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# $\mathcal{H}_\infty$-filtering for Markov jump linear systems with partial information on the jump parameter

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ABSTRACT

We study in this work the $\mathcal{H}_\infty$-filtering problem in a partial information context. We suppose that the state of the Markov chain $\theta(k)$ is not available to the filter, but only an estimation coming from a detector and represented by $\hat{\theta}(k)$. We present two main results related to the synthesis of filters that depend only on $\hat{\theta}(k)$ such that the $\mathcal{H}_\infty$ norm in relation to the estimation error is limited: the case in which the transition and detection probabilities are not exactly known, but belong to distinct convex sets; and the Bernoulli case in which we derive necessary and sufficient conditions for the filter synthesis. All the results are given in terms of linear matrix inequalities and are illustrated by two numerical examples of systems prone to faults.



## 1. Introduction

A great interest in systems subject to abrupt changes has arisen in recent years, a trend that is particularly evident due to the great number of critical applications that require a high level of reliability such as nuclear power generation, aircraft transportation, automotive systems, among others. In this context, the necessity of properly detecting and readily acting on occurrences such as component failures is of paramount importance for the safety and reliability of the process. For that, the study of fault-tolerant control systems (FTCS) and its direct application in active fault-tolerant control systems (AFTCS) provides an ever-growing framework to cope with system failures (abrupt changes), see, for instance, Hwang et al. (2010); Mahmoud et al. (2003). Regarding faults that are inherently stochastic, the use of appropriate theoretical tools for modeling and acting on these situations is required and, in such cases, Markov jump linear systems (MJLS) have been successfully used for both objectives. There is by now a large body of works in MJLS theory, such as Boukas (2006); Costa et al. (2005); 2013); Dragan et al. (2013); relevant works in the usage of MJLS in AFTCS as in Aberkane et al. (2008); Costa et al. (2015); Mahmoud et al. (2003); Siqueira and Terra (2009); Siqueira et al. (2007); and in Networked Control Systems (NCS) as well, see Hespanha et al. (2007) for a general survey in the area, and Geromel and Gabriel (2015); Gonçalves et al. (2010) for the application of MJLS in NCS.

For MJLS, the availability of the Markov mode $\theta(k)$ is an important aspect that must be taken into account in the project. The situation in which it is supposed that the application has complete access to $\theta(k)$ is called the *complete observation* case, as considered, for instance, in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering in Fioravanti et al. (2008) and Gonçalves et al. (2009), as well as in the filtering for the Bernoulli case in Fioravanti et al. (2015) and in the dynamic output feedback in Geromel et al. (2009), where all the previous results are given in terms of linear matrix inequalities (LMI) conditions. Still regarding the complete observation case, we have the $\mathcal{H}_2$ separation principle in Costa and Tuesta (2004) (the jointly problem of finding an $\mathcal{H}_2$ state feedback controller and a filter) based on coupled Riccati equations, and the Linear Minimum Mean Square Error (LMMSE) filter in Costa (1994) given in terms of recursive equations. However, the literature becomes more scarce whenever dealing with problems in which it is not always possible to directly measure $\theta(k)$, that is in fact a reasonable assumption. In this respect, two main approaches are commonly used in the literature, the *cluster* and *mode-independent* cases, where in the former approach it is considered that $\theta(k)$ can be grouped in disjoint sets dubbed clusters of observation, and it is assumed that the controller would always be aware of in which cluster the Markov mode is currently into; and in the latter approach the controller/filter has no access whatsoever to the Markov mode $\theta(k)$. The cluster setting was first proposed in do Val et al. (2002) for

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the $\mathcal{H}_2$ state-feedback control, and was applied also to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering in Gonçalves et al. (2010), as well as to the filtering problem under the Bernoulli condition in Fioravanti et al. (2015). Finally, regarding the mode-independent case, the $\mathcal{H}_\infty$ filtering was studied, e.g., in de Souza et al. (2006), and in the aforementioned Fioravanti et al. (2015) and Gonçalves et al. (2010).

Recently, it was considered in Costa et al. (2015) the so-called *detector approach* in which it is assumed that the Markov mode $\theta(k)$ cannot be directly measured, but instead there is a detector $\hat{\theta}(k)$ that provides estimated values of $\theta(k)$. This framework spanned a considerable number of recent works such as Costa et al. (2015) and Todorov et al. (2015), where the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems were tackled for discrete-time Markov chains, and in Rodrigues et al. (2015) and Stadtmann and Costa (2016) for the continuous-time case. Regarding filtering problems, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases were studied respectively in de Oliveira and Costa (2017) and de Oliveira and Costa (2016) for discrete-time Markov chains, and in Rodrigues et al. (2016) for continuous-time Markov chains. We can say that the importance of the detector approach is twofold. On the one hand it was shown in Costa et al. (2015) that the detector framework *generalizes* the most important cases involving MJLS, and thus it can be considered as an unifying approach in the study of the complete observation case (the detector provides a perfect estimation of $\theta(k)$), the cluster case (the detector indicates in which set $\theta(k)$ belongs to), and the mode-independent case (the detector cannot provide any reliable information regarding $\theta(k)$), as discussed above. On the other hand, the detector framework is closely related to AFTCS in the sense that, whenever the Markov states represent nominal and faulty modes of operation, we have that $\hat{\theta}(k)$ can be considered a *failure detector*. This is appealing to AFTCS applications and, in particular, to Fault Detection and Isolation algorithms (FDI) that lies at the very heart of this type of control problems, see, for instance, Hwang et al. (2010).

The objective of the present work is to extend the results of the $\mathcal{H}_\infty$ filtering problem under the detector approach obtained in de Oliveira and Costa (2016), and thereby we summarize our main contributions as follows:

- We propose LMI conditions for synthesizing filters that depend only on $\hat{\theta}(k)$ such that the $\mathcal{H}_\infty$ norm of the estimation error is less than a given value $\gamma$, considering that *both the transition and detection probabilities are uncertain* and described by appropriate convex sets. This is a generalization of the results presented in de Oliveira and Costa (2016);
- We derive *necessary and sufficient* conditions for the synthesis of $\mathcal{H}_\infty$ filters depending on $\hat{\theta}(k)$ whenever the transition probabilities do not depend on the current mode of operation $\theta(k)$ (the Bernoulli condition). This is the extension for the detector approach of Fioravanti et al. (2015).

In order to illustrate the usage of our framework in systems prone to faults, we present two numerical examples that are inserted in the AFTCS and NCS contexts: an unmanned aircraft whose sensors are subject to failures, and a classical mass-spring-damper system that sends its signal through a noisy and lossy channel.

This paper is organized as follows: in Section 2, we define the notation that is used throughout our work. In Section 3, we state the basic definitions and theoretical tools, such as the stochastic stability, the $\mathcal{H}_\infty$ norm and the bounded-real lemma for the detector case studied in Todorov et al. (2015), which is the basic tool we use to derive our main results. In Section 4, we present our first contribution, that is a sufficient condition for synthesizing a filter that depends only on the estimated mode $\hat{\theta}(k)$ for systems with partially known transition and detection probabilities (polytopic uncertainty), such that the $\mathcal{H}_\infty$ norm in relation to the estimation error is less than a given value $\gamma$. We tackle the Bernoulli case in Section 5, in which we present a necessary and sufficient condition for synthesizing the $\mathcal{H}_\infty$ filter depending on $\hat{\theta}(k)$. In order to illustrate our results, we present two numerical examples in Section 6 that are related to systems subject to faults and state our final remarks in Section 7. The proofs of our results are shown in the Appendix.

## 2. Notation

The notation used throughout is standard. As usual the real $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$, $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ represents the normed bounded linear space of all $m \times n$ real matrices and, for simplicity, we set $\mathbb{B}(\mathbb{R}^n) = \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$. The superscript $'$ indicates the transpose of a matrix, the $n$-dimensional identity matrix is represented by $I_n$ (or just $I$ for notational simplicity), the $m \times n$ null matrix by $0_{m \times n}$ (or similarly by just 0), the block diagonal matrix is denoted by $\mathbf{diag}(\cdot)$, and for any $R \in \mathbb{B}(\mathbb{R}^n)$ we define that $\mathrm{Her}(R) = R + R'$. For $N$ and $M$ positive integers, the sets $\mathbb{N}$ and $\mathbb{M}$ are defined as $\mathbb{N} = \{1, 2, 3..., N\}$ and $\mathbb{M} = \{1, 2, 3, ..., M\}$, respectively. Furthermore, the set $\mathbb{H}^{n,m}$ represents the linear space of all $N$-sequence of real matrices $V = \{V_1, V_2, ..., V_N\}$, $V_i \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $i \in \mathbb{N}$ and we adopt, for simplicity, $\mathbb{H}^n = \mathbb{H}^{n,n}$ and $\mathbb{H}^{n+} = \{V \in \mathbb{H}^n; V_i \geq 0,\ i = 1, \dots, N\}$. For $P, V \in \mathbb{H}^n$, we write that $P \geq V$ ($P > V$) if $P_i \geq V_i$ ($P_i > V_i$) for each $i = 1, \dots, N$. The operator $T$ is said to be positive if $T(V) \in \mathbb{B}(\mathbb{H}^{n+})$ given that $V \in \mathbb{H}^{n+}$. On the probabilistic space $(\Omega, \mathcal{F}, P)$, $\mathbf{E}(\cdot)$ represents the expected value operator. The notation $l_2^r(\Omega, \mathcal{F}, \{\mathcal{F}_k\}, P)$ represents the space of all discrete-time signals $\mathcal{F}_k$-adapted processes such that

$$\|w\|_2 = \sqrt{\sum_{k=0}^{\infty} \mathbf{E}(\|w(k)\|^2)} < \infty.$$

## 3. Preliminaries

On a probabilistic space $(\Omega, \mathcal{F}, P)$ we consider the following MJLS

$$\mathcal{G} : \begin{cases} x(k+1) = A_{\theta(k)}x(k) + J_{\theta(k)}w(k), \\ y(k) = L_{\theta(k)}x(k) + H_{\theta(k)}w(k), \\ z(k) = C_{\theta(k)}x(k) + E_{\theta(k)}w(k), \\ x(0) = x_0, \theta(0) = \theta_0, \end{cases} \tag{1}$$

where $x(\cdot) \in \mathbb{R}^n$ is the state variable, $y(\cdot) \in \mathbb{R}^p$ is the measured output, and $z(\cdot) \in \mathbb{R}^q$ is the output to be estimated. The signal $\theta(k)$ is a Markov chain taking values in $\mathbb{N} = \{1, 2, ..., N\}$ with transition probability $\mathbb{P} = [p_{ij}]$. Moreover, the signal $w$ is a stochastic disturbance such that $w = \{w(k) \in \mathbb{R}^r, k = 0, 1, 2, ...\} \in l_2^r(\Omega, \mathcal{F}, \{\mathcal{F}_k\}, P)$, as in Todorov et al. (2015).

As previously mentioned, we adopt the approach introduced in Costa et al. (2015) which considers that the state of the Markov chain $\theta(k)$ cannot be directly measured, but instead a detector provides the filter with an estimation of $\theta(k)$, denoted by $\hat{\theta}(k)$. The signal $\hat{\theta}(k)$ takes its values on the set $\mathbb{M}_i$, whenever $\theta(k) = i$, where $\mathbb{M}_i \subseteq \mathbb{M}$, and $\mathbb{M} = \{1, \dots, M\}$ is the set of all possible outcomes from the detector. Letting $\hat{\mathcal{F}}_k$ be the $\sigma$-field generated by $\{x(0), \theta(0), \hat{\theta}(0), w(0), \dots, x(k), \theta(k), w(k)\}$ (the detector output $\hat{\theta}(k)$ is not in $\hat{\mathcal{F}}_k$), we assume that

$$P(\hat{\theta}(k) = l \mid \hat{\mathcal{F}}_k) = P(\hat{\theta}(k) = l \mid \theta(k)) = \alpha_{\theta(k)l}, \quad l \in \mathbb{M}_{\theta(k)},$$
$$\sum_{l=1}^{M} \alpha_{il} = 1.$$

The joint Markov process composed by $(\theta(k), \hat{\theta}(k))$ is inspired on the classical example of a Hidden Markov Chain (HMC) studied, for example, in the book Ross (2010), in which $\theta(k)$ corresponds to the states of a not directly visible Markov chain, and the output $\hat{\theta}(k)$ corresponds to an observable sequence of emissions depending only on the state $\theta(k)$. Notice that, as usual for an HMC, we have the transition probability matrix $\mathbb{P}$ related to the Markov chain $\theta(k)$, and the detection probability matrix $\Upsilon = [\alpha_{il}]$, which defines the probability distribution of $\hat{\theta}(k)$ conditioned on the current value of the state of the Markov chain $\theta(k)$.

**Remark 1.** The detector-based approach generalizes the following important cases in MJLS as pointed out in Costa et al. (2015): (i) the complete observation case by setting $M = N$ and $\alpha_{ii} = 1$ for all $i \in \mathbb{N}$ yielding $\hat{\theta}(k) \equiv \theta(k)$, that is, $\theta(k)$ is exactly known, see, for instance, Gonçalves et al. (2009); (ii) the cluster case that is achieved by choosing a suitable set $\mathbb{M}$ such that $(M \leq N)$ and grouping $\mathbb{N}$ in $M$ disjoint sets $\mathbb{N}_l$. By means of an appropriate function $c : \mathbb{N} \to \mathbb{M}$ such that $c(i) = l$ for all $i \in \mathbb{N}_l$, we set in our framework that $\mathbb{M}_i = \{c(i)\}$ and $\alpha_{ic(i)} = 1$, and so $c$ would represent in which cluster the current mode is into, see, for instance, do Val et al. (2002); Gonçalves et al. (2010); and (iii) the mode-independent case, by setting $M = 1$ and $\alpha_{i1} = 1$ for all $i \in \mathbb{N}$, so that the detector would not provide any information regarding the current mode of operation $\theta(k)$, see, e.g., Gonçalves et al. (2010).

In this work, we study the following detector-based filter:

$$\mathcal{F}_f : \begin{cases} x_f(k+1) = A_{f\hat{\theta}(k)} x_f(k) + B_{f\hat{\theta}(k)} y(k), \\ z_f(k) = C_{f\hat{\theta}(k)} x_f(k) + D_{f\hat{\theta}(k)} y(k). \end{cases} \tag{2}$$

where $x_f(k) \in \mathbb{R}^n$, $x_f(0) = 0$, and $z_f(k) \in \mathbb{R}^q$. We point out that the matrices in (2) depend only on $\hat{\theta}(k)$ and also that the index $f$ denotes a given sequence of filter matrices (2) represented by

$$f = \left\{ \left[ \begin{array}{c|c} A_{fl} & B_{fl} \\ \hline C_{fl} & D_{fl} \end{array} \right] \in \mathbb{B}(\mathbb{R}^{n+p}, \mathbb{R}^{n+q}) \text{ for all } l \in \mathbb{M} \right\}. \tag{3}$$

The error signal is given as follows

$$e(k) = z(k) - z_f(k),$$

and so, by connecting (1) in (2), we get the following extended system

$$\mathcal{G}_f : \begin{cases} \tilde{x}(k+1) = \tilde{A}_{\theta(k)\hat{\theta}(k)} \tilde{x}(k) + \tilde{J}_{\theta(k)\hat{\theta}(k)} w(k), \\ e(k) = \tilde{C}_{\theta(k)\hat{\theta}(k)} \tilde{x}(k) + \tilde{E}_{\theta(k)\hat{\theta}(k)} w(k), \end{cases} \tag{4}$$

where $\tilde{x}(k)$ is defined as $\tilde{x}(k)' = \begin{bmatrix} x(k)' & x_f(k)' \end{bmatrix}$, and $\tilde{A}_{il}, \tilde{J}_{il}, \tilde{C}_{il}, \tilde{E}_{il}$, are given, for $i \in \mathbb{N}$, $l \in \mathbb{M}_i$, by:

$$\left[ \begin{array}{c|c} \tilde{A}_{il} & \tilde{J}_{il} \\ \hline \tilde{C}_{il} & \tilde{E}_{il} \end{array} \right] = \left[ \begin{array}{cc|c} A_i & 0 & J_i \\ B_{fl} L_i & A_{fl} & B_{fl} H_i \\ \hline C_i - D_{fl} L_i & -C_{fl} & E_i - D_{fl} H_i \end{array} \right]. \tag{5}$$

For the sake of uniformity, throughout this paper the indexes $i, j \in \mathbb{N}$ are only used for representing particular modes of the Markov chain $\theta(k)$, and the index $l \in \mathbb{M}$ only for particular values of the detector output $\hat{\theta}(k)$.

In the next definition we present the concept of stochastic stability of the extended system (4) that is used throughout our work.

**Definition 1** (Stochastic stability, Costa et al. (2015)). System (4) is said to be *stochastically stable* with $w(k) \equiv 0$ if $\|\tilde{x}\|_2^2 = \sum_{k=0}^{\infty} \mathbf{E}(\|\tilde{x}(k)\|^2) < \infty$ holds for every $x_0$ with finite second moment and $\theta_0$.

We notice from the extended system (4) that the MJLS (1) is not influenced by the detector-based filter (2), so that in order to have the stochastic stability of the extended system (4) we need to assume that the MJLS (1) is stochastically stable, as stated in the next assumption.

**Assumption 1.** The MJLS (1) is stochastically stable.

**Remark 2.** In the case where the Markov chain $\theta(k)$ is perfectly measured, the extended system in (4) can be replaced by the stochastic dynamic equation of the error signal $e(k)$, with the state estimator written in an observer form, so that it could be stochastically stable even if (1) is stochastically unstable see, for instance Gonçalves et al. (2009). However, as aforementioned, if we consider the extended system as in (4), Assumption 1 becomes a necessary condition for the stochastic stability of (4). We point out that this assumption is also used in other works involving the detector approach, see, for instance, de Oliveira and Costa (2016) and de Oliveira and Costa (2017), and the cluster observation, see, for instance, Gonçalves et al. (2010).

Considering Definition 1 we define the set of admissible filters as follows:

$$\mathcal{K} = \{f \text{ as in (3) such that (4) is stochastically stable}\}.$$

Thus in order to access the stochastic stability of (4) we define the following operators $\mathcal{E}, \mathcal{L} \in \mathbb{B}(\mathbb{H}^{2n})$, required to present Lyapunov-like coupled equations for our problem. For $i, j \in \mathbb{N}$, $\tilde{V} = \{\tilde{V}_1, \ldots, \tilde{V}_N\} \in \mathbb{H}^{2n}$, set

$$\mathcal{E}_i(\tilde{V}) = \sum_{j=1}^{N} p_{ij} \tilde{V}_j,$$

$$\mathcal{L}_i(\tilde{V}) = \sum_{l \in \mathbb{M}_i} \alpha_{il} \tilde{A}_{il}' \mathcal{E}_i(\tilde{V}) \tilde{A}_{il}.$$

The next lemma, introduced in Costa et al. (2015), presents necessary and sufficient conditions for determining if (4) is stochastically stable, based on Lyapunov-like coupled equations using the operator $\mathcal{L}$.

**Lemma 1** (Stability conditions, Costa et al. (2015)). *The following statements are equivalent:*

(i) *System (4) is stochastically stable.*
(ii) *There exists $\tilde{P} \in \mathbb{H}^{2n}$, $\tilde{P} > 0$, such that $\tilde{P}_i - \mathcal{L}_i(\tilde{P}) > 0$ for all $i \in \mathbb{N}$.*

**Proof.** See Costa et al. (2015). □

We now present the concept of the $\mathcal{H}_\infty$ norm for MJLS in the next definition.

**Definition 2** ($\mathcal{H}_\infty$ norm, Gonçalves et al. (2009); Todorov et al. (2015)). Consider that (4) is stochastically stable. For $\tilde{x}_0 = 0$, the $\mathcal{H}_\infty$ norm of (4) is defined as follows

$$\|\mathcal{G}_f\|_\infty = \sup_{w \in \mathcal{W}, \theta_0 \in \mathbb{N}} \frac{\|e\|_2}{\|w\|_2},$$

where $\mathcal{W} = \{w \in l_2^r(\Omega, \mathcal{F}, \{\mathcal{F}_k\}, P) \; ; \; \|w\|_2 > 0\}$.

For the deterministic case ($N = 1$ and $p_{11} = 1$), we point out that Definition 2 reduces to the usual $\mathcal{H}_\infty$ norm. Additionally the calculation of an upper bound $\gamma$ of the $\mathcal{H}_\infty$ norm can be carried out by means of the following inequality set for a given $\gamma > 0$:

$$\begin{bmatrix} \tilde{P}_i & \bullet \\ 0 & \gamma^2 I \end{bmatrix} > \sum_{l \in \mathbb{M}_i} \alpha_{il} \begin{bmatrix} \tilde{M}_{il} & \bullet \\ \tilde{N}_{il} & \tilde{S}_{il} \end{bmatrix}, \tag{6}$$

$$\begin{bmatrix} \tilde{M}_{il} & \bullet & \bullet & \bullet \\ \tilde{N}_{il} & \tilde{S}_{il} & \bullet & \bullet \\ \tilde{A}_{il} & \tilde{J}_{il} & \mathcal{E}_i(\tilde{P})^{-1} & \bullet \\ \tilde{C}_{il} & \tilde{E}_{il} & 0 & I \end{bmatrix} > 0, \tag{7}$$

for all $i \in \mathbb{N}, l \in \mathbb{M}_i$. This is formally stated in the next theorem that presents the bounded-real lemma for the detector case, taken from Todorov et al. (2015).

**Theorem 1** (The bounded-real lemma, Todorov et al. (2015)). *System (4) is stochastically stable with $\|\mathcal{G}_f\|_\infty < \gamma$ if there exist $\tilde{P}_i > 0$, $\tilde{M}_{il} > 0$ and $\tilde{S}_{il} > 0$ and $\tilde{N}_{il}$ such that the inequality set (6)-(7) hold for all $i \in \mathbb{N}, l \in \mathbb{M}_i$.*

**Proof.** See Todorov et al. (2015). □

Given the previous discussion regarding the stochastic stability and the definition of the $\mathcal{H}_\infty$ norm, we summarize our main goal as follows: we intend to synthesize filters in the form given in (2), that is, depending only on the estimated value of the Markov chain $\hat{\theta}(k)$, such that the estimation error defined by the extended system (4) is stochastically stable and respects a minimum upper bound $\gamma$ for the $\mathcal{H}_\infty$ norm. By defining $\|\mathcal{G}_*\|_\infty = \inf_{f\in\mathcal{K}} \|\mathcal{G}_f\|_\infty$, considering $\gamma^2$ in (6) as a variable and setting $\delta = \gamma^2$, as well as $\xi = \{\delta, A_{fl}, B_{fl}, C_{fl}, D_{fl}, \tilde{P}_i, \tilde{M}_{il}, \tilde{N}_{il}, \tilde{S}_{il}\}$ and $\Xi = \{\xi$ such that (6) – (7) holds$\}$, a possible approach for dealing with our goal can be formally stated by the optimization problem

$$\|\mathcal{G}_*\|_\infty^2 \le \inf_{\xi\in\Xi}\{\delta;\ \text{ subject to (6)-(7)}\}, \tag{8}$$

where the last inequality comes from Theorem 1. We point out that (8) is a hard to solve problem due to the nonlinearities involving the factor $\mathcal{E}_i(\tilde{P})^{-1}$ in (7). Bearing this in mind, we present in Section 4 conditions for achieving a $\hat{\theta}(k)$-dependent filter considering that the transition probability matrix $\mathbb{P}$ and the detection probability matrix $\Upsilon$ are uncertain. Furthermore in Section 5 we study the synthesis of filters considering the special case in which the Markov chain is a Bernoulli process, which allows us to derive necessary and sufficient LMI conditions for synthesizing filters depending only on $\hat{\theta}(k)$ such that $\|\mathcal{G}_f\|_\infty < \gamma$. We point out that the result of both sections are obtained by rewriting the optimization problem (8) through a linearization procedure of (6) and (7) presented in Gonçalves et al. (2009) and Gonçalves et al. (2010) in the case of Section 4, or by linearizing new conditions that are valid for the Bernoulli case in Section 5 that are based on Fioravanti et al. (2015).

## 4. The $\mathcal{H}_\infty$ filter for the detector case with uncertainty in $\mathbb{P}$ and $\Upsilon$

In this section we derive conditions for obtaining a suboptimal filter given as in (2) whose matrices depend on the estimated Markov state $\hat{\theta}(k)$, with uncertain matrices $\mathbb{P}$ and $\Upsilon$. For that, we define the convex sets

$$\mathbb{D}_{\mathbb{P}} = \left\{\mathbb{P}\,;\ \mathbb{P} = \sum_{s=1}^{\sigma}\eta_s\mathbb{P}_s = \sum_{s=1}^{\sigma}\eta_s\big[p_{ij}^{(s)}\big],\ \eta_s \ge 0,\ \sum_{s=1}^{\sigma}\eta_s = 1\right\}, \tag{9}$$

$$\mathbb{D}_{\Upsilon} = \left\{\Upsilon\,;\ \Upsilon = \sum_{t=1}^{\tau}\nu_t\Upsilon_t = \sum_{t=1}^{\tau}\nu_t\big[\alpha_{il}^{(t)}\big],\ \nu_t \ge 0,\ \sum_{t=1}^{\tau}\nu_t = 1\right\}, \tag{10}$$

and we consider that $\mathbb{P} \in \mathbb{D}_{\mathbb{P}}$ and $\Upsilon \in \mathbb{D}_{\Upsilon}$. Throughout the work we will associate the index $s$ with the set (9) of $\mathbb{P}$, and the index $t$ with the set (10) of $\Upsilon$. We point out that the case in which the matrices $\mathbb{P}$ and $\Upsilon$ are certainly known is a specialization of the uncertain case, and thus, as we are going to see in Remark 3 below, can be retrieved from the conditions that we will derive in Theorem 2.

We introduce the following set of LMI for a given $\gamma > 0$:

$$\begin{bmatrix} Z_i & \bullet & \bullet \\ Z_i & X_i & \bullet \\ 0 & 0 & \gamma^2 I \end{bmatrix} > \sum_{l\in\mathbb{M}_i}\alpha_{il}^{(t)}\begin{bmatrix} M_{il}^{(11)} & \bullet & \bullet \\ M_{il}^{(21)} & M_{il}^{(22)} & \bullet \\ N_{il}^{(11)} & N_{il}^{(12)} & S_{il} \end{bmatrix}, \tag{11}$$

$$\begin{bmatrix} M_{il}^{(11)} & \bullet & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(21)} & M_{il}^{(22)} & \bullet & \bullet & \bullet & \bullet \\ N_{il}^{(11)} & N_{il}^{(12)} & S_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i^{(s)}(Z)A_i & \mathcal{E}_i^{(s)}(Z)A_i & \mathcal{E}_i^{(s)}(Z)J_i & \mathcal{E}_i^{(s)}(Z) & \bullet & \bullet \\ R_lA_i + F_lL_i + G_l & R_lA_i + F_lL_i & R_lJ_i + F_lH_i & 0 & \mathrm{Her}(R_l) + \mathcal{E}_i^{(s)}(Z) - \mathcal{E}_i^{(s)}(X) & \bullet \\ C_i + K_lL_i + O_l & C_i + K_lL_i & E_i + K_lH_i & 0 & 0 & I \end{bmatrix} > 0, \tag{12}$$

for all $i \in \mathbb{N}, l \in \mathbb{M}_i$, $s \in \{1,\ldots,\sigma\}, t \in \{1,\ldots,\tau\}$, where $\mathcal{E}_i^{(s)}(V) = \sum_{j\in\mathbb{N}} p_{ij}^{(s)}V_j$. We define the set of variables of (11)–(12) as follows

$$\psi = \left\{X, Z, M_{il}^{(11)}, M_{il}^{(22)}, S_{il}, M_{il}^{(21)}, N_{il}^{(11)}, N_{il}^{(12)}, G_l, F_l, K_l, O_l, R_l\right\}, \tag{13}$$

as well as the set of all possible solutions of (11)–(12),

$$\Psi = \{\psi \text{ as defined in (13) such that (11)–(12) hold}\}. \tag{14}$$

The next theorem provides a sufficient condition for obtaining a filter as in (2) such that, for any $\mathbb{P} \in \mathbb{D}_{\mathbb{P}}$ and $\Upsilon \in \mathbb{D}_{\Upsilon}$, we have that $\|\mathcal{G}_f\|_\infty < \gamma$.

**Theorem 2.** *There exists a filter $f \in \mathcal{K}$ such that $\|\mathcal{G}_f\|_\infty < \gamma$, for a given $\gamma > 0$, $\mathbb{P} \in \mathbb{D}_{\mathbb{P}}$ and $\Upsilon \in \mathbb{D}_{\Upsilon}$, if there exists $\psi \in \Psi$, for $\Psi$ characterized as in (14). In the affirmative case, the filter matrices are given by $A_{fl} = -R_l^{-1}G_l, B_{fl} = -R_l^{-1}F_l, C_{fl} = -O_l$ and $D_{fl} = -K_l$ for all $l \in \mathbb{M}$.*

**Proof.** See the Appendix. □

The result provided by Theorem 2 allows us to calculate the best upper bound of $\|\mathcal{G}_f\|_\infty$ by considering the parameter $\gamma^2$ as a decision variable. Thus we set $\delta = \gamma^2$ in (11) and represent the set of variables of (11)–(12) by $\xi = \psi \cup \{\delta\}$. By defining the set of all possible solutions of (11)–(12) considering $\delta$ as a variable by $\Xi = \{\xi$ such that (11)–(12) hold$\}$, we rewrite the optimization problem (8) as follows:

$$\|\mathcal{G}_*\|_\infty^2 \le \inf_{\xi\in\Xi}\{\delta\,;\ \text{subject to } (11) - (12)\}. \tag{15}$$

The number of LMI conditions of (15) for the case in which $\alpha_{il} \ne 0, \forall i \in \mathbb{N}, l \in \mathbb{M}_i$ is $N\tau$ from (11) and $NM\sigma$ from (12). In relation to the number of matrix variables, we have $N(3M+2)$ symmetric matrices and $M(3N+5)$ full matrices of different dimensions, and one scalar variable $\delta$.

**Remark 3** (Complete observation and cluster case). The case of complete observation ($\hat{\theta}(k) = \theta(k)$) can be retrieved from Theorem 2 by setting $\alpha_{ii} = 1, i \in \mathbb{N}$ and $M = N$. The resulting LMI are equivalent to the ones found in Gonçalves et al. (2010) for the complete observation case, which are necessary and sufficient conditions. Similarly, considering the cluster case conditions and calculations for the detector approach that were stated in de Oliveira and Costa (2017), we retrieve the equivalent LMI given in Gonçalves et al. (2010) for the cluster case.

**Remark 4** (The case with $\mathbb{P}$ and $\Upsilon$ exactly known). As previously mentioned, the case without uncertainty in the transition and detection probabilities is easily derived from (11)–(12). Indeed, considering this situation and $\delta = \gamma^2$, the optimization problem (15) becomes

$$\|\mathcal{G}_*\|_\infty^2 \le \inf_{\xi\in\Xi}\{\delta\,;\ \text{subject to } (35) - (36)\}, \tag{16}$$

where (35)–(36) are presented in the Appendix, in the proof of Theorem 2. We note that (16) was studied in details in de Oliveira and Costa (2016), and also that the number of variables in (15) and (16) are the same, thus the only change in computational terms is the decreasing in the number of LMI conditions: $N$ from (35) and $NM$ from (36).

## 5. The $\mathcal{H}_\infty$ filter for the detector case under the Bernoulli condition

In this section we study the synthesis of $\hat{\theta}(k)$-filters under the Bernoulli condition:

$$p_j = p_{ij}, \forall i \in \mathbb{N}, \tag{17}$$

that is, all rows of $\mathbb{P}$ are numerically equal. Consider the following LMI set for a given $\gamma > 0$, for all $i \in \mathbb{N}, l \in \mathbb{M}_i$:

$$\tilde{Q} > \sum_i p_i \tilde{P}_i, \tag{18}$$

$$\begin{bmatrix} \tilde{P}_i & \bullet \\ 0 & \gamma^2 I \end{bmatrix} > \sum_{l \in \mathbb{M}_i} \alpha_{il} \begin{bmatrix} \tilde{M}_{il} & \bullet \\ \tilde{N}_{il} & \tilde{S}_{il} \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} \tilde{M}_{il} & \bullet & \bullet & \bullet \\ \tilde{N}_{il} & \tilde{S}_{il} & \bullet & \bullet \\ \tilde{Q}\tilde{A}_{il} & \tilde{Q}\tilde{J}_{il} & \tilde{Q} & \bullet \\ \tilde{C}_{il} & \tilde{E}_{il} & 0 & I \end{bmatrix} > 0. \tag{20}$$

Applying the congruence transformation $\mathbf{diag}(I_{2n}, I_r, Q^{-1}, I_q)$ to (20), we have that (20) can be equivalently written as

$$\begin{bmatrix} \tilde{M}_{il} & \bullet & \bullet & \bullet \\ \tilde{N}_{il} & \tilde{S}_{il} & \bullet & \bullet \\ \tilde{A}_{il} & \tilde{J}_{il} & \tilde{Q}^{-1} & \bullet \\ \tilde{C}_{il} & \tilde{E}_{il} & 0 & I \end{bmatrix} > 0. \tag{21}$$

The next proposition states an equivalence between the existence of a solution for (6)–(7) and for (18)–(20) whenever (17) holds. This result is the extension of the Bernoulli case in the detector approach for the complete observation case in Fioravanti et al. (2015).

**Proposition 1.** *Suppose that (17) holds. The following statements are equivalent:*

(i) *There exist $A_{fl}$, $B_{fl}$, $C_{fl}$, $D_{fl}$, $\tilde{P}_i$, $\tilde{M}_{il}$, $\tilde{S}_{il}$, and $\tilde{N}_{il}$ such that the inequality set (6)–(7) is satisfied.*

(ii) *There exist $A_{fl}$, $B_{fl}$, $C_{fl}$, $D_{fl}$, $\tilde{Q}$, $\tilde{P}_i$, $\tilde{M}_{il}$, $\tilde{S}_{il}$, and $\tilde{N}_{il}$ such that the LMI set (18)–(20) is satisfied.*

**Proof.** See the Appendix. □

Consider the following LMI set for a given $\gamma > 0$,

$$\begin{bmatrix} Z & \bullet \\ Z & X \end{bmatrix} > \sum_{i \in \mathbb{N}} p_i \begin{bmatrix} P_i^{(11)} & \bullet \\ P_i^{(21)} & P_i^{(22)} \end{bmatrix}, \tag{22}$$

$$\begin{bmatrix} P_i^{(11)} & \bullet & \bullet \\ P_i^{(21)} & P_i^{(22)} & \bullet \\ 0 & 0 & \gamma^2 I \end{bmatrix} > \sum_{l \in \mathbb{M}_i} \alpha_{il} \begin{bmatrix} M_{il}^{(11)} & \bullet & \bullet \\ M_{il}^{(21)} & M_{il}^{(22)} & \bullet \\ N_{il}^{(11)} & N_{il}^{(12)} & S_{il} \end{bmatrix}, \tag{23}$$

$$\begin{bmatrix} M_{il}^{(11)} & \bullet & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(21)} & M_{il}^{(22)} & \bullet & \bullet & \bullet & \bullet \\ N_{il}^{(11)} & N_{il}^{(12)} & S_{il} & \bullet & \bullet & \bullet \\ ZA_i & ZA_i & ZJ_i & Z & \bullet & \bullet \\ XA_i + F_l L_i + G_l & XA_i + F_l L_i & XJ_i + F_l H_i & Z & X & \bullet \\ C_i + K_l L_i + O_l & C_i + K_l L_i & E_i + K_l H_i & 0 & 0 & I \end{bmatrix} > 0, \tag{24}$$

for all $i \in \mathbb{N}, l \in \mathbb{M}_i$. We define also the set of all variables $\psi = \{X > 0, Z > 0, P_i^{(11)} > 0, P_i^{(22)} > 0, M_{il}^{(11)} > 0, M_{il}^{(22)} > 0, S_{il} > 0, P_i^{(21)}, M_{il}^{(21)}, N_{il}^{(11)}, N_{il}^{(12)}, G_l, F_l, K_l, O_l\}$ and $\Psi = \{\psi$ such that (22)–(24) hold$\}$. The next theorem establishes a necessary and sufficient condition for achieving an $\mathcal{H}_\infty$ filter in the form of (2) for the detector case under the Bernoulli condition. This result is the detector extension of Fioravanti et al. (2015) and relies in the transformations as presented therein.

**Theorem 3.** *There is a filter $f \in \mathcal{K}$ such that $\|\mathcal{G}_f\|_\infty < \gamma$ if and only if $\psi \in \Psi$. In the affirmative case, the filter matrices are given by $A_{fl} = (Z - X)^{-1} G_l$, $B_{fl} = (Z - X)^{-1} F_l$, $C_{fl} = -O_l$ and $D_{fl} = -K_l$, for all $l \in \mathbb{M}$.*

**Proof.** See the Appendix. □

Under the extra assumption that $p_i > 0$ for all $i \in \mathbb{N}$, it can be shown that the bounded-real lemma in Theorem 1 becomes necessary and sufficient. Set $\delta = \gamma^2$ in (23), and define $\xi = \psi \cup \{\delta\}$ as well as $\Xi = \{\xi\,;$ such that (22)-(24) hold$\}$. Suppose that there exists a detector-based filter $f \in \mathcal{K}$ with $\|\mathcal{G}_f\|_\infty = \gamma_0$. Then from Theorem 1 and Proposition 1 there exists $\psi_\epsilon$ such that $\psi_\epsilon \cup \{\gamma_\epsilon^2\} \in \Xi$, where $\gamma_\epsilon = \gamma_0 + \epsilon$ for $\epsilon > 0$, and thus $\sqrt{\inf_{\xi \in \Xi}\{\delta\,;\ \text{subject to (22)} - \text{(24)}\}} < \gamma_0 + \epsilon = \|\mathcal{G}_f\|_\infty + \epsilon$ for any $\epsilon > 0$. Taking the limit as $\epsilon$ goes to zero we get that $\sqrt{\inf_{\xi \in \Xi}\{\delta\,;\ \text{subject to (22)} - \text{(24)}\}} \le \|\mathcal{G}_f\|_\infty$. On the other hand from Theorem 1 and Proposition 1 again we have that for any $\psi$ such that $\psi \cup \{\gamma^2\} \in \Xi$ we have that there exists a detector-based filter $f \in \mathcal{K}$ with $\|\mathcal{G}_f\|_\infty < \gamma$, and thus $\|\mathcal{G}_*\|_\infty \le \sqrt{\inf_{\xi \in \Xi}\{\delta\,;\ \text{subject to (22)} - \text{(24)}\}}$. Combining the results we have that the optimum $\mathcal{H}_\infty$ filter can be obtained via the following convex optimization problem:

$$\|\mathcal{G}_*\|_\infty^2 = \inf_{\xi \in \Xi}\{\delta\,;\ \text{subject to (22)} - \text{(24)}\}. \tag{25}$$

The number of LMI conditions of (25) for the case in which $\alpha_{il} \neq 0, \forall i \in \mathbb{N}, l \in \mathbb{M}_i$ is $N(M+1)+1$ from (22), (23) and (24). Notice that this is an increase of one LMI in relation to (16). Regarding the matrix variables, we have $N(3M+2)+2$ symmetric and $N + M(3N+4)$ full matrices of different dimensions. We stress that, although the number of conditions and variables increases, the optimization problem (25) always yields the optimum $\mathcal{H}_\infty$ cost, in contrast with the guaranteed costs provided by (16) (the optimization problem (16) only yields the exact $\mathcal{H}_\infty$ norm for the complete observation case, see Remark 3).

## 6. Numerical examples

In this section we study the estimation of the states of the controlled lateral dynamics of an unmanned aircraft in the first example, considering that the sensors that provide the signal to the controller are subject to errors. In this context, we study the behavior of the $\mathcal{H}_\infty$ filter in three forms: (i) varying the parameters of $\Upsilon$; (ii) considering that $\Upsilon$ is uncertain, as described in (10); (iii) applying a sinusoidal exogenous input to (4) and studying the behavior of the estimation error and the ratio $\|e\|_2/\|w\|_2$ for this case. Additionally, we study the simplified conditions of Theorem 3 for the Bernoulli case by means of a classical mass-spring-damper system taken from Fioravanti et al. (2015), considering that the measured output is transmitted through a channel subject to noise and faults. In all examples, we denote $\gamma = \sqrt{\delta}$, the upper bound obtained by (15) or (16) and $\gamma^* = \sqrt{\delta^*}$, the actual $\mathcal{H}_\infty$ norm calculated by the bounded-real lemma with filter matrices given by the respective optimization problem (in the Bernoulli case, $\gamma^*$ is the same value as obtained by (25)).

### 6.1. Unmanned aircraft

In the first example, we study the $\mathcal{H}_\infty$ filtering of the controlled lateral dynamics of an unmanned aircraft taken from Ducard (2009). Matrices $(A_c, B_c)$ from the continuous model in Ducard (2009) are shown below.

$$\left[A_c \mid B_c\right] = \left[\begin{array}{cccc|cc} -11.4540 & 2.7185 & -19.4399 & 0.0000 & 78.4002 & -2.7282 \\ 0.5068 & -2.9875 & 23.3434 & 0.0000 & -3.4690 & 13.9685 \\ 0.0922 & -0.9957 & -0.4680 & 0.3256 & 0.0000 & 0.0000 \\ 1.0000 & 0.0926 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{array}\right].$$

The states are the roll rate ($\Delta p$), the yaw rate ($\Delta r$), the sideslip angle ($\Delta\beta$), and the roll angle ($\Delta\phi$), whereas the control inputs are the aileron ($\Delta\delta_{aileron}$) and the rudder ($\Delta\delta_{rudder}$). This plant is inherently unstable, and thus it is considered that the system is controlled by a state feedback controller with gain given by

$$K = \begin{bmatrix} -0.8777 & -0.0163 & -0.0971 & -1.0441 \\ -0.0082 & -0.8093 & -0.0701 & 0.0274 \end{bmatrix},$$

so that the closed-loop system is stable. It is considered that the channel that sends the states $\Delta r$, $\Delta\beta$, and $\Delta\phi$ to the controller may fail, so that for this faulty mode of operation the closed-loop system would become asymptotically unstable. For modeling this system as a MJLS, we define a Markov chain with two states: a nominal state (1) and a faulty state (2). Thus, the output matrices are given by

$$L_1 = I_4,\ L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H_1 = H_2 = 0_{4\times 2},$$

and the closed-loop matrices by $A_i = A_c + B_c(KL_i) = A_c + B_cK_i$, where $K_i = KL_i$, $i = 1, 2$. The objective is to retrieve the states of the discretized closed-loop system, and so we set $C_i = I_4$, $E_i = 0_{4\times 2}$ for all $i \in \mathbb{N}$. The discrete closed-loop matrices obtained through a zero-order hold of sampling period $T_s = 0.05$ s are given by

$$\left[A_1 \mid A_2\right] = \left[\begin{array}{cccc|cccc} 0.0078 & 0.0313 & -0.2828 & -0.9762 & 0.0193 & 0.0376 & -0.2025 & -0.0027 \\ 0.0255 & 0.4772 & 0.7629 & 0.0468 & 0.0379 & 0.8398 & 1.0335 & 0.0087 \\ -0.0000 & -0.0348 & 0.9543 & 0.0106 & -0.0002 & -0.0452 & 0.9495 & 0.0159 \\ 0.0121 & 0.0047 & -0.0097 & 0.9621 & 0.0124 & 0.0056 & -0.0059 & 1.0000 \end{array}\right].$$

Thus, for the nominal mode of operation given by $\theta(k) = 1$, we have the asymptotically stable closed-loop system, and for the faulty mode of operation $\theta(k) = 2$, the unstable closed-loop system. We consider a transition matrix given by

$$\mathbb{P} = \begin{bmatrix} 0.9000 & 0.1000 \\ 0.4000 & 0.6000 \end{bmatrix}.$$

Note that although the subsystem $\theta(k) = 2$ is unstable, the resulting MJLS is still stochastically stable, an interesting feature of MJLS that was studied, for example, in Costa et al. (2005), and as a consequence Assumption 1 holds. We consider that the disturbances are applied in the control input, yielding the following discretized $J_i$ matrices

$$\left[J_1 \mid J_2\right] = \left[\begin{array}{cc|cc} 0.9310 & -0.0126 & 0.9572 & -0.0143 \\ -0.0251 & 0.4930 & -0.0396 & 0.6396 \\ 0.0048 & -0.0137 & 0.0052 & -0.0164 \\ 0.0363 & 0.0004 & 0.0367 & 0.0006 \end{array}\right].$$

First, we assume that the detection matrix $\Upsilon$ is described by

$$\Upsilon = \begin{bmatrix} \rho_1 & 1-\rho_1 \\ 1-\rho_2 & \rho_2 \end{bmatrix}, \tag{26}$$

where $0 \le \rho_i \le 1$. Fig. 1 shows the costs $\gamma$ obtained via the optimization problem (16) and the actual $\mathcal{H}_\infty$ norm of (4) calculated by the bounded-real lemma in function of $\rho_i$, $i \in \{1, 2\}$.

We notice that the guaranteed cost surface lies superimposed over the actual $\mathcal{H}_\infty$ norm surface, in which there are only two coincident points in $\rho_1 = \rho_2 = 1$ and $\rho_1 = \rho_2 = 0$, with costs given by $\gamma^* = \gamma = 1.1480$. That is exactly the complete observation case, in which the detector can distinguish perfectly between the two modes of operation. We also have that the region described by $\rho_1 = 1 - \rho_2$ in Fig. 1, or equivalently whenever the rows of (26) are equal, corresponds to the worst-case scenario under the detector approach. That happens due to the inability of the detector to distinguish between mode 1 and 2, and thus it is equivalent to the mode-independent situation, with guaranteed cost $\gamma = 1.3592$ and actual norm $\gamma^* = 1.3487$. In this case, the filter matrices are equal for all $l \in \mathbb{M}$:

$$\left[\begin{array}{c|c} A_{fl} & B_{fl} \\ \hline C_{fl} & D_{fl} \end{array}\right] = \left[\begin{array}{cccc|cccc} -0.0579 & -0.0008 & 0.1026 & 1.0363 & 0.0698 & 0.0240 & -0.1059 & -1.3057 \\ 0.1927 & 0.3122 & 2.1917 & -0.5697 & 0.1462 & 0.1897 & -1.4705 & 0.6379 \\ -0.1013 & 0.0496 & 0.6032 & 0.1629 & 0.0502 & -0.0846 & 0.3532 & -0.1570 \\ -0.0357 & -0.0037 & 0.0356 & 1.0174 & 0.0526 & 0.0081 & -0.0345 & -0.0270 \\ \hline -0.0000 & 0.0000 & -0.0000 & 0.0000 & 1.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.2145 & 0.0578 & 0.0624 & -1.2730 & 0.0493 & 0.9392 & -0.0606 & 1.3176 \\ -0.0753 & 0.0526 & 0.5774 & 0.3040 & 0.0231 & -0.0520 & 0.4222 & -0.3129 \\ -0.0354 & -0.0034 & 0.0377 & 0.9476 & 0.0411 & 0.0034 & -0.0377 & 0.0533 \end{array}\right].$$

Additionally the special case of $\rho = \rho_1 = \rho_2$ in (26) is shown in Fig. 2, that corresponds to the line $\rho_1 = \rho_2$ of Fig. 1. We note that,

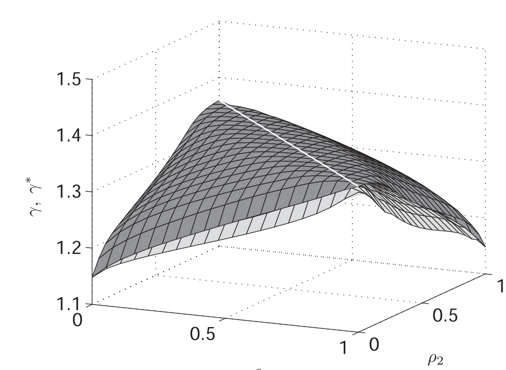


**Fig. 1.** Guaranteed cost $\gamma$ (darker surface) via the optimization problem (16) and actual $\mathcal{H}_\infty$ norm $\gamma^*$ (brighter surface) of (4) in function of $\rho_i$, $i \in \{1, 2\}$.

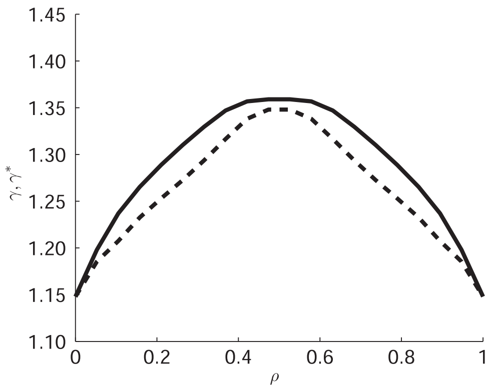


**Fig. 2.** Guaranteed cost $\gamma$ (full line) via the optimization problem (15) and actual $\mathcal{H}_\infty$ norm (dashed line) of (4) with $\Upsilon$ given as in (26) in function of $\rho = \rho_1 = \rho_2$.

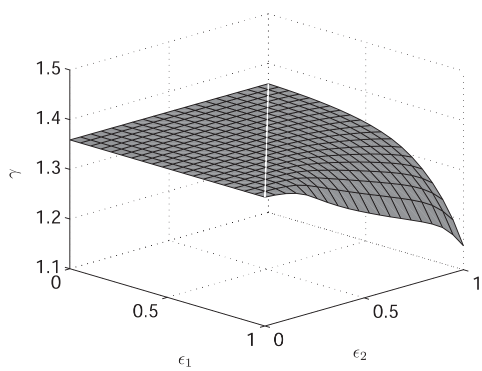


**Fig. 3.** Guaranteed cost $\gamma$ (darker surface) via the optimization problem (15) with $\Upsilon$ given as in (27) in function of $\epsilon_i$, $i \in \{1, 2\}$.

as previously discussed, the cases in which $\rho = 0$ and $\rho = 1$ are characterized by the perfect observation case and so the guaranteed cost $\gamma$ and the exact norm $\gamma^*$ are numerically equal. On the other hand the worst cost is yielded by $\rho = 0.5$ and corresponds to the mode-independent case as discussed in Remark 1, since in this case all rows of (26) are equal, and thus the detector cannot give any reliable information regarding the Markov chain.

Second, we consider that the detection matrix $\Upsilon$ is uncertain, whose vertices described as in (10) are given by

$$\left[\Upsilon_1 \;\middle|\; \Upsilon_2\right] = \left[\begin{array}{cc|cc} 1 & 0 & \epsilon_1 & 1-\epsilon_1 \\ 0 & 1 & 1-\epsilon_2 & \epsilon_2 \end{array}\right], \tag{27}$$

for $0 \le \epsilon_i \le 1$, $i \in \{1, 2\}$. Fig. 3 shows the costs $\gamma$ obtained via the optimization problem (15) in function of $\epsilon_i$, $i \in \{1, 2\}$. We notice a similar behavior in the surface shown in Figs. 1 and 3 in the range $\epsilon_1 + \epsilon_2 > 1$, where the guaranteed costs of both surfaces in Figs. 1 and 3 are numerically equal. However, in the range $\epsilon_1 + \epsilon_2 \le 1$, the calculated guaranteed costs become constant, characterized by the value of the mode-independent cost $\gamma = 1.3592$. This can be explained by writing $\Upsilon$ according to the definition in (10), as follows

$$\Upsilon = \nu_1 \Upsilon_1 + \nu_2 \Upsilon_2 = \begin{bmatrix} 1 - \nu_2(1-\epsilon_1) & \nu_2(1-\epsilon_1) \\ \nu_2(1-\epsilon_2) & 1 - \nu_2(1-\epsilon_2) \end{bmatrix}, \tag{28}$$

with $\nu_1 + \nu_2 = 1$ and $0 \le \nu_2(1-\epsilon_i) \le 1$, $i \in \{1, 2\}$. Recalling that the mode-independent cost is achieved whenever all rows of $\Upsilon$ are equal, as previously discussed, we set $1 - \nu_2(1-\epsilon_1) = \nu_2(1-\epsilon_2)$ in (28), leading to

$$\nu_2 = \frac{1}{2 - (\epsilon_1 + \epsilon_2)}.$$

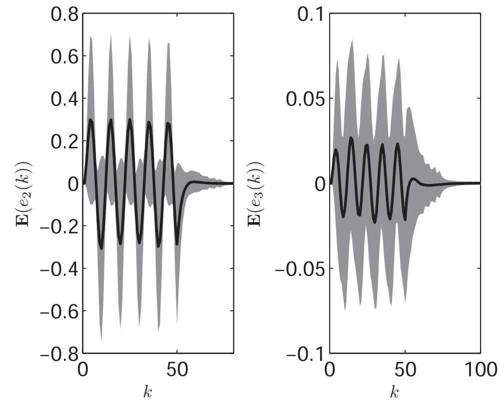


**Fig. 4.** Mean curves of $e_2(k)$ and $e_3(k)$ and their respective standard deviation envelopes (gray) in function of time for a Monte Carlo simulation of 2000 rounds.

We point out that the condition above is valid whenever $\epsilon_1 + \epsilon_2 \le 1$, as $0 \le \nu_2 \le 1$. Due to the very nature of the polytopic uncertainty problem, the solution will always yield the worst-case scenario, that is precisely the mode-independent case. On the other hand, for $\epsilon_1 + \epsilon_2 > 1$, $\Upsilon_2$ becomes dominant in the robust problem ($\nu_2 = 1$ for all this range), yielding the similar shape as in Fig. 1.

Finally, in order to illustrate the behavior of the estimation error and the value of the $\mathcal{H}_\infty$ norm, we set $\rho_1 = \rho_2 = 0.7$ in (26), take the initial distribution of $\theta(k)$ as the stationary distribution, and apply in (4) the following exogenous input:

$$w_2(k) = \begin{cases} \sin(0.6150k), & 0 \le k < 50, \\ 0, & k \ge 50, \end{cases} \tag{29}$$

as well as $w_1(k) = 0, k \ge 0$. Fig. 4 shows the mean curves of $e_2(k)$ and $e_3(k)$ and their respective standard deviation envelopes (gray) in function of time for a Monte Carlo simulation of 2000 rounds.

We note that the exogenous input in (29) causes small perturbations on the estimation error, that are limited by $|e_2(k)| < 0.8$ and $|e_3(k)| < 0.1$, $\forall k$. We have that $\|e\|_2/\|w\|_2 \approx 0.5037$, whereas $\gamma^* = 1.2871$, and so $\|e\|_2/\|w\|_2 \le \gamma^*$, as expected. We point out that the exogenous input in (29) is not the worst-disturbance for MJLS, for example as given in Seiler and Sengupta (2003) for the complete observation case, and thus we expect lower values of $\|e\|_2/\|w\|_2$ for sinusoidal signals.

### 6.2. Classical mass-spring-damper system

The second example is taken from Fioravanti et al. (2015), consisting of a classical mass-spring-damper system whose states are transmitted through a lossy channel, modeled by a Markov chain. There are three states: (1) the transmission is successful with a nominal noise level; (2) the transmission is successful with an augmented noise level; (3) the transmission is lost. In this case, the system is discretized via a zero order hold of period $T = 0.5$ s in each input, as in Fioravanti et al. (2015), yielding the following system matrices:

$$[A_i \;|\; J_i] = \left[\begin{array}{cccc|cc} -0.7562 & 0.5086 & 0.0791 & 0.1435 & 0.0974 & 0 \\ 0.2092 & 0.5604 & 0.0718 & 0.4012 & 0.0559 & 0 \\ -1.6559 & 0.0736 & -0.7588 & 0.5113 & -1.7562 & 0 \\ -0.1463 & -1.2887 & 0.2556 & 0.5140 & 0.2092 & 0 \end{array}\right],$$

and $L_i = L_2 = [0\ \ 0\ \ 1\ \ 0]$, $L_3 = 0_{1\times 4}$, as well as $H_1 = [0\ \ 0.1]$, $H_2 = [0\ \ 1]$ and $H_3 = 0_{1\times 2}$. We want to estimate the state $x_2$ and so we set $C_i = [0\ \ 1\ \ 0\ \ 0]$ and $E_i = [0\ \ 0]$ for all $i \in \mathbb{N}$. The transition

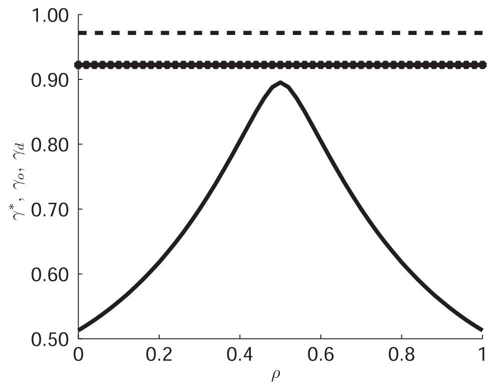


**Fig. 5.** $\gamma^*$ (continuous line), $\gamma_o$ (marked line) and $\gamma_d$ (dashed line) in function of $\rho$.

probability matrix is given by

$$\mathbb{P} = \begin{bmatrix} 0.5000 & 0.3000 & 0.2000 \\ 0.5000 & 0.3000 & 0.2000 \\ 0.5000 & 0.3000 & 0.2000 \end{bmatrix}$$

allowing us to use the results of Theorem 3 for the Bernoulli case. We also set $\Upsilon$ as follows

$$\Upsilon = \begin{bmatrix} \rho & 1-\rho & 0 \\ 1-\rho & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

for $0 \leq \rho \leq 1$. The detector structure in (30) can precisely sense the packet loss that occurs in the state (3), but may not perfectly distinguish between modes (1) and (2). We consider three different scenarios:

(i) We calculate $\gamma^*$ varying $\rho$ in (30) through the optimization problem (25);

(ii) We calculate the mode-independent cost by setting

$$\Upsilon = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix},$$

that is, all rows of $\Upsilon$ are equal.

(iii) We obtain the deterministic $\mathcal{H}_\infty$ filter for the nominal mode ($\theta(k) = 1$), and calculate the $\mathcal{H}_\infty$ costs of (4) using Theorem 1. In short, we study the behavior of the deterministic $\mathcal{H}_\infty$ filter of the nominal mode applied also to the fault and augmented noise situations as previously described.

Fig. 5 shows $\gamma^*$ in scenario (i), the mode-independent cost $\gamma_o$ in scenario (ii) and the calculated costs for the deterministic filter $\gamma_d$ in scenario (iii), in function of $\rho$.

First, we note that the worst costs are yielded by the deterministic $\mathcal{H}_\infty$ filter, due to the lack of information on the underlying stochastic process (for comparison, if only the nominal mode would take place, we would have $\gamma_d = 0.1451$). In contrast, the mode-independent filter outperforms the deterministic filter due to the information of the Markov chain, even if it does not vary according to $\hat{\theta}(k)$. Considering $\gamma^*$ we notice that if we take $\rho = 0$ or $\rho = 1$, we retrieve the complete observation case, as previously discussed in the first example. On the other hand, due to the structure of $\Upsilon$ in (30), the worst-case cost of $\gamma^*$ is in fact a cluster case scenario. This behavior is discussed in details in de Oliveira and Costa (2017) and is related to the structure of $\Upsilon$, where it was shown that it is not possible for the detector to tell the difference between modes whenever their respective detection probabilities are equal. Thus, specifically for $\rho = 0.5$, we have the clusters $\mathbb{N}^{(1)} = \{1, 2\}$ and $\mathbb{N}^{(2)} = \{3\}$, yielding the optimal cost $\gamma^* = 0.8953$, that is the same value obtained as in Fioravanti et al. (2015) for the same configuration of clusters.

## 7. Conclusion

In this work we studied the $\mathcal{H}_\infty$ filtering problem in a context of partial information of the Markov chain $\theta(k)$. We assume that there exists a detector $\hat{\theta}(k)$ that provides the only information regarding the hidden stochastic process $\theta(k)$, and propose a sufficient condition for obtaining $\hat{\theta}(k)$-dependent filters such that the $\mathcal{H}_\infty$ norm of the extended systems is less than a given level $\gamma$, including the cases in which the transition and the detection probabilities are not exactly known. Additionally, we derive a necessary and sufficient condition for synthesizing $\mathcal{H}_\infty$ filters for the case in which the Markov chain satisfies the Bernoulli condition, leading to the exact value of the $\mathcal{H}_\infty$ norm. In order to illustrate our results, we present two examples that are related to system prone to faults: an unmanned aircraft with sensor failures, and a system whose output is transmitted through a lossy channel.

A possible future work would be the study of the jointly problem of synthesizing $\mathcal{H}_\infty$ controllers and filters that depend only on the detector, which is an yet unsolved and more challenging problem. This is related to the dynamic output feedback control for the cluster case, that is still not solved in the lines of a single linearization procedure, as it is done in Geromel et al. (2009). In addition, the application of the detector approach in real systems, especially in systems prone to faults, would constitute an important step toward the development of this theory. Finally, the usage of this particular detector structure for residual generation in the context of AFTCS seems an interesting option for fault-detection and isolation algorithms.

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## Appendix

In this section we present the proofs of Theorem 2, Proposition 1, and Theorem 3. In what follows it will be useful to consider the following linearization procedure, based on the technique of slack variables introduced in Gonçalves et al. (2010), and the linearization method of Gonçalves et al. (2009) for dealing with the non-linearity of $\mathcal{E}_i(\tilde{P})^{-1}$ in (7). For $\tilde{P} = \{\tilde{P}_1, \dots, \tilde{P}_N\} \in \mathbb{H}^{2n}$, $\tilde{P}_i > 0$, we set the partitions of $\tilde{P}_i$, and its inverse $\tilde{P}_i^{-1}$, as follows:

$$\tilde{P}_i = \begin{bmatrix} X_i & \bullet \\ U_i' & \hat{X}_i \end{bmatrix}, \quad \tilde{P}_i^{-1} = \begin{bmatrix} Y_i & \bullet \\ V_i' & \hat{Y}_i \end{bmatrix}. \tag{31}$$

We also define the matrix $\mathcal{T}_i$ such that

$$\mathcal{T}_i = \begin{bmatrix} I & I \\ V_i' Y_i^{-1} & 0 \end{bmatrix}, \tag{32}$$

that leads to

$$\mathcal{T}_i' \tilde{P}_i \mathcal{T}_i = \begin{bmatrix} Z_i & Z_i \\ Z_i & X_i \end{bmatrix}, \tag{33}$$

where $Z_i = Y_i^{-1}$. In order to linearize $\mathcal{E}_i(\tilde{P})^{-1}$ of condition (7) in the decision variables of (33), which is a difficult task since $\mathcal{E}_i(\tilde{P})^{-1}$ is the inverse of a linear combination of $\tilde{P}_i$, we follow the procedure given in Gonçalves et al. (2009) and consider a partition for $\mathcal{E}_i(\tilde{P})^{-1}$ and the matrix $\mathcal{G}_i$ as follows:

$$\mathcal{E}_i(\tilde{P})^{-1} = \begin{bmatrix} \hat{T}_{1i} & \hat{T}_{2i} \\ \hat{T}_{2i}' & \hat{T}_{3i} \end{bmatrix}, \quad \mathcal{G}_i = \begin{bmatrix} \hat{T}_{1i}^{-1} & \mathcal{E}_i(X) \\ 0 & \mathcal{E}_i(U)' \end{bmatrix},$$

such that

$$\mathcal{G}_i'\mathcal{E}_i(\tilde{P})^{-1}\mathcal{G}_i = \begin{bmatrix} \hat{T}_{1i}^{-1} & \hat{T}_{1i}^{-1} \\ \hat{T}_{1i}^{-1} & \mathcal{E}_i(X) \end{bmatrix}. \tag{34}$$

By choosing a suitable structure of $\tilde{P}_i$ in (31) we can write (34) as the linear combination of (33) through the operator $\mathcal{E}_i(\cdot)$ and consequently linearize $\mathcal{E}_i(\tilde{P})^{-1}$, a process that will become clear in the proof of Theorem 2 below.

**Proof of Theorem 2.** Suppose that inequalities (11)–(12) hold. Notice that (11) is affine with respect to $\alpha_{il}^{(t)}$, and so is (12) with respect to $p_{ij}^{(s)}$. Thus, multiplying (11) and (12), respectively, by $\nu_t$ and $\eta_s$ (from (9) and (10)), and summing them up for $s \in \{1, \dots, \sigma\}$ and $t \in \{1, \dots, \tau\}$, we get that

$$\begin{bmatrix} Z_i & \bullet & \bullet \\ Z_i & X_i & \bullet \\ 0 & 0 & \gamma^2 I \end{bmatrix} > \sum_{l \in \mathbb{M}_i} \alpha_{il} \begin{bmatrix} M_{il}^{(11)} & \bullet & \bullet \\ M_{il}^{(21)} & M_{il}^{(22)} & \bullet \\ N_{il}^{(11)} & N_{il}^{(12)} & S_{il} \end{bmatrix}, \tag{35}$$

$$\begin{bmatrix} M_{il}^{(11)} & \bullet & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(21)} & M_{il}^{(22)} & \bullet & \bullet & \bullet & \bullet \\ N_{il}^{(11)} & N_{il}^{(12)} & S_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Z)A_i & \mathcal{E}_i(Z)A_i & \mathcal{E}_i(Z)J_i & \mathcal{E}_i(Z) & \bullet & \bullet \\ R_lA_i + F_lL_i + G_l & R_lA_i + F_lL_i & R_lJ_i + F_lH_i & 0 & \mathrm{Her}(R_l) + \mathcal{E}_i(Z) - \mathcal{E}_i(X) & \bullet \\ C_i + K_lL_i + O_l & C_i + K_lL_i & E_i + K_lH_i & 0 & 0 & I \end{bmatrix} > 0. \tag{36}$$

We set the partitions of $\tilde{M}_{il}$ and $\tilde{N}_{il}$ in (6) and of $\tilde{P}_i$ in (31), respectively, as follows

$$\tilde{M}_{il} = \begin{bmatrix} M_{il}^{(22)} & \bullet \\ M_{il}^{(21)'} - M_{il}^{(22)} & M_{il}^{(11)} + M_{il}^{(22)} - M_{il}^{(21)} - M_{il}^{(21)'} \end{bmatrix}, \quad \tilde{S}_{il} = S_{il},$$

$$\tilde{N}_{il}' = \begin{bmatrix} N_{il}^{(12)'} \\ N_{il}^{(11)'} - N_{il}^{(12)'} \end{bmatrix}, \quad \tilde{P}_i = \begin{bmatrix} X_i & \bullet \\ Z_i - X_i & X_i - Z_i \end{bmatrix}. \tag{37}$$

Notice that

$$\begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \begin{bmatrix} M_{il}^{(11)} & \bullet \\ M_{il}^{(21)'} & M_{il}^{(22)} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} = \tilde{M}_{il}$$

so that, from (12), we have that $\tilde{M}_{il} > 0$. From (37), we get that $U_i = Z_i - X_i$, $\hat{X}_i = -U_i$ and $V_i'Y_i^{-1} = I$. Furthermore, from the left hand side of (35) we notice that $X_i > Z_i$ and so $\mathcal{E}_i(X) - \mathcal{E}_i(Z)$ is invertible, that allows us to write that $R_l(\mathcal{E}_i(X) - \mathcal{E}_i(Z))^{-1}R_l' > R_l + R_l' + \mathcal{E}_i(Z) - \mathcal{E}_i(X)$ (see de Oliveira et al. (1999)) and so

$$\begin{bmatrix} M_{il}^{(11)} & \bullet & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(21)} & M_{il}^{(22)} & \bullet & \bullet & \bullet & \bullet \\ N_{il}^{(11)} & N_{il}^{(12)} & S_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Z)A_i & \mathcal{E}_i(Z)A_i & \mathcal{E}_i(Z)J_i & \mathcal{E}_i(Z) & \bullet & \bullet \\ R_lA_i + F_lL_i + G_l & R_lA_i + F_lL_i & R_lJ_i + F_lH_i & 0 & R_l(\mathcal{E}_i(X) - \mathcal{E}_i(Z))^{-1}R_l' & \bullet \\ C_i + K_lL_i + O_l & C_i + K_lL_i & E_i + K_lH_i & 0 & 0 & I \end{bmatrix} > 0 \tag{38}$$

also holds. Similar to Gonçalves et al. (2010), defining the matrix $\mathcal{Q}_{il}$ as,

$$\mathcal{Q}_{il} = \begin{bmatrix} I_n & I_n \\ 0_{n \times n} & \left(R_l^{-1}\right)'(\mathcal{E}_i(X) - \mathcal{E}_i(Z)) \end{bmatrix},$$

and applying the congruence transformation $\mathbf{diag}(I_n, I_n, I_r, \mathcal{Q}_{il}, I_q)$ to (38), we obtain that

$$\begin{bmatrix} M_{il}^{(11)} & \bullet & \bullet & \bullet & \bullet & \bullet \\ M_{il}^{(21)} & M_{il}^{(22)} & \bullet & \bullet & \bullet & \bullet \\ N_{il}^{(11)} & N_{il}^{(12)} & S_{il} & \bullet & \bullet & \bullet \\ \mathcal{E}_i(Z)A_i & \mathcal{E}_i(Z)A_i & \mathcal{E}_i(Z)J_i & \mathcal{E}_i(Z) & \bullet & \bullet \\ X_{il}^{(1)} + \mathcal{E}_i(U)A_{fl} & X_{il}^{(1)} & X_{il}^{(2)} & \mathcal{E}_i(Z) & \mathcal{E}_i(X) & \bullet \\ C_i - D_{fl}L_i - C_{fl} & C_i - D_{fl}L_i & E_i - D_{fl}H_i & 0 & 0 & I \end{bmatrix} > 0 \tag{39}$$

where, recalling that $G_l = -R_lA_{fl}$, $F_l = -R_lB_{fl}$, $O_l = -C_{fl}$, $K_l = -D_{fl}$ and $U_i = Z_i - X_i$, we define the auxiliary matrices $X_{il}^{(1)} = \mathcal{E}_i(X)A_i + \mathcal{E}_i(U)B_{fl}L_i$, $X_{il}^{(2)} = \mathcal{E}_i(X)J_i + \mathcal{E}_i(U)B_{fl}H_i$. As derived in Gonçalves et al. (2009), we have that $\hat{T}_{1i}^{-1} = \mathcal{E}_i(X) - \mathcal{E}_i(U)\mathcal{E}_i(\hat{X})^{-1}\mathcal{E}_i(U')$ and so, with the particular choice of $\tilde{P}_i$, in (37), we have that $\mathcal{E}_i(U) = -\mathcal{E}_i(\hat{X})$. Thus, $\hat{T}_{1i}^{-1} = \mathcal{E}_i(X) + \mathcal{E}_i(U) = \mathcal{E}_i(Z)$, and so Eqs. (35) and (39) can be rewritten, considering the choice of $\tilde{M}_{il}, \tilde{N}_{il}$ and $\tilde{S}_{il}$ as in (37) and the linearized forms (33) and (34), as

$$\begin{bmatrix} \mathcal{T}_i'\tilde{P}_i\mathcal{T}_i & 0 \\ 0 & \gamma^2 I \end{bmatrix} > \sum_{l \in \mathbb{M}_i} \alpha_{il} \begin{bmatrix} \mathcal{T}_i'\tilde{M}_{il}\mathcal{T}_i & \bullet \\ \tilde{N}_{il}'\mathcal{T}_i & \tilde{S}_{il} \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{T}_i'\tilde{M}_{il}\mathcal{T}_i & \bullet & \bullet & \bullet \\ \tilde{N}_{il}'\mathcal{T}_i & \tilde{S}_{il} & \bullet & \bullet \\ \mathcal{G}_i'\tilde{A}_{il}\mathcal{T}_i & \mathcal{G}_i'\tilde{J}_{il} & \mathcal{G}_i'\mathcal{E}_i(\tilde{P})^{-1}\mathcal{G}_i & \bullet \\ \tilde{C}_{il}\mathcal{T}_i & \tilde{E}_{il} & 0 & I \end{bmatrix} > 0.$$

Applying respectively the congruence transformations $\mathbf{diag}(\mathcal{T}_i^{-1}, I_r)$ and $\mathbf{diag}(\mathcal{T}_i^{-1}, I_r, \mathcal{G}_i^{-1}, I_q)$ to the inequalities above, we get (6) and (7) with system matrices given by (5) (thus $f \in \mathcal{K}$), and $\tilde{P}_i$, $\tilde{M}_{il}$, $\tilde{N}_{il}$ and $\tilde{S}_{il}$ as in (37), completing the proof. □

**Proof of Proposition 1.** We show the equivalence between (i) and (ii) below.

- $(ii) \Rightarrow (i)$: If (18) holds then $\left(\sum_j p_j\tilde{P}_j\right)^{-1} > \tilde{Q}^{-1}$, and thus (21) can be written as (7) for $\mathcal{E}_i(\tilde{P}) = \sum_j p_j\tilde{P}_j$.
- $(i) \Rightarrow (ii)$: Considering that (7) holds for (17), we have that $\mathcal{E}_i(\tilde{P}) = \mathcal{E}(\tilde{P})$. Notice that it is always possible to find $\Delta > 0$ such that

$$\begin{bmatrix} \tilde{M}_{il} & \bullet & \bullet & \bullet \\ \tilde{N}_{il} & \tilde{S}_{il} & \bullet & \bullet \\ \tilde{A}_{il} & \tilde{J}_{il} & \mathcal{E}(\tilde{P})^{-1} - \Delta I & \bullet \\ \tilde{C}_{il} & \tilde{E}_{il} & 0 & I \end{bmatrix} > 0$$

also holds. Defining $\tilde{Q}$ respecting $\mathcal{E}(\tilde{P}) < \tilde{Q} < (\mathcal{E}(\tilde{P})^{-1} - \Delta I)^{-1}$, we directly recuperate (18). On the other hand, we have that $\tilde{Q} < (\mathcal{E}(\tilde{P})^{-1} - \Delta I)^{-1}$ implies that $\tilde{Q}^{-1} > \mathcal{E}(\tilde{P})^{-1} - \Delta I$, and thus (21) holds, completing the proof. □

Similarly to the procedures involving $\tilde{P}_i$ in (31), we set the partitions of matrices $\tilde{Q}$ in (18) and (19), and its inverse $\tilde{Q}^{-1}$, as follows

$$\tilde{Q} = \begin{bmatrix} X & \bullet \\ U' & \hat{X} \end{bmatrix}, \quad \tilde{Q}^{-1} = \begin{bmatrix} Y & \bullet \\ V' & \hat{Y} \end{bmatrix}. \tag{40}$$

We define matrix $\mathcal{T}$ as

$$\mathcal{T} = \begin{bmatrix} I & I \\ V'Y^{-1} & 0 \end{bmatrix} \tag{41}$$

so that it is easy obtain that

$$(\mathcal{T}'\tilde{Q})\mathcal{T} = \begin{bmatrix} Y^{-1} & 0 \\ X & U \end{bmatrix}\begin{bmatrix} I & I \\ V'Y^{-1} & 0 \end{bmatrix} = \begin{bmatrix} Y^{-1} & Y^{-1} \\ Y^{-1} & X \end{bmatrix}. \tag{42}$$

The proof of Theorem 3 is shown below.

**Proof of Theorem 3.** For the necessity, considering that (18)–(20) hold for system matrices given by (5) we apply the congruence transformation $\mathcal{T}$ to (18) in order to obtain (22), for $Y^{-1} = Z$, and

$$\mathcal{T}'\tilde{P}_i\mathcal{T} = \begin{bmatrix} P_i^{(11)} & \bullet \\ P_i^{(21)} & P_i^{(22)} \end{bmatrix}.$$

In addition, applying the congruence transformation $\mathbf{diag}(\mathcal{T}, I_r)$ to (19) yields (23) for $P_i^{(11)}$, $P_i^{(21)}$ and $P_i^{(22)}$ as previously defined, as well as

$$\mathcal{T}'\tilde{M}_{il}\mathcal{T} = \begin{bmatrix} M_{il}^{(11)} & \bullet \\ M_{il}^{(21)} & M_{il}^{(22)} \end{bmatrix}, \quad \tilde{N}_{il}\mathcal{T} = \begin{bmatrix} N_{il}^{(11)} & N_{il}^{(12)} \end{bmatrix},$$

and $S_{il} = \tilde{S}_{il}$. Finally, we apply the congruence transformation $\mathbf{diag}(\mathcal{T}, I_r, \mathcal{T}, I_q)$ in (20) in order to obtain that

$$\begin{bmatrix} \mathcal{T}'\tilde{M}_{il}\mathcal{T} & \bullet & \bullet & \bullet \\ \tilde{N}_{il}\mathcal{T} & \tilde{S}_{il} & \bullet & \bullet \\ \mathcal{T}'\tilde{Q}\tilde{A}_{il}\mathcal{T} & \mathcal{T}'\tilde{Q}\tilde{J}_{il} & \mathcal{T}'\tilde{Q}\mathcal{T} & \bullet \\ \tilde{C}_{il}\mathcal{T} & \tilde{E}_{il} & 0 & I \end{bmatrix} > 0, \tag{43}$$

and so we set

$$\mathcal{T}'\tilde{Q}\tilde{A}_{il}\mathcal{T} = \begin{bmatrix} ZA_i & ZA_i \\ XA_i + F_lL_i + G_l & XA_i + F_lL_i \end{bmatrix}, \quad \mathcal{T}'\tilde{Q}\tilde{J}_{il} = \begin{bmatrix} ZJ_i \\ XJ_i + F_lH_i \end{bmatrix},$$

and $\tilde{C}_{il}\mathcal{T} = \begin{bmatrix} C_i + K_lL_i + O_l & C_i + K_lL_i \end{bmatrix}$, where $F_l = UB_{fl}$, $G_l = UA_{fl}V'Y^{-1}$, $K_l = -D_{fl}$ and $O_l = -C_{fl}V'Y^{-1}$. Finally, since the product $\mathcal{T}'Q\mathcal{T}$ yields (42) for $Z = Y^{-1}$, we have that (43) yields (24).

On the other hand, for the sufficiency, we set

$$Q = \begin{bmatrix} X & \bullet \\ Z - X & X - Z \end{bmatrix},$$

that yields $V'Y^{-1} = I$, as well as $\tilde{M}_{il}, \tilde{N}_{il}$ and $\tilde{S}_{il}$ as in (37), and

$$\tilde{P}_i = \begin{bmatrix} P_i^{(22)} & \bullet \\ P_i^{(21)'} - P_i^{(22)} & P_i^{(11)} + P_i^{(22)} - P_i^{(21)} - P_i^{(21)'} \end{bmatrix}.$$

In this case, applying the congruence transformations $\mathcal{T}^{-1}$ to (22) yields (18), and $\mathbf{diag}(\mathcal{T}^{-1}, I_r)$ to (23) yields (19). Besides, this choice of variable allow us to write (24) as in (43). Thus, applying the congruence transformation $\mathbf{diag}(\mathcal{T}^{-1}, I_r, \mathcal{T}^{-1}, I_q)$ to (24) yields (20) with system matrices given in (5), concluding also that $f \in \mathcal{K}$. □

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