Nonsynchronous Model Reduction for Uncertain 2-D Markov Jump Systems

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***Abstract*—Mode information is of great significance when investigating the Markov jump systems (MJSs). However, it is common in practical scenarios that the mode information is not completely accessible, which probably induces nonsynchronization problems. Taking this into consideration, in this article, we study nonsynchronous $\mathcal{H}_\infty$ model order reduction for 2-D MJSs with model uncertainty. The considered 2-D system and reduced-order model are characterized by the Roesser model. The nonsynchronization phenomenon between the original system and the reduced-order model is dealt with under the framework of the hidden Markov model. By appropriately selecting the Lyapunov function, the asymptotic mean-square stability and the $\mathcal{H}_\infty$ performance of the error system are analyzed, and sufficient conditions are proposed. Based on this, an efficient design method for nonsynchronous model order reduction is further proposed with the help of a projection lemma. Finally, the correctness and effectiveness of the designed reduced-order model are verified through some simulations.**

***Index Terms*—$\mathcal{H}_\infty$ model order reduction, Markov jumping, nonsynchronization, uncertain 2-D system.**

I. Introduction

The 2-D Markov jump system (2-D MJS) is a combination of the 2-D system and Markov jump system (MJS), which switches among a family of 2-D subsystems under the control of a Markov chain. The research on the 2-D system can date back to the 1970s. Givone and Roesser [1] have put forward a mutidimentional model, that is, the well-known Roesser model, when studying iterative circuits. The 2-D system is much different from the 1-D system, whose state evolves in two independent directions. As a matter of fact, the 2-D system is no less important than 1-D systems since it has a large range of applications. The 2-D model can be adopted to describe a number of practical processes, ranging from thermal process, water stream heating, to a stationary random field [2]. In [3], the Roesser model is applied in image processing. The recursive process control has been investigated in [4] using the 2-D system method. Later, the Markov chain is introduced into 2-D systems considering that there may exist abrupt changes in the system structure or parameters. 2-D MJS has intrigued widespread interest in the academia [5]–[7]. To list some, work [5] has presented a design method of the H_2 fault detection filter for 2-D MJSs with a random measurement missing phenomenon. In [6], a nonfragile state-feedback control scheme has been proposed for 2-D MJSs with state delays.

The mathematical model plays an important role in the traditional control area. A large number of model-based control and filtering methods have been proposed [8]–[14]. These days, there is an increasing number of large-scale systems put into service. It is sure that the orders of the model will rocket with the increase of system scale. However, high-order models will cause inconvenience in the analysis and synthesis for large-scale systems. The order of the controller may be as high as or even higher than the controlled system [15]. As we can imagine, although a “good” controller can be designed based on a high-order model, it may be rather difficult to achieve. As a consequence, it is rather important to find reduced-order models to approximate the high-order models. Many important results have been reported based on different approaches [16]–[20]. Zhou *et al.* [18] have investigated model order reduction for 2-D systems based on the balanced truncation and singular perturbation theory. The work [19] has presented novel identification and model order reduction algorithms for 2-D systems based on extended impulse response Gramians. The main contribution of [20] is finding an H_∞ reduced-order approximation method for 2-D digital filters with the help of a linear matrix inequality (LMI) approach. Compared with other methods, the LMI-based method is very powerful, which can be used to deal with many tough issues, for example, singular systems [12], time-delay systems [21], uncertain systems [22]–[24], and so on. Control systems are often faced with the inaccuracy of physical measurement or random variations in parameters. This will result in uncertainties in system models, which has a great influence on the stability and performance of the overall control system. Therefore, systems with uncertainties arouse many concerns. A robust Kalman filter has been designed in [22] for uncertain 2-D time-varying systems using the least square method. The

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model order reduction problem has been researched for 1-D and 2-D systems with polytopic uncertainties, respectively, in [23] and [24]. However, there is still much room left for a vast improvement in the study on model order reduction for uncertain 2-D systems.

As for MJSs, the trajectory of mode switching is very important. It has been proved that the overall system may be unstable even if all the subsystems are stable due to the stochastic switchings in MJSs. So, it is crucial to catch and make full use of the mode information. In ideal cases, the controller or filter can obtain full access to the modes of MJSs. Thus, we can devise synchronous controllers or filters for MJSs according to some criteria. For example, Aberkane and Dragan [25], Hou and Ma [26], and Luan *et al.* [27] have done some excellent works on synchronous control or filtering for MJSs. However, it is rather difficult to achieve such ideal conditions. There is more or less information loss during the process of mode transmission. Then, one simple solution overcoming this drawback is to use mode-independent controllers or filters [28], [29] instead of synchronous ones. Although the mode-independent method is friendly to use, it completely ignores mode information and then will cause conservatism to the results. Then, how can we get out of this dilemma? Nonsynchronous method provides an excellent solution for this question. Recently, there are increasing achievements on nonsynchronous issues. Note that the existing methods used to describe nonsynchronization mainly fall into three categories: 1) delay-dependent description method can be used if the nonsynchronization is induced by time-delay [30], [31]; 2) piecewise homogeneous Markov chain is another choice. Taking the filtering problem for example, the switchings of the filter are controlled by a piecewise homogeneous Markov chain subject to the original system's mode [32]; and 3) in the hidden Markov model, the switchings of the controller follow some conditional probabilities given the mode of the controlled system [33]–[39]. Among these three descriptions, the hidden Markov model is the most remarkable since it can well characterize both the nonsynchronization and the connection between the original system and the controller or filter through conditional probabilities. Moreover, the hidden Markov model covers mode-independent, synchronous, and nonsynchronous cases simultaneously, and hence is a very unified framework.

Given all the above, the research on nonsynchronous model reduction of uncertain 2-D MJSs is of great significance. However, this subject has not been sufficiently investigated yet. Some important works on model reduction have been published [16]–[20]. However, Li *et al.* [16] and Zeng *et al.* [17] studied this issue only for 1-D systems. Moreover, nonsynchronization and model uncertainties have not been considered in these works. On the other hand, nonsynchronous control and filtering problems have been extensively investigated based on the hidden Markov model [33]–[39]. The work [40] has investigated the nonsynchronous model reduction problem for 1-D MJSs with uncertain probabilities. However, as far as we know, there is little work concerning nonsynchronous model order reduction for 2-D MJSs with model uncertainties. This intrigues our interest in this research. What should be emphasized is that 2-D systems are much different from and more complex than 1-D systems. Moreover, the problem complexity is even aggravated by nonsynchronization and model uncertainty. If these problems are not properly dealt with, the obtained results may contain large conservatism. Therefore, it is challenging to relieve the complexity and reduce conservatism.

The main purpose of this article is to design a nonsynchronous reduced-order model for uncertain 2-D MJSs. The considered 2-D system takes the form of the Roesser model and suffers from model uncertainties. The nonsynchronization phenomenon is appropriately modeled by a unified framework, that is, the hidden Markov model. Note that it is the first time that the nonsynchronous model order reduction problem is considered for uncertain 2-D MJSs. The contributions of this work are mainly two-fold: first, by virtue of the Lyapunov function method, sufficient conditions are proposed for ensuring asymptotic mean-square stability and $\mathcal{H}_\infty$ performance of the resulting hidden Markov jump error system; second, based on the LMI technique, an efficient design method of the reduced-order model is proposed, where a considerable number of decision variables are reduced by resorting to Projection lemma. It is worth noting that the proposed method is also applicable to synchronous or mode-independent model reduction problems. The efficiency of the proposed method is finally validated by numerical simulations.

II. Preliminaries

Consider the following 2-D MJS with uncertain parameters:

$$\mathcal{S}_O\colon \begin{cases} \begin{bmatrix} x^h(k+1,i) \\ x^v(k,i+1) \end{bmatrix} = & \mathcal{A}_{\epsilon_{k,i}} \begin{bmatrix} x^h(k,i) \\ x^v(k,i) \end{bmatrix} + \mathcal{B}_{\epsilon_{k,i}} w(k,i) \\ y(k,i) = & \mathcal{C}_{\epsilon_{k,i}} \begin{bmatrix} x^h(k,i) \\ x^v(k,i) \end{bmatrix} + \mathcal{D}_{\epsilon_{k,i}} w(k,i). \end{cases} \tag{1}$$

The above 2-D system ($\mathcal{S}_O$) is defined on the probability space $(\Omega, \mathscr{F}, \mathscr{P})$ and established in the form of the Roesser model. The state of ($\mathcal{S}_O$) includes two components: 1) the horizontal state $x^h(k,i) \in \mathbb{R}^{n_h}$ and 2) the vertical state $x^v(k,i) \in \mathbb{R}^{n_v}$, evolving in two directions, k and i. The disturbance input $w(k,i) \in \mathbb{R}^{n_w}$ belongs to the l_2 space, that is, $w(k,i) \in l_2\{[0,\infty),[0,\infty)\}$. $y(k,i) \in \mathbb{R}^{n_y}$ denotes the system output. In the 2-D system ($\mathcal{S}_O$), there is a noticeable parameter $\epsilon_{k,i}$ which denotes the mode of system ($\mathcal{S}_O$). Under the control of $\epsilon_{k,i}$, system ($\mathcal{S}_O$) switches among a set of subsystems or modes $\mathcal{N} = \{1,2,\ldots,r\}$ in a Markov jumping style

$$\begin{cases} \Pr\{\epsilon_{k+1,i} = s | \epsilon_{k,i} = f\} = \lambda_{fs}^h \\ \Pr\{\epsilon_{k,i+1} = s | \epsilon_{k,i} = f\} = \lambda_{fs}^v \end{cases} \tag{2}$$

where λ_{fs}^h and λ_{fs}^v are transition probabilities from the fth mode to the sth mode in two directions. Certainly, λ_{fs}^h and λ_{fs}^v satisfy

$$\begin{cases} \lambda_{fs}^h \geq 0, & \sum_{s=1}^r \lambda_{fs}^h = 1 \\ \lambda_{fs}^v \geq 0, & \sum_{s=1}^r \lambda_{fs}^v = 1 \end{cases} \tag{3}$$

for $\forall f, s \in \mathcal{N}$. We define $\Phi^h = [\lambda_{fs}^h]$ and $\Phi^v = [\lambda_{fs}^v]$ as transition probability matrices. To be concise, we will use "f" as shorthand for $\epsilon_{k,i}$, "s" for $\epsilon_{k+1,i}$ or $\epsilon_{k,i+1}$, then the system matrices can be succinctly written as $\mathcal{A}_f$, $\mathcal{B}_f$, $\mathcal{C}_f$, and $\mathcal{D}_f$. These system matrices contain additive uncertainties as follows:

$$\begin{cases} \mathcal{A}_f = A_f + \Delta A_f \\ \mathcal{B}_f = B_f + \Delta B_f \\ \mathcal{C}_f = C_f + \Delta C_f \\ \mathcal{D}_f = D_f + \Delta D_f \end{cases} \tag{4}$$

where A_f, B_f, C_f, and D_f are deterministic, real valued, and pregiven matrices. The uncertain parts ΔA_f, ΔB_f, ΔC_f, and ΔD_f are assumed to be admissible, that is, they satisfy

$$\begin{bmatrix} \Delta A_f & \Delta B_f \\ \Delta C_f & \Delta D_f \end{bmatrix} = \begin{bmatrix} R_f^{(1)} \\ R_f^{(2)} \end{bmatrix} S_f(k,i) \begin{bmatrix} T_f^{(1)} & T_f^{(2)} \end{bmatrix} \tag{5}$$

and

$$S_f^\top(k,i) S_f(k,i) \le I \tag{6}$$

where $R_f^{(1)}$, $R_f^{(2)}$, $T_f^{(1)}$, and $T_f^{(2)}$ are preknown matrices, and $S_f(k,i) \in \mathbb{R}^{n_1 \times n_2}$ is uncertain and satisfies (6). The notation "$\top$" denotes transpose operation. I denotes an identity matrix with appropriate dimensions. In this article, all mentioned matrices and vectors are appropriately dimensioned if not specified.

To approximate system $(\mathcal{S}_O)$, we plan to establish a reduced-order model as follows:

$$\mathcal{S}_R : \begin{cases} \begin{bmatrix} \bar{x}^h(k+1,i) \\ \bar{x}^v(k,i+1) \end{bmatrix} = \bar{A}_{\nu_{k,i}} \begin{bmatrix} \bar{x}^h(k,i) \\ \bar{x}^v(k,i) \end{bmatrix} + \bar{B}_{\nu_{k,i}} w(k,i) \\ \bar{y}(k,i) = \bar{C}_{\nu_{k,i}} \begin{bmatrix} \bar{x}^h(k,i) \\ \bar{x}^v(k,i) \end{bmatrix} + \bar{D}_{\nu_{k,i}} w(k,i). \end{cases} \tag{7}$$

Similarly, $\bar{x}^h(k,i) \in \mathbb{R}^{\bar{n}_h}$ and $\bar{x}^h(k,i) \in \mathbb{R}^{\bar{n}_h}$ represent the horizontal state and vertical state of the reduced-order model $(\mathcal{S}_R)$, respectively. Note that $\bar{n}_h \le n_h$, $\bar{n}_v \le n_v$, and $\bar{n}_h + \bar{n}_v < n_h + n_v$. $\bar{y}(k,i)$ denotes the output of the reduced-order model, which has the same dimension as $y(k,i)$. The matrices $\bar{A}_{\nu_{k,i}}$, $\bar{B}_{\nu_{k,i}}$, $\bar{C}_{\nu_{k,i}}$, and $\bar{D}_{\nu_{k,i}}$ are unknown and are to be determined. $\nu_{k,i}$ denotes the mode of $(\mathcal{S}_R)$, which takes values in a limited set $\mathcal{M} = \{1, 2, \ldots, m\}$ and switches depending on the original system $(\mathcal{S}_O)$'s mode $\epsilon_{k,i}$, as follows:

$$\Pr\{\nu_{k,i} = l | \epsilon_{k,i} = f\} = \theta_{fl} \tag{8}$$

where $\theta_{fl} \ge 0$, $\sum_{l=1}^m \theta_{fl} = 1$ hold for $\forall f \in \mathcal{N}$, $l \in \mathcal{M}$. Equation (8) indicates the probability that the lth mode of $(\mathcal{S}_R)$ is activated when the fth mode of $(\mathcal{S}_O)$ is active. Define $\Gamma = [\theta_{fl}]$ as the conditional probability matrix. In sequel, we will use s instead of $\nu_{k,i}$ for simplicity.

Remark 1: Note that the reduced-order model $(\mathcal{S}_R)$ fails to obtain the precise mode information of the original system $(\mathcal{S}_O)$, which induces nonsynchronization between the modes of $(\mathcal{S}_O)$ and $(\mathcal{S}_R)$. However, they are not completely split. $(\mathcal{S}_O)$'s mode could influence the mode switches of $(\mathcal{S}_R)$ through conditional probability (8). Thus, $(\epsilon_{k,i}, \nu_{k,i}, \Gamma)$ forms a hidden Markov model, in which, $\epsilon_{k,i}$ is a hidden part and $\nu_{k,i}$ is an observation of $\epsilon_{k,i}$. The hidden Markov model is a unified framework for investigating mode-independent, synchronous, or nonsynchronous control/filtering/model order reduction problems for MJSs [33].

Define

$$\tilde{x}^h(k,i) = \begin{bmatrix} x^h(k,i) \\ \bar{x}^h(k,i) \end{bmatrix}, \quad \tilde{x}^v(k,i) = \begin{bmatrix} x^v(k,i) \\ \bar{x}^v(k,i) \end{bmatrix}$$
$$\tilde{y}(k,i) = y(k,i) - \bar{y}(k,i). \tag{9}$$

Then, by combining $(\mathcal{S}_O)$ and $(\mathcal{S}_R)$, we can obtain the following error system:

$$\mathcal{S}_E : \begin{cases} \begin{bmatrix} \tilde{x}^h(k+1,i) \\ \tilde{x}^v(k,i+1) \end{bmatrix} = \tilde{\mathcal{A}}_{fl} \begin{bmatrix} \tilde{x}^h(k,i) \\ \tilde{x}^v(k,i) \end{bmatrix} + \tilde{\mathcal{B}}_{fl} w(k,i) \\ \tilde{y}(k,i) = \tilde{\mathcal{C}}_{fl} \begin{bmatrix} \tilde{x}^h(k,i) \\ \tilde{x}^v(k,i) \end{bmatrix} + \tilde{\mathcal{D}}_{fl} w(k,i) \end{cases} \tag{10}$$

where

$$\begin{aligned} &\tilde{\mathcal{A}}_{fl} = \Lambda \check{\mathcal{A}}_{fl} \Lambda^\top, \ \tilde{\mathcal{B}}_{fl} = \Lambda \check{\mathcal{B}}_{fl}, \ \tilde{\mathcal{C}}_{fl} = \check{\mathcal{C}}_{fl} \Lambda^\top, \ \tilde{\mathcal{D}}_{fl} = \check{\mathcal{D}}_{fl} \\ &\check{\mathcal{A}}_{fl} = \check{A}_{fl} + \Delta \check{A}_f, \ \check{\mathcal{B}}_{fl} = \check{B}_{fl} + \Delta \check{B}_f \\ &\check{\mathcal{C}}_{fl} = \check{C}_{fl} + \Delta \check{C}_f, \ \check{\mathcal{D}}_{fl} = \check{D}_{fl} + \Delta \check{D}_f \\ &\check{A}_{fl} = \begin{bmatrix} A_f & 0 \\ 0 & \bar{A}_l \end{bmatrix}, \ \Delta \check{A}_f = \begin{bmatrix} \Delta A_f & 0 \\ 0 & 0 \end{bmatrix} \\ &\check{B}_{fl} = \begin{bmatrix} B_f \\ \bar{B}_l \end{bmatrix}, \ \Delta \check{B}_f = \begin{bmatrix} \Delta B_f \\ 0 \end{bmatrix} \\ &\check{C}_{fl} = \begin{bmatrix} C_f & -\bar{C}_l \end{bmatrix}, \ \Delta \check{C}_f = \begin{bmatrix} \Delta C_f & 0 \end{bmatrix} \\ &\check{D}_{fl} = D_f - \bar{D}_l, \ \Delta \check{D}_f = \Delta D_f \\ &\Lambda = \begin{bmatrix} I_{n_h} & 0 & 0 & 0 \\ 0 & 0 & I_{\bar{n}_h} & 0 \\ 0 & I_{n_v} & 0 & 0 \\ 0 & 0 & 0 & I_{\bar{n}_v} \end{bmatrix}. \end{aligned}$$

Note that $\Lambda^\top \Lambda = \Lambda \Lambda^\top = I$. For simplicity, we further make the following abbreviations:

$$\tilde{x}^\dagger(k,i) = \begin{bmatrix} \tilde{x}^h(k+1,i) \\ \tilde{x}^v(k,i+1) \end{bmatrix}, \ \tilde{x}(k,i) = \begin{bmatrix} \tilde{x}^h(k,i) \\ \tilde{x}^v(k,i) \end{bmatrix}. \tag{11}$$

The boundary condition of the error system $(\mathcal{S}_E)$ is defined as

$$\begin{cases} \tilde{\mathcal{X}}_b = \{\tilde{x}^h(0,i), \tilde{x}^v(k,0) | k, i = 0, 1, 2, \ldots\} \\ \Upsilon_b = \{\epsilon_{0,i}, \epsilon_{k,0} | k, i = 0, 1, 2, \ldots\} \\ \Theta_b = \{\theta_{0,i}, \theta_{k,0} | k, i = 0, 1, 2, \ldots\}. \end{cases} \tag{12}$$

In addition, it is called a zero boundary condition if $\tilde{\mathcal{X}}_b = 0$. Next, we will present an assumption on $\tilde{\mathcal{X}}_b$ and a definition to carry forward our work.

Assumption 1 [34]: Assume that the following condition always holds for $\tilde{\mathcal{X}}_b$:

$$\lim_{K \to \infty} \mathbb{E}\left\{ \sum_{k=0}^{K} \left(||\tilde{x}^h(0,k)||^2 + ||\tilde{x}^v(k,0)||^2 \right) \right\} < \infty \tag{13}$$

where $||\cdot||$ denotes the Euclidean vector norm and $\mathbb{E}\{\cdot\}$ denotes mathematical expectation.

Definition 1 [34]: The uncertain 2-D MJS $(\mathcal{S}_E)$ is said to be asymptotically mean square stable with an $\mathcal{H}_\infty$ noise attenuation performance γ if the following conditions hold.

1) For any boundary condition $(\tilde{\mathcal{X}}_b, \ \Upsilon_b, \ \Theta_b)$, and $w(k,i) \equiv 0$

$$\lim_{k+i \to \infty} \mathbb{E}\left\{ ||\tilde{x}(k,i)||^2 \right\} = 0 \tag{14}$$

holds.

2) Under zero boundary condition $\tilde{\mathcal{X}}_b = 0$ and $w(k,i) \in l_2\{[0,\infty),[0,\infty)\}$, the following condition holds:

$$\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\mathbb{E}\Big\{||\tilde{y}(k,i)||^2\Big\} < \gamma^2\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}||w(k,i)||^2 \quad (15)$$

where γ is a positive scalar.

We end this section by concluding the problem that we are interested in this article: design a nonsynchronous reduced-order model ($\mathcal{S}_R$) for uncertain 2-D MJS ($\mathcal{S}_O$), which could result in an asymptotically mean square stable error system ($\mathcal{S}_E$) with an $\mathcal{H}_\infty$ noise attenuation performance γ.

III. Main Results

In this section, we will present an effective model order reduction method for uncertain 2-D MJS ($\mathcal{S}_O$). To begin with, we will analyze the asymptotic mean-square stability and $\mathcal{H}_\infty$ noise attenuation performance for the error system ($\mathcal{S}_E$) with $\bar{A}_l$, $\bar{B}_l$, $\bar{C}_l$, and $\bar{D}_l$ known.

Theorem 1: Consider the uncertain 2-D MJS ($\mathcal{S}_E$) under Assumption 1. If there exist positive scalars $\gamma > 0$ and $\sigma > 0$, and positive-definite matrices $P_f = \text{diag}\{P_f^h, P_f^v\} > 0$, and $G_{fl} > 0$, such that for $\forall f \in \mathcal{N}$, $l \in \mathcal{M}$, the following conditions hold:

$$\sum_{l=1}^{m}\theta_{fl}G_{pl} < P_f \quad (16)$$

$$\begin{bmatrix} Z_{fl} & \mathscr{R}_f & \sigma\mathscr{T}_f^\top \\ * & -\sigma I & 0 \\ * & * & -\sigma I \end{bmatrix} < 0 \quad (17)$$

where diag{} denotes the block-diagonal matrix, and

$$Z_{fl} = \begin{bmatrix} -\varLambda^\top\bar{P}_f^{-1}\varLambda & 0 & \check{A}_{fl} & \check{B}_{fl} \\ * & -I & \check{C}_{fl} & \check{D}_{fl} \\ * & * & -\varLambda^\top G_{fl}\varLambda & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix}$$

$$\mathscr{R}_f = \Big[\big[R_f^{(1)\top} \;\; 0\big] \;\; R_f^{(2)\top} \;\; 0 \;\; 0\Big]^\top$$

$$\mathscr{T}_f = \Big[0 \;\; 0 \;\; \big[T_f^{(1)} \;\; 0\big] \;\; T_f^{(2)}\Big]$$

$$\bar{P}_f = \begin{bmatrix} \sum_{s=1}^{r}\lambda_{fs}^h P_s^h & 0 \\ 0 & \sum_{s=1}^{r}\lambda_{fs}^v P_s^v \end{bmatrix}$$

then system ($\mathcal{S}_E$) is asymptotically mean square stable with an $\mathcal{H}_\infty$ noise attenuation performance γ.

Proof: We will begin this proof with some derivations from conditions (16) and (17). First, we can find that (17) is equivalent to

$$Z_{fl} + \sigma^{-1}\mathscr{R}_f\mathscr{R}_f^\top + \sigma\mathscr{T}_f^\top\mathscr{T}_f < 0 \quad (18)$$

by using the Schur complement. Furthermore, (18) is equivalent to

$$Z_{fl} + \mathscr{R}_f S_f(k,i)\mathscr{T}_f + \mathscr{T}_f^\top S_f^\top(k,i)\mathscr{R}_f^\top < 0 \quad (19)$$

since $S_f^\top(k,i)S_f(k,i) \le I$, that is

$$\begin{bmatrix} -\varLambda^\top\bar{P}_f^{-1}\varLambda & 0 & \check{\mathcal{A}}_{fl} & \check{\mathcal{B}}_{fl} \\ * & -I & \check{\mathcal{C}}_{fl} & \check{\mathcal{D}}_{fl} \\ * & * & -\varLambda^\top G_{fl}\varLambda & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \quad (20)$$

holds. Premultiplying and postmultiplying (20) by $\text{diag}\{\varLambda, I, \varLambda, I\}$ and its transpose will not change the positive definiteness of (20). Hence, the following matrix inequality holds:

$$\begin{bmatrix} -\bar{P}_f^{-1} & 0 & \tilde{\mathcal{A}}_{fl} & \tilde{\mathcal{B}}_{fl} \\ * & -I & \tilde{\mathcal{C}}_{fl} & \tilde{\mathcal{D}}_{fl} \\ * & * & -G_{fl} & 0 \\ * & * & * & -\gamma^2 I. \end{bmatrix} < 0. \quad (21)$$

Using the Schur complement to (21), we can obtain

$$\varPi_{fl}^{(1)} + \varPi_{fl}^{(2)} - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} < \begin{bmatrix} G_{fl} & 0 \\ 0 & 0 \end{bmatrix} \quad (22)$$

where

$$\varPi_{fl}^{(1)} = \begin{bmatrix} \tilde{\mathcal{A}}_{fl}^\top \\ \tilde{\mathcal{B}}_{fl}^\top \end{bmatrix}\bar{P}_f\big[\tilde{\mathcal{A}}_{fl} \;\; \tilde{\mathcal{B}}_{fl}\big]$$

$$\varPi_{fl}^{(2)} = \begin{bmatrix} \tilde{\mathcal{C}}_{fl}^\top \\ \tilde{\mathcal{D}}_{fl}^\top \end{bmatrix}\big[\tilde{\mathcal{C}}_{fl} \;\; \tilde{\mathcal{D}}_{fl}\big].$$

Besides, (21) also implies the following inequality holds:

$$\tilde{\mathcal{A}}_{fl}^\top\bar{P}_f\tilde{\mathcal{A}}_{fl} < G_{fl}. \quad (23)$$

Note that (22) and (23) will play a crucial part in the further proof.

We carry on the proof by introducing the following Lyapunov function:

$$V(k,i) = \tilde{x}^\top(k,i)P_f\tilde{x}(k,i) \quad (24)$$

where $P_f = \text{diag}\{P_{\epsilon_{k,i}}^h, P_{\epsilon_{k,i}}^v\}$. Denote $P_s = \text{diag}\{P_{\epsilon_{k+1,i}}^h, P_{\epsilon_{k,i+1}}^v\}$. Then, we calculate its backward difference as follows:

$$\begin{aligned} \nabla V(k,i) &= \tilde{x}^{\dagger\top}(k,i)P_s\tilde{x}^\dagger(k,i) - \tilde{x}^\top(k,i)P_f\tilde{x}(k,i) \\ &= \tilde{x}^{h\top}(k+1,i)P_{\epsilon_{k+1,i}}^h\tilde{x}^h(k+1,i) \\ &\quad - \tilde{x}^{h\top}(k,i)P_{\epsilon_{k,i}}^h\tilde{x}^h(k,i) \\ &\quad + \tilde{x}^{v\top}(k,i+1)P_{\epsilon_{k,i+1}}^v\tilde{x}^v(k,i+1) \\ &\quad - \tilde{x}^{v\top}(k,i)P_{\epsilon_{k,i}}^v\tilde{x}^v(k,i). \end{aligned} \quad (25)$$

Based on (25), we could further figure out the sum of $\nabla V(k,i)$

$$\begin{aligned} \sum_{k=0}^{k_0}\sum_{i=0}^{i_0}\nabla V(k,i) &= \sum_{i=0}^{i_0}\Big\{\tilde{x}^{h\top}(k_0+1,i)P_{\epsilon_{k_0+1,i}}^h\tilde{x}^h(k_0+1,i) \\ &\qquad - \tilde{x}^{h\top}(0,i)P_{\epsilon_{0,i}}^h\tilde{x}^h(0,i)\Big\} \\ &\quad + \sum_{k=0}^{k_0}\Big\{\tilde{x}^{v\top}(k,i_0+1)P_{\epsilon_{k,i_0+1}}^v\tilde{x}^v(k,i_0+1) \\ &\qquad - \tilde{x}^{v\top}(k,0)P_{\epsilon_{k,0}}^v\tilde{x}^v(k,0)\Big\} \end{aligned} \quad (26)$$

where $k_0 > 0$ and $i_0 > 0$ are arbitrary integers. We let both k_0 and i_0 turn to infinity. Then, it can be inferred from (26) and the positive definiteness of P_f that

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\nabla V(k,i) &\geq -\sum_{k=0}^{\infty}\Big\{\tilde{x}^{h\top}(0,k)P^h_{\epsilon_{0,k}}\tilde{x}^h(0,k)\\ &\qquad + \tilde{x}^{v\top}(k,0)P^v_{\epsilon_{k,0}}\tilde{x}^v(k,0)\Big\}\\ &\geq -\xi_1\sum_{k=0}^{\infty}\Big\{||\tilde{x}^h(0,k)||^2 + ||\tilde{x}^v(k,0)||^2\Big\}\end{aligned} \tag{27}$$

where $\xi_1 > 0$ is the maximum eigenvalue of P_f $\forall f \in \mathcal{N}$. In addition, (26) also indicates that

$$\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\nabla V(k,i) \geq 0 \tag{28}$$

holds under zero boundary condition $\tilde{\mathcal{X}}_b = 0$.

Now, we are ready to analyze the asymptotic mean square stability and $\mathcal{H}_\infty$ performance for the error system ($\mathcal{S}_E$). We recalculate the backward difference $\nabla V(k,i)$ and take its expectation considering the dynamics of ($\mathcal{S}_E$)

$$\begin{aligned}&\mathbb{E}\{\nabla V(k,i)\}\\ &= \mathbb{E}\Big\{\tilde{x}^{\dagger\top}(k,i)P_s\tilde{x}^{\dagger}(k,i) - \tilde{x}^\top(k,i)P_f\tilde{x}(k,i)\big|\tilde{x}(k,i),\epsilon_{k,i}=f\Big\}\\ &= \mathbb{E}\left\{F^\top(k,i)\sum_{l=1}^{m}\theta_{fl}\Pi^{(1)}_{fl}F(k,i) - \tilde{x}^\top(k,i)P_f\tilde{x}(k,i)\right\}\end{aligned} \tag{29}$$

where $F(k,i) = [\tilde{x}^\top(k,i)\ \ w^\top(k,i)]^\top$. Letting $w(k,i) = 0$ and recalling (16) and (23), we will have

$$\begin{aligned}\mathbb{E}\{\nabla V(k,i)\} &= \mathbb{E}\left\{\tilde{x}^\top(k,i)\left(\sum_{l=1}^{m}\theta_{fl}\tilde{A}^\top_{fl}\bar{P}_f\tilde{A}_{fl} - P_f\right)\tilde{x}(k,i)\right\}\\ &< \mathbb{E}\left\{\tilde{x}^\top(k,i)\left(\sum_{l=1}^{m}\theta_{fl}G_{fl} - P_f\right)\tilde{x}(k,i)\right\}\\ &\leq -\xi_2\mathbb{E}\Big\{||\tilde{x}(k,i)||^2\Big\}\end{aligned} \tag{30}$$

where $-\xi_2$ is the maximum eigenvalue of $(\sum_{l=1}^{m}\theta_{fl}G_{fl} - P_f)$, $\forall f \in \mathcal{N}$, and $\xi_2 > 0$. As a result of (27) and (30), the following inequality holds:

$$\begin{aligned}&\mathbb{E}\left\{\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}||\tilde{x}(k,i)||^2\right\}\\ &< -\frac{1}{\xi_2}\mathbb{E}\left\{\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\nabla V(k,i)\right\}\\ &\leq \frac{\xi_1}{\xi_2}\mathbb{E}\left\{\sum_{k=0}^{\infty}\Big(||\tilde{x}^h(0,k)||^2 + ||\tilde{x}^v(k,0)||^2\Big)\right\}.\end{aligned} \tag{31}$$

Taking Assumption 1 into consideration, we can obtain

$$\mathbb{E}\left\{\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}||\tilde{x}(k,i)||^2\right\} < \infty \tag{32}$$

which implies (14) holds. Therefore, the asymptotic mean square stability of ($\mathcal{S}_E$) is proved.

Next, we analyze the $\mathcal{H}_\infty$ performance by considering the following index with $\tilde{\mathcal{X}}_b = 0$:

$$\begin{aligned}J &= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\mathbb{E}\Big\{\tilde{y}^\top(k,i)\tilde{y}(k,i) - \gamma^2 w^\top(k,i)w(k,i)\Big\}\\ &\leq \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\mathbb{E}\Big\{\tilde{y}^\top(k,i)\tilde{y}(k,i) - \gamma^2 w^\top(k,i)w(k,i)\Big\}\\ &\quad + \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\mathbb{E}\{\nabla V(k,i)\}\\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\mathbb{E}\Bigg\{F^\top(k,i)\sum_{l=1}^{m}\theta_{fl}\Big(\Pi^{(1)}_{fl} + \Pi^{(2)}_{fl}\\ &\qquad - \mathrm{diag}\{0,\gamma^2 I\}\Big)F(k,i) - \tilde{x}^\top(k,i)P_f\tilde{x}(k,i)\Bigg\}\end{aligned} \tag{33}$$

which, together with (16) and (22), could yield "$J < 0$" in a similar line as (30). Thus, (15) holds. So far, we have proved that ($\mathcal{S}_E$) is asymptotically mean square stable with a certain $\mathcal{H}_\infty$ performance based on conditions (16) and (17). This is the end of the proof. ∎

Remark 2: In the proof of Theorem 1, we resort to the matrix G_{fl} to reduce complexity induced by the nonsynchronization. The matrix G_{fl} helps reduce the matrix dimension of the sufficient condition (17) by separating most of the variables from conditional probability θ_{fl}, which will greatly facilitate the reduced-order model design.

Next, we will further handle the nonlinearity in the conditions of Theorem 1 such that we can turn to LMI boxes to find the solutions of $\bar{A}_l$, $\bar{B}_l$, $\bar{C}_l$, and $\bar{D}_l$ in ($\mathcal{S}_R$). Introduce a slack matrix as follows:

$$L_{fl} = \begin{bmatrix} X^{(1)}_{fl} & \Xi Y_l\\ X^{(2)}_{fl} & Y_l\end{bmatrix} \tag{34}$$

and further denote

$$L^{(1)}_{fl} = \begin{bmatrix} X^{(1)}_{fl}\\ X^{(2)}_{fl}\end{bmatrix},\ L^{(2)}_{l} = \begin{bmatrix} 0 & \Xi Y_l\\ 0 & Y_l\end{bmatrix} \tag{35}$$

where $X^{(1)}_{fl} \in \mathbb{R}^{n\times n}$, $Y_l \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $\Xi = \big[I_{\bar{n}}\ 0_{\bar{n}\times(n-\bar{n})}\big]^\top$, $n = n_h + n_v$, and $\bar{n} = \bar{n}_h + \bar{n}_v$. In order to facilitate further deduction, we denote the Lyapunov matrix $P_f = \mathrm{diag}\{P^h_f, P^v_f\}$ as

$$P^h_f = \begin{bmatrix} P^h_{11,f} & P^h_{12,f}\\ P^{h\top}_{12,f} & P^h_{22,f}\end{bmatrix},\ P^v_f = \begin{bmatrix} P^v_{11,f} & P^v_{12,f}\\ P^{v\top}_{12,f} & P^v_{22,f}\end{bmatrix} \tag{36}$$

where $P^h_{11,f} \in \mathbb{R}^{n_h\times n_h}$, $P^h_{22,f} \in \mathbb{R}^{\bar{n}_h\times\bar{n}_h}$, $P^v_{11,f} \in \mathbb{R}^{n_v\times n_v}$, and $P^v_{22,f} \in \mathbb{R}^{\bar{n}_v\times\bar{n}_v}$, and define $\hat{P}_f \triangleq \Lambda^\top P_f \Lambda$

$$\hat{P}_f = \begin{bmatrix} P^h_{11,f} & 0 & P^h_{12,f} & 0\\ 0 & P^v_{11,f} & 0 & P^v_{12,f}\\ P^{h\top}_{12,f} & 0 & P^h_{22,f} & 0\\ 0 & P^{v\top}_{12,f} & 0 & P^v_{22,f}\end{bmatrix}. \tag{37}$$

Then, we can rewrite $\Lambda^\top \bar{P}_f \Lambda$ as $\Lambda^\top \bar{P}_f \Lambda = \sum_{s=1}^m \mathcal{I}_{fs}\hat{P}_s$, where $\mathcal{I}_{fs}$ is a diagonal matrices concerning transition probabilities (2)

$$\mathcal{I}_{fs} = \begin{bmatrix} \lambda_{fs}^h I_{n_h} & 0 & 0 & 0 \\ 0 & \lambda_{fs}^v I_{n_v} & 0 & 0 \\ 0 & 0 & \lambda_{fs}^h I_{\bar{n}_h} & 0 \\ 0 & 0 & 0 & \lambda_{fs}^v I_{\bar{n}_v} \end{bmatrix}.$$

We denote $\hat{\mathcal{P}}_f \triangleq \sum_{s=1}^m \mathcal{I}_{fs}\hat{P}_s$.

Theorem 2: Consider the uncertain 2-D MJS ($\mathcal{S}_E$) under Assumption 1. If there exist positive scalars $\hat{\gamma} > 0$ and $\sigma > 0$, matrices $\hat{A}_l$, $\hat{B}_l$, $\hat{C}_l$, and $\hat{D}_l$, and positive-definite matrices $\hat{P}_f > 0$ and $\hat{G}_{fl} > 0$, such that for $\forall f \in \mathcal{N}$ and $l \in \mathcal{M}$, the following conditions hold:

$$\sum_{l=1}^m \theta_{fl}\hat{G}_{pl} < \hat{P}_f \tag{38}$$

$$\hat{\mathscr{M}}_f^\top \Psi_{fl}\hat{\mathscr{M}}_f < 0,\ \hat{\mathscr{N}}^\top \Psi_{fl}\hat{\mathscr{N}} < 0 \tag{39}$$

where

$$\Psi_{fl} = \begin{bmatrix} \Psi_{fl}^{(11)} & 0 & \Psi_l^{(13)} & \Psi_l^{(14)} & 0 \\ * & -I & \Psi_{fl}^{(23)} & \Psi_{fl}^{(24)} & \Psi_f^{(25)} \\ * & * & -\hat{G}_{fl} & 0 & \Psi_f^{(35)} \\ * & * & * & -\hat{\gamma} I & \Psi_f^{(45)} \\ * & * & * & * & -\sigma I \end{bmatrix}$$

$$\Psi_{fl}^{(11)} = \hat{\mathcal{P}}_f - L_l^{(2)} - L_l^{(2)\top}$$

$$\Psi_l^{(13)} = \begin{bmatrix} 0 & \Xi\hat{A}_l \\ 0 & \hat{A}_l \end{bmatrix},\ \Psi_l^{(14)} = \begin{bmatrix} \Xi\hat{B}_l \\ \hat{B}_l \end{bmatrix}$$

$$\Psi_{fl}^{(23)} = \begin{bmatrix} C_f & -\hat{C}_l \end{bmatrix}, \Psi_{fl}^{(24)} = D_f - \hat{D}_l, \Psi_f^{(25)} = \begin{bmatrix} R_f^{(2)} & 0 \end{bmatrix}$$

$$\Psi_f^{(35)} = \begin{bmatrix} 0 & \sigma T_f^{(1)\top} \\ 0 & 0 \end{bmatrix},\ \Psi_f^{(45)} = \begin{bmatrix} 0 & \sigma T_f^{(2)\top} \end{bmatrix}$$

$$\hat{\mathscr{M}}_f = \begin{bmatrix} \hat{\mathscr{M}}_f^{(1)} \\ I_{n+2\bar{n}+n_y+n_w+n_1+n_2} \end{bmatrix}$$

$$\hat{\mathscr{M}}_f^{(1)} = \begin{bmatrix} 0_{n\times\bar{n}} & 0_{n\times\bar{n}_y} & A_f & 0_{n\times\bar{n}} & B_f & R_f^{(1)} & 0_{n\times n_2} \end{bmatrix}$$

$$\hat{\mathscr{N}} = \begin{bmatrix} 0_{(n+\bar{n})\times(n+\bar{n}+n_y+n_w+n_1+n_2)} \\ I_{n+\bar{n}+n_y+n_w+n_1+n_2} \end{bmatrix}$$

then system ($\mathcal{S}_E$) is asymptotically mean square stable and has a certain $\mathcal{H}_\infty$ noise attenuation level. Moreover, if the LMIs (38) and (39) have feasible solutions, then the matrices in the reduced-order model ($\mathcal{S}_R$) are given by

$$\bar{A}_l = Y_l^{-1}\hat{A}_l,\ \bar{B}_l = Y_l^{-1}\hat{B}_l,\ \bar{C}_l = \hat{C}_l,\ \bar{D}_l = \hat{D}_l \tag{40}$$

and the corresponding $\mathcal{H}_\infty$ performance is $\gamma = \sqrt{\hat{\gamma}}$.

Proof: We will prove Theorem 2 by demonstrating that (38) and (39) are sufficient conditions for (16) and (17). First, it is not difficult to establish equivalence between (16) and (38) by conducting a congruence operation on (16) with Λ and denoting $\hat{P}_f \triangleq \Lambda^\top P_f \Lambda$, $\hat{G}_{fl} \triangleq \Lambda^\top G_{fl}\Lambda$.

On the other hand, according to the projection lemma, (39) holds, if and only if there exists matrix $L_{fl}^{(1)}$ such that the following inequality holds:

$$\Psi_{fl} + \mathscr{M}_f^\top L_{fl}^{(1)\top}\mathscr{N} + \mathscr{N}^\top L_{fl}^{(1)}\mathscr{M}_f < 0 \tag{41}$$

where

$$\mathscr{M}_f = \begin{bmatrix} -I & \hat{\mathscr{M}}_f^{(1)} \end{bmatrix}$$

$$\mathscr{N} = \begin{bmatrix} I_{n+\bar{n}} \\ 0_{(n+\bar{n}+n_y+n_w+n_1+n_2)\times(n+\bar{n})} \end{bmatrix}^\top.$$

Note that $\mathscr{M}_f\hat{\mathscr{M}}_f = 0$ and $\mathscr{N}\hat{\mathscr{N}} = 0$. We rewrite (41) as

$$\begin{bmatrix} \bar{\Psi}_{fl}^{(11)} & 0 & \bar{\Psi}_{fl}^{(13)} & \bar{\Psi}_{fl}^{(14)} & L_{fl}\bar{\Psi}_{fl}^{(15)} \\ * & -I & \Psi_{fl}^{(23)} & \Psi_{fl}^{(24)} & \Psi_f^{(25)} \\ * & * & -\hat{G}_{fl} & 0 & \Psi_f^{(35)} \\ * & * & * & -\hat{\gamma} I & \Psi_f^{(45)} \\ * & * & * & * & -\sigma I \end{bmatrix} < 0 \tag{42}$$

where

$$\bar{\Psi}_{fl}^{(11)} = \hat{\mathcal{P}}_f - L_{fl} - L_{fl}^\top,\ \bar{\Psi}_{fl}^{(13)} = \begin{bmatrix} X_{fl}^{(1)}A_f & \Xi\hat{A}_l \\ X_{fl}^{(2)}A_f & \hat{A}_l \end{bmatrix}$$

$$\bar{\Psi}_{fl}^{(14)} = \begin{bmatrix} X_{fl}^{(1)}B_f + \Xi\hat{B}_l \\ X_{fl}^{(2)}B_f + \hat{B}_l \end{bmatrix},\ \bar{\Psi}_{fl}^{(15)} = \begin{bmatrix} R_f^{(1)} & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that (42) implies that L_{fl} and Y_l are nonsingular. Furthermore, based on the following notations:

$$\hat{A}_l \triangleq Y_l\bar{A}_l,\ \hat{B}_l \triangleq Y_l\bar{B}_l,\ \hat{C}_l \triangleq \bar{C}_l,\ \hat{D}_l \triangleq \bar{D}_l,\ \hat{\gamma} \triangleq \gamma^2 \tag{43}$$

and the well-known fact that $\hat{\mathcal{P}}_f - L_{fl} - L_{fl}^\top \geq -L_{fl}\hat{\mathcal{P}}_f^{-1}L_{fl}^\top$, (42) is sufficient to guarantee the following inequality holds:

$$\begin{bmatrix} -L_{fl}\hat{\mathcal{P}}_f^{-1}L_{fl}^\top & 0 & L_{fl}\check{A}_{fl} & L_{fl}\check{B}_{fl} & L_{fl}\bar{\Psi}_{fl}^{(15)} \\ * & -I & \check{C}_{fl} & \check{D}_{fl} & \Psi_f^{(25)} \\ * & * & -\hat{G}_{fl} & 0 & \Psi_f^{(35)} \\ * & * & * & -\gamma^2 I & \Psi_f^{(45)} \\ * & * & * & * & -\sigma I. \end{bmatrix} < 0 \tag{44}$$

Recall that $\hat{G}_{fl} = \Lambda^\top G_{fl}\Lambda$ and $\hat{\mathcal{P}}_f = \Lambda^\top \bar{P}_f \Lambda$. Then, premultiplying and postmultiplying (44) with diag$\{L_{fl}^{-1}, I, I, I, I\}$ and its transpose could directly yield (17). Thus, the sufficiency of (38) and (39) ensuring (16) and (17) is proved. Finally, the matrices in the reduced-order model ($\mathcal{S}_R$) can be given by (40) according to (43). This completes the proof. ■

Remark 3: Inspired by [16], the projection lemma is applied in the above proof. Note that condition (39) is the result of using the projection lemma to (42). In fact, the LMI (42) can be regarded as an alternative to condition (39) since they are equivalent. However, the advantage of (39) over (42) is that the former contains fewer decision variables. Compared to (42), (39) excludes $L_{fl}^{(1)}$ from its decision variable list. As a consequence, the proposed design method is efficient.

IV. Numerical Example

In this section, we will apply the proposed model order reduction method to a two-mode 2-D MJS with state dimensions $n_h = n_v = 2$. The system parameters will be present in the following compact form:

$$\left[\begin{array}{c|c|c} A_f & B_f & R_f^{(1)} \\ \hline C_f & D_f & R_f^{(2)} \\ \hline T_f^{(1)} & T_f^{(2)} & \end{array}\right], \; f \in \{1, 2\}. \tag{45}$$

Then, the definite values for (45) are given as follows:

The 1st mode, $f = 1$

$$\left[\begin{array}{cccc|c|c} -0.5 & 0.75 & 1 & 0 & 1 & 0.1 \\ 0 & 0.21 & 0 & 0 & 0 & 0 \\ 0.2625 & 0 & -0.5 & 0.75 & 1 & 0.1 \\ 0 & 0 & -0.05 & -0.025 & 0 & 0 \\ \hline 0.1 & 0.2 & -0.3 & 0 & 0.5 & 0.1 \\ \hline 0.2 & 0.1 & 0.1 & -0.2 & -0.35 & \end{array}\right].$$

The 2nd mode, $f = 2$

$$\left[\begin{array}{cccc|c|c} -0.25 & 0.55 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.22 & -0.34 & -1 & 0.1 \\ 0.35 & 0 & -0.45 & 0.15 & 0 & 0 \\ 0 & 0 & -0.15 & -0.075 & -1 & 0.1 \\ \hline -0.1 & 0.1 & 0.1 & 0.2 & -0.2 & -0.1 \\ \hline 0.05 & 0.24 & -0.11 & 0.1 & 0.1 & \end{array}\right].$$

The uncertainty $S_f(k, i) \in \mathbb{R}$ is set as follows:

$$S_f(k, i) = \begin{cases} 0.1\omega_1(k, i), & f = 1 \\ 0.1\sin(k + i), & f = 2. \end{cases}$$

The disturbance input is $w(k, i) = 0.8^{k+i}\omega_2(k, i)$. $\omega_1(k, i)$ and $\omega_2(k, i)$ are both random noises with values between -1 and 1. The boundary condition is set as $x^h(0, i) = [-0.1 \;\; -0.1]^\top$ for $0 \le i \le 15$, $x^h(0, i) = [0 \;\; 0]^\top$ for $i > 15$, $x^v(k, 0) = [0.1 \; 0.1]^\top$ for $0 \le k \le 15$, and $x^v(k, 0) = [0 \;\; 0]^\top$ for $k>15$. The system switches between two modes according to the following transition probability matrices:

$$\Phi^h = \begin{bmatrix} 0.6 & 0.4 \\ 0.1 & 0.9 \end{bmatrix}, \; \Phi^v = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}.$$

For the system described above, we will design a two-mode reduced-order model by solving an optimization problem, that is, minimizing $\hat{\gamma}$ subject to (38) and (39). Then, we can figure out the matrices $\bar{A}_l$, $\bar{B}_l$, $\bar{C}_l$, and $\bar{D}_l$ through (40), and obtain the optimal $\mathcal{H}_\infty$ performance $\gamma^* \triangleq \sqrt{\hat{\gamma}}$. The following is the conditional probability matrix for the reduced-order model:

$$\Gamma = \begin{bmatrix} 0.2 & 0.8 \\ 0.9 & 0.1 \end{bmatrix}. \tag{46}$$

As described in Remark 1, the proposed design method is also applicable to the synchronous case or mode-independent case by letting $\mathcal{N} = \mathcal{M}$, $\Gamma = I$ or $\mathcal{M} = \{1\}$, $\Gamma = [1 \;\cdots\; 1]^\top$. We obtain the optimal $\mathcal{H}_\infty$ performance γ^* for all the three cases, which are presented in Table I. Table I shows that the synchronous case achieves the best $\mathcal{H}_\infty$ performance since the reduced-order model knows the exact mode information of the original system. On the contrary, in the mode-independent case, the reduced-order model has no access to mode information and, hence, has the worst $\mathcal{H}_\infty$ performance. The nonsynchronous case falls in between since the mode information is partially accessed. Besides, comparing the cases with different dimensions, we can find that the case with lower $\bar{n}_h$ and $\bar{n}_v$ has worse $\mathcal{H}_\infty$ performance due to that more information of the original system is discarded. All these observations are consistent with our expectations.

TABLE I
Optimal $\mathcal{H}_\infty$ Performance γ^* for Different Cases

	Synchronous	Nonsynchronous	Mode-independent
$\bar{n}_h = 1, \bar{n}_v = 1$	1.2246	1.3152	1.4240
$\bar{n}_h = 1, \bar{n}_v = 2$	0.8178	1.0909	1.4027
$\bar{n}_h = 2, \bar{n}_v = 1$	0.8297	1.1094	1.4165

TABLE II
Comparisons Between Two Methods

	$\bar{n}_h = 1, \bar{n}_v = 1$	$\bar{n}_h = 1, \bar{n}_v = 2$	$\bar{n}_h = 2, \bar{n}_v = 1$
γ^* with (38)(39)	1.3152	1.0909	1.1094
γ^* with (38)(42)	1.3152	1.0909	1.1094
Nod in (38)(39)	136	196	196
Nod in (38)(42)	232	308	308
Reduction	41.4%	36.4%	36.4%

On the other hand, Remark 3 has mentioned that the condition (39) involves less decision variables compared with (42), which helps improve computation efficiency. We try to validate this point through some simulations. See the results in Table II, where *Nod* denotes the number of decision variables in LMIs, and the percentage of *Nod* reduction is calculated as follows:

$$\text{Reduction} = \frac{[\text{Nod in (38)(42)}] - [\text{Nod in (38)(39)}]}{[\text{Nod in (38)(42)}]} \times 100\%.$$

We can observe from Table II that the method based on (38) and (39) achieves the same $\mathcal{H}_\infty$ performance as the method based on (38) and (42) does; however, the decision variables in the former are much less than that in the latter. The results show that the proposed method in this article can save a considerable amount of computation resources without introducing any conservatism.

Next, we will further investigate the case with $\bar{n}_h = 1$, $\bar{n}_v = 1$. With the parameters given above and the conditional probability (46), we solve the LMIs (38) and (39) and obtain the matrices $\bar{A}_l$, $\bar{B}_l$, $\bar{C}_l$, and $\bar{D}_l$, $l = 1, 2$ as follows:

$$\left[\begin{array}{c|c} \bar{A}_1 & \bar{B}_1 \\ \hline \bar{C}_1 & \bar{D}_1 \end{array}\right] = \left[\begin{array}{cc|c} -0.4777 & 0.1661 & -1.5863 \\ -0.0757 & -0.0842 & 1.2428 \\ \hline 0.0066 & 0.1707 & 0.3072 \end{array}\right]$$

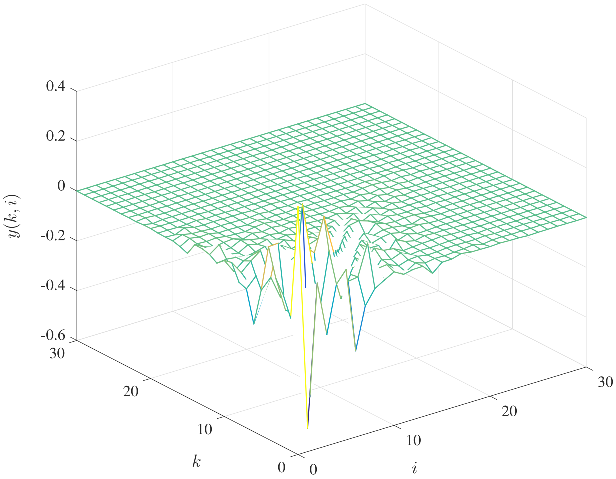

Fig. 1. Output of the original system $\mathcal{S}_O$.

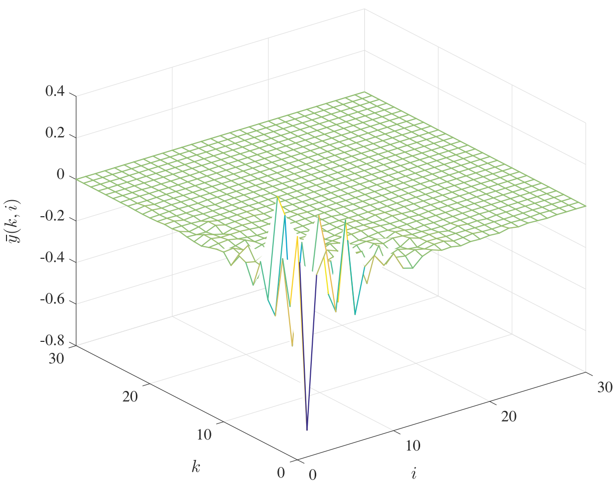

Fig. 2. Output of the reduced-order model $\mathcal{S}_R$.

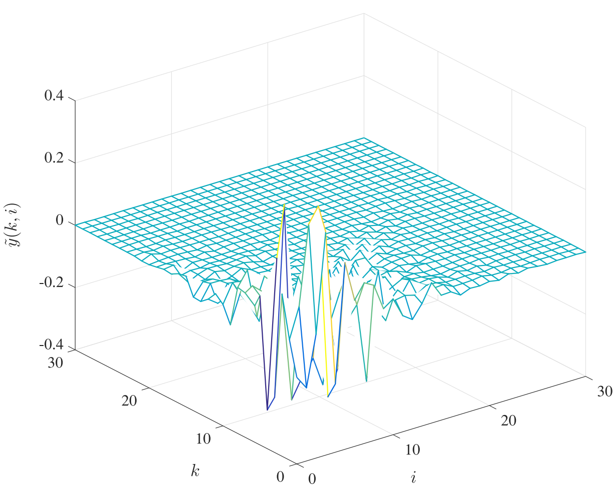

Fig. 3. Error $\tilde{y}(k,i)$.

$$\left[\begin{array}{c|c}\bar{A}_2 & \bar{B}_2 \\ \hline \bar{C}_2 & \bar{D}_2\end{array}\right] = \left[\begin{array}{cc|c}-0.6241 & 0.9045 & -0.9467 \\ -0.0809 & 0.1399 & 1.0553 \\ \hline 0.0012 & -0.5689 & 0.3012\end{array}\right].$$

The boundary condition of the reduced-order model is set as $\bar{x}(0,i) = \bar{x}(k,0) = [0 \quad 0]^\top$, $k, i = 0, 1, 2, \ldots$ Then, we

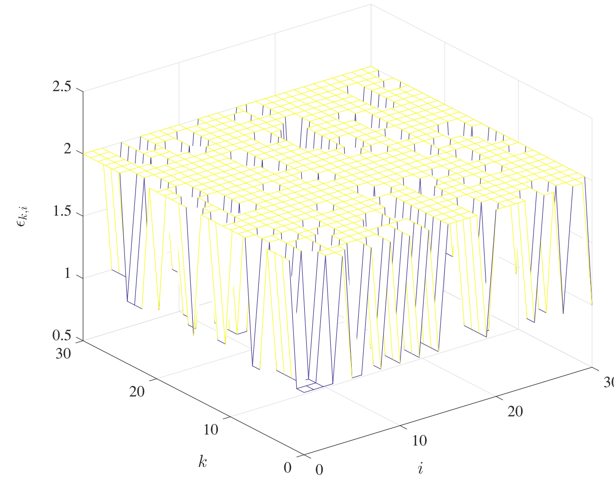

Fig. 4. Mode switchings of $\mathcal{S}_O$.

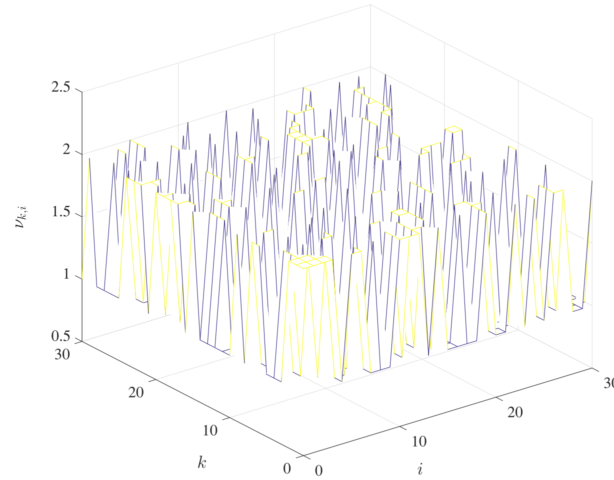

Fig. 5. Mode switchings of $\mathcal{S}_R$.

obtain the output trajectories of the original system and the reduced-order model, which are exhibited in Figs. 1 and 2. Fig. 3 further shows the trajectory of the error $\tilde{y}(k,i)$. Besides, Figs. 4 and 5, respectively, show the mode switchings of the original system and the reduced-order model, which demonstrates the nonsynchronization phenomenon between them. We can see that $\bar{y}(k,i)$ is close to $y(k,i)$, and the error $\tilde{y}(k,i)$ is small, which further verifies that the proposed design method is correct and effective.

V. Conclusion

This article was devoted to investigating nonsynchronous $\mathcal{H}_\infty$ model order reduction problem for uncertain 2-D MJSs. The 2-D system of interest was modeled in the form of the Roesser model, and there exist uncertainties in the system parameters. The mode of the designed reduced-order model switches asynchronously with the original 2-D MJSs due to which only partial mode information of the original system can be accessed by the reduced-order model. The nonsynchronization was modeled by the hidden Markov Model, namely, the mode switchings of the reduced-order model are dependent on the original system's mode according to certain conditional probabilities. Based on such a nonsynchronous framework, we have proposed a sufficient condition ensuring the asymptotic mean-square stability of the error system with a certain $\mathcal{H}_\infty$ performance by applying the Lyapunov function method. Then, we have further proposed a design method for the matrix parameters in the reduced-order model. Finally, a numerical example provides potent evidence for the effectiveness of the proposed method. Furthermore, based on the derived results, reduced-order control and reduced-order resilient filtering for 2-D MJSs would be interesting topics, which will be our future work.

References

[1] D. D. Givone and R. P. Roesser, "Multidimensional linear iterative circuits—General properties," *IEEE Trans. Comput.*, vol. C-21, no. 10, pp. 1067–1073, Oct. 1972.

[2] L. V. Hien and H. Trinh, "Stability analysis and control of two-dimensional fuzzy systems with directional time-varying delays," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 3, pp. 1550–1564, Jun. 2018.

[3] R. P. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Autom. Control*, vol. AC-20, no. 1, pp. 1–10, Feb. 1975.

[4] E. Rogers, K. Galkowski, and D. H. Owens, *Control Systems Theory and Applications for Linear Repetitive Processes* (Lecture Notes in Control and Information Sciences), vol. 349. Heidelberg, Germany: Springer, 2007, p. 466.

[5] L. Wu, X. Yao, and W. X. Zheng, "Generalized H_2 fault detection for two-dimensional Markovian jump systems," *Automatica*, vol. 48, no. 8, pp. 1741–1750, 2012.

[6] M. Dai, Z. Huang, J. Xia, B. Meng, J. Wang, and H. Shen, "Non-fragile extended dissipativity-based state feedback control for 2-D Markov jump delayed systems," *Appl. Math. Comput.*, vol. 362, pp. 1–11, Dec. 2019.

[7] Y. Luo, Z. Wang, G. Wei, and F. E. Alsaadi, "Nonfragile l_2-l_∞ fault estimation for Markovian jump 2-D systems with specified power bounds," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 50, no. 5, pp. 1964–1975, May 2020.

[8] Y. Luo, Z. Wang, J. Liang, G. Wei, and F. E. Alsaadi, "H_∞ control for 2-D fuzzy systems with interval time-varying delays and missing measurements," *IEEE Trans. Cybern.*, vol. 47, no. 2, pp. 365–377, Feb. 2017.

[9] S. Dong, G. Chen, M. Liu, and Z.-G. Wu, "Cooperative adaptive H_∞ output regulation of continuous-time heterogeneous multi-agent Markov jump systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, early access, Mar. 9, 2021, doi: 10.1109/TCSII.2021.3064944.

[10] S. Dong, G. Chen, M. Liu, and Z.-G. Wu, "Robust adaptive H_∞ control for networked uncertain semi-Markov jump nonlinear systems with input quantization," *Sci. China Inf. Sci.*, vol. 65, no. 8, 2022, Art. no. 3148252, doi: 10.1007/s11432-020-3187-8.

[11] L. V. Hien, H. M. Trinh, and P. N. Pathirana, "On l_1-gain control of 2-D positive Roesser systems with directional delays: Necessary and sufficient conditions," *Automatica*, vol. 112, pp. 1–10, Feb. 2020.

[12] Z. Feng and P. Shi, "Sliding mode control of singular stochastic Markov jump systems," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 4266–4273, Aug. 2017.

[13] W. Chen, D. Ding, H. Dong, and G. Wei, "Distributed resilient filtering for power systems subject to denial-of-service attacks," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 8, pp. 1688–1697, Aug. 2019.

[14] D. Ding, Z. Wang, and Q.-L. Han, "A set-membership approach to event-triggered filtering for general nonlinear systems over sensor networks," *IEEE Trans. Autom. Control*, vol. 65, no. 4, pp. 1792–1799, Apr. 2020.

[15] C. De Villemagne and R. E. Skelton, "Controller reduction using canonical interactions," *IEEE Trans. Autom. Control*, vol. 33, no. 8, pp. 740–750, Aug. 1988.

[16] X. Li, J. Lam, H. Gao, and P. Li, "Improved results on H_∞ model reduction for Markovian jump systems with partly known transition probabilities," *Syst. Control Lett.*, vol. 70, pp. 109–117, Aug. 2014.

[17] Y. Zeng, H.-K. Lam, and L. Wu, "Hankel-norm-based model reduction for stochastic discrete-time nonlinear systems in interval type-2 T–S fuzzy framework," *IEEE Trans. Cybern.*, early access, Jan. 17, 2020, doi: 10.1109/TCYB.2019.2950565.

[18] K. Zhou, J. L. Aravena, G. Gu, and D. Xiong, "2-D Model reduction by quasi-balanced truncation and singular perturbation," *IEEE Trans. Circuits Syst. II, Analog Digital Signal Process.*, vol. 41, no. 9, pp. 593–602, Sep. 1994.

[19] C. Xiao, V. Sreeram, W. Q. Liu, and A. N. Venetsanopoulos, "Identification and model reduction of 2-D systems via the extended impulse response Gramians," *Automatica*, vol. 34, no. 1, pp. 93–101, 1998.

[20] C. Du, L. Xie, and Y. C. Soh, "H_∞ reduced-order approximation of 2-D digital filters," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 48, no. 6, pp. 688–698, Jun. 2001.

[21] Y. He, M. Wu, G.-P. Liu, and J.-H. She, "Output feedback stabilization for a discrete-time system with a time-varying delay," *IEEE Trans. Autom. Control*, vol. 53, no. 10, pp. 2372–2377, Nov. 2008.

[22] D. Zhao, S. X. Ding, H. R. Karimi, Y. Li, and Y. Wang, "On robust Kalman filter for two-dimensional uncertain linear discrete time-varying systems: A least squares method," *Automatica*, vol. 99, pp. 203–212, Jan. 2019.

[23] L. Zhang, P. Shi, E.-K. Boukas, and C. Wang, "H_∞ model reduction for uncertain switched linear discrete-time systems," *Automatica*, vol. 44, no. 11, pp. 2944–2949, 2008.

[24] H. Gao, J. Lam, C. Wang, and S. Xu, "H_∞ model reduction for uncertain two-dimensional discrete systems," *Opt. Control Appl. Methods*, vol. 26, no. 4, pp. 199–227, 2005.

[25] S. Aberkane and V. Dragan, "H_∞ filtering of periodic Markovian jump systems: Application to filtering with communication constraints," *Automatica*, vol. 48, no. 12, pp. 3151–3156, 2012.

[26] T. Hou and H. Ma, "Exponential stability for discrete-time infinite Markov jump systems," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 4241–4246, Dec. 2016.

[27] X. Luan, S. Zhao, and F. Liu, "H_∞ control for discrete-time Markov jump systems with uncertain transition probabilities," *IEEE Trans. Autom. Control*, vol. 58, no. 6, pp. 1566–1572, Jun. 2013.

[28] M. Dolgov and U. D. Hanebeck, "Static output-feedback control of Markov jump linear systems without mode observation," *IEEE Trans. Autom. Control*, vol. 62, no. 10, pp. 5401–5406, Oct. 2017.

[29] C. E. de Souza, A. Trofino, and K. A. Barbosa, "Mode-independent H_∞ filters for Markovian jump linear systems," *IEEE Trans. Autom. Control*, vol. 51, no. 11, pp. 1837–1841, Nov. 2006.

[30] Z. Fei, S. Shi, C. Zhao, and L. Wu, "Asynchronous control for 2-D switched systems with mode-dependent average dwell time," *Automatica*, vol. 79, pp. 198–206, May 2017.

[31] L. Zhang and H. Gao, "Asynchronously switched control of switched linear systems with average dwell time," *Automatica*, vol. 46, no. 5, pp. 953–958, 2010.

[32] Z.-G. Wu, P. Shi, H. Su, and J. Chu, "Asynchronous l_2-l_∞ filtering for discrete-time stochastic Markov jump systems with randomly occurred sensor nonlinearities," *Automatica*, vol. 50, no. 1, pp. 180–186, 2014.

[33] Z.-G. Wu, P. Shi, Z. Shu, H. Su, and R. Lu, "Passivity-based asynchronous control for Markov jump systems," *IEEE Trans. Autom. Control*, vol. 62, no. 4, pp. 2020–2025, Apr. 2017.

[34] Z.-G. Wu, Y. Shen, P. Shi, Z. Shu, and H. Su, "H_∞ control for 2-D Markov jump systems in Roesser model," *IEEE Trans. Autom. Control*, vol. 64, no. 1, pp. 427–432, Jan. 2019.

[35] L. V. Hien and H. Trinh, "Observer-based control of 2-D Markov jump systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 64, no. 11, pp. 1322–1326, Nov. 2017.

[36] Y.-Y. Tao, Z.-G. Wu, and Y. Guo, "Two-dimensional asynchronous sliding-mode control of Markov jump Roesser systems," *IEEE Trans. Cybern.*, early access, Jul. 28, 2020, doi: 10.1109/TCYB.2020.3006329.

[37] Y. A. Zabala and O. L. V. Costa, "A detector-based approach for the constrained quadratic control of discrete-time Markovian jump linear systems," *IEEE Trans. Autom. Control*, vol. 65, no. 3, pp. 1211–1217, Mar. 2020.

[38] L. Wang, Z.-G. Wu, and Y. Shen, "Asynchronous mean stabilization of positive jump systems with piecewise-homogeneous Markov chain," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, early access, Mar. 8, 2021, doi: 10.1109/TCSII.2021.3064388.

[39] S. Dong, M. Liu, Z.-G. Wu, and K. Shi, "Observer-based sliding mode control for Markov jump systems with actuator failures and asynchronous modes," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, early access, Oct. 13, 2020, doi: 10.1109/TCSII.2020.3030703.

[40] Y. Shen, Z.-G. Wu, P. Shi, and C. K. Ahn, "Model reduction of Markovian jump systems with uncertain probabilities," *IEEE Trans. Autom. Control*, vol. 65, no. 1, pp. 382–388, Jan. 2020.

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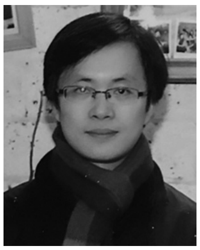

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