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H_∞ Control for Discrete-time Markov Jump Systems with Uncertain Transition Probabilities

Xiaoli Luan, Shunyi Zhao, and Fei Liu

***Abstract*—In this paper, the H_∞ control problem for a class of discrete-time Markov jump systems (MJSs) with uncertain transition probabilities (TPs) is investigated. The uncertain information of transition probabilities is quantized by Gaussian transition probability density function (PDF). In light of the proposed descriptions, the MJSs with Gaussian PDF of TPs cover the systems with precisely known and partially known TPs as two special cases. Sufficient conditions for the existence of H_∞ controller of the underlying systems are derived in term of linear matrix inequalities. A numerical example is presented to illustrate the effectiveness and potential of the developed theoretical results.**

***Index Terms*—Markov jump systems, uncertain transition probabilities, H_∞ control.**

I. Introduction

Markov jump systems (MJSs) can be considered as a finite collection of dynamic systems with transition governed by a stochastic process. As an essential system parameter, the transition probabilities (TPs) depict the random uncertainties of the transition between possible system behavior patterns. Under the assumption that TPs are precisely known a priori, the analysis and synthesis problems have been extensively investigated, see for examples [1], [2], [3], [4], [5], [6]and the references therein.

In practical applications, TPs are generally determined by physical experiments or numerical simulation, and only the estimated values of TPs are available. Recently, some studies on MJSs with uncertain TPs have been investigated in [7], [8], [9], where two types of uncertainties are considered. One is polytopic description where the TP matrix is assumed to be in a convex hull with known vertices. The other is described in an element-wise way. In this way, the elements of TP matrix are measured in practice and the error bounds are given at the same time. Then, the robust methodologies are utilized to deal with the normbounded or polytopic uncertainties presumed in the TPs.

On the other hand, considering that some elements in TP matrix are costly to obtain, Zhang studied the MJSs with partially known TPs in [10], [11]. In contrast with the uncertain TPs discussed in [7], [9], the concept of partially known TPs does not require any knowledge of the unknown elements. More recently, Goncalves and Fioravanti investigated a kind of polytopic uncertain model [12], which is more general than the partly known element one, in the sense that the latter is a special case of the former. However, it is important to point out that the measurement of TPs is a random process varying with the measuring time. So it is more reasonable to characterize the uncertain time-varying information of TPs from stochastic viewpoint. As indicated in [13], for continuous type of random variable, Gaussian probability density functions (PDFs) are often used to parameterize TPs. However, the issues on such systems with Gaussian PDFs have not been investigated up until now, although there is some attention on the Markov Process containing Gaussian TPs only in purely mathematics field [14], [15], [16].

Therefore, the main purpose of this paper is to complement the existing results by describing the uncertainties of TPs with a Gaussian stochastic process. The paper considers the H_∞ control problem for MJSs with random uncertain TPs. Gaussian PDF is utilized to characterize the relative likelihood for uncertain TPs to occur at a given constant. In this way, the uncertain information of TPs can be quantized by the variance of Gaussian PDF. Compared with the uncertainty model proposed in aforementioned literatures, this paper deals with the uncertain TPs from a random of view. By modeling the TPs with Guassian PDF, not only systems with precisely known or partially known TPs can be viewed as two special cases of the ones tackled here, but also the uncertain range of TPs can be quantized by the variance of Gaussian PDF.

Briefly, the paper is organized as follows. Section 2 introduces the problem definitions and calculates the desirable TP matrix from the continuous PDF. The mode-dependent and H_∞ controller design method is given in Section 3. The objective of Section 4 is to demonstrate the effectiveness of the proposed algorithm with a simulation example. Finally, Section 5 concludes this paper.

In the sequel, the following notion will be used: R^n and $R^{n\times m}$ denote n-dimensional Euclidean space, and the set of all the $n\times m$ real matrices, A^{T} (or x^{T}) and A^{-1} denote the transpose and the inverse of matrix or vector , $\|A\|$ denotes the Euclidean norm of matrix A, $E\{\cdot\}$ denotes the mathematics statistical expectation of the stochastic process or vector, $l_2[\ 0,\ \ \infty\ )$ is the space of summable infinite sequence over $[\ 0,\ \ \infty\ )$ and for $w=\{w(k)\}\in l_2[\ 0,\ \ \infty\ )$, it's norm is given by $\|w\|_2=\sqrt{\sum_{k=0}^{\infty}|w|^2}$. For sequence $z=\{z(k)\}\in l_2[\ 0,\ \ \infty\ )$, it's norm is given by $\|z\|_{E_2}=\sqrt{E[\sum_{k=0}^{\infty}|z|^2]}$. $P>0$ stands for a positive-definite matrix, I is the unit matrix with appropriate dimensions, " $*$" means the symmetric terms in a symmetric matrix.

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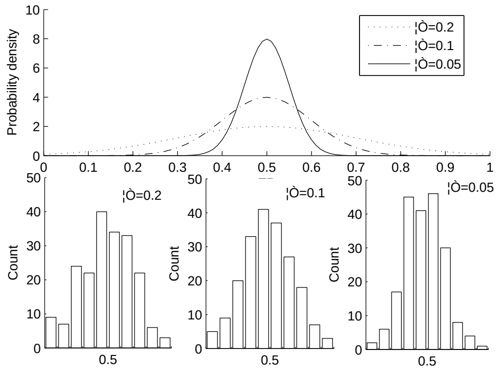

Fig. 1: Density functions for $(0.5, 0.2)$,$(0.5, 0.1)$ and $(0.5, 0.05)$

II. Problem Statement

Consider the following discrete-time Markov jump system:

$$\begin{array}{l} x(k+1) = A(r_k)x(k) + B(r_k)u(k) + B_w(r_k)w(k) \\ z(k) = C(r_k)x(k) + D(r_k)u(k) + D_w(r_k)w(k) \end{array} \tag{1}$$

where $x(k) \in R^n$ is the vector of state variables, $z(k) \in R^l$ is the controlled output, $u(k) \in R^m$ is the controlled input, $w(k) \in l_2^p[\ 0 \quad +\infty \ )$ is the external disturbances, $A(r_k)$, $B(r_k)$, $B_w(r_k)$, $C(r_k)$, $D(r_k)$ and $D_w(r_k)$ are known mode-dependent constant matrices with appropriate dimensions. For notational simplicity, in the following sections, we denote the system matrices as A_i, B_i, B_{wi}, C_i, D_i and D_{wi} . The random process $\{r_k, k \geq 0\}$ is a time-varying Markov chain taking values on a finite set $M = \{1, 2, \cdots, s\}$ with mode transition probabilities

$$\pi_{ij}^{(\xi_k)} = P_r(r_k = j | r_{k-1} = i, \xi_k) \tag{2}$$

where $\pi_{ij}^{(\xi_k)}$ denotes the transition probability from mode i at time $k-1$ to mode j at time k.

In this paper, the stochastic variation of TPs is assumed to be continuous, then we use a set of random variables indexed by Gaussian stochastic process $\{\xi_k, k \in K\}$ to depict the continuous random characteristics of TPs. The truncated Gaussian PDF of continuous random variables $\pi_{ij}^{(\xi_k)}$ can be expressed as follows:

$$p(\pi_{ij}^{(\xi_k)}) = \frac{\frac{1}{\sqrt{\sigma_{ij}}} f(\frac{\pi_{ij}^{(\xi_k)} - \mu_{ij}}{\sqrt{\sigma_{ij}}})}{F(\frac{1-\mu_{ij}}{\sqrt{\sigma_{ij}}}) - F(\frac{0-\mu_{ij}}{\sqrt{\sigma_{ij}}})} \tag{3}$$

where $f(\cdot)$ is the PDF of the standard normal distribution, $F(\cdot)$ is the cumulative distribution function of $f(\cdot)$, μ_{ij} and σ_{ij} are the known mean and variance of Gaussian PDF respectively. The matrix of transition probability density can be further defined as:

$$N = \begin{bmatrix} n(\mu_{11}, \sigma_{11}) & n(\mu_{12}, \sigma_{12}) & \cdots & n(\mu_{1s}, \sigma_{1s}) \\ n(\mu_{21}, \sigma_{21}) & n(\mu_{22}, \sigma_{22}) & \cdots & n(\mu_{2s}, \sigma_{2s}) \\ \vdots & \vdots & \ddots & \vdots \\ n(\mu_{s1}, \sigma_{s1}) & n(\mu_{s2}, \sigma_{s2}) & \cdots & n(\mu_{ss}, \sigma_{ss}) \end{bmatrix} \tag{4}$$

where $n(\mu_{ij}, \sigma_{ij}) = p(\pi_{ij}^{(\xi_k)})$ denotes the truncated Gaussian transition PDF of $\pi_{ij}^{(\xi_k)}$, which is assumed to be known in advance.

In order to explicitly illustrate how the truncated Gaussian PDFs describe relative likelihood for random time-varying TPs to occur at a given constant, the graphs of density functions for several different (μ, σ) pairs are presented in Fig. 1.

From Fig.1, it is easily seen that large values of σ yield graphs that are quite spread out about μ, whereas small values of σ yield graphs with a high peak about μ and most of the areas under the graph quite close to μ. Thus the value of variance parameter σ can be utilized to quantize the time-varying uncertain information of TPs. If the relative likelihood for random time-varying TPs to occur at a given constant is high, the value of σ can be selected to be small. Otherwise, we can select a larger σ to represent the bigger uncertainties of TPs.

Remark 2.1: For $i, j \in M$, if σ tends to zero, which means there is no uncertainties on the mean μ, the traditional MJSs with constant TPs in [17], [18] are special cases of our systems. And conversely, if σ turns to infinity, which means the uncertainties of TPs is so large that it is almost not accessible to TPs, our systems include the MJSs with partially known TPs in [19], [20], [21], [22] as special cases. Therefore, the MJSs with Gaussian PDF of TPs are more general, and can cover most of jump systems presented in existing literatures.

Next, we will focus our attention on the analysis of transition PDF to obtain the expected TP matrix. As stated in the previous content, the random variables $\pi_{ij}^{(\xi_k)}$ included in TP matrix are continuous. Then, the expectation of continuous random variable $\pi_{ij}^{(\xi_k)}$ is as follows:

$$\begin{aligned} \hat{\pi}_{ij}^{(\xi_k)} &= E(\pi_{ij}^{(\xi_k)}) = \int_0^1 \pi_{ij}^{(\xi_k)} p(\pi_{ij}^{(\xi_k)}) d\pi_{ij}^{(\xi_k)} \\ &= \mu_{ij} + \frac{f(\frac{0-\mu_{ij}}{\sqrt{\sigma_{ij}}}) - f(\frac{1-\mu_{ij}}{\sqrt{\sigma_{ij}}})}{F(\frac{0-\mu_{ij}}{\sqrt{\sigma_{ij}}}) - F(\frac{1-\mu_{ij}}{\sqrt{\sigma_{ij}}})} \sqrt{\sigma_{ij}} \end{aligned} \tag{5}$$

As a result, the desired TP matrix can be obtained

$$\Pi' = \begin{bmatrix} E(\pi_{11}^{(\xi_k)}) & E(\pi_{12}^{(\xi_k)}) & \cdots & E(\pi_{1s}^{(\xi_k)}) \\ E(\pi_{21}^{(\xi_k)}) & E(\pi_{22}^{(\xi_k)}) & \cdots & E(\pi_{2s}^{(\xi_k)}) \\ \vdots & \vdots & \ddots & \vdots \\ E(\pi_{s1}^{(\xi_k)}) & E(\pi_{s2}^{(\xi_k)}) & \cdots & E(\pi_{ss}^{(\xi_k)}) \end{bmatrix} \tag{6}$$

with

$$\sum_j^s E(\pi_{ij}^{(\xi_k)}) = 1, E(\pi_{ij}^{(\xi_k)}) \geq 0, 1 \leq i, j \leq s$$

III. H_∞ Controller Design

The previous section has obtained the desired TP matrix, this section will be devoted to studying the control problem of system (1). To this end, we construct the following state feedback controller for the discrete-time MJS (1):

$$u(k) = -K(r_k, \xi_k)x(k) \tag{7}$$

where $K_{i,\xi_k} \triangleq K(r_k, \xi_k)$ are constant gains to be designed for each mode $i \in M$. Substituting (7) into (1) yields the dynamics of the closed-loop system as follows:

$$\begin{array}{l} x_{k+1} = \bar{A}_{i,\xi_k} x_k + B_{wi} w_k \\ z_k = \bar{C}_{i,\xi_k} x_k + D_{wi} w_k \end{array} \tag{8}$$

where

$$\bar{A}_{i,\xi_k} = A_i - B_i K_{i,\xi_k}, \quad \bar{C}_{i,\xi_k} = C_i - D_i K_{i,\xi_k}$$

Before establishing our main results, let us recall the following definitions.

Definition 1. System (8) with $w_k \equiv 0$ is said to be stochastically stable if

$$E\left\{\sum_{k=0}^{\infty} \|x_k\|^2\right\} < \infty$$

for any initial condition $x_0 \in R^n$ and $r_0 \in M$.

Definition 2. Given a positive scalar γ , system (8) is said to be stochastically stabilizable with γ disturbance attenuation if under zero initial condition, $\|z\|_2 < \gamma \|w\|_2$ holds for all nonzero $w_k \in l_2[0, \infty)$.

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Therefore the purpose of the paper is to design a mode-dependent H_∞ state feedback controller such that the resulting closed-loop system (8) is stochastically stabilizable and has a prescribed H_∞ performance index.

Now we are in a position to present sufficient conditions for the existence of the mode-dependent H_∞ controller.

Theorem 3.1: For jump system (1), there exists a state feedback controller (7) such that the closed-loop system (8) possesses the $\gamma-$disturbance attenuation property, if for each $i \in M$, there exist mode-dependent positive-definite matrices X_{i,ξ_k} and mode-dependent matrices Y_{i,ξ_k} satisfying the following coupled linear matrix inequalities:

$$\begin{bmatrix} -X_{i,\xi_k} & 0 & (C_iX_{i,\xi_k}-D_iY_{i,\xi_k})^{\mathrm{T}} & U_{1i}^{\mathrm{T}} \\ & -\gamma^2 I & D_{wi}^{\mathrm{T}} & U_{2i}^{\mathrm{T}} \\ * & & -I & 0 \\ * & & * & -Z \end{bmatrix} < 0 \tag{9}$$

where

$$U_{1i}^{\mathrm{T}} = \begin{bmatrix} \sqrt{E\left(\pi_{i1}^{(\xi_k)}\right)}\widehat{A}_{i,\xi_k}^{\mathrm{T}}, & \cdots, & \sqrt{E\left(\pi_{is}^{(\xi_k)}\right)}\widehat{A}_{i,\xi_k}^{\mathrm{T}} \end{bmatrix}$$

$$U_{2i}^{\mathrm{T}} = \begin{bmatrix} \sqrt{E\left(\pi_{i1}^{(\xi_k)}\right)}B_{wi}^{\mathrm{T}}, & \cdots, & \sqrt{E\left(\pi_{is}^{(\xi_k)}\right)}B_{wi}^{\mathrm{T}} \end{bmatrix}$$

$$Z = \begin{bmatrix} X_{1,\xi_k}, & \cdots, & X_{s,\xi_k} \end{bmatrix}, \ \widehat{A}_{i,\xi_k} = A_iX_{i,\xi_k} - B_iY_{i,\xi_k}$$

A stabilizing controller to provide $\gamma-$disturbance attenuation can be constructed as $K_{i,\xi_k} = Y_{i,\xi_k}X_{i,\xi_k}^{-1}$

Proof. For the closed-loop system (8), choose a stochastic Lyapunov function candidate as $V(x_k, r_k, \xi_k) = x^{\mathrm{T}}P(r_k, \xi_k)x$, where $P_{i,\xi_k} \triangleq P(r_k, \xi_k)$ is mode-dependent and variation dependent positive definite symmetric matrix for each i. Then, one has

$$\begin{aligned} \Delta V(x_k) &= E\left[V(x_{k+1}, r_{k+1}, \xi_{k+1})|\, x_k, r_k, \xi_k\right] - V(x_k, r_k, \xi_k) \\ &= x_{k+1}^{\mathrm{T}}\bar{P}_{j,\xi_k}x_{k+1} - x_k^{\mathrm{T}}P_{i,\xi_k}x_k \\ &= x_k^{\mathrm{T}}\left(\bar{A}_{i,\xi_k}^{\mathrm{T}}\bar{P}_{j,\xi_k}\bar{A}_{i,\xi_k} - P_{i,\xi_k}\right)x_k \\ &\quad +2x_k^{\mathrm{T}}\bar{A}_{i,\xi_k}^{\mathrm{T}}\bar{P}_{j,\xi_k}B_{wi}w_k + w_k^{\mathrm{T}}B_{wi}^{\mathrm{T}}\bar{P}_{j,\xi_k}B_{wi}w_k \end{aligned} \tag{10}$$

where $\bar{P}_{j,\xi_k} \triangleq \sum_{j=1}^{s} E\left(\pi_{ij}^{(\xi_k)}\right)P_{j,\xi_k}$.

In the following, we assume zero initial condition $V(x_k)|_{k=0} = 0$, and define

$$J \triangleq E\left\{\sum_{k=0}^{\infty}\left[z_k^{\mathrm{T}}z_k - \gamma^2 w_k^{\mathrm{T}}w_k\right]\right\}$$

Since $V(x_k)|_{k=0} = 0$, we have

$$\begin{aligned} J &\le E\left\{\sum_{k=0}^{\infty}\left[z_k^{\mathrm{T}}z_k - \gamma^2 w_k^{\mathrm{T}}w_k + \Delta V(x_k)\right]\right\} \\ &= \sum_{k=0}^{\infty}\zeta_k^{\mathrm{T}}\Omega\zeta_k \end{aligned}$$

where

$$\zeta_k \triangleq \begin{bmatrix} x_k^{\mathrm{T}} & w_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

$$\Omega \triangleq \begin{bmatrix} \bar{A}_{i,\xi_k}^{\mathrm{T}}\bar{P}_{j,\xi_k}\bar{A}_{i,\xi_k} - P_{i,\xi_k} + \bar{C}_{i,\xi_k}^{\mathrm{T}}\bar{C}_{i,\xi_k} \\ \bar{A}_{i,\xi_k}^{\mathrm{T}}\bar{P}_{j,\xi_k}B_{wi} + \bar{C}_{i,\xi_k}^{\mathrm{T}}D_{wi} \\ B_{wi}^{\mathrm{T}}\bar{P}_{j,\xi_k}B_{wi} + D_{wi}^{\mathrm{T}}D_{wi} - \gamma^2 I \end{bmatrix}$$

If $\Omega < 0$ holds, we know from (10) that for $w_k \equiv 0$,

$$\begin{aligned} \Delta V(x_k) &= x_k^{\mathrm{T}}\left(\bar{A}_{i,\xi_k}^{\mathrm{T}}\bar{P}_{j,\xi_k}\bar{A}_{i,\xi_k} - P_{i,\xi_k}\right)x_k \\ &\le -\lambda_{\min}\left[-\left(\bar{A}_{i,\xi_k}^{\mathrm{T}}\bar{P}_{j,\xi_k}\bar{A}_{i,\xi_k} - P_{i,\xi_k}\right)\right]x_k^{\mathrm{T}}x_k \\ &\le -\beta\left\|x_k\right\|^2 \end{aligned}$$

where $\beta = \inf\left\{\lambda_{\min}\left[-\left(\bar{A}_{i,\xi_k}^{\mathrm{T}}\bar{P}_{j,\xi_k}\bar{A}_{i,\xi_k} - P_{i,\xi_k}\right)\right]\right\}$. From the above equation, we obtain that for any $T \ge 1$,

$$\begin{aligned} E\left\{\sum_{k=0}^{T}\left\|x_k\right\|^2\right\} &\le \tfrac{1}{\beta}E\left\{\sum_{k=0}^{T}\left[-\Delta V(x_k)\right]\right\} \\ &= \tfrac{1}{\beta}E\left\{V(x_0, r_0) - E\left[V(x_{T+1}, r_{T+1})|\, x_T, r_T\right]\right\} \\ &\le \tfrac{1}{\beta}E\left[V(x_0, r_0)\right] \end{aligned}$$

which implies that $E\left\{\sum_{k=0}^{T}\left\|x_k\right\|^2\right\} \le \frac{1}{\beta}E\left[V(x_0, r_0)\right] \le \infty$. Thus, the system is stochastically stable from Definition 1. On the other hand, $\Omega < 0$ results in $J < 0$, i.e., $\|z\|_2 < \gamma\|w\|_2$.

Letting $X_{i,\xi_k} \triangleq P_{i,\xi_k}^{-1}$, $Y_{i,\xi_k} \triangleq K_{i,\xi_k}X_{i,\xi_k}$ and pre-and post-multiplying Ω by $diag\left\{\begin{array}{cc} X_{i,\xi_k} & I \end{array}\right\}$, we find that $\Omega < 0$ is equivalent to the coupled LMIs (9). This completes the proof.

Remark 3.1: It should be pointed out that the result in Theorem 3.1 is only suitable for the case where $\sigma_{ij} \ne \infty$. If $\sigma_{ij} = 0$, the LMI (9) in Theorem 3.1 can be easily extended to H_∞ control of discrete-time jump linear systems with precisely known TPs.

Remark 3.2: For the situation where some elements in TP matrix are not accessible, e.g. $\sigma_{ij} = \infty, i = 1 or j = 2$ and $E(\pi_{ij}^{(\xi_k)}), i = 1 or j = 2$ does not exist, we can refer to the methods proposed in [23], [24] and get the following result:

$$\begin{bmatrix} -X_\kappa^i & 0 & L_\kappa^i\widehat{A}_{i,\xi_k} & L_\kappa^iB_{wi} \\ & -\pi_\kappa^i I & \pi_\kappa^i\hat{C}_{i,\xi_k} & \pi_\kappa^iD_{wi} \\ * & & -\pi_\kappa^iX_{i,\xi_k} & 0 \\ * & & * & -\pi_\kappa^i\gamma^2 I \end{bmatrix} < 0 \forall j \in M_\kappa^i \tag{11}$$

$$\begin{bmatrix} -X_j & 0 & \widehat{A}_{i,\xi_k} & X_jB_{wi} \\ & -I & \hat{C}_{i,\xi_k} & D_{wi} \\ * & & -X_{i,\xi_k} & 0 \\ * & & * & -\gamma^2 I \end{bmatrix} < 0 \forall j \in M_{u\kappa}^i \tag{12}$$

where

$$X_\kappa^i = diag\left\{\begin{array}{cccc} X_{\kappa_1^i}, & X_{\kappa_2^i}, & \cdots, & X_{\kappa_m^i} \end{array}\right\}$$

$$L_\kappa^i = \begin{bmatrix} \sqrt{\pi_{i\kappa_1^i}}I, & \sqrt{\pi_{i\kappa_2^i}}I, & \cdots, & \sqrt{\pi_{i\kappa_m^i}}I \end{bmatrix}$$

$$\hat{C}_{i,\xi_k} = \left(C_iX_{i,\xi_k} - D_iY_{i,\xi_k}\right), \ \pi_\kappa^i \triangleq \sum\nolimits_{j\in M_\kappa^i} E\left(\pi_{ij}^{(\xi_k)}\right)$$

$$M_\kappa^i \triangleq \left\{j : E\left(\pi_{ij}^{(\xi_k)}\right) is\ known\right\}$$
$$M_{u\kappa}^i \triangleq \left\{j : E\left(\pi_{ij}^{(\xi_k)}\right) is\ unknown\right\}$$

$$M_\kappa^i = \left\{\begin{array}{cccc} \kappa_1^i, & \kappa_2^i, & \cdots, & \kappa_m^i \end{array}\right\}, 1 \le m \le s$$

Remark 3.3: If all the elements in TP matrix are unknown, e.g. $\sigma_{ij} = \infty, i, j \in M$, then the following LMI holds:

$$\begin{bmatrix} -X_j & 0 & \widehat{A}_{i,\xi_k} & X_jB_{wi} \\ & -I & \hat{C}_{i,\xi_k} & D_{wi} \\ * & & -X_{i,\xi_k} & 0 \\ * & & * & -\gamma^2 I \end{bmatrix} < 0 \tag{13}$$

Remark 3.4: The main contribution of this paper lies in that uncertainty on the TPs is described in terms of Gaussian PDF. Then by analyzing the properties of Gaussian probability distribution, the *desired TP matrix is obtained before the controller design, which leads to the conservativeness of the main results. A possible direction for reducing conservativeness is to blend the mean and variance of Gaussian PDF into the controller design. As a consequence, more information of Gaussian PDF will be retained and integrated into the analysis of the system performance and robustness.*

IV. Numerical example

Example 1. The main feature of this paper is the way of dealing with the uncertainty in the transition probabilities rather than the controller design. Therefore, for highlighting the merits of our method and showing our systems are more general than the existing ones, we use an example from [20]. It consists of four operation modes with nominal transition probability matrix given by

$$\Pi = \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.3 & 0.2 \\ 0.1 & 0.1 & 0.5 & 0.3 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix}$$

In order to fit our purposes, the following PDF matrix is introduced to describe the uncertainty on TP matrix Π:

$$N = \begin{bmatrix} n(0.3,\sigma) & n(0.2,\sigma) & n(0.1,\sigma) & n(0.4,\sigma) \\ n(0.3,\sigma) & n(0.2,\sigma) & n(0.3,\sigma) & n(0.2,\sigma) \\ n(0.1,\sigma) & n(0.1,\sigma) & n(0.5,\sigma) & n(0.3,\sigma) \\ n(0.2,\sigma) & n(0.2,\sigma) & n(0.1,\sigma) & n(0.5,\sigma) \end{bmatrix}$$

where the values of mean are taken for the nominal elements of corresponding TP matrix, and the same value σ for different elements is used for a simple discussion.

According to equation (5), we can get the following relationship between σ and TP uncertainties:

$$\Delta\pi_{ij} = \frac{f(\frac{0-\mu_{ij}}{\sqrt{\sigma}}) - f(\frac{1-\mu_{ij}}{\sqrt{\sigma}})}{F(\frac{0-\mu_{ij}}{\sqrt{\sigma}}) - F(\frac{1-\mu_{ij}}{\sqrt{\sigma}})}\sqrt{\sigma} \tag{14}$$

From equation (14), it can be directly obtained that for $\sigma = 0$, $\Delta\pi_{ij} = 0$ holds. On the contrary, when σ turns to infinity, $\Delta\pi_{ij}$ tends to infinity. So our system can cover the two extremes, namely completely known and unknown TPs, as special cases.

Another benefit of using Gaussian PDF to describe the TP uncertainties lies in that the uncertain range can be quantized by the value of variance parameter σ. Table I shows the obtained TP matrix with different values of σ.

It can be shown from Table I that when σ increases, the uncertainties on TP matrix become bigger. This result is also in accordance with equation (5).

To calculate the controller gains, the following parameters are used in the simulation:

$$A_1 = \begin{bmatrix} 0 & -0.45 \\ 0.9 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -0.29 \\ 0.9 & 1.26 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & -0.45 \\ 1.1 & 0.88 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & -0.29 \\ 0.8 & 1.32 \end{bmatrix}$$
$$B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^{\mathrm{T}}$$
$$B_{w1} = B_{w2} = B_{w3} = B_{w4} = \begin{bmatrix} 0 & 0.2 \end{bmatrix}^{\mathrm{T}}$$
$$C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} 0.5 & 0.4 \end{bmatrix}$$
$$D_1 = D_2 = D_3 = D_4 = 0.9$$
$$D_{w1} = D_{w2} = D_{w3} = D_{w4} = 0.5$$

Giving $\sigma = 0.1$, the desired controller corresponding to the optimal H_∞ performance index can be solved with the parameters described by:

$$K_{1,\xi_k} = \begin{bmatrix} 05556 & 0.4444 \end{bmatrix}$$
$$K_{2,\xi_k} = \begin{bmatrix} -0.5553 & -0.4334 \end{bmatrix}$$
$$K_{3,\xi_k} = \begin{bmatrix} 0.7697 & 0.5070 \end{bmatrix}$$
$$K_{4,\xi_k} = \begin{bmatrix} -0.5212 & -0.4685 \end{bmatrix}$$

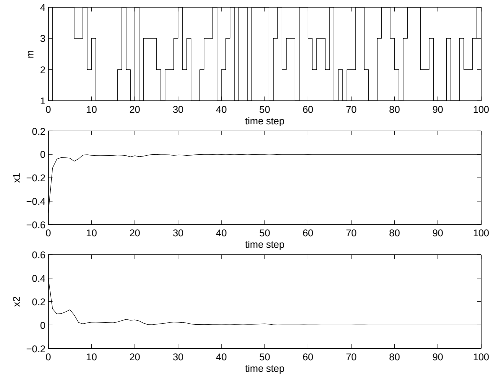

Fig. 2: State response of closed-loop system

Applying the obtained controllers, the state response of the resulting closed-loop system is given in Fig.2, where the initial condition is $x_0 = \begin{bmatrix} -0.5 & 0.4 \end{bmatrix}^{\mathrm{T}}$, and noise signal $w(k) = 0.5\exp(-0.1k)\sin(0.01\pi k)$. From the curves in Fig 2, one can see that despite the random uncertain transition probabilities, the designed H_∞ controller is feasible and effective ensuring that the closed-loop system is stable and yielding the optimal disturbance rejection performance.

Remark 4.1: As described above, the minimum value of σ corresponds to the case with no uncertainty, and increasing values of σ correspond to higher uncertainty level. When some values of σ turn to infinity, the proposed Gaussian PDF model covers the one in [18] as a special case. In this case, we can use LMIs (11)-(12) in Remark 3.2 to calculate the controller gain. In summary, Gaussian PDF model provides a uniform framework in describing TPs of MJSs, which covers the existing MJSs with exactly known TPs, partially known TPs and uncertain TPs in a given set as special cases.

Remark 4.2: In many practical problems, the estimation of TPs is far from accurate. Hence, estimation errors (uncertainties) are limiting factors in applying MJSs to real-world problems. Compared with the uncertainty of TPs lie in a given set, most typically a polytope, Gaussian distribution is often a more reasonable model for statistical uncertainty due to various central limit theorems. We will provide another practical example to demonstrate the practical usefulness of the theoretical study.

Example 2. The second example is adapted from [25],which is an application of MJS to economic system to address the problem of income determination and the business cycle.

A state-space version relating government expenditure and national income is employed as follows:

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -\alpha & 1-s+\alpha \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

where $x_2(k)$ stands for the national income ($x_1(k)$ differs from $x_2(k)$ only by a one-step lag), s is the marginal propensity to save, α is an accelerator coefficient and $u(k)$ is the government expenditure. The parameters s and α are grouped in three classes, according to Table II.

TABLE I: TP matrix with different values of σ

$\sigma = 0.01$				$\sigma = 0.05$			
0.29917	0.19945	0.10248	0.3989	0.29036	0.19862	0.12444	0.38658
0.29994	0.20006	0.29994	0.20006	0.2969	0.2031	0.2969	0.2031
0.10495	0.10495	0.49381	0.29629	0.12171	0.12171	0.47261	0.28398
0.19881	0.19881	0.10559	0.49679	0.19766	0.19766	0.12383	0.48086
$\sigma = 0.1$				$\sigma = 0.5$			
0.26661	0.22881	0.19604	0.30854	0.25793	0.24159	0.22582	0.27466
0.26907	0.23093	0.26907	0.23093	0.25818	0.24182	0.25818	0.24182
0.19381	0.19381	0.3488	0.26358	0.22556	0.22556	0.29126	0.25762
0.22734	0.22734	0.19478	0.35054	0.24144	0.24144	0.22569	0.29143

TABLE II: Multiplier-accelerator modes (from [25])

Mode	Name	Description
1	Norm	s and α in mid-range
2	Boom	s in low range (or α in high)
3	Slump	s in high range (or α in low)

The parameters for the model are given as follows:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2.5 & 3.2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -43.7 & 45.4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ 5.3 & -5.2 \end{bmatrix}, B_1 = B_2 = B_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$C_1 = \begin{bmatrix} 1.5477 & -1.0976 \\ -1.0976 & 1.9145 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 3.1212 & -0.5082 \\ -0.5082 & 2.7824 \\ 0 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1.8385 & -1.2728 \\ -1.2728 & 1.6971 \\ 0 & 0 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 1.6125 \end{bmatrix}^{\mathrm{T}}$$

$$D_2 = \begin{bmatrix} 0 & 0 & 1.0794 \end{bmatrix}^{\mathrm{T}}, D_3 = \begin{bmatrix} 0 & 0 & 1.0540 \end{bmatrix}^{\mathrm{T}}$$

Similar with the first example, we use the following Gaussian PDF matrix to describe the uncertainty on TP matrix

$$N = \begin{bmatrix} n(0.67,\sigma) & n(0.17,\sigma) & n(0.16,\sigma) \\ n(0.30,\sigma) & n(0.47,\sigma) & n(0.23,\sigma) \\ n(0.26,\sigma) & n(0.10,\sigma) & n(0.64,\sigma) \end{bmatrix}$$

Different from the first example, in this example, instead of stressing the generality of Gaussian PDF model, we show the potentials of the proposed approaches in practical systems. Therefore, simply giving $\sigma = 0.1$ and according to (5), we get the following TP matrix

$$\Pi^{'} = \begin{bmatrix} 0.4848 & 0.2594 & 0.2557 \\ 0.3167 & 0.3964 & 0.2869 \\ 0.2949 & 0.2343 & 0.4708 \end{bmatrix}$$

With the obtained TP matrix, the robust controller is given by

$$K_{1,\xi_k} = \begin{bmatrix} -2.3663 & 2.4337 \end{bmatrix}$$
$$K_{2,\xi_k} = \begin{bmatrix} -4.1980 & 3.7854 \end{bmatrix}$$
$$K_{3,\xi_k} = \begin{bmatrix} 4.0855 & -4.3665 \end{bmatrix}$$

For all three cases the initial condition $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$ is assumed to be exactly known. The trajectories of the closed-loop system are plotted in Fig. 2. The simulation results imply that the desired goal is well achieved.

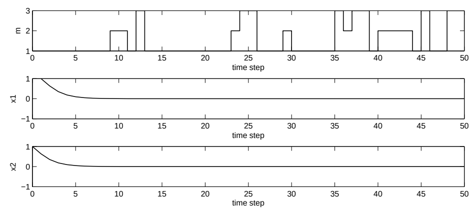

Fig. 3: State response of closed-loop system

V. CONCLUSION

A uniform framework solving H_∞ control problem for a class of Markov jump systems with random uncertain TPs has been presented. Gaussian stochastic process is utilized to descript the stochastically changing uncertain characteristic of TP matrix, which is commonly encountered in practical systems. The continuous random variables included in TP matrix are depicted by Gaussian PDFs, which can cover the systems with precisely known and partially known TPs as two special cases. Then, the criterion for the existence of the admissible controller is obtained. The results can also be extended to the jump linear systems with parametric uncertainties and time-delays. A possible direction for future work is to investigate the stabilization problem for MJS with non-accessible mode information within LMI framework.

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