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Feedback control of a class of linear discrete systems with jump parameters and quadratic cost criteria

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Feedback control of a class of linear discrete systems with jump parameters and quadratic cost criteria†

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The optimal control for a discrete-time linear system with quadratic criterion functional is derived for the case where the system parameters are subject to a discrete-time, discrete-state Markov process. Utilization of the results is demonstrated in a numerical example using Samuelson's multiplier–accelerator macroeconomic model.

1. Introduction

In Sworder (1969) a stochastic maximum principle was developed for a class of continuous-time linear systems whose parameters were subject to sudden value changes in accordance with a continuous-time, discrete-state Markov process. Application of the indicated algorithm to the problem when the cost functional was quadratic produced an explicit optimal feedback control law which utilized the solution of a coupled set of Ricatti-type equations. It is the purpose of the present paper to derive the optimal control law and minimum expected cost for a discrete-time linear system with quadratic criterion functional when the system parameters are governed by a discrete-time, discrete-state Markov process. The optimization is accomplished by the techniques of dynamic programming (Bellman 1961). Utilization of the results is demonstrated in a numerical example involving an economic process.

2. System description

It will be assumed that the system to be controlled is given by the linear, vector difference equation

$$\left.\begin{aligned} x(k+1) &= A(k)x(k)+B(k)u(k), \quad 0 \leqslant k \leqslant N \\ x(0) &= x_0 \\ y(k) &= Cx(k) \end{aligned}\right\} \qquad (1)$$

where $x(k)$ is the n-dimensional state, $u(k)$ is the scalar actuating signal and $y(k)$ is the scalar output. Equation (1) is intended to serve as a model of a system subject to abrupt changes in its coefficient matrices; e.g. sudden changes in an economic process. Consider the situation where the number

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of parameter values is finite, say s. We wish to model the parameter processes with a discrete-time, discrete-state Markov jump process. Denote the event that

$$[A(k), B(k)] = [A_i, B_i] \tag{2}$$

by $r(k)=i$ and let S be the state space of r. Then it is assumed that there exists an $s \times s$ matrix P with elements p_{ij} such that

$$\left.\begin{aligned} \text{Prob}\,[r(k+1)=j\,|\,r(k)=i] &= p_{ij} \\ \text{Prob}\,[r(0)=i] &= p_i \\ \sum_{j=1}^{s} p_{ij} = 1, \quad 0 \leqslant p_{ij} &\leqslant 1 \end{aligned}\right\} \tag{3}$$

The actuating signal utilizes feedback of both the state and the operating mode

$$u(k) = \bar{u}(k, x(k), r(k)) \tag{4}$$

A control policy $\bar{u}$ must be selected from an admissible set $U = R$.

The cost criterion is given by

$$J[u] = \mathscr{E} \sum_{k=0}^{N-1} x(k)' Q_{r(k)} x(k) + 2u(k) M_{r(k)} x(k) + b_{r(k)} u^2(k) \tag{5}$$

where $Q_{r(k)} \geqslant 0$ is symmetric, $Q_{r(k)} - b_{r(k)}{}^{-1} M_{r(k)}{}' M_{r(k)} \geqslant 0$, $b_{r(k)} > 0$, and $\mathscr{E}$ is the expectation operator taken with respect to both the parameters of the system and cost criterion.

3. Optimal controller

The optimal controller in this application will be that element $\bar{u}^* \in U$ that minimizes $V(k_0, x_0, r(k_0))$ where

$$\left.\begin{aligned} V(k, x, i) = J[\bar{u}^*] = \min_{\bar{u} \in U} \mathscr{E} \Bigg\{ &\sum_{l=k}^{N-1} x(l, \bar{u})' Q_{r(l)} x(l, u) \\ &+ 2\bar{u}(l, x(l, \bar{u}), r(l)) M_{r(l)} x(l, \bar{u}) \\ &+ b_{r(l)} \bar{u}^2(l, x(l, \bar{u}) r(l)) \,|\, x(k) = x, r(k) = i \Bigg\}. \\ &\qquad k = 0, 1, \ldots, N-1 \\ V(N, x(N), r(N)) = 0 \qquad & \end{aligned}\right\} \tag{6}$$

形式

We shall find an explicit expression for $\bar{u}^*$ by applying the formalism of dynamic programming to (6), relying on the global property of this technique, i.e. comparing the optimal decision with all other decisions at every stage of the process, to lead to optimality conditions which are sufficient. Noteworthy references concerning the application in general are contained in Varaiya (1972) and Gunkel and Franklin (1963).

Using the optimality principle, write (6) as

$$\left.\begin{aligned} V(k, x, i) = \min_{\bar{u} \in U} \mathscr{E}\, \{[x(k)'Q_i x(k) + 2\bar{u}(k)M_i x(k) + b_i \bar{u}^2(k)] \\ + V(k+1, x(k+1), r(k+1)=j)\,|\,k, r(k)=i\} \\ k = 0, 1, \ldots, N=1 \\ V(N, x(N), r(N)) = 0 \end{aligned}\right\} \tag{7}$$

where in (7) we have simplified our notation.

As a trial solution to (7) let

$$V(k, x, i) = x(k)'K(k, i)x(k) \tag{8}$$

By substituting (8) and (1) into (7), obtain

$$\begin{aligned} x(k)'K(k, i)x(k) = \min_{\bar{u} \in U} \{x(k)'Q_i x(k) + 2\bar{u}(k)M_i x(k) + b_i \bar{u}^2(k) \\ + [A_i x(k) + B_i \bar{u}(k)]' \,.\, \mathscr{E}[K(k+1, r(k+1)=j)\,|\,k, r(k)=i] \\ \times [A_i x(k) + B_i \bar{u}(k)]\} \end{aligned} \tag{9}$$

where, since

$$\left.\begin{aligned} \mathscr{E}[K(k+1, r(k+1)=j)\,|\,k, r(k)=i] = \sum_{j=1}^{s} K_j(k+1, r(k+1)) \\ \times \operatorname{Prob}'[r(k+1)=j\,|\,r(k)=i] \\ = \sum_{j=1}^{s} K_j(k+1)p_{ij} \end{aligned}\right\} \tag{10}$$

$$\begin{aligned} x(k)'K(k, i)x(k) = \min_{\bar{u} \in U} \Bigg\{ x'(k)Q_i(k) + 2\bar{u}(k)M_i x(k) + b_i \bar{u}^2 \\ + [\bar{u}(k)B_i' + x(k)'A_i'] \sum_{j=1}^{s} K_j(k+1)p_{ij} \\ \times [A_i x(k) + B_i \bar{u}(k)] \Bigg\} \end{aligned} \tag{11}$$

Differentiating (11) with respect to $\bar{u}(k)$ yields the optimal control

$$\begin{aligned} \bar{u}^*(k, x, i) = -\left[b_i + B_i' \sum_{j=1}^{s} K_j(k+1)p_{ij}B_i \right]^{-1} \\ \times \left[M_i + B_i' \sum_{j=1}^{s} K_j(k+1)p_{ij}A_i \right] x(k) \end{aligned} \tag{12}$$

Substituting (12) in (11) produces, following some algebra, the equation for $K(k, i)$:

$$\left.\begin{aligned}K_i(k) = {} & \left[A_i' \sum_{j=1}^{s} K_j(k+1) p_{ij} A_i + Q_i\right] \\ & - \left[M_i + B_i' \sum_{j=1}^{s} K_j(k+1) p_{ij} A_i\right]' \\ & \times \left[b_i + B_i' \sum_{j=1}^{s} K_j(k+1) p_{ij} B_i\right]^{-1} \\ & \times \left[M_i + B_i' \sum_{m=1}^{s} K_j(k+1) p_{ij} A_i\right], \\ & \qquad\qquad 0 \leqslant k < N-1 \\ K_{r(N)}(N) = {} & 0\end{aligned}\right\} \tag{13}$$

Observe that for a deterministic process, where $s=1$, $i=j$, $p_{ij}=1$ and thus

$$\sum_{j=1}^{s} K_j(k+1) p_{ij} = K(k+1)$$

(12) and (13) reduce to their deterministic counterparts (Dorato and Levis (1971).

The minimum expected cost at $k=0$ is

$$V(0, x(0), r(0)) = x_0' K_{r(0)}(0) x_0 \tag{14}$$

For an infinite optimization interval, $t=N=\infty$, we get, upon setting $K_i(k) = K_i(k+1) = K_{i\infty}$,

$$\begin{aligned}\bar{u}^*(k, x, i) = {} & -\left[b_i + B_i' \sum_{j=1}^{s} K_{j\infty} p_{ij} B_i\right]^{-1} \\ & \times \left[M_i + B_i' \sum_{j=1}^{s} K_{j\infty} p_{ij} A_i\right] x(k)\end{aligned} \tag{15}$$

$$\begin{aligned}K_{j\infty} = {} & \left[A_i' \sum_{j=1}^{s} K_{j\infty} p_{ij} A_i + Q_i\right] \\ & - \left[M_i' + B_i' \sum_{j=1}^{s} K_{j\infty} p_{ij} A_i\right]' \left[b_i + B_i' \sum_{j=1}^{s} K_{j\infty} p_{ij} B_i\right]^{-1} \\ & \times \left[M_i + B_i' \sum_{j=1}^{s} K_{j\infty} p_{ij} A_i\right]\end{aligned} \tag{16}$$

with minimum expected cost at $k=0$ of

$$V(0, x(0), r(0)) = x_0' K_{r(0)\infty} x_0 \tag{17}$$

In § 1 it was observed that the cost criterion matrices Q_i, M_i, b_i are jump parameters. Obviously, the results derived are valid for discrete-time systems with specified constant Q, M, b matrices. Similarly, expressions for a quadratic cost with no cross-product term are obtained simply by deleting terms involving M in (6), (7), (9), (11)–(13), (15) and (16). The case in which the dynamical matrices are independent from one time increment to the next was studied by Gunkel and Franklin (1963) and a similar solution is given in (13) with $p_{ij}=p_j$ for all i; i.e. the components of $A(k)$, $B(k)$ are random variables and $p(r(k+1)=j|r(k)=i)=p(r(k+1)=j)\forall i$.

4. Example

To illustrate utilization of these results, we choose a simple economic system based on Samuelson's (1939) multiplier-accelerator model which appears in difference equation form :

$$C_t=cY_{t-1} \tag{18}$$

$$I_t=w(Y_{t-1}-Y_{t-2}) \tag{19}$$

$$Y_t=C_t+I_t+G_t \tag{20}$$

where

$C=$ consumption expenditure,
$Y=$ national income,
$I=$ induced private investment,
$G=$ government expenditure,
$c=(1-s)$,
$=$ marginal propensity to consume,
$=$ slope of the consumption versus income curve,
$s=$ marginal propensity to save,
$\frac{1}{s}=$ the multiplier,
$w=$ accelerator coefficient,
$t=$ subscript for time,
$=kT=k$, $(T=1)$.

The interaction between the multiplier and accelerator was made famous in the referenced article by Samuelson. There are recent papers on the subject by Evans (1969) and Kogiku (1968). As can be seen, the model is highly aggregated, intended primarily for use as a theoretical tool rather than as a realistic representation of the economy. We use it to demonstrate simply an application of our results.

It is shown by Blair (1974) that (18)–(20) can be expressed in state-space form as a regulator problem

$$x(k+1)+Ax(k)+Bu(k) \tag{21}$$

with output

$$y(k-1)=Cx(k)=x_2(k) \tag{22}$$

where

$$A=\begin{bmatrix} 0 & 1 \\ -w & 1-s+w \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C=[0 \quad 1] \tag{23}$$

The coefficients s and w were computed for the U.S. economy for all years 1929 to 1971, based on data by the U.S. Department of Commerce (1971). To remove bias, all cost quantities were factored for population growth and inflation placing s and w on a per capita, annual time-scale basis in terms of 1958 dollars. These results were used directly without smoothing or other manipulation to reflect more realistically the actual behaviour of the coefficients. Thus, the ranges for s and w employed here depart significantly from the classical ranges generally proposed by Keynesian and post-Keynesian economists ; i.e. $0<s<1$ and $0<w<3$ (Ackley 1969, p. 489 ; Matthews 1959, pp. 14, 22 ; Allen 1968, p. 192 ; Allen 1959, p. 332). The relationship $c=1-s$ is maintained.

Upon examination, it was found that the parameters s and w could be grouped, individually, in three natural classes or modes. As a convenience, the following terminology is adopted.

Mode i	Terminology	Description
1	' Norm '	s (or w) in mid-range
2	' Boom '	s in low range (or w in high)
3	' Slump '	s in high range (or w in low)

As a rationale for this terminology, we might expect the marginal propensity to save s to decline in ' good ' times and increase in the ' bad '. On the other hand, the acceleration coefficient would be expected to exhibit opposite tendencies. We stress that this terminology is qualitative, not quantitative, and is primarily intended to facilitate reference.

In this example, we examine the case where w is taken as stochastic and s deterministic.

i	Stochastic w_i	Corresponding s_i
1 (' Norm ')	2·5	0·3
2 (' Boom ')	43·7	$-0·7$
3 (' Slump ')	$-5·3$	0·9

The discrete-time, discrete-state transition matrix P_w for this case is obtained by computing the transition probabilities from the data at the sampling instants. This yields

$$P_w = \begin{bmatrix} 0·67 & 0·17 & 0·16 \\ 0·30 & 0·47 & 0·23 \\ 0·26 & 0·10 & 0·64 \end{bmatrix} \tag{24}$$

where the elements p_{ij} equal the probability that a transition from state i to state j will occur at $t=kT$, $(T=1)$. (See eqn. (3).)

The A_i matrices corresponding to (23) are

$$A_i = \begin{bmatrix} 0 & 1 \\ -2\cdot5 & 3\cdot2 \end{bmatrix}, \quad \text{` Norm ', with eigenvalues } 1\cdot360 \text{ and } 1\cdot840 \tag{25}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -43\cdot7 & 45\cdot4 \end{bmatrix}, \quad \text{` Boom ', with eigenvalues } 0\cdot984 \text{ and } 44\cdot416 \tag{26}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ 5\cdot3 & -5\cdot2 \end{bmatrix}, \quad \text{` Slump ', with eigenvalues } -6\cdot07 \text{ and } 0\cdot870 \tag{27}$$

The Q_i, M_i and b_i matrices were taken to be

$$\left.\begin{aligned} Q_1 = \begin{bmatrix} 3\cdot60 & -3\cdot80 \\ -3\cdot80 & 4\cdot87 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 238\cdot9 & -254\cdot2 \\ -254\cdot2 & 270\cdot5 \end{bmatrix} \\ Q_3 = \begin{bmatrix} 5 & -4\cdot5 \\ -4\cdot5 & 4\cdot5 \end{bmatrix} \end{aligned}\right\} \tag{28}$$

$$M_1 = [-0\cdot7 \quad 1\cdot7], \quad M_2 = [-5\cdot58 \quad 6\cdot40], \quad M_3 = [0\cdot735 \quad -0\cdot505] \tag{29}$$

$$b_1 = 2\cdot6, \quad b_2 = 1\cdot165, \quad b_3 = 1\cdot111 \tag{30}$$

The conditions on eqn. (5) are satisfied. In addition, the conditions of the following theorem (Blair 1974) are met for each of the three modes taken deterministically.

Theorem

If (A, B) is completely controllable and $(C, A - Bb^{-1}M)$ is completely observable, then there exists a unique solution $K_\infty > 0$, to the infinite time deterministic version of (16) and it yields an asymptotically stable closed-loop system.

A digital computer was used to obtain a solution to the infinite time problem for this example by iterating eqn. (13) to a steady state. The results for the stochastic system are

$$K_{1\infty} = \begin{bmatrix} 16\cdot3 & -12\cdot5 \\ -12\cdot5 & 26\cdot5 \end{bmatrix} \tag{31}$$

P1

$$K_{2\infty} = \begin{bmatrix} 1972\cdot5 & -1985\cdot4 \\ -1984\cdot5 & 2013\cdot5 \end{bmatrix} \tag{32}$$

$$K_{3\infty} = \begin{bmatrix} 28\cdot3 & -33\cdot2 \\ -33\cdot2 & 54\cdot9 \end{bmatrix} \tag{33}$$

The minimum expected costs and the optimal control policy can be obtained by applying (31)–(33) to (14) and (15). To illustrate, take an initial condition vector corresponding to one unit of output. Then $x_0' = (0, 1)$ and

$$\left.\begin{aligned} x_0' K_{1\infty} x_0 &= 26{\cdot}5 \\ x_0' K_{2\infty} x_0 &= 2013{\cdot}5 \\ x_0' K_{3\infty} x_0 &= 54{\cdot}9 \end{aligned}\right\} \tag{34}$$

The corresponding solutions for the three possible deterministic systems each taken as a separate system and with no consideration of parameter changes are :

$$K_{1\infty} = \begin{bmatrix} 14{\cdot}6 & -12{\cdot}3 \\ -12{\cdot}3 & 16{\cdot}3 \end{bmatrix} \tag{35}$$

$$K_{2\infty} = \begin{bmatrix} 1975{\cdot}0 & -1986{\cdot}2 \\ -1986{\cdot}2 & 2051{\cdot}2 \end{bmatrix} \tag{36}$$

$$K_{3\infty} = \begin{bmatrix} 27{\cdot}7 & -32{\cdot}0 \\ -32{\cdot}0 & 39{\cdot}3 \end{bmatrix} \tag{37}$$

The minimum costs and optimal controls for these separate deterministic systems can be obtained by applying these results to the deterministic versions of eqns. (14) and (15). Again, to illustrate, we can take $x_0' = (0, 1)$ and obtain

$$\left.\begin{aligned} x_0' K_{1\infty} x_0 &= 16{\cdot}3 \\ x_0' K_{2\infty} x_0 &= 2051{\cdot}2 \\ x_0' K_{3\infty} x_0 &= 39{\cdot}3 \end{aligned}\right\} \tag{38}$$

In (34) observe a reduction in the relatively high expected cost of being in mode 2 as compared to the deterministic result (38) for the same mode. This can be explained by the fact that the stochastic controls ' recognize ' the probabilities of jumping to less expensive modes and reduce their gain accordingly. Similarly, the higher gains or costs of stochastic control in modes 1 and 3 as compared to the deterministic can be attributed to ' recognition ' by the stochastic of the probability of jumping to the relatively ' bad ' mode 2.

' Bad ', as used here, refers, of course, to the cost of control not to the economic mode represented. Mode 2, recall, represents a ' boom ' in the economic process. The fact that mode 2 is more costly to control is due primarily to its greater instability as evidenced by the eigenvalues of A_2.

5. Conclusions

The feedback control of a class of linear discrete time stochastic systems has been studied. The results are the counterparts of those obtained earlier by Sworder (1969) for a class of linear continuous-time stochastic systems. As in the latter case, when a quadratic cost criterion is employed, the resulting matrix Ricatti-type equations are identical in form to those for a deterministic system except that they are stochastically intercoupled. It is anticipated that the present results will be useful when it is appropriate to model a linear dynamical system by a discrete time process with randomly jumping parameters.

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