozin's work clarified many confused concepts and results and gave a nice explanation of the relationship among various stochastic stability concepts. For systems with randomly varying parameters (or variable structure), the work on stability can be traced back to Rosenbloom [6]. Bergen [7] studied the first and second moment stability of a randomly switched linear system which switches at a given sequence of time points and the system's modes at each switch point form a sequence of independently identically distributed (i.i.d.) random variables. Bergen's result was later refined by Bharucha [8]. A summary of the results in [8] appeared in Kozin's surey [1]. Darkhorskii and Leibovich [9] investigated systems where the time intervals between jumps are i.i.d. random variables and the systems' modes are governed by a finite state Markov chain. This extended the results of Bharucha. However, the necessary and sufficient condition for stability claimed in [9] turns out to be only sufficient. A counterexample to necessity can be easily constructed. The approach adopted by all the above work used Kronecker products of matrices.

Recently, Mariton [32], [33], [37] used a stochastic Lyapunov function approach, developed by Kushner [2], to obtain a sufficient condition for mean square stability of the systems in the form of (1.4) with $D(\cdot) \equiv 0$, and he also derived in [36] a sufficient condition for the almost sure stability of jump linear systems by obtaining a simple upper bound on the top Lyapunov exponent and a necessary and sufficient conditions for mean square stability via a Kronecker product approach. If the noise term in (1.4) is present, then the types of asymptotic stability properties defined in [1] cannot be achieved; the best that can be obtained is bounded variance of the state process. In this case, the expected value of the quadratic cost is expected to tend to infinity for the infinite-time horizon problem.

In this paper, we study the stochastic stability properties of the closed-loop system (1.4). We concentrate on the "homogeneous" part of (1.4) (i.e., $D(\cdot) \equiv 0$). By analyzing the stochastic properties of the transition matrix for the jump linear systems, we will show that the second moment stability concepts, namely, mean square stability, stochastic stability and exponential mean square stability (see definitions in next section) are all equivalent, and any one of them implies almost sure sample stability. In particular, we will establish a necessary and sufficient condition for exponential mean square stability via the Lyapunov function approach. For one-dimensional systems, we show that the region of δ-moment stability in the parameter space of the system tends monotonically to that of almost sure sample stability as δ goes to 0^+. Both a Lyapunov function approach and a direct method will be applied to establish the results. We will also discuss some recent results on Lyapunov exponents for the study of almost sure stability. The discrete-time case is treated separately in a companion paper [16].

The paper is organized as follows: in Section II, basic definitions for stability and the Lyapunov exponent associated with the sample state process are introduced. In Section III, we investigate the moment stability properties of the system and their relationship with almost sure sample stability. The Lyapunov exponent method is briefly discussed in Section IV and is intended only as an introduction to the type of work which are recently being investigated. Finally, Section V contains concluding remarks.

II. Definitions and Comments

Consider the jump linear system in the following form:

$$\begin{cases} \dot{x}(t) = A(r(t))x(t), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \quad (2.1)$$

which may be regarded as the "homogeneous" system of (1.4). In (2.1), $\{r(t), t \geq 0\}$ is the finite-state time homogeneous Markov process described previously. Let $(p_1, p_2, \cdots, p_s)$ denote an initial distribution of $r(t)$. For simplicity, we take the initial state x_0 as a fixed nonrandom constant vector. The underlying probability space is denoted by $(\Omega, \mathscr{F}, P)$ where Ω is the space of elementary events, $\mathscr{F}$ is a σ-field, and P is the probability measure. $\mathscr{F}$ can be interpreted as the collection of events (subsets of Ω) which are P-measurable. The solution process $x(t, x_0, \omega)$ (or simply, $x(t, x_0)$ or $x(t)$) is then a random process defined on $(\Omega, \mathscr{F}, P)$, as indicated by the dependence on $\omega \in \Omega$. The standard vector norm in $\mathbb{R}^n$ will be denoted by $\|\cdot\|$. The matrix norm is the operator norm induced by the standard vector norm and will be denoted by the same notation $\|\cdot\|$. We make the following definitions:

Definition 2.1: For system (2.1), the equilibrium point 0 is

I) *asymptotically mean square stable*, if for any $x_0 \in \mathbb{R}^n$ and initial distribution $(p_1, \cdots, p_s)$ of $r(t)$,

$$\lim_{t \to +\infty} E\left\{\|x(t, x_0, \omega)\|^2\right\} = 0;$$

II) *exponentially mean square stable*, if for any $x_0 \in \mathbb{R}^n$ and initial distribution $(p_1, \cdots, p_s)$ of $r(t)$, there exists constants $\alpha, \beta > 0$ such that

$$E\left\{\|x(t, x_0, \omega)\|^2\right\} \leq \alpha \|x_0\|^2 e^{-\beta t}, \qquad \forall t \geq 0;$$

III) *stochastically stable*, if for any $x_0 \in \mathbb{R}^n$ and initial distribution $(p_1, \cdots, p_s)$ of $r(t)$

$$\int_0^{+\infty} E\left\{\|x(t, x_0, \omega)\|^2\right\} dt < +\infty;$$

IV) *almost surely (asymptotically) stable*, if for any $x_0 \in \mathbb{R}^n$ and initial distribution $(p_1, \cdots, p_s)$ of $r(t)$

$$P\left\{\lim_{t\to+\infty} \|x(t, x_0, \omega)\| = 0\right\} = 1.$$

In the above, $E\{\cdot\}$ denotes the expectation operator with respect to the underlying probability measure P. □

The definitions above are "stronger" than usual in the sense that they require the relations to hold for any initial distribution $(p_1, \cdots, p_s)$ of $r(t)$. They should not, however, be confused with the strong forms of stability discussed in Kozin's paper [1]. They are similar to the weak stability concepts discussed in Has'minskii's work [3, p. 25]. We believe that the requirement on the intial distribution in the above definitions is reasonable from a practical perspective. That is, stability properties of the system should be independent of the intial state of the form process just as they are independent of the intial state x_0 of the system. Note that in the case when each pair of states of $r(t)$ communicates, the requirement on the intial distribution in the above definitions is not necessary.

One of the main objectives of this paper is to establish the fact that I)–III) are equivalent, and they imply IV). The definition I) for asymptotic mean square stability is weaker than the asymptotic mean square stability of Has'minskii [3]. However, it is strong enough for (2.1), since we will show the equivalence of I), II), and III). According to Kozin [4], for the linear homogeneous systems concerned here, IV) is equivalent to almost sure sample asymptotic Lyapunov stability. Clearly, II) implies I) and III). It follows from Kozin [4] that III) or II) implies IV).

The definition III) for stochastic stability was introduced by Ji and Chizeck [27]. The importance of this definition of stochastic stability is that it corresponds naturally to a finite cost for the infinite horizon JLQ problem. There exists a stronger result for JLQ problems, such as testable necessary and sufficient conditions for the existence of a state-feedback optimal control law, when the stabilization condition is interpreted in the sense of stochastic stability III) [27].

Although the second moment stability definitions I)–III) are most naturally associated with the optimal control of the system subject to a quadratic cost in the form of (1.3), as pointed out by Kozin [1], almost sure sample stability is of great practical importance, because it is the sample paths rather than the moments of the state process which are observed in practice. It is well known that moment stability requirements can be very conservative, when compared to almost sure sample stability; see [5] for several examples. Unfortunately, the study of almost sure stability is difficult because it is usually hard to obtain practical results on sample functions. However, the recent development of the Lyapunov exponent method provides a way to study almost sure stability concepts. although this method is powerful for the study of almost sure sample stability properties of linear stochastic systems, there is a gap between theory and application. It is expected that computational methods will result as the development of the method continues.

For system (2.1) with a given intial distribution $p = (p_1, p_2, \cdots, p_s)$ of $r(t)$, the Lyapunov exponent $\lambda_\omega(x_0)$ associated with a sample solution $x(t, x_0, \omega)$ of (2.1) (identified with the intial state x_0) is defined by

$$\lambda_\omega(x_0) = \lim_{t\to+\infty} \frac{1}{t} \log \|x(t, x_0, \omega)\| \tag{2.2}$$

if the indicated limit exists. For fixed $x_0 \in \mathbb{R}^n$, $\lambda_\omega(x_0)$ is real-valued random variable on the probability space $(\Omega, \mathscr{F}, P)$, as indicated by the subscript ω. For $\omega \in \Omega$ fixed, it is a real-valued function of x_0. Note that $\lambda_\omega(x_0)$ is also implicitly dependent on the initial distribution p of $r(t)$. From (2.2), we see that

$$\|x(t, x_0, \omega)\| \approx \exp(\lambda_\omega(x_0)t), \qquad \text{for } t \text{ large}.$$

Thus, the sign of $\lambda_\omega(x_0)$ determines the asymptotic stability of the sample solution $x(t, x_0, \omega)$ and the magnitude $|\lambda_\omega(x_0)|$ gives the exponential rate of convergence (or divergence) of the sample solution. The system (2.1) will be almost surely stable, if for any x_0 and p, $\lambda_\omega(x_0) < 0$ almost surely and not almost surely stable, if for some x_0 and p, $\lambda_\omega(x_0) > 0$ with positive probability. Before going any further, let us look at a simple example.

Example 2.2: Consider the scalar s-form jump linear system given by

$$\begin{cases} \dot{x}(t) = a(r(t, \omega))x(t), & t \ge 0 \\ x(0) = x_0 \in \mathbb{R} \setminus \{0\} \end{cases} \tag{2.3}$$

where $r(t, \omega) \in \{1, 2, \cdots, s\}$ is assumed to be ergodic with the infinitesimal matrix Q and unique stationary (invariant) distribution $(\pi_1, \cdots, \pi_s)$. The solution process is given by

$$x(t, x_0, \omega) = x_o \exp\left[\int_0^t a(r(u, \omega))\, du\right].$$

Therefore, it follows from the law of large numbers [34, p. 220] that independent of the intial state $x_0 \in \mathbb{R} \setminus \{0\}$ and intial distribution p of $r(t)$, the Lyapunov exponent is given by

$$\begin{aligned} \lambda_\omega(x_0) &= \lim_{t\to+\infty} \frac{1}{t} \log \|x(t, x_0, \omega)\| \\ &= \lim_{t\to+\infty} \frac{1}{t} \log |x_0| + \lim_{t\to+\infty} \frac{1}{t} \int_0^t a(r(u, \omega))\, du \\ &= E_\pi\{a(r(0, \omega))\} = \sum_{i=1}^s \pi_i a_i \qquad \text{a.s.} \end{aligned}$$

where E_π is the expectation with respect to the stationary distribution $(\pi_1, \cdots, \pi_s)$ of $r(t, \omega)$ and $a_i = a(i)$ for all $i \in S$. Hence, the system is almost surely stable, if

$$\pi_1 a_1 + \pi_2 a_2 + \cdots + \pi_s a_s < 0. \tag{2.4}$$

It can be shown that if $\sum_{i=1}^{s}\pi_i a_i = 0$, then

$$P\left\{\lim_{\tau\to+\infty}\sup_{t\geq\tau}|x(t,x_0,\omega)|>0\right\}>0.$$

Therefore, (2.4) is also necessary for almost sure stability. □

We remark here that for a higher dimensional systems, if the flow of the system commutes, a similar condition as in (2.4) can be obtain. Actually, all the results developed for one-dimensional jump linear systems in this work can be directly extended to higher dimensional jump linear systems with commuting flow. In [36], a sufficient condition for almost sure stability of jump linear systems is developed. For one-dimensional systems like (2.3), the sufficient condition is the same as (2.4), which we have shown to also be necessary for this class of systems. This fact was not recognized in [36] and we will shown later in Section IV that the approach used in [36] will not lead to necessary and sufficient conditions for higher dimensional jump linear systems, even if the flow of the systems commutes.

In the above example, it turns out that the random variable $\lambda_\omega(x_0)$ is actually a constant almost surely, independent of the intial distribution p and intial state $x_0\in\mathbb{R}\setminus\{0\}$. This simple and elegant property of the Lyapunov exponents actually holds for all the general jump linear systems in the form of (2.1) with Q satisfying an irreducibility condition. We will present this result in Section IV.

III. Equivalence of Moment Stability Properties and Their Relationship with Almost Sure Stability

In this section, we first prove the following theorem concerning the stability properties defined in the previous section for the jump linear system (2.1).

Theorem 3.1: For the jump linear systems (2.1)

a) the asymptotic mean square stability I), the exponential mean square stability II) and the stochastic stability III) are equivalent;

b) any form of moment stability in the sense of I)–III) in Definition 2.1 implies almost sure stability IV). The converse is not true in general.

This problem establishes the equivalence of second moment stability properties and the fact that each of them is sufficient for almost sure stability of (2.1). As discussed in [1], in general, these results are not true for systems with arbitrary underlying random processes. Clearly, II) implies I) and III). Therefore to show a), we need only to prove that III) implies II) and I) implies III). The fact that moment stability implies almost sure sample stability was observed earlier in Bergen [7] and Bharucha [8], for a special class of randomly switched systems. A general result for systems in the form (2.1) with $r(t)$ an arbitrary stationary random process satisfying certain separability and boundness conditions was obtained by Kozin [4]. In particular, for (2.1), Kozin's results give the fact that II) or III) implies IV). As reported in [1], Bharucha [8] was able to establish the fact that asymptotic mean square stability implies exponential mean square stability for a class of randomly switched systems using Kronecker products of matrices. Our approach is to use a sojourn description of the form process $r(t)$, to write the fundamental matrix of the system as a product of random matrices, and to use this representation to show that I) implies III). Then a stochastic Lyapunov function method developed by Kushner [2] is used to show that III) implies II). This will establish a) of the theorem. We will then study scalar jump linear systems, to illustrate the difference of the stability domains in the space of the parameters of the system for the mean square stability and almost sure stability.

Before proving Theorem 3.1, we first establish some preliminaries. Consider the system (2.1) with $r(t)\in S=\{1,2,\cdots,s\}$ having the infinitesimal matrix $Q=(q_{ij})_{s\times s}$. Suppose the initial distribution of $r(t)$ is given by $(p_1,p_2,\cdots,p_s)$ and let $A_j=A(j)$ for all $j\in S$. Define

$$\left.\begin{aligned} p_{ii}&=0\\ q_i&=-q_{ii}=\sum_{l\neq i}q_{il}\\ p_{ij}&=\frac{q_{ij}}{q_i}\end{aligned}\right\},\qquad \text{for } i,j\in S \text{ and } i\neq j. \quad (3.1)$$

Let $\{r_k;\ k=1,2,\cdots,\}$ be the Markov chain defined on the state space S with stationary one-step transition matrix $(p_{ij})_{s\times s}$ and initial distribution $(p_1,\cdots,p_s)$. This chain is referred to as the *embedded Markov chain* of $r(t)$. We have the following sojourn description of the process $r(t)$ [35, p. 254].

Starting in state $r(0)=i$, the process sojourns there for a duration of time that is exponentially distributed with parameter q_i. The process then jumps to state $j\neq i$ with probability p_{ij}; the sojourn time in state j is exponentially distributed with parameter q_j; and so on. The sequence of states visited by the process, denoted by $i_1,i_2,\cdots$, is the embedded Markov chain $\{r_k,\ k=1,2,\cdots,\}$. Conditioning on $i_1,i_2,\cdots$, the successive sojourn times $\tau^{(1)},\tau^{(2)},\cdots$, are independent exponentially distributed random variables with parameters $q_{i_1},q_{i_2},\cdots$, denoted by $\tau_{i_1},\tau_{i_2},\cdots$.

Let $\mathscr{B}_{\mathbb{R}}$ be the Borel σ-field on $\mathbb{R}$ and let μ_j be the unique probability measure on the measurable space $(\mathbb{R},\mathscr{B}_{\mathbb{R}})$ induced by the exponential distribution function F_j with parameter q_j, i.e., by

$$F_j(y)=\begin{cases}1-e^{-q_j y}, & \text{if } y\geq 0\\ 0, & \text{if } y<0.\end{cases}\qquad(3.2)$$

The joint process $\{(r_k,\tau^{(k)})\}_{k=1}^{+\infty}$ is also a Markov process defined on the state-space $S\times[0,+\infty)$ which has a finite-diemnsional distribution given by the following: for any Borel set $H_j\in\mathscr{B}_{\mathbb{R}}$ and $i_j\in S$ for $j=1,2,\cdots,k$

$$\begin{aligned}P\{r_j=i_j,\ \tau^{(j)}\in H_j: j=1,2,\cdots,k\}\\ &=P\{r_j=i_j,\ \tau^{(1)}\in H_1,\ \tau^{(j)}\in H_j: j=2,3,\cdots,k\mid r_1\\ &=i_1\}p_{i_1}\\ &=P\{r_j=i_j,\ \tau^{(j)}\in H_j: j=2,3,\cdots,k\mid r_1=i_1,\\ &\quad\tau^{(1)}\in H_1\}P\{\tau^{(1)}\in H_1\mid r_1=i_1\}p_{i_1}\end{aligned}$$

$$= P\{r_j = i_j,\ \tau^{(2)} \in H_2,\ \tau^{(j)} \in H_j : j = 3, 4, \cdots, k \mid r_2 = i_2\} p_{i_1 i_2} \mu_{i_1}(H_1) p_{i_1}$$
$$\vdots$$
$$= p_{i_1} p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{k-1} i_k} \mu_{i_1}(H_1) \cdots \mu_{i_k}(H_k). \quad (3.3)$$

In arriving at the formula (3.3), we have used the Markovian property of $\{(r_k, \tau^{(k)}) : k = 1, 2, \cdots,\}$ and the independence property of the sojourn times when conditioning on $i_1, i_2, \cdots$. Let $t_k = \tau^{(1)} + \tau^{(2)} + \cdots + \tau^{(k)}$ be the waiting time for kth jump. It follows from the sojourn description of $r(t)$ above, the fundamental matrix $\Phi(t)$ of (2.1) may be written as

$$\Phi(t) = e^{A_{r_{k+1}}(t-t_k)} e^{A_{r_k}\tau^{(k)}} \cdots e^{A_{r_1}\tau^{(1)}},$$
$$\text{for } t \in [t_k, t_{k+1}),\ k = 0, 1, \cdots, \quad (3.4)$$

where $t_0 \overset{\text{def}}{=} 0$. From (3.3) and (3.4), the mean square of the state process at time t_k can be computed as

$$E\{\|x(t_k, x_0, \omega)\|^2\}$$
$$= x_0' E\{\Phi'(t_k)\Phi(t_k)\} x_0$$
$$= x_0' E\{e^{A'_{r_1}\tau^{(1)}} \cdots e^{A'_{r_k}\tau^{(k)}} e^{A_{r_k}\tau^{(k)}} \cdots e^{A_{r_1}\tau^{(1)}}\} x_0$$
$$= x_0' \Big\{ \sum_{(i_1, i_2, \cdots, i_k)} p_{i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k}$$
$$\cdot \int_{\mathbb{R}^k} e^{A'_{i_1} y_1} \cdots e^{A'_{i_k} y_k} e^{A_{i_k} y_k} \cdots e^{A_{i_1} y_1}$$
$$\cdot \mu_{i_1}(dy_1) \mu_{i_2}(dy_2) \cdots \mu_{i_k}(dy_k) \Big\} x_0 \quad (3.5)$$

where the integration is carried out against the product measure and the summation is taken over all possible $(i_1, i_2, \ldots, i_k)$. Similarly

$$E \int_0^{+\infty} \|x(t, x_0, \omega)\|^2\, dt$$
$$= x_0' E\Big\{ \int_0^{+\infty} \Phi'(s)\Phi(s)\, ds \Big\} x_0$$
$$= \sum_{k=0}^{+\infty} x_0' E\Big\{ \int_{t_k}^{t_{k+1}} e^{A'_{r_1}\tau^{(1)}}$$
$$\cdots e^{A'_{r_k}\tau^{(k)}} e^{A'_{r_{k+1}}(s-t_k)} e^{A_{r_{k+1}}(s-t_k)} e^{A_{r_k}\tau^{(k)}}$$
$$\cdots e^{A_{r_1}\tau^{(1)}}\, ds \Big\} x_0$$
$$\overset{\text{def}}{=} x_0' \sum_{k=0}^{+\infty} H(k) x_0 \quad (3.6)$$

where

$$H(k) = E\Big\{ e^{A'_{r_1}\tau^{(1)}} \cdots e^{A'_{r_k}\tau^{(k)}}$$
$$\cdot \Big[\int_0^{\tau^{(k+1)}} e^{A'_{r_{k+1}}s} e^{A_{r_{k+1}}s}\, ds \Big] e^{A_{r_k}\tau^{(k)}} \cdots e^{A_{r_1}\tau^{(1)}} \Big\}. \quad (3.7)$$

We need the following Lemma. Its proof appears in the Appendix.

Lemma 3.2: For any $F \in \mathbb{R}^{q \times l}$, $G \in \mathbb{R}^{l \times l}$ and $d \in \mathbb{R}^l$, if $FG^k d$ tends to 0 as $k \to +\infty$, then $\sum_{k=0}^{+\infty} FG^k d$ exists. □

Proposition 3.3: For the system (2.1), the asymptotic mean square stability I) implies the stochastic stability III). □

Proof: Suppose that the system (2.1) is asymptotically mean square stable (I). Since t_k goes to $+\infty$ almost surely as k tends to $+\infty$, for any $\{i_j \in S : j = 1, 2, \cdots, k\}$ satisfying $P\{r_j = i_j : j = 1, 2, \cdots, k\} > 0$, the integral in (3.5) is bounded. Thus, by Fubini's Theorem

$$0 < \int_{\mathbb{R}^k} e^{A'_{i_1} y_1} \cdots e^{A'_{i_k} y_k} e^{A_{i_k} y_k}$$
$$\cdots e^{A_{i_1} y_1} \mu_{i_1}(dy_1) \cdots \mu_{i_k}(dy_k)$$
$$= \int_{\mathbb{R}} \Big[e^{A'_{i_1} y_1} \Big[\int_{\mathbb{R}} e^{A'_{i_2} y_2}$$
$$\cdots \Big[\int_{\mathbb{R}} e^{A'_{i_k} y_k} e^{A_{i_k} y_k} \mu_{i_k}(dy_k) \Big]$$
$$\cdots \Big] e^{A_{i_2} y_2} \mu_{i_2}(dy_2) e^{A_{i_1} y_1} \Big] \mu_{i_1}(dy_1)$$
$$\overset{\text{def}}{=} L_{i_1} L_{i_2} \cdots L_{i_k}(I) < +\infty \quad (3.8)$$

where I is the identity matrix and $L_j : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ for $j \in S$ is the linear transformation defined by

$$L_j(M) = \int_{\mathbb{R}} e^{A'_j t} M e^{A_j t} \mu_j(dt)$$
$$= \int_0^{+\infty} q_j e^{A'_j t} M e^{A_j t} e^{-q_j t}\, dt. \quad (3.9)$$

Note that L_j is a positive linear operator and preserves the positive definiteness of matrices. From (3.8), for each $j \in S$ satisfying $P\{r_i = j\} > 0$ for some $i \geq 1$, asymptotic mean square stability guarantees that L_j is well-defined, i.e., $\|L_j\| < +\infty$. Since $(p_1, \ldots, p_s)$ can be arbitrary, L_j is well-defined for every $j \in S$.

Let "$\bar{\ }$" denote the linear isomorphism from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n^2}$ defined by (transforming M into a column vector by sequentially listing the columns m_i of M)

$$M = (m_1, m_2, \cdots, m_n) \leftrightarrow \overline{M} = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix}.$$

Let $\tilde{L}_j$ be the $n^2 \times n^2$ matrix induced by L_j under the linear isomorphism "$\bar{\ }$", i.e., let M be an arbitrary $n \times n$ matrix, then $\tilde{L}_j$ is the unique matrix defined by the relationship

$$\overline{L_j(M)} = \tilde{L}_j \overline{M}.$$

Let "*" denote the inverse of "$\bar{\ }$". Define $G \in \mathbb{R}^{sn^2 \times sn^2}$ by

$$G = \begin{pmatrix} p_{11}\tilde{L}_1 & p_{12}\tilde{L}_2 & \cdots & p_{1s}\tilde{L}_s \\ p_{21}\tilde{L}_1 & p_{22}\tilde{L}_2 & \cdots & p_{2s}\tilde{L}_s \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1}\tilde{L}_1 & p_{s2}\tilde{L}_2 & \cdots & p_{ss}\tilde{L}_s \end{pmatrix} \tag{3.10}$$

Then, it follows from (3.5) and (3.8) that

$$\begin{aligned}
\overline{E\{\Phi'(t_k)\Phi(t_k)\}} &= \overline{\sum_{(i_1, i_2, \ldots, i_k)} p_{i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} L_{i_1} L_{i_2} \cdots L_{i_k}(I)} \\
&= \sum_{(i_1, i_2, \ldots, i_k)} p_{i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \tilde{L}_{i_1} \tilde{L}_{i_2} \cdots \tilde{L}_{i_k} \bar{I} \\
&\qquad \text{(by linearity)} \\
&= \sum_{(i_1, i_2, \ldots, i_k)} (p_{i_1}\tilde{L}_{i_1})(p_{i_1 i_2}\tilde{L}_{i_2}) \cdots (p_{i_{k-1} i_k}\tilde{L}_{i_k})\bar{I} \\
&= (p_1 \tilde{L}_1 p_2 \tilde{L}_2 \cdots p_s \tilde{L}_s) G^{k-2} \begin{pmatrix} \bar{I} \\ \bar{I} \\ \vdots \\ \bar{I} \end{pmatrix} \overset{\text{def}}{=} F G^{k-2} d, \\
&\qquad k \geq 2
\end{aligned} \tag{3.11}$$

where the sums are taken over all possible $(i_1, i_2, \cdots, i_k)$. Thus, asymptotic mean square stability implies that

$$\lim_{k \to +\infty} F G^{k-2} d = 0. \tag{3.12}$$

From (3.3) and (3.7), we have

$$\begin{aligned}
H(k) &= E\Bigg\{ e^{A'_{r_1} \tau^{(1)}} \\
&\quad \times \cdots e^{A'_{r_k} \tau^{(k)}} \left[\int_0^{\tau^{(k+1)}} e^{A'_{r_{k+1}} y} e^{A'_{r_{k+1}} y}\, dy \right] e^{A_{r_k} \tau^{(k)}} \\
&\quad \times \cdots e^{A_{r_1} \tau^{(1)}} \Bigg\} \\
&= \sum_{(i_1, i_2, \cdots, i_{k+1})} p_{i_1} p_{i_1 i_2} \\
&\quad \times \cdots p_{i_k i_{k+1}} L_{i_1} \\
&\quad \times \cdots L_{i_k}\left(\int_{\mathbb{R}} \int_0^f e^{A'_{i_{k+1}} y} e^{A_{i_{k+1}} y}\, dy \mu_{i_{k+1}}(df) \right).
\end{aligned} \tag{3.13}$$

Integrating by parts and using the fact that L_j is well-defined, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^f e^{A'_{i_{k+1}} y} e^{A_{i_{k+1}} y}\, dy \mu_{i_{k+1}}(df) \\
&= \frac{1}{q_{i_{k+1}}} \int_{\mathbb{R}} e^{A'_{i_{k+1}} y_{k+1}} e^{A_{i_{k+1}} y_{k+1}} \mu_{i_{k+1}}(dy_{k+1}) \\
&= \frac{1}{q_{i_{k+1}}} L_{i_{k+1}}(I).
\end{aligned} \tag{3.14}$$

Using (3.11), (3.13), (3.14), (3.6), and (3.7), we have

$$\begin{aligned}
&E\left\{ \int_0^{+\infty} \|x(t, x_0, \omega)\|^2\, dt \right\} \\
&= x'_0 \Bigg\{ \sum_{k=0}^{+\infty} \sum_{(i_1, \cdots, i_{k+1})} \frac{1}{q_{i_{k+1}}} p_{i_1} p_{i_1 i_2} \\
&\quad \times \cdots p_{i_k i_{k+1}} L_{i_1} \\
&\quad \times \cdots L_{i_{k+1}}(I) \Bigg\} x_0 \\
&\leq \max_{1 \leq j \leq s} \left\{ \frac{1}{q_j} \right\} \\
&\quad x'_0 \left(\sum_{k=0}^{+\infty} F G^k d \right)^* x_0 + C_0
\end{aligned} \tag{3.15}$$

where C_0 is a constant. Lemma 3.2, (3.12), and (3.15) yield the desired result. □

A direct corollary to Proposition 3.3 is a sufficient condition for mean square stability and thus stochastic stability.

Corollary 3.4: A sufficient condition for mean square stability I) is that the matrix G has all its eigenvalues inside the unit circle. □

A similar condition for mean square stability for a class of randomly varying parameter systems was obtained by Darkhovskii and Leibovich [9] using Kronecker products of matrices. However, the necessary and sufficient condition claimed in that paper is only sufficient. In order to establish a) of Theorem 3.1, we need only to show III) implies II). This will be accomplished by using a stochastic Lyapunov function approach.

Proposition 3.5: A necessary and sufficient condition for stochastic stability III) of system (2.1) is that there exist positive definite matrices M_j for $j = 1, 2, \cdots, s$ such that

$$-q_i M_i + \sum_{j \neq i} q_{ij} M_j + A'_i M_i + M_i A_i = -I \tag{3.16}$$

for $i = 1, 2, \cdots, s$. Where in (3.16), I is the identity matrix.

Furthermore, stochastic stability III) implies exponential mean square stability II). □

The condition (3.16) was obtained only as a sufficient condition for exponential mean square stability in [37] via Lyapunov function approach.

Proof: The joint process $\{(x(t, x_0), r(t)) : t \geq 0\}$ is a time homogeneous Markov process with the infinitesimal generator $\mathscr{L}$ acting on smooth functions $f(x, r) \overset{def}{=} (f(x, 1), \cdots f(x, s))'$, given by

$$\mathscr{L} = Q + \operatorname{diag}\left\{ x' A'_1 \frac{\partial}{\partial x}, \cdots, x' A'_s \frac{\partial}{\partial x} \right\}. \tag{3.17}$$

Suppose (2.1) is stochastic stable, i.e.,

$$E \int_0^{+\infty} \|x(t, x_0, \omega)\|^2\, dt < +\infty \tag{3.18}$$

for all $x_0 \in \mathbb{R}^n$. Define the function Ψ on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \times S$ by

$$\Psi(T, t, x, j) \overset{\text{def}}{=} E\left\{ \int_t^T x'(\tau, \omega) x(\tau, \omega)\, dt \middle| x(t, \omega) = x, r(t) = j \right\}. \tag{3.19}$$

By the time homogeneous property, we have with a slight abuse of notation that

$$\begin{aligned}\Psi(T, t, x, j) &= \Psi(T - t, x, j) \\ &= E\left\{ \int_0^{T-t} x'(\tau, \omega) x(\tau, \omega)\, d\tau \middle| x(0) = x, r(0) = j \right\} \\ &= x' E\left\{ \int_0^{T-t} \Phi'(\tau)\Phi(\tau)\, d\tau \middle| r(0) = j \right\} x \\ &\overset{\text{def}}{=} x' M(T - t, j) x. \end{aligned} \tag{3.20}$$

Since the system is stochastically stable, $M(\cdot, j)$ is a monotonically increasing and positive definite matrix-valued function bounded from above. Thus

$$M_j \overset{\text{def}}{=} \lim_{T \to +\infty} M(T, j) \tag{3.21}$$

exists. Here, M_j is also positive definite. Let T be arbitrarily fixed. For any $T > s > t > 0$

$$\begin{aligned}&\frac{d}{dt} E\Psi(T - t, x, j)\,|\text{evaluated along the system trajectory} \\ &= \lim_{s \downarrow t} \frac{1}{s - t}\left[E\{\Psi(T - s, x(s), r(s)) | x(t) = x, r(t) = j\} - \Psi(T - t, x, j)\right] \\ &= \frac{\partial}{\partial t}\Psi(T - t, x, j) + (\mathscr{L}\Psi)(T - t, x, j) \\ &= x' \frac{\partial}{\partial t} M(T - t, x, j) x - q_j x' M(T - t, j) x \\ &\quad + \sum_{l \neq j} q_{jl} x' M(T - t, l) x \\ &\quad + x'(A_j' M(T - t, j) + M(T - t, j) A_j) x \\ &= x' \left[\frac{\partial}{\partial t} M(T - t, j) - q_j M(T - t, j) \right. \\ &\quad + \sum_{l \neq j} q_{jl} M(T - t, l) + A_j' M(T - t, j) \\ &\quad \left. + M(T - t, j) A_j \right] x. \end{aligned} \tag{3.22}$$

On the other hand, by (3.19)

$$\begin{aligned}&E\{\Psi(T - s, x(s), r(s)) \mid x(t) = x, r(t) = j\} - \Psi(T - t, x, j) \\ &= E\{\Psi(T - s, x(s), r(s)) - \Psi(T - t, x, j) \mid x(t) = x, r(t) = j\} \\ &= E\left\{ E\left\{ \int_s^T x'(\tau) x(\tau)\, d\tau \mid x(s), r(s) \right\} \right. \\ &\quad -E\left\{ \int_t^T x'(\tau) x(\tau)\, d\tau \mid x(t) = x, r(t) = j \right\} \left. \mid x(t) = x, r(t) = j \right\} \\ &= E\left\{ \int_s^T x'(\tau) x(\tau)\, d\tau - \int_t^T x'(\tau) x(\tau)\, d\tau \mid x(t) = x, r(t) = j \right\} \\ &= -E\left\{ \int_0^{s-t} x'(\tau) x(\tau)\, d\tau \mid x(0) = x, r(0) = j \right\} \\ &= -x' E\left\{ \int_0^{s-t} \Phi'(\tau)\Phi(\tau)\, d\tau \mid r(0) = j \right\} x. \end{aligned} \tag{3.23}$$

However

$$\begin{aligned}&\lim_{s \downarrow t} \frac{1}{s - t} E\left\{ \int_0^{s-t} \Phi'(\tau)\Phi(\tau)\, d\tau \middle| r(0) = j \right\} \\ &= \lim_{s \downarrow t} E\left\{ \frac{1}{s - t} \int_0^{s-t} \Phi'(\tau)\Phi(\tau)\, d\tau \middle| r(0) = j \right\} \\ &= E\{I \mid r(0) = j\} = I. \end{aligned} \tag{3.24}$$

Let T go to $+\infty$ in (3.22) and note that $\partial/\partial t M(T - t, j)$ tends to 0 as T goes to $+\infty$. Then (3.22), (3.23), and (3.24) give (3.16). This proves necessity. For sufficiency, let $M(j) = M_j$ for $j = 1, 2, \cdots, s$ solve (3.16). Define a stochastic Lyapunov function $V(x, r)$ by

$$V(x, r) = x' M(r) x. \tag{3.25}$$

Then

$$\begin{aligned}(\mathscr{L}V)(x, i) &= -\lambda_i V(x, i) \\ &\quad + \sum_{j \neq i} \lambda_{ij} V(x, j) + x' A_i' \frac{\partial}{\partial x} V(x, i) \\ &= x' \left[-\lambda_i M_i + \sum_{j \neq i} \lambda_{ij} M_j \right. \\ &\quad \left. + A_i' M_i + M_i A_i \right] x = -x' x \end{aligned} \tag{3.26}$$

by (3.16). Thus

$$\begin{aligned}(\mathscr{L}V)(x, i) &= -\frac{x' x}{V(x, i)} V(x, i) \\ &\leq -\alpha V(x, i), \qquad \forall i \in S, x \neq 0 \end{aligned} \tag{3.27}$$

where α is the positive constant given by

$$\alpha = \min_{x,i} \frac{x'x}{V(x,i)} = \min_{x,i} \frac{x'x}{x'M_i x} = \left[\max_{x,i}\left(\frac{x}{\|x\|}\right)' M_i \left(\frac{x}{\|x\|}\right)\right]^{-1}.$$

By Dynkin's formula [2, p. 10]

$$\begin{aligned} E\{V(x(t,x_0,\omega),r(t))\} - V(x_0,r(0)) &= E\left\{\int_0^t (\mathscr{L}V)(x(s,x_0,\omega),r(s))\,ds\right\} \\ &\le -\alpha\int_0^t EV(x(s,x_0,\omega),r(s))\,ds. \end{aligned}$$

Applying the Gronwall–Bellman lemma, we have

$$E\{V(x(t;x_0,\omega),r(t))\} \le V(x_0,r(0))e^{-\alpha t}.$$

This implies that (2.1) is exponentially mean square stable and therefore stochastically stable. This completes the proof of sufficiency. The last statement of Proposition 3.5 is already proved in the proof of the sufficiency. This completes the proof of the proposition. □

Having established Proposition 3.3 and 3.5, we have proved a) of Theorem 3.1. Proposition 3.5 presents a testable necessary and sufficient condition for the three equivalent moment stability properties I), II), and III). A similar condition is established by Ji and Chizeck [27] for stochastic stabilizability in a different format. As mentioned before, Kozin showed that stochastic stability III) or exponential mean square stability II) of (2.1) implies almost sure stability IV). Since we have established the equivalence of I), II), and III), mean square stability also implies almost sure stability. It is well known that the converse statement is not true, i.e., second moment stability is stronger than sample stability. However, as observed by Has'minskii [3], Kozin and Sugimoto [17], and Arnold [13], for a certain class of linear stochastic systems, sample stability properties are inherited by *δ-moment stability* properties for small δ ($\delta\downarrow 0^+$). We next prove a theorem of this sort for one-dimensional jump linear systems. We conjecture that the result holds for general ***n***-dimensional jump linear systems as well. But, so far, we have not been able to obtain a rigorous proof of the result for the general case.

Consider the one-dimensional jump linear system as follows

$$\begin{cases} \dot{x}(t) = a(r(t))x(t), & t \ge 0 \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{3.28}$$

where $r(t)$ is the Markov process given before and we assume that each pair of states of $r(t)$ communicates. Let $a_i = a(i)$ for $i \in S$ and let $(\pi_1,\cdots,\pi_s)$ be the unique stationary distribution of $r(t)$ (note that $r(t)$ is ergodic in this case).

Definition 3.6: For system (3.28), the equilibrium 0 is said to be asymptotically δ-moment stable ($\delta > 0$), if for some (any) initial distribution $(p_1,\cdots,p_s)$ and any x_0

$$\lim_{t\to+\infty} E\left\{\|x(t,x_0,\omega)\|^\delta\right\} = 0.$$

□

As demonstrated in Example 2.2, the region of almost sure stability for (3.28) in the parameter space $\mathbb{R}^s$ ($(a_1,\cdots,a_s)\in\mathbb{R}^s$) is defined by the open half space

$$\Sigma^a \overset{\text{def}}{=} \left\{(a_1,\cdots,a_s)\in\mathbb{R}^s : \pi_1 a_1 + \pi_2 a_2 + \cdots + \pi_s a_s < 0\right\}. \tag{3.29}$$

Let Σ^δ denote the region of δ-moment stability, i.e.,

$$\Sigma^\delta \overset{\text{def}}{=} \{(a_1,\cdots,a_s)\in\mathbb{R}^s : (3.28) \text{ is } \delta\text{-moment stable}\}. \tag{3.30}$$

We have the following result.

Theorem 3.7: For system (3.28)

a) for any $0<\delta\le\alpha<+\infty$, α-moment stability implies δ-moment stability and δ-moment stability implies almost sure stability. That is, $\Sigma^\alpha \subset \Sigma^\delta \subset \Sigma^a$.

b) the region of δ-moment stability Σ^δ monotonically tends to that of almost sure stability Σ^a as δ goes to 0. That is, $\lim_{\delta\to 0^+}\Sigma^\delta = \bigcup_{\delta>0}\Sigma^\delta = \Sigma^a$. □

Before proving the theorem, we need the following basic results.

Lemma 3.8: The system (3.28) is δ-moment stable if and only if the following equation

$$(Q + \delta\,\mathrm{diag}\{a_1,\cdots,a_s\})Z = -b \tag{3.31}$$

with $b = (1,1,\cdots,1)' \in \mathbb{R}^s$ has a solution $Z = (z_1, z_2,\cdots,z_s)$ satisfying $z_j > 0$ for all $j\in S$. □

Proof: The proof of the lemma is similar to the proof of Proposition 3.5. For the proof of necessity, assume $x_0 > 0$ and define

$$\phi_\delta(t) = \exp\left[\delta\int_0^t a(r(\tau))\,d\tau\right]$$

and

$$M(T-t,j) = E\left\{\int_0^{T-t}\phi_\delta(\tau)\,d\tau \middle| r(0)=j\right\}, \qquad j\in S.$$

Let $\Psi(T-t,x,j) = M(T-t,j)x^\delta$ for $x\ge 0$. For sufficiency, take the Lyapunov function

$$V(x,r) = M(r)|x|^\delta$$

with

$$M(j) = \lim_{T\to+\infty} M(T-t,j) = E\left\{\int_0^{+\infty}\phi_\delta(t)\,dt \middle| r(0)=j\right\}. \tag{3.32}$$

M_j is well-defined because we can show that δ-moment stability implies that the last term in (3.32) is finite, using a similar argument as in the proof of Proposition 3.3. The details of the proof will not be repeated here. □

Lemma 3.9: For $n \geq 2$, let

$$H \overset{\text{def}}{=} \begin{pmatrix} -h_1 & h_{12} & \cdots & h_{1n} \\ h_{21} & -h_2 & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & -h_n \end{pmatrix} \overset{\text{def}}{=} (H_1, H_2, \cdots, H_n) \tag{3.33}$$

where H_j is the jth column of H. Let $b = (1, 1, \cdots, 1)' \in \mathbb{R}^n$. Suppose the entries of H satisfy $h_i > 0$, $h_{ij} \geq 0$, and $\sum_{m \neq i} h_{im} \leq h_i$ for all $i, j \in N = \{1, 2, \cdots, n\}$ with $\sum_{j \neq k} h_{kj} < h_k$ for some $k \in N$. Then

a) $\det(H) = (-1)^n \eta$ for some $\eta > 0$;

b) $\det(H_1, \cdots, H_{j-1}, b, H_{j+1}, \cdots, H_n) = (-1)^{n-1} \gamma_j$ for some $\gamma_j > 0$ and for all j. □

The proof of the lemma is presented in the Appendix. The next lemma relates the unique stationary distribution of $r(t)$ to the minors of its infinitesimal matrix. Its proof also appears in the Appendix.

Lemma 3.10: Suppose each pair of the states of the form process $r(t)$ communicate. Let $(\pi_1, \cdots, \pi_s)$ be the unique stationary distribution of $r(t)$. For each $i \in S$, let Q_i be the $(s-1) \times (s-1)$ matrix obtained by deleting the ith row and ith column of Q, the infinitesimal matrix of $r(t)$ and let $\alpha_i = \det(Q_i)$. Then there exists a constant $c > 0$ such that $\pi_i / \alpha_i = (-1)^{s-1} c$ for all $i \in S$. □

Proof of Theorem 3.7: It is obvious that

$$|x(t, x_0)|^\delta = |x_0|^\delta \exp\left[\delta \int_0^t a(r(\tau))\, d\tau\right] = |x_0|^\delta \phi_\delta(t). \tag{3.34}$$

By using arguments similar to the proof of Proposition 3.3, we can easily show that δ-moment stability (iff $E\{\phi_\delta(t)\} \to 0$ as $t \to +\infty$) implies that

$$E \int_0^{+\infty} \phi_\delta(t)\, dt < +\infty$$

and this implies exponential δ-moment stability, i.e., $E\{\phi_\delta(t)\} < \zeta \exp(-\beta t)$ for some $\zeta, \beta > 0$. Thus, by Jensen's inequality [34, p. 33] and the law of large numbers [34, p. 220], we obtain that δ-moment stability implies

$$\begin{aligned} -\beta &\geq \lim_{t \to +\infty} \frac{1}{t} \log E\{\phi_\delta(t)\} \\ &= \lim_{t \to +\infty} \frac{1}{t} \log E\left\{|x(t, x_0)|^\delta\right\} \\ &= \lim_{t \to +\infty} \frac{1}{t} \log E\left\{\exp\left[\delta \int_0^t a(r(\tau))\, d\tau\right]\right\} \\ &\geq \lim_{t \to +\infty} \log\exp\left[E\left\{\frac{1}{t}\delta \int_0^t a(r(\tau))\, d\tau\right\}\right] \\ &= \lim_{t \to +\infty} \delta E\left\{\frac{1}{t} \int_0^t a(r(\tau))\, d\tau\right\} \\ &= \delta(\pi_1 a_1 + \pi_2 a_2 + \cdots + \pi_s a_s) \text{ a.s.} \end{aligned} \tag{3.35}$$

This yields the almost sure stability of the system, since the exponent is negative. Now, consider that for any $0 < \delta \leq \alpha < +\infty$, by Jensen's inequality

$$E\{\phi_\alpha(t)\} = E\{\phi_\delta^{\alpha/\delta}(t)\} \geq \left(E\{\phi_\delta\}\right)^{\alpha/\delta}. \tag{3.36}$$

It follows from (3.34) and (3.36) that α-moment stability implies δ-moment stability. This proves a) of Theorem 3.7. To show b), we show that for arbitrarily given $(a_1, \cdots, a_s) \in \sum^a$, there exists a small $\delta > 0$ such that (3.31) has a positive solution $Z = (z_1, \cdots, z_s) > 0$. By Lemma 3.8, this will prove that fact that the system (3.28) with parameters $(a_1, \cdots, a_s)$ is also δ-moment stable. However, since $(a_1, \cdots, a_s) \in \sum^a$ is arbitrarily chosen, this shows b) of Theorem 3.7.

Let $(a_1, \cdots, a_s) \in \sum^a$. Then, $\lambda \overset{\text{def}}{=} \pi_1 a_1 + \pi_2 a_2 + \cdots + \pi_s a_s < 0$. Consider that

$$\det(Q + \delta \operatorname{diag}(a_1, \cdots, a_s)) = y_s \delta^s + \cdots + y_2 \delta^2 + y_1 \delta \tag{3.37}$$

where $y_1 = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_s a_s$ with α_i being given in Lemma 3.10. By Lemma 3.10, we know that

$$\det(Q + \delta \operatorname{diag}(a_1, \cdots, a_s)) = y_1 \delta + O(\delta^2) = (-1)^{s-1} c^{-1} \lambda \delta + O(\delta^2). \tag{3.38}$$

In (3.38), $c > 0$ is the constant given in Lemma 3.10. Thus, for δ sufficiently small, $\det(Q + \delta \operatorname{diag}(a_1, \cdots, a_s)) \neq 0$ and (3.31) has a unique solution $Z = (z_i, z_2, \cdots, z_s)$ given by

$$z_j = z_j(\delta) = \frac{-\det(F_j(\delta))}{\det(Q + \delta \operatorname{diag}(a_1, \cdots, a_s))}. \tag{3.39}$$

Where in (3.39), $F_j(\delta)$ is the matrix obtained by replacing the jth column of the matrix $Q + \delta \operatorname{diag}(a_1, \cdots, a_s)$ by b. It is immediate that

$$\det(F_j(\delta)) = \det(G_j) + O(\delta) \tag{3.40}$$

with G_j being the matrix obtained by replacing the jth column of Q by b. Since each pair of states of $r(t)$ communicates, by b) of Lemma 3.9, we have

$$\det(G_j) = (-1)^{s-1} \gamma_j, \qquad \text{for some } \gamma_j > 0. \tag{3.41}$$

Therefore, from (3.38), (3.39), (3.40), and (3.41), we have

$$z_j = \frac{-\left((-1)^{s-1}\gamma_j + O(\delta)\right)}{(-1)^{s-1} c^{-1} \lambda \delta + O(\delta^2)} = \frac{(-1)^s \gamma_j + O(\delta)}{(-1)^s c^{-1}(-\lambda)\delta + O(\delta^2)}. \tag{3.42}$$

Since $-\lambda > 0$, we see that we can always take δ sufficient small so that $z_j > 0$ for all $j \in S$. This proves b) and completes the proof. □

We conclude this section with an example.

Example 3.11: Consider the two form scalar jump linear system

$$\begin{cases} \dot{x}(t) = a(r(t))x(t), & t \geq 0 \\ x(0) = x_0 \in \mathbb{R} \end{cases} \quad (3.43)$$

where $r(t) \in \{1, 2\}$ has the infinitesimal matrix

$$Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix}, \quad 0 < q < +\infty \quad (3.44)$$

with unique stationary distribution $\pi = (1/2, 1/2)$. Let $a_i = a(i)$ for $i = 1, 2$. We can directly compute the δ-moment of solution process using the sojourn descritpion of $r(t)$ given before. For any $\delta > 0$, using the finite-dimensional distribution (3.3), we have

$$E\{|x(t_k, x_0)|^\delta\} = |x_0|^\delta \sum_{(i_1, \cdots, i_k)} p_{i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \prod_{j=1}^{k} q_{i_j} \int_0^{+\infty} e^{\delta a(i_j)t} e^{-q_{i_j}t}\, dt. \quad (3.45)$$

Since

$$p_i = \tfrac{1}{2}, \quad p_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \neq j, \end{cases} \quad q_j = q, \forall i, j \in \{1, 2\}$$

it follows that

$$E\{|x(t_{2l}, x_0)|^\delta\} = \left(\frac{q}{q - \delta a_1}\right)^l \left(\frac{q}{q - \delta a_2}\right)^l |x_0|^\delta. \quad (3.46)$$

Together (3.45) and (3.46) imply that a necessary condition for δ-moment stability is

$$\delta a_1 < q$$
$$\delta a_2 < q$$
$$\left(\frac{q}{q - \delta a_1}\right)\left(\frac{q}{q - \delta a_2}\right) < 1. \quad (3.47)$$

It is easy to show that (3.47) is also sufficient for δ-moment stability. This necessary and sufficient condition can also be directly derived from (3.31) in Lemma 3.8.

From Example 2.2, a necessary and sufficient condition for almost sure stability is $a_1 + a_2 < 0$. Fig. 1 illustrates the stability regions in the (a_1, a_2) parameter space for almost sure and δ-moment stability. The stability region for δ-moment stability increases as δ decreases and it tends to the region for almost sure stability as δ goes to 0^+. □

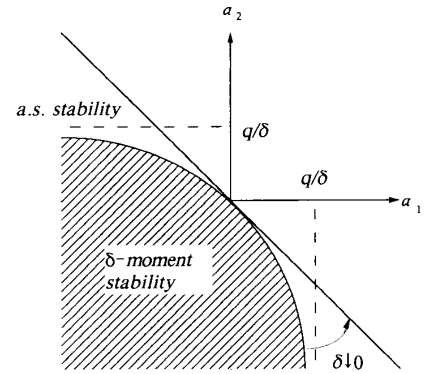

Fig. 1. Stability regions for system (3.43).

IV. Lyapunov Exponents and Almost Sure Stability

For the jump linear system (2.1), the equivalent second moment stability properties I), II), and III) imply almost sure stability IV). Example (3.11) illustrates the difference between moment and sample stability. It turns out that mean square stability is much more conservative than almost sure stability. In this section, we present some recent results on Lyapunov exponents for the study of almost sure stability.

The Lyapunov exponent, as defined in (2.2), is naturally associated with the asymptotic growth (or decay) property of a sample solution process of (2.1). Once the sign of the (top) exponent is determined, in an almost sure sense, the almost sure stability property is obtained.

The study of Lyapunov exponents was initiated by Lyapunov a century ago. A collection of efforts that have been made are presented in the book [10]. A methematically oriented survey is given by Arnold and Wihstutz [11]. It is a topic iof intensive current research. A detailed discussion of Lyapunov exponents is beyond our purpose here and the interested reader is referred to [10]. We will concentrate on the results which can be directly applied to the jump linear system (2.1).

By definition, the Lyapunov exponent is given by

$$\lambda_\omega(x_0) = \lim_{t \to +\infty} \frac{1}{t} \log \|x(t, x_0, \omega)\| \quad (4.1)$$

if the indicated limit exists. $\lambda_\omega(x_0)$ is a random variable for fixed $x_0 \in \mathbb{R}^n$ and a function of x_0 for fixed $\omega \in \Omega$. However, as demonstrated by Example 2.1, $\lambda_\omega(x_0)$ for an ergodic one-dimensional system is almost surely a constant and also is independent of $x_0 \neq 0$. This property of the Lyapunov exponents actually holds in general for the jump linear system (2.1) with the form process $r(t)$ satisfying an irreducibility condition. The following theorem is a direct corollary to the nonrandom spectrum theorem proved in [12].

Theorem 4.1: For system (2.1), suppose that each pair of states of $r(t)$ communicates. Then, there are q real constants

$$+\infty > \bar{\lambda}_1 > \bar{\lambda}_2 > \cdots > \bar{\lambda}_q > -\infty$$

and a sequence of subspaces $\{V_j\}_{j=1}^{q+1}$ of $\mathbb{R}^n$ satisfying

$$\mathbb{R}^n \stackrel{\text{def}}{=} V_1 \supset V_2 \supset \cdots \supset V_q \supset V_{q+1} \stackrel{\text{def}}{=} \{0\}$$

such that

a) $\lambda_\omega(x_0) = \bar{\lambda}_j$ a.s. if and only if $x_0 \in V_j \setminus V_{j+1}$ for $j = 1, 2, \cdots, q$;

b) the subspace V_j is $A(i)$-invariant for all i and j, i.e.,

$$A(i)V_j \subset V_j, \quad \forall i \in S \text{ and } j \in \{1, 2, \cdots, q\};$$

c) all the results and quantities above are independent of the initial distribution $(p_1, p_2, \cdots, p_s)$ of the form process $r(t)$.

This theorem completely characterizes the qualitative properties of the Lyapunov exponents of the jump linear system (2.1). The top exponent $\overline{\lambda}_1$ determines the almost sure stability of (2.1). Since according to b) of the theorem, V_j is an invariant subspace of (2.1), the restriction of (2.1) to V_j makes sense. The almost sure stability of the restricted system is determined by its own top exponent $\overline{\lambda}_j$. Readers are encouraged to refer to [12] and the references cited therein, for more detailed discussions on the theorem and related results.

The question to ask is how can we compute the number $\overline{\lambda}_1$, or at least determine its sign? The key idea behind most computation procedures developed so far is due to Has'minskii [18], which can be described briefly as follows: introduce the polar coordinate $\theta(t) = x(t)/\|x(t)\|$ and $\rho(t) = \log\|x(t)\|$. Then, (2.1) is equivalent to

$$\begin{cases} \dot{\theta}(t) = \left[A(r(t)) - \theta'(t)A(r(t))\theta(t)I\right]\theta(t) \\ \dot{\rho}(t) = \theta'(t)A(r(t))\theta(t). \end{cases} \tag{4.2}$$

We see that $\{(\theta(t), r(t)) : t \ge 0\}$ is jointly a time homogeneous Markov process defined on the compact state-space $U^{n-1} \times S$ with $U^{n-1} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : \|x\| = 1\}$ being the unit sphere in $\mathbb{R}^n$. If $\{(\theta(t), r(t)) : t \ge 0\}$ is ergodic with a unique invariant distribution $f(\theta, i)\,d\theta$, then it follows from the law of large numbers [33, p. 220] that for any initial distribution of $r(t)$ and any $x_0 \in \mathbb{R}^n \setminus \{0\}$

$$\begin{aligned} \lambda_\omega(x_0) &= \lim_{t \to +\infty} \frac{1}{t} \log\|x(t, x_0)\| = \lim_{t \to +\infty} \frac{1}{t}\rho(t) \\ &= \lim_{t \to +\infty} \frac{1}{t} \int_0^t \theta'(u)A(r(u))\theta(u)\,du \\ &= \sum_{i=1}^{s} \int_{U^{n-1}} \varphi' A(i)\varphi f(\varphi, i)\,d\varphi \text{ a.s.} \end{aligned} \tag{4.3}$$

Has'minskii's work was later refined by Pinsky [19]. The formula (4.3) presents a prototype formula for the computation of the Lyapunov exponent. Unfortunately, the determination of the invariant density function $f(\theta, i)$ is usually very difficult task and represents a challenging problem at the current time, even for a simple but nontrivial system such as the random harmonic oscillator [14], [15].

The random harmonic oscillator is two-dimensional jump linear system with a two state form process, given by

$$\dot{x}(t) = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \sigma\xi(t)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]x(t) \tag{4.4}$$

where $\xi(t) \in \{-1, +1\}$ is the random telegraph process and $|\sigma| < 1$ is a parameter signifying the noise strength. By Theorem 4.1, we know that $\lambda_\omega(x_0) = \overline{\lambda}_1$ a.s. for any $x_0 \in \mathbb{R}^2 \setminus \{0\}$. The positivity of $\overline{\lambda}_1$, thus the almost sure instability of (4.4) and an analytic series expansion of $\overline{\lambda}_1$ in the power of the noise strength σ are derived in [14], [15] using a stochastic averaging procedure. A significant effort is now being devoted toward developing effective computational procedures for the exponents of linear stochastic systems.

The Lyapunov exponent concept is also useful when studying moment stability of linear stochastic systems. One may define a Lyapunov exponent for the mth moment as

$$\lambda_m(x_0) = \lim_{t \to +\infty} \frac{1}{t} \log E\left\{\|\,\|x(t, x_0, \omega)\|^m\right\}. \tag{4.5}$$

Arnold [13] has obtained some basic properties of $\overline{\lambda}_m(x_0)$ as a function of (m, x_0) and has also established some interesting results on the relationship between moment and almost sure stability properties for a system in the form of (2.1) with $r(t)$ being a diffusion process, see [13] for details.

As mentioned previously, second moment stability is often conservative. But it is naturally associated with the optimal linear quadratic control problem. Almost sure stability is a natural criteria to use for stability, since it is a sample property. However, a suitable form of the cost function to be optimized to guarantee almost sure stability (not necessarily guaranteeing second moment stability) is not available. This problem must be solved in order to develop significant results in optimal controller design directly based on almost sure stability (stabilizability).

Next, we present an example to illustrate results developed in this work.

Example 4.2: Consider the jump linear system (2.1) with a two form process $r(t) \in \{-1, +1\}$ which has an infinitesimal matrix in the form

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and initial distribution (p_1, p_2). The parameter matrices are

$$A(-1) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A(+1) = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Second Moment Stability: It follows from Theorem 3.1 and Proposition 3.5 that a necessary and sufficient condition for second moment stability in the sense of I), II), and III) is that the matrix equation (3.16) has a positive definite solution

$$M_1 = \begin{pmatrix} \alpha_1^{(1)} & \beta^{(1)} \\ \beta^{(1)} & \alpha_2^{(1)} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} \alpha_1^{(2)} & \beta^{(2)} \\ \beta^{(2)} & \alpha_2^{(2)} \end{pmatrix}.$$

For this syetm, $q_j = q_{ij} = 1$ for $i, j = 1, 2$. The equation (3.16) can be written in the form of a system of linear equations:

$$\begin{pmatrix} -5 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_2^{(1)} \\ \beta^{(1)} \\ \alpha_1^{(2)} \\ \alpha_2^{(2)} \\ \beta^{(2)} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}. \tag{4.6}$$

Equation (4.6) has a unique solution given by

$$(-1, -49, 6, -6, -148, 24)'.$$

Thus, we have

$$M_1 = \begin{pmatrix} -1 & 6 \\ 6 & -49 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} -6 & 24 \\ 24 & -148 \end{pmatrix}$$

which are not positive definite. Hence, the system is not stable in the sense of I), II), and III).

Almost Sure Stability: All common invariant subspaces of $A(-1)$ and $A(+1)$ are easily determined as

$$\{0\}, \quad W \stackrel{\text{def}}{=} \text{span}\{(1,0)'\}, \qquad \mathbb{R}^2.$$

For any $x_0 \in W \setminus \{0\}$, since W is invariant

$$x(t, x_0, \omega) = x_0 \exp\left(\int_0^t a(r(\tau))\, d\tau\right)$$

where $a(-1) = -2$ and $a(+1) = 1/2$. By the law of large numbers [34, p. 220]

$$\begin{aligned} \bar{\lambda}_\omega(x_0) &= \lim_{t \to +\infty} \frac{1}{t} \log \|x(t, x_0, \omega)\| \\ &= \lim_{t \to +\infty} \frac{1}{t} \int_0^t a(r(\tau))\, d\tau \\ &= \frac{1}{2}(-2) + \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{3}{4} \stackrel{\text{def}}{=} \bar{\lambda}_2 \end{aligned}$$

$$\text{a.s.} \quad \forall x_0 \in W \setminus \{0\}. \tag{4.7}$$

By Theorem 4.1, we know that there is $\bar{\lambda}_1$ such that

$$\bar{\lambda}_\omega(x_0) = \bar{\lambda}_1 \qquad \text{a.s.} \quad \forall x_0 \in \mathbb{R}^2 \setminus W. \tag{4.8}$$

In order to determine $\bar{\lambda}_1$, we need invoke the regularity property of the Lyapunov exponent [11, p. 3], i.e.,

$$\bar{\lambda}_1 + \bar{\lambda}_2 = \lim_{t \to +\infty} \frac{1}{t} \log|\det \Phi(t)| \qquad \text{a.s.}$$

where $\Phi(t)$ is the fundamental matrix of the system. By the Liouville formula

$$\begin{aligned} \lim_{t \to +\infty} \frac{1}{t} \log|\det \Phi(t)| &= \lim_{t \to +\infty} \frac{1}{t} \text{tr}\{A(r(\tau))\}\, d\tau \\ &= \frac{1}{2}(-1-2) + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) \\ &= -1 \qquad \text{a.s.} \end{aligned} \tag{4.9}$$

In arriving at (4.9), we used the law of large numbers again. Thus, from (4.8) and (4.9), we have $\bar{\lambda}_1 = -1/4$. Since $\bar{\lambda}_2 < \bar{\lambda}_1 < 0$, the system is almost surely stable. □

In [36], Mariton derives sufficient conditions for the almost sure stability (or instability) by obtaining a simple upper bound on the top exponent for jump linear systems. In the next example, we illustrate that the results we have presented for almost sure stability are stronger than those obtained in [36].

Example 4.2: Earlier we commented that for scalar systems the sufficient condition obtained in [36] is also necessary but not for higher dimensional systems. Let us consider the jump linear systems:

$$\begin{cases} \dot{x}(t) = A(r(t))x(t), & t \geq 0 \\ x_0 = x(0) \end{cases} \tag{4.8}$$

with $A(i)A(j) = A(j)A(i)$ for all $i, j \in \{1, 2, \cdots, s\}$. Consider the example given in [36], $A(1) = \text{diag}(1, -2)$ and $A(2) = I = \text{diag}(1, 1)$ with $\pi_1 = 1/3$ and $\pi_2 = 2/3$. According to [36], no conclusion for almost sure stability or instability of (4.8) can be reached using the result of the paper. However, using the Lyapunov exponent approach as outlined in the current paper, the following conclusions can be easily obtained: Note that $\mathbb{R}^2 = E_1 \oplus E_2$ with $E_1 = \text{span}\{(1,0)'\}$ and $E_2 = \text{span}\{(0,1)'\}$.

1) For $x_0 \in E_1$, $\lambda_\omega(x_0) = 1 \times \frac{1}{3} + 1 \times \frac{2}{3} = 1$ a.s. and the system is thus almost surely unstable on E_1.

2) For $x_0 \in E_2$, $\lambda_\omega(x_0) = -2 \times \frac{1}{3} + 1 \times \frac{2}{3} = 0$ a.s. and the systems is also almost surely unstable on E_2.

3) Applying Theorem 4.1, we have for $x_0 \in \mathbb{R}^2 \setminus E_2$, $\lambda_\omega(x_0) = 1$ a.s. Hence, we conclude that the system is almost surely unstable. □

V. Concluding Remarks

In this paper, we have studied the stochastic stability properties of jump linear systems. It is shown that asymptotic mean square stability I), exponential mean square stability II) and stochastic stability III) are equivalent and they imply almost sure stability IV) of the system. A testable necessary and sufficient condition for I), II), and III) is proved. We have also studied the relationship between almost sure stability and δ-moment stability for one-dimensional jump linear syatems, and the results can be directly extended to jump linear systems with commuting flow. The Lyapunov exponent method for the study of almost sure stability IV) is discussed and a theorem which characterizes the qualitative properties of Lyapunov exponents of the jump linear systems is stated. It is shown that these results can provide more detailed information about the almost sure stability or instability of the systems, than previously available sufficient conditions.

A possible direction for future work is to generalize these results to the case where the form process $r(t)$ is defined on a continuous parameter space and to generalize the results of Theorem 3.7 to higher dimensional systems. This requires a more involved effort than is presented in this paper.

Appendix

Proof of Lemma 3.2: Let T be a nonsingular matrix which transforms G to its Jordan canonical form, i.e., $T^{-1}GT = \text{diag}(G_1, G_2, \cdots, G_m)$ with

$$G_j = \begin{pmatrix} \eta_j & 1 & 0 & \cdots & 0 \\ 0 & \eta_j & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & & \eta_j \end{pmatrix} \in \mathbb{R}^{q_j \times q_j}, \quad j = 1, 2, \cdots, m. \tag{A.1}$$

Assume that T is chosen so that

$$\begin{cases} |\eta_j| \geq 1, & \text{for } 1 \leq j \leq m_1 \\ |\eta_j| < 1, & \text{for } m_1 < j \leq m. \end{cases}$$

Let $\hat{F} = FT = (\hat{F}_1, \hat{F}_2, \cdots, \hat{F}_m)$ and $\hat{d} = T^{-1}d = (\hat{d}_1', \hat{d}_2', \cdots, \hat{d}_m')'$. Then

$$FG^k d = FT^{-1}TG^kTT^{-1}d = \hat{F}G^k\hat{d} = \sum_{j=1}^{m_1} \hat{F}_j G_j^k \hat{d}_j + \sum_{j=m_1+1}^{m} \hat{F}_j G_j^k \hat{d}_j \tag{A.2}$$

and

$$\sum_{k=1}^{+\infty} FG^k d = \lim_{N\to+\infty} \sum_{k=1}^{N}\sum_{j=1}^{m_1} \hat{F}_j G_j^k \hat{d}_j + \lim_{N\to+\infty} \sum_{k=1}^{N}\sum_{j=m_1+1}^{m} \hat{F}_j G_j^k \hat{d}_j = \lim_{N\to+\infty} \sum_{k=1}^{N}\sum_{j=1}^{m_1} \hat{F}_j G_j^k \hat{d}_j + \sum_{j=m_1+1}^{m} \hat{F}_j \sum_{k=1}^{+\infty} G_j^k \hat{d}_j. \tag{A.3}$$

The last term in (A.3) is well-defined because $|\eta_j| < 1$ for $m_1 + 1 \leq j \leq m$. To show

$$\left\| \sum_{k=1}^{+\infty} FG^k d \right\| < +\infty$$

it is thus enough to show

$$\lim_{N\to+\infty} \left\| \sum_{k=1}^{N}\sum_{j=1}^{m_1} \hat{F}_j G_j^k \hat{d}_j \right\| < +\infty. \tag{A.4}$$

Taking the limit $k \to +\infty$ in (A.2) and using the hypothesis $\lim_{k\to+\infty} FG^k d = 0$, we have

$$\lim_{k\to+\infty} \sum_{j=1}^{m_1} \hat{F}_j G_j^k d_j = 0. \tag{A.5}$$

However, we may write (A.5) as

$$\lim_{k\to+\infty} \sum_{j=1}^{m_1} \hat{F}_j G_j^k \hat{d}_j = \lim_{k\to+\infty} \sum_{j=1}^{m_1} \eta_j^k \left(f_{j1} + k\eta_j^{-1} f_{j2} + \cdots + \frac{k(k-1)\cdots(k-q_j-1)}{(q_j-1)!} \eta_j^{-q_j} f_{jq_j} \right) = 0 \tag{A.6}$$

where $f_{j1}, f_{j2}, \cdots, f_{jq_j} \in \mathbb{R}^q$. Without loss of generality, we assume $\eta_j \neq \eta_i$ for $i \neq j$ (otherwise, we may collect terms in (A.6) which have same η_j and redefine the vectors $f_{j1}, \cdots f_{jq_j}$). Since $|\eta_j| \geq 1$ for $1 \leq j \leq m_1$, (A.6) implies that

$$\lim_{k\to+\infty} \left(f_{j1} + k\eta_j^{-1} f_{j2} + \cdots + \frac{k(k-1)\cdots(k-q_j-1)}{(q_j-1)!} \eta_j^{-q_j} f_{jq_j} \right) = 0, \quad \forall 1 \leq j \leq m_1. \tag{A.7}$$

Observe that the coefficients of f_{jl} in (A.7) each has a different order of k. Therefore, (A.7) implies that $f_{jl} = 0$ for all $1 \leq j \leq m_1$ and $1 \leq l \leq q_j$. Thus,

$$\sum_{j=1}^{m_1} \hat{F}_j G_j^k \hat{d}_j \equiv 0, \qquad \forall k \geq 1.$$

From this, (A.4) follows and the proof is completed. □

Proof of Lemma 3.9:

a) We use mathematical induction to the dimension n of H. The result is clearly true when $n = 2$. Suppose it is true when $n = m - 1$. In the case when $n = m$. Let $g_i = h_i - \Sigma_{j\neq i} h_{ji}$. By the assumption of the lemma, we have $g_i \geq 0$ for all $i \in N \overset{\text{def}}{=} \{1, 2, \cdots, n\}$ and $g_k > 0$ for some $k \in N$. Define

$$D(\delta) = \det(\delta\bar{H} + \tilde{H}), \qquad \text{for } \delta \in [0, 1]. \tag{A.8}$$

Where in (A.8)

$$\bar{H} = \operatorname{diag}(g_1, g_2, \cdots, g_m)$$

$$\tilde{H} = \begin{pmatrix} \sum_{j\neq 1} h_{1j} & -h_{12} & \cdots & -h_{1n} \\ -h_{21} & \sum_{j\neq 2} h_{2j} & \cdots & -h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -h_{n1} & -h_{n2} & \cdots & \sum_{j\neq n} h_{nj} \end{pmatrix}.$$

Let $D_i(\delta)$ be the matrix obtained by deleting the ith row and ith column of $D(\delta)$, for $i = 1, 2, \cdots, n$. Then, $\det(H) = (-1)^m \det(\bar{H} + \tilde{H}) = (-1)^m D(1)$. We prove a) by showing $D(1) > 0$ in the following. Consider that

$$\frac{d}{d\delta} D(\delta) = \frac{d}{d\delta} \det(\delta\bar{H} + \tilde{H}) = g_1 \det(D_1(\delta)) + g_2 \det(D_2(\delta)) + \cdots + g_m \det(D_m(\delta)). \tag{A.9}$$

It is easy to see that for any $\delta \in (0, 1)$ and $1 \leq j \leq m$, $-D_j(\delta)$ is a $(m-1) \times (m-1)$ matrix in the same form as H and satisfies the assumption of the lemma. Thus, by the induction hypothesis, we have for any $\delta \in (0, 1)$ and $1 \leq j \leq m$,

$$(-1)^{m-1} \det(D_j(\delta)) = (-1)^{m-1} \eta_j, \qquad \eta_j > 0. \tag{A.10}$$

Since not all g_j are zero, by (A.9) and (A.10), we have

$$\frac{d}{d\delta} D(\delta) > 0, \qquad \forall \delta \in (0, 1). \tag{A.11}$$

Thus, $D(1) > D(0) = \det(\tilde{H}) = 0$ (note that $\tilde{H}$ is singular). This proves a).

b) We use mathematical induction again. b) is clearly true when $n = 2$. Suppose it is true when $n = m - 1$. In the case when $n = m$, let $\hat{H} = (\hat{H}_1, \hat{H}_2, \cdots, \hat{H}_{m-1})$ be the $(m-1) \times (m-1)$ matrix obtained by deleting the first row and first column of H. Here, $\hat{H}_j$ is the jth column of $\hat{H}$. Let $\hat{b} = (1, 1, \cdots, 1)' \in \mathbb{R}^{m-1}$. Consider that

$$\begin{aligned}
&\det(b, H_2, \cdots, H_m) \\
&= \det\begin{pmatrix} 1 & h_{12} & \cdots & h_{1m} \\ 1 & -h_2 & \cdots & h_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & h_{m2} & \cdots & -h_m \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & h_{12} & \cdots & h_{1m} \\ 0 & -h_2 - h_{12} & \cdots & h_{2m} - h_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_{m2} - h_{12} & \cdots & -h_m - h_{1m} \end{pmatrix} \\
&= \det(\hat{H}) - h_{12}\det(\hat{b}, \hat{H}_2, \cdots, \hat{H}_{m-1}) - \cdots \\
&\quad - h_{1m}\det(\hat{H}_1, \cdots, \hat{H}_{m-2}, \hat{b}). \qquad \text{(A.12)}
\end{aligned}$$

Note that for any j, $(\hat{H}_1, \cdots, \hat{H}_{j-1}, \hat{b}, \hat{H}_{j+1}, \cdots, \hat{H}_{m-1})$ is in a form such that b) is true when applying the induction hypothesis, i.e.,

$$\begin{aligned}
&\det(\hat{H}_1, \cdots, \hat{H}_{j-1}, \hat{b}, \hat{H}_{j+1}, \cdots, \hat{H}_{m-1}) \\
&\quad = (-1)^{m-2}\gamma_j', \quad \gamma_j' > 0, \\
&\quad \forall j = 1, 2, \cdots, m-1. \qquad \text{(A.13)}
\end{aligned}$$

where $\gamma_1 = \eta + h_{12}\gamma_1' + \cdots + h_{1m}\gamma_{m-1}' > 0$. Note that for each j, there exists a nonsingular matrix $T \in \mathbb{R}^{m \times m}$ such that

$$\begin{aligned}
&T^{-1}(H_1, \cdots, H_{j-1}, b, H_{j+1}, \cdots, H_m)T \\
&\quad = (b, H_2', \cdots, H_m')
\end{aligned}$$

where $(b, H_2', \cdots, H_m')$ is in the same form as $(b, H_2, \cdots, H_m)$ and satisfies the condition of the lemma. Therefore, we have b). □

Proof of Lemma 3.10: Since each pair of states of $r(t)$ communicates, by Lemma 3.9, it is easy to see that $\alpha_i = \det(Q_i) \neq 0$. The unique stationary distribution of $r(t)$ must satisfy the equation $\pi Q = 0$. Equivalently

$$Q'\pi' = \begin{pmatrix} -q_1 & q_{21} & \cdots & q_{s1} \\ q_{12} & -q_2 & \cdots & q_{s2} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1s} & q_{2s} & \cdots & -q_s \end{pmatrix}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_s \end{pmatrix} = 0.$$

This implies that for any $i \in S$

$$Q_i'\begin{pmatrix} \pi_1 \\ \vdots \\ \pi_{i-1} \\ \pi_{i+1} \\ \vdots \\ \pi_s \end{pmatrix} = -\pi_i\begin{pmatrix} q_{i1} \\ \vdots \\ q_{i(i-1)} \\ q_{i(1+1)} \\ \vdots \\ q_{is} \end{pmatrix}.$$

Solving this system of linear equations, we have for any $i, j \in S$ with $i \neq j$

$$\pi_j = \frac{\pi_i}{\alpha_i} \times \det\begin{bmatrix} -q_1 & \cdots & q_{(j-1)1} & -q_{i1} & q_{(j+1)1} & \cdots & q_{(i-1)1} & q_{(i+1)1} & \cdots & q_{s1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ q_{1(i-1)} & \cdots & q_{(j-1)(i-1)} & -q_{i(i-1)} & q_{(j+1)(i-1)} & \cdots & -q_{i-1} & q_{(i+1)(i-1)} & \cdots & q_{s(i-1)} \\ q_{1(i+1)} & \cdots & q_{(j-1)(i+1)} & -q_{i(i+1)} & q_{(j+1)(i+1)} & \cdots & q_{(i-1)(i+1)} & -q_{i+1} & \cdots & q_{s(i+1)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ q_{1s} & \cdots & q_{(j-1)s} & -q_{is} & q_{(j+1)s} & \cdots & q_{(i-1)s} & q_{(i+1)s} & \cdots & -q_s \end{bmatrix}.$$

Thus, by a), (A.12), and (A.14), we obtain

$$\begin{aligned}
\det(b, H_2, \cdots, H_m) &= (-1)^{m-1}(\eta + h_{12}\gamma_1' + \cdots \\
&\quad + h_{1m}\gamma_{m-1}') \overset{\text{def}}{=} (-1)^{m-1}\gamma_1
\end{aligned}$$

We show that the determinant on the right-hand side is α_j. To do this, we show that the matrix inside the brackets can be transformed to Q_j' by elementary transformations which preserve the determinant.

First, add all rows to the jth row and then, multiply the jth row and the jth column of the resulting matrix by -1, we end up with

$$\pi_j = \frac{\pi_i}{\alpha_i} \times \det\begin{bmatrix} -q_1 & \cdots & q_{(j-1)1} & q_{i1} & q_{(j+1)1} & \cdots & q_{(i-1)1} & q_{(i+1)1} & \cdots & q_{s1} \\ q_{12} & \cdots & q_{(j-1)2} & q_{i2} & q_{(j+1)2} & \cdots & q_{(i-1)2} & q_{(i+1)2} & \cdots & q_{s2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ q_{1i} & \cdots & q_{(j-1)i} & -q_i & q_{(j+1)i} & \cdots & q_{(i-1)i} & q_{(i+1)i} & \cdots & q_{si} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ q_{1(i-1)} & \cdots & q_{(j-1)(i-1)} & q_{i(i-1)} & q_{(j+1)(i-1)} & \cdots & -q_{i-1} & q_{(i+1)(i-1)} & \cdots & q_{s(i-1)} \\ q_{1(i+1)} & \cdots & q_{(j-1)(i+1)} & q_{i(i+1)} & q_{(j+1)(i+1)} & \cdots & q_{(i-1)(i+1)} & -q_{i+1} & \cdots & q_{s(i+1)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ q_{1s} & \cdots & q_{(j-1)s} & q_{is} & q_{(j+1)s} & \cdots & q_{(i-1)s} & q_{i+1)s} & \cdots & -q_s \end{bmatrix}.$$

Now, it is easy to see that the matrix inside the determinant can be transformed to Q_j' by a sequence of simultaneous row and column exchanges. This completes the proof of Lemma 3.10. □

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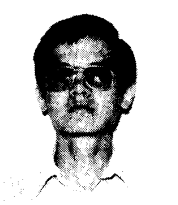

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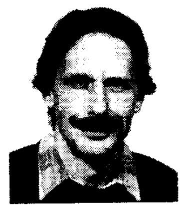

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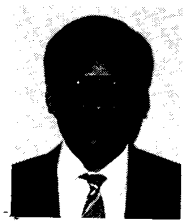

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