Automatica 146 (2022) 110594

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Time-varying gain controller synthesis of piecewise homogeneous semi-Markov jump linear systems☆

Yongxiao Tian [a], Huaicheng Yan [a,d,*], Hao Zhang [b], Meng Wang [a], Jianjun Yi [c]

[a] Key Laboratory of Smart Manufacturing in Energy Chemical Process, Ministry of Education, East China University of Science and Technology, Shanghai, 200237, China
[b] Department of Control Science and Engineering, Tongji University, Shanghai, 200092, China
[c] School of Mechanical and Power Engineering, East China University of Science and Technology, Shanghai, 200237, China
[d] School of Information Science and Engineering, Chengdu University, Chengdu 610106, China

ARTICLE INFO

Article history:
Received 26 August 2021
Received in revised form 19 March 2022
Accepted 11 July 2022
Available online 23 September 2022

Keywords:
Elapsed time
Mean-square stability
Piecewise homogeneous
Semi-Markov kernel
Semi-Markov jump linear systems
Time-varying gain controller

ABSTRACT

This paper is devoted to the time-varying gain controller synthesis problem for a class of discrete-time piecewise homogeneous semi-Markov jump linear systems (SMJLSs). Both sojourn-time probability mass functions and embedded Markov chain transition probability are simultaneously regulated by high-level Markov chain, which leads to a more general piecewise homogeneous semi-Markov kernel. A novel class of multivariate-dependent Lyapunov function is constructed, which is mode-dependent, elapsed-time-dependent, and variation-dependent. The objective of this paper is to design two classes of time-varying gain controllers, i.e., elapsed-time-dependent controller and finite memory controller under bound sojourn time. Numerically testable stabilization criteria are established for discrete-time piecewise homogeneous SMJLSs via the proposed novel Lyapunov function and two classes of controllers. Furthermore, two desired stabilizing controllers are designed such that the closed-loop system is mean-square stable. Finally, the proposed control strategies are applied to a numerical example and a practical example of solar thermal receiver to show the effectiveness and applicability of the theoretical results.

1. Introduction

Stochastic jumping systems, which are a class of multi-modes systems or processes that are regulated by a stochastic switching law, can be used to describe the phenomenon of system parameter abrupt change caused by system failures or external interferences (Balluchi, Benvenuti, Di Benedetto, & Sangiovanni-Vincentelli, 2013; Basin & Maldonado, 2014; Boukas, 2006; Tian, Yan, Zhang, Zhan, & Peng, 2020a; Vargas, Costa, Acho, & do Val, 2017). In particular, Markov/semi-Markov jump systems as a special branch of stochastic jumping systems, which are drawing much attention due to a wide range of potential applications in recent years. Mainly because the multi-modes systems governed by a Markov/semi-Markov chain or process can be used to model random faults/failures in practical industrial systems. Substantially, semi-Markov jump systems (SMJSs), which systems mode sojourn time obeys more general random distribution that is memory distribution, can be regarded as generalized format to Markov jump systems (MJSs). Memory distribution make SMJSs' transition probabilities (TPs) time-varying (*time-interval-dependent*), which is more consistent with wider practical situations. All of these make the SMJSs have more extensive practical applications compared with the conventional MJSs. Such as communication networks (Ma, Djouadi, & Li, 2012), transportation systems (Mudge & Al-Sadoun, 1985), and biological systems (Papadopoulou, 2013), semi-Markov model has been demonstrated to be more effective. Recently, a great number of elegant results on the analysis and synthesis problems of MJSs/SMJSs have been reported, please see Bolzern, Colaneri, and De Nicolao (2014), Jiang, Kao, Gao, and Yao (2017), Jiang, Kao, Karimi, and Gao (2018), Li, Shi, Yao, and Wu (2016), Mu and Hu (2020), Shen, Wu, Shi, Shu, and Karimi (2019), Tian, Yan, Zhang, Cheng, and Shen (2021), Tian, Yan, Zhang, Zhan, and Peng (2020b), Xue, Yan, Zhang, Shen, and Shi (2021), Zhang, Leng, and Colaneri (2016a), Zhang, Yang, and Colaneri (2016b) and the references therein.

However, most of the existing literatures do not consider that the transition information of stochastic jump systems may

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Bert Tanner under the direction of Editor Christos G. Cassandras.

* Corresponding author at: Key Laboratory of Smart Manufacturing in Energy Chemical Process, Ministry of Education, East China University of Science and Technology, Shanghai, 200237, China.

E-mail addresses: tianyongxiao1991@163.com (Y. Tian), hcyan@ecust.edu.cn (H. Yan), zhang_hao@tongji.edu.cn (H. Zhang), mengwang@ecust.edu.cn (M. Wang), jjyi@ecust.edu.cn (J. Yi).

https://doi.org/10.1016/j.automatica.2022.110594
0005-1098/

depend on the jumping initial instant (*time-point-dependent*), that is, the random process which only considers the homogeneous property (homogeneous Markov/semi-Markov chain or process), see, e.g. Bolzern et al. (2014), Jiang et al. (2018), Ning, Zhang, and Lam (2018), Xu, Wu, Su, Shi, and Que (2019), Xue et al. (2021), Zhang, Cai, and Shi (2019) and Zhang et al. (2016b). In these literatures, the transition information is independent of jumping initial instants, and the random sojourn time obeys homogeneous memoryless or memory random distributions. In reality, in addition to memory distribution, there are many factors affecting the TPs of MJSs/SMJSs, and homogeneous Markov/semi-Markov processes are not suitable in many applications. It has been shown in Costa, do Val, and Geromel (1999), El Ghaoui and Scorletti (1996) and Xiong, Lam, Gao, and Ho (2005) that certain TPs may lead to system instability or at least deteriorated performance.

The non-homogeneous stochastic jumping systems, whose systems modes jumping dynamics depend on the initial time, have drawn extensive attention in practical application fields, such as, electric transportations (Rolink & Rehtanz, 2013), economic problems (D'Amico, 2009). Related researchers have obtained some interesting results via several powerful techniques for non-homogeneous MJSs/SMJSs in control fields (Aberkane, 2012; Liu, Zhang, & Zheng, 2017; Mathieu, Foucher, Dellamonica, & Daures, 2007; Ning, Zhang, Mesbah, & Colaneri, 2020; Papadopoulou & Vassiliou, 1994; Vassiliou & Papadopoulou, 1992; Vasslllou & Tsantas, 1984; Zhang, Cai, Tan, & Shi, 2020). Note that for the systems with general time-varying (non-homogeneous) TPs, the difficulties in developing the related theories are inevitable as encountered in general time-varying dynamic systems. Based on the idea of approximate simulation, finite consecutive homogeneous Markov processes (or chains) with different intervals can be used to replace non-homogeneous Markov process (or chains), we called this type is "piecewise homogeneous". Fortunately, theory results have been developed by aforementioned idea for piecewise homogeneous MJSs in Zhang (2009). It has been shown that SMJSs have superior application prospects in contrast to MJSs, and much efforts have been put into this research in view of the practical importance of developing numerically testable stability conditions. Aiming at solving random sojourn time does not necessarily follow a given random distribution problem, the semi-Markov kernel (SMK) approach has been proposed in Ning et al. (2018) and Zhang et al. (2016a), which captures the information of the sojourn-time probability mass functions (ST-PMFs) depending on both the current and next system modes. In Zhang et al. (2016a) and Ning et al. (2018), numerically tested stability and stabilization criteria have been established via restrict the upper (lower) bound of all the system modes sojourn time gives rise to the concept of σ-error mean-square stability. Considering that the time-varying transfer information is more in line with the actual situation, the non-homogeneous SMJSs with both *time-interval-dependent* and *time-point-dependent* transition information have more research value. Inspired by the idea of Zhang (2009), we may propose a class of finite piecewise homogeneous semi-Markov process (or chain). Correspondingly, non-homogeneous SMJSs can transfer into piecewise homogeneous SMJSs. The SMK is equipped with "piecewise homogeneous" property and have the so-called "piecewise homogeneous SMK", which can be categorized into two cases, one is that either ST-PMFs or TPs are piecewise homogeneous, while the other one is that both ST-PMFs and TPs are piecewise homogeneous.

On the other hand, it has been verified that the time-varying gain control scheme is more powerful and less conservative than the time-invariant approach. In summary, there are two main forms of time-varying gain state feedback controller. One is that the controller gain changes with the change of time, which is called the elapsed-time-dependent control strategy. This class of controller mainly used in switching system (deterministic or stochastic), some results see in Fei, Shi, Wang, and Wu (2017) and Zhang, Zhuang, and Shi (2015). The other one is the control input using previous system state measurements during a finite time interval, which is called finite horizon/frequency memory control strategy and has been developed in Ebihara, Peaucelle, and Arzelier (2011), Kwon and Han (2004), Trégouët, Peaucelle, Arzelier, and Ebihara (2013), Wang, Qiu, and Feng (2018) and Yuan and Wu (2017). The authors in Ebihara et al. (2011) and Trégouët et al. (2013) introduced the so-called periodically memory state feedback controller. A finite memory output feedback control scheme was developed for linear time invariant (LTI) systems in time domain Kwon and Han (2004) and LTI systems in frequency domain Wang et al. (2018), respectively, which shows that this class of control scheme is more robust against parameter uncertainties and achieves faster convergence rate than the previous works. This type of controller is mainly to improve the performance of the resulting control system by enriching the dynamics of the controller and includes the traditional state feedback controller as a special case. Also, the time-varying gain controller bring some complexities and difficulties to the control synthesis problem. Nevertheless, to the best of the authors' knowledge, very few results have tackled the stabilization problem of discrete-time piecewise homogeneous semi-Markov jump linear systems (SMJLSs) with time-varying gain state feedback controller, which motivates us to fill this gap.

In light of the observations mentioned earlier, in this paper, we are dedicated to the stabilization analysis for discrete-time piecewise homogeneous SMJLSs via two classes of time-varying gain controllers. The main contributions of this paper include the following three aspects: *Firstly*, a more general stabilization analysis framework is developed for discrete-time piecewise homogeneous SMJLSs, in which ST-PMFs and TPs are considered to be simultaneously piecewise homogeneous. *Secondly*, two new results on time-varying gain state feedback controllers synthesis of piecewise homogeneous SMJLSs are proposed by employing a novel multivariate-dependent Lyapunov function, which is shown to be more effective method in solving the time-varying gain control problem. *Thirdly*, compared with the existing results on SMJSs, our study is more general and practical than those with homogeneous of sojourn and transition information. Besides, this paper also defines the concept of piecewise homogeneous semi-Markov chain, which completes the relevant definition of semi-Markov chain.

Notations: Throughout this paper, $\mathbb{R}^n$ denotes the $n-$dimension Euclidean space, $\mathbb{R}^{n\times m}$ is set of all $n \times m$ real matrices. Define diag{A,B} $= \left[\begin{smallmatrix} A & 0 \\ * & B \end{smallmatrix}\right]$, where $*$ represents symmetry element. $\lambda_{\max}\{A\}$ and $\lambda_{\min}\{A\}$ stand for the eigenvalue of matrix A with maximum real part and minimum real part, respectively. $\mathbf{He}(A)$ is used to present $A + A^T$, $\mathcal{T} = \{1, 2, \dots, T\}$, $\mathcal{M} = \{1, 2, \dots, M\}$, $\mathcal{N} = \{1, 2, \dots, N\}$. $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ represent the sets of positive real numbers and non-negative real numbers, respectively. $\mathbb{Z}_{\geq \Upsilon}$ and $\mathbb{Z}_{[\Upsilon_1,\Upsilon_2]}$ denote the sets of integers $\{\gamma \in \mathbb{Z} | \gamma \geq \Upsilon\}$ and $\{\gamma \in \mathbb{Z} | \Upsilon_1 \leq \gamma \leq \Upsilon_2\}$, respectively. $\|\cdot\|$, $\mathbb{E}\{\cdot\}$ and $\mathbb{P}\{\cdot\}$ signify, respectively, Euclidean vector norm, the expected value operator and the probability measure. $\otimes$ means the Kronecker product of matrices. $\mathbf{I}_m$ and $\mathbf{0}_{n\times m}$ are identity matrix of order m and zero matrix of order $n \times m$, respectively.

2. Preliminaries and problem formulation

To introduce the piecewise homogeneous semi-Markov chain formally, the following three stochastic processes are necessary. On a fixed complete probability space $\aleph = (\Psi, \mathcal{F}, \mathbb{P})$, where Ψ

is the sample space, $\mathcal{F}$ is the σ-algebra of subsets of Ψ, and $\mathbb{P}$ is the probability measure on $\mathcal{F}$. We consider the stochastic process $\{R_n\}_{n\in\mathbb{Z}_{\geq 0}}$ taking values in $\mathcal{T}$, the stochastic process $\{k_n\}_{n\in\mathbb{Z}_{\geq 0}}$, and the stochastic process $\{s_n\}_{n\in\mathbb{Z}_{\geq 0}}$ taking values in $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\geq 1}$ respectively, where R_n is the index of system mode at the nth jump, k_n denotes the time at the nth jump and $s_n = k_{n+1} - k_n$ represents the sojourn-time of mode R_n between the nth jump and $(n+1)$th jump. It is noted that $k_0 = 0$, and k_n increases monotonically with n.

The following definitions, lemmas will be adopted in the rest of this paper. For more details refer to Barbu and Limnios (2006), D'Amico (2009), Janssen and Manca (2006) and Ning et al. (2020).

2.1. Mathematical definition of piecewise homogeneous semi-Markov chain

For clarity of further discussion, let us first recall the following definitions on non-homogeneous semi-Markov chain (SMC).

Definition 1. A binary stochastic process $\{(R_n, k_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ is said to be a discrete-time non-homogeneous Markov renewal chain (MRC), if for $\forall i, j \in \mathcal{T}(i \neq j)$, $\forall n \in \mathbb{Z}_{\geq 0}$, $\forall t \in \mathbb{Z}_{\geq 0}$, and $\forall \nu \in \mathbb{Z}_{\geq 1}$, $\mathbb{P}(R_{n+1} = j, s_n = \nu | R_0, k_0; \ldots; R_n = i, k_n = t) = \mathbb{P}(R_{n+1} = j, s_n = \nu | R_n = i, k_n = t)$.

Definition 2. Given a discrete-time non-homogeneous MRC $\{(R_n, k_n)\}_{n\in\mathbb{Z}_{\geq 0}}$, (i) stochastic process $\{R_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is called the non-homogeneous embedded Markov chain (EMC) of the non-homogeneous MRC $\{(R_n, k_n)\}_{n\in\mathbb{Z}_{\geq 0}}$; (ii) the stochastic process $\{r_k\}_{k\in\mathbb{Z}_{\geq 0}}$ is said to be a non-homogeneous SMC associated with the non-homogeneous MRC $\{(R_n, k_n)\}_{n\in\mathbb{Z}_{\geq 0}}$, if $r_k = R_{N(k)}$, $\forall k \in \mathbb{Z}_{\geq 0}$, where $N(k) \triangleq \max\{n \in \mathbb{Z}_{\geq 0} | k_n \leq k\}$.

Then, let $\rho_{ij}(\nu, t) = \mathbb{P}(s_n = \nu | R_{n+1} = j, R_n = i, k_n = t)$, $\chi_{ij}(t) = \mathbb{P}(R_{n+1} = j | R_n = i, k_n = t)$ and $\pi_{ij}(\nu, t) = \mathbb{P}(R_{n+1} = j, s_n = \nu | R_n = i, k_n = t)$, it is easy to find that $\pi_{ij}(\nu, t) = (\mathbb{P}(R_{n+1} = j, R_n = i, s_n = \nu, k_n = t)/\mathbb{P}(R_{n+1} = j, R_n = i, k_n = t))(\mathbb{P}(R_{n+1} = j, R_n = i, k_n = t)/\mathbb{P}(R_n = i, k_n = t)) = \rho_{ij}(\nu, t)\chi_{ij}(t)$. They are called non-homogeneous ST-PMF, TPs of non-homogeneous EMC and non-homogeneous SMK, respectively. And in particular $\chi_{ii}(t) = 0$, $\rho_{ii}(\nu, t) = 0$, $\pi_{ij}(\nu, t) \in [0, 1]$, and $\sum_{\nu=1}^{\infty}\sum_{j\in\mathcal{T}} \pi_{ij}(\nu, t) = 1$, $\forall i \in \mathcal{T}$, $\forall t \in \mathbb{Z}_{\geq 0}$, $\forall \nu \in \mathbb{Z}_{\geq 1}$.

Remark 1. Based on the idea of Ref. Zhang (2009), we propose a piecewise homogeneous SMK to replace *time-point-dependent* SMK (non-homogeneous SMK), which implies the considered non-homogeneous SMK is time varying but invariant within an interval. It needs to introduce two high-level homogeneous Markov chains (HHMC) ϕ_k, φ_k taking value in finite set $\mathcal{M}$, $\mathcal{N}$, respectively. The piecewise homogeneous SMK has the following description: $\Pi(\nu) = [\pi_{ij}^{(\phi_{k_{n+1}}, \varphi_{k_{n+1}})}(\nu)]_{i,j\in\mathcal{T}}$, $\pi_{ij}^{(\phi_{k_{n+1}}, \varphi_{k_{n+1}})}(\nu) = \rho_{ij}^{(\phi_{k_{n+1}})}(\nu)\chi_{ij}^{(\varphi_{k_{n+1}})}$, where $\pi_{ij}^{(\phi_{k_{n+1}}, \varphi_{k_{n+1}})}(\nu) \in [0, 1]$, $\chi_{ii}^{(\varphi_{k_{n+1}})} = 0$, $\rho_{ii}^{(\phi_{k_{n+1}})}(\nu) = 0$, $\sum_{\nu=1}^{\infty}\sum_{j\in\mathcal{T}} \pi_{ij}^{(\phi_{k_{n+1}}, \varphi_{k_{n+1}})}(\nu) = 1$.

The HHMC TPs are as follows

$\mathbb{P}\{\phi_{k_{n+1}} = b | \phi_{k_{n+1}-1} = a\} = \omega_{ab}$, $\forall a, b \in \mathcal{M}$,

$\mathbb{P}\{\varphi_{k_{n+1}} = q | \varphi_{k_{n+1}-1} = p\} = \varrho_{pq}$, $\forall p, q \in \mathcal{N}$.

These imply the variations of EMC's TPs and ST-PMFs in different interval follow a Markov chain.

The concept of piecewise homogeneous SMC is given below.

Definition 3. A binary stochastic process $\{(R_n, k_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ is said to be a discrete-time piecewise homogeneous MRC, if for any $j \in \mathcal{T}$, $\forall \nu \in \mathbb{Z}_{\geq 1}$ and $\forall n \in \mathbb{Z}_{\geq 0}$,

$$\mathbb{P}(R_{n+1} = j, s_n = \nu | R_0, k_0; R_1, k_1; \ldots; R_n, k_n) \tag{1}$$

$$= \mathbb{P}(R_{n+1} = j, s_n = \nu | R_n = i) \tag{2}$$

$$= \pi_{ij}^{(\phi_{k_{n+1}}, \varphi_{k_{n+1}})}(\nu) \tag{3}$$

$$= \frac{\mathbb{P}(R_{n+1} = j, s_n = \nu, R_n = i)}{\mathbb{P}(R_{n+1} = j, R_n = i)} \frac{\mathbb{P}(R_{n+1} = j, R_n = i)}{\mathbb{P}(R_n = i)} \tag{4}$$

$$= \rho_{ij}^{(\phi_{k_{n+1}})}(\nu)\chi_{ij}^{(\varphi_{k_{n+1}})}, \tag{5}$$

where $\pi_{ij}^{(\phi_{k_{n+1}}, \varphi_{k_{n+1}})}(\nu)$, $\rho_{ij}^{(\phi_{k_{n+1}})}(\nu)$ and $\chi_{ij}^{(\varphi_{k_{n+1}})}$ are called piecewise homogeneous SMK, piecewise homogeneous ST-PMFs and TPs of piecewise homogeneous EMC, respectively.

Definition 4. Consider a discrete-time piecewise homogeneous MRC $\{(R_n, k_n)\}_{n\in\mathbb{Z}_{\geq 0}}$, the stochastic process $\{r_k\}_{k\in\mathbb{Z}_{\geq 0}}$ is said to be a piecewise homogeneous SMC associated with the piecewise homogeneous MRC $\{(R_n, k_n)\}_{n\in\mathbb{Z}_{\geq 0}}$, if $r_k = R_{N(k)}$, $\forall k \in \mathbb{Z}_{\geq 0}$, where $N(k) \triangleq \max\{n \in \mathbb{Z}_{\geq 0} | k_n \leq k\}$.

Remark 2. In order to simplify the description, we do not give the definition of piecewise homogeneous EMC, which can refer to Definitions 1–2. In addition, the form of the piecewise homogeneous SMK and the meaning of ϕ_k, φ_k are given in Remark 1.

Remark 3. The concept of piecewise homogeneous was first introduced in Zhang (2009), which can be treated as a tradeoff between the homogeneous case and non-homogeneous case. In this paper, we introduce a more general piecewise homogeneous SMC, which is characterized by the piecewise homogeneous EMC and the piecewise homogeneous ST-PMFs. Obviously, if (3)–(5) are not regulated by two high level of Markov chains ϕ_k, φ_k, then the stochastic process $\{(R_n, k_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ will degenerate into a homogeneous MRC. Besides, in this paper, ϕ_k, φ_k are assumed to be independent on σ−algebra $\mathcal{F}_{k-1} = \sigma\{R_1, R_2, \ldots, R_{k-1}\}$, ϕ_k and φ_k are mutually independent random processes.

2.2. System description and objectives

For a fixed complete probability space $\aleph$, we consider a class of discrete-time piecewise homogeneous SMJLSs described by

$$x(k+1) = A_{r_k}x(k) + B_{r_k}u(k), \tag{6}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are the state and the control input, respectively. The stochastic process $\{r_k\}_{k\in\mathbb{Z}_{\geq 0}}$ is a piecewise homogeneous SMC, and A_i and B_i are known real matrices, for $r_k = i$.

In the following, two classes of time varying gain controllers are considered.

2.2.1. Elapsed-time-dependent controller

Construct the following elapsed-time-dependent controller that also depends on the system mode and HHMC mode:

$$u(k) = K_{i,a,p}(\iota)x(k), \tag{7}$$

$\forall r(k) = i \in \mathcal{T}, \phi(k) = a \in \mathcal{M}, \varphi(k) = p \in \mathcal{N}$,

where $K_{i,a,p}(\iota)$ is a mode-dependent, elapsed-time-dependent and variation-dependent (MD-ED-VD) controller gain, and $\iota = k - k_n$, $\forall n \in \mathbb{Z}_{\geq 0}$ denotes the elapsed time since the system switches to the current mode. Then, with the combination of (7), the resulting closed-loop system is presented as follows:

$$\tilde{x}(k+1) = \mathscr{A}_{i,a,p}(\iota)\tilde{x}(k), \tag{8}$$

where $\tilde{x}(k) = x(k)$ and $\mathscr{A}_{i,a,p}(\iota) = A_i + B_iK_{i,a,p}(\iota)$. It can be deduced that

$$\tilde{x}(k_n + \iota) = \prod_{\ell=0}^{\iota-1} \mathscr{A}_{i,a,p}(\iota)\tilde{x}(k_n), \ \iota \in \mathbb{Z}_{[0,k_{n+1}-k_n-1]}. \tag{9}$$

2.2.2. Finite memory controller

A finite memory controller is given as the following form:

$$u(k)=\sum_{\iota=0}^{\hbar_i-1}K_{i,a,p}(\iota)x(k-\iota), \tag{10}$$

$\forall r(k-\iota)=i\in\mathcal{T},\phi(k-\iota)=a\in\mathcal{M},\varphi(k-\iota)=p\in\mathcal{N}$,

where $K_{i,a,p}(\iota)$ is a MD-ED-VD controller gain, and $\iota=k-k_n$, $\forall n\in\mathbb{Z}_{\geq 0}$, denotes the elapsed time since the system switches to the current mode. $\hbar_i$ is called finite horizon for ith mode, and it does not exceed the sojourn time ν.

Then, combining the system (6) yield the closed-loop system as follows:

$$\tilde{x}(k+1)=\mathcal{A}_{i,a,p}\tilde{x}(k), \tag{11}$$

where $\mathcal{A}_{i,a,p}=\begin{bmatrix}\mathbf{0}_{(\hbar_i-1)n\times n} & \mathbf{I}_{(\hbar_i-1)n}\\ & \bar{A}_{i,a,p}\end{bmatrix}$, with $\bar{A}_{i,a,p}=[B_iK_{i,a,p}(\hbar_i-1),\dots,B_iK_{i,a,p}(1),A_i+B_iK_{i,a,p}(0)]$, and $\tilde{x}(k)=[x^\top(k-\hbar_i+1),\dots,x^\top(k-1),x^\top(k)]$ is the augmented state vector.

Remark 4. In order to reduce the conservatism of the controller synthesis criteria, we design two classes of time-dependent controllers subject to discrete-time piecewise homogeneous SMJLSs. These two classes of controllers have both the same and different points. The same point is that the controller gain depends on the system mode elapsed time. The difference is that the controller (7) only feedback the system state values at the current moment, while the controller (10) using all state feedback values in a finite range from a past moment to the current moment.

Definition 5. A function $\varpi(\cdot):\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}_\infty$ function if it is continuous, strictly increasing, unbounded and $\varpi(0)=0$.

Definition 6. The closed-loop system (8) or (11) is said to be mean-square stable if, for every initial conditions $\tilde{x}(0)$, $r(0)\in\mathcal{T}$, and the upper bound $\delta_i\in\mathbb{Z}_{\geq 1}$ of sojourn time for the ith mode, $\forall i\in\mathcal{T}$, the following equation holds

$$\lim_{k\to\infty}\mathbb{E}\left\{\|\tilde{x}(k)\|^2\right\}\Big|_{\tilde{x}(0),r_0(s_n\in\mathbb{Z}_{[1,\delta_i]}|R_n=i)}=0. \tag{12}$$

Remark 5. The definition of σ–error mean square stability is given by limiting the upper bound of sojourn time subject to homogeneous SMJSs (Zhang et al., 2016a), in which the σ is defined as $\sigma\triangleq|\ln(\mathbb{F}_i(\delta_i))|$. In fact, $\mathbb{F}$ is a cumulative mass function (CMF) of sojourn time, denoted as $\mathbb{F}_i(\delta_i)=\mathbb{P}(s_n\leq\delta_i|r_n=i)=\sum_{\nu=1}^{\delta_i}\sum_{j\in\mathcal{T}}\pi_{ij}(\nu)$. However, in a piecewise homogeneous SMJLSs, the error σ is in the following form:

$$\begin{aligned}\sigma&=\sum_{i\in\mathcal{T}}|\ln(\mathbb{F}_i(\delta_i))|=\sum_{i\in\mathcal{T}}|\ln(\sum_{\nu=1}^{\delta_i}\sum_{j\in\mathcal{T}}\sum_{b\in\mathcal{M}}\sum_{q\in\mathcal{N}}\rho_{ij}^{(b)}(\nu)\chi_{ij}^{(q)})|\\&=\sum_{i\in\mathcal{T}}|\ln(\sum_{\nu=1}^{\delta_i}\sum_{j\in\mathcal{T}}\sum_{b\in\mathcal{M}}\sum_{q\in\mathcal{N}}\pi_{ij}^{(b,q)}(\nu))|,\end{aligned}$$

where σ is time-varying and uncertain, owing to the time-varying TPs of the associated piecewise homogeneous EMC and piecewise homogeneous ST-PMFs information. Thus, in this paper, index σ will not be considered.

Lemma 1 (Ning et al., 2018)**.** *Consider a discrete-time stochastic jumping nonlinear closed-loop system $\tilde{x}(k+1)=f(\tilde{x}(k),r_k)$, where $\tilde{x}(k)$ denotes the system state, and r_k is a piecewise homogeneous stochastic process regulated by high-level Markov chains ϕ_k, φ_k. The jumping instants are denoted by $k_0,k_1,\dots,k_s,\dots$ with $k_0=0$. The system is mean-square stable, if there exist a set of smooth energy functions $V(\tilde{x}(k),r_k,\phi_k,\varphi_k,k-k_s):\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ and three $\mathcal{K}_\infty$ functions $\varpi_1,\varpi_2,\varpi_3$, such that for any initial conditions $\tilde{x}(0)\in\mathbb{R}^n$, $r_0\in\mathcal{T}$, $\phi_0\in\mathcal{M}$, $\varphi_0\in\mathcal{N}$ and given finite $h_i\in\mathbb{R}_{>0}$, $\forall r_{k_s}=i\in\mathcal{T}$, $\forall\phi_{k_s}=a\in\mathcal{M}$, $\forall\varphi_{k_s}=p\in\mathcal{N}$, the following inequalities hold*

$$\varpi_1(\|\tilde{x}(k)\|)\leq V(\tilde{x}(k),r_k,\phi_k,\varphi_k,k-k_s)\leq\varpi_2(\|\tilde{x}(k)\|), \tag{13}$$

$$V(\tilde{x}(k),r_k,\phi_k,\varphi_k,k-k_s)\leq h_iV(\tilde{x}(k_s),r_k,\phi_k,\varphi_k),0, \tag{14}$$

$$\begin{aligned}&\mathbb{E}\{V(\tilde{x}(k_{s+1}),r_{k_{s+1}},\phi_{k_{s+1}},\varphi_{k_{s+1}},0)\}|_{\tilde{x}(0),r_0,\phi_0,\varphi_0}\\&\quad-V(\tilde{x}(k_s),r_{k_s},\phi_{k_s},\varphi_{k_s},0)\leq-\varpi_3(\|\tilde{x}(k_s)\|).\end{aligned} \tag{15}$$

3. Main results

In this section, the stabilization problem of system (6) via two classes of time-varying gain controllers will be explored.

3.1. Stabilization analysis via elapsed-time-dependent controller

At the beginning, a sufficient stabilization condition via elapsed-time-dependent controller will be shown.

Theorem 1. *Given finite constants $h_i\in\mathbb{R}_{>0}$, $\delta_i\in\mathbb{Z}_{\geq 1}$, $i\in\mathcal{T}$, the closed-loop system (8) with piecewise homogeneous SMC is mean-square stable if there exist a set of matrices $P_{i,a,p}(\iota)>0$, $i\in\mathcal{T}$, $a\in\mathcal{M}$, $p\in\mathcal{N}$, $\iota\in\mathbb{Z}_{[0,\delta_i-1]}$, $\nu\in\mathbb{Z}_{[1,\delta_i]}$, such that for $\forall i\in\mathcal{T}$, $\forall a\in\mathcal{M}$, $\forall p\in\mathcal{N}$, $\forall\iota\in\mathbb{Z}_{[2,\delta_i-1]}$ ($\delta_i\in\mathbb{Z}_{\geq 3}$),*

$$(\prod_{\ell=0}^{\iota-1}\mathscr{A}_{i,a,p}(\ell))^\top P_{i,a,p}(\iota)\prod_{\ell=0}^{\iota-1}\mathscr{A}_{i,a,p}(\ell)-h_iP_{i,a,p}(0)<0, \tag{16}$$

and for $\forall$ $i\in\mathcal{T}$, $\forall a\in\mathcal{M}$, $\forall p\in\mathcal{N}$, $\forall\nu\in\mathbb{Z}_{[1,\delta_i]}$ ($\delta_i\in\mathbb{Z}_{\geq 1}$),

$$\sum_{\nu=1}^{\delta_i}(\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\ell))^\top\mathcal{H}_i^{(b,q)}(\nu)\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\ell)-P_{i,a,p}(0)<0, \tag{17}$$

where

$$\mathcal{H}_i^{(b,q)}(\nu)=\sum_{b\in\mathcal{M}}\sum_{q\in\mathcal{M}}\sum_{j\in\mathcal{T}}\omega_{ab}\varrho_{pq}\pi_{ij}^{(b,q)}(\nu)P_{j,b,q}(0)/\bar{\upsilon}_i^{(b,q)},$$

$$\bar{\upsilon}_i^{(b,q)}=\sum_{\nu=1}^{\delta_i}\sum_{b\in\mathcal{M}}\sum_{q\in\mathcal{N}}\sum_{j\in\mathcal{T}}\omega_{ab}\varrho_{pq}\pi_{ij}^{(b,q)}(\nu).$$

Proof. Choose a stochastic Lyapunov functional candidate, which is MD-ETD-VD function, as

$$V(\tilde{x}(k),i,a,p,\iota)=\tilde{x}^\top(k)P_{i,a,p}(\iota)\tilde{x}(k), \tag{18}$$

$\forall i\in\mathcal{T},a\in\mathcal{M},p\in\mathcal{N}$.

For $\forall r_k=i$, $\phi_k=a$, $\varphi_k=p$, $\forall k\in\mathbb{Z}_{[k_n,k_{n+1}-1]}$, it holds that

$$\lambda_1\|\tilde{x}(k)\|^2\leq V(\tilde{x}(k),i,a,p,\iota)\leq\lambda_2\|\tilde{x}(k)\|^2, \tag{19}$$

where $\lambda_1=\inf_{i\in\mathcal{T},a\in\mathcal{M},p\in\mathcal{N},\iota\in\mathbb{Z}_{[0,\delta_i-1]}}\{\lambda_{\min}(P_{i,a,p}(\iota))\}$ and $\lambda_2=\sup_{i\in\mathcal{T},a\in\mathcal{M},p\in\mathcal{N},\iota\in\mathbb{Z}_{[0,\delta_i-1]}}\{\lambda_{\max}(P_{i,a,p}(\iota))\}$. Obviously, λ_1 and λ_2 are $\mathcal{K}_\infty$ function, choose $\varpi_1(\|\tilde{x}(k)\|)=\lambda_1\|\tilde{x}(k)\|^2$ and $\varpi_2(\|\tilde{x}(k)\|)=\lambda_2\|\tilde{x}(k)\|^2$, thus (13) holds.

For $\forall k\in\mathbb{Z}_{[k_n,k_{n+1}-1]}$, it can be found that

$$\begin{aligned}&V(\tilde{x}(k_n+\iota),i,a,p,\iota)-h_iV(\tilde{x}(k_n),i,a,p,0)\\&=\tilde{x}^\top(k_n)[(\prod_{\ell=0}^{\iota-1}\mathscr{A}_{i,a,p}(\iota))^\top P_{i,a,p}(\iota)\prod_{\ell=0}^{\iota-1}\mathscr{A}_{i,a,p}(\iota)\\&\quad-h_iP_{i,a,p}(0)]\tilde{x}(k_n)<0,\end{aligned}$$

which implies (14).

Let $R_n = i, R_{n+1} = j, \forall i \neq j, \nu = k_{n+1} - k_n$. Then

$$
\begin{aligned}
&\mathbb{E}\{V(\tilde{x}(k_{n+1}), j, b, q, 0)\}|_{\tilde{x}(k_n, R_n = i, \phi_{k_n} = a, \varphi_{k_n} = p, s_n \in \mathbb{Z}_{[1\,\delta_i]})}\\
&\quad - V(\tilde{x}(k_n), i, a, p, 0)\\
=&\tilde{x}^\top(k_{n+1})\{\sum_{j\in\mathcal{T}, b\in\mathcal{M}, q\in\mathcal{N}} P_{j,b,q}(0)\mathscr{P}_1\}\tilde{x}(k_{n+1})\\
&\quad - \tilde{x}^\top(k_n)P_{i,a,p}(0)\tilde{x}(k_n)\\
=&\tilde{x}^\top(k_{n+1})\Big\{\sum_{j\in\mathcal{T}, b\in\mathcal{M}, q\in\mathcal{N}} P_{j,b,q}(0)\frac{\mathscr{P}_2}{\mathscr{P}_3} \times \frac{\mathscr{P}_3}{\mathscr{P}_4}\Big\}\\
&\quad \times \tilde{x}(k_{n+1}) - \tilde{x}^\top(k_n)P_{i,a,p}(0)\tilde{x}(k_n)\\
=&\tilde{x}^\top(k_{n+1})\Big\{\sum_{\nu=1}^{\infty}\sum_{j\in\mathcal{T}} P_{j,b,q}(0)\sum_{b\in\mathcal{M}}\omega_{ab}\sum_{q\in\mathcal{N}}\varrho_{pq}\pi_{ij}^{(b,q)}(\nu)\Big\}\\
&\quad \times \tilde{x}(k_{n+1}) - \tilde{x}^\top(k_n)P_{i,a,p}(0)\tilde{x}(k_n)\\
=&\tilde{x}^\top(k_n)\Big\{\sum_{\nu=1}^{\delta_i}\Big[\frac{\sum_{b\in\mathcal{M}}\omega_{ab}\sum_{q\in\mathcal{N}}\varrho_{pq}\sum_{j\in\mathcal{T}}\pi_{ij}^{(b,q)}(\nu)}{\bar{\upsilon}_i^{(b,q)}}\\
&\quad \times (\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\ell))^\top P_{j,b,q}(0)\prod_{\ell=0}^{\iota-1}\mathscr{A}_{i,a,p}(\ell)\Big]\Big\}\tilde{x}(k_n)\\
&\quad - \tilde{x}^\top(k_n)P_{i,a,p}(0)\tilde{x}(k_n)\\
\le& -\lambda_{\min}\Big(-\sum_{\nu=1}^{\delta_i}(\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\ell))^\top\mathcal{H}_i^{(b,q)}(\nu)\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\ell)\\
&\quad + P_{i,a,p}(0)\Big)\|\tilde{x}(k_n)\|^2\\
=& -\lambda_3\|\tilde{x}(k_n)\|^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_3 =& \inf_{\substack{i\in\mathcal{T}, a, b\in\mathcal{M},\\ b,q\in\mathcal{N}, \nu\in\mathbb{Z}_{[1,\delta_i]}}}\Big\{\lambda_{\min}\Big(\Big[-\sum_{\nu=1}^{\delta_i}(\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\ell))^\top\mathcal{H}_i^{(b,q)}(\nu)\\
&\times \prod_{\ell=0}^{\iota-1}\mathscr{A}_{i,a,p}(\ell) + P_{i,a,p}(0)\Big]\Big)\Big\},\\
\mathscr{P}_1 =& \mathbb{P}\{R_{k+1} = j, s_n = \nu, \phi_{k_{n+1}} = b, \varphi_{k_{n+1}} = q | R_k = i,\\
&\quad \phi_{k_n} = a, \varphi_{k_n} = p\},\\
\mathscr{P}_2 =& \mathbb{P}\{R_{k+1} = j, s_n = \nu, \phi_{k_{n+1}} = b, \varphi_{k_{n+1}} = q,\\
&\quad R_k = i, \phi_{k_n} = a, \varphi_{k_n} = p\},\\
\mathscr{P}_3 =& \mathbb{P}\{\phi_{k_{n+1}} = b, \varphi_{k_{n+1}} = q, R_k = i, \phi_{k_n} = a, \varphi_{k_n} = p\},\\
\mathscr{P}_4 =& \mathbb{P}\{R_k = i, \phi_{k_n} = a, \varphi_{k_n} = p\}.
\end{aligned}
$$

We choose $\varpi_3(\|\tilde{x}(k_n)\|) = \lambda_3\|\tilde{x}(k_n)\|^2$, it is easy to find that $\varpi_3(\cdot) \in \mathcal{K}_\infty$, one has (15).

Above all, system (8) is mean-square stable. This proof is completed. □

Remark 6. Theorem 1 gives a more general stabilization criterion for a piecewise homogeneous SMJLSs, which is still valid if two high-level Markov chain φ_k and ϕ_k are the same. Furthermore, when ST-PMFs and EMC only one is piecewise homogeneous, the numerical testable criteria see Corollary 1.

Due to existing the continued product of matrix $\mathscr{A}_{i,a,p}(\ell)$, it is difficult to obtain stabilization conditions by the techniques for linear matrix inequalities based on Theorem 1 directly. Next, the control synthesis problem will be addressed for the closed-loop piecewise homogeneous SMJLSs (8) by eliminating the continued product, and a tractable condition is exhibited for the existence of the controller (7), such that the closed-loop piecewise homogeneous SMJLSs (8) is mean square stable.

Theorem 2. *Given finite constants $h_i \in \mathbb{R}_{>0}$, $\delta_i \in \mathbb{Z}_{\ge 1}$, and scalars ω_ζ, $i \in \mathcal{T}$, $\zeta = 1, 2, 3$, the closed-loop system (8) with piecewise homogeneous SMC is mean-square stable if there exist sets of matrices $\bar{Q}_{i,a,p}(\iota,\kappa) > 0$, $\tilde{\mathcal{J}}_i^{(b,q)}(\nu,\epsilon_1) > 0$ and $\bar{\mathcal{J}}_i^{(b,q)}(\delta_i) > 0$, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$, $\iota \in \mathbb{Z}_{[0,\delta_i-1]}$, $\kappa \in \mathbb{Z}_{[0,\iota]}$, $\epsilon_1 \in \mathbb{Z}_{[0,\nu-1]}$, $\nu \in \mathbb{Z}_{[1,\delta_i]}$, and sets of matrices $U_{i,a,p}(\iota)$ and X, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$, $\iota \in \mathbb{Z}_{[0,\delta_i-1]}$, such that for $\forall i \in \mathcal{T}$, $\forall a \in \mathcal{M}$, $\forall p \in \mathcal{N}$, $\forall \iota \in \mathbb{Z}_{[2,\delta_i-1]}$ ($\delta_i \in \mathbb{Z}_{\ge 3}$), $\kappa \in \mathbb{Z}_{[1,\iota-1]}$, the followings are satisfied*

$$
\begin{aligned}
&\begin{bmatrix} \bar{Q}_{i,a,p}(\iota,\kappa+1) & * \\ \mathbf{0}_{n\times n} & -\bar{Q}_{i,a,p}(\iota,\kappa) \end{bmatrix}\\
&+ \mathbf{He}\{\Lambda_1 \otimes [-X^\top \quad \mathfrak{A}_{i,a,p}(\kappa)]\} < 0,
\end{aligned} \tag{20}
$$

$\forall i \in \mathcal{T}$, $\forall a \in \mathcal{M}$, $\forall p \in \mathcal{N}$, $\forall \iota \in \mathbb{Z}_{[1,\delta_i-1]}$ ($\delta_i \in \mathbb{Z}_{\ge 2}$), there holds

$$
\begin{aligned}
&\begin{bmatrix} \bar{Q}_{i,a,p}(\iota,1) & * \\ \mathbf{0}_{n\times n} & -h_i\bar{Q}_{i,a,p}(0,0) \end{bmatrix}\\
&+ \mathbf{He}\{\Lambda_2 \otimes [-X^\top \quad \mathfrak{A}_{i,a,p}(0)]\} < 0,
\end{aligned} \tag{21}
$$

and for $\forall i \in \mathcal{T}$, $\forall a, b \in \mathcal{M}$, $\forall p, q \in \mathcal{N}$, $\forall \epsilon_1 \in \mathbb{Z}_{[0,\delta_i-1]}$, ($\delta_i \in \mathbb{Z}_{\ge 1}$), the followings are satisfied

$$
\begin{bmatrix} \bar{\Phi}_{i,a,p}(\epsilon_1) & * \\ \bar{\mathbf{Q}}_1[\mathbf{I}_n \ \mathbf{0}_{n\times n}]\frac{\hat{\pi}_i^{(b,q)}(\epsilon_1+1)}{\sqrt{\bar{\upsilon}_i^{(b,q)}}} & -\bar{\mathbf{Q}}_1 \end{bmatrix} < 0, \tag{22}
$$

$$
\bar{\mathcal{J}}_i^{(b,q)}(0) - \bar{Q}_{i,a,p}(0,0) < 0, \tag{23}
$$

where

$$
\begin{aligned}
&\hat{\boldsymbol{\pi}}_i^{(b,q)}(\epsilon_1)\\
=&\sqrt{\omega_{a1}\varrho_{p1}\pi_{i1}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots, \sqrt{\omega_{a1}\varrho_{pN}\pi_{i1}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots,\\
&\sqrt{\omega_{aM}\varrho_{p1}\pi_{i1}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots, \sqrt{\omega_{aM}\varrho_{pN}\pi_{i1}^{(b,q)}(\epsilon_1)}\mathbf{I}_n,\\
&\sqrt{\omega_{a1}\varrho_{p1}\pi_{i2}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots, \sqrt{\omega_{a1}\varrho_{pN}\pi_{i2}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots,\\
&\sqrt{\omega_{aM}\varrho_{p1}\pi_{i2}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots, \sqrt{\omega_{aM}\varrho_{pN}\pi_{i2}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots,\\
&\sqrt{\omega_{a1}\varrho_{p1}\pi_{iT}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots, \sqrt{\omega_{a1}\varrho_{pN}\pi_{iT}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots,\\
&\sqrt{\omega_{aM}\varrho_{p1}\pi_{iT}^{(b,q)}(\epsilon_1)}\mathbf{I}_n, \ldots, \sqrt{\omega_{aM}\varrho_{pN}\pi_{iT}^{(b,q)}(\epsilon_1)}\mathbf{I}_n,\\
&\bar{\mathbf{Q}}_1 = diag\{\bar{Q}_{1,1,1}(0,0), \ldots, \bar{Q}_{1,1,N}(0,0), \ldots, \bar{Q}_{1,M,1}(0,0),\\
&\quad \ldots, \bar{Q}_{1,M,N}(0,0), \bar{Q}_{2,1,1}(0,0), \ldots, \bar{Q}_{2,1,N}, \ldots,\\
&\quad \bar{Q}_{2,M,1}(0,0), \ldots, \bar{Q}_{2,M,N}(0,0), \ldots, \bar{Q}_{T,1,1}(0,0), \ldots,\\
&\quad \bar{Q}_{T,1,N}(0,0), \ldots, \bar{Q}_{T,M,1}(0,0), \ldots, \bar{Q}_{T,M,N}(0,0)\},\\
&\mathcal{I}(\epsilon_1) = \mathbf{I}_n, \epsilon_1 \in \mathbb{Z}_{[1,\delta_i-1]}, \ \bar{\mathcal{J}}_i^{(b,q)}(\epsilon_1) = \sum_{\nu=\epsilon_1+1}^{\delta_i}\tilde{\mathcal{J}}_i^{(b,q)}(\nu,\epsilon_1),\\
&\tilde{\mathcal{J}}_i^{(b,q)}(\nu,\epsilon_1) = X^\top\mathcal{J}_i^{(b,q)}(\nu,\epsilon_1)X, \ \mathcal{I}(\delta_i) = 0,\\
&\bar{\Phi}_{i,a,p}(\epsilon_1) = \bar{\Theta}_{i,a,p}(\epsilon_1) + \mathbf{He}\{\Lambda_3 \otimes [-X^\top \quad \mathfrak{A}_{i,a,p}(\epsilon_1)]\},\\
&\bar{\Theta}_{i,a,p}(\epsilon_1) = \begin{bmatrix} \mathcal{I}(\epsilon_1+1)\bar{\mathcal{J}}_i^{(b,q)}(\epsilon_1+1) & * \\ \mathbf{0}_{n\times n} & -\bar{\mathcal{J}}_i^{(b,q)}(\epsilon_1) \end{bmatrix},\\
&\mathfrak{A}_{i,a,p}(\epsilon_1) = A_iX + B_iU_{i,a,p}(\epsilon_1), \ \Lambda_\zeta = [1, \omega_\zeta]^\top, \zeta = 1, 2, 3.
\end{aligned}
$$

The desired controller gains of (7) are $K_{i,a,p}(\iota) = U_{i,a,p}(\iota)X^{-1}$, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$, $\iota \in \mathbb{Z}_{[0,\delta_i-1]}$.

Proof. We prove the theorem via three steps. Firstly, through variable substitution, equivalent conditions are introduced to remove the continued product in inequality (16). Secondly, the continued product in inequality (17) will also be removed by similar technique. Finally, by means of matrix congruent transformation, the inequalities conditions that guarantee conditions (16)–(17) satisfy can be obtained with the admissible controller gain.

Step 1: Set $Q_{i,a,p}(\iota,\iota)=P_{i,a,p}(\iota)$, (16) is equivalent to

$$\begin{cases}(\prod_{\ell=0}^{\iota-1}\mathscr{A}_{i,a,p}(\ell))^{\top}Q_{i,a,p}(\iota,\iota)\prod_{\ell=0}^{\iota-1}\mathscr{A}_{i,a,p}(\ell) & \\ -\mathscr{A}_{i,a,p}^{\top}(0)Q_{i,a,p}(\iota,1)\mathscr{A}_{i,a,p}(0)<0, & \text{(a)}\\ \mathscr{A}_{i,a,p}^{\top}(0)Q_{i,a,p}(\iota,1)\mathscr{A}_{i,a,p}(0) & \\ -h_iQ_{i,a,p}(0,0)<0. & \text{(b)}\end{cases} \tag{24}$$

On the other hand, for the (24)(a), it follows that

$$\sum_{\kappa=1}^{\iota-1}[(\prod_{\ell=0}^{\kappa-1}\mathscr{A}_{i,a,p}(\ell))^{\top}((\mathscr{A}_{i,a,p}(\kappa))^{\top}Q_{i,a,p}(\iota,\kappa+1)\mathscr{A}_{i,a,p}(\kappa)$$
$$-\ Q_{i,a,p}(\iota,\kappa))\prod_{\ell=0}^{\kappa-1}\mathscr{A}_{i,a,p}(\ell)]<0, \tag{25}$$

and accordingly

$$(\mathscr{A}_{i,a,p}(\kappa))^{\top}Q_{i,a,p}(\iota,\kappa+1)\mathscr{A}_{i,a,p}(\kappa)-Q_{i,a,p}(\iota,\kappa)<0. \tag{26}$$

It is easy to find that (26) and (24)(b) are equivalent to

$$\begin{bmatrix}\mathscr{A}_{i,a,p}(\kappa)\\ \mathbf{I}_n\end{bmatrix}^{\top}\begin{bmatrix}Q_{i,a,p}(\iota,\kappa+1) & *\\ \mathbf{0}_{n\times n} & -Q_{i,a,p}(\iota,\kappa)\end{bmatrix}\begin{bmatrix}\mathscr{A}_{i,a,p}(\kappa)\\ \mathbf{I}_n\end{bmatrix}<0, \tag{27}$$

$$\begin{bmatrix}\mathscr{A}_{i,a,p}(0)\\ \mathbf{I}_n\end{bmatrix}^{\top}\begin{bmatrix}Q_{i,a,p}(\iota,1) & *\\ \mathbf{0}_{n\times n} & -h_iQ_{i,a,p}(0,0)\end{bmatrix}\begin{bmatrix}\mathscr{A}_{i,a,p}(0)\\ \mathbf{I}_n\end{bmatrix}<0. \tag{28}$$

By projection lemma (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994), there exist two slack variables $\mathcal{G}_1,\mathcal{G}_2$ with appropriate dimension, such that the following inequalities hold

$$\begin{bmatrix}Q_{i,a,p}(\iota,\kappa+1) & *\\ \mathbf{0}_{n\times n} & -Q_{i,a,p}(\iota,\kappa)\end{bmatrix}$$
$$+\ \mathbf{He}\{\mathcal{G}_1^{\top}[-\mathbf{I}_n\quad \mathscr{A}_{i,a,p}(\kappa)]\}<0, \tag{29}$$

$$\begin{bmatrix}Q_{i,a,p}(\iota,1) & *\\ \mathbf{0}_{n\times n} & -h_iQ_{i,a,p}(0,0)\end{bmatrix}$$
$$+\ \mathbf{He}\{\mathcal{G}_2^{\top}[-\mathbf{I}_n\quad \mathscr{A}_{i,a,p}(0)]\}<0. \tag{30}$$

Step 2: Set $\mathcal{J}_i^{(b,q)}(\nu,\nu)=\mathcal{H}_i^{(b,q)}(\nu)$, (17) is equivalent to

$$\begin{cases}\sum_{\nu=1}^{\delta_i}((\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\iota))^{\top}\mathcal{J}_i^{(b,q)}(\nu,\nu)\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\iota) & \\ -\mathcal{J}_i^{(b,q)}(\nu,0))<0, & \text{(a)}\\ \hat{\mathcal{J}}_i^{(b,q)}(0)-Q_{i,a,p}(0,0)<0. & \text{(b)}\end{cases} \tag{31}$$

For the (31)(a), it holds that

$$\sum_{\nu=1}^{\delta_i}((\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\iota))^{\top}\mathcal{J}_i^{(b,q)}(\nu,\nu)\prod_{\ell=0}^{\nu-1}\mathscr{A}_{i,a,p}(\iota)-\mathcal{J}_i^{(b,q)}(\nu,0))$$
$$=\sum_{\epsilon_1=0}^{\delta_i-1}\Bigg[(\prod_{\ell=0}^{\epsilon_1-1}\mathscr{A}_{i,a,p}(\ell))^{\top}\sum_{\nu=\epsilon_1+1}^{\delta_i}(\mathscr{A}_{i,a,p}^{\top}(\epsilon_1)\mathcal{J}_i^{(b,q)}(\nu,\epsilon_1+1) \tag{32}$$
$$\times\ \mathscr{A}_{i,a,p}(\epsilon_1)-\mathcal{J}_i^{(b,q)}(\nu,\omega_1))\prod_{\ell=0}^{\epsilon_1-1}\mathscr{A}_{i,a,p}(\ell)\Bigg]<0.$$

Then, it yields that

$$\sum_{\nu=\epsilon_1+1}^{\delta_i}(\mathscr{A}_{i,a,p}^{\top}(\epsilon_1)\mathcal{J}_i^{(b,q)}(\nu,\epsilon_1+1)\mathscr{A}_{i,a,p}(\epsilon_1)$$
$$-\ \mathcal{J}_i^{(b,q)}(\nu,\omega_1))<0. \tag{33}$$

Similar to **Step 1**, there exists a slack variable $\mathcal{G}_3$ with appropriate dimension, such that the following inequality holds

$$\begin{bmatrix}\mathcal{I}(\epsilon_1+1)\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1+1)+\mathcal{J}_i^{(b,q)}(\epsilon_1+1,\epsilon_1+1) & *\\ \mathbf{0}_{n\times n} & -\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1)\end{bmatrix}$$
$$+\ \mathbf{He}\{\mathcal{G}_3^{\top}[-\mathbf{I}_n\quad \mathscr{A}_{i,a,p}(\epsilon_1)]\}<0, \tag{34}$$

combining with $\mathcal{J}_i^{(b,q)}(\epsilon_1+1,\epsilon_1+1)=\sum_{b\in\mathcal{M}}\sum_{q\in\mathcal{N}}\sum_{j\in\mathcal{T}}\omega_{ab}\varrho_{pq}\pi_{ij}^{(b,q)}(\epsilon_1+1)Q_{j,b,q}(0,0)/\bar{\upsilon}_i^{(b,q)}$. By Schur complement lemma, (34) implies the following

$$\begin{bmatrix}\Phi_{i,a,p}(\epsilon_1) & *\\ \mathbf{Q}_1[\mathbf{I}_n\ \mathbf{0}_{n\times n}]\frac{\hat{\boldsymbol{\pi}}_i^{(b,q)}(\epsilon_1+1)}{\sqrt{\bar{\upsilon}_i^{(b,q)}}} & -\mathbf{Q}_1\end{bmatrix}<0, \tag{35}$$

where $\Phi_{i,a,p}(\epsilon_1)=\Theta_{i,a,p}+\mathbf{He}\{\mathcal{G}_3^{\top}[-\mathbf{I}_n\quad \mathscr{A}_{i,a,p}(\epsilon_1)]\}$, $\Theta_{i,a,p}(\epsilon_1)=\begin{bmatrix}\mathcal{I}(\epsilon_1+1)\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1+1) & *\\ \mathbf{0}_{n\times n} & -\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1)\end{bmatrix}$, with $\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1)=\sum_{\nu=\epsilon_1+1}^{\delta_i}\mathcal{J}^{(b,q)}(\nu,\epsilon_1)$, and

$$\begin{aligned}\mathbf{Q}_1=\mathrm{diag}\{&Q_{1,1,1}(0,0),\ldots,Q_{1,1,N}(0,0),\ldots,Q_{1,M,1}(0,0),\\ &\ldots,Q_{1,M,N}(0,0),Q_{2,1,1}(0,0),\ldots,Q_{2,1,N}(0,0),\\ &\ldots,Q_{2,M,1}(0,0),\ldots,Q_{2,M,N}(0,0),\ldots,Q_{T,1,1}(0,0),\\ &\ldots,Q_{T,1,N}(0,0),\ldots,Q_{T,M,1}(0,0),\ldots,Q_{T,M,N}(0,0)\}.\end{aligned}$$

Step 3: By defining $\bar{Q}_{i,a,p}(0,0)=X^{\top}Q_{i,a,p}(0,0)X$, $\bar{Q}_{i,a,p}(\iota,\kappa)=X^{\top}Q_{i,a,p}(\iota,\kappa)X$, $\bar{\mathcal{J}}_i^{(b,q)}(\epsilon_1)=X^{\top}\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1)X$.

We specify the slack matrix $\mathcal{G}_\zeta$, as $\mathcal{G}_\zeta^{\top}=[\Lambda_\zeta\otimes G^{\top}]$, $\zeta=1,2,3$ ($G\in\mathbb{R}^{n\times n}$ is a nonsingular matrix) and define $X=G^{-1}$, by taking the congruence transformations diag$\{X^{\top},X^{\top}\}$ to (29), (30), diag$\{X^{\top},X^{\top},\underbrace{X^{\top},\ldots,X^{\top}}_{MNT}\}$ to (35), $X^{\top}$ to (31)(b), they give rise to the matrices inequalities (20)–(23), respectively. This proof is completed. □

If only ST-PMF (or EMC) is piecewise homogeneous, Theorem 2 leads to the results given as follows.

Corollary 1. *Given finite constants $h_i\in\mathbb{R}_{>0}$, $\delta_i\in\mathbb{Z}_{\geq 1}$, and scalars ω_ζ, $i\in\mathcal{T}$, $\zeta=1,2,3$, the closed-loop system (8) with piecewise homogeneous SMC is mean-square stable if there exist sets of matrices $\bar{Q}_{i,a}(\iota,\kappa)>0$, $\tilde{\mathcal{J}}_i^{(b)}(\nu,\epsilon_1)>0$ and $\bar{\mathcal{J}}_i^{(b)}(\delta_i)>0$, $i\in\mathcal{T}$, $a,b\in\mathcal{M}$, $\iota\in\mathbb{Z}_{[0,\delta_i-1]}$, $\kappa\in\mathbb{Z}_{[0,\iota]}$, $\epsilon_1\in\mathbb{Z}_{[0,\nu-1]}$, $\nu\in\mathbb{Z}_{[1,\delta_i]}$, and sets of matrices $U_{i,a}(\iota)$ and X, $i\in\mathcal{T}$, $a\in\mathcal{M}$, $\iota\in\mathbb{Z}_{[0,\delta_i-1]}$, such that for $\forall i\in\mathcal{T}$, $\forall a\in\mathcal{M}$, $\forall\iota\in\mathbb{Z}_{[2,\delta_i-1]}$ ($\delta_i\in\mathbb{Z}_{\geq 3}$), $\kappa\in\mathbb{Z}_{[1,\iota-1]}$, the followings are satisfied*

$$\begin{bmatrix}\bar{Q}_{i,a}(\iota,\kappa+1) & *\\ \mathbf{0}_{n\times n} & -\bar{Q}_{i,a}(\iota,\kappa)\end{bmatrix}$$
$$+\ \mathbf{He}\{\Lambda_1\otimes[-X^{\top}\quad \mathfrak{A}_{i,a}(\kappa)]\}<0, \tag{36}$$

$\forall i\in\mathcal{T}$, $\forall a\in\mathcal{M}$, $\forall\iota\in\mathbb{Z}_{[1,\delta_i-1]}$ ($\delta_i\in\mathbb{Z}_{\geq 2}$), there holds

$$\begin{bmatrix}\bar{Q}_{i,a}(\iota,1) & *\\ \mathbf{0}_{n\times n} & -h_i\bar{Q}_{i,a}(0,0)\end{bmatrix}$$
$$+\mathbf{He}\{\Lambda_2\otimes[-X^{\top}\quad \mathfrak{A}_{i,a}(0)]\}<0, \tag{37}$$

and for $\forall i\in\mathcal{T}$, $\forall a,b\in\mathcal{M}$, $\forall\epsilon_1\in\mathbb{Z}_{[0,\delta_i-1]}$, ($\delta_i\in\mathbb{Z}_{\geq 1}$), the followings are satisfied

$$\begin{bmatrix}\bar{\Phi}_{i,a}(\epsilon_1) & *\\ \bar{\mathbf{Q}}_2[\mathbf{I}_n\ \mathbf{0}_{n\times n}]\frac{\check{\boldsymbol{\pi}}_i^{(b)}(\epsilon_1+1)}{\sqrt{\bar{\upsilon}_i^{(b)}}} & -\bar{\mathbf{Q}}_2\end{bmatrix}<0, \tag{38}$$

$$\bar{\mathcal{J}}_i^{(b)}(0)-\bar{Q}_{i,a}(0,0)<0, \tag{39}$$

where

$$\breve{\pi}_i^{(b)}(\epsilon_1) = \sqrt{\omega_{a1}\pi_{i1}^{(b)}(\epsilon_1)}\mathbf{I}_n, \ldots, \sqrt{\omega_{a1}\pi_{iT}^{(b)}(\epsilon_1)}\mathbf{I}_n,$$
$$\sqrt{\omega_{a2}\pi_{i1}^{(b)}(\epsilon_1)}\mathbf{I}_n, \ldots, \sqrt{\omega_{a2}\pi_{iT}^{(b)}(\epsilon_1)}\mathbf{I}_n, \ldots,$$
$$\sqrt{\omega_{aM}\pi_{i1}^{(b)}(\epsilon_1)}\mathbf{I}_n, \ldots, \sqrt{\omega_{aM}\pi_{iT}^{(b)}(\epsilon_1)}\mathbf{I}_n,$$
$$\bar{\mathbf{Q}}_2 = diag\{\bar{Q}_{1,1}(0,0), \ldots, \bar{Q}_{1,M}(0,0), \bar{Q}_{2,1}(0,0), \ldots, \bar{Q}_{2,M}(0,0), \ldots, \bar{Q}_{\mathcal{T},1}(0,0), \ldots, \bar{Q}_{\mathcal{T},M}(0,0)\},$$

and other notations are the same as in Theorem 2. Moreover, the desired controller gains of (7) are $K_{i,a}(\iota) = U_{i,a}(\iota)X^{-1}$, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $\iota \in \mathbb{Z}_{[0,\delta_i-1]}$.

3.2. Stabilization analysis via finite memory controller

Next, a sufficient stabilization condition via finite memory controller will be shown.

Theorem 3. *Given finite constants $h_i \in \mathbb{R}_{>0}$, $\delta_i, \hbar_i \in \mathbb{Z}_{\geq 1}$, $i \in \mathcal{T}$, the closed-loop system (11) with piecewise homogeneous SMC is mean-square stable if there exist a set of matrices $P_{i,a,p}(\iota) > 0$, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$, $\iota \in \mathbb{Z}_{[0,\delta_i-1]}$, such that for $\forall i \in \mathcal{T}$, $\forall a \in \mathcal{M}$, $\forall p \in \mathcal{N}$, $\forall \iota \in \mathbb{Z}_{[1,\delta_i-1]}$ ($\delta_i \in \mathbb{Z}_{\geq 2}$),*

$$(\mathcal{A}_{i,a,p}^\top)^\iota P_{i,a,p}(\iota)\mathcal{A}_{i,a,p}^\iota - h_i P_{i,a,p}(0) < 0, \tag{40}$$

and for $\forall$ $i \in \mathcal{T}$, $\forall a, b \in \mathcal{M}$, $\forall p, q \in \mathcal{N}$, $\forall \nu \in \mathbb{Z}_{[1,\delta_i]}$ ($\delta_i \in \mathbb{Z}_{\geq 1}$),

$$\sum_{\nu=1}^{\delta_i} (\mathcal{A}_{i,a,p}^\top)^\nu \mathcal{H}_i^{(b,q)}(\nu)\mathcal{A}_{i,a,p}^\nu - P_{i,a,p}(0) < 0, \tag{41}$$

the notations have been defined in Theorem 1.

Proof. The proof details can be referred to Theorem 1, which is omitted here for saving space. □

Remark 7. It is worth noting that the dimension of Lyapunov matrix in Theorem 3 is different from that in Theorem 1. Due to the expansion of augmented system dimension, the Lyapunov matrix $P_{i,a,p}(\iota) \in \mathbb{R}^{\hbar_i n}$ in Theorem 3. For simplicity, we still use symbol $P_{i,a,p}(\iota)$ to present Lyapunov matrix in Theorem 3.

Theorem 4. *Given finite constants $h_i \in \mathbb{R}_{>0}$, $\delta_i, \hbar_i \in \mathbb{Z}_{\geq 1}$, and scalars $\omega_\zeta(\iota)$, $i \in \mathcal{T}$, $\zeta = 1, 2$, $\iota = 0, 1, \ldots, \hbar_i - 1$, the closed-loop system (11) with piecewise homogeneous SMC is mean-square stable if there exist sets of matrices $\bar{Q}_{i,a,p}(\iota, \kappa) > 0$, $\bar{\mathcal{J}}_i^{(b,q)}(\nu, \epsilon_1) > 0$ and $\bar{\mathcal{J}}_i^{(b,q)}(\delta_i) > 0$, $i \in \mathcal{T}$, $a, b \in \mathcal{M}$, $p, q \in \mathcal{N}$, $\iota \in \mathbb{Z}_{[0,\delta_i-1]}$, $\kappa \in \mathbb{Z}_{[0,\iota]}$, $\epsilon_1 \in \mathbb{Z}_{[0,\nu-1]}$, $\nu \in \mathbb{Z}_{[1,\delta_i]}$, and sets of matrices $U_{i,a,p}(\iota)$ and X, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$, $\iota \in \mathbb{Z}_{[0,\hbar_i-1]}(\hbar_i \leq \delta_i)$, such that for $\forall i \in \mathcal{T}$, $\forall a \in \mathcal{M}$, $\forall p \in \mathcal{N}$, $\forall \iota \in \mathbb{Z}_{[1,\delta_i-1]}$ ($\delta_i \in \mathbb{Z}_{\geq 2}$), $\kappa \in \mathbb{Z}_{[0,\iota-1]}$, the followings are satisfied*

$$\mathfrak{I}^\top \begin{bmatrix} \bar{Q}_{i,a,p}(\iota, \kappa+1) & * \\ \mathbf{0}_{\hbar_i n \times \hbar_i n} & -\bar{Q}_{i,a,p}(\iota, \kappa) \end{bmatrix} \mathfrak{I} + \mathbf{He}\{\tilde{\Lambda}_1 \otimes [-X^\top \quad \hat{A}_{i,a,p}]\} < 0, \tag{42}$$

$$\bar{Q}_{i,a,p}(\iota, 0) - h_i \bar{Q}_{i,a,p}(0,0) < 0, \tag{43}$$

and for $\forall i \in \mathcal{T}$, $\forall a, b \in \mathcal{M}$, $\forall p, q \in \mathcal{N}$, $\forall \epsilon_1 \in \mathbb{Z}_{[0,\delta_i-1]}$ ($\delta_i \in \mathbb{Z}_{\geq 1}$), the followings are satisfied

$$\begin{bmatrix} \bar{\Xi}_{i,a,p}(\epsilon_1) & * \\ \bar{\mathbf{Q}}_1 [\mathbf{I}_{\hbar_i n} \ \mathbf{0}_{\hbar_i n \times \hbar_i n}] \mathfrak{I} \frac{\hat{\pi}_i^{(b,q)}(\epsilon_1+1)}{\sqrt{\bar{\upsilon}_i^{(b,q)}}} & -\bar{\mathbf{Q}}_1 \end{bmatrix} < 0, \tag{44}$$

$$\bar{\mathcal{J}}_i^{(b,q)}(0) - \bar{Q}_{i,a,p}(0,0) < 0, \tag{45}$$

where

$$\hat{A}_{i,a,p} = [B_i U_{i,a,p}(\hbar_i - 1), \ldots, B_i U_{i,a,p}(1), A_i X + B_i U_{i,a,p}(0)],$$
$$\tilde{\Lambda}_\zeta = [1, \omega_\zeta(\hbar_i - 1), \ldots, \omega_\zeta(0)]^\top, \ \zeta = 1, 2,$$
$$\bar{\Xi}_{i,a,p}(\epsilon_1) = \mathfrak{I}^\top \bar{\Theta}_{i,a,p}(\epsilon_1)\mathfrak{I} + \mathbf{He}\{\tilde{\Lambda}_2 \otimes [-X^\top \quad \hat{A}_{i,a,p}]\},$$
$$\bar{\Theta}_{i,a,p}(\epsilon_1) = \begin{bmatrix} \mathcal{I}(\epsilon_1+1)\bar{\mathcal{J}}_i^{(b,q)}(\epsilon_1+1) & * \\ \mathbf{0}_{\hbar_i n \times \hbar_i n} & -\bar{\mathcal{J}}_i^{(b,q)}(\epsilon_1) \end{bmatrix},$$
$$\mathfrak{I} = \begin{bmatrix} \mathbf{0}_{(\hbar_i-1)n \times 2n} & \mathbf{I}_{(\hbar_i-1)n} \\ \mathbf{I}_{(\hbar_i+1)n} & \end{bmatrix},$$

the other notations are already defined in Theorem 2. The desired controller gains of (10) can be calculated as $K_{i,a,p}(\iota) = U_{i,a,p}(\iota)X^{-1}$, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$, $\iota \in \mathbb{Z}_{[0,\hbar_i-1]}$, $\hbar_i \leq \delta_i$.

Proof. Similar to Theorem 2, the powers of $\mathcal{A}_{i,a,p}$ of (40) and (41) can be eliminated, which yields that

$$\mathcal{A}_{i,a,p}^\top Q_{i,a,p}(\iota, \kappa+1)\mathcal{A}_{i,a,p} - Q_{i,a,p}(\iota, \kappa) < 0, \tag{46}$$
$$Q_{i,a,p}(\iota, 0) - h_i Q_{i,a,p}(0,0) < 0, \tag{47}$$

and

$$\sum_{\nu=\epsilon_1+1}^{\delta_i} (\mathcal{A}_{i,a,p}^\top \mathcal{J}_i^{(b,q)}(\nu, \epsilon_1+1)\mathcal{A}_{i,a,p} - \mathcal{J}_i^{(b,q)}(\nu, \epsilon_1)) < 0, \tag{48}$$
$$\hat{\mathcal{J}}_i^{(b,q)}(0) - Q_{i,a,p}(0,0) < 0. \tag{49}$$

It is easy to find that (46) is equivalent to

$$\begin{bmatrix} \mathcal{A}_{i,a,p} \\ \mathbf{I}_{\hbar_i n} \end{bmatrix}^\top \begin{bmatrix} Q_{i,a,p}(\iota, \kappa+1) & * \\ \mathbf{0}_{\hbar_i n \times \hbar_i n} & -Q_{i,a,p}(\iota, \kappa) \end{bmatrix} \begin{bmatrix} \mathcal{A}_{i,a,p} \\ \mathbf{I}_{\hbar_i n} \end{bmatrix} < 0. \tag{50}$$

Notice a fact that

$$\begin{bmatrix} \mathcal{A}_{i,a,p} \\ \mathbf{I}_{\hbar_i n} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0}_{(\hbar_i-1)n \times 2n} & \mathbf{I}_{(\hbar_i-1)n} \\ \mathbf{I}_{(\hbar_i+1)n} & \end{bmatrix}}_{\mathfrak{I}} \begin{bmatrix} \bar{A}_{i,a,p} \\ \mathbf{I}_{\hbar_i n} \end{bmatrix} < 0. \tag{51}$$

Then, (50) can be rewritten as the following form

$$\begin{bmatrix} \bar{A}_{i,a,p} \\ \mathbf{I}_{\hbar_i n} \end{bmatrix}^\top \mathfrak{I}^\top \begin{bmatrix} Q_{i,a,p}(\iota, \kappa+1) & * \\ \mathbf{0}_{\hbar_i n \times \hbar_i n} & -Q_{i,a,p}(\iota, \kappa) \end{bmatrix} \mathfrak{I} \begin{bmatrix} \bar{A}_{i,a,p} \\ \mathbf{I}_{\hbar_i n} \end{bmatrix} < 0. \tag{52}$$

By projection lemma (Boyd et al., 1994), there exists slack variable $\tilde{\mathcal{G}}_1$, such that the following inequality holds

$$\mathfrak{I}^\top \begin{bmatrix} Q_{i,a,p}(\iota, \kappa+1) & * \\ \mathbf{0}_{\hbar_i n \times \hbar_i n} & -Q_{i,a,p}(\iota, \kappa) \end{bmatrix} \mathfrak{I} + \mathbf{He}\{\tilde{\mathcal{G}}_1^\top [-\mathbf{I}_n \quad \bar{A}_{i,a,p}]\} < 0. \tag{53}$$

In order to linearize the condition (53), we specify the slack matrix $\tilde{\mathcal{G}}_1$ as $\tilde{\mathcal{G}}_1^\top = [\tilde{\Lambda}_1 \otimes G^\top]$ ($G \in \mathbb{R}^{n \times n}$ is a nonsingular matrix) and define $X = G^{-1}$, $\Gamma_1 = \text{diag}\underbrace{\{X^\top, \ldots, X^\top\}}_{\hbar_i+1}$. Performing a congruent transformations Γ_1 to (53), $X^\top$ to (47), one can obtain (42) and (43), respectively.

Using the above skills, there exists a slack variable $\tilde{\mathcal{G}}_2$, such that the following inequality holds

$$\mathfrak{I}^\top \begin{bmatrix} \mathcal{I}(\epsilon_1+1)\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1+1) + \mathcal{J}_i^{(b,q)}(\epsilon_1+1, \epsilon_1+1) & * \\ \mathbf{0}_{\hbar_i n \times \hbar_i n} & -\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1) \end{bmatrix} \mathfrak{I} + \mathbf{He}\{\tilde{\mathcal{G}}_2^\top [-\mathbf{I}_n \quad \bar{A}_{i,a,p}]\} < 0, \tag{54}$$

which guarantees (48) hold.

By Schur complement lemma, one has

$$\begin{bmatrix} \Xi_{i,a,p} & * \\ \mathbf{Q}_1 [\mathbf{I}_{\hbar_i n} \ \mathbf{0}_{\hbar_i n \times \hbar_i n}] \mathfrak{I} \frac{\hat{\pi}_i^{(b,q)}(\epsilon_1+1)}{\sqrt{\bar{\upsilon}_i^{(b,q)}}} & -\mathbf{Q}_1 \end{bmatrix} < 0, \tag{55}$$

Table 1
Comparison of the number and dimension of free variables and computational time between Theorems 2 and 4.

Condition	Number and dimension of free variables	Computational time (Example 2)
Theorem 2	$MN\sum_{i=1}^{T}(\delta_i^2+2\delta_i)+1$ n-dimension free variables	0.0786 s
Theorem 4	$MN\sum_{i=1}^{T}(\delta_i^2+\delta_i)$ $(\hbar_i n)$-dimension free variables $MN\sum_{i=1}^{T}\hbar_i+1$ n-dimension free variables	0.8408 s

where $\Xi_{i,a,p}(\epsilon_1)=\mathfrak{I}^\top\Theta_{i,a,p}(\epsilon_1)\mathfrak{I}+\mathbf{He}\{\tilde{\mathcal{G}}_2[-\mathbf{I}_n\quad \bar{A}_{i,a,p}]\}$, $\Theta_{i,a,p}(\epsilon_1)=\begin{bmatrix}\mathcal{I}(\epsilon_1+1)\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1+1) & *\\ \mathbf{0}_{\hbar_i n\times\hbar_i n} & -\hat{\mathcal{J}}_i^{(b,q)}(\epsilon_1)\end{bmatrix}$.

We specify the slack matrix $\mathcal{G}_2$ as $\tilde{\mathcal{G}}_2^\top=[\tilde{\Lambda}_2\otimes G^\top]$ and define $\Gamma_2=\mathrm{diag}\{\underbrace{X^\top,\ldots,X^\top}_{\hbar_i+1},\underbrace{X^\top,\ldots,X^\top}_{MNT}\}$.

Performing a congruent transformations Γ_2 to (55), $X^\top$ to (49), one can obtain (44) and (45), respectively.

The proof is completed. □

Remark 8. It is noted that the dimension of the augmented system matrix is determined by the finite horizon $\hbar_i$. The dimension of augmented system matrix increases as the finite horizon increases. A larger value of $\hbar_i$ means a larger finite memory interval, and the performance of the corresponding memory controller will also be improved. However, this also leads to an increase in the dimension of the Lyapunov matrix $P_{i,a,p}(\iota)$ and computation burden inevitably. Hence, it is necessary to allow for the trade-off between the computation burden and the large of the values of $\hbar_i$.

Remark 9. Similar to Remark 7, in order to reduce the number of the notations, we still use the matrices $\bar{Q}_{i,a,p}(\iota)$, $\bar{\mathcal{J}}_i^{(b,q)}(\nu,\epsilon_1)\in\mathbb{R}^{\hbar_i n}$ in Theorem 4.

Remark 10. In fact, when choose the same sojourn-time upper bound, the dimensions of the closed-loop system matrix subject to Theorem 4 are significantly larger than those in Theorem 2. It can be readily seen in Table 1 that both the number of free variables and the dimension of free variables in Theorem 4 are significantly larger than those in Theorem 2. Hence, Theorem 2 is simpler than Theorem 4 when $\hbar_i$ equals to the sojourn-time upper bound.

Similar to Corollary 1, when only ST-PMF (or EMC) is piecewise homogeneous, Theorem 4 leads to the results given as follows.

Corollary 2. *Given finite constants $h_i\in\mathbb{R}_{>0}$, $\delta_i,\hbar_i\in\mathbb{Z}_{\geq1}$, and scalars $\omega_\zeta(\iota)$, $i\in\mathcal{T}$, $\zeta=1,2$, $\iota=0,1,\ldots,\hbar_i-1$, the closed-loop system (11) with piecewise homogeneous SMC is mean-square stable if there exist sets of matrices $\bar{Q}_{i,a}(\iota,\kappa)>0$, $\bar{\mathcal{J}}_i^{(b)}(\nu,\epsilon_1)>0$ and $\bar{\mathcal{J}}_i^{(b)}(\delta_i)>0$, $i\in\mathcal{T}$, $a,b\in\mathcal{M}$, $\iota\in\mathbb{Z}_{[0,\delta_i-1]}$, $\kappa\in\mathbb{Z}_{[0,\iota]}$, $\epsilon_1\in\mathbb{Z}_{[0,\nu-1]}$, $\nu\in\mathbb{Z}_{[1,\delta_i]}$, and sets of matrices $U_{i,a}(\iota)$ and X, $i\in\mathcal{T}$, $a\in\mathcal{M}$, $\iota\in\mathbb{Z}_{[0,\hbar_i-1]}(\hbar_i\leq\delta_i)$, such that for $\forall i\in\mathcal{T}$, $\forall a\in\mathcal{M}$, $\forall\iota\in\mathbb{Z}_{[1,\delta_i-1]}$ $(\delta_i\in\mathbb{Z}_{\geq2})$, $\kappa\in\mathbb{Z}_{[0,\iota-1]}$, the followings are satisfied*

$$\mathfrak{I}^\top\begin{bmatrix}\bar{Q}_{i,a}(\iota,\kappa+1) & *\\ \mathbf{0}_{\hbar_i n\times\hbar_i n} & -\bar{Q}_{i,a}(\iota,\kappa)\end{bmatrix}\mathfrak{I}+\mathbf{He}\{\tilde{\Lambda}_1\otimes[-X^\top\quad \hat{A}_{i,a}]\}<0, \tag{56}$$

$$\bar{Q}_{i,a}(\iota,0)-h_i\bar{Q}_{i,a}(0,0)<0, \tag{57}$$

and for $\forall i\in\mathcal{T}$, $\forall a,b\in\mathcal{M}$, $\forall\epsilon_1\in\mathbb{Z}_{[0,\delta_i-1]}$ $(\delta_i\in\mathbb{Z}_{\geq1})$, the followings are satisfied

$$\begin{bmatrix}\bar{\Xi}_{i,a}(\epsilon_1) & *\\ \bar{\mathbf{Q}}_2[\mathbf{I}_{\hbar_i n}\ \mathbf{0}_{\hbar_i n\times\hbar_i n}]\mathfrak{I}\frac{\breve{\pi}_i^{(b)}(\epsilon_1+1)}{\sqrt{\bar{\upsilon}_i^{(b)}}} & -\bar{\mathbf{Q}}_2\end{bmatrix}<0, \tag{58}$$

$$\bar{\mathcal{J}}_i^{(b)}(0)-\bar{Q}_{i,a}(0,0)<0, \tag{59}$$

and other notations are the same as in Theorem 4. Moreover, the desired controller gains of (10) are $K_{i,a}(\iota)=U_{i,a}(\iota)X^{-1}$, $i\in\mathcal{T}$, $a\in\mathcal{M}$, $\iota\in\mathbb{Z}_{[0,\hbar_i-1]}$, $\hbar_i\leq\delta_i$.

3.3. Some corollaries on homogeneous system case and time-invariant controller case

If we utilize a class of Lyapunov function independent of the high-level Markov chains, i.e., binary stochastic process $\{(R_n,k_n)\}_{n\in\mathbb{N}}$ is homogeneous. In this time, we choose

$$V(\tilde{x}(k),i,\iota)=\tilde{x}^\top(k)P_i(\iota)\tilde{x}(k),\ \forall i\in\mathcal{T}, \tag{60}$$

then Theorems 2 and 4 will reduce to Corollary 3 presented as follows.

Corollary 3. *Given finite constants $h_i\in\mathbb{R}_{>0}$, $\delta_i,\hbar_i\in\mathbb{Z}_{\geq1}$, $i\in\mathcal{T}$, and scalars $\omega_1,\omega_2,\omega_3,\omega_\zeta(\iota)$, $\zeta=1,2$, $\iota=0,1,\ldots,\hbar_i-1$, the system (6) with homogeneous SMC under two classes of time-varying gain controllers is mean square stable*

- *(elapsed-time-dependent controller) if there exist sets of matrices $\bar{Q}_i(\iota,\kappa)>0$, $\bar{\mathcal{J}}_i(\nu,\epsilon_1)>0$ and $\bar{\mathcal{J}}_i(\delta_i)>0$, $i\in\mathcal{T}$, $\iota\in\mathbb{Z}_{[0,\delta_i-1]}$, $\kappa\in\mathbb{Z}_{[0,\iota]}$, $\epsilon_1\in\mathbb{Z}_{[0,\nu-1]}$, $\nu\in\mathbb{Z}_{[1,\delta_i]}$, and sets of matrices $U_i(\iota)$ and X, $i\in\mathcal{T}$, $\iota\in\mathbb{Z}_{[0,\delta_i-1]}$, such that for $\forall i\in\mathcal{T}$, $\forall\iota\in\mathbb{Z}_{[2,\delta_i-1]}$ $(\delta_i\in\mathbb{Z}_{\geq3})$, $\kappa\in\mathbb{Z}_{[1,\iota-1]}$, the followings are satisfied*

$$\begin{bmatrix}\bar{Q}_i(\iota,\kappa+1) & *\\ \mathbf{0}_{n\times n} & -\bar{Q}_i(\iota,\kappa)\end{bmatrix}+\mathbf{He}\{\Lambda_1\otimes[-X^\top\quad \mathfrak{A}_i(\kappa)]\}<0, \tag{61}$$

$\forall i\in\mathcal{T}$, $\forall\iota\in\mathbb{Z}_{[1,\delta_i-1]}$ $(\delta_i\in\mathbb{Z}_{\geq2})$, there holds

$$\begin{bmatrix}\bar{Q}_i(\iota,1) & *\\ \mathbf{0}_{n\times n} & -h_i\bar{Q}_i(0,0)\end{bmatrix}+\mathbf{He}\{\Lambda_2\otimes[-X^\top\quad \mathfrak{A}_i(0)]\}<0, \tag{62}$$

and for $\forall i\in\mathcal{T}$, $\forall\epsilon_1\in\mathbb{Z}_{[0,\delta_i-1]}$, $(\delta_i\in\mathbb{Z}_{\geq1})$, the followings are satisfied

$$\begin{bmatrix}\bar{\Phi}_i(\epsilon_1) & *\\ \bar{\mathbf{Q}}_3[\mathbf{I}_n\ \mathbf{0}_{n\times n}]\frac{\bar{\pi}_i(\epsilon_1+1)}{\sqrt{\bar{\upsilon}_i}} & -\bar{\mathbf{Q}}_3\end{bmatrix}<0, \tag{63}$$

$$\bar{\mathcal{J}}_i(0)-\bar{Q}_i(0,0)<0. \tag{64}$$

- *(finite memory controller) if there exist sets of matrices $\bar{Q}_i(\iota,\kappa)>0$, $\bar{\mathcal{J}}_i(\nu,\epsilon_1)>0$ and $\bar{\mathcal{J}}_i(\delta_i)>0$, $i\in\mathcal{T}$, $\iota\in\mathbb{Z}_{[0,\delta_i-1]}$, $\kappa\in\mathbb{Z}_{[0,\iota]}$, $\epsilon_1\in\mathbb{Z}_{[0,\nu-1]}$, $\nu\in\mathbb{Z}_{[1,\delta_i]}$, and sets of matrices $U_i(\iota)$ and X, $i\in\mathcal{T}$, $\iota\in\mathbb{Z}_{[0,\hbar_i-1]}(\hbar_i\leq\delta_i)$, such that for $\forall i\in\mathcal{T}$, $\forall\iota\in\mathbb{Z}_{[1,\delta_i-1]}$ $(\delta_i\in\mathbb{Z}_{\geq2})$, $\kappa\in\mathbb{Z}_{[0,\iota-1]}$, the followings are satisfied*

$$\mathfrak{I}^\top\begin{bmatrix}\bar{Q}_i(\iota,\kappa+1) & *\\ \mathbf{0}_{\hbar_i n\times\hbar_i n} & -\bar{Q}_i(\iota,\kappa)\end{bmatrix}\mathfrak{I}+\mathbf{He}\{\tilde{\Lambda}_1\otimes[-X^\top\quad \hat{A}_i]\}<0, \tag{65}$$

$$\bar{Q}_i(\iota,0)-h_i\bar{Q}_i(0,0)<0, \tag{66}$$

and for $\forall i \in \mathcal{T}$, $\forall \epsilon_1 \in \mathbb{Z}_{[0,\delta_i-1]}$ $(\delta_i \in \mathbb{Z}_{\geq 1})$, *the followings are satisfied*

$$\begin{bmatrix} \bar{\Xi}_i(\epsilon_1) & * \\ \bar{\mathbf{Q}}_3[\mathbf{I}_{\hbar_i n} \ \mathbf{0}_{\hbar_i n\times \hbar_i n}]\mathfrak{I}\frac{\tilde{\boldsymbol{\pi}}_i(\epsilon_1+1)}{\sqrt{\bar{\upsilon}_i}} & -\bar{\mathbf{Q}}_3 \end{bmatrix} < 0, \tag{67}$$

$$\bar{\mathcal{J}}_i(0) - \bar{Q}_i(0,0) < 0, \tag{68}$$

where

$$\bar{\mathbf{Q}}_3 = diag\{\bar{Q}_1(0,0), \bar{Q}_2(0,0), \ldots, \bar{Q}_T(0,0)\},$$
$$\tilde{\boldsymbol{\pi}}_i(\epsilon_1) = [\sqrt{\pi_{i1}(\epsilon_1)}\mathbf{I}_m, \sqrt{\pi_{i2}(\epsilon_1)}\mathbf{I}_m, \ldots, \sqrt{\pi_{iT}(\epsilon_1)}\mathbf{I}_m],$$
$$m = n, \hbar_i n,$$

and other notations have been defined in Theorems 2 and 4. Moreover, the desired controller gains of (7)/(10) are $K_i(\iota) = U_i(\iota)X^{-1}$, $i \in \mathcal{T}$, $\iota \in \mathbb{Z}_{[0,\hbar_i-1]}$, $\hbar_i \leq \delta_i$.

Remark 11. It is should be pointed that system (6) become a homogeneous SMJLSs in Corollary 3, and the form of ST-PMF changes from $\rho_{ij}^{(\phi_{k_{n+1}})}(\nu)$ to $\rho_{ij}(\nu)$, and TPs of EMC changes form $\chi_{ij}^{(\varphi_{k_{n+1}})}$ to χ_{ij}. If the sojourn time of each mode obeys geometric distribution, that is, $\rho_{ij}(\nu) = \eta_i(1-\eta_i)^{\nu-1}$, $\eta_i \in (0,1)$, then system (6) will become an standard homogeneous Markov jump linear systems (MJLSs). As a result, the criteria developed for piecewise homogeneous SMJLSs are also applicable to homogeneous SMJLSs and MJLSs.

If we utilize a class of elapsed-time-independent controller for system (6), i.e., time invariant state-feedback controller (as a special case of controller (7) and (10)), the form is described as follows:

$$u(k) = K_{i,a,p}x(k), \tag{69}$$

then Theorems 2 and 4 will reduce to Corollary 4 presented as follows.

Corollary 4. *Given finite constants* $h_i \in \mathbb{R}_{>0}$, $\delta_i \in \mathbb{Z}_{\geq 1}$, $i \in \mathcal{T}$, *and scalars* ω_ζ, $\zeta = 1, 2$, *the system (6) with piecewise homogeneous SMC under controller (69) is mean square stable, if there exist sets of matrices* $\bar{Q}_{i,a,p}(\iota,\kappa) > 0$, $\bar{\mathcal{J}}_i^{(a,p)}(\nu,\epsilon_1) > 0$ *and* $\bar{\mathcal{J}}_i^{(a,p)}(\delta_i) > 0$, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$, $\iota \in \mathbb{Z}_{[0,\delta_i-1]}$, $\kappa \in \mathbb{Z}_{[0,\iota]}$, $\epsilon_1 \in \mathbb{Z}_{[0,\nu-1]}$, $\nu \in \mathbb{Z}_{[1,\delta_i]}$, *and sets of matrices* $U_{i,a,p}$ *and* X, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$, *such that for* $\forall i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$, $\forall \iota \in \mathbb{Z}_{[1,\delta_i-1]}$ $(\delta_i \in \mathbb{Z}_{\geq 2})$, $\kappa \in \mathbb{Z}_{[0,\iota-1]}$, *the followings are satisfied*

$$\begin{bmatrix} \bar{Q}_{i,a,p}(\iota,\kappa+1) & * \\ \mathbf{0}_{n\times n} & -\bar{Q}_{i,a,p}(\iota,\kappa) \end{bmatrix} + \mathbf{He}\{\Lambda_1 \otimes [-X^\top \quad \mathfrak{A}_{i,a,p}]\} < 0, \tag{70}$$

$$\bar{Q}_{i,a,p}(\iota,0) - h_i\bar{Q}_{i,a,p}(0,0) < 0, \tag{71}$$

and for $\forall i \in \mathcal{T}$, $\forall a,b \in \mathcal{M}$, $\forall p,q \in \mathcal{N}$, $\forall \epsilon_1 \in \mathbb{Z}_{[0,\delta_i-1]}$, $(\delta_i \in \mathbb{Z}_{\geq 1})$, *the followings are satisfied*

$$\begin{bmatrix} \tilde{\Phi}_{i,a,p}(\epsilon_1) & * \\ \bar{\mathbf{Q}}_1[\mathbf{I}_n \ \mathbf{0}_{n\times n}]\frac{\hat{\pi}_i^{(b,q)}(\epsilon_1+1)}{\sqrt{\bar{\upsilon}_i^{(b,q)}}} & -\bar{\mathbf{Q}}_1 \end{bmatrix} < 0, \tag{72}$$

$$\bar{\mathcal{J}}_i^{(b,q)}(0) - \bar{Q}_{i,a,p}(0,0) < 0. \tag{73}$$

where $\tilde{\Phi}_{i,a,p}(\epsilon_1) = \bar{\Theta}_{i,a,p}(\epsilon_1) + \mathbf{He}\{\Lambda_2 \otimes [-X^\top \quad \mathfrak{A}_{i,a,p}]\}$ *and other notations have been defined in Theorem 2. The desired controller gains of (69) are* $K_{i,a,p} = U_{i,a,p}X^{-1}$, $i \in \mathcal{T}$, $a \in \mathcal{M}$, $p \in \mathcal{N}$.

4. Illustrative examples

In this section, we provide a numerical example to test the stabilization criteria of Theorem 2. After that, we apply our proposed control synthesis criteria of Theorem 4 to a solar thermal receiver to demonstrate the applicability of the developed theoretical results.

Example 1. Consider the system (6) with piecewise homogeneous SMC under controller (7) of three modes, the parameters as follows:

$$A_1 = \psi\begin{bmatrix} -0.4227 & -0.7710 \\ 0.4600 & -1.6912 \end{bmatrix}, \ A_2 = \psi\begin{bmatrix} -0.9084 & 1.4536 \\ -1.0901 & -0.7266 \end{bmatrix},$$
$$A_3 = \psi\begin{bmatrix} 0.2772 & -0.5313 \\ 1.3938 & -1.5770 \end{bmatrix}, \ B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $\psi > 0$ characterizes the distance between system eigenvalues and unit circle. Without loss of generality, the positive constants h_i are given by [1.8 1.2 1.5]. Two high-level Markov chains ϕ_k and φ_k are considered to be the same stochastic process, and taking value in index set $\{1,2,3\}$.

Three TPs of piecewise homogeneous EMC are given by:

$$\begin{bmatrix} 0 & 0.85 & 0.15 \\ 0.2 & 0 & 0.8 \\ 0.4 & 0.6 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0.79 & 0.21 \\ 0.45 & 0 & 0.55 \\ 0.65 & 0.35 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0.36 & 0.64 \\ 0.25 & 0 & 0.75 \\ 0.3 & 0.7 & 0 \end{bmatrix}.$$

Piecewise homogeneous ST-PMFs are

$$\text{ST-PMFs}_1 = \begin{bmatrix} 0 & \frac{0.6^\nu\cdot 0.4^{10-\nu}\cdot 10!}{(10-\nu)!\cdot\nu!} & 0.2\cdot 0.8^{\nu-1} \\ 0.1^{(t-1)^2} - 0.1^{t^2} & 0 & \frac{0.5^8\cdot 8!}{(8-\nu)!\nu!} \\ 0.4\cdot 0.6^{\nu-1} & 0.3^{(\nu-1)^{0.8}} - 0.3^{\nu^{0.8}} & 0 \end{bmatrix},$$

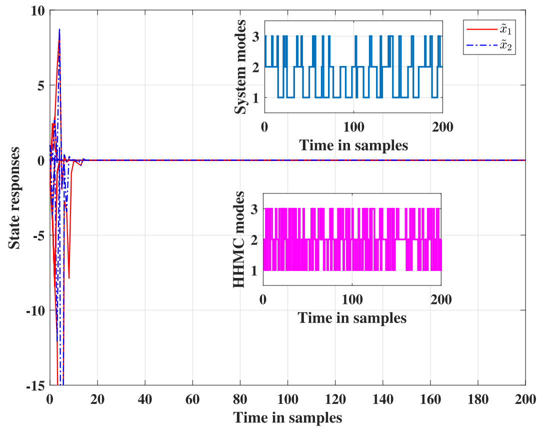
$$\text{ST-PMFs}_2 = \begin{bmatrix} 0 & \frac{0.6^\nu\cdot 0.4^{10-\nu}\cdot 10!}{(10-\nu)!\cdot\nu!} & 0.5^\nu \\ 0.1^{(t-1)^2} - 0.1^{t^2} & 0 & \frac{0.5^8\cdot 8!}{(8-\nu)!\nu!} \\ 0.6\cdot 0.4^{\nu-1} & 0.3^{(\nu-1)^{0.5}} - 0.3^{\nu^{0.5}} & 0 \end{bmatrix},$$

$$\text{ST-PMFs}_3 = \begin{bmatrix} 0 & \frac{0.5^{10}\cdot 10!}{(10-\nu)!\cdot\nu!} & 0.3\cdot 0.7^{\nu-1} \\ 0.1^{(t-1)^2} - 0.1^{t^2} & 0 & \frac{0.5^8\cdot 8!}{(8-\nu)!\nu!} \\ 0.4\cdot 0.6^{\nu-1} & 0.3^{(\nu-1)^{0.8}} - 0.3^{\nu^{0.8}} & 0 \end{bmatrix}.$$

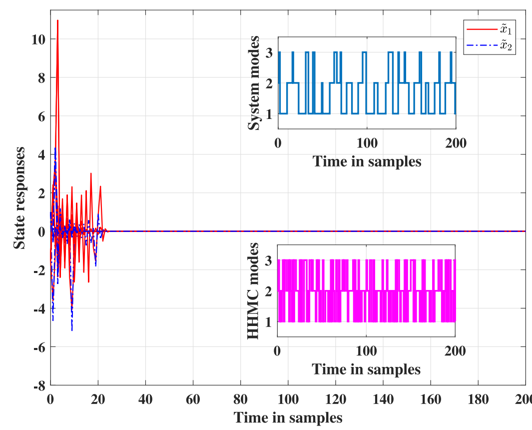
The HHMC TPs matrix is assumed as:

$$\text{HHMC TPs} = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.3 & 0.4 & 0.3 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}. \tag{74}$$

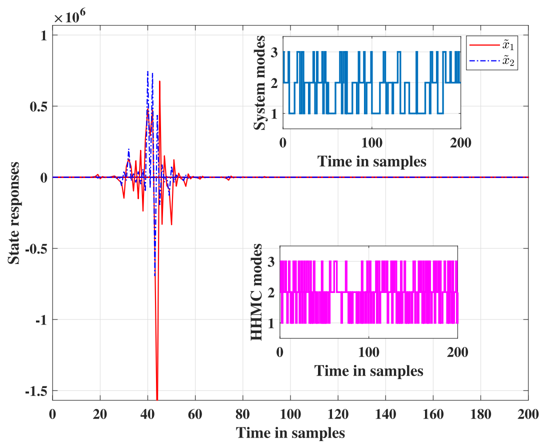
Now, we demonstrate the performance of the proposed control scheme (7). Under the initial condition $\tilde{x}(0) = [-1.5, 1]^\top$, and choose $\Lambda_\zeta = [1,0]^\top$, $\zeta = 1,2,3$. Fig. 1(a) plots 10 realizations of state responses by randomly generating jumping sequences subject to the upper bounds of sojourn time for the three modes are specified as $\delta_1 = 10$, $\delta_2 = 8$, $\delta_3 = 6$, when $\psi = 1$. Fig. 1(b) shows 10 realizations of state responses by randomly generating jumping sequences subject to the upper bounds of sojourn time for the three modes are specified as $\delta_1 = 8$, $\delta_2 = 8$, $\delta_3 = 7$, when $\psi = 1.27$. Fig. 1(c) gives 10 realizations of state responses by randomly generating jumping sequences subject to the upper bounds of sojourn time for the three modes are specified as $\delta_1 = 9$, $\delta_2 = 5$, $\delta_3 = 4$, when $\psi = 1.40$. Fig. 1(d) plots 10 realizations of state responses by randomly generating jumping sequences subject to the upper bounds of sojourn time for the three modes are specified as $\delta_1 = 9$, $\delta_2 = 5$, $\delta_3 = 4$, when $\psi = 1.41$. It is easily found from Fig. 1(a)–(c) that all the curves converge to zero, which verifies the effectiveness of the designed controller with piecewise homogeneous SMK. It can be seen from Fig. 1(d) that the closed-loop system is not stable when $\psi > 1.40$. Therefore, our developed stabilization criteria are effective in the judgement of stabilization of the system. Also, by applying controllers (7)



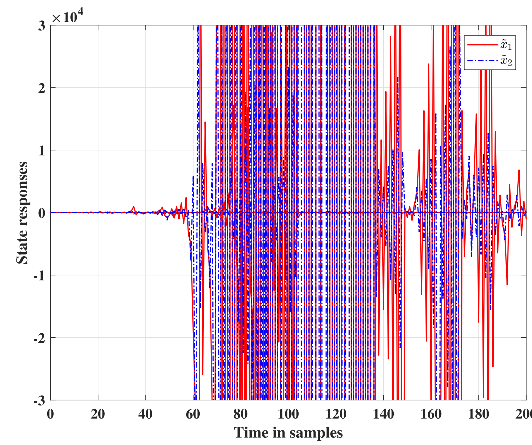
(a) $\psi = 1,\ \delta_1 = 10, \delta_2 = 8, \delta_3 = 6$.



(b) $\psi = 1.27,\ \delta_1 = 8, \delta_2 = 8, \delta_3 = 7$.

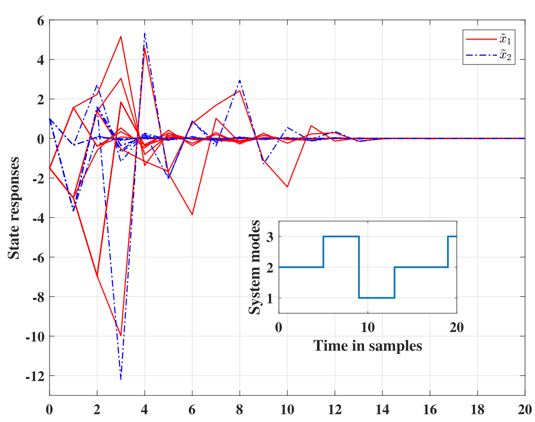


(c) $\psi = 1.40,\ \delta_1 = 9, \delta_2 = 5, \delta_3 = 4$.

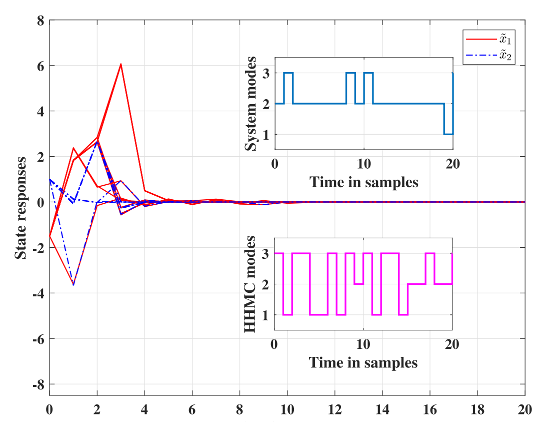


(d) $\psi = 1.41,\ \delta_1 = 9, \delta_2 = 5, \delta_3 = 4$.

Fig. 1. Closed-loop system states evolution of $\tilde{x}(t)$ by controller (7) subject to 10 different randomly generated jumping sequences.



(a) Applying the controller (7) by Corollary 3($\psi = 1$, $\delta_1 = 10, \delta_2 = 8, \delta_3 = 6$).



(b) Applying the controller (69) by Corollary 4($\psi = 1$, $\delta_1 = 10, \delta_2 = 8, \delta_3 = 6$).

Fig. 2. Closed-loop system states evolution of $\tilde{x}(t)$ subject to 10 different randomly generated jumping sequences.

and (69) to homogeneous system case and time-invariant controller case, respectively. We can obtain the state responses of the corresponding closed loop system with 10 randomly generated jumping sequences, as shown in Fig. 2(a)–(b), respectively. The state responses in both figures converge to 0.

Example 2. Considering a solar thermal receiver system borrowed from Costa, Fragoso, and Marques (2006). The plant is basically composed of a set of adjustable mirrors, the heliostats, surrounding a tower with a boiler (see Fig. 3). By controlling the feedwater flow rate to the boiler, the outlet steam temperature

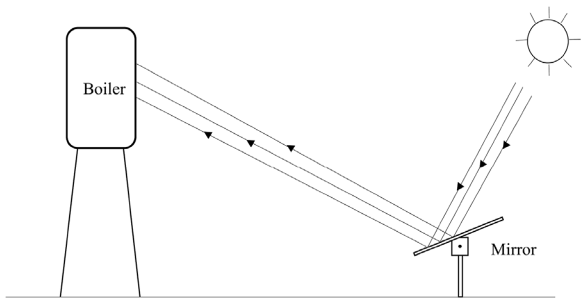


Fig. 3. A solar thermal receiver plant.

can be kept in a suitable range. Cloud movement over the heliostats can cause sudden changes in insolation, which leads to the stochastic nature of the system dynamics. From insolation data collected at the site, it was established that the mean duration of a cloudy period was approximately 2.3 min, while the mean interval of direct insolation was 4.3 min. Based on this information, two operation modes were defined: (1) sunny; and (2) cloudy.

The solar thermal receiver system is described by system (6) with the proposed controllers, and the parameters given in Table 2 for $r_k \in \{1, 2\}$. Two high-level Markov chains ϕ_k and φ_k are considered to be the same stochastic process, and taking value in index set $\{1, 2, 3\}$.

Table 2
Parameters for the solar thermal receiver model.

	Sunny	Cloudy
Plant Parameters	$A_1 = 0.8353$	$A_2 = 0.9646$
	$B_1 = 0.0915$	$B_2 = 0.0982$

The TPs of piecewise homogeneous EMC as follows:

$$\mathrm{TPs}_1 = \mathrm{TPs}_2 = \mathrm{TPs}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Piecewise homogeneous ST-PMFs are

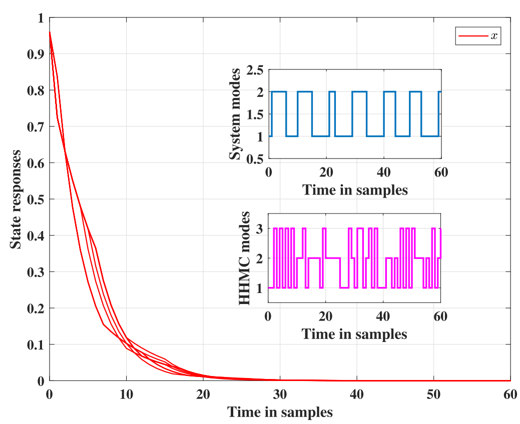
$$\text{ST-PMFs}_1 = \begin{bmatrix} 0 & \frac{0.6^{\nu} \cdot 0.4^{10-\nu} \cdot 10!}{(10-\nu)! \cdot \nu!} \\ \frac{e^{-5} \cdot 5^{\nu}}{\nu!} & 0 \end{bmatrix},$$

$$\text{ST-PMFs}_2 = \begin{bmatrix} 0 & \frac{0.6^{\nu} \cdot 0.4^{10-\nu} \cdot 10!}{(10-\nu)! \cdot \nu!} \\ \frac{e^{-6} \cdot 6^{\nu}}{\nu!} & 0 \end{bmatrix},$$

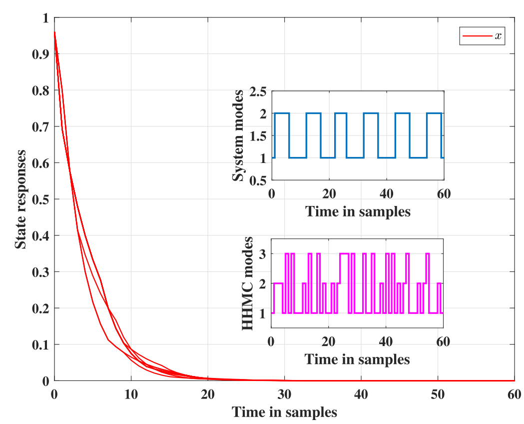
$$\text{ST-PMFs}_3 = \begin{bmatrix} 0 & \frac{0.5^{10} \cdot 10!}{(10-\nu)! \cdot \nu!} \\ \frac{e^{-5} \cdot 5^{\nu}}{\nu!} & 0 \end{bmatrix},$$

and the HHMC TPs matrix is assumed as (74).

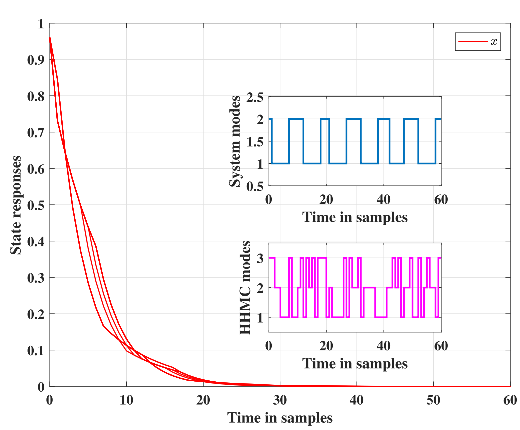
Choose $[h_1, h_2] = [500, 0.03]$, $\delta_1 = 6$, $\delta_2 = 5$, and $\hbar_i = 2$, $\tilde{\Lambda}_{\zeta} = [1, 0, 0]^{\top}$, $\zeta = 1, 2$. By applying controllers (7), (10) and (69), respectively, under the initial condition $x(0) = 0.96$, Fig. 4 depicts the system state responses of the closed-loop solar thermal receiver system with 10 generated jumping sequences



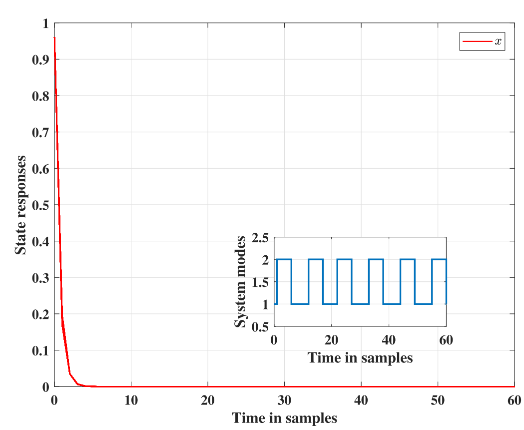
(a) Applying the controller (7) by Theorem 2.



(b) Applying the controller (10) by Theorem 4.



(c) Applying the controller (69) by Corollary 4.



(d) Applying the controller (10) by Corollary 3(finite memory controller).

Fig. 4. Closed-loop solar thermal receiver system states evolution of $x(t)$ subject to 10 different randomly generated jumping sequences.

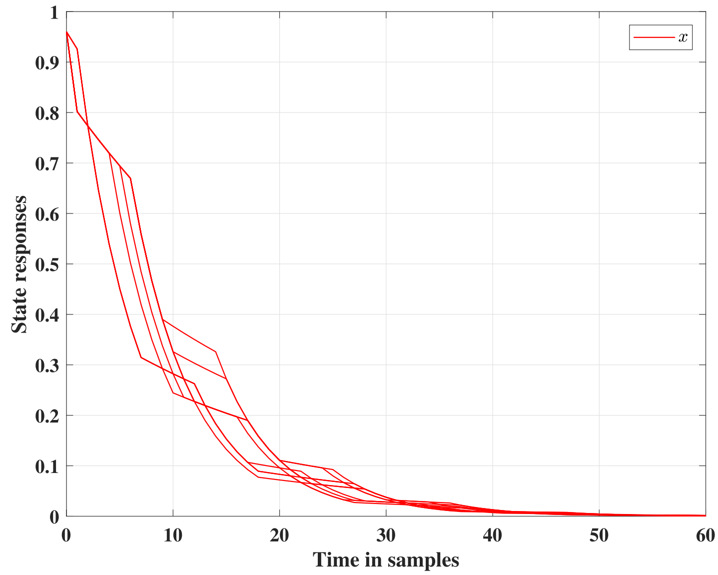


Fig. 5. Open-loop solar thermal receiver system states evolution of $x(t)$ subject to 10 different randomly generated jumping sequences.

subject to the given distributions. It is easily found that, compared with the open loop system curves in Fig. 5, all the curves in Fig. 4 can converge to 0 faster. Under equal conditions, it can be clearly observed from Figs. 4–5 that the designed controller is effective. By the inspection of Fig. 4(a)–(c), it is easily observed that the closed-loop solar thermal receiver system curves with applying controller (10) achieve faster convergence rate than that with applying controller (7)/(69). Besides, subfigure of Fig. 4 gives the possible time sequences evolution of the values of system modes and HHMC modes.

Also, to verify the applicability of Corollary 3 in real systems, controller (10) is applied to homogeneous SMJLSs case. It is easily found that, compared with the open loop system curves in Fig. 5, all the curves in Fig. 4(d) can converge to 0 faster.

5. Conclusion

In this paper, the stabilization analysis problem for a class of discrete-time piecewise homogeneous SMJLSs have been addressed via time-varying gain state feedback controllers. A more general scenario is considered that the SMK is piecewise homogeneous, which covers conventional homogeneous SMK and TPs of piecewise homogeneous MJSs. The mode switchings are governed by a piecewise homogeneous semi-Markov chain with finite sojourn time. By virtue of MD-ETD-VD Lyapunov matrix, numerically testable stabilization criterion has been obtained for the closed-loop resulting systems and two classes of time-varying gain controllers design methods are also given. Finally, a numerical example and a solar thermal receiver system are given to shown the effectiveness and applicability of the proposed time-varying gain controller synthesis methodology. The future works will extend to studying the state estimation problem of discrete-time piecewise homogeneous SMJLSs.

Acknowledgments

This work is supported by National Natural Science Foundation of China (62103146, 62073143, 61922063, 62003139), Program of Shanghai Academic Research Leader, China (19XD1421000), Shanghai Shuguang Project, China (18SG18), China Postdoctoral Science Foundation Project (2020TQ0096, 2021M690056), Shanghai and Hong Kong-Macao-Taiwan Science and Technology Cooperation Project, China (19510760200), Shanghai Natural Science Foundation, China (20ZR1415200), and Innovation Program of Shanghai Municipal Education Commission, China (2021-01-07-00-02-E00107).

References

Aberkane, S. (2012). Bounded real lemma for non-homogeneous Markovian jump linear systems. *IEEE Transactions on Automatic Control*, *58*(3), 797–801.

Balluchi, A., Benvenuti, L., Di Benedetto, M. D., & Sangiovanni-Vincentelli, A. (2013). The design of dynamical observers for hybrid systems: Theory and application to an automotive control problem. *Automatica*, *49*(4), 915–925.

Barbu, V., & Limnios, N. (2006). Empirical estimation for discrete-time semi-Markov processes with applications in reliability. *Nonparametric Statistics*, *18*(7–8), 483–498.

Basin, M. V., & Maldonado, J. J. (2014). Optimal controller for uncertain stochastic linear systems with Poisson noises. *IEEE Transactions on Industrial Informatics*, *10*(1), 267–275.

Bolzern, P., Colaneri, P., & De Nicolao, G. (2014). Stochastic stability of positive Markov jump linear systems. *Automatica*, *50*(4), 1181–1187.

Boukas, E.-K. (2006). *Stochastic switching systems: Analysis and design*. Springer.

Boyd, S., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory, Volume 15*. Philadelphia, PA: SIAM.

Costa, O. L. V., Fragoso, M. D., & Marques, R. P. (2006). *Discrete-time Markov jump linear systems*. Springer Science & Business Media.

Costa, O. L. V., do Val, J. B. R., & Geromel, J. C. (1999). Continuous-time state-feedback H_2-control of Markovian jump linear systems via convex analysis. *Automatica*, *35*(2), 259–268.

D'Amico, G. (2009). The crossing barrier of a non-homogeneous semi-Markov chain. *Stochastics. An International Journal of Probability and Stochastic Processes*, *81*(6), 589–600.

Ebihara, Y., Peaucelle, D., & Arzelier, D. (2011). Periodically time-varying memory state-feedback controller synthesis for discrete-time linear systems. *Automatica*, *47*(1), 14–25.

El Ghaoui, L., & Scorletti, G. (1996). Control of rational systems using linear-fractional representations and linear matrix inequalities. *Automatica*, *32*(9), 1273–1284.

Fei, Z., Shi, S., Wang, Z., & Wu, L. (2017). Quasi-time-dependent output control for discrete-time switched system with mode-dependent average dwell time. *IEEE Transactions on Automatic Control*, *63*(8), 2647–2653.

Janssen, J., & Manca, R. (2006). *Applied semi-Markov processes*. Springer Science & Business Media.

Jiang, B., Kao, Y., Gao, C., & Yao, X. (2017). Passification of uncertain singular semi-Markovian jump systems with actuator failures via sliding mode approach. *IEEE Transactions on Automatic Control*, *62*(8), 4138–4143.

Jiang, B., Kao, Y., Karimi, H. R., & Gao, C. (2018). Stability and stabilization for singular switching semi-Markovian jump systems with generally uncertain transition rates. *IEEE Transactions on Automatic Control*, *63*(11), 3919–3926.

Kwon, W. H., & Han, S. (2004). Receding horizon finite memory controls for output feedback controls of state-space systems. *IEEE Transactions on Automatic Control*, *49*(11), 1905–1915.

Li, H., Shi, P., Yao, D., & Wu, L. (2016). Observer-based adaptive sliding mode control for nonlinear Markovian jump systems. *Automatica*, *64*, 133–142.
Liu, M., Zhang, L., & Zheng, W. X. (2017). Fault reconstruction for stochastic hybrid systems with adaptive discontinuous observer and non-homogeneous differentiator. *Automatica*, *85*, 339–348.
Ma, X., Djouadi, S. M., & Li, H. (2012). State estimation over a semi-Markov model based cognitive radio system. *IEEE Transactions on Wireless Communication*, *11*(7), 2391–2401.
Mathieu, E., Foucher, Y., Dellamonica, P., & Daures, J. P. (2007). Parametric and non homogeneous semi-Markov process for HIV control. *Methodology and Computing in Applied Probability*, *9*(3), 389–397.
Mu, X., & Hu, Z. (2020). Stability analysis for semi-Markovian switched singular stochastic systems. *Automatica*, *118*, Article 109014.
Mudge, T. N., & Al-Sadoun, H. B. (1985). A semi-Markov model for the performance of multiple-bus systems. *IEEE Transactions on Computers*, *100*(10), 934–942.
Ning, Z., Zhang, L., & Lam, J. (2018). Stability and stabilization of a class of stochastic switching systems with lower bound of sojourn time. *Automatica*, *92*, 18–28.
Ning, Z., Zhang, L., Mesbah, A., & Colaneri, P. (2020). Stability analysis and stabilization of discrete-time non-homogeneous semi-Markov jump linear systems: A polytopic approach. *Automatica*, *120*, Article 109080.
Papadopoulou, A. A. (2013). Some results on modeling biological sequences and web navigation with a semi-Markov chain. *Communications in Statistics. Theory and Methods*, *42*(16), 2853–2871.
Papadopoulou, A. A., & Vassiliou, P. C. (1994). Asymptotic behavior of nonhomogeneous semi-Markov systems. *Linear Algebra and its Applications*, *210*, 153–198.
Rolink, J., & Rehtanz, C. (2013). Large-scale modeling of grid-connected electric vehicles. *IEEE Transactions on Power Delivery*, *28*(2), 894–902.
Shen, Y., Wu, Z. G., Shi, P., Shu, Z., & Karimi, H. R. (2019). H_∞ Control of Markov jump time-delay systems under asynchronous controller and quantizer. *Automatica*, *99*, 352–360.
Tian, Y., Yan, H., Zhang, H., Cheng, J., & Shen, H. (2021). Asynchronous output feedback control of hidden semi-markov jump systems with random mode-dependent delays. *IEEE Transactions on Automatic Control*, http://dx.doi.org/10.1109/TAC.2021.3110006.
Tian, Y., Yan, H., Zhang, H., Zhan, X., & Peng, Y. (2020a). Dynamic output-feedback control of linear semi-Markov jump systems with incomplete semi-Markov kernel. *Automatica*, *117*, Article 108997.
Tian, Y., Yan, H., Zhang, H., Zhan, X., & Peng, Y. (2020b). Resilient static output feedback control of linear semi-Markov jump systems with incomplete semi-Markov kernel. *IEEE Transactions on Automatic Control*, *66*(9), 4274–4281.
Trégouët, J. F., Peaucelle, D., Arzelier, D., & Ebihara, Y. (2013). Periodic memory state-feedback controller: New formulation, analysis, and design results. *IEEE Transactions on Automatic Control*, *58*(8), 1986–2000.
Vargas, A. N., Costa, E. F., Acho, L., & do Val, J. B. (2017). Switching stochastic nonlinear systems with application to an automotive throttle. *IEEE Transactions on Automatic Control*, *63*(9), 3098–3104.
Vassiliou, P. C., & Papadopoulou, A. A. (1992). Non-homogeneous semi-Markov systems and maintainability of the state sizes. *Journal of Applied Probability*, *29*(3), 519–534.
Vasslllou, P. C., & Tsantas, N. (1984). Stochastic control in non-homogeneous Markov systems. *International Journal of Computational Methods*, *16*(3), 139–155.
Wang, M., Qiu, J., & Feng, G. (2018). Finite frequency memory output feedback controller design for T-S fuzzy dynamical systems. *IEEE Transactions on Fuzzy Systems*, *26*(6), 3301–3313.
Xiong, J., Lam, J., Gao, H., & Ho, D. W. (2005). On robust stabilization of Markovian jump systems with uncertain switching probabilities. *Automatica*, *41*(5), 897–903.
Xu, Z., Wu, Z. G., Su, H., Shi, P., & Que, H. (2019). Energy-to-peak filtering of semi-Markov jump systems with mismatched modes. *IEEE Transactions on Automatic Control*, *65*(10), 4356–4361.
Xue, M., Yan, H., Zhang, H., Shen, H., & Shi, P. (2021). Dissipativity-based filter design for Markov jump systems with packet loss compensation. *Automatica*, *133*, Article 109843.
Yuan, C., & Wu, F. (2017). Exact-memory and memoryless control of linear systems with time-varying input delay using dynamic IQCs. *Automatica*, *77*, 246–253.
Zhang, L. (2009). H_∞ Estimation for discrete-time piecewise homogeneous Markov jump linear systems. *Automatica*, *45*(11), 2570–2576.
Zhang, L., Cai, B., & Shi, Y. (2019). Stabilization of hidden semi-Markov jump systems: Emission probability approach. *Automatica*, *101*, 87–95.
Zhang, L., Cai, B., Tan, T., & Shi, Y. (2020). Stabilization of non-homogeneous hidden semi-Markov Jump systems with limited sojourn-time information. *Automatica*, *117*, Article 108963.
Zhang, L., Leng, Y., & Colaneri, P. (2016a). Stability and stabilization of discrete-time semi-Markov jump linear systems via semi-Markov kernel approach. *IEEE Transactions on Automatic Control*, *61*(2), 503–508.
Zhang, L., Yang, T., & Colaneri, P. (2016b). Stability and stabilization of semi-Markov jump linear systems with exponentially modulated periodic distributions of sojourn time. *IEEE Transactions on Automatic Control*, *62*(6), 2870–2885.
Zhang, L., Zhuang, S., & Shi, P. (2015). Non-weighted quasi-time-dependent H_∞ filtering for switched linear systems with persistent dwell-time. *Automatica*, *54*, 201–209.

Yongxiao Tian received the Ph.D. degree in control science and engineering with the East China University of Science and Technology, Shanghai, China, in 2020. He is currently a Postdoctoral Researcher with East China University of Science and Technology, Shanghai. His current research interests include stochastic jump systems, and networked control systems.

Huaicheng Yan received his B.Sc. degree in automatic control from Wuhan University of Technology, China, in 2001, and the Ph.D. degree in control theory and control engineering from Huazhong University of Science and Technology, China, in 2007. From 2007 to 2009, he was a Postdoctoral Fellow with the Chinese University of Hong Kong. Currently, he is a Professor with the School of Information Science and Engineering, East China University of Science and Technology, Shanghai, China. He has published over 200 papers in the peer-reviewed international journals. He is an associate editor for *IEEE Transactions on Neural Networks and Learning Systems*, *International Journal of Robotics and Automation* and *IEEE Open Journal of Circuits and Systems, etc*. His research interests include networked control systems, multi-agent systems, stochastic systems and robotics.

Hao Zhang received the B.Sc. degree in automatic control from Wuhan University of Technology, Wuhan, China, in 2001 and received Ph.D. degree in control theory and control engineering from Huazhong University of Science and Technology Wuhan, China, in 2007. Currently, she is a Professor with the School of Electronics and Information Engineering, Tongji University, Shanghai, China. From December 2011 to December 2013. She is also a Postdoctoral Fellow with the City University of Hong Kong, Hong Kong. Prof. Zhang was a recipient of the National Science Fund for Excellent Young Scholars of China and also a Changjiang Young Scholar Distinguished Professor in 2019. She is an associate editor for *IEEE Intelligent Transportation Systems Magazine*. Her research interests include network-based control systems and multi-agent systems.

Meng Wang received the B.Eng. degree in Automation from Northeastern University at Qinhuangdao, China, in 2011, and the M.Eng. degree in Control Science and Engineering from Harbin Institute of Technology, Harbin, China, in 2013, and the Ph.D. degree in the Department of Mechanical and Biomedical Engineering, City University of Hong Kong, Kowloon, Hong Kong SAR, China, in 2018. He is currently an Associate Professor with School of Information Science and Engineering, East China University of Science and Technology, Shanghai, China. His research interests include robust control and filtering, fuzzy systems and control, and their engineering applications.

Jianjun Yi received the B.S. degree in mechanical manufacturing engineering, the M.S. degree in automotive engineering, and the Ph.D. degree in mechatronics engineering from the Dalian University of Technology (DLUT), Dalian, China, in 1991, 1996, and 1999, respectively. In 2001, he joined the East China University of Science and Technology (ECUST), Shanghai, China, where he has developed research activities in mechatronics, robotics, image processing, intelligent control, and the Internet of Things (IoT). He has been involved in teaching activities in automatic control and mechatronic engineering, as an Associate Professor since 2001 and a Professor since 2008. His current technical areas include the IoT, image processing, intelligent control, robotics, and wireless sensor networks (WSNs).