A FREQUENCY THEOREM FOR THE CASE IN WHICH THE STATE AND CONTROL SPACES ARE HILBERT SPACES, WITH AN APPLICATION TO SOME PROBLEMS IN THE SYNTHESIS OF OPTIMAL CONTROLS. I

V. A. Yakubovich

UDC 62.50+517.9+519.9

INTRODUCTION

The frequency theorem [1-4] states that, under certain weak assumptions, a necessary and sufficient condition for the existence of a Hermitian matrix H satisfying

$$\mathrm{Re}[x^*H(Ax+bu)]+\mathcal{F}(x,u)\geqslant 0,\quad \forall x,\quad u, \tag{0.1}$$

is

$$\mathcal{F}[(i\omega I-A)^{-1}bu,\ u]\geqslant 0,\ \forall u,\ \forall\omega\in(-\infty,\ +\infty). \tag{0.2}$$

Here x and u are complex vectors and, in general, have different dimensions (we denote their dimensions by N and n), A is a square $(N \times N)$-matrix and b is a rectangular $(N \times n)$-matrix with complex elements, F(x, u) is a Hermitian form, and I is the unit matrix. If A has imaginary eigenvalues $i\omega_j$ then $\omega \neq \omega_j$ in (0.2). The frequency theorem also states that, if (0.2) holds, there are matrices $H = H^*$, h, and $\varkappa$ of dimensions $N \times N$, $N \times n$, and $n \times n$, respectively, such that

$$\mathrm{Re}[x^*H(Ax+bu)]+\mathcal{F}(x,u)=|h^*x-\varkappa u|^2,\ \forall x,u. \tag{0.3}$$

The assumption under which the assertion holds is that either A is Hurwitz or the pair A, b is Kalman-controllable. (In this formulation the theorem is proved in [4], where a procedure is also derived for the determination of H, h, and $\varkappa$.)

The frequency theorem is used in the study of nonlinear systems of differential equations, in the synthesis of optimal controls (cf. [3-8], which contain bibliographies), and in the synthesis of systems from spectral characteristics of the output process [9-11].

The relations obtained from (0.3) by equating the coefficients of $x_jx_k^*$, $x_ju_k^*$, and $u_ju_k^*$ are called Lur'e equations. (Cf. §1 below for more details.) The solvability of these equations for a special form F(x, u) served as a criterion for the stability of the nonlinear system under consideration in the first consideration of absolute stability (cf. [6, 12-14], which also contain bibliographies).

We shall extend the frequency theorem to the infinite-dimensional case with x and u vectors in Hilbert spaces, A and b bounded linear operators; we also consider some optimal-control problems (investigated for the finite-dimensional case in [15-18]) whose solutions are simply obtained applying the frequency theorem. The proof of the frequency theorem given below uses ideas from optimal-control theory.

There are several proofs of the frequency theorem for the finite-dimensional case: a proof in [1] based on the successive lowering of the dimensions of the spaces; a proof in [2, 3] based on a result concerning the factorization of a matrix polynomial; an "algebraic" proof in [19, 20] (significantly different); a proof in [9] based on the Hahn–Banach theorem and the moment method. Our method, in spite of the infinite-dimensionality of the spaces **X** and U, is the simplest. In the proof we use considerations and methods from optimal-control theory [21-26]. To avoid references to sources in which these results are stated in a form different from that required here (in particular where finite spaces are considered), we state the required results with proofs.

Translated from Sibirskii Matematicheskii Zhurnal, Vol. 15, No. 3, pp. 639-668, May-June, 1974. Original article submitted July 16, 1973.

In conclusion we indicate another important reference [27], where an algorithm for the synthesis of an optimal control in the problem of minimizing a quadratic functional was first obtained [for $n = 1$ and a positive-definite form $\mathcal{F}(x, u)$]. This algorithm can be considered as furnishing a solution of Lur'e's equations for the indicated special case.

§1. Statement of the Results

Let $\mathbf{X} = \{x\}$ and $\mathbf{U} = \{u\}$ be real or complex Hilbert spaces (either both real or both complex), whose elements will be called the states (x) and the controls (u). It is convenient to define the scalar product in $\mathbf{X}$ as follows:

$$(x_1, x_2) = x_2^* x_1$$

(this product is linear in x_1 and antilinear in x_2 if $\mathbf{X}$ is a complex space†). The norm in $\mathbf{X}$ is taken to be

$$|x| = \sqrt{x^* x}.$$

The scalar product and the norm in $\mathbf{U}$ are written similarly:

$$u_1^* u_2 \text{ and } |u| = \sqrt{u^* u}.$$

We assume that a bounded linear operator

$$A : \mathbf{X} \to \mathbf{X}, \quad b : \mathbf{U} \to \mathbf{X}$$

and a quadratic form $\mathcal{F}(x, u)$ (a Hermitian form if $\mathbf{X}$ and $\mathbf{U}$ are complex) are given on $\mathbf{X} \times \mathbf{U}$. A quadratic (Hermitian) form is, as usual, a function

$$\mathcal{F}(x, u) = \begin{pmatrix} x \\ u \end{pmatrix}^* \hat{F} \begin{pmatrix} x \\ u \end{pmatrix},$$

where $\hat{F} = \hat{F}^*$ is a bounded, linear, self-adjoint operator [an operator of the form $\mathcal{F}(x, u)$] on the Hilbert space $\mathbf{X} \times \mathbf{U}$ of elements $\begin{pmatrix} x \\ u \end{pmatrix}$ with scalar product

$$\begin{pmatrix} x_1 \\ u_1 \end{pmatrix}^* \begin{pmatrix} x_2 \\ u_2 \end{pmatrix} = x_1^* x_2 + u_1^* u_2.$$

We always assume that no points of the spectrum of A are on the imaginary axis. Let $\mathbf{X}$ and $\mathbf{U}$ be complex spaces. Then the expression

$$\mathcal{F}[(i\omega I_X - A)^{-1} bu, u] \tag{1.1}$$

(ω is a real number) has a meaning‡; it is clearly a quadratic form on $\mathbf{U}$ for each fixed ω, i.e.,

$$\mathcal{F}[(i\omega I_X - A)^{-1} bu, u] = u^* \Pi(i\omega) u, \tag{1.2}$$

where $\Pi(i\omega) : \mathbf{U} \to \mathbf{U}$ is a bounded self-adjoint operator. This operator will often be used in the sequel. If $\mathbf{X}$ and $\mathbf{U}$ are real spaces, (1.1) is not defined. In this case we write $\mathbf{X}_c$ and $\mathbf{U}_c$ for the complex extensions.††

The linear operators A and b can be extended in the obvious way to the complex spaces $\mathbf{X}_c$, $\mathbf{U}_c$, by putting $A(x_1 + ix_2) = Ax_1 + iAx_2$, $b(u_1 + iu_2) = bu_1 + ibu_2$. The form $\mathcal{F}(x, u)$ is extended similarly to $\mathbf{X}_c \times \mathbf{U}_c$; the same symbol will be used for the extended form. We note that $\mathcal{F}(x, u) = \mathcal{F}(x_1, u_1) + \mathcal{F}(x_2, u_2)$ (here $x = x_1 + ix_2 \in \mathbf{X}_c$, $u = u_1 + iu_2 \in \mathbf{U}_c$, $x_j \in \mathbf{X}$, $u_j \in \mathbf{U}$). With these definitions, the expression (1.1) has a meaning. The operator $\Pi(i\omega) : \mathbf{U}_c \to \mathbf{U}_c$ is again defined by (1.2). We shall assume that the indicated extensions of $\mathbf{X}$, $\mathbf{U}$ to $\mathbf{X}_c$, $\mathbf{U}_c$ have been made, and we write $\mathbf{X} = \mathbf{X}_c$, $\mathbf{U} = \mathbf{U}_c$, except when it is necessary to distinguish between $\mathbf{X}$, $\mathbf{U}$ and $\mathbf{X}_c$, $\mathbf{U}_c$.

† Hence the operation $*$ can be considered as a correspondence between X and the dual space X^*; this can be established by using the general form of a linear functional in Hilbert space.

‡ In (1.1) and below, we write I_Z and 0_Z for any Hilbert space Z, for the unit and zero operators in Z, and $\mathbf{0}_Z$ for the zero vector in Z.

†† The space $\mathbf{X}_c$ is the space of vectors $x = x_1 + ix_2$ for $x_1, x_2 \in X$ with the natural linear operations over the complex-number field and with the scalar product $(x_1 + ix_2)^*(x_1' + ix_2') = x_1^* x_1' + x_2^* x_2' + i(x_1^* x_2' - x_2^* x_1')$.

Let $Z = \{z\}$ be any Hilbert space of elements z and let $K = K^*$ be a bounded self-adjoint operator on Z. We use the following notation:

$K \geq 0$ indicates that $z^*Kz \geqslant 0, \forall z \in Z$,

$K > 0$ indicates that $z^*Kz > 0, \forall z \in Z, z \neq 0$,

$K \gg 0$ indicates that $\exists \delta > 0: z^*Kz \geqslant \delta |z|^2, \forall z \in Z$.

The relations

$$K \leqslant 0, K < 0, K \ll 0.$$

are defined similarly. A self-adjoint operator $K \gg 0$ ($K \ll 0$) will as usual be called positive-definite (negative-definite). If $K \gg 0$ ($K \ll 0$), the form z^*Kz will be called positive-definite (negative-definite). We write $K_1 \geq K_2$, $K_1 > K_2$, $K_1 \gg K_2$, respectively, if

$$K_1 - K_2 \geqslant 0, K_1 - K_2 > 0, K_1 - K_2 \gg 0.$$

Definition 1. An operator A is called Hurwitz, if its spectrum is located in the open left half-plane.

THEOREM 1. Let A be Hurwitz. For the existence of a bounded self-adjoint operator H such that $\operatorname{Re} x^*H(Ax + bu) + \mathscr{F}(x, u)$ is a positive-definite form in x and u, i.e.,

$$\operatorname{Re} x^*H(Ax+bu)+\mathscr{F}(x, u) \gg 0, \tag{1.3}$$

it is necessary and sufficient that, for some $\delta > 0$,†

$$\mathscr{F}[(i\omega I_X - A)^{-1}bu, u] \geqslant \delta |u|^2, \forall u, \forall \omega \in (-\infty, +\infty). \tag{1.4}$$

Suppose that (1.4) holds and that $\varkappa : U \to U$ is an arbitrary bounded linear operator such that $\mathscr{F}(0, u) = |\varkappa u|^2$. (Such an operator clearly exists.) Then there are bounded linear operators $H = H^* : X \to X$ and $h^* : X \to U$ such that

$$\operatorname{Re} x^*H(Ax+bu)+\mathscr{F}(x, u) = |\varkappa(u - h^*x)|^2 \quad (\forall x, u). \tag{1.5}$$

The operator $B = A + bh^* : X \to X$ has the following property: for $a \in X$, $|e^{Bt}a| \in L_2(0, \infty)$. The operators $H = H^*$ and h with the above properties are uniquely determined.

For the existence of operators $H = H^*$ and h such that (1.5) holds, it is necessary that (1.4) hold with $\delta = 0$, i.e., that $\Pi(i\omega) \geq 0$, $\forall \omega \in (-\infty, +\infty)$.

Let

$$\mathscr{F}(x, u) = \frac{1}{2} x^*F_0x + \operatorname{Re} x^*fu + u^*\Gamma u,$$

where $F_0 = F_0^* : X \to X$, $f : U \to X$, $\Gamma = \Gamma^* : U \to U$ are bounded linear operators. Letting $\omega \to \infty$ in (1.4) we find that $\Gamma = \Pi(i\infty) \gg 0$.

Comparison of operational coefficients for "similar" terms in (1.5) shows that (1.5) is equivalent to the following equations in $H = H^*$ and h:

$$\begin{cases} HA + A^*H + F_0 - 2h\Gamma h^* = 0, \\ Hb + f + 2h\Gamma = 0. \end{cases}$$

As in the finite-dimensional case [4-6, 13, 14], these equations are called Lur'e's equations. They can be reduced to a single quadratic equation in $H = H^*$ (by the substitution in the first equation of the value of h obtained from the second equation) or a single quadratic equation in h. To obtain the equation for h we introduce the operator $\mathfrak{A}$ defined by the relation

$$\mathfrak{A}(H) = \int_0^\infty e^{A^*t}He^{At}dt$$

and such that $\mathscr{H} = \mathfrak{A}(H)$ is the solution of the equation $A^*H + HA = -\mathscr{H}$. Then the quadratic equation for h is

$$2\mathfrak{A}(hh^*)b - 2h\Gamma + [f + \mathfrak{A}(F_0)b] = 0.$$

† If X and U are real spaces, the Re in (1.3) and (1.5) can be omitted.

It is in this form [for a special form $\mathscr{F}(x, u)$ and for $\mathbf{X} = R^n$, $\mathbf{U} \leftarrow R^1$] that a vector quadratic "resolving equation" is obtained in [12] for the construction of Lyapunov functions of a special type. [The operator H is given in terms of h by the formula $H = \mathfrak{A}(F_0) + 2\mathfrak{A}(hh^*)$.] Theorem 1 implies that the condition

$$\exists\delta>0:\ \Pi(i\omega)\geqslant\delta I_{\mathrm{U}}>0, \quad \forall\omega$$

is sufficient to ensure the solvability of Lur'e's equations; it also implies that, if $\Gamma \gg 0$, the condition $\Pi(i\omega) \geq 0$, $\forall\omega$ is necessary for the solvability of Lur'e's equations.

There are many different solutions $\{H, h\}$ of Lur'e's equations. (If $\mathbf{X} = R^n$, $\mathbf{U} = R^1$, there are 2^n solutions h, with multiplicity taken into account.) However, Theorem 1 implies that a solution $\{H, h\}$ with the property indicated in Theorem 1 (concerning $B = A + bh^*$) is unique. For the case in which $\mathbf{X}$ and $\mathbf{U}$ are finite-dimensional, a convenient algorithm is given in [4] for finding the required H and h ($\Gamma \geq 0$), and also in [29] (for the case of sign-definite Γ).†

In control problems the operator A is not usually Hurwitz. In this connection we introduce the following definition.

Definition 2. A pair of operators A, b is said to be stabilizable if there is a bounded linear operator such that $C = A + bc^*$ is Hurwitz.

Clearly, any pair A, b with a Hurwitz operator A is stabilizable. The fact that a pair A, b can be stabilized means that, for the system

$$\frac{dx}{dt}=Ax+bu$$

there is a feedback

$$u=c^*x$$

such that the resulting system is asymptotically stable. If $\mathbf{X}$ and $\mathbf{U}$ are finite-dimensional, any Kalman-controllable pair is stabilizable (cf. [3]). This also holds in the infinite-dimensional case (in which controllability is defined as in the finite-dimensional case). However, the only proof of this result known to the author is too long to be included in this article.

We consider the commonly encountered case in which $\mathbf{U}$ is finite-dimensional, $\mathbf{X}$ is infinite-dimensional, and the "nonempty" invariant subspace of A generated by the part of the spectrum in the right half-plane is finite-dimensional (we recall that no points of the spectrum of A are on the imaginary axis). Let A_1, b_1 be the finite-dimensional operators induced by A and b on the "unstable" invariant subspace. Let A_1, b_1 be Kalman controllable. It is easily verified that, under these circumstances, A, b is stabilizable (a control system is stabilizable if its unstable part is controllable).

The following result includes Theorem 1.

THEOREM 2 (a frequency theorem). In Theorem 1, let the condition that A be Hurwitz be replaced by the condition that no points of the spectrum of A be on the imaginary axis and that the pair A, b be stabilizable. Then all the assertions of Theorem 1 remain in force.

COROLLARY 1. Suppose that no points of the spectrum of A are on the imaginary axis and the pair A, b is stabilizable. Then the operational function $\Pi(i\omega)$ introduced above has the factorization

$$\Pi(i\omega)=\xi(i\omega)^*\Gamma\xi(i\omega), \tag{1.6}$$

where $\Gamma = \Pi(i\infty) \gg 0$,

$$\xi(i\omega)=I_u+h^*(A-i\omega I_{\mathbf{X}_c})^{-1}b, \tag{1.7}$$

and $h^*: \mathbf{U} \to \mathbf{U}$ is a bounded linear operator such that $B = A + bh^*$ has the property $|e^{Bt}a| \in L_2(0, \infty)$, $\forall a \in \mathbf{X}$.

In fact, if we use the vector x determined by the relation $Ax + bu = i\omega x$ in (1.5), we obtain an equality between quadratic forms which implies (1.6) and (1.7).

† Lur'e's equations with sign-definite Γ are encountered in the synthesis of optimal controls for linear differential games.

It can be shown that, if the pair (A, b) is assumed to be controllable [for $z \in X_c$, $z^*(A-i\omega I_X)^{-1}b = 0$, $\forall \omega \in (-\infty, +\infty)$ implies $z = 0$], there is an operator h such that (1.6), (1.7) are satisfied and B has the indicated property.

Theorems 1 and 2 are used in studying continuous control systems. For discrete systems described by equations $x_{n+1} = Ax_n + bu_n$, $n = 0, 1, 2, 3, \ldots$, (1.3) is replaced by

$$(Ax+bu)^*H(Ax+bu)-x^*Hx+\mathcal{F}(x, u)\gg 0, \tag{1.8}$$

and (1.5) is replaced by

$$(Ax+bu)^*H(Ax+bu)-x^*Hx+\mathcal{F}(x, u)=|\varkappa u-h^*x|^2. \tag{1.9}$$

For the discrete case the frequency theorem is as follows.

THEOREM 3. Suppose that either the spectrum of A is in $|\lambda| < 1$ or no points of the spectrum of A are on the unit circle, and the pair A, b is discretely stabilizable, i.e., there is a bounded linear operator $c: U \to X$ such that the spectrum of $C = A + bc^*$ is in $|\lambda| < 1$. For the existence of a bounded self-adjoint operator $H = H^*: X \to X$ satisfying (1.8), it is necessary and sufficient that, for some $\delta > 0$,

$$\mathcal{F}[(\lambda I_X-A)^{-1}bu, u]\geqslant\delta|u|^2, \forall u, \forall|\lambda|=1. \tag{1.10}$$

Now assume that (1.10) holds, and let λ_0, $|\lambda_0| = 1$ be arbitrary (if U and X are real, either $\lambda_0 = 1$ or $\lambda_0 = -1$). Let $\varkappa_0: U \to U$ be any bounded operator such that

$$\mathcal{F}[(\lambda_0 I_X-A)^{-1}bu, u]=|\varkappa_0 u|^2, \forall u \tag{1.11}$$

(such operators clearly exist). Then there are bounded linear operators

$$H=H^*: X\to X;\ h: U\to X,\ \varkappa: U\to U,$$

such that (1.9) holds and

$$\varkappa-h^*(\lambda_0 I_X-A)^{-1}b=\varkappa_0. \tag{1.12}$$

Relation (1.12) is a normalization condition. Namely, if x and u satisfy $Ax + bu = \lambda_0 x$, (1.9) implies that

$$\begin{gathered}\mathcal{F}(x, u)=|\varkappa u-h^*x|^2, \text{ i.e.}\\ \mathcal{F}[(\lambda_0 I_{X_c}-A)^{-1}bu, u]=|[\varkappa-h^*(\lambda_0 I_{X_c}-A)^{-1}bu]|^2.\end{gathered} \tag{1.13}$$

The operator for the form on the left side of this relation is known; it is used in the definition of $\varkappa_0$ [cf. (1.11)]. Relation (1.12) follows from (1.13).

COROLLARY. Let the conditions of Theorem 3 be satisfied and let $\Pi(\lambda): U_c \to U_c$ be the Hermitian operational function defined for all λ, $|\lambda| = 1$ by the relation

$$\mathcal{F}[(\lambda I_X-A)^{-1}bu, u)=u^*\Pi(\lambda)u, \quad \forall u.$$

Suppose that

$$\exists\delta>0,\ \forall|\lambda|=1: \Pi(\lambda)\geqslant\delta I_u.$$

Then $\Pi(\lambda)$ has the factorization

$$\Pi(\lambda)=\xi(\lambda)^*\xi(\lambda),$$

where

$$\xi(\lambda)=\varkappa+h^*(A-\lambda I_X)^{-1},$$

and $\varkappa: U \to U$ and $h: X \to U$ are bounded linear operators.

Remark. If (1.10) holds, in Theorem 3 there are also operators $H = H^*: X \to X$, $h: U \to X$, and $\varkappa: U \to U$ such that (1.9) and (1.12) hold, and $B = A + bh^*$ satisfies $\sum_{n=0}^{\infty} |B^n a|^2 < \infty$. Operators H, h, and $\varkappa$ with these properties are uniquely determined. We do not reproduce the proof because of lack of space. This proof is obtained by carrying out the proof of Theorem 3 in the same fashion as the proof of Theorems 1 and 2 (cf. §2, 3).

We now give a sketch of the proof of Theorem 1. It is shown in [4, 15] that the existence of the representation (1.5) leads to the solution of the problem of the existence of an optimal control. Consider the system

$$\frac{dx}{dt}=Ax+bu, \quad x(0)=a, \tag{1.14}$$

where A is Hurwitz. Let $L_2[\mathbf{U}, [0, \infty)]$ be the Hilbert space of functions $u(t): [0, \infty) \to \mathbf{U}$ such that

$$\|u(\cdot)\|^2=\int_0^\infty |u(t)|^2\, dt,$$

is finite, and also consider the Hilbert space $L_2[\mathbf{X}, [0, \infty)]$ of functions x(t) with values in $\mathbf{X}$. It is easily seen (cf. §1) that, for functions $u(\cdot) \in L_2[\mathbf{U}, [0, \infty)]$ (admissible controls), (1.6) has a solution x(t) and $x(\cdot) \in L_2[\mathbf{X}, [0, \infty)]$ [the first relation (1.6) is satisfied almost everywhere]. We call x(t) the solution corresponding to the control u(t). For admissible $u = u(\cdot)$, the functional

$$J(u)=\int_0^\infty \mathscr{F}[x(t), u(t)]\,dt \tag{1.15}$$

has a meaning, where $\mathscr{F}(x, u)$ is the form introduced above on $\mathbf{X} \times \mathbf{U}$ and x(t) is the solution corresponding to the control $u = u(t)$.

If $\mathbf{X}$ and $\mathbf{U}$ are real Hilbert spaces, an admissible control u(t) and the corresponding solution x are in $\mathbf{U}$ and $\mathbf{X}$ and not in their complex extensions.

An admissible control $u^0 = u^0(\cdot)$ is called optimal if, for any admissible control $u = u(\cdot)$,

$$J(u^0)\leqslant J(u). \tag{1.16}$$

Assume for the moment that the assertion of Theorem 1 holds and, in particular, that we have the representation (1.5) for some $H = H^*$ and h. Consider the function

$$V(x)=\frac{1}{2}x^*Hx. \tag{1.17}$$

Substituting any admissible control $u = u(t)$ and the corresponding solution $x = x(t)$ in (1.5) and integrating from $t = 0$ to $t = \infty$, we obtain

$$J(u)=V(a)+\int_0^\infty |\varkappa[u(t)-h^*x(t)]|^2\, dt. \tag{1.18}$$

Here we have used the relation $dV[x(t)]/dt = \mathrm{Re}\, x^*H(Ax + bu)$ and the fact that $|x(t)| \to 0$ for $t \to \infty$, which follows from the conditions $|x(t)| \in L_2(0, \infty)$, $|dx/dt| \in L_2(0, \infty)$.

By virtue of the assertion of Theorem 1 (concerning $B = A + bh^*$), the control $u = u^0(t)$ defined by the relation

$$u^0(t)=h^*x^0(t), \tag{1.19}$$

is admissible and has the form $u^0(t) = h^*e^{Bt}a$ [in (1.19) $x^0(t)$ is the corresponding solution]. It follows from (1.18) that (1.19) is an optimal control, and it also follows from (1.18) that

$$V(a)=J(u^0) \tag{1.20}$$

is the value of the functional (1.15) for the optimal control, i.e., the minimum possible value of (1.15).

Formula (1.19) realizes the synthesis of an optimal control. Formulas for the synthesis of optimal controls, i.e., formulas of the form $u^0(t) = f[x^0(t), t]$ (where $f[x, t]$ is independent of $a = x(0)$) expressing the value of an optimal control in terms of the value of the corresponding solution are convenient since they can be realized without the knowledge of the initial vector $a = x(0)$ and, in general, without the knowledge of the state of the system x(t) at the initial time. Thus, by realizing the formula (1.19), i.e., realizing the feedback $u = h^*x$, we obtain a system $dx/dt = Bx$ with the optimal property for arbitrary $x(0) = a \in \mathbf{X}$. The resulting control $u^0(t) = h^*e^{Bt}a$, as a function of the time, is automatically optimal for arbitrary $a \in \mathbf{X}$.

The relation between the optimal-control problem and the representation (1.5) described above is used in [4, 15, 17, 18] for the synthesis of an optimal control in the finite-dimensional case (when the spaces $\mathbf{X}$ and $\mathbf{U}$ are finite-dimensional). Below, in the infinite-dimensional case, we first proceed in the opposite direction. Theorem 1 will be obtained below as a corollary of the solution of the synthesis problem, and this solution will not be based on the representation (1.5) (as it was above). Subsequently, in the solution of other synthesis problems, the representation (1.5), thus proved, will be used. To recapitulate: we first prove the existence and uniqueness of an optimal control for the simple problem considered above [under the assumption that (1.4) holds], independently of the existence of the representation (1.5). We prove that the function given by (1.20) is a quadratic form. By the same token (1.17) will determine the operator $H = H^*$. It will be established that the optimal control can be expressed linearly in terms of the corresponding solution, i.e., (1.19) holds with some bounded linear operator $h: U \to X$. Finally, it will be proved that, for the H and h we have found, (1.5) holds and H and h have the properties indicated in Theorem 1.

We now consider optimal-control problems that are somewhat more complex than the auxiliary problem introduced above. Let the system be described by (1.14) where A is not necessarily Hurwitz. Consider the set $\mathfrak{M}_a$ of functions $u(t): [0, \infty) \to U$ possessing the property $u(\cdot) \in L_2[U, [0, \infty)]$, and let $x(\cdot) \in L_2[X[0, \infty)]$, where $x(t)$ is the solution of (1.6) with $u = u(t)$. The stabilizability of the pair A, b means that $\mathfrak{M}_a$ is not empty.

If $u(\cdot) \in \mathfrak{M}_a$, the functional (1.15) is finite. A control $u^0(\cdot) \in \mathfrak{M}_a$ is called optimal in $\mathfrak{M}_a$ if (1.16) holds for all $u(\cdot) \in \mathfrak{M}_a$.

THEOREM 4. Suppose that the spectrum of A has no points on the imaginary axis, the pair A, b is stabilizable, and (1.4) holds, i.e.,

$$\exists \delta>0,\ \forall \omega \in (-\infty, \infty): \ \Pi(i\omega) \geqslant \delta I_U. \tag{1.21}$$

Then there is an optimal control in $\mathfrak{M}_a$ that is unique and has the form $u^0(t) = h^* x^0(t)$, where $x^0(t)$ is the corresponding solution and h is the operator defined in Theorem 2, i.e., the operator such that (1.5) holds and $B = A + bh^*$ has the properties described in Theorem 1. Moreover, $J(u^0) = (1/2) a^* H a$, where $H = H^*$ is the operator in (1.5).

For the existence of an optimal control it is necessary that $\Pi(i\omega) \geq 0$, $\forall \omega \in (-\infty, +\infty)$. If the last condition is violated, $\inf_{u \in \mathfrak{M}_a} J(u) = -\infty$.

Hence the solution is again given by (1.19), which determines the synthesized optimal control (simultaneously for all $a \in X$, i.e., for all classes $\mathfrak{M}_a$).† Relation (1.15) implies that, as a function of the time, the optimal control has the form

$$u^0(t) = h^* e^{Bt} a.$$

To illustrate the above method in a more complicated case, we consider the analogous problem on a finite time interval with one fixed end. Let the functional determining the quality of the control be

$$J_T(u) = \int_0^T F[x(t), u(t)]\,dt, \tag{1.22}$$

where $F(x, u)$ is the form defined above and $x(t)$ is the corresponding solution of (1.14).

We now introduce the space $L_2[U, [0, T]]$ of admissible controls $u(t): [0, T] \to U$ for which $|u(t)| \in L_2(0, T)$; the space $L_2[X, [0, T]]$ is defined similarly. A control u^0 is now called optimal if

$$J_T(u^0) \leqslant J_T(u), \quad \forall u(\cdot) \in L_2[U, [0, T]]. \tag{1.23}$$

THEOREM 5. Suppose that there are no points of the spectrum of A on the imaginary axis, that the pair A, b is stabilizable, and that (1.21) holds. Let $H = H^*$, h, and $\varkappa$ be the unique operators for which (1.5) holds and let $B = A + bh^*$ have the properties described in Theorem 1. Assume that the inverse H^{-1} exists and is bounded, and that the operator

$$W(t) = H^{-1} - \int_0^{T-t} e^{Bs} b \Gamma^{-1} b e^{B^* s}\,ds \tag{1.24}$$

† Uniqueness implies the coincidence (almost everywhere) of the values of the controls as functions of the time. Of course, in general there may be another synthesis of the optimal control, i.e., we can have $u^0 = \varphi[x^0]$, where $\varphi(x) \neq h^* x$.

has a bounded inverse for $0 \le t \le T$.† Then there is an optimal control minimizing $J_T[u]$ given by the formula

$$u^0(t) = g(t)^* x^0(t), \tag{1.25}$$

where $x^0(t)$ is the corresponding solution and

$$g(t) = h + \frac{1}{2} e^{B^*(T-t)} W(t)^{-1} e^{B(T-t)} b \Gamma^{-1}. \tag{1.26}$$

Moreover,

$$J(u^0) = \frac{1}{2} a^* [H - e^{B^*T} W(0)^{-1} e^{BT}] a. \tag{1.27}$$

If $T > 0$ is sufficiently large, the frequency condition (1.2) is "almost necessary" for the existence of an optimal control; if the condition $\mathcal{F}[(i\omega I - A)^{-1} bu, u] \ge 0$ is violated, i.e., $\exists \omega_0 \in R'$, $\exists v \in \mathbf{U}: \mathcal{F}[(i\omega_0 I - A)^{-1} \cdot bv, v] < 0$, there is no optimal control and $\inf J = -\infty$.

Since the operator (1.26) is independent of $a = x(0)$, (1.25) realizes the synthesis of the optimal control.

Theorems 4 and 5 are proved in §4 by the method described above, which is based on the use of (1.5). Many other problems in the synthesis of optimal controls in various problems of the minimization of quadratic functionals can be solved similarly. For example, the results in [15, 17, 28] can be converted to apply to the infinite-dimensional case.

§2. Proof of Theorem 1

Consider the system (1.14) in which A is Hurwitz. For arbitrary $a \in \mathbf{X}$ and an arbitrary $u(t): [0, \infty) \to \mathbf{U}$ such that $|u(t)| \in L(0, T)$, $\forall T > 0$, there is a solution $x(t)$ [unique in the class of absolutely continuous functions $x(t): (0, \infty) \to \mathbf{X}$] given by the formula

$$x(t) = e^{At} a + \int_0^t e^{A(t-\tau)} bu(\tau)\, d\tau. \tag{2.1}$$

Since A is bounded and Hurwitz, there are constants $C > 0$ and $\alpha > 0$ such that

$$|e^{At}| \le C e^{-\alpha t}. \tag{2.2}$$

This well-known inequality can be derived, for example, from the formula

$$e^{At} = \int_{\mathfrak{K}} e^{\zeta t} (\zeta I - A)^{-1} d\zeta,$$

where $\mathfrak{K}$ is a closed contour containing the spectrum of A, and can thus be assumed to lie in the region $\operatorname{Re} \zeta \le -\alpha < 0$. (This last condition determines α.)

Relations (2.1) and (2.2) imply that, if $u(\cdot) \in L_2[\mathbf{U}, [0, \infty)]$, we have $x(\cdot) \in L_2[\mathbf{X}, [0, \infty)]$.

Consider the problem posed in §1 concerning the existence of an optimal control $u^0(\cdot) \in L_2[\mathbf{U}, [0, \infty)]$ minimizing (1.15).

Our first object is to prove the existence of an optimal control [under the condition (1.4)].

LEMMA 1. Let $\mathfrak{U} = \{u\}$ be any Hilbert space (real or complex) and let $J(u)$ be any quadratic function on $\mathfrak{U}$ i.e.,‡

$$J(u) = u^* R u + 2 \operatorname{Re}(r^* u) + \rho, \tag{2.3}$$

where R is a bounded self-adjoint operator on $\mathfrak{U}$, $r \in \mathfrak{U}$, and ρ is a real number. For there to be a point $u^0 \in \mathfrak{U}$ satisfying $J(u^0) \le J(u)$ for arbitrary $u \in \mathfrak{U}$ it is necessary and sufficient that either

$$R \gg 0, \tag{2.4}$$

†As above, the operator $\Gamma: \mathbf{U} \to \mathbf{U}$ in (1.24) is of the form $F(0, u)$. Relation (2.21) implies that $\Gamma \gg 0$. If $H \gg 0$, the existence of a bounded inverse $W(t)^{-1}$, $\forall 0 \le t \le T$ is equivalent to the existence of a bounded inverse $W(T)^{-1}$. If $H \ll 0$, the bounded inverse $W(t)^{-1}$, $\forall 0 \le t \le T$ obviously exists.

‡If $\mathfrak{U}$ is a real Hilbert space, $\operatorname{Re}(r^*, u) = r^* u$ in (2.3).

or

$$R \geqslant 0 \tag{2.5}$$

and that the equation

$$Ru^0 + r = 0 \tag{2.6}$$

have a solution $u^0 \in \mathfrak{U}$. Any solution u^0 of (2.6) is satisfactory [it is unique if (2.4) holds]. The minimum $J(u^0)$ of the functional is determined by the relations

$$J(u^0) = \rho + r^* u^0 = \rho - (u^0)^* R u^0. \tag{2.7}$$

If (2.5) does not hold, $\inf J(u) = -\infty$.

Proof. If (2.5) does not hold, i.e., if there is $v \in \mathfrak{U}$ for which $v^*Rv < 0$, then $J(\lambda v) \to -\infty$ for $\lambda \to +\infty$; hence $\inf J(u) = -\infty$. Thus (2.5) is a necessary condition for the existence of the point u^0. Relations (1.16) and $J(u) = J(u^0) + 2\mathrm{Re}\,(Ru^0 + r,\ u-u^0) + O(|u-u^0|^2)$ imply that $\mathrm{Re}\,(Ru^0 + r,\ u-u^0) = 0$, $\forall u \in \mathfrak{U}$ and (2.6) follows.

Suppose that (2.5) holds and (2.6) has a solution u^0 (for example if $R \gg 0$). It follows from (2.6) and (2.3) that (2.7) holds and

$$J(u) = (u-u^0)^* R (u-u^0) + J(u^0). \tag{2.8}$$

Relations (2.5) and (2.8) imply $J(u) \geq J(u^0)$. If (2.4) holds, a bounded inverse R^{-1} exists, i.e., u^0 is unique.

Now consider the functional (1.15). Formula (2.1) can be written

$$x(\cdot) = x_a(\cdot) + hu(\cdot),\ \text{where } x_a(t) = e^{At}a, \tag{2.9}$$

and h is a bounded linear operator on $L_2[\mathbf{U}, [0, \infty)]$ into $L_2[\mathbf{X}, [0, \infty)]$. Hence (1.15) is a quadratic functional on $\mathfrak{U} = L_2[\mathbf{U}, [0, \infty)]$, i.e., it can be expressed in the form (2.3). The corresponding quadratic form

$$J_0(u) = u(\cdot)^* R u(\cdot) \tag{2.10}$$

is obtained by replacing $x_a(\cdot)$ in (2.9) by zero, i.e., for $a = 0_X \in \mathbf{X}$. Hence

$$J_0(u) = \int_0^\infty \mathcal{F}[x(t), u(t)]\,dt, \tag{2.11}$$

where x(t) is determined from u(t) by the system

$$\frac{dx}{dt} = Ax + bu, \quad x(0) = 0_x. \tag{2.12}$$

LEMMA 2. Let $\mathfrak{U} = L_2[\mathbf{U}, [0, \infty)]$, let A be Hurwitz, and let J(u) be the quadratic functional (1.5) on $\mathfrak{U}$. Let $\Pi(i\omega)$ be the bounded self-adjoint operator on U_C defined by (1.2),† i.e.,

$$\mathcal{F}[(i\omega I_X - A)^{-1} bu, u] = u^* \Pi(i\omega) u.$$

We assume that $\exists \delta > 0$, $\forall \omega : \Pi(i\omega) \geq \delta I_U$. Then $R \gg 0$ in the representation (2.3).

Proof. Consider the Hilbert space $L_2[\mathbf{X}_C, (-\infty, +\infty)]$ of functions $x(t) : (-\infty, +\infty) \to \mathbf{X}_C$ such that $|x(t)| \in L_2(-\infty, +\infty)$, and also the Hilbert space $L_2[\mathbf{U}_C, (-\infty, +\infty)]$ defined analogously. Let $L_2[\mathbf{X}_C, (-\infty, +\infty)]$ be the space of Fourier transforms

$$\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega t} x(t)\,dt$$

The integral converges as a principal value in the mean-square sense, i.e.,

$$\int_{-\infty}^{\infty} \left| \tilde{x}(\omega) - \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-i\omega t} x(t)\,dt \right|^2 d\omega \to 0 \ \text{ for } \ \Lambda \to \infty.$$

† We recall (cf. §1) that here U_C is the complex extension of the space $\mathbf{U}$. (If $\mathbf{U}$ is a complex Hilbert space $U_C = \mathbf{U}$.) Below, $\mathbf{X}_C$ is the complex extension of $\mathbf{X}$.

Let $\tilde{L}_2[U_C, (-\infty, +\infty)]$ be the space of functions $\tilde{u}(\omega)$, $-\infty < \omega < +\infty$ with values in U_C defined similarly.

We set up a relation between each function $x(\cdot) \in L_2[X, [0, \infty)]$ and the function $x(\cdot) \in L_2[X, (-\infty, +\infty)]$ denoted by the same symbol; $x(t) = 0_X$ for $t < 0$. Imbed the space $L_2[U_C, (0, \infty)]$ in $L_2[U_C, (-\infty, +\infty)]$ similarly. Let $x(\cdot) \in L_2[X, [0, \infty)]$ and $u(\cdot) \in L_2[U, [0, \infty)]$ satisfy (2.12) [we assume that $x(t)$ is absolutely continuous and that the first relation (2.12) holds almost everywhere]. Then the functions $x(\cdot) \in L_2[X_C, (-\infty, +\infty)]$, $u(\cdot) \in L_2[U, (-\infty, +\infty)]$ also satisfy (2.12), and $x(t)$ is absolutely continuous on $(-\infty, +\infty)$. Hence their Fourier transforms

$$\tilde{x}(\cdot) \in \tilde{L}_2[X_c, (-\infty, +\infty)] \text{ and } \tilde{u}(\cdot) \in \tilde{L}_2[U_c, (-\infty, +\infty)]$$

satisfy

$$i\omega\tilde{x} = A\tilde{x} + b\tilde{u}. \tag{2.13}$$

The quadratic form (2.11) can be written

$$J_0(u) = \int_{-\infty}^{\infty} \mathcal{F}[x(t), u(t)]dt = \int_{-\infty}^{\infty} \mathcal{F}[\tilde{x}(\omega), \tilde{u}(\omega)]d\omega. \tag{2.14}$$

[Here we have used Parseval's formula and the boundedness of the operator of the form $\mathcal{F}(x, u)$.] It follows from (2.13), (2.14), and the definition of $\Pi(i\omega)$ that

$$J_0(u) = \int_{-\infty}^{\infty} \tilde{u}(\omega)^* \Pi(i\omega)\tilde{u}(\omega)d\omega. \tag{2.15}$$

Hence, by virtue of the condition of Lemma 2,

$$J_0(u) = u(\cdot)^* Ru(\cdot) \geqslant \delta \int_{-\infty}^{\infty} |\tilde{u}(\omega)|^2 d\omega = \delta \int_{-\infty}^{\infty} |u(t)|^2 dt = \delta \int_{-\infty}^{\infty} |u(t)|^2 dt = \delta \|u(\cdot)\|^2,$$

i.e., $R \geq \delta I_U$, which is the required result.

Lemmas 1 and 2 imply the validity of the assertions made above.

LEMMA 3. If (1.4) holds, there is an optimal control minimizing (1.7).

The following lemma is used only in the proof of Theorem 2 (and not in the proof of Theorem 1), but it is more convenient to state it here.

LEMMA 4. Suppose that A is Hurwitz and $\Pi(i\omega)$ is defined as in Lemma 2. If the condition $\Pi(i\omega) \geq 0$ is not satisfied, i.e., if

$$\exists \omega_0 \in R^1, \ \exists v \in U: \ v^* \Pi(i\omega_0) v < 0, \tag{2.16}$$

then

$$\inf J(u) = -\infty \tag{2.17}$$

(here the inf is over the set of admissible controls), i.e., there is no optimal control.

Proof. Lemma 1 implies that it is sufficient to find a control $u(\cdot) \in L_2[U, [0, \infty)]$ for which $J_0(u) < 0$. In this case (2.17) follows from Lemma 1. We first assume that X and U are complex spaces.

Let

$$\left.\begin{aligned} u(t) &= e^{i\omega_0 t} v \quad \text{for } 0 \leqslant t \leqslant T, \\ u(t) &= 0 \quad \text{for } t \geqslant T. \end{aligned}\right\} \tag{2.18}$$

The corresponding solution is

$$x(t) = e^{i\omega_0 t} x_0 - e^{At} x_0 \quad \text{for } 0 \leqslant t \leqslant T,$$
$$x(t) = e^{A(t-T)} \ (e^{i\omega_0 T} x_0 - e^{AT}, \ x_0) \quad \text{for } t \geqslant T$$

where

$$x_0 = (i\omega_0 I - A)^{-1} bv.$$

For $T \to \infty$ we have

$$\int_T^{\infty} \mathcal{F}[x(t), u(t)]dt = \int_0^{\infty} \mathcal{F}[x(T+s), 0_U]ds \to \int_0^{\infty} \mathcal{F}[e^{As} x_0, 0_U]ds;$$

hence this integral is a bounded function of T for $T > 0$.

Since

$$\begin{pmatrix} x \\ u \end{pmatrix} = e^{i\omega_0 t}\begin{pmatrix} x_0 \\ v \end{pmatrix} + \begin{pmatrix} e^{-Atx_0} \\ 0_U \end{pmatrix}$$

on [0, T], the fact that A is Hurwitz implies that

$$\int_0^T \mathcal{F}[x(t), u(t)]dt = \int_0^T \mathcal{F}(x_0, v)dt + O(1) = v^*\Pi(i\omega_0)vT + O(1).$$

Hence

$$J_0(u) = \int_0^T \mathcal{F}[x(t), u(t)]dt + \int_T^\infty \mathcal{F}[x(t), u(t)]dt < 0$$

for sufficiently large $T > 0$. Thus (2.18) furnishes the required control if T is large enough, and our assertion is proved for the complex case.

Now let **X**, **U** be real spaces, so that A, b and the operator of the form $\mathcal{F}(x, u)$ are real. (These operators are on $\mathbf{X}_C$, $\mathbf{U}_C$ and the fact that A is real implies that, for $x \in \mathbf{X}_C$ we have $\overline{Ax} = A\bar{x}$, where $\bar{x}$ is the complex conjugate of x: $\bar{x} = x' - ix''$ if $x = x' + ix''$, $x' \in \mathbf{X}$, $x'' \in \mathbf{X}$). The foregoing construction of u(t) and x(t) is now not satisfactory, since their values are in $\mathbf{U}_C$ and $\mathbf{X}_C$ and not in **U** and **X**. Let

$$x(t) = x_1(t) + ix_2(t),\ u(t) = u_1(t) + iu_2(t),$$

where $x_j(t) \in \mathbf{X}$, $u_j(t) \in \mathbf{U}$. Clearly each of the pairs $\{x_1, u_1\}$, $\{x_2, u_2\}$ satisfies (2.12). Since $\mathcal{F}(x, u) = \mathcal{F}(x_1, u_1) + \mathcal{F}(x_2, u_2)$ and $J_0(u) = J_0(u_1) + J_0(u_2) < 0$, either $J_0(u_1) < 0$ or $J_0(u_2) < 0$. Hence there is a real pair x_j, u_j (where either $j = 1$ or $j = 2$) such that $J_0(u_j) < 0$, which is the required result, and Lemma 4 follows.

Lemma 4 can also be proved by using a variant of the Payley–Wiener theorem.

Remark. It follows from the results already proved that, if (2.16) holds, there is a number $T > 0$ and a function $u(t) \in L_2[\mathbf{U}, [0, T]]$ such that

$$\int_0^T \mathcal{F}[x(t), u(t)]dt < 0, \tag{2.19}$$

where x(t) is the corresponding solution of the equation $dx/dt = Ax + bu$, $x(0) = 0_X$.

Lemma 1 and (2.19) imply that if (2.16) holds and $T > 0$ is large enough the minimization of the functional (1.22) leads to $\inf J(u) = -\infty$, where the lower bound is over all $u(\cdot) \in L_2[\mathbf{U}, [0, T]]$. We have thus proved the final part of Theorem 5 (the necessity of the frequency condition if $T > 0$ is sufficiently large).

We now turn to our problem. Consider again (1.14) and assume that the conditions of Theorem 1 are satisfied. Results already proven imply that, for arbitrary $a \in \mathbf{X}$, there is an optimal control u^0. We write this control and the corresponding solution of (1.14) as

$$u^0(t, a),\ x^0(t, a). \tag{2.20}$$

The value of $J(u^0)$ naturally depends on a, and we write

$$V(a) = J(u^0) \tag{2.21}$$

and consider the dependence of the function $V(a): \mathbf{X} \to R^1$ on a.

LEMMA 5. Let (1.4) be satisfied. The function $V(a)$ defined by (2.21) is a quadratic form, i.e., there is a bounded, linear, self-adjoint operator $H: \mathbf{X} \to \mathbf{X}$ such that

$$V(a) = \frac{1}{2}a^*Ha \tag{2.22}$$

(the coefficient 1/2 is introduced for convenience in later calculations).

Proof. In the definition of J(u) above, the vector a in (1.14) was fixed and J(u) was defined on $L_2[\mathbf{U}, [0, \infty)]$. Now, with a taking more than one value, we assume that it is defined on $\mathbf{X} \times L_2[\mathbf{U}, [0, \infty)]$. It follows from (2.1) and (1.7) that J(u) is a quadratic form in a and $u(\cdot)$ with a bounded operator. This means that the operator $R = R^*$ in (2.3) is independent of a and the vector $r \in L_2[\mathbf{U}, [0, \infty)]$ depends linearly on a:

$$r=Sa, \tag{2.23}$$

where $S: X \to L_2[\mathbf{U}, [0, \infty)]$ is a bounded linear operator, and $\rho = \rho(a)$ is a quadratic form with a bounded operator. Lemma 2 implies that $R \gg 0$, and Lemma 1 implies that the optimal control is unique and given by (2.6). Hence

$$u^0(\cdot)=-R^{-1}Sa, \tag{2.24}$$

where $R^{-1}S: \mathbf{X} \to L_2[\mathbf{U}[0, \infty)]$ is a bounded linear operator. It follows from (2.7) and (2.24) that

$$J(u^0)=\rho(a)+r^*u^0=\rho(a)-a^*S^*RSa$$

is a quadratic form with a bounded operator, and Lemma 5 follows.

We now investigate the functions (2.20). We shall prove that the optimal control and the corresponding solution for the functional

$$J(u,t_0)=\int_{t_0}^{\infty}\mathcal{F}(x,u)\,dt, \tag{2.25}$$

where $u = u(t) \in L_2[\mathbf{U}, [t_0, \infty)]$ and $x(t)$ is the solution of

$$\frac{dx}{dt}=Ax+bu, \qquad x(t_0)=a'. \tag{2.26}$$

can be expressed in terms of these functions. (The problem considered previously is obtained from this problem by putting $t_0 = 0$.) In (2.26) a' is an arbitrary (fixed) vector of $\mathbf{X}$. In fact the substitutions

$$t=t'+t_0,\ x(t)=x'(t-t_0),\ u(t)=u'(t-t_0)$$

reduce our new problem to the original problem. Hence, if (1.4) holds, there is an optimal control [for which $J(u) \geq J(u^0)$, $\forall u \in L_2[\mathbf{U}, (t_0, \infty)]$]. This control and the corresponding solution are

$$u^0(t-t_0,a'),\quad x^0(t-t_0,a') \tag{2.27}$$

and Lemma 3 implies that the optimal control is unique.

LEMMA 6. The functions (2.20) possess the semigroup property

$$\begin{cases} u^0(t+s,a)=u^0[t,x^0(s,a)], \\ x^0(t+s,a)=x^0[t,x^0(s,a)], \qquad \forall s\geqslant 0,\ t\geqslant 0, \end{cases} \tag{2.28}$$

the first relation here holding almost everywhere on $0 \leq t < \infty$.

Proof. Consider the inequality (1.16). From what we have already proven we conclude that $u^0 = u^0(t, a)$ is the unique function (considered as an element of $L_2[\mathbf{U}, [0, \infty)]$) for which (1.16) holds with an arbitrary admissible function $u = u(t)$. Suppose that an admissible $u(t)$ satisfies

$$u(t)=u^0(t)\quad \text{for}\quad 0\leqslant t\leqslant t_0.$$

Then (1.16) becomes an analogous inequality with the functional (1.15) (with lower limit of integration $t = 0$) replaced by the functional (2.25) (with lower limit of integration $t = t_0$). The pairs $\{u(t), x(t)\}$ and $\{u^0(t, a), x^0(t, a)\}$ satisfy (2.26) with $a' = x'(t_0, a)$. Hence $u^0(t, a)$ and $x^0(t, a)$ are the optimal control and the corresponding solution for the functional (2.25) with the condition (2.26). As already proved, they are also the functions (2.27) with $a' = x^0(t_0, a)$, and they are unique. Hence

$$u^0(t,a)=u^0[t-t_0,x^0(t_0,a)],\quad x^0(t,a)=x^0[t-t_0,x^0(t_0,a)].$$

These relations are converted into relations (2.28) if t_0 is replaced by s and t is replaced by $t + t_0$. They were obtained as equalities for elements in $L_2[\mathbf{U}[0, \infty)]$ and $L_2[\mathbf{X}, [0, \infty)]$. Since $x(t)$ is absolutely continuous, the second equation (2.28) holds for all t, $0 \leq t < \infty$, while the first holds only almost everywhere.

Remark 1. Our proof is rather general. Lemma 6 (which is well known) also holds when (1.6) is replaced by a nonlinear equation

$$\frac{dx}{dt}=f(x,u),\qquad x(0)=a;$$

[here $u(t) \in D \subset U$, $\forall t \geq 0$] and $\mathcal{F}(x, u)$ in (1.15) is an arbitrary function. It is only necessary that, besides having the natural conditions [ensuring that the functional (1.15) is finite and certain other properties], the optimal control be unique. We now consider our problem. The following lemma is a corollary of a well-known theorem in dynamic programing (the author regrets that he cannot supply a precise reference). For the sake of completeness we give a proof.

LEMMA 7. Let $H = H^*$ be the bounded self-adjoint operator defined in Lemma 5, and let

$$\mathcal{G}(x, u) = \operatorname{Re} x^* H(Ax + bu) + \mathcal{F}(x, u). \tag{2.29}$$

For arbitrary $a \in X$, the following relations are satisfied almost everywhere on $[0, \infty)$ [namely for values of t for which $dx^0(t, a)/dt$) exists†]:

$$\mathcal{G}[x^0(t, a), v] \geqslant 0, \quad \forall v \in U, \tag{2.30}$$

$$\mathcal{G}[x^0(t, a), u^0(t, a)] = 0. \tag{2.31}$$

Proof. Let $x^0 = x^0(t, a)$ and let t_0 be a value of t for which dx^0/dt exists. Take any number and any $v \in U$. Let $x^0(t_0, a) = x_0'$ and $x'(t)$ be the solution of (2.26) on the interval $[t_0, t_0 + \delta)$ for $a' = x_0'$ and the control $u(t) = v$. Let $x'(t_0 + \delta) = x_1'$. For (1.14) consider the control

$$u(t) = \begin{cases} u^0(t, a) & \text{on } [0, t_0), \\ v & \text{on } [t_0, t_0+\delta), \\ u^0(t - t_0 - \delta, x_1') & \text{on } [t_0 + \delta, \infty). \end{cases}$$

The corresponding solution is

$$x(t) = \begin{cases} x^0(t, a) & \text{on } [0, t_0) \\ x'(t) & \text{on } [t_0, t_0+\delta) \\ x^0(t - t_0 - \delta, x_1') & \text{on } [t_0 + \delta, \infty) \end{cases}$$

[x(t) is clearly continuous]. For this control, the functional (1.15) takes the value

$$J(u) = \int_0^{t_0} \mathcal{F}[x^0(t, a), u^0(t, a)]dt + \int_{t_0}^{t_0+\delta} \mathcal{F}[x'(t), v]dt + V(x_1'). \tag{2.32}$$

Here we have used the relation

$$\int_{t_0+\delta}^{\infty} \mathcal{F}[x^0(t - t_0 - \delta, x_1'), u^0(t - t_0 - \delta, x_1')]dt = V(x'),$$

which follows from the definition (2.21) of $V(a)$. For the optimal control $u^0 = u^0(t, a)$ we have

$$J(u^0) = \int_0^{t_0} \mathcal{F}[x^0(t, a), u^0(t, a)]dt + V[x^0(t_0, a)]. \tag{2.33}$$

This relation is a consequence of (2.28) since, by virtue of (2.28),

$$\int_{t_0}^{\infty} \mathcal{F}[x^0(t, a), u^0(t, a)]dt = \int_0^{\infty} \mathcal{F}[x^0(t_0 + s, a), u^0(t_0 + s, a)]dt = \int_0^{\infty} \mathcal{F}[x^0(s, x_0'), u^0(s, x_0')]ds = V(x_0'),$$

where $x' = x(t_0, a)$. Relations (2.32), (2.33), and the inequality $J(u) \geq J(u^0)$ imply

$$V(x') - V(x_0') + \int_{t_0}^{t_0+\delta} \mathcal{F}[x'(t), v]dt \geqslant 0. \tag{2.34}$$

Since $x_1' = x'(t_0 + \delta)$, $x_0' = x'(t_0)$, where $x'(t)$ is the solution of (2.27) with $u(t) = v$, we have

$$\frac{V(x_1') - V(x_0')}{\delta} \to \left. \frac{dV[x'(t)]}{dt} \right|_{t=t_0} = 2 \operatorname{Re} x_0' H(Ax_0' + bv)$$

for $\delta \to 0$. Dividing (2.34) by $\delta > 0$ and taking the limit for $\delta \to 0$, we obtain (2.30) for $t = t_0$. Differentiation of both sides of (2.33) with respect to t_0 [the left side of this relation, which is equal to $V(a)$, is independent of t_0] yields (2.32) for $t = t_0$ and the lemma follows.

† For $t = 0$ only the right derivative need exist.

LEMMA 8. Let

$$\mathscr{G}_0(u) = u^*\Gamma u + 2\,\mathrm{Re}\, g^* u + \gamma \tag{2.35}$$

be a quadratic function on $\mathbf{U}$ (so that $\Gamma = \Gamma^*$ is a bounded operator, $g \in \mathbf{U}$, and ρ is a real number). Suppose that $\Gamma \gg 0$, $\mathscr{G}_0(u) \geq 0$, $\forall u \in \mathbf{U}$, and that there is $u^0 \in \mathbf{U}$ for which $\mathscr{G}_0(u^0) = 0$. Let $\varkappa$ be an arbitrary bounded linear operator satisfying $\varkappa^*\varkappa = \Gamma$. Then

$$\mathscr{G}_0(u) = |\varkappa(u-u^0)|^2, \tag{2.36}$$

and

$$u^0 = -\Gamma^{-1} g. \tag{2.37}$$

Proof. The operator $\varkappa^{-1}$ clearly exists and is bounded. It follows from (2.35) that

$$\mathscr{G}_0(u) = \gamma - g^*\Gamma^{-1} g + |\Gamma^{1/2}(u + \Gamma^{-1} g)|^2. \tag{2.38}$$

Since $\mathscr{G}_0(u) \geq 0$, $\forall u$, we have $\gamma - g^*\Gamma^{-1}g \geq 0$. Since $\mathscr{G}_0(u^0) = 0$, $\gamma - g^*\Gamma^{-1}g = 0$, (2.37) holds, and the representation (2.38) is converted into (2.36).

LEMMA 9. Suppose that (1.4) holds and $H = H^*$ is the operator defined in Lemma 5. Let

$$\mathscr{F}(x, u) = x^* F_0 x + 2\,\mathrm{Re}\, x^* f u + u^*\Gamma u.$$

($F_0 = F_0^*$ and f are bounded operators, $F_0 : \mathbf{X} \to \mathbf{X}$, $f : \mathbf{U} \to \mathbf{X}$). Let

$$h = Hb + f \tag{2.39}$$

(so that $h^* : \mathbf{X} \to \mathbf{U}$ is a bounded linear operator) and let $\varkappa$ be any bounded linear operator such that $\varkappa^*\varkappa = \Gamma$. Then the form (2.29) has the representation

$$\mathscr{G}(x, u) = |\varkappa(u - h^* x)|^2, \tag{2.40}$$

and, for almost all t, the functions $u^0(t, a)$ and $x^0(t, a)$ satisfy

$$u^0(t, a) = h^* x^0(t, a). \tag{2.41}$$

Proof. Lemma 7 implies that the form (2.29) satisfies (2.30) and (2.31). Apply Lemma 8 to the form $\mathscr{G}_0(u) = \mathscr{G}[x^0(t, a), u]$ (t and a are fixed). Comparison of (2.29) and (2.35) yields

$$g = f^* x^0(t, a) + b^* H x^0(t, a).$$

Relation (2.37) implies (2.41), where h is given by (2.39). Relation (2.41) is obtained for all t for which $dx^0(t, a)/dt$ exists (cf. Lemma 7). Since $u^0(t, a)$ can vary on a set of zero measure, we assume that (2.41) holds everywhere. It follows from (2.36) and (2.41) that

$$\mathscr{G}[x^0(t, a), u] = |\varkappa[u - x^0(t, a)]|^2$$

for all t. Since $a \in \mathbf{X}$ is arbitrary, we obtain (2.40) for $t = 0$, and this completes the proof of the lemma.

We now complete the proof of Theorem 1. Let (1.4) hold. Lemma 9 implies (1.5), and (3.41) and (1.14) imply that $x^0(t, a)$ satisfies $dx/dt = Bx$ almost everywhere and that $|x^0(t, a)| \in L_2(0, \infty)$. Hence $|e^{Bt} a| \in L_2(0, \infty)$, $\forall a \in \mathbf{X}$, which is the assertion of the theorem.

We now prove that operators H, h with the indicated properties are unique. Suppose that, besides the pair H, h, there is a pair H_1, h_1 of operators with the same properties for which (1.5) holds. As proved in §1, the control $u^0(t, a)$ and the corresponding solution $x^0(t, a)$ satisfy $u^0(t, a) = h_1^* x^0(t, a)$. The uniqueness of the pair $u^0(t, a)$, $x^0(t, a)$ (Lemma 3) implies that $h_1^* x^0(t, a) = h^* x(t, a)$; in particular $h_1^* a = h^* a$, $\forall a$. Hence $h_1 = h$. Relation (1.5) with $u = 0$ yields $\mathrm{Re}\, x^* H A x = |\varkappa h^* x|^2 - \mathscr{F}(x, 0) = \mathrm{Re}\, x^* H_1 A x$, $\forall x$, i.e., H and H_1 both satisfy the equation $HA + A^*H = -P$, where $P = P^*$ is a bounded linear operator. Since A is Hurwitz, the solution H of this equation is unique and $H = H_1$.

We shall prove that there is a bounded linear self-adjoint operator H (in general distinct from that determined above) such that (1.3) holds. Let $\mathscr{F}_1(x, u) = \mathscr{F}(x, u) + \delta[|x|^2 + |u|^2]$, where $\delta > 0$ is so small that $\mathscr{F}_1[(i\omega I_X - A)^{-1} bu, u] \leq -\delta_1 |u|^2$, $\forall \omega \in (-\infty, +\infty)$, $\forall u \in \mathbf{U}$. It follows from (1.4) that such a δ exists. It follows from results already proved that operators $H = H^*$ and h exist with the indicated properties such that

$$\mathrm{Re}\, x^* H(Ax + bu) + \mathscr{F}_1(x, u) = |\varkappa(u - h^* x)|^2.$$

Thus (1.3) holds.

We have proved that condition (1.4) is sufficient to ensure the existence of an operator $H = H^*$ with the indicated properties such that (1.3) holds. The necessity of this condition is evident. Let $\mathbf{X} = \mathbf{X}_C$, $\mathbf{U} = \mathbf{U}_C$, i.e., $\mathbf{X}$ and $\mathbf{U}$ are complex Hilbert spaces. Let (1.3) hold, i.e., there is $\delta > 0$ such that

$$\operatorname{Re} x^* H(Ax+bu)+\mathscr{F}(x, u) \geqslant \delta[|x|^2+|u|^2], \qquad \forall x, u. \tag{2.42}$$

Suppose that x and u are related by the equation

$$Ax+bu=i\omega x \quad (\omega \in R^1).$$

Then (2.4) implies (1.4).

Suppose that $\mathbf{X} \neq \mathbf{X}_C$, $\mathbf{U} \neq \mathbf{U}_C$ [thus the Re can be omitted in (1.3) and (2.42)]. Let $\mathbf{Y} = \mathbf{X} \times \mathbf{U}$. Any quadratic form $\mathscr{K}(y)$ on the real Hilbert space $\mathbf{Y}$ can be extended to the complex extension $\mathbf{Y}_C$ according to the formula $\mathscr{K}_C(y_1 + iy_2) = \mathscr{K}(y_1) + \mathscr{K}(y_2)$. Then the inequality $\mathscr{K}(y) \gg 0$ on $\mathbf{Y}$ implies that $\mathscr{K}_C(y) \gg 0$ on $\mathbf{Y}_C$. By virtue of our condition

$$\mathscr{K}(y)=x^* H(Ax+bu)+\mathscr{F}(x, u) \gg 0$$

for $y = \binom{x}{u} \in \mathbf{Y}$. For complex $y \in \mathbf{Y}_C$, we have

$$\mathscr{K}_c(y)=\operatorname{Re} x^* H(Ax+bu)+\mathscr{F}_c(x, u) \gg 0,$$

where H, A, and b are the extensions of the original operators to complex space, i.e., for example $A(x_1 + ix_2) = Ax_1 + iAx_2$. Hence

$$\mathscr{F}_c[(i\omega I_X-A)^{-1} bu, u] \ll 0, \quad \forall \omega \in R^1, \quad \forall u \in \mathbf{U}.$$

This inequality coincides with (1.4) since, in (1.4), the form $\mathscr{F}_C(x, u)$ is denoted by $\mathscr{F}(x, u)$. This completes the proof of Theorem 1.

§3. Proof of Theorems 2 and 3

We first prove Theorem 2. In §2 we assumed that A was Hurwitz. We now assume that no points of the spectrum of A are on the imaginary axis, and the pair A, b is stabilizable. It follows from the proof of Theorem 1 in §2 that the proof of the necessity of (1.4) and the derivation of (1.3) from (1.5) remain valid in the case under consideration. We shall prove that, in the present case, the validity of the representation (1.5) follows from the same assertion for the case in which A is Hurwitz. This will prove Theorem 2.

Let $c: \mathbf{U} \to \mathbf{X}$ be an operator such that $C = A + bc_*$ is Hurwitz. Instead of u consider the variable $v = u - c^* x$. The form

$$\mathscr{F}(x, u)=\mathscr{F}_1(x, v) \tag{3.1}$$

is clearly a quadratic form in x and v with a bounded operator. We shall prove that the Hermitian operator $\Pi_1(i\omega): \mathbf{U}_C \to \mathbf{U}_C$ determined by the relation

$$\mathscr{F}_1[(I_X-i\omega C)^{-1} bv, v]=v^* \Pi_1(i\omega) v, \tag{3.2}$$

satisfies

$$\exists \delta_1>0, \quad \forall \omega: \Pi_1(i\omega) \geqslant \delta_1 I_u. \tag{3.3}$$

Then there are operators $H = H^*$ and h_1 such that

$$\operatorname{Re} x^* H(Cx+bv)+\mathscr{F}_1(x, v)=-|\varkappa(v-h_1^* x)|^2. \tag{3.4}$$

Since the left side of (3.4) is equal to

$$\operatorname{Re} x^* H(Ax+bv)+\mathscr{F}(x, u),$$

the representation (3.4) is converted into (1.5) with $h = h_1 + c$.

Hence it suffices to prove (3.3). Express $\Pi_1(i\omega)$ in terms of the operator $\Pi(i\omega)$ determined by (1.2), i.e., by the relation

$$\mathscr{F}[(I_{X_c}-i\omega A)^{-1} bu, u]=-u^* \Pi(i\omega) u. \tag{3.5}$$

The equality

$$Ax+bu=i\omega x$$

is equivalent to

$$Cx+bv=i\omega x.$$

Hence, for x, u, and v related by these equalities,

$$x=C_{i\omega}^{-1}bv=A_{i\omega}^{-1}bu, \tag{3.6}$$

where

$$C_{i\omega}=i\omega I_X-C, \quad A_{i\omega}=i\omega I_X-A. \tag{3.7}$$

By virtue of our conditions the operators (3.7) have bounded inverses. Relations (3.1), (3.2), and (3.5) imply that

$$v^*\Pi_1(i\omega)v=u^*\Pi(i\omega)u, \tag{3.8}$$

where u and v satisfy $u = v + c^*x$ and (3.6). Hence

$$u=\zeta(i\omega)v, \quad v=\zeta_1(i\omega)u, \tag{3.9}$$

where

$$\zeta(i\omega)=I_U+c^*C_{i\omega}^{-1}b, \quad \zeta_1(i\omega)=I_U-c^*A_{i\omega}^{-1}b \tag{3.10}$$

are bounded linear operators on **U** into **U**. It follows from (3.9) that $\zeta(i\omega)\zeta_1(i\omega) = I_U$, i.e., each of the operators (3.10) has a bounded inverse and

$$\zeta_1(i\omega)=\zeta(i\omega)^{-1}.$$

Using (3.9) for u in (3.8) and the fact that v is arbitrary, we obtain

$$\Pi_1(i\omega)=\zeta(i\omega)^*\Pi(i\omega)\zeta(i\omega). \tag{3.11}$$

By assumption (1.4) holds, i.e., $\Pi(i\omega) \geq \delta I_U$; hence

$$\Pi_1(i\omega)\geqslant\delta\zeta(i\omega)^*\zeta(i\omega). \tag{3.12}$$

We shall prove that

$$\zeta(i\omega)^*\zeta(i\omega)\geqslant\varepsilon(\omega)=1/|\zeta_1(i\omega)|^2. \tag{3.13}$$

Let $\zeta = \zeta(i\omega)$, $\zeta_1 = \zeta_1(i\omega)$. If $u \in$ **U** and $v = \zeta u$ are arbitrary, $u = \zeta_1 v$, $|u| \leq |\zeta_1| \cdot |v|$. Thus

$$u^*\zeta^*\zeta u=|\zeta u|^2=|v|^2\geqslant\frac{1}{|\zeta_1|^2}|u|^2, \tag{3.14}$$

and (3.13) follows. The second formula in (3.10) implies that $\varepsilon(\omega) > 0$, $\forall\omega \in (-\infty, +\infty)$ and $\varepsilon(\omega) \to 1$ when $|\omega| \to \infty$. The required relation (3.3) for $\varepsilon(\omega) \geq \varepsilon_0 > 0$, $\forall\omega \in (-\infty, +\infty)$ follows from (3.13) and (3.12) and this proves Theorem 2.

We now prove Theorem 3. We assume that (1.10) holds and prove the existence of operators H, h, and $\varkappa$ with the indicated properties such that (1.9) holds. To this end we assume for the moment that the operators H, h, $\varkappa$ exist and we transform the identity (1.9). Let λ_0 be an arbitrary number, $|\lambda_0| = 1$, and, for real **X** and **U**, either $\lambda_0 = 1$ or $\lambda_0 = -1$. Let

$$A_0=\frac{A+\lambda_0 I_X}{A-\lambda_0 I_X}, \tag{3.15}$$

$$x=\lambda_0^{-1}(A_0-I_X)(y/\sqrt{2}-bu/2). \tag{3.16}$$

Since $A_0-I_X = 2\lambda_0(A-\lambda_0 I_X)^{-1}$, the operator A_0-I_X is nonsingular, the replacement of y, u by x, u is reversible, and

$$A=\lambda_0\frac{A_0+I_X}{A_0-I_X}. \tag{3.17}$$

Moreover,

$$Ax+bu=y_1-v, \quad x=\lambda_0^{-1}(y_2-v),$$

where

$$y_1=(A_0+I_X)y/\sqrt{2},\quad y_2=(A_0-I_X)y/\sqrt{2},\quad v=(A_0-I_X)bu/2.$$

The first two terms in (1.9) transform into

$$y_1^*Hy_1-y_2^*Hy_2+2(y_2-y_1)^*Hv=2\operatorname{Re} y^*H(A_0y+b_0u),$$

where

$$b_0=-\frac{1}{\sqrt{2}}(A_0-I_X)b.$$

Hence (1.9) transforms into

$$2\operatorname{Re} y^*H(A_0y+b_0u)+F_0(y,u)=|\varkappa_0(u-h_0^*y)|^2, \tag{3.18}$$

where

$$F_0(y,u)=F(x,u),\quad \varkappa_0(u-h^*y)=\varkappa(u-h^*x), \tag{3.19}$$

and the variables $x \in \mathbf{X}$, $y \in \mathbf{X}$, and $u \in \mathbf{U}$ are related by (3.16). The operator $\varkappa_0$ satisfies

$$\mathcal{F}_0(0,u)=|\varkappa_0 u|^2,\quad \forall u,$$

[this follows from (3.16)] and coincides with (1.11). Moreover $\varkappa_0$ can be expressed in terms of $\varkappa$ and h by the relation (1.12). This relation is obtained by expressing x in terms of y in the second identity of (3.19) [according to (3.16)] and comparing coefficients of u. Expressing y in terms of x and equating coefficients of u, we obtain

$$\varkappa=\varkappa_0(1-h_0^*b/\sqrt{2}). \tag{3.20}$$

The operators h and h_0 are related by the equality

$$h^*=\varkappa_0h_0^*(A-\lambda_0I), \tag{3.21}$$

which follows from (3.19) and (3.16). The identity (3.18) is equivalent to (1.9), since the inverse transformations convert (3.18) into (3.9). Here $\varkappa$ and h are expressed in terms of $\varkappa_0$ and h_0 by using (3.20) and (3.21).

Hence, to prove the existence of operators H, h, and $\varkappa$ satisfying (1.9), it is sufficient to establish the existence of operators $H:\mathbf{X}\to\mathbf{X}$ and $h_0:\mathbf{X}\to\mathbf{U}$ satisfying (3.18).

Let $\varkappa_0$ be defined as indicated in Theorem 3 and let (3.21) hold. To prove the existence of the required H, h_0 we apply Theorem 2. We first assume that A is "discretely stable," i.e., the spectrum of A is in the region $|\lambda| < 1$. Then (3.15) implies that the spectrum of A_0 lies in the region $\operatorname{Re}\lambda < 0$, i.e., A_0 is Hurwitz. It follows from Theorem 1 that there are operators $H = H^*$, h_0 satisfying (3.18) if

$$\mathcal{F}_0(y,u)\geqslant\delta|u|^2 \tag{3.22}$$

for all u and all y, satisfying

$$A_0y+bu=i\omega y \tag{3.23}$$

for $\omega \in (-\infty, +\infty)$. Relation (3.23) is equivalent to the relation

$$Ax+bu=\lambda x \tag{3.24}$$

for $\lambda = \lambda_0(i\omega + 1)(i\omega-1)$. When ω ranges over the real axis, λ ranges over the unit circle. Inequality (3.22) is transformed into (1.10). Hence (3.22) holds. It follows that there are H, h_0 satisfying (3.18), and thus there are H, h, and $\varkappa$ satisfying (1.9). In the case of real $\mathbf{X}$ and $\mathbf{U}$, the operators H, h_0 are real (i.e., although defined in $\mathbf{X}_c$, $\mathbf{U}_c$ they do not leave the space $\mathbf{X}$). Relations (3.20) and (3.22) imply in this case that h, $\varkappa$ are real.

Let part of the spectrum of A be in the region $|\lambda| > 1$, but suppose that the pair A, b is discretely stabilizable; the spectrum of $C = A + bc^*$ is in $|\lambda| < 1$. The substitution $u = v + c^*x$ in (1.9) yields an analogous identity with A and u replaced by C and v and with the new form $\mathcal{F}_1(x, v) = \mathcal{F}(x, u)$. As in the proof of Theorem 2, we prove that this form satisfies $\mathcal{F}_1[\lambda I_{X_c}-C)^{-1}bv, v] \geq \delta |v|^2$, $v \in \mathbf{U}$, $|\lambda| = 1$. Hence, by virtue of results already proved, there are operators H, h, and $\varkappa$ satisfying the transformed identity, and so satisfying the original identity.

As in the proof of Theorem 2 we conclude that the existence of H, h, and $\varkappa$ satisfying (1.9) implies that condition (1.10) is sufficient to ensure the existence of an operator $H = H^*$ satisfying (1.8).

The necessity of condition (1.10) is obvious. In fact let (1.8) hold, i.e., let

$$(Ax+bu)^*H(Ax+bu)-x^*Hx+\mathcal{F}(x,u)\geqslant\delta[|x|^2+|u|^2],\ \forall x,u.$$

Substituting the expression for x in terms of u obtained from $Ax + bu = \lambda x$, we conclude that (1.10) holds, and Theorem 3 follows.

§4. Proof of Theorems 4 and 5

We use the method described in §1 (cf. [4]).

We first prove Theorem 4. Suppose that (1.21) holds and that $H = H^*$ and h are operators which, by virtue of Theorem 2, satisfy (1.5). Substituting $u = u(t) \in \mathfrak{M}_a$ and $x = x(t)$ in (1.5), where x(t) is the corresponding solution, and integrating from $t = 0$ to $t = \infty$, we find, as in §1, that (1.18) holds. The properties of B imply that the control $u = h^*x = e^{Bt}a$ is admissible, and (1.18) implies that it is optimal.

The uniqueness of an optimal control is proved in §2 when A is Hurwitz. Suppose that A is not Hurwitz but $C = A + bc^*$ is Hurwitz (see Definition 2). The substitution $u = v + c^*x$ yields a system with a unique optimal control. Let $u_1^0(t)$, $x_1^{(0)}(t)$, $u_2^0(t)$, and $x_2^{(0)}(t)$ be distinct pairs of optimal controls and the corresponding solutions and let $\Delta u = u_1^{(0)}-u_2^{(0)}$, $\Delta x = x_1^0-x_2^0$. Then the $v_j^0 = u_j^0-c^*x_j^0$ are optimal controls of the new system and $v_1^0 = v_2^0$ almost everywhere. Hence $\Delta u = c^*\Delta x$, and (1.14) implies that $d\Delta x/dt = A\Delta x + b\Delta u = C\Delta x$ and $\Delta x(0) = 0_X$. Thus $\Delta x(t) \equiv 0_X$, $\Delta u \equiv 0_u$.

It remains to prove the necessity of the condition $\Pi(i\omega) \geq 0$, $\omega \in (-\infty, +\infty)$. If A is Hurwitz, this is proved in §2 (Lemma 4). Suppose that A is not Hurwitz but $C = A + bc^*$ is Hurwitz. The substitution $u = v + c^*x$ yields a system with a Hurwitz operator C with the new functional

$$J_1(v)=\int_0^\infty \mathcal{F}_1[x(t),v(t)]dt, \tag{4.1}$$

where $F_1(x, v) = F(x, v)$. The corresponding controls $u = u(t)$ and $v = v(t)$ satisfy

$$J_1(v)=J(u).$$

It was proved in §2 that the operator $\Pi(i\omega)$ for the new system [which we denote by $\Pi_1(i\omega)$] is related to the original operator by the equality (3.11). Hence violation of the condition $\Pi(i\omega) \geq 0$ implies the violation of the condition $\Pi_1(i\omega) \geq 0$ (for the same ω). Lemma 4 in §2 implies that $\inf J_1(v) = -\infty$ when the inf is over admissible controls v. It follows from (4.1) that $\inf J(a) = -\infty$, where the inf is over all $u \in \mathfrak{M}_a$. This proves Theorem 4.

We now consider the proof of Theorem 5.† We assume that all the conditions of Theorem 5 are satisfied.

Let $G(t) = G(t)^* : \mathbf{X} \to \mathbf{X}$ and $g(t) : \mathbf{U} \to \mathbf{X}$ be bounded linear operators depending on $t \in [0, T]$, whose properties will be described below. We shall try to choose these operators so that the function

$$V(t,x)=\frac{1}{2}x^*G(t)x \tag{4.2}$$

satisfies the following relation, analogous to the relation (1.5) in the frequency theorem:

$$\frac{dV(t,x)}{dt}+\mathcal{F}(x,u)=|\varkappa[u-g(t)^*x]|^2. \tag{4.3}$$

Here dV(t, x)/dt is the derivative obtained by using (1.14):

$$\frac{dV(t,x)}{dt}=\frac{1}{2}x^*\frac{dG(t)}{dt}x+\operatorname{Re}x^*G(t)(Ax+bu) \tag{4.4}$$

(the derivative dG/dt must exist). We also require that

$$G(T)=0. \tag{4.5}$$

† Here we follow the reasoning in [29]. We note that the final part of Theorem 5 (the necessity of the frequency condition for sufficiently large T) was proved above (cf. the remark after Lemma 4, §2).

Substitution of u = u(t) and the corresponding solution x(t) of (1.14) in (4.3) and integration from t = 0 to t = T yield the following formula, analogous to (1.18):

$$J_T(u)=V(0,a)+\int_0^T |\varkappa[u(t)-g(t)^*x(t)]|^2\,dt. \tag{4.6}$$

It follows that if

$$u^0(t)=g(t)^*x^0(t) \tag{4.7}$$

is an admissible control, it is optimal. Hence we arrive at the problem of finding operators G(t) = G(t)* and g(t) with the properties indicated above.

The identity (4.3) can be transformed into an operational Riccati equation for G(t) = G(t)*. If G(t) = const and g ≡ const, (4.3) coincides with (1.5) which, according to our condition, is satisfied by H, h, and $\varkappa$:

$$\operatorname{Re} x^*H(Ax+bu)+\mathcal{F}(x,u)=|\varkappa(u-h^*x)|^2.$$

Hence G(t) ≡ H = const and g(t) ≡ h = const form a particular solution of the Riccati equation. A knowledge of a particular solution leads to the complete solution. The calculation below is essentially the determination of the solution of the Riccati equation.

We use the notation

$$v=u-h^*x,\qquad g_1(t)=g(t)-h,\qquad G_1(t)=G(t)-H. \tag{4.8}$$

Subtracting (4.6) from (4.3) and using (4.4) and the relation Ax + bu = Bx + bv, we obtain

$$\frac{1}{2}x^*\frac{dG_1}{dt}x+\operatorname{Re} x^*G_1(Bx+bv)=|\varkappa(v-g_1^*x)|^2-|\varkappa v|^2. \tag{4.9}$$

The converse clearly holds: (4.9) implies (4.3). Hence it is sufficient to find G_1 and g_1 satisfying (4.9), and then use (4.8) to calculate G and g. Rewrite (4.9) as two equations:

$$\frac{1}{2}\frac{dG_1}{dt}+G_1B+B^*G_1=g_1\Gamma^{-1}g_1^*, \tag{4.10}$$

$$G_1b+2g_1\Gamma=0. \tag{4.11}$$

Substitution of (4.11) for g_1 in (4.10) yields the "degenerate" Riccati equation

$$\frac{dG_1}{dt}+G_1B+B^*G_1=G_1b\Gamma^{-1}b^*G_1; \tag{4.12}$$

which is transformed by the substitution $G_1 = -Z^{-1}$ into the linear equation

$$\frac{dZ}{dt}=BZ+ZB^*+b\Gamma^{-1}b \tag{4.13}$$

(it is known that Z^{-1} exists). Relations (4.5) and (4.8) yield $G_1(t) = -H$ and

$$Z(T)=H^{-1} \tag{4.14}$$

(Theorem 5 implies that H^{-1} exists). It follows from (4.13) and (4.14) that

$$Z(t)=e^{B(t-T)}\left[H^{-1}-\int_0^{T-t} e^{Bs}b\Gamma^{-1}b^*e^{B^*s}\,ds\right]e^{B^*(t-T)}. \tag{4.15}$$

Theorem 5 implies that this operator has a bounded inverse. Hence $G_1(t) = -Z(t)^{-1}$ satisfies (4.10), and G(t) and g(t), as determined by (4.11) and (4.8), satisfy (4.3). Hence (4.7) is admissible and is thus optimal. Formula (4.7) coincides with (1.25), (1.26).

For $J(u^0) = V(0, a) = (1/2)a^*G(0)a$, (1.27) is a consequence of (4.8) and (4.15) and Theorem 5 follows.

LITERATURE CITED

1. V. A. Yakubovich, "Solution of certain matrix inequalities encountered in automatic-control theory," Dokl. Akad. Nauk SSSR, 143, No. 6, 1304-1307 (1962).

2. R. E. Kalman, "Liapunov functions for the problem of Lur'e in automatic control," Proc. Nat. Acad. Sci. USA, 49 (1963).
3. V. M. Popov, Hyper-Stability of Automatic Systems [in Russian], Nauka, Moscow (1970).
4. V. A. Yakubovich, "The frequency theorem in control theory," Sib. Matem. Zh., 14, No. 2, 384-419 (1973).
5. M. A. Aizerman and F. P. Gantmakher, Absolute Stability of Control Systems [in Russian], Izd-vo AN SSSR, Moscow (1963).
6. F. R. Gantmakher and V. A. Yakubovich, "Absolute stability of nonlinear control systems," in: Transactions of the Second All-Union Congress on Theoretical and Applied Mechanics [in Russian], Nauka, Moscow (1965).
7. D. Siljak, Nonlinear Systems, John Wiley and Sons, New York (1969).
8. E. B. Lee and L. Markus, Foundation of Optimal Control Theory, London (1967).
9. P. Faurre, Identification par Minimisation d'une Representation Markovienne de Processus Aléaire, Lecture Notes in Mathematics (A collection of informal reports and seminars), Vol. 132, Springer-Verlag, Berlin (1970), pp. 85-106.
10. P. Faurre and J. P. Marmorat, "Un algorithme de réalisation stochastique," Compt. Rend. Acad. Sci., Ser.A, 268, 978-981 (1969).
11. P. Faurre, Identification of Markovian Representations for Stochastic Processes, Fourth Hawaii International Conference on System Sciences (January 12-14, 1971), pp. 576-578.
12. A. I. Lur'e, Some Nonlinear Problems in Automatic-Control Theory [in Russian], Gostekhizdat, Moscow (1951).
13. A. M. Letov, Stability of Nonlinear Control Systems [in Russian], Gostekhizdat, Moscow (1962).
14. S. Lefshets, Stability of Nonlinear Automatic-Control Systems [in Russian], Mir, Moscow (1967).
15. V. A. Andreev, Yu. F. Kazarinov, and V. A. Yakubovich, "Synthesis of optimal controls for linear nonhomogeneous systems in problems of minimizing quadratic functionals," Dokl. Akad. Nauk SSSR, 199, No. 2, 258-261 (1971).
16. V. A. Yakubovich, "The synthesis of optimal controls in linear differential games on a finite time interval with a quadratic payoff function," ibid., 200, No. 3, 548-551 (1971).
17. V. A. Andreev, Yu. F. Kazarinov, and V. A. Yakubovich, "The synthesis of optimal controls for linear inhomogeneous systems in the problem of minimizing the mean value of a quadratic functional," ibid., 202, No. 6, 1247-1250 (1972).
18. V. A. Andreev, "The synthesis of optimal controls for inhomogeneous linear systems with a quadratic quality criterion," Sib. Matem. Zh., 13, No. 3, 698-702 (1972).
19. R. E. Kalman, A New Algebraic Method in Stability Theory, Transactions of the Fifth International Conference on Nonlinear Vibrations [in Russian], Vol. 2, Izd. In-ta Matem. AN UkrSSR, Kiev (1970).
20. V. A. Yakubovich, "Solution of an algebraic problem in control theory," Dokl. Akad. Nauk SSSR, 193, No. 1, 57-60 (1970).
21. N. N. Krasovskii, Appendix IV in the book The Theory of Motion Stability [in Russian] by I. G. Malkin, Nauka, Moscow (1966).
22. N. N. Krasovskii, The Theory of Motion Control [in Russian], Nauka, Moscow (1968).
23. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes [in Russian], Fizmatgiz (1961).
24. V. G. Boltyanskii, Mathematical Methods of Optimal Control [in Russian], Nauka, Moscow (1969).
25. R. E. Kalman, The General Theory of Control Systems, Transactions of the First IFAK Congress [in Russian], Vol. 1, Izd-vo AN SSSR (1961).
26. R. Bellman, I. Glicksberg, and O. Gross, Some Problems in Mathematical Control Theory [Russian translation], IL, Moscow (1962).
27. A. I. Lur'e, "A minimum quality criterion for control systems," Izv. Akad. Nauk SSSR, Otd. Tekhn. Nauk, Tekhnicheskaya Kibernetika, No. 4, 140-146 (1963).
28. V. A. Andreev and A. I. Shepelyavyi, "The synthesis of optimal controls for amplitude-sampled-data systems in the problem of minimizing the mean value of a quadratic functional," Sib. Matem. Zh., 14, No. 2, 250-276 (1973).
29. V. A. Yakubovich, "The synthesis of optimal controls in a linear differential game on a finite time interval with a quadratic payoff functional," Dokl. Akad. Nauk SSSR, 200, No. 3, 548-551 (1971).