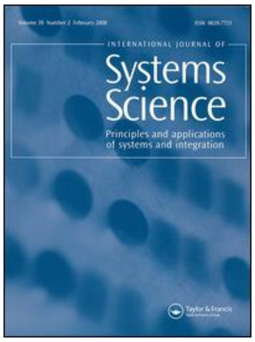

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Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering for Markov jump linear systems

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ABSTRACT
We study in this work the mixed $\mathcal{H}_2/\mathcal{H}_\infty$-filtering problem for Markov jump linear systems in a partial observation context. Rather than the Markov chain $\theta(k)$, we consider that only an estimation $\hat{\theta}(k)$ coming from a detector is available to the filter. We present a sufficient condition for the synthesis of a filter depending only on $\hat{\theta}(k)$ such that the estimation errors for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problems are bounded by given scalars. The robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering considering that the transition and detection probabilities are uncertain, as well as the complete observation, cluster and mode-independent cases, are also studied. The results are given in terms of linear matrix inequalities and are illustrated by a numerical example.

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Optimal filtering; switched systems; hidden Markov models; linear matrix inequalities

1. Introduction

Dynamic systems that present abrupt changes in their behaviour have been a topic of great interest in the literature. Particularly, systems subject to failures originate great concern, specially in critical applications such as nuclear power plants and aircraft operation. More recently, the use of communication networks in the control loop has also led to the investigation of such changes due to the imperfect characteristics of the communication channel. In this context, the research on Active Fault-Tolerant Control Systems (AFTCS) and Networked Control Systems (NCS) is focused on sensing the failure and acting on the faulty system in order to achieve a minimum degree of performance and safety, see, for instance, Mahmoud, Jiang, and Zhang (2003), Hespanha, Naghshtabrizi, and Xu (2007), and references therein. Due to the abrupt dynamic changes caused by faults, switched systems are appealing for modelling such phenomena, and whenever those changes are stochastic, the use of Markov jump linear systems (MJLS) becomes a powerful tool. The theory of MJLS is already consolidated in the literature, as can be seen, for instance, in Costa, Fragoso, and Marques (2005), Boukas (2006), Dragan, Morozan, and Stoica (2013), and more recent results in Markov jump systems can be found, for instance, in Shen, Park, and Ye (2016) and Song, He, Liu, Niu, and Ding (2016).

Regarding the filtering problem in MJLS, Fioravanti, Gonçalves, and Geromel (2008) and Gonçalves, Fioravanti, and Geromel (2009) tackled the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problem by means of linear matrix inequalities (LMI) mainly considering that the Markov mode is available to the filter, although some discussion of filters that do not depend on the mode is also given. Costa and Tuesta (2004) studied the so-called separation principle for the $\mathcal{H}_2$ control and filtering, where the optimal $\mathcal{H}_2$ filter is obtained through coupled Riccati equations (CARE), and also Geromel, Gonçalves, and Fioravanti (2009) studied the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ dynamic output feedback problem in the LMI formulation. All the main results of the previous papers assume that the Markov state $\theta(k)$ is available to the filter, an assumption that cannot always be fulfilled, and so there are some alternatives that deal with the situation of *partial observation* of $\theta(k)$. The *cluster* case, introduced in DoVal, Geromel, and Gonçalves (2002), considers that the Markov modes can be grouped in disjoint sets called *clusters* so that the controller would know the cluster in which the Markov chain is currently operating. This result was extended to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problem in Gonçalves, Fioravanti, and Geromel (2010) and Gonçalves, Geromel, and Fioravanti (2011), where the latter work dealt also with uncertain transition probabilities. For the case where there is no information at all regarding the underlying Markov chain, the so-called mode-independent case, we can cite the aforementioned Fioravanti et al. (2008), Gonçalves et al. (2009, 2010, 2011) for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering

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problems; Fioravanti, Gonçalves, and Geromel (2015) for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cluster and mode-independent Bernoulli filters; Costa and Guerra (2002) considered the robust and mode-independent quadratic filtering in the LMI formulation, the $\mathcal{H}_\infty$ mode-independent filter of de Souza, Trofino, and Barbosa (2006) and Morais, Palma, Peres, and Oliveira (2018) studied the robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problem in an unified approach for uncertain transition probabilities.

More recently, it was studied in Costa, Fragoso, and Todorov (2015) the so-called *detector approach*, or *Hidden Markov model* approach, that possesses a strong *liaison* with the AFTCS theory. In this particular model, it is assumed that the Markov chain $\theta(k)$ is hidden and there is only an estimation represented by $\hat{\theta}(k)$, given by some failure detector. Particularly, the relation between $(\theta(k), \hat{\theta}(k))$ is inspired on the hidden Markov chain theory, see for instance, Ross (2010), and, as shown in Costa et al. (2015), this approach encompasses all the previous situations regarding the availability of the Markov chain: the complete, cluster and mode-independent cases. There is by now a considerable body of works that use this formulation, such as the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ state-feedback control problems studied in Costa et al. (2015) and Todorov, Fragoso, and Costa (2018), and also the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback control problem in de Oliveira and Costa (2018). The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering were tackled in de Oliveira and Costa (2017a) and de Oliveira and Costa (2017b), respectively. Moreover, the passivity control problem was analysed in Wu, Shi, Shu, Su, and Lu (2017), and the asynchronous $l_2 - l_\infty$ filtering was studied in Wu, Shi, Su, and Chu (2014) considering an independent Markov chain for the filter, rather than the detector approach. For the continuous-time MJLS, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ state-feedback control were treated in Stadtmann and Costa (2017) and Rodrigues, Todorov, and Fragoso (2015), and the $\mathcal{H}_\infty$ filtering problem in Rodrigues, Todorov, and Fragoso (2016).

In terms of the performance of the filter, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theory is appealing since that, for the first problem, a quadratic functional of the estimation error is minimised if a white noise sequence is applied in the system, whereas in the second problem, the effects of norm-limited exogenous inputs could be mitigated by considering the minimisation of the input-to-output gain for the worst-case disturbance. Furthermore, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering could potentially combine the best of both worlds in engineering applications such as in aerospace application, see for instance, Rotstein, Sznaier, and Idan (1996). In this work, the states of a jet-plane are estimated through a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter considering that the measurements are subject to white noise (the $\mathcal{H}_2$ problem), and the aircraft structure is affected by downbursts (the $\mathcal{H}_\infty$ problem), a threatening type of windshear. We can mention a few works that laid the basis and developed this theory for the case without jumps such as Khargonekar, Rotea, and Baeyens (1996), Palhares and Peres (2001), Yang and Hung (2002), and Gao, Lam, Xie, and Wang (2005). However, when it comes to Markov jump systems (MJS), the focus on multiple performance criteria seems to have changed to the mixed $\mathcal{H}_\infty$ and passivity performance level problem, e.g. in Mathiyalagan, Park, and Sakthivel (2014), Shen, Wu, and Park (2015), Shen, Su, and Park (2016), where Shen et al. (2015) consider semi-Markov jump systems and Shen et al. (2016), singular Markov jump systems and the synthesis of both mode-dependent and mode-independent filters; and in a different sense, the $l_2 - l_\infty$ performance index (l_2-to-peak gain) of the aforementioned Wu et al. (2014). Even though there are some studies concerning the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem for MJLS, see for instance Sheng, Zhang, and Gao (2014), Costa et al. (2005) and de Oliveira and Costa (2018), the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem, as formulated for the case without jumps, has received less attention in the MJS setting, to the knowledge of the authors.

Bearing in mind the previous discussion, this work aims to study the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem for MJLS under the detector framework. Our formulation is flexible enough to cope with separate exogenous inputs and estimated outputs related to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problems similarly to Yang and Hung (2002). We present a sufficient condition for the synthesis of filters depending only on $\hat{\theta}(k)$ that ensures that the estimation errors with respect to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problems are bounded by given $\gamma > 0$ and $\delta > 0$. Moreover, we study the synthesis of robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters in the sense that the transition and detection probabilities are uncertain. As a by-product of the detector approach, we are also able to state synthesis conditions for the perfect observation, cluster and mode-independent cases. Finally we illustrate our results by means of a numerical example in the context of NCS, where the data is transmitted through a noisy and faulty channel.

This work is organised as follows. Section 2 introduces our notation, Section 3 starts the preliminary studies by presenting the system, the detector formulation, the filter structure, the definitions of stochastic stability, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, and the general problem formulation. We present our main result in Section 4 that consists of LMI conditions for the design of filters depending only on $\hat{\theta}(k)$ such that the $\mathcal{H}_2$ norm with respect to its related estimated output is bounded, and also that the $\mathcal{H}_\infty$ norm for its respective estimated output is also bounded. Moreover, we tackle the robust situation in Section 5 and the cluster and complete observation cases in Section 6. The numerical example that illustrates our results is presented in Section 7. Our final remarks are stated in Section 8 and the proofs of Theorem 4.1, Theorem 5.1 and Proposition 6.1 are presented in the appendix.

2. Notation

The notation used throughout is standard. The real n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The linear space of all $m \times n$ linear operators is represented by $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ and, for simplicity, we set $\mathbb{B}(\mathbb{R}^n) = \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$. The superscript $'$ indicates the transpose of a matrix. The identity operator of size n is represented by I_n, the null operator, by $0_{n\times m}$, the trace operator by $\mathrm{Tr}(\cdot)$, the block diagonal matrix is denoted by $\mathbf{diag}(\cdot)$, and $\mathrm{Her}(G) := G + G'$ for $G \in \mathbb{B}(\mathbb{R}^n)$. For a partitioned symmetric matrix, the symbol $(\bullet)$ represents symmetric blocks. For N and M positive integers, the sets $\mathbb{N}$ and $\mathbb{M}$ are defined by $\mathbb{N} := \{1, 2, 3, \dots, N\}$ and $\mathbb{M} := \{1, 2, 3, \dots, M\}$, respectively. Furthermore, the set $\mathbb{H}^{n,m}$ represents the linear space of all N-sequence of real matrices $V = (V_1, V_2, \dots, V_N)$, $V_i \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $i \in \mathbb{N}$ and we adopt, for simplicity, $\mathbb{H}^n := \mathbb{H}^{n,n}$ and $\mathbb{H}^{n+} := \{V \in \mathbb{H}^n; V_i \geq 0, i = 1, \dots, N\}$. For $P, V \in \mathbb{H}^{n+}$, we write that $P \geq V$ ($P > V$) if $P_i \geq V_i$ ($P_i > V_i$) for each $i = 1, \dots, N$. For given matrices V_s, $s \in \{1, \dots, \sigma\}$, we also define the set

$$\mathbb{D}(V_1, \dots, V_\sigma) := \left\{ V;\ V = \sum_{s=1}^{\sigma} \eta_s V_s = \sum_{s=1}^{\sigma} \eta_s \left[p_{ij}^{(s)}\right], \right. \left. \eta_s \geq 0, \sum_{s=1}^{\sigma} \eta_s = 1 \right\}. \tag{1}$$

On the probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_k\}$, $\mathbf{E}(\cdot)$ represents the expected value operator. The notation $l_2^r(\Omega, \mathcal{F}, \{\mathcal{F}_k\}, P)$ represents the space of all discrete-time signals $\mathcal{F}_k$-adapted processes such that

$$\|z\|_2 := \sqrt{\sum_{k=0}^{\infty} \mathbf{E}(\|z(k)\|^2)} < \infty.$$

3. Preliminaries

On a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_k\}$, we consider the following general Markov jump linear system (MJLS)

$$\mathcal{G} : \begin{cases} x(k+1) &= A_{\theta(k)} x(k) + J_{\theta(k)}^{(2)} w(k) + J_{\theta(k)}^{(\infty)} v(k), \\ y(k) &= L_{\theta(k)} x(k) + H_{\theta(k)}^{(2)} w(k) + H_{\theta(k)}^{(\infty)} v(k), \\ z_2(k) &= C_{\theta(k)}^{(2)} x(k) + E_{\theta(k)}^{(2)} w(k), \\ z_\infty(k) &= C_{\theta(k)}^{(\infty)} x(k) + E_{\theta(k)}^{(\infty)} v(k), \\ x(0) &= 0, \theta(0) = \theta_0, \end{cases} \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state variable and $y(k) \in \mathbb{R}^p$ is the measured output. Besides $w(k) \in \mathbb{R}^{r_2}$ and $v(k) \in \mathbb{R}^{r_\infty}$ are exogenous input signals, and $z_2(k) \in \mathbb{R}^{q_2}$ and $z_\infty(k) \in \mathbb{R}^{q_\infty}$ are output signals to be estimated related to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance criteria, respectively. The variable $\theta(k)$ is a discrete-time homogeneous Markov chain taking its values in the set $\mathbb{N}$ with transition probability matrix given by $\mathbb{P} := [p_{ij}]$ and initial probability distribution $\mu_i := P(\theta_0 = i)$, $i \in \mathbb{N}$.

In this work, we consider the so-called *detector*, or *hidden Markov model* (HMM), approach in which we assume that we do not have access to the Markov chain$\theta(k)$, but only to an observable part denoted by $\hat{\theta}(k)$. In this context, we introduce the set of all possible outcomes of $\hat{\theta}(k)$ by $\mathbb{M}$ and, given $\hat{\mathcal{F}}_k = \sigma\{x(0), w(0), v(0), \theta(0), \hat{\theta}(0), \dots, x(k), w(k), v(k), \theta(k)\}$ and $\hat{\mathcal{F}}_0 = \sigma\{x(0), w(0), v(0), \theta(0)\}$, we assume that

$$P(\hat{\theta}(k) = l \mid \hat{\mathcal{F}}_k) = P(\hat{\theta}(k) = l \mid \theta(k) = i) = \alpha_{\theta(k)l}.$$

The set of possible outcomes of the detector for a given Markov mode $\theta(k)$ is denoted by $\mathbb{M}_{\theta(k)}$, and so it is clear that $\bigcup_{i\in\mathbb{N}} \mathbb{M}_i = \mathbb{M}$. We also define the detection matrix $\Upsilon := [\alpha_{il}]$.

Remark 3.1: The joint process $(\theta(k), \hat{\theta}(k))$ is called a hidden Markov chain (HMC), whose theory is well established in the literature (Ross, 2010). Particularly this partial observation setting applied to Markov jump systems was studied in detail, for instance, in Costa et al. (2015), where it is shown that the complete observation, the cluster and mode-independent cases are generalised by the detector approach.

The goal is to synthesise filters that depend only on the observable part $\hat{\theta}(k)$ as in de Oliveira and Costa (2017b) and de Oliveira and Costa (2017a), but with the following structure,

$$\mathcal{F} : \begin{cases} x_f(k+1) &= A_{f\hat{\theta}(k)} x_f(k) + B_{f\hat{\theta}(k)} y(k), \\ z_{f2}(k) &= C_{f\hat{\theta}(k)}^{(2)} x_f(k), \\ z_{f\infty}(k) &= C_{f\hat{\theta}(k)}^{(\infty)} x_f(k), \\ x_f(0) &= 0, \hat{\theta}(0) = \hat{\theta}_0, \end{cases} \tag{3}$$

that is, a strictly proper filter (the filter's output does not depend on $y(k)$), where $x_f(k) \in \mathbb{R}^n$, $z_{f2}(k) \in \mathbb{R}^{q_2}$ and

$z_{f\infty}(k) \in \mathbb{R}^{q_\infty}$. We define the error signals as follows:

$$e_2(k) = z_2(k) - z_{f2}(k),$$
$$e_\infty(k) = z_\infty(k) - z_{f\infty}(k).$$

Connecting (3) to the output of (2) yields the extended system,

$$\mathcal{G}_F : \begin{cases} \tilde{x}(k+1) & = A_{\theta(k)\hat{\theta}(k)}\tilde{x}(k) + J^{(2)}_{\theta(k)\hat{\theta}(k)}w(k) \\ & \quad + J^{(\infty)}_{\theta(k)\hat{\theta}(k)}v(k), \\ e_2(k) & = C^{(2)}_{\theta(k)\hat{\theta}(k)}\tilde{x}(k) + E^{(2)}_{\theta(k)}w(k), \\ e_\infty(k) & = C^{(\infty)}_{\theta(k)\hat{\theta}(k)}\tilde{x}(k) + E^{(\infty)}_{\theta(k)}v(k), \end{cases} \tag{4}$$

for $\tilde{x}(k) \in \mathbb{R}^{2n}$ defined as $\tilde{x}(k)' = \begin{bmatrix} x(k)' & x_f(k)' \end{bmatrix}$, and $A_{il}, J^{(2)}_{il}, J^{(\infty)}_{il}, C^{(2)}_{il}$, and $C^{(\infty)}_{il}$ are given, for all $i \in \mathbb{N}$, $l \in \mathbb{M}_i$, by

$$\left[\begin{array}{c|c|c} A_{il} & J^{(2)}_{il} & J^{(\infty)}_{il} \\ \hline C^{(2)}_{il} & E^{(2)}_{i} & 0 \\ \hline C^{(\infty)}_{il} & 0 & E^{(\infty)}_{i} \end{array}\right] = \left[\begin{array}{cc|c|c} A_i & 0 & J^{(2)}_i & J^{(\infty)}_i \\ B_{fl}L_i & A_{fl} & B_{fl}H^{(2)}_i & B_{fl}H^{(\infty)}_i \\ \hline C^{(2)}_i & -C^{(2)}_{fl} & E^{(2)}_i & 0 \\ \hline C^{(\infty)}_i & -C^{(\infty)}_{fl} & 0 & E^{(\infty)}_i \end{array}\right]. \tag{5}$$

The extended structure composed by the filter matrices is defined by

$$F_l = \left[\begin{array}{c|c} A_{fl} & B_{fl} \\ \hline C^{(2)}_{fl} & 0 \\ \hline C^{(\infty)}_{fl} & 0 \end{array}\right], F_l \in \mathbb{B}\left(\mathbb{R}^{n+p}, \mathbb{R}^{n+q_2+q_\infty}\right), \quad \text{for all } l \in \mathbb{M}. \tag{6}$$

Remark 3.2: In this paper, we use the same filter structure as in Gao et al. (2005), Palhares and Peres (2001), and more specifically Yang and Hung (2002), for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem. The idea of using strictly proper filters is to avoid the presence of crossed terms between w and v in the outputs of z_2 and z_∞.

We introduce next the definition of stochastic stability used throughout our work.

Definition 3.1: System (4) is said to be stochastically stable for $w = 0$ and $v = 0$ if

$$\|\tilde{x}\|_2^2 = \sum_{k=0}^{\infty} \mathbf{E}(\|\tilde{x}(k)\|^2) < \infty, \tag{7}$$

for every θ_0 and every finite second moment $\tilde{x}_0 = \begin{bmatrix} x(0)' & x_f(0)' \end{bmatrix}'$.

Given the structure defined in (6), we define the set $\mathbb{F}$ of admissible filter structures $F = (F_1, \ldots, F_M)$:

$$\mathbb{F} = \left\{F : \text{such that (4) is stochastically stable}\right\}.$$

We need the following assumption.

Assumption 3.1: System (2) is stochastically stable.

Remark 3.3: For the complete observation case ($\hat{\theta}(k) = \theta(k)$), we have that Assumption 3.1 would not be necessary since the filter could be written in a Markov observer form and, in this case, the dynamic equation for the error could be stochastically stable even if the original system is not (see, for instance, Costa et al., 2005). However for the general case represented by (4), we adopt this similar approach used in Gonçalves et al. (2010), de Oliveira and Costa (2017a), de Oliveira and Costa (2017b) that requires Assumption 3.1.

Considering the following operators for $V \in \mathbb{H}^{2n}$,

$$\mathcal{E}_i(V) = \sum_{j \in \mathbb{N}} p_{ij}V_j,$$
$$\mathcal{L}_i(V) = \sum_{l \in \mathbb{M}_i} \alpha_{il}A'_{il}\mathcal{E}_i(V)A_{il},$$

for all $i, j \in \mathbb{N}$, and $\mathcal{E}, \mathcal{L} \in \mathbb{B}(\mathbb{H}^{2n})$, where A_{il} is given in (5) (and so it depends implicitly on the filter matrices), we have the following theorem borrowed from Costa et al. (2015).

Theorem 3.1 (Costa et al., 2015): *The following assertions are equivalent*:

(1) *System (4) is stochastically stable.*
(2) *There exists $P \in \mathbb{H}^{2n}$, $P > 0$, such that $P - \mathcal{L}(P) > 0$.*

We are now able to characterise the performance indexes used in this work.

Definition 3.2 (The $\mathcal{H}_2$ norm for MJLS): Consider that $F \in \mathbb{F}$, $\tilde{x}_0 = 0$ and $v = 0$. Let e_{2s} be the controlled output of (4) for the exogenous input $w(k)$ given by

$$w(k) = \begin{cases} w_s, & k = 0, \\ 0, & k > 0, \end{cases} \tag{8}$$

where w_s is the sth standard basis of $\mathbb{R}^{r_2}$. Then, the $\mathcal{H}_2$ norm of (4) is defined as follows:

$$\|\mathcal{G}_F\|_2^2 := \sum_{s=1}^{r_2} \|e_{2s}\|_2^2, \quad \|e_{2s}\|_2^2 := \sum_{k=0}^{\infty} \mathbf{E}\left(\|e_2(k)\|^2\right).$$

It is clear that the definition of the $\mathcal{H}_2$ norm for MJLS given in Definition 3.2 reduces to the usual $\mathcal{H}_2$ norm for the case without jumps, that is, $N = 1$. Furthermore, recalling that $\mu_i = P(\theta_0 = i)$, we introduce the following inequalities for a given filter structure,

$$\sum_{i\in\mathbb{N}}\sum_{l\in\mathbb{M}_i}\mu_i\alpha_{il}\mathrm{Tr}\,(W_{il}) < \gamma^2,$$
$$\begin{bmatrix} W_{il} & \bullet & \bullet \\ J_{il}^{(2)} & \mathcal{E}_i(P)^{-1} & \bullet \\ E_i^{(2)} & 0 & I \end{bmatrix} > 0, \tag{9}$$

$$P_i > \sum_{l\in\mathbb{M}_i}\alpha_{il}M_{il}^{(2)}, \quad \begin{bmatrix} M_{il}^{(2)} & \bullet & \bullet \\ A_{il} & \mathcal{E}_i(P)^{-1} & \bullet \\ C_{il}^{(2)} & 0 & I \end{bmatrix} > 0, \tag{10}$$

for all $l \in \mathbb{M}_i$, $i \in \mathbb{N}$.

Proposition 3.1 (Costa et al., 2015; de Oliveira & Costa, 2017a): *For a given filter structure F, if the inequality set (9)–(10) holds for $W_{il} > 0$, and $P \in \mathbb{H}^{2n}$,$P > 0$, then $F \in \mathbb{F}$ and $\|\mathcal{G}_F\|_2 < \gamma$.*

Remark 3.4: An alternative definition that can be found, for instance, in de Oliveira and Costa (2017a) can be stated for the $\mathcal{H}_2$ norm of (4) as follows: consider that $v(k) \equiv 0$ for all $k \in \{0, 1, 2, \ldots\}$, and $w(k)$ is a white noise sequence with covariance matrix given by the identity matrix, and independent of the joint HMC process $(\theta(k), \hat{\theta}(k))$ (see Remark 3.1) and that the Markov chain is ergodic, for more information, see Ross (2010). If $\nu = \mu$, where $\nu_i = \lim_{k\to\infty} P(\theta(k) = i)$, we have that

$$\|\mathcal{G}_F\|_2^2 = \lim_{k\to\infty}\mathbf{E}\,(\|e_2(k)\|^2).$$

We now change our focus to the second performance index considered in this work, described in the next definition.

Definition 3.3 (The $\mathcal{H}_\infty$ norm for MJLS): Consider that $F \in \mathbb{F}$, $\tilde{x}_0 = 0$ and $w = 0$. The $\mathcal{H}_\infty$ norm of (4) is given by

$$\|\mathcal{G}_F\|_\infty := \sup_{\theta_0, v\in\mathcal{V}}\frac{\|e_\infty\|_2}{\|v\|_2}, \tag{11}$$

where $\mathcal{V} = \{v \in l_2^{r_\infty}(\Omega, \mathcal{F}, \{\mathcal{F}_k\}, P)\ ;\ \|v\|_2 > 0\}$.

The bounded-real lemma for MJLS in the detector approach derived in Todorov et al. (2018) is presented in the next theorem.

Theorem 3.2 (Todorov et al., 2018): *System (4) is stochastically stable with $\|\mathcal{G}_F\|_\infty < \delta$ for a given $\delta > 0$ if the following inequality set holds for all $i \in \mathbb{N}$, $l \in \mathbb{M}_i$:*

$$\begin{bmatrix} Q_i & \bullet \\ 0 & \delta^2 I \end{bmatrix} > \sum_{l\in\mathbb{M}_i}\alpha_{il}\begin{bmatrix} M_{il} & \bullet \\ N_{il} & S_{il} \end{bmatrix}, \tag{12}$$

$$\begin{bmatrix} M_{il} & \bullet & \bullet & \bullet \\ N_{il} & S_{il} & \bullet & \bullet \\ A_{il} & J_{il}^{(\infty)} & \mathcal{E}(Q)^{-1} & \bullet \\ C_{il}^{(\infty)} & E_i^{(\infty)} & 0 & I \end{bmatrix} > 0, \tag{13}$$

for $Q \in \mathbb{H}^{2n}$, $Q > 0$.

Given the previous definitions and auxiliary results, we are now able to formally state the goal of this work, that is, to study the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering by minimising an auxiliary cost related to the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms of (4). For instance, one of the problems that will be considered is to design filters depending only on $\hat{\theta}(k)$ by minimising the upper bound γ of the $\mathcal{H}_2$ norm of (4), while guaranteeing that its $\mathcal{H}_\infty$ norm is bounded by a given δ. Thus we intend to synthesise admissible filters via the following optimisation problem:

$$\inf_{F\in\mathbb{F}}\{\gamma,\ \text{such that } \|G_F\|_2 < \gamma \quad \text{and} \quad \|G_F\|_\infty < \delta\}, \tag{14}$$

where $\|G_F\|_2$ is related to the error signal e_2 (see Definition 3.2) and $\|G_F\|_\infty$, to e_∞ (see Definition 3.3). A first approach for solving (14) is to minimise the objective function subject to (9)–(10) and (12)–(13) with system matrices given in (5),

$$\inf_{F\in\mathbb{F},P>0,W>0,Q>0}\{\gamma,\ \text{such that (9)} - \text{(10) and (12)} - \text{(13) holds}\}. \tag{15}$$

We point out that the feasibility set of (15) is not convex, and hence it is hard to solve. However, considering the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering design given in de Oliveira and Costa (2017a) and de Oliveira and Costa (2017b), whose transformations were laid by Gonçalves et al. (2010), it is possible to formulate sufficient conditions for problem (15) yielding an approximate problem, as we will see in the following sections.

Alternatively, the problems of finding an admissible filter that (1) minimises the upper bound δ of the $\mathcal{H}_\infty$ while guaranteeing that $\|\mathcal{G}_F\|_2 < \gamma$ (the so-called 'inverse' mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering); (2) minimises the linear combination $a_2\gamma^2 + a_\infty\delta^2$, for $a_2 \geq 0$, $a_\infty \geq 0$; are also of interest. As we are going to see in the following sections, our result provides an approximation to (14), as well as to the situations (1) and (2) just described.

4. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem

For the $\mathcal{H}_2$ filtering part of the problem, we introduce the following LMI for a given $\gamma > 0$

$$\sum_{i\in\mathbb{N}}\sum_{l\in\mathbb{M}_i}\mu_i\alpha_{il}\mathrm{Tr}\left(W_{il}\right) < \gamma^2, \tag{16}$$

$$\begin{bmatrix} W_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Y)J_i^{(2)} & \mathcal{E}_i(Y) & \bullet & \bullet \\ G_lJ_i^{(2)}+F_lH_i^{(2)} & 0 & \mathrm{Her}(G_l)+\mathcal{E}_i(Y)-\mathcal{E}_i(X) & \bullet \\ E_i^{(2)} & 0 & 0 & I \end{bmatrix} > 0, \tag{17}$$

$$\begin{bmatrix} Y_i & \bullet \\ Y_i & X_i \end{bmatrix} > \sum_{l\in\mathbb{M}_i}\alpha_{il}\begin{bmatrix} M_{il}^{(2,11)} & \bullet \\ M_{il}^{(2,21)} & M_{il}^{(2,22)} \end{bmatrix}, \tag{18}$$

$$\begin{bmatrix} M_{il}^{(2,11)} & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(2,21)} & M_{il}^{(2,22)} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Y)A_i & \mathcal{E}_i(Y)A_i & \mathcal{E}_i(Y) & \bullet & \bullet \\ G_lA_i+F_lL_i+R_l & G_lA_i+F_lL_i & 0 & \mathrm{Her}(G_l)+\mathcal{E}_i(Y)-\mathcal{E}_i(X) & \bullet \\ C_i^{(2)}+O_l^{(2)} & C_i^{(2)} & 0 & 0 & I \end{bmatrix} > 0, \tag{19}$$

for all $i \in \mathbb{N}, l \in \mathbb{M}_i$. On the other hand, the following LMI set for a given $\delta > 0$ refers to the $\mathcal{H}_\infty$ filtering part of the problem,

$$\begin{bmatrix} T_i & \bullet & \bullet \\ T_i & Z_i & \bullet \\ 0 & 0 & \delta^2 I \end{bmatrix} > \sum_{l\in\mathbb{M}_i}\alpha_{il}\begin{bmatrix} M_{il}^{(\infty,11)} & \bullet & \bullet \\ M_{il}^{(\infty,21)} & M_{il}^{(\infty,22)} & \bullet \\ N_{il}^{(\infty,11)} & N_{il}^{(\infty,12)} & S_{il} \end{bmatrix}, \tag{20}$$

$$\begin{bmatrix} M_{il}^{(\infty,11)} & \bullet & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(\infty,21)} & M_{il}^{(\infty,22)} & \bullet & \bullet & \bullet & \bullet \\ N_{il}^{(\infty,11)} & N_{il}^{(\infty,12)} & S_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(T)A_i & \mathcal{E}_i(T)A_i & \mathcal{E}_i(T)J_i^{(\infty)} & \mathcal{E}_i(T) & \bullet & \bullet \\ G_lA_i+F_lL_i+R_l & G_lA_i+F_lL_i & G_lJ_i^{(\infty)}+F_lH_i^{(\infty)} & 0 & \Xi_{il} & \bullet \\ C_i^{(\infty)}+O_l^{(\infty)} & C_i^{(\infty)} & E_i^{(\infty)} & 0 & 0 & I \end{bmatrix} > 0, \tag{21}$$

for all $i \in \mathbb{N}, l \in \mathbb{M}_i$, where $\Xi_{il} = \mathrm{Her}(G_l) + \mathcal{E}_i(T) - \mathcal{E}_i(Z)$. We define the set of variables of (16)–(19) as

$$\phi_2 = \left\{W_{il}, X_i, Y_i, G_l, F_l, R_l, O_l^{(2)}, M_{il}^{(2,11)}, M_{il}^{(2,21)}, M_{il}^{(2,22)}, i \in \mathbb{N}, l \in \mathbb{M}_i\right\} \cup \psi_2,$$

where $\psi_2 = \emptyset$ if $\gamma > 0$ is given, and $\psi_2 = \{\gamma_a\}, \gamma_a = \gamma^2$, if the upper bound γ is considered as a variable of the problem. Similarly, for (20)–(21), we define the set ϕ_∞ as follows:

$$\phi_\infty = \left\{T_i, Z_i, G_l, F_l, R_l, O_l^{(\infty)}, M_{il}^{(\infty,11)}, M_{il}^{(\infty,21)}, M_{il}^{(\infty,22)}, N_{il}^{(\infty,11)}, N_{il}^{(\infty,12)}, S_{il}, i \in \mathbb{N}, l \in \mathbb{M}_i\right\} \cup \psi_\infty,$$

where $\psi_\infty = \emptyset$ if $\delta > 0$ is given, and $\psi_\infty = \{\delta_a\}, \delta_a = \delta^2$, if δ is a variable. The set of variables of the joint $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem can be described by

$$\phi = \phi_2 \cup \phi_\infty,$$

and the set of all possible joint solutions of (16)–(19) and (20)–(21) is represented by

$$\Phi = \{\phi \text{ such that}(16) - (19) \text{ and } (20) - (21)\text{jointly hold}\}.$$

We point out that $\phi_2 \cap \phi_\infty = \{G_l, F_l, R_l, l \in \mathbb{M}\}$, that is, there is a coupling between the LMI sets (16)–(19) and (20)–(21). As we are going to see in Theorem 4.1, the variables G_l, F_l, R_l are associated with the filter matrices A_{fl} and B_{fl} that govern the evolution of the filter states for both outputs $e_2(k)$ and $e_\infty(k)$ in (3).

We have the following synthesis theorem.

Theorem 4.1: *If there exists $\phi \in \Phi$ for given $\gamma > 0$ and $\delta > 0$, then for $A_{fl} = -G_l^{-1}R_l, B_{fl} = -G_l^{-1}F_l, C_{fl}^{(2)} = -O_l^{(2)}$ and $C_{fl}^{(\infty)} = -O_l^{(\infty)}$, $l \in \mathbb{M}$, we have that $F \in \mathbb{F}$, $\|\mathcal{G}_F\|_2 < \gamma$ and $\|\mathcal{G}_F\|_\infty < \delta$.*

Proof: See Appendix 1. ■

Given the result in Theorem 4.1, we can write the following approximation for the optimisation problem (15):

$$\inf_{\phi\in\Phi} f(\gamma, \delta), \tag{22}$$

where $f(\gamma, \delta) = \gamma_a, \gamma_a = \gamma^2$ and $\delta > 0$ is given. We point out that the result of Theorem 4.1 also encompasses the so-called 'inverse' mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem by taking $f(\gamma, \delta) = \delta_a$, $\delta_a = \delta^2$, for a given $\gamma > 0$; as well as $f(\gamma, \delta) = a_2\gamma_a + a_\infty\delta_a$, for $a_2 \geq 0$ and $a_\infty \geq 0$.

Remark 4.1: It is important to point out that although the LMI set (16)–(19) is similar to that given in de Oliveira and Costa (2017a) and (20)–(21) is similar to that given in de Oliveira and Costa (2017b), they have to be solved jointly on the variables (G_l, R_l, F_l) which yield to the filter matrices A_{fl} and B_{fl}. This is a key point in the proof of Theorem 4.1 that allows us to solve a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem. Moreover, as we are going to see in the following sections, this formulation allows us to study the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering for the robust and cluster case, and also the perfect observation case.

5. Robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering

The LMI sets (16)–(19) and (20)–(21) are affine with respect to given probability matrices $\mathbb{P}_1,\ldots,\mathbb{P}_\sigma$ and $\Upsilon_1,\ldots,\Upsilon_\tau$, as similarly noted in Gonçalves et al. (2011) and de Oliveira and Costa (2017b), a fact that allows us to consider uncertain transition and detection probabilities. For that, we assume that $\mathbb{P}\in\mathbb{D}(\mathbb{P}_1,\ldots,\mathbb{P}_\sigma)$ and $\Upsilon\in\mathbb{D}(\Upsilon_1,\ldots,\Upsilon_\tau)$, recalling the definition of $\mathbb{D}(V_1,\ldots,V_\sigma)$ in (1). We say that system (4) is robust stochastically stable if there exists a filter as in (3) such that (7) holds for every $\mathbb{P}\in\mathbb{D}(\mathbb{P}_1,\ldots,\mathbb{P}_\sigma)$ and $\Upsilon\in\mathbb{D}(\Upsilon_1,\ldots,\Upsilon_\tau)$, every initial condition $\tilde{x}_0$ with finite second moment and every initial Markov mode θ_0. As in Assumption 3.1, we also have to assume that system (2) is robust stochastically stable with respect to uncertainties in $\mathbb{P}$. We define also the set $\mathbb{F}_r$ of all robust filter structures such that system (4) is robust stochastically stable. Then, it follows by changing

$$p_{ij}=p_{ij}^{(s)},\quad \alpha_{il}=\alpha_{il}^{(t)} \tag{23}$$

on (16)–(19) and (20)–(21) and solving this new LMI set for $i\in\mathbb{N}$, $l\in\mathbb{M}_i$, $s\in\{1,\ldots,\sigma\}$ and $t\in\{1,\ldots,\tau\}$, that we would have a sufficient condition for achieving robust filters. This result is stated in the next theorem.

Theorem 5.1: *For $s\in\{1,\ldots,\sigma\}$ and $t\in\{1,\ldots,\tau\}$, consider simultaneously the LMIs as (16)–(19) and (20)–(21) replacing p_{ij} and α_{il} by $p_{ij}^{(s)}$ and $\alpha_{il}^{(t)}$. Given that $\mathbb{P}\in\mathbb{D}(\mathbb{P}_1,\ldots,\mathbb{P}_\sigma)$ and $\Upsilon\in\mathbb{D}(\Upsilon_1,\ldots,\Upsilon_\tau)$, if there exists $\phi\in\Phi$, then for $A_{fl}=-G_l^{-1}R_l$, $B_{fl}=-G_l^{-1}F_l$, $C_{fl}^{(2)}=-O_l^{(2)}$ and $C_{fl}^{(\infty)}=-O_l^{(\infty)}$, $l\in\mathbb{M}$, we have that $F\in\mathbb{F}_r$, $\|\mathcal{G}_F\|_2<\gamma$ and $\|\mathcal{G}_F\|_\infty<\delta$.*

Proof: See Appendix 2. ■

6. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering for the cluster and complete observation cases

In the discussion that follows, we consider the most known settings related to the observation of the Markov chain $\theta(k)$, namely, the complete observation, cluster and mode-independent cases, as discussed in Introduction and in Remark 3.1. For that, we enunciate the following assumption stated in de Oliveira and Costa (2017a).

Assumption 6.1: The modes of the Markov chain can be partitioned into κ disjoint subsets $\mathbb{N}^s$, with $\bigcup_{s=1}^{\kappa}\mathbb{N}^s=\mathbb{N}$, such that for all $s\in\{1,\ldots,\kappa\}$, $i\in\mathbb{N}^s$, we have that $\mathbb{M}_i=\mathbb{M}^s$ for disjoint sets $\mathbb{M}^s$, with $\bigcup_{s=1}^{\kappa}\mathbb{M}^s=\mathbb{M}$, and $\alpha_{il}=\alpha_l^s$, for all $l\in\mathbb{M}^s$.

We introduce the following LMI set for the cluster case, for $s\in\{1,\ldots,\kappa\}$, $i\in\mathbb{N}^s$, and a given $\gamma>0$,

$$\sum_{i\in\mathbb{N}}\mu_i\mathrm{Tr}\left(\bar{W}_i\right)<\gamma^2, \tag{24}$$

$$\begin{bmatrix} \bar{W}_i & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Y)J_i^{(2)} & \mathcal{E}_i(Y) & \bullet & \bullet \\ G_sJ_i^{(2)}+F_sH_i^{(2)} & 0 & \mathrm{Her}(G_s)+\mathcal{E}_i(Y)-\mathcal{E}_i(X) & \bullet \\ E_i^{(2)} & 0 & 0 & I \end{bmatrix}>0, \tag{25}$$

$$\begin{bmatrix} Y_i & \bullet & \bullet & \bullet & \bullet \\ Y_i & X_i & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Y)A_i & \mathcal{E}_i(Y)A_i & \mathcal{E}_i(Y) & \bullet & \bullet \\ G_sA_i+F_sL_i+R_s & G_sA_i+F_sL_i & 0 & \mathrm{Her}(G_s)+\mathcal{E}_i(Y)-\mathcal{E}_i(X) & \bullet \\ C_i^{(2)}+O_s^{(2)} & C_i^{(2)} & 0 & 0 & I \end{bmatrix}>0, \tag{26}$$

as well as

$$\begin{bmatrix} T_i & \bullet & \bullet & \bullet & \bullet & \bullet \\ T_i & Z_i & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \delta^2 I & \bullet & \bullet & \bullet \\ \mathcal{E}_i(T)A_i & \mathcal{E}_i(T)A_i & \mathcal{E}_i(T)J_i^{(\infty)} & \mathcal{E}_i(T) & \bullet & \bullet \\ G_sA_i+F_sL_i+R_s & G_sA_i+F_sL_i & G_sJ_i^{(\infty)}+F_sH_i^{(\infty)} & 0 & \Xi_{si} & \bullet \\ C_i^{(\infty)}+O_s^{(\infty)} & C_i^{(\infty)} & E_i^{(\infty)} & 0 & 0 & I \end{bmatrix}>0, \tag{27}$$

where $\Xi_{si}=\mathrm{Her}(G_s)+\mathcal{E}_i(T)-\mathcal{E}_i(Z)$, for a given $\delta>0$. We define the set of variables of (24)–(26), and of (27), as follows:

$$\begin{aligned}\bar{\phi}_2 &= \Big\{\bar{W}_i, X_i, Y_i, G_s, F_s, R_s, O_s^{(2)}, i\in\mathbb{N}^s,\\ &\quad s\in\{1,\ldots,\kappa\}\Big\}\cup\psi_2 \quad \text{and}\\ \bar{\phi}_\infty &= \Big\{T_i, Z_i, G_s, F_s, R_s, O_s^{(\infty)}, i\in\mathbb{N}^s,\\ &\quad s\in\{1,\ldots,\kappa\}\Big\}\cup\psi_\infty,\end{aligned}$$

where, as before, $\psi_2=\emptyset$ if $\gamma>0$ is given, and $\psi_2=\{\gamma_a\}$, $\gamma_a=\gamma^2$, if it is considered as a variable. Similarly, $\psi_\infty=\emptyset$ if $\delta>0$ is given, and $\psi_\infty=\{\delta_a\}$, if $\delta_a=\delta^2$ is a variable. The set of variables of the joint $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem for the cluster case can be described by

$$\bar{\phi}=\bar{\phi}_2\cup\bar{\phi}_\infty,$$

and also the set of all possible joint solutions of (24)–(26) and (27) is given by

$$\bar{\Phi}=\left\{\bar{\phi}\ \text{such that (24)–(26) and (27) jointly hold}\right\}.$$

Proposition 6.1: *Given that Assumption 6.1 holds, $\phi \in \Phi$ if and only if $\bar{\phi} \in \bar{\Phi}$.*

Proof: See Appendix 3. ■

Considering the equivalence stated in Proposition 6.1, we redefine the optimisation problem (22) for the simplified conditions (24)–(26) and (27):

$$\inf_{\bar{\phi}\in\bar{\Phi}} f(\gamma,\delta). \tag{28}$$

Finally it is clear that the complete observation setting is a special case of the cluster scenario where $\mathbb{N}^s = \{s\}, s \in \mathbb{N}$. Similarly as stated in Gonçalves et al. (2010), by setting

$$G_s = G_i, F_s = F_i, R_s = R_i, O_s^{(2)} = O_i^{(2)}, O_s^{(\infty)} = O_i^{(\infty)},$$

in (24)–(26) and (27), we would retrieve the complete observation setting. Concerning the detector approach on this situation, and bearing in mind the discussion on Assumption 6.1, if $M = N$ we could also have that $\mathbb{M}^s = \{s\}$, $s \in \mathbb{M}$, as well as $\alpha_{ss} = 1$, that is, a perfect detector. Similarly the mode-independent case is obtained by fixing $G_s = G$, $F_s = F$, $R_s = R$, $O_s^{(2)} = O^{(2)}$, $O_s^{(\infty)} = O^{(\infty)}$,$s \in \{1,\dots,\kappa\}$.

Remark 6.1: The LMI set (24)–(26) and (27) are similar to that given in Gonçalves et al. (2010). The main difference resides in the fact that in Gonçalves et al. (2010), the resulting filter is proper and also that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problems are not considered to be jointly solved. On the other hand, we consider separated inputs and outputs for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems, and also different Lyapunov matrices in Proposition 3.1 and Theorem 3.2, an approach that flexibilises the results in (24)–(26) and (27).

7. Numerical example

The example is based on the one given in Fioravanti et al. (2015), consisting of a classical mass-spring-damper system whose states are transmitted through a channel subject to noise and failures. There are three possibilities: (1) the transmission is successful with a nominal noise level; (2) the intensity of the transmitted signal is diminished with an augmented noise level; (3) the transmission is lost. In this case, the system is discretised via a zero-order hold of period $T = 0.5$ s in each input, as in Fioravanti et al. (2015), yielding the following system matrices:

$$\left[\begin{array}{c|c|c} A_i & J_i^{(2)} & J_i^{(\infty)} \end{array}\right]$$

$$= \left[\begin{array}{cccc|ccccc|cc} -0.7562 & 0.5086 & 0.0791 & 0.1435 & 1 & 0 & 0 & 0 & 0 & 0.0974 & 0 \\ 0.2092 & 0.5604 & 0.0718 & 0.4012 & 0 & 1 & 0 & 0 & 0 & 0.0559 & 0 \\ -1.6559 & 0.0736 & -0.7588 & 0.5113 & 0 & 0 & 1 & 0 & 0 & -1.7562 & 0 \\ -0.1463 & -1.2887 & 0.2556 & 0.5140 & 0 & 0 & 0 & 1 & 0 & 0.2092 & 0 \end{array}\right].$$

The measurement matrices are given by

$$\left[\begin{array}{c|c|c} L_1 & H_1^{(2)} & H_1^{(\infty)} \\ \hline L_2 & H_2^{(2)} & H_2^{(\infty)} \\ \hline L_3 & H_3^{(2)} & H_3^{(\infty)} \end{array}\right] = \left[\begin{array}{cccc|ccccc|cc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0.1 \\ \hline 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

where the measured state $x_3(k)$ corresponds to the velocity of the first mass. We point out that, considering the $\mathcal{H}_2$ filtering problem, the values of the matrices $J_i^{(2)}$ introduce uncertainties on the states by injecting white noise sequences through the first components of $w(k)$, namely $w_i(k), i \in \{1,\dots,4\}$ (see Remark 3.4). On the other hand, the fifth component $w_5(k)$ models the noise on the channel. Similarly, for the $\mathcal{H}_\infty$ filtering problem, we would have that the component $v_1(k)$ is a disturbance in the system, and the component $v_2(k)$ models the noisy aspect of the channel. It is clear then that the signal-to-noise ratio for $\theta(k) = 1$ is adequate for the transmission, whereas for $\theta(k) = 2$, the quality of the signal being sent will degrade due to the lower intensity of the measured state and the greater intensity of the noise for both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problems.

The goal for this example is to synthesise a filter that, given the noisy and faulty measurements of the velocity of the first mass: (1) provides us the best possible estimate of $x_4(k)$, the velocity of the second mass, in the context of the $\mathcal{H}_2$ filtering problem; (2) provides us the estimation of $x_2(k)$, the position of the second mass, for the $\mathcal{H}_\infty$ filtering problem. Thus we set

$$\left[\begin{array}{c} C_i^{(2)} \\ \hline C_i^{(\infty)} \end{array}\right] = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \end{array}\right], \left[\begin{array}{c|c} E_i^{(2)} & E_i^{(\infty)} \end{array}\right]$$
$$= \left[\begin{array}{ccccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad i \in \mathbb{N}.$$

The transition probability matrix and the initial probability distribution are given by

$$\left[\begin{array}{c|c} \mathbb{P} & \mu' \end{array}\right] = \left[\begin{array}{ccc|c} 0.7000 & 0.2000 & 0.1000 & 0.2500 \\ 0.1000 & 0.5000 & 0.4000 & 0.3889 \\ 0.1000 & 0.4000 & 0.5000 & 0.3611 \end{array}\right],$$

and we point out that the initial probability distribution μ is taken as the stationary distribution of the Markov chain. In this case, we set Υ as follows:

$$\Upsilon = \begin{bmatrix} \rho & 1-\rho & 0 \\ 1-\rho & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{29}$$

for $0 \le \rho \le 1$. The detector structure in (29) can precisely sense the packet loss that occurs in the state (3), but may not know what happens if the system is in $\theta(k) = 1$ and $\theta(k) = 2$. We initially investigate three different situations:

- The cluster case, that is obtained by setting $\rho = 0.50$ in (29), see Assumption 6.1. Intuitively, this can be explained by noting that the detection probabilities are equal in (29) for $i \in \{1, 2\}$ and also that $\mathbb{M}^1 = \{1, 2\}$ and $\mathbb{M}^2 = \{3\}$. A possible interpretation is that the detector cannot distinguish between these two modes of operation.
- A case where the detector possesses a 'reasonable' chance of observing the correct noisy state of operation, and so we set $\rho = 0.90$.
- The detector can perfectly know in which mode the system is currently operating, and so we have that $\rho = 1.00$.

We obtain γ by means of (22) by varying δ from the approximate minimum value of $\delta_{\min}$ where (22) is feasible for each $\rho \in \{0.50, 0.90, 1.00\}$ up to $\delta = 4$. The results are shown in Figure 1. First we notice that as the value of ρ decreases, the value of the lower bound of $\delta_{\min}$ increases. That is to say that the quality of the detector has an influence on the minimum δ such that (22) has a solution. On the other hand, it is clear that the best upper bounds γ

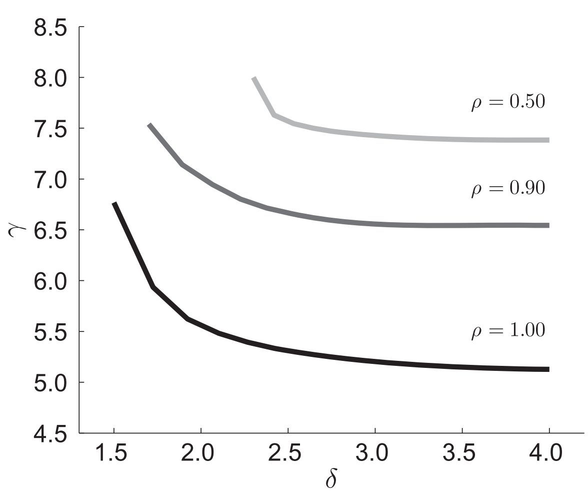

Figure 1. γ in function of δ for $\rho = 1.00$ (black line), $\rho = 0.90$ (grey line) and $\rho = 0.50$ (brighter grey line).

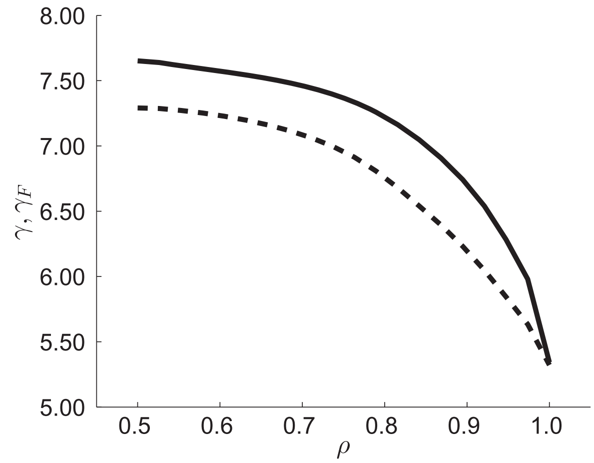

Figure 2. γ and $\gamma_F = \|\mathcal{G}_F\|_2$ in function of ρ.

are obtained for the complete observation case, that is, $\rho = 1.00$. Finally, as $\delta \to \infty$, the restriction on the $\mathcal{H}_\infty$ norm of (4) is untighten that leads to better guaranteed costs of γ. Those values converge to stationary values, that is, the upper bound δ will influence the problem only on a restricted interval. For comparison, by taking $\delta = 4$, we get that $\gamma = 5.12780$ and $\gamma_F = 5.12777$, that is, the guaranteed cost and the actual $\mathcal{H}_2$ norm are numerically close for this situation.

We now take $\delta = 2.4$, and by means of the optimisation problem (22), vary $\rho \in [0.5, 1.0]$. The result is shown in Figure 2. As we discussed in Figure 1, we note that for smaller values of ρ in this case, we obtain the worst results in terms of the upper bound γ, and also the actual $\mathcal{H}_2$ norm, γ_F. Besides for $\rho = 1.0$, we have that $\gamma = 5.3416$ and $\gamma_F = 5.3211$, and so for $\delta = 2.4$ we still get some conservatism for the complete observation case in relation to the $\mathcal{H}_2$ filtering problem.

Finally, we set $\delta = 2.4$ and $\rho = 0.90$ and calculate the filter matrices by means of (22). We have that $\gamma = 6.7047$, $\gamma_F = 6.1957$ and $\delta_F = 1.8051$, and also that

$$\left[\begin{array}{c|c} A_{f1} & B_{f1} \\ \hline A_{f2} & B_{f2} \\ \hline A_{f3} & B_{f3} \end{array}\right] = \left[\begin{array}{cccc|c} -0.7723 & 0.5189 & 0.0601 & 0.1294 & 0.0059 \\ 0.2127 & 0.5593 & -0.0224 & 0.3994 & 0.0950 \\ -1.5459 & 0.0259 & -0.1571 & 0.5962 & -0.5678 \\ -0.1560 & -1.2837 & 0.0190 & 0.4995 & 0.2268 \\ \hline -0.7759 & 0.4865 & 0.0450 & 0.1360 & 0.0068 \\ 0.2090 & 0.5561 & 0.0828 & 0.3992 & 0.0307 \\ -1.5340 & 0.1753 & -0.5686 & 0.5552 & 0.0242 \\ -0.1625 & -1.3038 & 0.2247 & 0.5078 & 0.0801 \\ \hline -0.7619 & 0.5091 & 0.0771 & 0.1445 & 0 \\ 0.2116 & 0.5602 & 0.0726 & 0.4008 & 0 \\ -1.6201 & 0.0707 & -0.7464 & 0.5052 & 0 \\ -0.1511 & -1.2883 & 0.2539 & 0.5148 & 0 \end{array}\right],$$

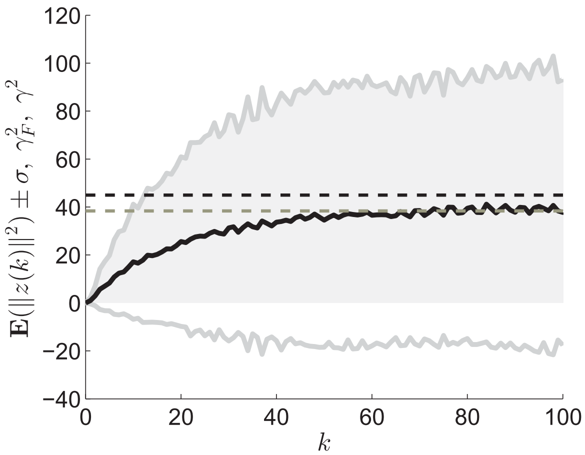

Figure 3. $\mathbf{E}(\|e_2(k)\|^2)$ (black line), γ_F^2 (dashed grey line) and γ^2 (dashed black line) in function of k for $\delta = 2.4$ and $\rho = 0.90$.

as well as

$$\begin{bmatrix} C_{f1}^{(2)} \\ \hline C_{f2}^{(2)} \\ \hline C_{f3}^{(2)} \end{bmatrix} = \begin{bmatrix} -0.0144 & 0.0041 & -0.0473 & 1.0654 \\ \hline -0.0384 & 0.0233 & 0.0313 & 1.0025 \\ \hline -0.0261 & 0.0021 & -0.0090 & 1.0044 \end{bmatrix}$$

and

$$\begin{bmatrix} C_{f1}^{(\infty)} \\ \hline C_{f2}^{(\infty)} \\ \hline C_{f3}^{(\infty)} \end{bmatrix} = \begin{bmatrix} 0.0612 & 0.9814 & 0.0463 & -0.0078 \\ \hline 0.4217 & 1.0407 & 0.1306 & -0.0180 \\ \hline 0.3434 & 0.9720 & 0.1188 & -0.0584 \end{bmatrix}.$$

By injecting the exogenous input $w(k)$, a white noise sequence of covariance matrix equal to I_5, for $\nu = 0$, and performing a Monte Carlo simulation of 2000 rounds, we calculate and plot in Figure 3 the mean value curve of $\|e_2(k)\|^2$ in function of k, as well as the square of the actual $\mathcal{H}_2$ norm, γ_F^2 and γ^2. Figure 3 illustrates the stochastic definition of the $\mathcal{H}_2$ norm discussed in Remark 3.4, that is, $\mathbf{E}(\|e_2(k)\|^2) \to \gamma_F^2$ as $k \to \infty$. In this case, the value of γ^2 illustrates the conservatism of the optimisation problem (22) for this situation.

8. Conclusion

We studied in this work the joint problem of finding filters depending only on an estimation $\hat{\theta}(k)$ of the underlying Markov chain such that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms with respect to the estimation error signals are bounded by given scalars, the so-called mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem. We separate the exogenous inputs and estimated outputs for both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems and proposed a sufficient synthesis condition in the LMI formulation such that the $\mathcal{H}_2$ norm with respect to the error estimation of $z_2(k)$ is bounded by γ, and the $\mathcal{H}_\infty$ norm of the error related to the output $z_\infty(k)$ is bounded by δ. Furthermore, we studied the synthesis of robust filters (with respect to uncertain transition and detection probabilities), and the complete, cluster and mode-independent cases as by products of the detector approach. A numerical example is given in order to illustrate our results.

For future works, the joint problem of synthesising filters and controllers is appealing and challenging. This is linked to the dynamical output feedback problem that has not been solved yet in the sense of simple congruence transformations as presented in Geromel et al. (2009). Furthermore, the framework of the detector approach applied to this problem could also yield the cluster and mode-independent cases as well.

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No potential conflict of interest was reported by the authors.

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Appendices

Appendix 1. Proof of Theorem 4.1

This proof is similar to that given in de Oliveira and Costa (2017a) and de Oliveira and Costa (2017b), and is based on Gonçalves et al. (2010). We define the following partitions for P_i and $P_i{}^{-1}$ in (9)–(10),

$$P_i = \begin{bmatrix} X_i & \bullet \\ U_i & \hat{Y}_i \end{bmatrix}, \quad P_i^{-1} = \begin{bmatrix} Y_i^{-1} & \bullet \\ V_i & \hat{T}_i \end{bmatrix}, \tag{A1}$$

along with

$$Q_i = \begin{bmatrix} Z_i & \bullet \\ \hat{U}_i & \hat{X}_i \end{bmatrix}, \quad Q_i^{-1} = \begin{bmatrix} T_i^{-1} & \bullet \\ \hat{V}_i & \hat{Z}_i \end{bmatrix}, \tag{A2}$$

in (12) and (13), so we stress that (A.1) and (A.2) are the Lyapunov matrices related to (9)–(10) and (12)–(13), respectively. Due to the positive definiteness of P and Q we have that T_i^{-1} and Z_i^{-1} can be inverted and thus we define the following matrices:

$$\mathcal{T}_i^{(2)} = \begin{bmatrix} I & I \\ V_iY_i & 0 \end{bmatrix}, \quad \mathcal{T}_i^{(\infty)} = \begin{bmatrix} I & I \\ \hat{V}_iT_i & 0 \end{bmatrix},$$

such that

$$\mathcal{T}^{(2)'}P_i\mathcal{T}^{(2)} = \begin{bmatrix} Y_i & \bullet \\ Y_i & X_i \end{bmatrix}, \quad \mathcal{T}^{(\infty)'}Q_i\mathcal{T}^{(\infty)} = \begin{bmatrix} T_i & \bullet \\ T_i & Z_i \end{bmatrix}. \tag{A3}$$

On the other hand, we partition $\mathcal{E}_i(P)^{-1}$ and $\mathcal{E}_i(Q)^{-1}$ such that

$$\mathcal{E}_i(P)^{-1} = \begin{bmatrix} \hat{P}_{1i} & \bullet \\ \hat{P}_{2i} & \hat{P}_{3i} \end{bmatrix}, \quad \mathcal{E}_i(Q)^{-1} = \begin{bmatrix} \hat{Q}_{1i} & \bullet \\ \hat{Q}_{2i} & \hat{Q}_{3i} \end{bmatrix}$$

and also we define the following matrices:

$$\mathcal{H}_i^{(2)} = \begin{bmatrix} \hat{P}_{1i}^{-1} & \mathcal{E}_i(X) \\ 0 & \mathcal{E}_i(U) \end{bmatrix}, \quad \mathcal{H}_i^{(\infty)} = \begin{bmatrix} \hat{Q}_{1i}^{-1} & \mathcal{E}_i(Z) \\ 0 & \mathcal{E}_i(\hat{U}) \end{bmatrix},$$

such that

$$\mathcal{H}_i^{(2)'}\mathcal{E}_i(P)^{-1}\mathcal{H}_i^{(2)} = \begin{bmatrix} \hat{P}_{1i}^{-1} & \bullet \\ \hat{P}_{1i}^{-1} & \mathcal{E}_i(X) \end{bmatrix}, \quad \mathcal{H}_i^{(\infty)'}\mathcal{E}_i(Q)^{-1}\mathcal{H}_i^{(\infty)} = \begin{bmatrix} \hat{Q}_{1i}^{-1} & \bullet \\ \hat{Q}_{1i}^{-1} & \mathcal{E}_i(Z) \end{bmatrix}. \tag{A4}$$

By choosing suitable structures for P and Q, we can write $\hat{P}_{1i}^{-1}$ and $\hat{Q}_{1i}^{-1}$ in terms of the problem variables, a procedure that will become clear in the next proof.

Proof of Theorem 4.1: Consider that $\phi \in \Phi$. We set P_i in (9)–(10) as follows:

$$P_i = \begin{bmatrix} X_i & \bullet \\ U_i & \hat{Y}_i \end{bmatrix} = \begin{bmatrix} X_i & \bullet \\ Y_i - X_i & X_i - Y_i \end{bmatrix},$$

which implies that $U_i = -\hat{Y}_i$, $U_i = Y_i - X_i$, $V_iY_i = I$ and $\hat{P}_{1i}^{-1} = \mathcal{E}_i(Y)$ for all $i \in \mathbb{N}$. Furthermore, as in Gonçalves et al. (2010), we define the matrix

$$\mathcal{D}_{il}^{(2)} := \begin{bmatrix} I & I \\ 0 & G_l^{-T}[\mathcal{E}_i(X) - \mathcal{E}_i(Y)] \end{bmatrix}.$$

For the $\mathcal{H}_2$ problem, we note that $G_l[\mathcal{E}_i(X) - \mathcal{E}_i(Y)]^{-1}G_l \geq \text{Her}(G_l) + \mathcal{E}_i(Y) - \mathcal{E}_i(X)$ (see, for instance, Daafouz & Bernussou, 2001) that allows us to infer that

$$\begin{bmatrix} W_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Y)J_i^{(2)} & \mathcal{E}_i(Y) & \bullet & \bullet \\ G_lJ_i^{(2)} + F_lH_i^{(2)} & 0 & G_l[\mathcal{E}_i(X) - \mathcal{E}_i(Y)]^{-1}G_l' & \bullet \\ E_i^{(2)} & 0 & 0 & I \end{bmatrix} > 0, \tag{A5}$$

$$\begin{bmatrix} M_{il}^{(2,11)} & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(2,21)} & M_{il}^{(2,22)} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Y)A_i & \mathcal{E}_i(Y)A_i & \mathcal{E}_i(Y) & \bullet & \bullet \\ G_lA_i + F_lL_i + R_l & G_lA_i + F_lL_i & 0 & G_l[\mathcal{E}_i(X) - \mathcal{E}_i(Y)]^{-1}G_l' & \bullet \\ C_i^{(2)} + O_l^{(2)} & C_i^{(2)} & 0 & 0 & I \end{bmatrix} > 0 \tag{A6}$$

also holds. Applying the congruence transformations **diag** $(I_{r_2}, \mathcal{D}_{il}^{(2)}, I_{q_2})$ and **diag**$(I_n, I_n, \mathcal{D}_{il}^{(2)}, I_{q_2})$ to the LMI in (A.5) and (A.6), and considering that $R_l = -G_lA_{fl}$, $F_l = -G_lB_{fl}$ and $C_{fl}^{(2)} = -O_l^{(2)}$, we obtain that

$$\begin{bmatrix} W_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Y)J_i^{(2)} & \mathcal{E}_i(Y) & \bullet & \bullet \\ \mathcal{E}_i(X)J_i^{(2)} + \mathcal{E}_i(U)B_{fl}H_i^{(2)} & \mathcal{E}_i(Y) & \mathcal{E}_i(X) & \bullet \\ E_i^{(2)} & 0 & 0 & I \end{bmatrix} > 0, \tag{A7}$$

$$\begin{bmatrix} M_{il}^{(2,11)} & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(2,21)} & M_{il}^{(2,22)} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Y)A_i & \mathcal{E}_i(Y)A_i & \mathcal{E}_i(Y) & \bullet & \bullet \\ \mathcal{E}_i(X)A_i + \mathcal{E}_i(U)B_{fl}L_i + \mathcal{E}_i(U)A_{fl} & \mathcal{E}_i(X)A_i + \mathcal{E}_i(U)B_{fl}L_i & \mathcal{E}_i(Y) & \mathcal{E}_i(X) & \bullet \\ C_i^{(2)} - C_{fl}^{(2)} & C_i^{(2)} & 0 & 0 & I \end{bmatrix} > 0. \tag{A8}$$

This allows us to rewrite the LMI in (A.7) and (A.8) in the following form:

$$\begin{bmatrix} W_{il} & \bullet & \bullet \\ \mathcal{H}_i^{(2)'} J_{il} & \mathcal{H}_i^{(2)'} \mathcal{E}_i(P)^{-1} \mathcal{H}_i^{(2)} & \bullet \\ E_i^{(2)} & 0 & I \end{bmatrix} > 0,$$

$$\begin{bmatrix} \mathcal{T}_i^{(2)'} M_{il}^{(2)} \mathcal{T}_i^{(2)} & \bullet & \bullet \\ \mathcal{H}_i^{(2)'} A_{il} \mathcal{T}_i^{(2)} & \mathcal{H}_i^{(2)'} \mathcal{E}_i(P)^{-1} \mathcal{H}_i^{(2)} & \bullet \\ C_{il}^{(2)} \mathcal{T}_i^{(2)} & 0 & I \end{bmatrix} > 0,$$

as well as the inequality (18) as $\mathcal{T}_i^{(2)'} P_i \mathcal{T}_i^{(2)} > \sum_{l \in \mathbb{M}_i} \alpha_{il} \mathcal{T}_i^{(2)'} M_{il}^{(2)} \mathcal{T}_i^{(2)}$, for system matrices shown in (5). Applying respectively the similarity transformations $\mathbf{diag}(I_{r_2}, \mathcal{H}_i^{(2)^{-1}}, I_{q_2})$, $\mathbf{diag}(\mathcal{T}_i^{(2)^{-1}}, \mathcal{H}_i^{(2)^{-1}}, I_{q_2})$ and $\mathcal{T}_i^{(2)^{-1}}$ to the previous inequalities, and considering (16), we obtain (9)–(10) with system matrices given in (5).

On the other hand, considering the $\mathcal{H}_\infty$ filtering, we set Q_i in (A.2) such that

$$Q_i = \begin{bmatrix} Z_i & \bullet \\ \hat{U}_i & \hat{X}_i \end{bmatrix} = \begin{bmatrix} Z_i & \bullet \\ T_i - Z_i & Z_i - T_i \end{bmatrix}. \tag{A9}$$

Similarly as before, we have that $G_l[\mathcal{E}_i(Z) - \mathcal{E}_i(T)]^{-1} G_l' \geq \mathrm{Her}(G_l) + \mathcal{E}_i(T) - \mathcal{E}_i(Z)$, and so

$$\begin{bmatrix} M_{il}^{(\infty,11)} & \bullet & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(\infty,21)} & M_{il}^{(\infty,22)} & \bullet & \bullet & \bullet & \bullet \\ N_{il}^{(\infty,11)} & N_{il}^{(\infty,12)} & S_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(T)A_i & \mathcal{E}_i(T)A_i & \mathcal{E}_i(T)J_i^{(\infty)} & \mathcal{E}_i(T) & \bullet & \bullet \\ G_lA_i + F_lL_i + R_l & G_lA_i + F_lL_i & G_lJ_i^{(\infty)} + F_lH_i^{(\infty)} & 0 & \Pi_{il} & \bullet \\ C_i^{(\infty)} + O_l^{(\infty)} & C_i^{(\infty)} & E_i^{(\infty)} & 0 & 0 & I \end{bmatrix} > 0, \tag{A10}$$

also holds, where $\Pi_{il} := G_l[\mathcal{E}_i(Z) - \mathcal{E}_i(T)]^{-1} G_l'$. Defining the matrix

$$\mathcal{D}_{il}^{(\infty)} = \begin{bmatrix} I & I \\ 0 & G_l^{-T}[\mathcal{E}_i(Z) - \mathcal{E}_i(T)] \end{bmatrix}$$

and applying the congruence transformation $\mathbf{diag}(I_n, I_n, I_{r_\infty}, \mathcal{D}_{il}^{(\infty)}, I_{q_\infty})$ to the LMI in (A.10) yields

$$\begin{bmatrix} M_{il}^{(\infty,11)} & \bullet & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(\infty,21)} & M_{il}^{(\infty,22)} & \bullet & \bullet & \bullet & \bullet \\ N_{il}^{(\infty,11)} & N_{il}^{(\infty,12)} & S_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(T)A_i & \mathcal{E}_i(T)A_i & \mathcal{E}_i(T)J_i^{(\infty)} & \mathcal{E}_i(T) & \bullet & \bullet \\ \bar{\Xi}_{il} + \mathcal{E}_i(\hat{U})A_{fl} & \bar{\Xi}_{il} & \mathcal{E}_i(Z)J_i^{(\infty)} + \mathcal{E}_i(\hat{U})B_{fl}H_i^{(\infty)} & \mathcal{E}_i(T) & \mathcal{E}_i(Z) & \bullet \\ C_i^{(\infty)} - C_{fl}^{(\infty)} & C_i^{(\infty)} & E_i^{(\infty)} & 0 & 0 & I \end{bmatrix} > 0, \tag{A11}$$

where $\bar{\Xi}_{il} = \mathcal{E}_i(Z)A_i + \mathcal{E}_i(\hat{U})B_{fl}L_i$. Considering the partition of Q_i in (A.9), we can rewrite (A.11) as follows:

$$\begin{bmatrix} \mathcal{T}_i^{(\infty)'} M_{il} \mathcal{T}_i^{(\infty)} & \bullet & \bullet & \bullet \\ N_{il} \mathcal{T}_i^{(\infty)} & S_{il} & \bullet & \bullet \\ \mathcal{H}_i^{(\infty)'} A_{il} \mathcal{T}_i^{(\infty)} & \mathcal{H}_i^{(\infty)'} J_{il}^{(\infty)} & \mathcal{H}_i^{(\infty)'} \mathcal{E}_i(Q)^{-1} \mathcal{H}_i^{(\infty)} & \bullet \\ C_{il}^{(\infty)} \mathcal{T}_i^{(\infty)} & E_i^{(\infty)} & 0 & I \end{bmatrix} > 0,$$

with system matrices given in (5). Additionally the first inequality of (20) can be rewritten as

$$\begin{bmatrix} \mathcal{T}_i^{(\infty)'} Q_i \mathcal{T}_i^{(\infty)} & \bullet \\ 0 & \delta^2 I \end{bmatrix} > \sum_{l \in \mathbb{M}_i} \alpha_{il} \begin{bmatrix} \mathcal{T}_i^{(\infty)'} M_{il} \mathcal{T}_i^{(\infty)} & \bullet \\ N_{il} \mathcal{T}_i^{(\infty)} & S_{il} \end{bmatrix}.$$

Applying the similarity transformations $\mathbf{diag}(\mathcal{T}_i^{(\infty)^{-1}}, I_{r_\infty}, \mathcal{H}_i^{(\infty)^{-1}}, I_{q_\infty})$ and $\mathbf{diag}(\mathcal{T}_i^{(\infty)^{-1}}, I_{r_\infty})$ to the previous inequalities yields (13) and (12) with system matrices in (5) completing the proof. ■

Appendix 2. Proof of Theorem 5.1

Proof of Theorem 5.1: It directly follows from the fact that (16)–(19) and (20)–(21) are affine with respect to $p_{ij}^{(s)}$ and $\alpha_{il}^{(t)}$. Thus, by multiplying the modified version of (16) by η_t and summing it up for all $t \in \{1, \ldots, \tau\}$, we have that the sum on the left-hand side is satisfied with the uncertain $\Upsilon \in \mathbb{D}(\Upsilon_1, \ldots, \Upsilon_\tau)$. The same reasoning applies for (18) and (20). Additionally, if we multiply the modified version of (17) by $\bar{\eta}_s$ and sum it up for all $s \in \{1, \ldots, \sigma\}$, we have that (17) holds for $\mathbb{P} \in \mathbb{D}(\mathbb{P}_1, \ldots, \mathbb{P}_\sigma)$. The claim follows after applying the same reasoning as above in (19) and (21). ■

Appendix 3. Proof of Proposition 6.1

Proof of Proposition 6.1: The proof is similar to that given in de Oliveira and Costa (2017a). First, given that $\phi \in \Phi$ and Assumption 6.1, we would like to obtain the LMIs (24)–(26) for the cluster problem. For that, considering the sums in Equations (18) and (20), we define $M_{is}^{(2,11)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s M_{il}^{(2,11)}$, $M_{is}^{(2,21)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s M_{il}^{(2,21)}$, $M_{is}^{(2,22)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s M_{il}^{(2,22)}$, $M_{is}^{(\infty,11)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s M_{il}^{(\infty,11)}$, $M_{is}^{(\infty,21)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s M_{il}^{(\infty,21)}$, $M_{is}^{(\infty,22)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s M_{il}^{(\infty,22)}$, $N_{is}^{(\infty,11)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s N_{il}^{(\infty,11)}$, $N_{is}^{(\infty,12)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s N_{il}^{(\infty,12)}$, $S_{is} := \sum_{l \in \mathbb{M}^s} \alpha_l^s S_{il}$, for all $i \in \mathbb{N}^s$. By multiplying (17), (19), (21), by α_l^s for all $i \in \mathbb{N}^s$, respectively summing each LMI up for all $l \in \mathbb{M}^s$, defining $\bar{W}_i := \sum_{l \in \mathbb{M}^s} \alpha_l^s W_{il}$, $G_s := \sum_{l \in \mathbb{M}^s} \alpha_l^s G_l$, $F_s := \sum_{l \in \mathbb{M}^s} \alpha_l^s F_l$, $R_s := \sum_{l \in \mathbb{M}^s} \alpha_l^s R_l$, $O_s^{(2)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s O_l^{(2)}$, and $O_s^{(\infty)} := \sum_{l \in \mathbb{M}^s} \alpha_l^s O_l^{(\infty)}$, and considering (18) and (20), we get (25)–(27). Finally, (16) directly implies (24) through the sum in $l \in \mathbb{M}^s$. Conversely, if $\bar{\phi} \in \bar{\Phi}$, by setting $W_{il} = \bar{W}_i$, we get that

$$\sum_{i \in \mathbb{N}} \mu_i W_{il} < \gamma^2$$

for all $l \in \mathbb{M}_i$, and thus by multiplying it by α_{il} and summing it up for all $l \in \mathbb{M}_i$, we get (16). Furthermore, setting

$G_l = G_s, F_l = F_s, R_l = R_s, O_l^{(2)} = O_s^{(2)}$ and$O_l^{(\infty)} = O_s^{(\infty)}$ for all $s \in \{1, \ldots, \kappa\}$, $i \in \mathbb{N}^s$ and $l \in \mathbb{M}^s$, and also,

$$\begin{bmatrix} M_{il}^{(2,11)} & \bullet \\ M_{il}^{(2,21)} & M_{il}^{(2,22)} \end{bmatrix} = \begin{bmatrix} Y_i & \bullet \\ Y_i & X_i \end{bmatrix} - \epsilon_2 I_{2n\times 2n}$$

and

$$\begin{bmatrix} M_{il}^{(\infty,11)} & \bullet & \bullet \\ M_{il}^{(\infty,21)} & M_{il}^{(\infty,22)} & \bullet \\ N_{il}^{(\infty,11)} & N_{il}^{(\infty,12)} & S_{il} \end{bmatrix} = \begin{bmatrix} T_i & \bullet & \bullet \\ T_i & Z_i & \bullet \\ 0 & 0 & \delta^2 I \end{bmatrix} - \epsilon_\infty I_{2n+r_\infty \times 2n+r_\infty}$$

for suitable $\epsilon_2 > 0$ and $\epsilon_\infty > 0$, we have that $\phi \in \Phi$ after straightforward manipulations. ■