[11] D. Swaroop, "String stability of Interconnected systems: An application to platooning in automated highway systems," Ph.D. dissertation, Dept. Mechanical Engineering, Univ. California, Berkeley, CA, Dec. 1994.

[12] D. Swaroop and D. Niemann, "Some new results on the oscillatory response of LTI systems," in *Proc. IEEE Conf. Decision Control*, Kobe, Japan, Dec. 1996.

[13] ——, "On the impulse response of LTI systems," in *Proc. Amer. Control Conf.*, Arlington, VA, June 2001.

[14] S. Darbha, "The design of controllers achieving a nonnegative closed loop impulse response for continuous-time systems," Automatica, February 2002, to be published.

[15] M. Vidyasagar, "On undershoot and nonminimum phase zeros," *IEEE Trans. Automat. Contr.*, vol. AC-31, p. 440, May 1986.

[16] ——, *Control System Synthesis: A Factorization Approach*. Cambridge, MA: MIT Press, 1985.

[17] D. V. Widder, "The inversion of the Laplace integral and the related moment problem," *Trans. Amer. Math. Soc.*, vol. 36, pp. 107–200, 1934.

[18] D. C. Youla, J. J. Bongiorno, Jr., and N. N. Lu, "Single-loop feedback stabilization of linear multi-variable dynamical plants," *Automatica*, vol. 10, pp. 159–173, 1974.

Stationary Filter for Linear Minimum Mean Square Error Estimator of Discrete-Time Markovian Jump Systems

O. L. V. Costa and S. Guerra

***Abstract*—We derive in this note a stationary filter for the linear minimum mean square error estimator (LMMSE) of discrete-time Markovian jump linear systems (MJLSs). We obtain the convergence of the error covariance matrix of the LMMSE to a stationary value under the assumption of mean square stability of the MJLS and ergodicity of the associated Markov chain. It is shown that there exists a unique solution for the stationary Riccati filter equation and, moreover, this solution is the limit of the error covariance matrix of the LMMSE. The advantage of this scheme is that it is very easy to implement and all calculations can be performed offline, leading to a linear time-invariant filter.**

***Index Terms*—Jump systems, Kalman filter, Markov parameters, Riccati equation.**

I. INTRODUCTION

In this note, we obtain a stationary linear minimum mean square error estimator (LMMSE) filter for discrete-time Markovian jump linear systems (MJLSs). The problem of optimal and suboptimal filtering for MJLSs has been addressed in [1], [9], [4], [7], [2], [12], and [13], among others, under the hypothesis of Gaussian distribution for the disturbances, and by [14] and [15] for the non-Gaussian case. Suboptimal algorithms have to be considered to limit the computational requirements, since the optimal estimator requires exponentially increasing memory and computation with time. In the aforementioned papers, the authors considered nonlinear suboptimal estimators, which require online calculations. In [11], the LMMSE for MJLSs was obtained. This filter has dimension Nn, where n is the dimension of the state variable and N is the number of states of the Markov chain. The advantage of this formulation is that it leads to a time-varying linear filter easy to implement and in which all calculations (the gain matrices) can be performed offline. Moreover it can be applied to a broader class of systems than the linear models with output uncertainty studied in [8] and [10].

In [11], the filter equation is a function of the error covariance matrix. This covariance was expressed as the difference between two recursive equations: one associated with the second moment matrix of the state variable and the other one associated with the second moment matrix of the estimator. Initially in this note, we write the error covariance matrix in terms of a recursive Riccati difference equation of dimension Nn, added with an additional term that depends on the second moment matrix of the state variable. This extra term would be zero for the case with no jumps. After that we present conditions to guarantee the convergence of the error covariance matrix to the stationary solution of an Nn dimensional algebraic Riccati equation. Moreover, we prove stability of the stationary filter. These results allow us to design a time-invariant (a fixed-gain matrix) stable suboptimal filter of LMMSE for MJLSs.

This note is organized as follows. In Section II, we present the problem formulation, some assumptions, notation, and the main theorem in [11], deriving the LMMSE. In Section III, we present the main results of this note. We obtain a recursive Riccati equation for the error covariance matrix and show its convergence to the stationary solution of an algebraic Riccati equation. In Section IV, we present some numerical examples. The note is concluded in Section V with some final comments. All proofs are presented in the Appendix.

II. PRELIMINARIES

We start by presenting some notation we will use throughout the note. We denote by $\mathbb{R}^{m\times n}$ the space of $m \times n$ real matrices and by $\mathbb{R}^m$ the space of m-dimensional real vectors. The superscript $'$ stands for transpose of a matrix. Define $\mathcal{H}^n = \{Q = (Q_1, \ldots, Q_N); Q_i \in \mathbb{R}^{n\times n}, i = 1, \ldots, N\}$ and $\mathcal{H}^{n+} = \{Q = (Q_1, \ldots, Q_N) \in \mathcal{H}^n; Q_i \geq 0, i = 1, \ldots, N\}$. For $Q = (Q_1, \ldots, Q_N) \in \mathcal{H}^{n+}$, $V = (V_1, \ldots, V_N) \in \mathcal{H}^{n+}$ we say that $Q \geq V$ if for each $i = 1, \ldots, N$ we have that $Q_i \geq V_i$. For a collection of N matrices $D_1, \ldots, D_N$, with $D_j \in \mathbb{R}^{n\times m}$, $\mathrm{diag}[D_j] \in \mathbb{R}^{Nn\times Nm}$ represents the diagonal matrix formed by D_j in the diagonal and zero elsewhere. $1_{\{\cdot\}}$ stands for the Dirac measure and $r_\sigma(\cdot)$ denotes spectral radius.

We will consider the following discrete-time Markovian jump-linear system:

$$x(k+1) = A_{\theta(k)}x(k) + C_{\theta(k)}\xi(k) \tag{1a}$$

$$y(k) = H_{\theta(k)}x(k) + G_{\theta(k)}\nu(k). \tag{1b}$$

Here, $\{x(k)\}$ denotes the $\mathbb{R}^n$-valued state sequence, $\{\xi(k)\}$ and $\{\nu(k)\}$ are random disturbances in $\mathbb{R}^{q_1}$ and $\mathbb{R}^{q_2}$, respectively, $\{y(k)\}$ is the $\mathbb{R}^m$-valued output sequence, $\{\theta(k)\}$ is a discrete-time Markov chain taking values in a finite state space $\{1, \ldots, N\}$, and with transition probability matrix $\mathbf{P} = [p_{ij}]$. We set $\pi_j(k) := P(\theta(k) = j)$. $A_i, C_i, H_i, G_i, i = 1, \ldots, N$ are matrices of appropriate dimensions. We will make the following assumptions.

- A1) $G_iG_i' > 0$ for all $i = 1, \ldots, N$.
- A2) $\{\xi(k)\}$ and $\{\nu(k)\}$ are null mean second-order, independent wide sense stationary sequences mutually independent with covariance matrices equal to the identity.
- A3) $x(0)1_{\{\theta(0)=i\}}, i = 1, \ldots, N$ are second-order random vectors with $E(x(0)x(0)'1_{\{\theta(0)=i\}}) = V_i$ and $E(x(0)1_{\{\theta(0)=i\}}) = \mu_i$ for $i = 1, \ldots, N$.
- A4) $x(0)$ and $\{\theta(k)\}$ are independent of $\{\xi(k)\}$ and $\{\nu(k)\}$.

Manuscript received February 13, 2001; revised November 26, 2001. Recommended by Associate Editor X. Zhou. This work was supported in part by CNPq (Brazilian National Research Council), CAPES, FAPESP (Research Council of the State of São Paulo), PRONEX, and IM-AGIMB.

The authors are with the Departamento de Engenharia de Telecomunicações e Controle, Escola Politécnica da Universidade de São Paulo, 05424 970 São Paulo, SP, Brazil (e-mail: oswaldo@lac.usp.br).

Publisher Item Identifier 10.1109/TAC.2002.800745.

TABLE I
SET OF PARAMETERS CONSIDERED IN THE SIMULATIONS

cases	p_{11}	p_{22}	a_1	a_2	c_1	c_2	h_1	h_2	g_1	g_2
01	0.975	0.95	0.995	0.99	0.1	0.1	1.0	1.0	5.0	5.0
02	0.995	0.99	0.995	0.995	0.5	0.5	1.0	0.8	0.8	0.8
03	0.975	0.95	0.995	0.995	0.1	5.0	1.0	1.0	1.0	1.0
04	0.975	0.95	0.995	0.25	1.0	1.0	1.0	1.0	1.0	1.0
05	0.975	0.95	0.995	0.25	0.1	0.1	1.0	1.0	5.0	5.0
06	0.975	0.95	0.995	0.25	0.1	5.0	1.0	1.0	5.0	5.0

Denote by $\mathcal{L}(y^k)$ the linear subspace spanned by $y^k := (y(k)' \dots y(0)')'$ (see [6]), that is, a random variable $r \in \mathcal{L}(y^k)$ if $r = \sum_{i=0}^{k} \alpha(i)' y(i)$ for some $\alpha(i) \in \mathbb{R}^m$, $i = 0, \dots, k$. For $k \geq 0$ and $j \in \{1, \dots, N\}$, define

$$z(k, j) := x(k) 1_{\{\theta(k)=j\}} \in \mathbb{R}^n$$

$$z(k) := \begin{pmatrix} z(k, 1) \\ \vdots \\ z(k, N) \end{pmatrix} \in \mathbb{R}^{Nn}.$$

Define also $\hat{z}(k|k-1)$ as the projection of $z(k)$ onto $\mathcal{L}(y^{k-1})$ and

$$\tilde{z}(k|k-1) := z(k) - \hat{z}(k|k-1).$$

The second-moment matrices associated to the aforementioned variables are

$$\begin{aligned} Z_i(k) &:= E(z(k, i) z(k, i)') \in \mathbb{R}^{n \times n}, & i = 1, \dots, N, \\ Z(k) &:= E(z(k) z(k)') = \operatorname{diag}(Z_i(k)) \in \mathbb{R}^{Nn \times Nn} \\ \hat{Z}(k|l) &:= E(\hat{z}(k|l) \hat{z}(k|l)') \in \mathbb{R}^{Nn \times Nn}, & 0 \leq l \leq k, \\ \tilde{Z}(k|l) &:= E(\tilde{z}(k|l) \tilde{z}(k|l)') \in \mathbb{R}^{Nn \times Nn}, & 0 \leq l \leq k. \end{aligned}$$

We consider the following augmented matrices:

$$A := \begin{bmatrix} p_{11} A_1 & \dots & p_{N1} A_N \\ \vdots & \dots & \vdots \\ p_{1N} A_1 & \dots & p_{NN} A_N \end{bmatrix} \in \mathbb{R}^{Nn \times Nn} \tag{2}$$

$$\begin{aligned} G(k) &:= [G_1 \pi_1(k)^{1/2} \dots G_N \pi_N(k)^{1/2}] \in \mathbb{R}^{m \times N q_2} \\ H &:= [H_1 \dots H_N] \in \mathbb{R}^{m \times Nn}. \end{aligned}$$

We present now the main result of [11], derived from geometric arguments as in [6].

Theorem 1: Consider the system represented by (1a) and (1b), with assumptions A1)–A4). Then the LMMSE $\hat{x}(k|k)$ is given by

$$\hat{x}(k|k) = \sum_{i=1}^{N} \hat{z}(k, i|k) \tag{3}$$

where $\hat{z}(k|k)$ satisfies the recursive equation

$$\begin{aligned} \hat{z}(k|k) &= \hat{z}(k|k-1) + \tilde{Z}(k|k-1) H' (H \tilde{Z}(k|k-1) H' \\ &\quad + G(k) G(k)')^{-1} (y(k) - H \hat{z}(k|k-1)) \\ \hat{z}(k|k-1) &= A \hat{z}(k-1|k-1), \ k \geq 1 \end{aligned} \tag{4}$$

$$\hat{z}(0|-1) = E(z(0)) = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix}.$$

The positive–semidefinite matrices $\tilde{Z}(k|k-1) \in \mathbb{R}^{Nn \times Nn}$ are obtained from

$$\tilde{Z}(k|k-1) = Z(k) - \hat{Z}(k|k-1) \tag{5}$$

where $Z(k) = \operatorname{diag}(Z_j(k))$ are given by the recursive equation

$$Z_j(k+1) = \sum_{i=1}^{N} p_{ij} A_i Z_i(k) A_i' + \sum_{i=1}^{N} p_{ij} \pi_i(k) C_i C_i', \ Z_j(0) = V_j, \quad j = 1, \dots, N \tag{6}$$

and $\hat{Z}(k|k-1)$ are given by the recursive equation

$$\begin{aligned} \hat{Z}(k|k) &= \hat{Z}(k|k-1) + \hat{Z}(k|k-1) H' \\ &\quad \cdot \left(H \tilde{Z}(k|k-1) H' + G(k) G(k)' \right)^{-1} \\ &\quad \cdot H \hat{Z}(k|k-1) \\ \hat{Z}(k|k-1) &= A \hat{Z}(k-1|k-1) A', \ \hat{Z}(0|-1) \\ &= E(z(0)) E(z(0))'. \end{aligned} \tag{7}$$

Remark 1: Notice that in Theorem 1 the inverse of $H\tilde{Z}(k|k-1)H' + G(k)G(k)'$ is well defined since for each $k = 0, 1, \dots$ there exists $\iota(k) \in \{1, \dots, N\}$ such that $\pi_{\iota(k)}(k) > 0$ and from condition A1)

$$\begin{aligned} & H \tilde{Z}(k|k-1) H' + G(k) G(k)' \\ & \quad \geq G(k) G(k)' \\ & \quad = \sum_{i=1}^{N} \pi_i(k) G_i G_i' \geq \pi_{\iota(k)}(k) G_{\iota(k)} G_{\iota(k)}' > 0. \end{aligned}$$

III. MAIN RESULTS

In Theorem 1, the term $\tilde{Z}(k|k-1)$ is expressed in (5) as the difference between $Z(k)$ and $\hat{Z}(k|k-1)$, which are obtained form the recursive equations (6) and (7). In the next lemma, we will write $\tilde{Z}(k|k-1)$ directly as a recursive Riccati equation, with an additional term that depends on the second moment matrices $Z_i(k)$. Notice that this extra term would be zero for the case in which there are no jumps ($N = 1$). We define $Q(k) := (Z_1(k), \dots, Z_N(k)) \in \mathcal{H}^{n+}$ and the linear operator $\mathcal{B}(\cdot)$: $\mathcal{H}^n \to \mathbb{R}^{Nn \times Nn}$ as follows: for $\Upsilon = (\Upsilon_1, \dots, \Upsilon_N) \in \mathcal{H}^n$

$$\mathcal{B}(\Upsilon) := \operatorname{diag} \left[\sum_{i=1}^{N} p_{ij} A_i \Upsilon_i A_i' \right] - A(\operatorname{diag}[\Upsilon_i]) A'.$$

Notice that if $\Upsilon = (\Upsilon_1, \dots, \Upsilon_N) \in \mathcal{H}^{n+}$ then $\mathcal{B}(\Upsilon) \geq 0$. Indeed, consider $v = (v_1' \dots v_N')' \in \mathbb{R}^{Nn}$. Then, it is easy to check that

$$\begin{aligned} & v' \mathcal{B}(\Upsilon) v \\ & \quad = \sum_{i=1}^{N} E_i((v_{\theta(1)} - E_i(v_{\theta(1)}))' A_i \Upsilon_i A_i' (v_{\theta(1)} - E_i(v_{\theta(1)}))) \\ & \quad \geq 0 \end{aligned}$$

where $E_i(\cdot)$ represents the expected value conditioned on $\theta(0) = i$. We also define

$$T(k) := A \tilde{Z}(k|k-1) H' \left(H \tilde{Z}(k|k-1) H' + G(k) G(k)' \right)^{-1} \tag{8}$$

[recall from Remark 1 that the inverse in (8) is well defined]. We have the following Lemma, proved in the Appendix.

Lemma 1: $\tilde{Z}(k|k-1)$ satisfies the following recursive Riccati equation:

$$\begin{aligned} \tilde{Z}(k+1|k) &= A \tilde{Z}(k|k-1) A' + \mathcal{B}(Q(k)) \\ &\quad + \operatorname{diag} \left[\sum_{i=1}^{N} \pi_i(k) p_{ij} C_i C_i' \right] - A \tilde{Z}(k|k-1) H' \end{aligned}$$

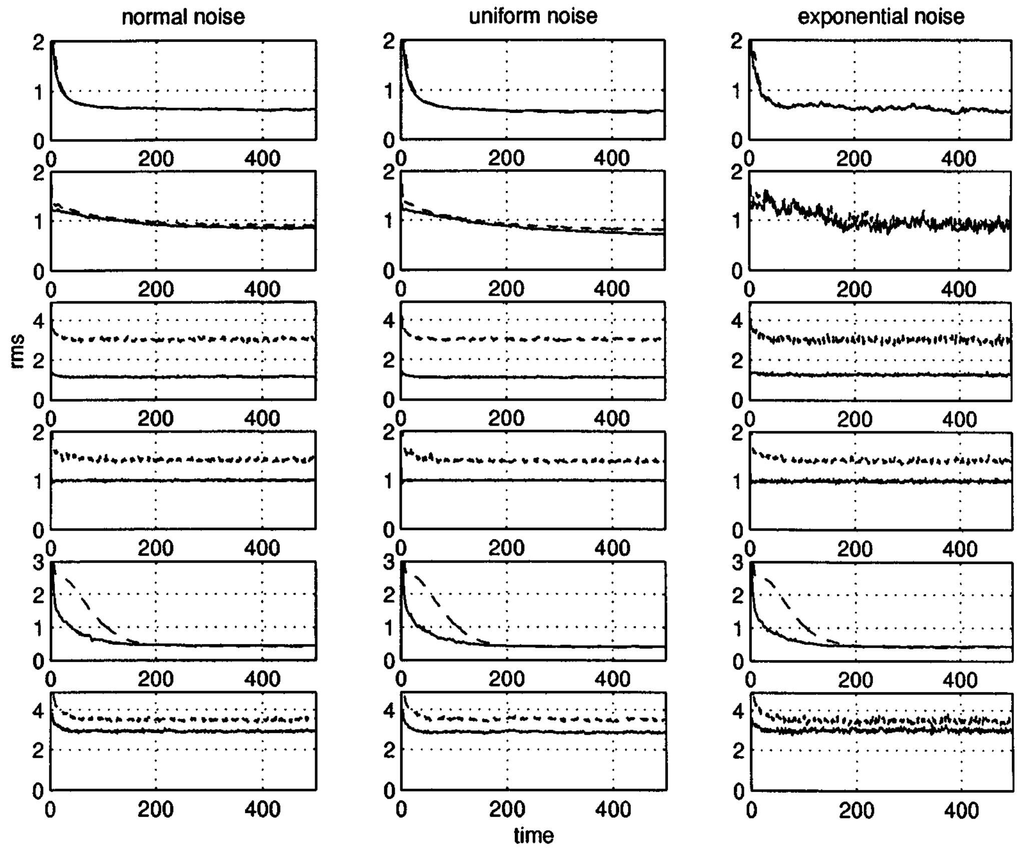

Fig. 1. Comparisons between (solid line) IMM and stationary (dash line) LMMSE filters.

$$\cdot \left(H\tilde{Z}(k|k-1)H' + G(k)G(k)'\right)^{-1}$$
$$\cdot H\tilde{Z}(k|k-1)A' \qquad (9)$$

where $Q(k) = (Z_1(k), \ldots, Z_N(k)) \in \mathcal{H}^{n+}$ are given by the recursive equation (6).

Note that for the case with no jumps ($N = 1$) we would have $\mathcal{B}(Q(k)) = 0$ and therefore (9) would reduce to the standard recursive Ricatti equation for the Kalman filter. Equations (6) and (9) describe a recursive Riccati equation for $\tilde{Z}(k|k-1)$. We will now establish its convergence when $k \to \infty$. We will assume that

H1) system (1a) is mean square stable (MSS) according to the definition in [5];

H2) the Markov chain is ergodic.

From hypothesis H2), it follows that $\lim_{k\to\infty} P(\theta(k) = i)$ exists and it is independent from $\theta(0)$. We define

$$\pi_i := \lim_{k\to\infty} P(\theta(k) = i) = \lim_{k\to\infty} \pi_i(k)$$
$$G := [G_1\pi_1^{1/2} \ldots G_N\pi_N^{1/2}] \in \mathbb{R}^{m\times Nq_2}.$$

From hypothesis H1) and H2), and [5, Prop. 8], $Q(k) \to Q$ as $k \to \infty$, where $Q = (Z_1, \ldots, Z_N) \in \mathcal{H}^{n+}$ is the unique solution that satisfies

$$Z_j = \sum_{i=1}^{N} p_{ij}(A_i Z_i A_i' + \pi_i C_i C_i'), \qquad j = 1, \ldots, N. \qquad (10)$$

In what follows, we define for any matrix $Z \in \mathbb{R}^{Nn\times Nn}$, $Z \geq 0$, $T(Z) \in \mathbb{R}^{Nn\times Nn}$ as

$$T(Z) := AZH'(HZH' + GG')^{-1}.$$

As in Remark 1, we have that $HZH' + GG' > 0$ and, thus, the aforementioned inverse is well defined.

The following Theorem, proved in the Appendix, establishes the asymptotic convergence of $\tilde{Z}(k|k-1)$.

Theorem 2: Suppose that hypothesis A1), H1), and H2) hold. Consider the algebraic Riccati equation given by

$$Z = AZA' + \operatorname{diag}\left[\sum_{i=1}^{N} \pi_i p_{ij} C_i C_i'\right]$$
$$-AZH'(HZH' + GG')^{-1}HZA' + \mathcal{B}(Q) \qquad (11)$$

where $Q = (Z_1, \ldots, Z_N) \in \mathcal{H}^{n+}$ satisfies (10). Then there exists a unique positive–semidefinite solution $P \in \mathbb{R}^{Nn\times Nn}$ to (11). Moreover, $r_\sigma(A - T(P)H) < 1$ and for any $Q(0) = (Z_1(0), \ldots, Z_N(0))$ with $Z_i(0) \geq 0$, $i = 1, \ldots, N$, and $\tilde{Z}(0|-1) = \operatorname{diag}[Z_i(0)] - E(z(0))E(z(0))' \geq 0$ we have that $\tilde{Z}(k+1|k)$ given by (6) and (9) satisfies

$$\tilde{Z}(k+1|k) \stackrel{k\to\infty}{\to} P.$$

IV. Numerical Examples

In this section, we present some numerical comparisons between the stationary LMMSE filter and the IMM filter [9]. Recall that to implement the stationary LMMSE filter we first have to solve the system of linear equations (10) with unique solution Z_i, $i = 1, \ldots, N$. After that, we plug $Q = (Z_1, \ldots, Z_N)$ into (11) and solve the corresponding algebraic Riccati equation to obtain P. The stationary LMMSE estimator $\hat{x}(k|k)$ is given by (3) where $\hat{z}(k|k)$ satisfies the (time-invariant) recursive equations (4) replacing $\tilde{Z}(k|k-1)$ by P. So the filter is very easy to implement and all calculations can be performed offline. For the numerical examples we considered $x(0)$ Gaussian with mean 10 and variance 10, $\theta(k) \in \{1, 2\}$, $\{\xi(k)\}$ and $\{\nu(k)\}$ are independent sequence of noises, and $\pi_1(0) = \pi_2(0) = 0.5$. We considered six cases for the parameters a_i, c_i, h_i, g_i, and p_{ij}, shown in Table I. For all cases, we performed 4000 Monte Carlo simulations from $k = 1, \ldots, 500$, with the values of $\theta(k)$ generated randomly. Both filters were compared under the same conditions.

The results obtained are in Fig. 1, showing the square root of the mean square error (rms) through time for each of the six cases studied. We considered three kinds of distributions for the noise: normal, uniform, and exponential. The first graphic represents case 1, and so on. The behavior of the LMMSE filter as well as the IMM do not change significantly for each kind of noise considered. We can see from the simulations that the IMM had a better performance than the stationary LMMSE in cases 3, 4, and 6, and started better in case 5, with the same asymptotic error. For cases 1 and 2, both filters had the same behavior. We believe that these simulations suggest that the stationary LMMSE can be a good alternative in situations where computing power is at a premium. Besides being very simple to implement, all calculations for the filter can be performed offline, so that the amount of online signal processing required is very modest.

V. Final Remarks

In this note, we have obtained sufficient conditions for the convergence of the error covariance matrix to a stationary value for the LMMSE of MJLSs. The LMMSE for MJLSs was obtained in [11]. In this note it was shown that if the MJLS MSS and the Markov chain is ergodic then the error covariance matrix will converge to the unique positive–semidefinite solution of an Nn-dimensional algebraic Riccati equation associated to the problem. Moreover, the filter error equation with the stationary gain will be stable.

The main advantage of this scheme is that it is very simple to implement, and all calculations can be performed offline. The resulting filter is a discrete-time invariant linear system. Another advantage is that we can consider uncertainties in the parameters of the system through, for instance, an LMI approach. The results of these studies will be presented in a forthcoming paper.

Appendix

In this appendix, we present the proofs of the results in Section III.

Proof. Lemma 1: From standard results on Riccati difference equations [6] the recursive equation (9) is equivalent to

$$\begin{aligned}\tilde{Z}(k+1|k) = &(A - T(k)H)\tilde{Z}(k|k-1)(A-T(k)H)' \\ &+ \mathcal{B}(Q(k)) + \operatorname{diag}\left[\sum_{i=1}^{N} \pi_i(k) p_{ij} C_i C_i'\right] \\ &+ T(k)G(k)G(k)'T(k)'. \end{aligned} \tag{12}$$

Writing (1a) in terms of $z(k)$, we obtain

$$z(k+1) = Az(k) + M(k+1)z(k) + \vartheta(k) \tag{13}$$

where

$$M(k+1,j) = [\,(1_{\{\theta(k+1)=j\}} - p_{1j})A_1 1_{\{\theta(k)=1\}} \quad \ldots \quad (1_{\{\theta(k+1)=j\}} - p_{Nj})A_N 1_{\{\theta(k)=N\}}\,]$$

$$M(k+1) = \begin{bmatrix} M(k+1,\,1) \\ \vdots \\ M(k+1,\,N) \end{bmatrix}$$

$$\vartheta(k) = \begin{bmatrix} 1_{\{\theta(k+1)=1\}} C_{\theta(k)} \xi(k) \\ \vdots \\ 1_{\{\theta(k+1)=N\}} C_{\theta(k)} \xi(k) \end{bmatrix}.$$

From (4) and (8), we obtain

$$\hat{z}(k+1|k) = A\hat{z}(k|k-1) + T(k)H\tilde{z}(k|k-1) + T(k)G_{\theta(k)}\nu(k) \tag{14}$$

and, thus, from (13) and (14) we get

$$\tilde{z}(k+1|k) = (A - T(k)H)\tilde{z}(k|k-1) + \vartheta(k) + M(k+1)z(k) - T(k)G_{\theta(k)}\nu(k). \tag{15}$$

Therefore, from (15), the recursive equation for $\tilde{Z}(k|k-1)$ is given by

$$\begin{aligned}&\tilde{Z}(k+1|k) \\ &= (A - T(k)H)\tilde{Z}(k|k-1)(A-T(k)H)' \\ &\quad + E(M(k+1)z(k)z(k)'M(k+1)') + E(\vartheta(k)\vartheta(k)') \\ &\quad + T(k)E(G_{\theta(k)}\nu(k)\nu(k)'G_{\theta(k)}')T(k)' \end{aligned} \tag{16}$$

and after some algebraic manipulations we obtain

$$E(M(k+1)z(k)z(k)'M(k+1)') = \mathcal{B}(Q(k)) \tag{17}$$

$$E(\vartheta(k)\vartheta(k)') = \operatorname{diag}\left[\sum_{i=1}^{N} p_{ij} C_i C_i' \pi_i(k)\right] \tag{18}$$

$$E(G_{\theta(k)}\nu(k)\nu(k)'G_{\theta(k)}) = G(k)G(k)'. \tag{19}$$

Replacing (17)–(19) into (16), we get (12). ■

In order to prove Theorem 2, we will need two auxiliary results. Let κ be such that $\inf_{\ell \geq \kappa} \pi_i(\ell) > 0$ for all $i = 1, \ldots, N$ (since $\pi_i(k) \overset{k\to\infty}{\to} \pi_i > 0$ we have that this number exists). Define $\alpha_i(k) = \inf_{\ell \geq k} \pi_i(\ell + \kappa)$. Obviously

$$\begin{aligned}\pi_i(k+\kappa) &\geq \alpha_i(k) \\ &\geq \alpha_i(k-1), \qquad k = 1,\,2,\,\ldots, \qquad i = 1,\,\ldots,\,N \end{aligned} \tag{20}$$

and $\alpha_i(k) \overset{k\to\infty}{\to} \pi_i$ exponentially fast. Define now $\overline{Q}(k) := (\overline{Z}_1(k),\,\ldots,\,\overline{Z}_N(k)) \in \mathcal{H}^{n+}$ with $\overline{Z}_j(0) = 0$, $j = 1,\,\ldots,\,N$ and

$$\overline{Z}_j(k+1) = \sum_{i=1}^{N} p_{ij}\left(A_i \overline{Z}_i(k) A_i' + \alpha_i(k) C_i C_i'\right).$$

In Lemma 2, recall that $Q := (Z_1,\,\ldots,\,Z_N) \in \mathcal{H}^{n+}$ is the unique solution that satisfies (10).

Lemma 2: $\overline{Q}(k) \overset{k\to\infty}{\to} Q$ and for each $k = 0,\,1,\,2,\,\ldots$

$$Q(k+\kappa) \geq \overline{Q}(k) \geq \overline{Q}(k-1). \tag{21}$$

Proof: From [5, Prop. 8], we get that $\overline{Q}(k) \overset{k\to\infty}{\to} Q$. Let us show now (21) by induction on k. For $k = 0$, the result is immediate, since $Q(\kappa) \geq 0 = \overline{Q}(0)$ and $\overline{Q}(1) \geq 0 = \overline{Q}(0)$. Suppose that (21) holds for k. Then, from (20) and (21), we have

$$\begin{aligned} Z_j(k+1+\kappa) &= \sum_{i=1}^{N} p_{ij}(A_i Z_i(k+\kappa)A_i' + \pi_i(k+\kappa)C_i C_i') \\ &\geq \sum_{i=1}^{N} p_{ij}\left(A_i \overline{Z_i}(k) A_i' + \alpha_i(k) C_i C_i'\right) \\ &= \overline{Z}_j(k+1) \\ &\geq \sum_{i=1}^{N} p_{ij}\left(A_i \overline{Z_i}(k-1) A_i' + \alpha_i(k-1) C_i C_i'\right) \\ &= \overline{Z}_j(k) \end{aligned}$$

completing the induction argument in (21). ■

Now define

$$R(k+1) = AR(k)A' + \mathcal{B}(\overline{Q}(k)) + \text{diag}\left[\sum_{i=1}^{N} \alpha_i(k) p_{ij} C_i C_i'\right] + AR(k)H'(HR(k)H' + \overline{G}(k)\overline{G}(k)')^{-1} HR(k)A'$$

where $R(0) = 0$ and $\overline{G}(k) = [G_1 \alpha_1(k)^{1/2} \ldots G_N \alpha_N(k)^{1/2}]$. Notice that from the definition of κ and condition A1) we have that the inverse of $HR(k)H' + \overline{G}(k)\overline{G}(k)'$ is well defined.

Lemma 3: For each $k = 0, 1, \ldots$

$$0 \leq R(k) \leq R(k+1) \leq \tilde{Z}(k+1+\kappa | k+\kappa). \tag{22}$$

Proof: Let us show (22) by induction on k. Setting $S(k) = AR(k)H'(HR(k)H' + \overline{G}(k)\overline{G}(k)')^{-1}$ it follows that, if $R(k) \leq \tilde{Z}(k+\kappa | k-1+\kappa)$, then from (20) and (21):

$$\begin{aligned} R(k+1) &= (A - T(k+\kappa)H)R(k)(A - T(k+\kappa)H)' \\ &\quad + \mathcal{B}(\overline{Q}(k)) + \text{diag}\left[\sum_{i=1}^{N} \alpha_i(k) p_{ij} C_i C_i'\right] \\ &\quad + T(k+\kappa)\overline{G}(k)\overline{G}(k)'T(k+\kappa)' \\ &\quad - (T(k+\kappa) - S(k))(HR(k)H' \\ &\quad + \overline{G}(k)\overline{G}(k)')(T(k+\kappa) - S(k))' \\ &\leq (A - T(k+\kappa)H)\tilde{Z}(k+\kappa | k-1+\kappa)(A - T(k+\kappa)H)' \\ &\quad + \mathcal{B}(Q(k+\kappa)) + \text{diag}\left[\sum_{i=1}^{N} \pi_i(k+\kappa) p_{ij} C_i C_i'\right] \\ &\quad + T(k+\kappa)G(k+\kappa)G(k+\kappa)'T(k+\kappa)' \\ &= \tilde{Z}(k+1+\kappa | k+\kappa). \end{aligned}$$

Obviously, $R(0) = 0 \leq \tilde{Z}(\kappa | \kappa - 1)$, showing that $R(k) \leq \tilde{Z}(k + \kappa | k-1+\kappa)$ for all $k = 0, 1, 2, \ldots$. Similarly, if $R(k) \geq R(k-1)$, then from (20) and (21)

$$\begin{aligned} R(k) &= (A - S(k)H)R(k-1)(A - S(k)H)' + \mathcal{B}(\overline{Q}(k-1)) \\ &\quad + \text{diag}\left[\sum_{i=1}^{N} \alpha_i(k-1) p_{ij} C_i C_i'\right] \\ &\quad + S(k)\overline{G}(k-1)\overline{G}(k-1)'S(k)' \\ &\quad - (S(k) - S(k-1))(HR(k-1)H' \\ &\quad + \overline{G}(k-1)\overline{G}(k-1)')(S(k) - S(k-1))' \\ &\leq (A - S(k)H)R(k)(A - S(k)H)' + \mathcal{B}(Q(k)) \\ &\quad + \text{diag}\left[\sum_{i=1}^{N} \alpha_i(k) p_{ij} C_i C_i'\right] + S(k)\overline{G}(k)\overline{G}(k)'S(k)' \\ &= R(K+1) \end{aligned}$$

and since $R(0) = 0 \leq R(1)$ the induction argument is completed for (22). ∎

Proof. Theorem 2: From the MSS of (1a) [Hypothesis H1)], we have from [5, Prop. 5] that $r_\sigma(A) < 1$ and thus according to standard results for algebraic Riccati equations there exists a unique positive–semidefinite solution $P \in \mathbb{R}^{Nn \times Nn}$ to (11) and moreover $r_\sigma(A - T(P)H) < 1$ (see [3]). Furthermore, P satisfies

$$P = (A - T(P)H)P(A - T(P)H)' + \mathcal{B}(Q) + \text{diag}\left[\sum_{i=1}^{N} \pi_i p_{ij} C_i C_i'\right] + T(P)GG'T(P)'. \tag{23}$$

Define $P(0) = \tilde{Z}(0 | -1)$ and

$$P(k+1) = (A - T(P)H)P(k)(A - T(P)H)' + \mathcal{B}(Q(k)) + \text{diag}\left[\sum_{i=1}^{N} \pi_i(k) p_{ij} C_i C_i'\right] + T(P)G(k)G(k)'T(P)'. \tag{24}$$

Let us show by induction on k that $P(k) \geq \tilde{Z}(k|k-1)$. Since

$$\begin{aligned} \tilde{Z}(k+1|k) &= \mathcal{B}(Q(k)) + \text{diag}\left[\sum_{i=1}^{N} \pi_i(k) p_{ij} C_i C_i'\right] \\ &\quad + (A - T(P)H)\tilde{Z}(k|k-1)(A - T(P)H)' \\ &\quad + T(P)G(k)G(k)'T(P)' - (T(k) - T(P)) \\ &\quad \cdot \left(H\tilde{Z}(k|k-1)H' + G(k)G(k)'\right)(T(k) - T(P))' \end{aligned} \tag{25}$$

we have from (24) and (25) that

$$\begin{aligned} &\left(P(k+1) - \tilde{Z}(k+1|k)\right) \\ &\quad = (A - T(P)H)\left(P(k) - \tilde{Z}(k|k-1)\right)(A - T(P)H)' \\ &\qquad + (T(k) - T(P))\left(H\tilde{Z}(k|k-1)H' + G(k)G(k)'\right) \\ &\qquad \cdot (T(k) - T(P))'. \end{aligned} \tag{26}$$

By definition, $P(0) = \tilde{Z}(0|-1)$. Suppose that $P(k) \geq \tilde{Z}(k|k-1)$. From (26), we have that $P(k+1) \geq \tilde{Z}(k+1|k)$. Therefore, we have shown by induction that $P(k) \geq \tilde{Z}(k|k-1)$ for all $k = 0, 1, 2, \ldots$. From H1) and H2), we have that $Q(k) \overset{k\to\infty}{\to} Q$, $G(k) \overset{k\to\infty}{\to} G$ and $\text{diag}[\sum_{i=1}^{N} \pi_i(k) p_{ij} C_i C_i'] \overset{k\to\infty}{\to} \text{diag}[\sum_{i=1}^{N} \pi_i p_{ij} C_i C_i']$ exponentially fast. From $r_\sigma(A - T(P)H) < 1$ and [5, Prop. 2] we get that $P(k) \overset{k\to\infty}{\to} \overline{P}$, where $\overline{P}$ satisfies

$$\overline{P} = (A - T(P)H)\overline{P}(A - T(P)H)' + \mathcal{B}(Q) + \text{diag}\left[\sum_{i=1}^{N} \pi_i p_{ij} C_i C_i'\right] + T(P)GG'T(P)' \tag{27}$$

and $\overline{P}$ is the unique solution to (27). Recalling that P satisfies (23), we have that P is also a solution to (27) and from uniqueness, $\overline{P} = P$. Therefore

$$\tilde{Z}(k|k-1) \leq P(k) \tag{28}$$

and $P(k) \overset{k\to\infty}{\to} P$. From (28) and (22) it follows that $0 \leq R(k) \leq R(k+1) \leq P(k+1+\kappa)$ and, thus, we can conclude that $R(k) \uparrow R$ whenever $k \to \infty$ for some $R \geq 0$. Moreover, from the fact that $\alpha_i(k) \overset{k\to\infty}{\to} \pi_i$ and $\overline{Q}(k) \overset{k\to\infty}{\to} Q$ we have that R satisfies (11). From uniqueness of the positive–semidefinite solution to (11), we can conclude that $R = P$. From (28) and (22), $R(k) \leq \tilde{Z}(k+\kappa|k-1+\kappa) \leq P(k+\kappa)$ and since $R(k) \uparrow P$, and $P(k) \to P$ as $k \to \infty$, we get that $\tilde{Z}(k|k-1) \overset{k\to\infty}{\to} P$. ∎

REFERENCES

[1] G. A. Ackerson and K. S. Fu, “On the estate estimation in switching environments,” *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 10–17, Feb. 1970.

[2] Y. Bar-Shalom and X. R. Li, *Estimation and Tracking. Principles, Techniques, and Software*. Norwood, MA: Artech House, 1993.

[3] F. M. Callier and Ch. A. Desoer, *Linear System Theory*. New York: Springer-Verlag, 1999.

[4] C. G. Chang and M. Athans, “State estimation for discrete systems with switching parameters,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-14, pp. 418–424, May 1978.

[5] O. L. V. Costa and M. D. Fragoso, “Stability results for discrete-time linear systems with markovian jumping parameters,” *J. Math. Anal. Appl.*, vol. 179, pp. 154–178, 1993.

[6] M. H. A. Davis and R. B. Vinter, *Stochastic Modeling and Control*. New York: Chapman & Hall, 1985.

[7] F. Dufour and R. J. Elliott, “Adaptive control of linear systems with markov perturbations,” *IEEE Trans. Automat. Contr.*, vol. 43, pp. 351–372, Mar. 1997.

[8] M. T. Hadidi and C. S. Schwartz, “Linear recursive state estimators under uncertain observations,” *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 944–948, Dec. 1979.

[9] H. A. P. Blom and Y. Bar-Shalom, "The interacting multiple model agorithm for systems with markovian switching coefficients," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 780–783, Aug. 1988.
[10] N. E. Nahi, "Optimal recursive estimation with uncertain observation," *IEEE. Trans. Inform. Theory*, vol. IT-15, pp. 457–462, July 1969.
[11] O. L. V. Costa, "Linear minimum mean square error estimation for discrete-time Markovian jump linear systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1685–1689, Aug. 1994.
[12] J. K. Tugnait, "Adaptive estimation and identification for discrete systems with Markov jump parameters," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 1054–1064, Oct. 1982.
[13] ——, "Detection and estimation for abruptly changing systems," *Automatica*, vol. 18, pp. 607–615, 1982.
[14] Q. Zhang, "Optimal filtering of discrete-time hybrid systems," *J. Optim. Theory Appl.*, vol. 100, pp. 123–144, 1999.
[15] ——, "Hybrid filtering for linear systems with non-Gaussian distturbances," *IEEE Trans. Automat. Contr.*, vol. 45, pp. 50–61, Jan. 2000.

Global Stabilization of Cascade Systems by C^0 Partial-State Feedback

Wei Lin and Radom Pongvuthithum

***Abstract*—This note shows how the *nonsmooth but continuous* feedback design approach developed recently for global stabilization of nonlinear systems with uncontrollable unstable linearization, and the notion and properties of the input-to-state stability (ISS) Lyapunov function can be effectively coupled, resulting in globally stabilizing C^0 partial-state feedback controllers for a class of cascade systems which may *not be smoothly stabilizable*, even locally.**

***Index Terms*—Cascade systems, global stabilization, non-Lipschitz continuous controllers, partial-state feedback, uncontrollable unstable linearization.**

I. Introduction and Discussion

In this note, we consider a class of nonlinear cascade systems described by equations of the form

$$\begin{aligned} \dot{z} &= f_0(z, x_1) \\ \dot{x}_1 &= d_1(t)x_2^{p_1} + f_1(z, x_1, x_2) \\ &\ \vdots \\ \dot{x}_{r-1} &= d_{r-1}(t)x_r^{p_{r-1}} + f_{r-1}(z, x_1, \ldots, x_r) \\ \dot{x}_r &= d_r(t)u^{p_r} + f_r(z, x_1, \ldots, x_r, u) \end{aligned} \tag{1.1}$$

where $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$ is the measurable state, $z \in \mathbb{R}^{n-r}$ the unmeasurable state and $u \in \mathbb{R}$ the control input, respectively, $p_i \geq 1$, $i = 1, \ldots, r$, are *odd* integers. The functions $f_i: \mathbb{R}^{n-r+i+1} \to \mathbb{R}$, $i = 1, \ldots, r$, are C^1 with $f_i(0, \ldots, 0) = 0$, and $f_0: \mathbb{R}^{n-r+1} \to \mathbb{R}^{n-r}$ is C^1 with $f_0(0, 0) = 0$, and $d_i: \mathbb{R} \to \mathbb{R}$ is C^0, representing an unknown time-varying parameter.

Manuscript received December 17, 2001; revised March 9, 2002. Recommended by Associate Editor M. Reyhanoglu. This work was supported in part by the National Science Foundation under Grants ECS-9875273, ECS-9906218, DMS-9972045, and DMS-0203387.

The authors are with the Department of Electrical Engineering and Computer Science, Case Western Reserve University, Cleveland, OH 44106 USA (e-mail: linwei@nonlinear.cwru.edu).

Publisher Item Identifier 10.1109/TAC.2002.800743.

The purpose of this note is to point out that by combining the *nonsmooth but continuous* feedback design approach proposed recently in [15] with the concepts and properties of input to state stability (ISS) and the ISS Lyapunov function [18], [17], [11], it is possible to construct explicitly, under appropriate conditions, a non-Lipschitz continuous *partial-state* feedback control law $u(x_1, \ldots, x_r)$ [i.e., use only the measurable state x of (1.1)], making the trivial solution $(z, x) = 0$ of the cascade system (1.1) globally strongly stable in the sense of Kurzweil [12], [15].

The problem of *nonsmooth* or *continuous* feedback stabilization arises naturally when a controlled plant cannot be dealt with, even locally, by any smooth feedback, due to the inherent nonlinearity of the system, e.g., the linearized system has uncontrollable modes associated with eigenvalues on the open right-half plane. The problem has received considerable attention since the original work of Kawski [9], [10], in which it was proved, among the other things, how a local C^0 non-Lipschitz stabilizer can be designed for the benchmark example—$\dot{x}_1 = x_2^3 + x_1, \dot{x}_2 = u$—which is not smoothly stabilizable. Since then, several papers [1]–[5], [9], [10], [19] have investigated the continuous feedback stabilization problem for nonlinear systems which may not have a controllable first approximation, but may be approximated by controllable, high-order homogeneous systems. The success of these developments relies on the powerful notions such as homogeneity with respect to a family of dilation and homogeneous approximation [9], [10], [7], [8]. However, all the stabilization results obtained in [1], [4], [3], [5], [9], and [10] are *local* and *nonconstructive*, due to the use of a homogeneous or nilpotent approximation and Hermes' robust stability theorem for homogeneous systems [7], [16].

Recently, we have developed a *constructive* feedback design method for global stabilization of a significant class of nonlinear systems such as (1.1) via *continuous* state feedback [14], [15]. In particular, using the theory of homogeneous systems [7]–[10], [3], [19], together with the tool of *adding a power integrator* [13], we presented in [14], [15] a systematic design algorithm which enables one to explicitly construct, in an iterative manner, a C^0 globally stabilizing state feedback controller as well as a C^1 control Lyapunov function which is positive definite and proper. Our *continuous* feedback strategy has overcome the obstacle such as uncontrollable unstable linearization of the system, and hence resulted in solutions to a variety of challenging control problems that have remained open and unsolved for decades [15].

All of the aforementioned contributions, however, only study the case where the entire system states are available for the design of feedback control laws. In this note, we consider a more realistic situation where only the partial-states of the cascade system (1.1), namely, $(x_1, \ldots, x_r)$, are measurable and can be used in the feedback design. The main objective of the note is two-folds: to give sufficient conditions and to develop a machinery for the explicit construction of *non-Lipschitz continuous partial-state* feedback control laws that render the cascade system (1.1) globally strongly stable [12], [15].

The class of cascade systems considered in this note is characterized by the following conditions.

Assumption 1.1: There exists a C^1 Lyapunov function $U_0(z)$, which is positive–definite and proper, such that

$$\frac{\partial U_0}{\partial z} f_0(z, x_1) \leq -\|z\|^2 + \alpha(x_1) \tag{1.2}$$

where $\alpha(x_1)$ is a C^2 positive–definite function with $\alpha(0) = 0$.

Assumption 1.2: For $i = 1, \ldots, r$, there are positive real numbers λ_i and μ_i satisfying

$$0 < \lambda_i \leq d_i(t) \leq \mu_i.$$