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H_∞-Control for Markovian Jumping Linear Systems with Parametric Uncertainty[1]

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Abstract. This paper studies the problem of H_∞-control for linear systems with Markovian jumping parameters. The jumping parameters considered here are two separable continuous-time, discrete-state Markov processes, one appearing in the system matrices and one appearing in the control variable. Our attention is focused on the design of linear state feedback controllers such that both stochastic stability and a prescribed H_∞-performance are achieved. We also deal with the robust H_∞-control problem for linear systems with both Markovian jumping parameters and parameter uncertainties. The parameter uncertainties are assumed to be real, time-varying, norm-bounded, appearing in the state matrix. Both the finite-horizon and infinite-horizon cases are analyzed. We show that the control problems for linear Markovian jumping systems with and without parameter uncertainties can be solved in terms of the solutions to a set of coupled differential Riccati equations for the finite-horizon case or algebraic Riccati equations for the infinite-horizon case. Particularly, robust H_∞-controllers are also designed when the jumping rates have parameter uncertainties.

Key Words. H_∞-control, Markovian jumping parameters, Riccati inequalities, state feedback, uncertain systems.

1. Introduction

Many physical systems have variable structures subject to random changes, which may result from abrupt phenomena such as component and

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interconnection failures, parameters shifting, tracking, and time required to measure some of the variables at different stages. Systems with this character may be modeled as hybrid ones, i.e., the state space of the systems contains both discrete and continuous states. Among this kind of systems, fault-tolerant control (FTC) systems have been a subject of great practical importance, which has attracted a lot of interest for the last three decades; see Refs. 1–3 and references therein. FTC systems have been developed in order to achieve high levels of reliability and performance in situations where the controlled system can have potentially damaging effects on the environment if failures of its components take place. Such a control system is designed to retain some portion of its control integrity in the event of a specified set of possible component failures[4] or large changes in the system operating conditions that resemble these failures.

FTC system design consists of two parts: passive design and active design. A passive FTC system can tolerate one or more component failures, while satisfactorily accomplishing its mission without reconfiguring itself. An active FTC system design involves automatically detecting and identifying the failed components and then reconfiguring the control law on-line in response to these decisions (Ref. 2). The dynamic behavior of active fault-tolerant control systems is governed by stochastic differential equations, because failures and failure detection decisions occur randomly. One of the most important stochastic differential equations is the one with Markovian jumping parameters, which has been widely used in jumping systems. As is well known, the dynamics of discrete and continuous states in jumping systems are respectively modeled by a finite-state Markov chain and linear differential equations subject to discrete process. Achievements have been made in jumping linear quadratic (JLQ) control theory, since the pioneering work on JLQ control (Ref. 4). The JLQ control problem was solved in Ref. 5 using the stochastic maximum principle for state feedback in the finite-horizon case. Reference 6 also obtained the same results using dynamic programming for both the finite-horizon and infinite-horizon cases. Reference 7 provided an approach to the output feedback JLQ control problem. The continuous-time partially observable situation was studied in Ref. 8. An analysis of the discrete-time version of the JLQ control problem was given in Ref. 9 in the absence of driving noise. Recently, the problems of controllability, stabilizability, and continuous-time JLQ control have been theoretically addressed in Ref. 10 and references therein. The stochastic stability properties of jumping linear systems have been investigated systematically in Ref. 11, showing that several normally used stochastic stability

[4]We mean a change in the operating behavior of a component such that the new behavior differs significantly from the nominal behavior for that component.

concepts are equivalent. Very recently, Ref. 12 tackled the JLQ problem when the discrete Markov process in systems is not directly observable and obtained necessary conditions for optimality. More recently, discrete-time LQ-optimal control problems for infinite Markov jump parameter systems have been studied in Ref. 13 in connection with both models without noise and with noise. The issue of the robust stability and control of Markov jumping linear continuous-time systems has been investigated in Refs. 14–16. In the counterpart of H_∞-control for systems with Markovian jumping parameters, results for continuous-time and discrete-time systems have been obtained in Refs. 17 and 18, respectively. Using averaging and aggregation techniques, Ref. 19 studied the H_∞-control design for large-scale jump linear systems where the form process admits strong and weak interactions. However to date, the design of H_∞-controllers for FTC linear systems has not been investigated fully.

On the other hand, the design of control systems that can handle model uncertainties has been one of the most challenging problems, receiving considerable attention from control engineers and scientists in the past decades. Various approaches to the problem of robust control design for uncertain dynamical systems have been proposed, for example, robust stabilization, sensitive minimization, H_∞-control, and loop-transfer recovery; see Refs. 20–23 and references therein.

Since the pioneering work on so-called H_∞-optimal control theory (Ref. 23), there has been a dramatic progress in H_∞-control theory in the past few years. Both continuous-time and discrete-time systems have been studied intensively. The essential idea of H_∞-control is to design a controller to optimize the closed-loop system performance for the worst exogenous input. The goal in H_∞-control is to design a controller such that the H_∞-norm of the transfer function from the disturbance input to the controlled output is minimized. Many familiar robust control problems can be recast as H_∞-control problems. It was shown in Ref. 24 that the state feedback H_∞-control can be solved in terms of an algebraic Riccati equation. In a seminal paper (Ref. 25), the state-space solution to the output feedback H_∞-control problem was developed. Similarly to the linear-quadratic Gaussian control problem, the output feedback H_∞-control can be solved in terms of two Riccati equations.

In this paper, the problem of fault-tolerant control of linear systems with Markovian jumping parameters is investigated in an H_∞-sense. We study control systems that have automatic failure monitoring capability in order to reorganize or reconfigure the control law in real time in response to failure indications. In the systems we are concerned with, random variations occurring in the system description are modeled as failures with Markovian transition characteristics, i.e., a stochastic differential equation description.

Also, there is an additional random process, i.e., failure detection and identification process, that induces random variations in the control law which changes the system description through the control reconfiguration strategy. Linear dynamic state feedback control design methodologies are proposed. Robust control for the above systems with norm-bounded, time-varying parameter uncertainty in the state matrix are also tackled. Our attention is on the problem of robust H_∞-control, where the controller is required to guarantee the prescribed H_∞-performance, irrespective of the parameter uncertainty. Both the finite-horizon and infinite-horizon cases are studied. The main results of this paper establish that the H_∞-fault-tolerant control problem for Markovian jumping systems can be solved in terms of the solutions to a set of coupled Riccati equations.

Notation. The notations in this paper are quite standard. The superscript T denotes the transpose; the notation $X \geq Y$ [respectively, $X > Y$], where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite [respectively, positive definite]; I is the identity matrix; $\mathscr{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure P; $L^2[0, T]$ stands for the space of square-integrable vector functions over the interval $[0, T]$; $\|\cdot\|$ refers to either the Euclidean vector norm or the matrix norm, which is the operator norm induced by the standard vector norm; $\|\cdot\|_2$ stands for the norm in $L^2[0, T]$, while $\|\cdot\|_{E_2}$ denotes the norm in $L^2((\Omega, \mathscr{F}, P), [0, T])$; $(\Omega, \mathscr{F}, P)$ is a probability space.

2. Problem Formulation and Preliminaries

Consider the following class of dynamical system in a fixed complete probability space $(\Omega, \mathscr{F}, P)$:

$$\dot{x}(t) = A(t, \eta_t)x(t) + B_1(t, \eta_t)w(t) + B_2(t, \eta_t)u(t, r_t),$$

$$x(0) = 0, \qquad r_0 = i, \qquad \eta_0 = j, \qquad t \in [0, T], \tag{1}$$

$$z(t) = C(t, \eta_t)x(t) + D(t, \eta_t)u(t, r_t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t, r_t) \in \mathbb{R}^k$ is control input, $w(t) \in \mathbb{R}^p$ is the disturbance input which belongs to $L^2[0, T]$, $z(t) \in \mathbb{R}^m$ is the controlled output which belongs to $L^2((\Omega, \mathscr{F}, P), [0, T)]$.

$\{r_t, t \geq 0\}$ denotes the state of the failure detection and identification process which monitors the state $\{\eta_t, t \geq 0\}$ of the random process describing the failures. Also, the random behavior of the process r_t is conditioned on the failure process state η_t. Furthermore, it is assumed that η_t and r_t are separable, measurable, continuous-time, discrete-state Markov process, with η_t being homogeneous, taking values on finite sets

$$\mathscr{X}=\{1,2,\ldots,\nu\} \quad \text{and} \quad \mathscr{S}=\{1,2,\ldots,\mu\},$$

respectively. Thus, the system description depends upon the true failure state η_t, while the input that is applied to the plant depends upon the control law used in response to the indication by the failure detection and identification process in which the system state is r_t.

As r_t is a Markov process when conditioned on η_t for single-sample failure detection and identification tests, the conditional probability p_{ij}^k that the process will jump from mode i at time t to mode j at time $t+\delta$, with $i, j \in \mathscr{S}$, given that $\eta_t = k \in \mathscr{X}$, is (see Ref. 6)

$$\begin{aligned} p_{ij}^k &= \Pr(r_{t+\delta}=j \mid r_t = i,\ \eta_t = k) \\ &= \begin{cases} \lambda_{ij}^k \delta + o(\delta), & \text{if } i \neq j, \\ 1 + \lambda_{ii}^k \delta + o(\delta), & \text{if } i = j, \end{cases} \end{aligned} \tag{3}$$

with transition probability rates $\lambda_{ij}^k \geq 0$, $i \neq j$, and

$$\lambda_{ii}^k = -\sum_{j=1, j\neq i}^{s} \lambda_{ij}^k. \tag{4}$$

For the failure process η_t, the transition probability from mode k at time t to mode l at time $t+\delta$, with $k, l \in \mathscr{X}$, is given by

$$\begin{aligned} p_{kl} &= \Pr(\eta_{t+\delta}=l \mid \eta_t = k) \\ &= \begin{cases} \alpha_{kl} \delta + o(\delta), & \text{if } k \neq l, \\ 1 + \alpha_{kk} \delta + o(\delta), & \text{if } k = l, \end{cases} \end{aligned} \tag{5}$$

with transition probability rates $\alpha_{kl} \geq 0$, $k \neq l$, and

$$\alpha_{kk} = -\sum_{j=1, j\neq k}^{s} \alpha_{kj} \tag{6}$$

where $\delta > 0$ and $\lim_{\delta \downarrow 0} o(\delta)/\delta = 0$. We can easily show that the joint process $\{x(t), r_t, \eta_t\}$ whose realizations satisfy (1) is a $(n+2)$-dimensional Markov process.

In the above, for $\eta_t = k$, $k \in \mathscr{X}$,

$$A(t,\eta_t) \triangleq A(t,k), \qquad B_1(t,\eta_t) \triangleq B_1(t,k), \qquad B_2(t,\eta_t) \triangleq B_2(t,k),$$
$$C(t,\eta_t) \triangleq C(t,k), \qquad D(t,\eta_t) \triangleq D(t,k)$$

are known, real, time-varying, piecewise-continuous between each jump, bounded matrices of appropriate dimensions that describe the nominal system. For notational simplicity, for a time-varying matrix $X(t)$, we will denote $X(t) = X$ sometimes, wherever no confusion arises regarding the time dependence of this quantity.

Remark 2.1. Note that (1)–(2) is a hybrid system in which the state $x(t)$ takes values continuously, while the states r_t and η_t take values discretely. This kind of system can be used to represent many important physical systems subject to random failures and structure changes, such as electric power systems (Ref. 26), control systems of a solar thermal central receiver (Ref. 27), communications systems (Ref. 28), aircraft flight controls (Ref. 29), control of nuclear power plants (Ref. 30), and manufacturing systems (Refs. 31–35).

Now, we present definitions on stochastic stability and stochastic stabilizability for general jumping parameter systems. Consider the jump linear system

$$\dot{x}(t) = A(\eta_t)x(t) + B(\eta_t)u(t, r_t),$$
$$x(0) = x_0, \qquad r_0 = i, \qquad \eta_0 = j, \qquad t \geq 0, \tag{7}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t, r_t)$ is the control input, $\{\eta_t, t \geq 0\}$ and $\{r_t, t \geq 0\}$ are finite Markov processes as in the system (1), $x_0 \in \mathbb{R}^n$ is an initial state as a fixed nonrandom constant vector; $A(\eta_t)$ and $B(\eta_t)$ are suitable constant matrices when $\eta_t = k$, $k \in \mathscr{Z}$.

Definition 2.1. See Ref. 11. For the system (7) with $B(\eta_t) \equiv 0$, the equilibrium point 0 is stochastically stable if, for any x_0 and $\eta_0 \in \mathscr{Z}$,

$$\int_0^\infty \mathscr{E}\{\|x(t, x_0)\|^2\}\, dt < +\infty.$$

Definition 2.2. See Ref. 10. The system (7) is stochastically stabilizable if, for any x_0, $r_0 \in \mathscr{S}$, and $\eta_0 \in \mathscr{Z}$, there exists a linear feedback gain $L(r_t, \eta_t)$ that is constant for each value of $r_t \in \mathscr{S}$ and $\eta_t \in \mathscr{Z}$ such that the control law

$$u(t) = -L(r_t, \eta_t)x(t),$$

with $\|L(r_t, \eta_t)\| < \infty$, ensures that the resulting closed-loop system is stochastically stable.

In this paper, associated with (1)–(6), we consider the problem of state feedback control for the FTC system (1). We assume that perfect state information is available for feedback, and we are concerned with designing a state feedback controller $\mathscr{G}(r_t, \eta_t)$,

$$u(t) = -L(t, r_t, \eta_t)x(t), \tag{8}$$

such that, for all nonzero $w(t) \in L^2$,

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2, \tag{9}$$

where $\gamma > 0$ is a prescribed level of disturbance attenuation to be achieved and

$$\|z(t)\|_{E_2} = \mathscr{E}\left\{\int_0^T z^T(t)z(t)\,dt\right\}^{1/2}.$$

When (9) is satisfied, the system (1)–(6) with controller (8) is said to have H_∞-performance (9) over the horizon $[0, T]$.

The control problem that we address in this paper is as follows: Design a state feedback controller $\mathscr{G}(r_t, \eta_t)$ such that:

(i) in the finite-horizon case, the system (1) with $\mathscr{G}(r_t, \eta_t)$ has performance (9) over a given horizon $[0, T]$;
(ii) in the infinite-horizon case, the system (1) with $\mathscr{G}(r_t, \eta_t)$ is stochastically stable and has performance (9) over $[0, \infty)$.

We make the following assumption on the system (1)–(6).

Assumption 2.1. For all $\eta_t = k \in \mathscr{Z}$ on $[0, T]$,

$$D^T(t, k)[C(t, k), D(t, k)] = [0, R(t, k)],$$
$$R(t, k) = R^T(t, k) > 0. \tag{10}$$

Remark 2.2. Assumption 2.1 means that the H_∞-control problem for the system of (1)–(6) is nonsingular, which is a standard assumption in the nonsingular H_∞-control problem for the nominal system (1)–(6) without jump parameters r_t and η_t (Ref. 25).

In the case of a control problem on an infinite horizon, $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C(\cdot)$, $D(\cdot)$ are constant matrices when $\eta_k = k$, $k \in \mathscr{Z}$; they are denoted by $A(k)$, $B_1(k)$, $B_2(k)$, $C(k)$, $D(k)$, respectively. We shall adopt the following assumption.

Assumption 2.2. For all $k \in \mathscr{Z}$,

(a) $(A(k)\ B_2(k))$ is stochastically stabilizable,
(b) $(C(k)\ A(k))$ is observable.

Remark 2.3. It should be noted that, under Assumption 2.2(b), Assumption 2.2(a) is necessary and sufficient for the existence of a controller such that the system (1)–(6) is stable; see Ref. 10.

The following result shows that the stochastic stability of the system (7) is equivalent to a set of v coupled algebraic Lyapunov equations.

Lemma 2.1. See Ref. 11. Consider the system (7), with $B(\eta_t) \equiv 0$, $\eta_t \in \mathscr{Z}$. Then, the following statements are equivalent:

(a) the system (7) is stochastically stable;
(b) for any given positive-definite matrices $N(k)$, $k \in \mathscr{Z}$, there exist positive-definite matrices $M(k)$, $k \in \mathscr{Z}$, satisfying

$$A^T(k)M(k) + M(k)A(k) + \sum_{j=1}^{v} \alpha_{kj} M(j) + N(k) = 0, \quad k \in \mathscr{Z}. \tag{11}$$

Remark 2.4. In Ref. 11, it has been proved that, for the system (7), the terms "stochastically stable," "asymptotically mean-square stable," and "exponentially mean-square stable" are equivalent, and any of them can imply "almost surely asymptotically stable." Lemma 2.1 also provides the necessary and sufficient conditions for asymptotical mean-square stability and exponential mean-square stability of the system (7), and sufficient conditions for almost surely asymptotical stability of the system (7). Also note that the left-hand side of (11) being less than zero also implies statement (a) in Lemma 2.1.

Similarly to Lemma 2.1, for the stochastic stabilizability of the system (7), we have the following result.

Lemma 2.2. See Ref. 3. Consider the system (7). Then, the following statements are equivalent:

(a) the system (7) is exponential mean-square stabilizable by a control law

$$u(t) = -L(i, k)x(t), \qquad i \in \mathscr{S}, k \in \mathscr{Z};$$

(b) for any given positive-definite matrices Q_{ik}, $i\in\mathcal{S}$ and $k\in\mathcal{Z}$, there exist positive-definite matrices $P_{ik}>0$, $i\in\mathcal{S}$ and $k\in\mathcal{Z}$, satisfying

$$\tilde{A}_{ik}^T P_{ik} + P_{ik}\tilde{A}_{ik} + \sum_{j\neq i}^{\mu} \lambda_{ij}^k P_{jk} + \sum_{j\neq k}^{\nu} \alpha_{jk} P_{ij} + Q_{ik}$$

$$=0, \qquad \forall i\in\mathcal{S}, k\in\mathcal{Z}, \tag{12}$$

where

$$\tilde{A}_{ik} = A(k) - B(k)L(i,k) - 0.5\sum_{j\neq i}^{\mu} \lambda_{ij}^k - 0.5\sum_{j\neq k}^{\nu} \alpha_{jk}, \qquad i\in\mathcal{S}, k\in\mathcal{Z}.$$

Remark 2.5. Just as Lemma 2.1, Lemma 2.2 shows that the stochastic stabilizability of the system (7) is related to the existence of positive-definite solutions to a set of $\nu\times\mu$ coupled algebraic Lyapunov equations. It should be noted that the left-hand side of (12) being less than zero is also a sufficient condition for exponential mean-square stabilizability of the system (7). Particularly, when $\mu=1$, Lemma 2.2 reduces to Theorem 1 of (Ref. 10).

3. Designing an H_∞-Controller

In this section, we investigate the H_∞-control problem for the system (1)–(6). It is shown that the above problem can be solved in terms of a set of $\nu\times\mu$ coupled Riccati inequalities. Both the finite-horizon and infinite-horizon cases are considered.

Our first result deals with the problem of H_∞-control for the system (1)–(6) on a finite horizon.

Theorem 3.1. Consider the system (1)–(6). Then, for a given $\gamma>0$, there exists a state feedback controller $u(t)$ such that

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2,$$

for all nonzero $w(t)\in L^2[0,T]$, if the following set of $\nu\times\mu$ coupled differential Riccati equations:

$$\begin{aligned}
&\dot{P}_{ik}(t) + A(t,k)^T P_{ik}(t) + P_{ik}(t)A(t,k)\\
&+P_{ik}(t)[\gamma^{-2}B_1(t,k)B_1^T(t,k) - B_2(t,k)R^{-1}(t,k)B_2^T(t,k)]P_{ik}(t)\\
&+\sum_{j=1}^{\mu}\lambda_{ij}^k P_{jk}(t) + \sum_{j=1}^{\nu}\alpha_{jk}P_{ij}(t) + C^T(t,k)C(t,k) = 0,\\
&P_{ik}(T)=0, \qquad i\in\mathcal{S}, \qquad k\in\mathcal{Z}, \qquad t\in[0,T],
\end{aligned} \tag{13}$$

has a solution $\{P_{ik}(t), i\in\mathscr{S}, k\in\mathscr{Z}\}$ on $[0, T]$. Moreover, a suitable control law is given by

$$\begin{aligned}
&u(t)=-L(t, r_t, \eta_t)x(t),\\
&L(t, r_t, \eta_t)=R^{-1}(t, \eta_t)B_2^T(t, \eta_t)P_{r_t,\eta_t}(t),\\
&t\in[0, T], \qquad r_t=i, \qquad \eta_t=k, \qquad i\in\mathscr{S}, \qquad k\in\mathscr{Z}.
\end{aligned}\tag{14}$$

Proof. Let

$$J(u) := \mathscr{E}\left\{\int_0^T z^T(t)z(t)-\gamma^2 w^T(t)w(t)\,dt\right\},\tag{15}$$

and let $\mathscr{L}^u(\cdot)$ be the infinitesimal operator of the process $\{x(t), r_t, \eta_t\}$ for the system (1) at the point $\{t, x, r_t, \eta_t\}$ (Ref. 3); that is, let $V=V(t, x, r_t, \eta_t)$. Then,

$$\begin{aligned}
\mathscr{L}^u V=&\,\partial V/\partial t+\dot{x}^T(t)(\partial V/\partial x)|_{r_t=i,\eta_t=k}\\
&+\sum_{j=1}^{\mu}\lambda_{ij}^k V(t, x, j, k)\\
&+\sum_{j=1}^{\nu}\alpha_{kj}V(t, x, i, j).
\end{aligned}\tag{16}$$

Hence, for $r_t=i$, $\eta_t=k$, $i\in\mathscr{S}$, $k\in\mathscr{Z}$, and $V(t, x, r_t, \eta_t)=x^T(t)P_{ik}(t)x(t)$, we have from (16)

$$\begin{aligned}
&\mathscr{L}^u[x^T(t)P_{ik}(t)x(t)]\\
&=x^T(t)\dot{P}_{ik}(t)x(t)+\dot{x}^T(t)P_{ik}(t)x(t)+x^T(t)P_{ik}(t)\dot{x}(t)\\
&+\sum_{j=1}^{\mu}\lambda_{ij}^k x^T(t)P_{jk}(t)x(t)+\sum_{j=1}^{\nu}\alpha_{jk}x^T(t)P_{ij}(t)x(t)\\
&=x^T(t)\dot{P}_{ik}(t)x(t)\\
&+[x^T(t)A^T(t, k)+w^T(t)B_1^T(t, k)\\
&+u^T(t)B_2^T(t, k)]P_{ik}(t)x(t)\\
&+x^T(t)P_{ik}(t)[A(t, k)x(t)+B_1(t, k)w(t)+B_2(t, k)u(t)]\\
&+\sum_{j=1}^{\mu}\lambda_{ij}^k x^T(t)P_{jk}(t)x(t)+\sum_{j=1}^{\nu}\alpha_{jk}x^T(t)P_{ij}(t)x(t).
\end{aligned}\tag{17}$$

Subtracting and adding (17) into (15), we have

$$J(u) = \mathscr{E}\left\{\left[\int_0^T z^T(t)z(t) + x^T(t)\dot{P}_{ik}(t)x(t) + x^T(t)A^T(t,k)P_{ik}(t)x(t) + w^T(t)B_1^T(t,k)P_{ik}(t)x(t) + u^T(t)B_2^T(t,k)P_{ik}(t)x(t) + x^T(t)P_{ik}(t)A(t,k)x(t) + x^T(t)P_{ik}(t)B_1(t,k)w(t) + x^T(t)P_{ik}(t)B_2(t,k)u(t) + \sum_{j=1}^{\mu} \lambda_{ij}^k x^T(t)P_{jk}(t)x(t) + \sum_{j=1}^{\nu} \alpha_{jk} x^T(t)P_{ij}(t)x(t) - \gamma^2 w^T(t)w(t)\right] dt\right\} - \mathscr{E}\left\{\int_0^T \mathscr{L}^u(x^T(t)P_{ik}(t)x(t))\, dt\right\}. \tag{18}$$

Substituting (2) into (18), we get

$$J(u) = \mathscr{E}\left\{\int_0^T \left[x^T(t)\Big(\dot{P}_{ik}(t) + A(t,k)^T P_{ik}(t) + P_{ik}(t)A(t,k) + P_{ik}(t)[\gamma^{-2}B_1(t,k)B_1^T(t,k) - B_2(t,k)R^{-1}(t,k)B_2^T(t,k)]P_{ik}(t) + \sum_{j=1}^{\mu} \lambda_{ij}^k P_{jk}(t) + \sum_{j=1}^{\nu} \alpha_{jk} P_{ij}(t) + C^T(t,k)C(t,k)\Big)x(t) + [u(t) + R^{-1}(t,k)B_2^T(t,k)P_{ik}(t)x(t)]^T R(t,k) \times [u(t) + R^{-1}(t,k)B_2^T(t,k)P_{ik}(t)x(t)] - \gamma^2(w(t) - \gamma^{-2}B_1^T(t,k)P_{ik}(t)x(t))^T \times (w(t) - \gamma^{-2}B_1^T(t)P_{ik}(t)x(t))\right] dt\right\} - \mathscr{E}\left\{\int_0^T \mathscr{L}^u(x^T(t)P_{ik}(t)x(t))\, dt\right\}. \tag{19}$$

By applying the Dynkin formula (Refs. 3, 36, 37), we have

$$\mathscr{E}\left\{\int_0^T \mathscr{L}^u(x^T(t)P_{ik}(t)x(t))\, dt\right\} = \mathscr{E}[x^T(T)P_{ik}(T)x(T)] - \mathscr{E}[x^T(0)P_{ik}(0)x(0)].$$

Finally, using the facts $x(0)=0$ and $P_{ik}(T)=0$ and the assumption $w(t)\equiv 0$, when we choose the controller $u(t)$ as that in (14), we get from (19)

$$J(u)\leq -\gamma^2\mathscr{E}\left\{\int_0^T (w(t)-\gamma^{-2}B_1^T(t,k)P_{ik}(t)x(t))^T \times (w(t)-\gamma^{-2}B_1^T(t,k)P_{ik}(t)x(t))\,dt\right\}<0,$$

which completes our proof. □

The next theorem considers the H_∞-control for the system (1)–(6) on an infinite horizon.

Theorem 3.2. Consider the system (1)–(6). Then, for a given $\gamma>0$, there exists a state feedback controller $u(t)$ such that the resulting closed-loop system is stochastically stable and

$$\|z(t)\|_{E_2}<\gamma\|w(t)\|_2,$$

for all nonzero $w(t)\in L^2[0,\infty)$, if the following set of $\mu\times\nu$ coupled algebraic Riccati equations:

$$A(k)^TP_{ik}+P_{ik}A(k)+P_{ik}[\gamma^{-2}B_1(k)B_1^T(k)-B_2(k)R^{-1}(k)B_2^T(k)]P_{ik} + \sum_{j=1}^{\mu}\lambda_{ij}^kP_{jk}+\sum_{j=1}^{\nu}\alpha_{jk}P_{ij}+C^T(k)C(k)=0, \qquad i\in\mathscr{S}, k\in\mathscr{X}, \tag{20}$$

has a solution $\{P_{ik}, i\in\mathscr{S}, k\in\mathscr{X}\}$. Moreover, a suitable control law is given by

$$u(t)=-L(r_t,\eta_t)x(t), \qquad L(r_t,\eta_t)=R^{-1}(\eta_t)B_2^T(\eta_t)P_{r_t,\eta_t},$$
$$t\in[0,\infty), \qquad r_t=i, \qquad \eta_t=k, \qquad i\in\mathscr{S}, \qquad k\in\mathscr{X}. \tag{21}$$

Proof. For the stochastic stability of the closed-loop system (1) with control law (21), we rewrite (20) as

$$A_{ik}^TP_{ik}+P_{ik}A_{ik}+P_{ik}[\gamma^{-2}B_1(k)B_1^T(k)+B_2(k)R^{-1}(k)B_2^T(k)]P_{ik} + \sum_{j=1}^{\mu}\lambda_{ij}^kP_{jk}+\sum_{j=1}^{\nu}\alpha_{jk}P_{ij}+C^T(k)C(k)=0, \qquad i\in\mathscr{S}, k\in\mathscr{X}, \tag{22}$$

where

$$A_{ik}=A(k)-B_2(k)L(i,k)=A(k)-B_2(k)R^{-1}(k)B_2^T(k)P_{ik}.$$

Now, using the facts that P_{ik}, $i \in \mathcal{S}$, $k \in \mathcal{Z}$, is positive definite and $(C(k), A(k))$, $k \in \mathcal{Z}$, is observable, the stochastic stability of the closed-loop system can be obtained (Ref. 10). For the performance,

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2,$$

for all nonzero $w(t) \in L^2[0, \infty)$, it can be proven using the same technique as that in Theorem 3.1 for the finite-horizon case. □

Remark 3.1. Theorems 3.1 and 3.2 provide sufficient conditions for the solvability of the H_∞-control problem (1)–(6) for the finite-horizon and infinite-horizon cases, respectively, related to $\mu \times \nu$ coupled differential and difference Riccati equations, respectively. Note that, when $r_t = \eta_t$, Theorems 3.1 and 3.2 reduce to the results obtained in Ref. 17. Also, when $r_t = \eta_t = 1$, Theorems 3.1 and 3.2 recover the solution for the standard H_∞-control problem on linear systems (Ref. 25).

Remark 3.2. It should be noted that, when $\gamma \to \infty$ and $r_t = \eta_t$, the μ coupled algebraic Riccati equations (20) reduce to the ones corresponding to jump LQG problem with weighting matrices $C^T(k)C(k)$ for the state and $R(k)$ for the input, respectively, which has been shown to be a necessary and sufficient condition to guarantee that the jump LQG problem has a solution (Ref. 10). Therefore, if γ is large enough, Eqs. (20) are expected to possess a positive-definite solution.

Remark 3.3. Theorems 3.1 and 3.2 are extensions of the results in Ref. 17, which reduce to those when $r_t = \eta_t$, for all $t \in [0, T]$, $r_t = i \in \mathcal{S}$. Also, it is obvious that, if the left-hand sides of (13) and (20) are less than zero, statements (a) in Theorems 3.1 and 3.2 are still true.

It should be noted that many computational algorithms for solving coupled algebraic Riccati equations are available now; see, e.g., Refs. 38 and 39.

Remark 3.4. If the transition probability rates λ_{ij}^k and α_{ij}, with $i, j \in \mathcal{S}$, $k \in \mathcal{Z}$, are sufficiently small (that is, the system jump from the ith mode to the jth mode is very slow, then from the proof of Theorem 3.1, we can conclude that the sufficient conditions (13) in Theorem 3.1 and (20) in

Theorem 3.2 reduce, respectively, to

$$\begin{aligned}
&\dot{P}_{ik}(t)+A(t,k)^T P_{ik}(t)+P_{ik}(t)A(t,k)\\
&+P_{ik}(t)[\gamma^{-2}B_1(t,k)B_1^T(t,k)-B_2(t,k)R^{-1}(t,k)B_2^T(t,k)]P_{ik}(t)\\
&+C^T(t,k)C(t,k)\leq 0,\\
&P_{ik}(T)=0, \qquad i\in\mathscr{S},\ k\in\mathscr{Z},\ t\in[0,T],
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&A(k)^T P_{ik}+P_{ik}A(k)+P_{ik}[\gamma^{-2}B_1(k)B_1^T(k)-B_2(k)R^{-1}(k)B_2^T(k)]P_{ik}\\
&+C^T(k)C(k)\leq 0, \qquad i\in\mathscr{S},\ k\in\mathscr{Z}.
\end{aligned} \tag{24}$$

Note that (23) and (24) are decoupled inequalities, which has advantages for computation.

Finally, we should point out that a mode-dependent control law is sometimes unrealistic, as it requires that the jumping processes $\{\eta_t\}$ and $\{r_t\}$ be measured exactly on-line. When

$$B_2(t,\eta_t)=B_2(t) \quad \text{and} \quad D(t,\eta_t)=D(t),$$

i.e., $B(t)$ and $D(t)$ are independent of the jump process $\{\eta_t\}$, we may have a controller that is mode-independent. A sufficient condition in this situation can be found by requiring that the matrices $P_{ik}(t)$, $i\in\mathscr{S}$, $k\in\mathscr{Z}$, in (13) and P_{ik}, $i\in\mathscr{S}$, $k\in\mathscr{Z}$, in (20) be all equal to $P(t)$ and P, respectively, and using the facts

$$\sum_{j=1}^{\nu}\lambda_{ij}^k=0, \qquad i\in\mathscr{S},\ k\in\mathscr{Z},$$

$$\sum_{j=1}^{\mu}\alpha_{ij}=0, \qquad i\in\mathscr{S},\ k\in\mathscr{Z}.$$

Therefore, the controller in Theorems 3.1 and 3.2 reduce to

$$u(t)=-R^{-1}(t)B_2(t)P(t)x(t), \tag{25}$$

$$u(t)=-R^{-1}B_2Px(t). \tag{26}$$

It should be noted that the controllers (25) and (26) are independent of the transition matrices; so, it is expected that the above-mentioned sufficient conditions are quite conservative.

4. Robust H_∞-Control Results

In this section, we study the problem of robust H_∞-control for the system (1)–(6) with parameter uncertainty. We establish sufficient conditions on the robust control problem, and show that the solution to the above problem depends upon the solution to a set of $\mu \times \nu$ coupled Riccati equations.

Consider the following uncertain dynamical system:

$$\dot{x}(t) = [A(t, \eta_t) + \Delta A(t, \eta_t)]x(t) + B_1(t, \eta_t)w(t) + B_2(t, \eta_t)u(t, r_t),$$

$$x(0) = 0, \qquad r_0 = i, \qquad \eta_0 = k, \qquad t \in [0, T], \tag{27}$$

$$z(t) = C(t, \eta_t)x(t) + D(t, \eta_t)u(t, r_t), \tag{28}$$

where $x(t)$, $u(t, r_t)$, $w(t)$, r_t, η_t, $A(t, \eta_t)$, $B_1(t, \eta_t)$, $B_2(t, \eta_t)$, $C(t, \eta_t)$, $D(t, \eta_t)$ are the same as in (1)–(2) and $\Delta A(t, \eta_t)$ is a real, time-varying matrix function representing the norm-bounded parameter uncertainty for each $r_t = i \in \mathscr{S}$. The admissible parameter uncertainties are assumed to be modeled as

$$\Delta A(t, \eta_t) = H(t, \eta_t)F(t, \eta_t)E(t, \eta_t); \tag{29}$$

here, for $\eta_t = k$, $k \in \mathscr{Z}$, $E(t, \eta_t) \in \mathbb{R}^{j \times n}$ and $H(t, \eta_t) \in \mathbb{R}^{n \times i}$ are known real, time-varying, piecewise-continuous matrices between each jump, which characterize how the uncertain parameter in $F(t, \eta_t)$ enters the nominal matrix $A(t, \eta_t)$; and $F(t, \eta_t)$, $\eta_t = k \in \mathscr{Z}$, is an unknown, time-varying matrix function satisfying

$$\|F(t, \eta_t)\|_2 \leq 1, \qquad \forall t \geq 0,\ \eta_t = k \in \mathscr{Z}, \tag{30}$$

with the element of $F(t, \eta_t)$ being Lebesgue measurable for any $\eta_t = k \in \mathscr{Z}$.

Remark 4.1. The parameter uncertainty structure as in (29)–(30), when $\eta_t = k$, $k \in \mathscr{Z}$, has been widely used in problems of robust control and robust filtering of uncertain systems (see, e.g., Refs. 40 and 41 and references therein), and many practical systems possess parameter uncertainties which can be either exactly modeled or overbounded by (30). Observe that the unknown matrix $F(t, \eta_t)$ in (29) can even be allowed to be state dependent [i.e., $F(t, \eta_t) = F(t, x(t), \eta_t)$], as long as (30) is satisfied.

When $t \to \infty$, i.e., the infinite horizon is concerned, $H(t, \eta_t)$, $F(t, \eta_t)$, $E(t, \eta_t)$ are constant matrices, with $\eta_t = k \in \mathscr{Z}$, denoted by $H(k)$, $F(k)$, $E(k)$.

In this section, we consider the problem of robust state feedback control for the uncertain Markovian jumping system (27)–(28). We design a state feedback controller $\mathscr{G}(r_t, \eta_t)$,

$$u(t) = -L(t, r_t, \eta_t)x(t), \tag{31}$$

such that, for a given $\gamma > 0$, for all nonzero $w(t) \in L^2$, and for all parameter uncertainties satisfying (30),

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2. \tag{32}$$

In this situation, the system (27)–(28) with controller (31) is said to have robust H_∞-performance (32) over the horizon $[0, T]$.

More specifically, our objective is to design a state feedback controller $\mathscr{G}(r_t, \eta_t)$, such that:

(i) in the finite-horizon case, the system (27)–(28) and (3)–(6) with $\mathscr{G}(r_t, \eta_t)$ has robust performance (32) over a given horizon $[0, T]$;
(ii) in the infinite-horizon case, the system (27)–(28) and (3)–(6) with $\mathscr{G}(r_t, \eta_t)$ is robust stochastically stable and has robust performance (32) over $[0, \infty)$.

Here, "robust stochastically stable" means that the uncertain system (27) is stochastically asymptotically stable about the origin for all admissible uncertainties.

First, let us recall a matrix inequality that will be needed in the proof of our robust control results.

Lemma 4.1. See Ref. 40. Let E, F, H be real, time-varying matrices of appropriate dimensions. Then, for any $\epsilon > 0$ and for all matrices F satisfying $\|F\|_2 \leq 1$,

$$HFE + E^T F^T H^T \leq (1/\epsilon) HH^T + \epsilon E^T E.$$

Before proceeding to the main results in this section, first we analyze the problem of the robust stochastic stability of the system (27).

Theorem 4.1. Consider the system (27) with $B_1(t, \eta_t) \equiv 0$ and $B_2(t, \eta_t) \equiv 0$, $\eta_t \in \mathscr{X}$. Then, the system is robust stochastically stable for all admissible uncertainties if, for a given selection of $\epsilon(k) > 0$, $k \in \mathscr{X}$, and any given positive-definite symmetric matrices $N(k)$, $k \in \mathscr{X}$, there exist positive-definite matrices $M(k)$, $k \in \mathscr{X}$, satisfying

$$\begin{aligned} &A^T(k)M(k) + M(k)A(k) + \sum_{j=1}^{v} \alpha_{kj} M(j) + N(k) \\ &+ [1/\epsilon(k)] M(k)H(k)H^T(k)M(k) + \epsilon(k)E^T(k)E(k) = 0. \end{aligned} \tag{33}$$

Proof. Suppose that the ν coupled algebraic Riccati equations (33) have solution $M(k)$, $k\in\mathscr{Z}$. Since $\|F(\eta_t)\|_2\leq 1$, $\eta_t=k\in\mathscr{Z}$, we have from Lemma 4.1 that

$$
\begin{aligned}
&[1/\epsilon(k)]M(k)H(k)H^T(k)M(k)+\epsilon(k)E^T(k)E(k)\\
&\geq E^T(k)F^T(k)H^T(k)M(k)+M(k)H(k)F(k)E(k). \qquad (34)
\end{aligned}
$$

Combining (33) with (34) leads to

$$
\begin{aligned}
&[A(k)+H(k)F(k)E(k)]^TM(k)+M(k)[A(k)+H(k)F(k)E(k)]\\
&+\sum_{j=1}^{\nu}\alpha_{kj}M(j)+N(k)\leq 0, \qquad k\in\mathscr{Z}, \qquad (35)
\end{aligned}
$$

for all admissible parameter uncertainties $F(\eta_t)$ satisfying (30). By Lemma 2.1, the desired result comes from (35) immediately. □

Remark 4.2. Similarly to Remark 2.4 in Section 2, the existence of positive-definite solutions $M(k)$, $k\in\mathscr{Z}$, to (33) also implies that the uncertain system (27) is robust asymptotically mean-square stable, exponentially mean-square stable, and almost surely asymptotically stable.

In connection with the robust stochastical stabilization problem of (27), we have the following theorem.

Theorem 4.2. The system (27) with $B_1(t,\eta_t)\equiv 0$ is robust stochastically stabilizable for all admissible uncertainties in (29)–(30), if there exists a control law $u(t)=-L(i,k)x(t)$, $i\in\mathscr{S}$, $k\in\mathscr{Z}$, such that, for a given selection of $\epsilon(k)>0$, $k\in\mathscr{Z}$, and any given positive-definite matrices N_{ik}, $i\in\mathscr{S}$, $k\in\mathscr{Z}$, there exist positive-definite matrices M_{ik}, $i\in\mathscr{S}$, $k\in\mathscr{Z}$, satisfying

$$
\begin{aligned}
&[A(k)-B_2(k)L(i,k)]^TM_{ik}+M_{ik}[A(k)-B_2(k)L(i,k)]\\
&+\sum_{j=1}^{\mu}\lambda_{ij}^kM_{jk}+\sum_{j=1}^{\nu}\alpha_{jk}M_{ij}+N_{ik}+[1/\epsilon(k)]\\
&\times M_{ik}H(k)H^T(k)M_{ik}+\epsilon(k)E^T(k)E(k)=0. \qquad (36)
\end{aligned}
$$

Proof. It can be carried out by exactly the same way as in Theorem 4.1 and applying Lemma 2.2. □

Now, we are in a position to derive a robust state feedback controller for the system (27)–(28) and (3)–(6) with H_∞-performance (9) over a finite horizon.

Theorem 4.3. Consider the system (27)–(28) with (3)–(6). Then, for a given $\gamma > 0$, there exists a state feedback controller $u(t)$ such that

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2,$$

for all nonzero $w(t) \in L^2[0, T]$ and all uncertainties satisfying (30), if for a given selection of $\epsilon(k) > 0$, $k \in \mathscr{Z}$, the following set of $\mu \times \nu$ coupled differential Riccati equations:

$$\begin{aligned}
&\dot{P}_{ik}(t) + A(t,k)^T P_{ik}(t) + P_{ik}(t)A(t,k) \\
&+ P_{ik}(t)[\gamma^{-2}B_1(t,k)B_1^T(t,k) - B_2(t,k)R^{-1}(t,k)B_2^T(t,k)]P_{ik}(t) \\
&+ \sum_{j=1}^{\mu} \lambda_{ij}^k P_{jk}(t) + \sum_{j=1}^{\nu} \alpha_{jk} P_{ij}(t) + C^T(t,k)C(t,k) \\
&+ [1/\epsilon(k)]P_{ik}(t)H(t,k)H^T(t,k)P_{ik}(t) + \epsilon(k)E^T(t,k)E(t,k) = 0, \\
&P_{ik}(T) = 0, \qquad i \in \mathscr{S}, k \in \mathscr{Z}, \qquad t \in [0, T],
\end{aligned} \tag{37}$$

has a solution $\{P_{ik}(t),\ i \in \mathscr{S},\ k \in \mathscr{Z}\}$ on $[0, T]$. Moreover, a suitable control law is given by

$$\begin{aligned}
&u(t) = -L(t, r_t, \eta_t)x(t), \\
&L(t, r_t, \eta_t) = R^{-1}(t, \eta_t)B_2^T(t, \eta_t)P_{r_t,\eta_t}(t), \\
&t \in [0, T], \qquad r_t = i, \qquad \eta_t = k, \qquad i \in \mathscr{S}, \qquad k \in \mathscr{Z}.
\end{aligned} \tag{38}$$

Proof. Suppose that the $\mu \times \nu$ coupled differential Riccati equations (37) have a solution $P_{ik}(t)$, $i \in \mathscr{S}$, $k \in \mathscr{Z}$. Since $\|F(t,k)\|_2 \le 1$, $k \in \mathscr{Z}$, using the same arguments as in Theorem 4.1, we have

$$\begin{aligned}
&\dot{P}_{ik}(t) + [A(t,k) + \Delta A(t,k)]^T P_{ik}(t) + P_{ik}(t)[A(t,k) + \Delta A(t,k)] \\
&+ P_{ik}(t)[\gamma^{-2}B_1(t,k)B_1^T(t,k) - B_2(t,k)R^{-1}(t,k)B_2^T(t,k)]P_{ik}(t) \\
&+ \sum_{j=1}^{\mu} \lambda_{ij}^k P_{jk}(t) + \sum_{j=1}^{\nu} \alpha_{jk} P_{ij}(t) + C^T(t,k)C(t,k) \le 0, \\
&P_{ik}(T) = 0, \qquad i \in \mathscr{S}, \qquad k \in \mathscr{Z}, \qquad t \in [0, T],
\end{aligned} \tag{39}$$

for all admissible parameter uncertainties $F(t, r_t)$, $r_t = k \in \mathscr{Z}$, satisfying (30). By Theorem 3.1, we have

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2. \qquad \square$$

Our next result deals with the robust H_∞-control problem for the system (27)–(28) and (3)–(6) on an infinite horizon.

Theorem 4.4. Consider the system (27)–(28) with (3)–(6). Then, for a given $\gamma > 0$, there exists a state feedback controller $u(t)$ such that the resulting closed-loop system is stochastically stable and

$$\|z(t)\|_{E_2} \leq \gamma \|w(t)\|_2,$$

for all nonzero $w(t) \in L^2[0, \infty)$ and all uncertainties satisfying (30), if for a given selection of $\epsilon(k) > 0$, $k \in \mathscr{L}$, the following set of $\mu \times \nu$ coupled algebraic Riccati equations:

$$A(k)^T P_{ik} + P_{ik} A(k) + P_{ik}[\gamma^{-2} B_1(k) B_1^T(k) - B_2(k) R^{-1}(k) B_2^T(k)] P_{ik}$$
$$+ \sum_{j=1}^{\mu} \lambda_{ij}^k P_{jk} + \sum_{j=1}^{\nu} \alpha_{jk} P_{ij} + C^T(k) C(k) + [1/\epsilon(k)] P_{ik} H(k) H^T(k) P_{ik}$$
$$+ \epsilon(k) E^T(k) E(k) = 0, \qquad i \in \mathscr{S}, k \in \mathscr{L}, \tag{40}$$

has a positive definite solution $\{P_{ik}, \ i \in \mathscr{S}, \ k \in \mathscr{L}\}$. Moreover, a suitable control law is given by

$$u(t) = -L(r_t, \eta_t) x(t), \qquad L(r_t, \eta_t) = R^{-1}(\eta_t) B_2^T(\eta_t) P_{r_t, \eta_t},$$
$$t \in [0, \infty), \qquad r_t = i, \qquad \eta_t = k, \qquad i \in \mathscr{S}, \qquad k \in \mathscr{L}.$$

Proof. It can be established by using the same technique as in Theorem 4.3 together with Theorem 3.2. □

Remark 4.3. The existence of scaling parameters $\epsilon(k)$, $k \in \mathscr{L}$, in Theorems 4.1–4.4 can be checked using convex optimization over LMIs (Ref. 42).

Remark 4.4. From Theorems 3.1 and 3.2, note that the $\mu \times \nu$ coupled differential Riccati equations (37) and algebraic Riccati equations (40) are the sufficient conditions to solve the following H_∞-control problem without parameter uncertainties for the finite-horizon and infinite-horizon cases, respectively:

$$\dot{x}(t) = A(t, \eta_t) x(t) + [B_1(t, \eta_t), [\gamma/\sqrt{\epsilon(\eta_t)}] H(t, \eta_t)] \hat{w}(t)$$
$$+ B_2(t, \eta_t) u(t, r_t),$$
$$x(0) = 0, \qquad r_0 = i, \qquad \eta_0 = k, \qquad t \in [0, T], \tag{41}$$

$$\hat{z}(t) = \begin{bmatrix} \sqrt{\epsilon(\eta_t)} E(t, \eta_t) \\ C(t, \eta_t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ D(t, \eta_t) \end{bmatrix} u(t, r_t),$$
$$r_t = i \in \mathscr{S}, \qquad \eta_t = k \in \mathscr{L}, \tag{42}$$

where

$$\hat{w}(t)=\begin{bmatrix} w(t) \\ \gamma^{-1}\sqrt{\epsilon(\eta_t)}F(t,\eta_t)E(t,\eta_t) \end{bmatrix}, \qquad \hat{z}(t)=\begin{bmatrix} z_1(t) \\ z(t) \end{bmatrix},$$

$$z_1(t)=\sqrt{\epsilon(\eta_t)}E(t,\eta_t)x(t).$$

It is easy to see that

$$\|\hat{z}(t)\|_{E_2}\geq\|z(t)\|_{E_2}.$$

Therefore, we conclude that, if we can solve the H_∞-control problem for the system (41)–(42) with (3)–(6), then we can also solve the robust H_∞-control problem for the system (27)–(28) with (3)–(6) via the same controller.

Remark 4.5. All the results that we present in this section can be extended to consider the robust control problem for the system (27)–(28) when the jumping rates in (3) and (5) contain some uncertainties, i.e.,

$$p_{ij}^k=\Pr(r_{t+\delta}=j\,|\,r_t=i,\ \eta_t=k)$$
$$=\begin{cases}(\lambda_{ij}^k+\Delta\lambda_{ij}^k)\delta+o(\delta), & \text{if } i\neq j,\\ 1+(\lambda_{ii}^k+\Delta\lambda_{ii}^k)\delta+o(\delta), & \text{if } i=j,\end{cases} \tag{43}$$

with transition probability rates $\lambda_{ij}^k+\Delta\lambda_{ij}^k\geq 0$, $i\neq j$, and

$$\lambda_{ii}^k+\Delta\lambda_{ii}^k=-\sum_{j=1,j\neq i}^{s}(\lambda_{ij}^k+\Delta\lambda_{ij}^k), \tag{44}$$

and

$$p_{kl}=\Pr(\eta_{t+\delta}=l\,|\,\eta_t=k)$$
$$=\begin{cases}(\alpha_{kl}+\Delta\alpha_{kl})\delta+o(\delta), & \text{if } k\neq l,\\ 1+(\alpha_{kk}+\alpha_{kk})\delta+o(\delta), & \text{if } k=l,\end{cases} \tag{45}$$

with transition probability rates $\alpha_{kl}+\Delta\alpha_{kl}\geq 0$, $k\neq l$, and

$$\alpha_{kk}+\Delta\alpha_{kk}=-\sum_{j=1,j\neq k}^{s}(\alpha_{kj}+\Delta\alpha_{kj}). \tag{46}$$

We assume that the uncertainties $\Delta\lambda_{ij}^k$ and $\Delta\alpha_{kl}$ satisfy, respectively,

$$\|\Delta\lambda_{ij}^k\|\leq s_{ij}^k, \quad \|\Delta\alpha_{kl}\|\leq q_{kl}, \qquad \forall i,j\in\mathscr{S},\ \forall k,l\in\mathscr{Z}, \tag{47}$$

where s_{ij}^k and q_{kl} are known scalars $\forall i,j\in\mathscr{S}$, $\forall k,l\in\mathscr{Z}$.

Similarly to Theorems 4.3 and 4.4, we have the following results.

Theorem 4.5. Consider the system (27)–(28) with (43)–(46). Then, for a given $\gamma > 0$, there exists a state feedback controller $u(t)$ such that

$$\|z(t)\|_{E_2} < \gamma \|w(t)\|_2,$$

for all nonzero $w(t) \in L^2[0, T]$ and all uncertainties satisfying (30) and (47) if, for a given selection of $\epsilon(k) > 0$, $k \in \mathscr{Z}$, the following set of $\mu \times \nu$ coupled differential Riccati equations:

$$\begin{aligned}
&\dot{P}_{ik}(t) + A(t,k)^T P_{ik}(t) + P_{ik}(t) A(t,k) \\
&+ P_{ik}(t)[\gamma^{-2} B_1(t,k) B_1^T(t,k) - B_2(t,k) R^{-1}(t,k) B_2^T(t,k)] P_{ik}(t) \\
&+ \sum_{j=1}^{\mu} (\lambda_{ij}^k + s_{ij}^k) P_{jk}(t) + \sum_{j=1}^{\nu} (\alpha_{jk} + q_{jk}) P_{ij}(t) + C^T(t,k) C(t,k) \\
&+ [1/\epsilon(k)] P_{ik}(t) H(t,k) H^T(t,k) P_{ik}(t) + \epsilon(k) E^T(t,k) E(t,k) = 0, \\
&P_{ik}(T) = 0, \qquad i \in \mathscr{S}, \qquad k \in \mathscr{Z}, \qquad t \in [0, T],
\end{aligned}$$

has a solution $\{P_{ik}(t), i \in \mathscr{S}, k \in \mathscr{Z}\}$ on $[0, T]$. Moreover, a suitable control law is given by

$$\begin{aligned}
&u(t) = -L(t, r_t, \eta_t) x(t), \\
&L(t, r_t, \eta_t) = R^{-1}(t, \eta_t) B_2^T(t, \eta_t) P_{r_t, \eta_t}(t), \\
&t \in [0, T], \qquad r_t = i, \qquad \eta_t = k, \qquad i \in \mathscr{S}, \qquad k \in \mathscr{Z}.
\end{aligned}$$

Proof. It can be worked out by using the same arguments as in Theorem 4.3. □

Theorem 4.6. Consider the system (27)–(28) with (43)–(46). Then, for a given $\gamma > 0$, there exists a state feedback controller $u(t)$ such that the resulting closed-loop system is stochastically stable and

$$\|z(t)\|_{E_2} \le \gamma \|w(t)\|_2,$$

for all nonzero $w(t) \in L^2[0, \infty)$ and all uncertainties satisfying (30) and (47) if, for a given selection of $\epsilon(k) > 0$, $k \in \mathscr{Z}$, the following set of $\mu \times \nu$ coupled algebraic Riccati equations:

$$A(k)^T P_{ik} + P_{ik}A(k) + P_{ik}[\gamma^{-2}B_1(k)B_1^T(k) - B_2(k)R^{-1}(k)B_2^T(k)]P_{ik}$$

$$+ \sum_{j=1}^{\mu} (\lambda_{ij}^k + s_{ij}^k)P_{jk} + \sum_{j=1}^{\nu} (\alpha_{jk} + q_{jk})P_{ij} + C^T(k)C(k)$$

$$+ [1/\epsilon(k)]P_{ik}H(k)H^T(k)P_{ik} + \epsilon(k)E^T(k)E(k) = 0, \qquad i \in \mathscr{S}, k \in \mathscr{X},$$

Proof. The desired result can be carried out along the same line as in the proof of Theorem 4.4. □

5. Conclusions

This paper has investigated the problems of H_∞-control for a class of fault-tolerant control linear systems with Markovian jumping parameters. We propose a control design method, using a Riccati equation approach, such that the required H_∞-performance can be achieved. Both the finite-horizon and infinite-horizon cases have been considered. Robust H_∞-control problems for the above systems comprising norm-bounded parameter uncertainty have also been studied. It has been shown that control policies for the above problems can be made by solving coupled differential Riccati equations or algebraic Riccati equations. The problem of robust H_∞-control with uncertain jumping rates is also studied.

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