Les Cahiers du GERAD ISSN: 0711–2440

On Reference Model Tracking for Markov Jump Systems

E.-K. Boukas

G–2007–103

December 2007

On Reference Model Tracking for Markov Jump Systems

El-Kébir Boukas

GERAD and Mechanical Engineering Department
École Polytechnique de Montréal
P.O. Box 6079, station Centre-ville
Montréal (Québec) Canada, H3C 3A7

el-kebir.boukas@polymtl.ca

December 2007

Les Cahiers du GERAD

G–2007–103

Abstract

This papers deals with the tracking problem for the Markov Jump systems with external finite energy disturbance. A state feedback controller that makes the state vector of the system track precisely a given state vector of a reference model is proposed. The tracking problem is formulated as an $\mathscr{H}_\infty$ control problem and an approach to synthesize the state feedback controller that quadratically stabilizes the augmented dynamics and at the same time rejects the external disturbance is developed. This approach is based on the solution of some linear matrix inequalities (LMIs). A numerical example is provided to show the usefulness of the developed results.

Key Words: Markov jump systems, stabilization, state feedback, $\mathscr{H}_\infty$ control, linear matrix inequalities.

Résumé

Cet article traite du problème de poursuite de modèle pour la classe des systèmes à sauts markoviens. Un contrôleur par retour d'état qui force l'état du système considéré à poursuivre celui d'un modèle de référence est employé. Le problème de poursuite est formulé sous forme d'un problème de commande $\mathscr{H}_\infty$ et une approche qui permet la synthèse du contrôleur qui stabilise le système augmenté et en même temps assure le rejet de perturbation est développée. La solution de ce problème dépend de la faisabilité d'un certain ensemble de LMIs. Un exemple numérique est fourni pour montrer l'utilité des résultats proposés.

1 Introduction

Systems with abrupt changes in their dynamics that results from causes like connections or disconnections of some components, failures in the components etc. represent an interesting class of industrial systems that unfortunately the linear invariant system is unable to describe precisely. Examples of such dynamical systems can be found in manufacturing systems, power systems, telecommunications systems, etc. The occurrence of the abrupt changes in this class of systems is always random. These practical systems have been modeled by the class of linear systems with Markovian jumps. This class of systems has two components in the state vector. The first component of this state vector takes values in $\mathbb{R}^n$ and evolves continuously in time and it represents the classical state vector that is usually used in the modern control theory. The second one takes values in a finite set and switches in a random manner between a finite number of states (see Mariton, 1990 and Boukas, 2005 and the references therein). This component is represented by a continuous-time Markov process taking values discretely in a finite space. Usually the state vector of the class of Markov jump systems is denoted by $(x(t), r_t)$.

In 1960, Krasovskii and Lidskii introduced the framework of the class of systems with Markovian jumps. This class of system was found to be appropriate to model many practical systems and since that, it has attracted a lot of researchers from the control and operations research communities. Beside the theoretical contributions on the stability, stabilization, filtering problems, etc. of this class of systems, we witnessed its use in modeling a variety of practical systems mainly those with abrupt changes in their structure. More specifically, Boukas (2005) covers most all these problems by establishing LMI conditions to solve them. de Souza and Fragoso (1993) established conditions for $\mathscr{H}_\infty$ control problem for linear Markovian jump systems. The robust case of this class of systems has been tackled in Shi and Boukas (1997) where LMI conditions were developed to synthesize the stabilizing controller. The output stabilization of the Markovian jump systems has been tackled in de Farias *et al.* (2000), Boukas (2005) where LMI conditions where developed to solve this problem. In Wang *et al.* (2002), the stabilization problem for the class of Markovian jump systems with time-delay and external disturbance has been tackled and solved using the LMI setting. Results on filtering can be found in Zhang (2000) and Boukas (2005). In Boukas and Liu (2001), Sethi and Zhang (1994) the framework of the class of Markovian jump systems has been used in manufacturing systems with random breakdowns to deal with the production and maintenance planning. For more details on what it has been done on the class of Markovian jump systems, we refer the reader to Boukas (2005) and the references therein.

Regarding the stabilization problem most of all what has been reported in the literature consider the case of controllers with mode dependent gains which requires the knowledge of the mode at each time we want to switch the controller gain and the instant at which the switch occurs in addition to the knowledge of the state vector $x(t)$. For more details on this, we refer the reader to Boukas (2005) and the references therein where different approaches have been proposed to solve the stabilization problem. But practically, this is not always possible since the mode is not always accessible and neither the instant of the switch, which restricts the use of such controllers. It is possible to estimate the mode as it was done by Zhang (2000) and therefore continue to use the controller. But if we are interested by real time applications this is not possible unless the size of the system is small.

The aim of this paper is to synthesize a state feedback controller to make the state of the linear dynamic system with Markovian jumps track the state vector of a given reference model. The reference model we will consider does not depend on the system mode. This problem arises in many practical systems. As an example, we cite the case of a production system with breakdowns where we search in general to force the total production to track the total demand of the system despite changes in the production capacity that may vary randomly in time (Boukas, 2006).

To the best of our knowledge this problem has not been fully investigated and the methodology we are proposing in this paper to tackle the tracking problem for the class of Markov jump systems has never tackled before. Our solution to this consists of formulating it as an $\mathscr{H}_\infty$ stabilization which allows us to use the $\mathscr{H}_\infty$ theory (see Boukas, 2005 and the references therein).

The rest of the paper is organized as follows. In Section 2, the tracking problem is formulated. Section 3 presents the main contribution of the paper. The problem is formulated as an $\mathscr{H}_\infty$ control problem and then solved using some known results. In Section 4, a numerical example is provided to show the validity of the proposed methodology to handle the tracking problem of the class of systems we are considering.

Throughout this paper, the following notations will be used. $A^\top$ denotes the transposition of the matrix A. For symmetric and positive-definite matrices X and Y the notation $X > Y$ (respectively $X < Y$) means that $(X - Y)$ is positive-definite (resp. negative-definite). $\mathbb{I}$ denotes the identity matrix with appropriate dimension that may be understood from the context.

2 Problem statement

Let us consider a dynamical system defined on a fundamental probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that its dynamics is described by the following differential equations:

$$\begin{cases} \dot{x}_p(t) = A(r_t)x_p(t) + B(r_t)u(t) + B_w(r_t)\omega(t), \\ x_p(0) = x_{p0}, \end{cases} \tag{1}$$

where $x_p(t) \in \mathbb{R}^n$ is the state vector, $x_{p0} \in \mathbb{R}^n$ is the initial state, $u(t) \in \mathbb{R}^m$ is the control input, $\omega(t) \in \mathbb{R}^l$ is an exogenous input that is supposed to have finite energy, $\{r_t, t \geq 0\}$ is the continuous-time Markov process taking values in a finite space $\mathscr{S} = \{1, 2, \cdots, N\}$, and describes the evolution of the mode at time t, $A(r_t) \in \mathbb{R}^{n\times n}$, $B(r_t) \in \mathbb{R}^{n\times m}$ and $B_w(r_t) \in \mathbb{R}^{n\times l}$ are known matrices with appropriate dimensions.

The switching of the Markov process $\{r_t, t \geq 0\}$ between the different modes is supposed to be described by the following probability transitions:

$$\mathbb{P}\left[r_{t+h} = j | r_t = i\right] = \begin{cases} \lambda_{ij} h + o(h), & \text{when } r_t \text{ jumps from } i \text{ to } j \ , \\ 1 + \lambda_{ii} h + o(h), & \text{otherwise}, \end{cases} \tag{2}$$

where λ_{ij} is the transition rate from mode i to mode j with $\lambda_{ij} \geq 0$ when $i \neq j$ and $\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$ and $o(h)$ is such that $\lim_{h\to 0} \frac{o(h)}{h} = 0$.

The objective of this paper is to design a state feedback controller that forces the state vector of the system, $x_p(t)$, to track the state vector, $x_m(t)$, of a reference model that is governed by the following dynamics:

$$\dot{x}_m(t) = A_m x_m(t) + B_m v(t), x_m(0) = m_{m0}, \tag{3}$$

where $x_m(t) \in \mathbb{R}^n$ is the state vector of the reference model and $v(t) \in \mathbb{R}^m$ is a given finite energy reference input, A_m and B_m are known matrices with appropriate dimensions.

Remark 2.1 *Notice that the dynamics we want to track is independent of the system mode. In the rest of this paper we will assume the complete access to the state vector, and to the mode when it is necessary for feedback.*

The aim of this paper is to design a state feedback control law that permits the state vector of the Markov jump system (1)-(2) to track precisely the solution of the reference model described by (3). To solve this problem, we formulate it as an $\mathscr{H}_\infty$ control problem and synthesize a state feedback controller with mode dependent gains that stabilizes an augmented dynamics obtained from the system and the reference model. The mode-independent state feedback controller is also tackled.

Before closing this section, let us recall a lemma that we will be using in the rest of the paper.

Lemma 2.1 *(Schur complement (Boukas, 2005)) The linear matrix inequality*

$$\begin{bmatrix} H & S^\top \\ S & R \end{bmatrix} > 0,$$

is equivalent to

$$R > 0, H - S^\top R^{-1} S > 0,$$

where $H = H^\top$, $R = R^\top$ and S is a matrix with appropriate dimension.

3 Main results

The problem we are dealing with in this paper can be seen as a tracking problem that pushes all the components of the state vector, $x_p(t)$, to follow precisely their counterparts in the state vector $x_m(t)$. For this purpose, let us define the error between the state vector $x_p(t)$ and the state vector $x_m(t)$ at time t as follows:

$$e(t) = x_p(t) - x_m(t) \tag{4}$$

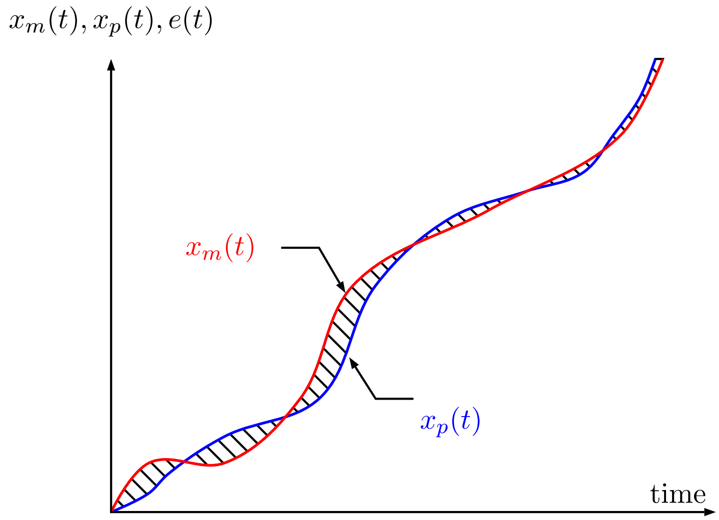

Figure 1: Behaviors of $x(t)$ and $x_m(t)$ versus time

Figure 1 shows in $\mathbb{R}$ the behaviors of the state variable $x_p(t)$ and the variable $x_m(t)$ of the reference model and their error in function of time t. Since we search to minimize this error and track precisely the state vector of the reference model, an integral action of this error is used, i.e $\dot{x}_e(t) = e(t) = x_p(t) - x_m(t)$. The augmented dynamics is then given by the following differential equations:

$$P : \begin{cases} \dot{\eta}(t) = \tilde{A}(r_t)\eta(t) + \tilde{B}(r_t)u(t) + \tilde{B}_w(r_t)w(t), \eta(0) = \eta_0 \\ e(t) = \tilde{C}(r_t)\eta(t) \end{cases} \tag{5}$$

where:

$$\eta(t) = \begin{bmatrix} x_p(t) \\ x_e(t) \\ x_m(t) \end{bmatrix}, w(t) = \begin{bmatrix} \omega(t) \\ v(t) \end{bmatrix}, \tilde{A}(r_t) = \begin{bmatrix} A(r_t) & 0 & 0 \\ \mathbb{I} & 0 & -\mathbb{I} \\ 0 & 0 & A_m \end{bmatrix}$$

$$\tilde{B}(r_t) = \begin{bmatrix} B(r_t) \\ 0 \\ 0 \end{bmatrix}, \tilde{B}_w(r_t) = \begin{bmatrix} B_\omega(r_t) & 0 \\ 0 & 0 \\ 0 & B_m \end{bmatrix}, \tilde{C}(r_t) = \begin{bmatrix} \mathbb{I} & 0 & -\mathbb{I} \end{bmatrix}$$

This augmented dynamics has an external disturbance, $w(t)$, that has finite energy since $\omega(t)$ and $v(t)$ have finite energy. We can then, design a stabilizing controller based on $\mathscr{H}_\infty$

theory. This controller will forces the error to go to zero in the steady state regime, which means that the state vector, $x(t)$, follows precisely the state vector, $x_m(t)$, of the reference model.

Remark 3.1 *The state vector $\eta(t)$ of the augmented dynamics belongs to $\mathbb{R}^{3n}$ which makes it bigger than the original problem. But since the results we will develop are in the LMI framework, this will not affect the resolution.*

The tracking problem is then brought to an $\mathscr{H}_\infty$ control problem of the augmented dynamics. The problem consists in some sense of determining a stabilizing controller for the augmented dynamics and at the same time guarantees the disturbance rejection with a desired level $\gamma > 0$. The controller we will design is given by the following expression:

$$u(t) = K(r_t)\eta(t), \tag{6}$$

where $K(i)$ is the controller gain that has to be determine for all $i \in \mathscr{S}$.

The structure of the closed-loop with such controller is illustrated in Figure 2 where P is the dynamics of the augmented system.

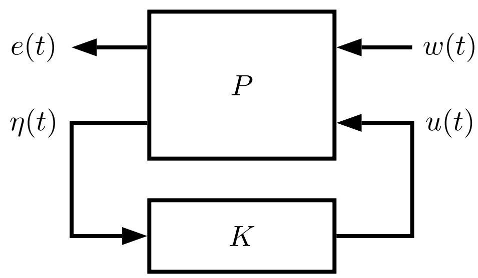

Figure 2: Block diagram

Before giving our main results, let us recall some definitions that we will use in this paper.

Definition 3.1 *(Boukas, 2005) System (5) with $u(t) \equiv 0$ is said to be internally quadratically stochastically stable if there exists a set of symmetric and positive-definite matrices $\tilde{P} = (\tilde{P}(1), \cdots, \tilde{P}(N)) > 0$, satisfying the following holds for each $i \in \mathscr{S}$:*

$$\tilde{A}^\top(i)\tilde{P}(i) + \tilde{P}(i)\tilde{A}(i) + \sum_{j=1}^{N} \lambda_{ij}\tilde{P}(j) < 0. \tag{7}$$

Definition 3.2 *(Boukas, 2005) Let γ be a given positive constant. System (5) with $u(t) \equiv 0$ is said to be stochastically stable with $\gamma-$disturbance attenuation if there exists a constant $M(\eta_0)$ with $M(0) = 0$, such that the following holds:*

$$\|e\|_2 \triangleq \mathbb{E}\left[\int_0^\infty e^\top(t)e(t)dt\right]^{1/2} \leq \gamma\left[\|w\|_2^2 + M(\eta_0)\right]^{\frac{1}{2}}. \tag{8}$$

Remark 3.2 *In the rest of the paper we will use the word stable either for quadratic stochastic stability and internal quadratic stochastic stability.*

The following theorem gives the condition we should satisfy to guarantee the stability of the free augmented dynamics ($u(t) = 0, \forall t \geq 0$) and at the same time reject the disturbance with a certain level $\gamma > 0$.

Theorem 3.1 *Let γ be a given positive constant. If there exists a set of symmetric and positive-definite matrices $\tilde{P} = (\tilde{P}(1), \cdots, \tilde{P}(N)) > 0$, with $P(i) \in \mathbb{R}^{3n \times 3n}$, such that the following LMI holds for each $i \in \mathscr{S}$:*

$$\begin{bmatrix} \tilde{A}^\top(i)\tilde{P}(i) + \tilde{P}(i)\tilde{A}(i) + \sum_{j=1}^{N} \lambda_{ij}\tilde{P}(j) + \tilde{C}^\top(i)\tilde{C}(i) & \tilde{P}(i)\tilde{B}_w(i) \\ \tilde{B}_w^\top(i)\tilde{P}(i) & -\gamma^2\mathbb{I} \end{bmatrix} < 0, \tag{9}$$

then system (5) with $u(t) \equiv 0$ is quadratically stable and satisfies the following:

$$\|e\|_2 \leq \left[\gamma^2\|w\|_2^2 + \eta_0^\top\tilde{P}(r_0)\eta_0\right]^{\frac{1}{2}}, \tag{10}$$

which means that the system with $u(t) = 0$ for all $t \geq 0$ is stable with γ-disturbance attenuation.

Proof: From (9) and using Schur complement, we get the following inequality

$$\tilde{A}^\top(i)\tilde{P}(i) + \tilde{P}(i)\tilde{A}(i) + \sum_{j=1}^{N} \lambda_{ij}\tilde{P}(j) + \tilde{C}^\top(i)\tilde{C}(i) < 0.$$

which implies the following since $\tilde{C}^\top(i)\tilde{C}(i) \geq 0$, we have:

$$\tilde{A}^\top(i)\tilde{P}(i) + \tilde{P}(i)\tilde{A}(i) + \sum_{j=1}^{N} \lambda_{ij}\tilde{P}(j) < 0.$$

Based on Definition 3.1, this proves that the system under study is internally quadratically stable.

Let us now prove that (10) is satisfied. To this end, let us define the following performance function:

$$J_T = \mathbb{E}\left[\int_0^T [e^\top(t)e(t) - \gamma^2 w^\top(t)w(t)]dt\right].$$

To prove (10), it suffices to establish that J_∞ is bounded, i.e:

$$J_\infty \leq V(\eta_0, r_0) = \eta_0^\top\tilde{P}(r_0)\eta_0.$$

If at time t, $\eta(t) = \eta$ and $r(t) = i$, for $i \in \mathscr{S}$, and $V(\eta(t), i) = \eta^\top(t)\tilde{P}(i)\eta(t)$, the infinitesimal operator acting on $V(.)$ and emanating from the point (η, i) at time t is given by (Boukas, 2005):

$$\begin{aligned}
\mathscr{L}V(\eta(t), i) &= \lim_{h\to 0} \frac{1}{h} \left\{\mathbb{E}\left[V(\eta(t+h), r(t+h))|\eta(t) = \eta, r(t) = i\right] - V(\eta(t), i)\right\} \\
&= \eta^\top(t) \left[\tilde{A}^\top(i)\tilde{P}(i) + \tilde{P}(i)\tilde{A}(i) + \sum_{j=1}^{N} \lambda_{ij}\tilde{P}(j)\right] \eta(t) \\
&\quad +\eta^\top(t)\tilde{P}(i)\tilde{B}_w(i)w(t) + w^\top(t)\tilde{B}_w^\top(i)\tilde{P}(i)\eta(t),
\end{aligned}$$

and

$$\begin{aligned}
e^\top(t)e(t) - \gamma^2 w(t)w(t) &= \left[\tilde{C}(i)\eta(t)\right]^\top \left[\tilde{C}(i)\eta(t)\right] - \gamma^2 w(t)w(t) \\
&= \eta^\top(t)\tilde{C}^\top(i)\tilde{C}(i)\eta(t) - \gamma^2 w^\top(t)w(t)
\end{aligned}$$

which implies the following equality:

$$e^\top(t)e(t) - \gamma^2 w^\top(t)w(t) + \mathscr{L}V(\eta(t), r_t) = \zeta^\top(t)\Theta(r_t)\zeta(t),$$

with

$$\begin{aligned}
\Theta(i) &= \begin{bmatrix} \tilde{A}^\top(i)\tilde{P}(i) + \tilde{P}(i)\tilde{A}(i) + \sum_{j=1}^{N} \lambda_{ij}\tilde{P}(j) + \tilde{C}^\top(i)\tilde{C}(i) & \tilde{P}(i)\tilde{B}_w(i) \\ \tilde{B}_w^\top(i)\tilde{P}(i) & -\gamma^2\mathbb{I} \end{bmatrix} \\
\zeta^\top(t) &= \begin{bmatrix} \eta^\top(t) & w^\top(t) \end{bmatrix}.
\end{aligned}$$

Therefore,

$$J_T = \mathbb{E}\left[\int_0^T [e^\top(t)e(t) - \gamma^2 w^\top(t)w(t) + \mathscr{L}V(\eta(t), r_t)]dt\right] - \mathbb{E}\left[\int_0^T \mathscr{L}V(\eta(t), r_t)]dt\right]$$

Using now Dynkin's formula, i.e:

$$\mathbb{E}\left[\int_0^T \mathscr{L}V(\eta(t), r_t)dt|\eta_0, r_0\right] = \mathbb{E}[V(\eta(T), r_T)|\eta_0, r_0] - V(\eta_0, r_0).$$

we get

$$J_T = \mathbb{E}\left[\int_0^T \zeta^\top(t)\Theta(r_t)\zeta(t)dt\right] - \mathbb{E}[V(\eta(T), r_T)|\eta_0, r_0] + V(\eta_0, r_0).$$

Since $\Theta(i) < 0$ for each $i \in \mathscr{S}$, and $\mathbb{E}[V(\eta(T), r_T)|\eta_0, r_0] \geq 0$, the last relation implies the following:

$$J_T \leq V(\eta_0, r_0),$$

which yields $J_\infty \leq V(\eta_0, r_0)$, i.e., $\|e\|_2^2 - \gamma^2\|w\|_2^2 \leq \eta_0^\top \tilde{P}(r_0)\eta_0$.

This gives the desired results:

$$\|e\|_2 \leq \left[\gamma^2\|w\|_2^2 + \eta_0^\top \tilde{P}(r_0)\eta_0\right]^{\frac{1}{2}}$$

which ends the proof of the theorem. □

Let us develop the LMI condition that allows us to synthesize the state feedback controller that stabilizes the augmented system and at the same time rejects the disturbance rejection.

Using the results of Theorem 3.1 and Schur complement, the closed-loop dynamics will be stable and guarantees the disturbance rejection of level $\gamma > 0$ if there exists a symmetric and positive-definite matrix, $\tilde{P}(i) > 0$ such that the following holds:

$$\tilde{A}_{cl}^\top(i)\tilde{P}(i) + \tilde{P}(i)\tilde{A}_{cl}(i) + \sum_{j=1}^{N}\lambda_{ij}\tilde{P}(j) + \tilde{C}^\top(i)\tilde{C}(i) + \gamma^{-2}\tilde{P}(i)\tilde{B}_w(i)\tilde{B}_w^\top(i)\tilde{P}(i) < 0$$

where $\tilde{A}_{cl}(i) = \tilde{A}(i) + \tilde{B}(i)K(i)$.

This inequality matrix is nonlinear in the design parameters $\tilde{P}(i)$ and $K(i)$. To put it into LMI form, let $\tilde{X}(i) = \tilde{P}^{-1}(i)$. Pre- and post-multiply this inequality by $\tilde{X}(i)$ gives:

$$\begin{aligned}\tilde{X}(i)\tilde{A}^\top(i) + \tilde{A}(i)\tilde{X}(i) + \tilde{X}(i)\tilde{K}^\top(i)\tilde{B}^\top(i) + \tilde{B}(i)\tilde{K}(i)\tilde{X}(i)\\ + \sum_{j=1}^{N}\lambda_{ij}\tilde{X}(i)\tilde{X}^{-1}(j)\tilde{X}(i) + \tilde{X}(i)\tilde{C}^\top(i)\tilde{C}(i)\tilde{X}(i) + \gamma^{-2}\tilde{B}_w(i)\tilde{B}_w^\top(i) < 0\end{aligned}$$

Let $\mathcal{S}_i(\tilde{X})$ and $\mathcal{X}_i(\tilde{X})$ be defined by:

$$\begin{aligned}\mathcal{S}_i(\tilde{X}) &= \left[\sqrt{\lambda_{i1}}\tilde{X}(i), \cdots, \sqrt{\lambda_{ii-1}}\tilde{X}(i), \sqrt{\lambda_{ii+1}}\tilde{X}(i), \cdots, \sqrt{\lambda_{iN}}\tilde{X}(i)\right],\\ \mathcal{X}_i(\tilde{X}) &= \operatorname{diag}\left[\tilde{X}(1), \ldots, \tilde{X}(i-1), \tilde{X}(i+1), \cdots, \tilde{X}(N)\right].\end{aligned}$$

Using these definitions we can show that the following holds:

$$\sum_{j=1}^{N}\lambda_{ij}\tilde{X}(i)\tilde{X}^{-1}(j)\tilde{X}(i) = \lambda_{ii}\tilde{X}(i) + \mathcal{S}_i(\tilde{X})\mathcal{X}_i^{-1}(\tilde{X})\mathcal{S}_i^\top(\tilde{X})$$

Letting now $Y(i) = K(i)\tilde{X}(i)$ and using Schur complement lemma, we get:

$$\begin{bmatrix} J(i) & \tilde{X}(i)\tilde{C}^\top(i) & \tilde{B}_w(i) & \mathcal{S}_i(\tilde{X}) \\ \tilde{C}(i)\tilde{X}(i) & -\mathbb{I} & 0 & 0 \\ \tilde{B}_w^\top(i) & 0 & -\gamma^2\mathbb{I} & 0 \\ \mathcal{S}_i^\top(\tilde{X}) & 0 & 0 & -\mathcal{X}_i(\tilde{X}) \end{bmatrix} < 0 \tag{11}$$

with $J(i) = \tilde{X}(i)\tilde{A}^\top(i) + \tilde{A}(i)\tilde{X}(i) + Y^\top(i)\tilde{B}^\top(i) + \tilde{B}(i)\tilde{Y}(i) + \lambda_{ii}\tilde{X}(i)$.

The following theorem gives a design method for the unconstrained state feedback controller that guarantees the tracking problem of the class of systems we are studying.

Theorem 3.2 *Let γ be a positive constant. If there exist a set of symmetric and positive-definite matrices $\tilde{X} = (\tilde{X}(1), \cdots, \tilde{X}(N)) > 0$ and a matrix Y such that the set of LMIs (11) holds for every $i \in \mathscr{S}$, then system (5) under the controller (6) with $K(i) = Y(i)\tilde{X}^{-1}(i)$ is stable and moreover the closed-loop system satisfies the disturbance rejection of level $\gamma > 0$.*

¿From the practical point of view, the controller that quadratically stabilizes the system and at the same time guarantees the minimum disturbance rejection is of great interest. This controller can be obtained by solving the following optimization problem:

$$\mathrm{P}: \begin{cases} \min_{\substack{\nu>0,\\ \tilde{X}=(\tilde{X}(1),\cdots,\tilde{X}(N))>0,\\ Y=(Y(1),\cdots,Y(N))}} \nu \\ s.t: \\ \begin{bmatrix} J(i) & \tilde{X}(i)\tilde{C}^\top(i) & \tilde{B}_w(i) & \mathcal{S}_i(\tilde{X}) \\ \tilde{C}(i)\tilde{X}(i) & -\mathbb{I} & 0 & 0 \\ \tilde{B}_w^\top(i) & 0 & -\nu\mathbb{I} & 0 \\ \mathcal{S}_i^\top(\tilde{X}) & 0 & 0 & -\mathcal{X}_i(\tilde{X}) \end{bmatrix} < 0 \end{cases}$$

where the LMI in the constraints is obtained from (11) by replacing γ^2 by ν.

The following corollary gives the results on the design of the controller that quadratically stabilizes the system (1) and simultaneously guarantees the smallest disturbance rejection level.

Corollary 3.1 *Let $\nu > 0$, $\tilde{X} = (\tilde{X}(1), \cdots, \tilde{X}(N)) > 0$ and $Y = (Y(1), \cdots, Y(N))$ be the solution of the optimization problem P. Then, the controller (6) with $K(i) = Y(i)\tilde{X}^{-1}(i)$ quadratically stabilizes the class of production systems we are considering and moreover the closed-loop system satisfies the disturbance rejection of level $\sqrt{\nu}$.*

In all what we developed earlier in this paper we supposed the complete access to the system mode. Practically, this is hard to get and we need to estimate the mode to continue to apply the previous results. In this section, we will relax this assumption and try to synthesize controllers with common gain for all the modes that do not require the knowledge of this mode. The constant gain state feedback controller that we will design has the following form:

$$u(t) = \mathscr{K}\eta(t) \tag{12}$$

Remark 3.3 *In the rest of the paper, we will continue to assume the complete access of the state vector $\eta(t)$ for feedback as we did previously.*

Before designing this controller, let us give the conditions that we should verify when a constant matrix independent on the mode is used in the Lyapunov function expression. The corresponding results is summarized by the following corollary:

Corollary 3.2 *Let γ be a given positive constant. If there exists a symmetric and positive-definite matrix $\tilde{P} > 0$ such that the following LMI holds for each $i \in \mathscr{S}$:*

$$\begin{bmatrix} \tilde{A}^\top(i)\tilde{P} + \tilde{P}\tilde{A}(i) + \tilde{C}^\top(i)\tilde{C}(i) & \tilde{P}\tilde{B}_w(i) \\ \tilde{B}_w^\top(i)\tilde{P} & -\gamma^2\mathbb{I} \end{bmatrix} < 0, \tag{13}$$

then system (5) with $u(t) \equiv 0$ is quadratically stable and satisfies the following:

$$\|e\|_2 \leq \left[\gamma^2\|w\|_2^2 + \eta_0^\top\tilde{P}\eta_0\right]^{\frac{1}{2}}, \tag{14}$$

which means that the system with $u(t) = 0$ for all $t \geq 0$ is stable with γ-disturbance attenuation.

Proof: Let us consider the following candidate Lyapunov function:

$$V(\eta(t), i) = \eta^\top(t)\tilde{P}\eta(t)$$

where $\tilde{P}$ is symmetric and positive-definite matrix.

Following the same steps as for Theorem 3.1 and using the fact that $\sum_{j=1}^N \lambda_{ij} = 0$, we can show the results of this corollary. □

Let us now use the results of this corollary and focus on the design of the state feedback controller with constant gain. For this purpose notice that the closed-loop system under this controller will be quadratically stable if there exists a symmetric and positive-definite matrix $\tilde{P}$ such that following holds for each $i \in \mathscr{S}$:

$$\tilde{A}^\top(i)\tilde{P} + \tilde{P}\tilde{A}(i) + \mathscr{K}\tilde{B}^\top(i)\tilde{P} + \tilde{P}\tilde{B}(i)\mathscr{K} + \tilde{C}^\top(i)\tilde{C}(i) + \gamma^{-2}\tilde{P}\tilde{B}_w(i)\tilde{B}_w^\top(i)\tilde{P} < 0$$

which is nonlinear. To transform it into an LMI, let $\tilde{X} = \tilde{P}^{-1}$. Pre- and post-multiply this inequality by $\tilde{X}$ gives:

$$\tilde{X}\tilde{A}^\top(i) + \tilde{A}(i)\tilde{X} + \tilde{X}\mathscr{K}^\top\tilde{B}^\top(i) + \tilde{B}(i)\mathscr{K}\tilde{X} + \tilde{X}\tilde{C}^\top(i)\tilde{C}(i)\tilde{X} + \gamma^{-2}\tilde{B}_w(i)\tilde{B}_w^\top(i) < 0$$

Letting $K = \mathscr{K}\tilde{X}$ and using Schur complement we get the required LMI design conditions for the design of the constant gain state feedback controller. The following corollary summarizes such results:

Corollary 3.3 *Let γ be a positive constant. If there exist symmetric and positive-definite matrix $\tilde{X} > 0$ and a matrix K such that the following LMI holds for every $i \in \mathscr{S}$:*

$$\begin{bmatrix} \tilde{X}\tilde{A}^\top(i) + \tilde{A}(i)\tilde{X} + K^\top\tilde{B}^\top(i) + \tilde{B}(i)K & \tilde{X}\tilde{C}^\top(i) & \tilde{B}_w(i) \\ \tilde{C}(i)\tilde{X} & -\mathbb{I} & 0 \\ \tilde{B}_w^\top(i) & 0 & -\gamma^2\mathbb{I} \end{bmatrix} < 0 \tag{15}$$

then system (5) under the controller (12) with $\mathscr{K} = K\tilde{X}^{-1}$ is stable and moreover the closed-loop system satisfies the disturbance rejection of level $\gamma > 0$.

The results of this corollary will allow us to determine of the controller gain, $\mathscr{K}$, but there is no guarantee that the control will not exceed its bounds ($|u_k(t)| < \bar{u}_k$, where $\bar{u}_k$ is a given positive constant) that are imposed by physics and are hard constraints that we should always satisfy. We should then add to the previous LMI extra conditions that forces the control law to satisfy the bounds all the time. For this purpose let the ellipsoid set $\mathscr{D}$ be defined by:

$$\mathscr{D} = \{\eta \in \mathbb{R}^{3n} | \eta^\top \tilde{X}^{-1} \eta \leq 1\}$$

where $\tilde{X}$ is a symmetric and positive-definite matrix.

Remark 3.4 *The same remark can be applied to Theorem 3.1. In the rest of the paper we will focus on the mode independent controller. The developed results can be extended easily to the mode dependent controller.*

When the initial condition is known (which is the case in our problem), we can find an upper bound of the control component, $u_k(t) = (\mathscr{K}\eta(t))_k\,, k = 1, 2, \cdots, n$ as follows. Take the matrices $\tilde{X}$ and K obtained by solving the LMI (15) and add to this the following condition:

$$\eta_0^\top \tilde{X}^{-1} \eta_0 \leq 1$$

which gives using Schur complement:

$$\begin{bmatrix} 1 & \eta_0^\top \\ \eta_0 & \tilde{X} \end{bmatrix} \geq 0$$

This guarantees that the control vector will always remain in the ellipsoid $\mathscr{D}$ and hence,

$$\begin{aligned}
\max_{t \geq 0} |u_k(t)|^2 &= \max_{t \geq 0} |\left(\mathscr{K}\eta(t)\right)_k|^2 \\
&= \max_{t \geq 0} |\left(K\tilde{X}^{-1}\eta(t)\right)_k|^2 \\
&\leq \max_{\eta \in \mathscr{D}} |\left(K\tilde{X}^{-1}\eta\right)_k|^2 \\
&\leq \|\left(K\tilde{X}^{-\frac{1}{2}}\right)_k\|_2^2 \|\tilde{X}^{-\frac{1}{2}}\eta\|_2^2 = \left(K\tilde{X}^{-1}K^\top\right)_{kk} \eta^\top \tilde{X}^{-1}\eta \\
&= \left(K\tilde{X}^{-1}K^\top\right)_{kk}
\end{aligned}$$

The constraint $|u_k(t)| \leq \bar{u}_k$ will be enforced if there exists a symmetric and positive-definite matrix such that:

$$\begin{bmatrix} Z & K \\ K^\top & \tilde{X} \end{bmatrix} \geq 0$$

with $Z_{kk} \leq \bar{u}_k^2$.

Taking care of these developments, the following corollary will allow the design of the required controller that stabilizes the augmented system and guarantees the disturbances rejection.

Corollary 3.4 *Let γ be a positive constant. If there exist a symmetric and positive-definite matrix $\tilde{X} > 0$ and a matrix K such that the following LMI holds:*

$$\begin{cases} \begin{bmatrix} \tilde{X}\tilde{A}^\top + \tilde{A}\tilde{X} + K^\top \tilde{B}^\top + \tilde{B}\tilde{Y} & \tilde{X}\tilde{C}^\top & \tilde{B}_w \\ \tilde{C}\tilde{X} & -\mathbb{I} & 0 \\ \tilde{B}_w^\top & 0 & -\gamma^2 \mathbb{I} \end{bmatrix} < 0 \\ \begin{bmatrix} Z & K \\ K^\top & \tilde{X} \end{bmatrix} \geq 0, \begin{bmatrix} 1 & \eta_0^\top \\ \eta_0 & \tilde{X} \end{bmatrix} \geq 0 \end{cases} \tag{16}$$

with $Z_{kk} \leq \bar{u}_k^2, k = 1, 2, \cdots, n$, then the system (5) under the controller (12) with $\mathscr{K} = K\tilde{X}^{-1}$ is stable and moreover the closed-loop system satisfies the disturbance rejection of level $\gamma > 0$.

Similarly, the state feedback controller with constant gain that quadratically stabilizes the system and at the same time guarantees the minimum disturbance rejection is obtained by solving the following optimization problem:

$$\text{P1} : \begin{cases} \min_{\substack{\nu > 0, \\ \tilde{X} > 0, \\ K}} \nu \\ s.t: \\ \begin{bmatrix} \tilde{X}\tilde{A}^\top + \tilde{A}\tilde{X} + K^\top \tilde{B}^\top + \tilde{B}\tilde{Y} & \tilde{X}\tilde{C}^\top & \tilde{B}_w \\ \tilde{C}\tilde{X} & -\mathbb{I} & 0 \\ \tilde{B}_w^\top & 0 & -\nu \mathbb{I} \end{bmatrix} < 0 \\ \begin{bmatrix} Z & K \\ K^\top & \tilde{X} \end{bmatrix} \geq 0, \begin{bmatrix} 1 & \eta_0^\top \\ \eta_0 & \tilde{X} \end{bmatrix} \geq 0, Z_{kk} \leq \bar{u}_k^2, k = 1, 2, \cdots, n \end{cases}$$

where the first LMI in the constraints is obtained from (16) by replacing γ^2 by ν.

The following corollary gives the results on the design of the controller that quadratically stabilizes the system (1) and simultaneously guarantees the smallest disturbance rejection level.

Corollary 3.5 *Let $\nu > 0$, $\tilde{X} > 0$ and K be the solution of the optimization problem P1. Then, the controller (12) with $\mathscr{K} = K\tilde{X}^{-1}$ quadratically stabilizes the class of production systems we are considering and moreover the closed-loop system satisfies the disturbance rejection of level $\sqrt{\nu}$.*

4 Numerical example

To illustrate the effectiveness of the developed results, let us consider a two modes system with state variable in $\mathbb{R}^2$. The required matrices are given by:

- mode # 1:

$$A(1) = \begin{bmatrix} 1.0 & 2.0 \\ 2.0 & 1.0 \end{bmatrix}, B(1) = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}, B_w(1) = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$$

- mode # 2:

$$A(2) = \begin{bmatrix} 1.0 & 1.5 \\ 1.5 & -1.0 \end{bmatrix}, B(2) = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}, B_w(2) = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}$$

The switching between the two modes is supposed to be described by:

$$\Lambda = \begin{bmatrix} -1.0 & 1.0 \\ 2.0 & -2.0 \end{bmatrix}$$

The reference model that we would like to track is described by the following matrices:

$$A_m = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -1.3 \end{bmatrix}, B_m = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}.$$

Letting $\gamma = 3.0$ and solving the LMIs (11), we get:

$$\tilde{X}(1) = \begin{bmatrix} 14.7248 & -15.8391 & -1.2633 & -0.5075 & 11.0499 & -16.6493 \\ -15.8391 & 54.8591 & 1.0289 & -0.0796 & -10.7689 & 25.4238 \\ -1.2633 & 1.0289 & 107.4514 & -21.2320 & -0.4697 & 1.2186 \\ -0.5075 & -0.0796 & -21.2320 & 87.2374 & -1.3166 & 0.8595 \\ 11.0499 & -10.7689 & -0.4697 & -1.3166 & 9.3108 & -12.1520 \\ -16.6493 & 25.4238 & 1.2186 & 0.8595 & -12.1520 & 31.2984 \end{bmatrix},$$

$$\tilde{X}(2) = \begin{bmatrix} 27.2638 & -13.9391 & -0.9893 & -0.2976 & 6.9717 & -11.5052 \\ -13.9391 & 32.7401 & 1.2340 & -0.3471 & -14.0069 & 26.7412 \\ -0.9893 & 1.2340 & 107.2373 & -21.1536 & -0.2826 & 1.1296 \\ -0.2976 & -0.3471 & -21.1536 & 87.2535 & -0.9138 & 0.3165 \\ 6.9717 & -14.0069 & -0.2826 & -0.9138 & 7.4193 & -11.6742 \\ -11.5052 & 26.7412 & 1.1296 & 0.3165 & -11.6742 & 27.6392 \end{bmatrix},$$

$$Y(1) = \begin{bmatrix} -148.7794 & -627.6578 & 11.6550 & 6.2656 & -110.6331 & 216.0183 \end{bmatrix},$$

$$Y(2) = \begin{bmatrix} -530.9206 & -210.9653 & -9.2069 & -3.5175 & 99.8401 & -171.5375 \end{bmatrix}.$$

which gives the following gains:

$$K(1) = \begin{bmatrix} -33.2505 & -25.1395 & -0.1567 & -0.0247 & 22.4528 & 18.3594 \end{bmatrix},$$
$$K(2) = \begin{bmatrix} -30.4456 & -7.7640 & -0.1949 & 0.0731 & 28.4246 & 0..6450 \end{bmatrix}.$$

A simulation program has been written to simulate the behavior of the two state $x_1(t)$ and $x_2(t)$ with respect to time. The reference model is chosen to be stable with initial conditions not equal to zero as it is the case for the system states. The simulation results show that the state vector of the system after the transient regime tracks precisely the one of the model reference and the error is zero as expected. Based on the results of Theorem 3.2 we can see that the system is stochastically stable under the design controller and reject the disturbance rejection of level $\gamma = 3$.

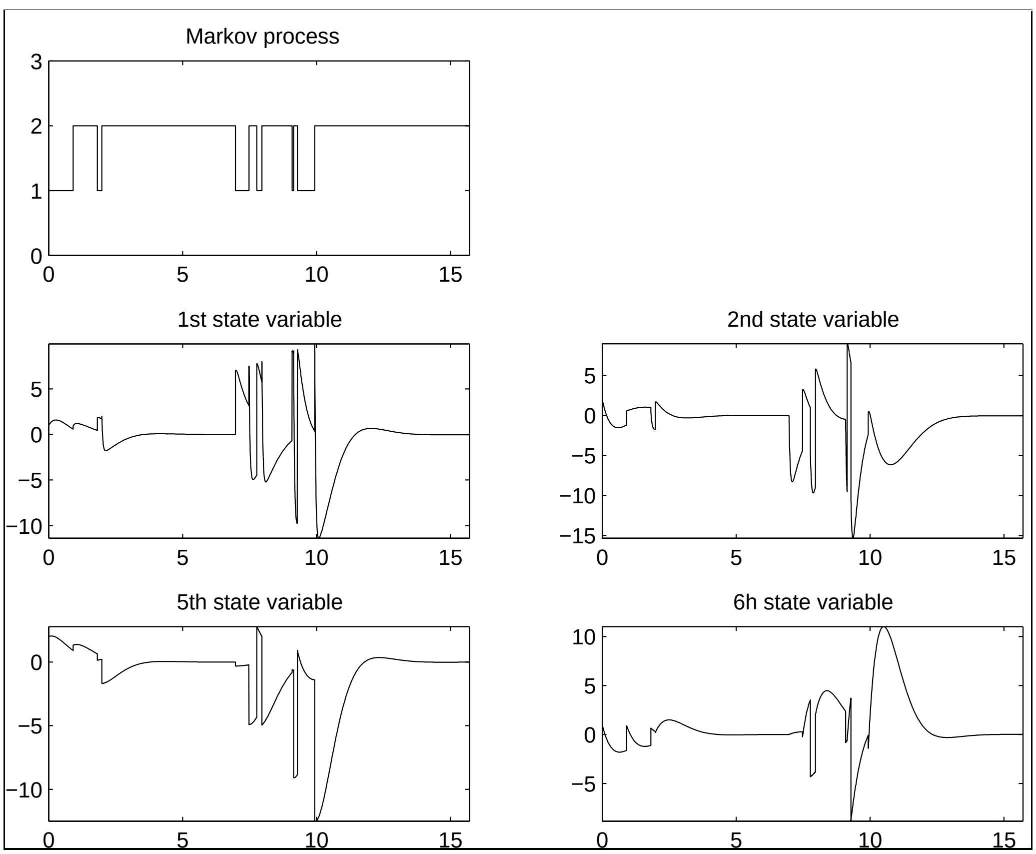

Figure 3: State $r(t)$, $x_p(t)$ and $x_m(t)$ versus time t

5 Conclusion

This paper dealt with the tracking problem for the class of Markov jump systems. The dynamics we would like to track is supposed to be described by a system of linear differential equations that does not depend on the system mode. This problem has been solved using the $\mathscr{H}_\infty$ theory. A design method based on LMI approach is proposed to synthesize the state feedback controller that forces the state vector of the system to track precisely the one of the reference model.

References

Krasovskii, N.N., and Lidskii, E.A. (1961), "Analysis design of controller in systems with random attributes, Part 1 and Part 2," *Automatic Remote Control*, 22, 1021–1025 and 1141–1146.

Boukas, E.K. (2006), "Manufacturing Systems: LMI Approach," *IEEE Transactions On Automatic Control*, 51(6), 1014–1018.

Boukas, E.K. (2005), *Stochastic Hybrid Systems: Analysis and Design*, Boston: Birkhauser.

Boukas, E.K., and Liu, Z.K. (2001), "Production and Maintenance Control for Manufacturing Systems," *IEEE Transactions on Automatic Control*, 46(9), 1455–1460.

de Souza, C.E., and Fragoso, M.D. (1993), "$\mathscr{H}_\infty$ Control for Linear Systems with Markovian Jumping Parameters," *Control-Theory and Advanced Technology*, 9(2), 457–466.

de Farias, D.P., Geromel, J.C., do Val, J.B.R., and Costa, O.L.V. (2000), "Output Feedback Control of Markov Jump Linear Systems in Continuous-Time," *IEEE Transactions on Automatic Control*, 45(5), 944–949.

Mariton, M. (1990), *Jump linear Systems in Automatic Control*, New-York: Marcel Dekker.

Shi, P., and Boukas, E.K. (1997), "$\mathscr{H}_\infty$ Control for Markovian Jumping Linear Systems with Parametric Uncertainty," *Journal of Optimization Theory and Apllications*, 95, 75–99.

Wang, Z., Qiao, H., and Burnham, K.J. (2002), "On Stabilization of Bilinear Uncertaing Time-Delay Systems with Markovian Jumping Parameters," *IEEE Transactions on Automatic Control*, 47(4), 640–646.

Sethi, S.P., and Zhang, Q. (1994), *Hierarchical Decision Making in Stochastic Manufacturing Systems*, Boston: Birkhauser.

Zhang, Q. (2000), "Hybrid Filtering for Linear Systems with Non-Gaussian Disturbances," *IEEE Transactions on Automatic Control*, 45(1), 50–61.