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Mode-Independent $\mathcal{H}_\infty$ Filters for Hybrid Markov Linear Systems

Carlos E. de Souza, Alexandre Trofino and Karina A. Barbosa

***Abstract*— This paper addresses the problem of $\mathcal{H}_\infty$ filtering for continuous-time linear systems with Markovian jumping parameters. The main contribution of the paper is to provide a method for designing an asymptotically stable linear time-invariant $\mathcal{H}_\infty$ filter for systems where the jumping parameter is not accessible. The cases where the transition rate matrix of the Markov process is either exactly known, or unknown but belongs to a given polytope, are treated. The robust $\mathcal{H}_\infty$ filtering problem for systems with polytopic uncertain matrices is also considered and a filter design method based on a Lyapunov function that depends on the uncertain parameters is developed. The proposed filter designs are given in terms of linear matrix inequalities.**

I. INTRODUCTION

The $\mathcal{H}_\infty$ filtering approach has been the subject of extensive research over the past decade. In $\mathcal{H}_\infty$ filtering, the noise sources are arbitrary deterministic signals with bounded energy, or average power, and a filter is sought which ensures a prescribed upper-bound on the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error; see, e.g. [8]-[10], [15], [17], [18] and the references therein. This filtering approach is very appropriate to applications where the statistics of the noise signals are not exactly known.

Over the past few years, Markovian jump linear (MJL) systems have been attracting an increasing attention. This class of systems is very appropriate to model plants whose structure is subject to random abrupt changes due to, for instance, random component failures, abrupt environment disturbance, changes of the operating point of a linearized model of a nonlinear system, etc. A number of control and filtering problems related to these systems has been analysed by several authors (see, e.g. [2]-[14], [16], [19]-[21] and the references therein). In particular with regard to filtering, minimum linear mean square filtering schemes have been studied in, e.g. [4], [5] and [13], whereas $\mathcal{H}_\infty$ filtering has been investigated in, for example, [2], [6], [8]-[10] and [21]. With exception of [6], which deals with discrete-time MJL systems, a common feature of the existing $\mathcal{H}_\infty$ filtering results is that the jumping parameter is assumed to be accessible and is required for the implementation of the filter, which is a MJL system as well. To the best of the authors' knowledge, to date the design of a time-invariant $\mathcal{H}_\infty$ filter for continuous-time MJL systems with a *non-accessible* jumping parameter has not yet been addressed.

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C.E. de Souza is with the Department of Systems and Control, Laboratório Nacional de Computação Científica (LNCC/MCT), Av. Getúlio Vargas 333, 25651-070 Petrópolis, RJ, Brazil csouza@lncc.br

A. Trofino is with the Department of Automation and Systems, Universidade Federal de Santa Catarina, PO Box 476, 88040-900 Florianópolis, SC, Brazil trofino@das.ufsc.br

K.A. Barbosa is with the Department of Automation and Systems, Universidade Federal de Santa Catarina, PO Box 476, 88040-900 Florianópolis, SC, Brazil karinab@das.ufsc.br

This paper addresses the problem of $\mathcal{H}_\infty$ filtering for continuous-time MJL systems where the jumping parameter is not accessible. Attention is focused on the design of an asymptotically stable linear time-invariant filter (i.e. *mode-independent*), that ensures mean square stability for the estimation error dynamics and a prescribed upper-bound on the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error. Robust $\mathcal{H}_\infty$ filters are also derived for MJL systems subject to polytopic-type parameter uncertainty in either the transition rate matrix of the Markov process, or in the matrices of the state-space model for the possible modes of operation of the system. In the later case, we propose a robust $\mathcal{H}_\infty$ filter design based on a Lyapunov function that depends on the uncertain parameters. The filter designs are given in terms of linear matrix inequality (LMIs). The potentials of the proposed $\mathcal{H}_\infty$ filtering methods are demonstrated via two examples.

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ and I_n denote the n-dimensional Euclidean space, the set of $n \times m$ real matrices and the $n \times n$ identity matrix, respectively, $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix and $\mathcal{L}_2$ denotes the space of square integrable vector functions over $[0,\infty)$. For a real matrix S, $S > 0$ ($S < 0$) means that S is symmetric and positive definite (negative definite). For a symmetric block matrix, the symbol $\star$ denotes the transpose of the blocks outside the main diagonal block, and $\mathbf{E}[\,\cdot\,]$ stands for mathematical expectation.

II. PROBLEM STATEMENT

Fix an underlying probability space $(\Omega, \mathcal{F}, I\!\!P)$ and consider the stochastic system:

$$(\mathcal{S}):\quad \dot{x}(t) = A(\theta_t)x(t) + B(\theta_t)w(t) \qquad (1)$$

$$y(t) = C(\theta_t)x(t) + D(\theta_t)w(t) \qquad (2)$$

$$z(t) = L(\theta_t)x(t) \qquad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^{n_w}$ is the noise signal (including process and measurement noises), which is assumed to be an arbitrary signal in $\mathcal{L}_2$, $y(t) \in \mathbb{R}^{n_y}$ is the measurement, and $z(t) \in \mathbb{R}^{n_z}$ is the signal to be estimated. $\{\theta_t\}$ is a homogeneous Markov process with right continuous trajectories and taking values on the finite set $\Xi = \{1, 2, \ldots, \sigma\}$ with stationary transition probabilities:

$$I\!\!P\{\theta_{t+h} = j \,|\, \theta_t = i\} = \begin{cases} \lambda_{ij}h + o(h), & i \neq j \\ 1 + \lambda_{ii}h + o(h), & i = j \end{cases}$$

where $h > 0$ and $\lambda_{ij} \geq 0$ is the transition rate from the state i to j, $i \neq j$, and

$$\lambda_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{\sigma} \lambda_{ij}. \tag{4}$$

The set Ξ comprises the various operation modes of system $(\mathcal{S})$ and for each possible value of $\theta_t = i$, $i \in \Xi$, the matrices associated with the 'i-th mode' will be denoted by

$$A_i := A(\theta_t = i), \;\; B_i := B(\theta_t = i), \;\; C_i := C(\theta_t = i), \\ D_i := D(\theta_t = i), \;\; L_i := L(\theta_t = i)$$

where A_i, B_i, C_i, D_i and L_i are constant matrices for any $i \in \Xi$.

No assumption is made on the accessibility of the jumping process $\{\theta_t\}$. This is in contrast with existing approaches of $\mathcal{H}_\infty$ filtering for MJL systems, such as those of [8]-[10], which require the jumping parameter to be available for the filter implementation.

The filtering problem to be addressed is to obtain an estimate, $\hat{z}(t)$, of $z(t)$ via a causal *mode-independent* linear filter which provides a uniformly small estimation error, $\tilde{z}(t) := z(t) - \hat{z}(t)$, for all $w \in \mathcal{L}_2$. Attention is focused on the design of a linear time-invariant, asymptotically stable, filter of order n with state-space realization

$$\begin{aligned} \mathcal{F}: \; \dot{\hat{x}}(t) &= A_f \hat{x}(t) + B_f y(t), \quad \hat{x}(0) = 0 \\ \hat{z}(t) &= C_f \hat{x} \end{aligned} \tag{5}$$

where the matrices $A_f \in \mathbb{R}^{n \times n}$, $B_f \in \mathbb{R}^{n \times n_y}$ and $C_f \in \mathbb{R}^{n_z \times n}$ are to be found.

It follows from (1)-(5) that the dynamics of the estimation error, $\tilde{z}(t)$, can be described by the following state-space model

$$(\mathcal{S}_e): \; \dot{\xi}(t) = \tilde{A}(\theta_t)\xi(t) + \tilde{B}(\theta_t)w(t) \tag{6}$$

$$\tilde{z}(t) = \tilde{C}(\theta_t)\xi(t) \tag{7}$$

where

$$\tilde{A}(\theta_t) = \begin{bmatrix} A(\theta_t) & 0 \\ B_f C(\theta_t) & A_f \end{bmatrix}, \quad \tilde{B}(\theta_t) = \begin{bmatrix} B(\theta_t) \\ B_f D(\theta_t) \end{bmatrix},$$

$$\tilde{C}(\theta_t) = \begin{bmatrix} L(\theta_t) & -C_f \end{bmatrix}, \;\; \xi = \begin{bmatrix} x^T & \hat{x}^T \end{bmatrix}^T.$$

Also, let $\tilde{A}_i$, $\tilde{B}_i$ and $\tilde{C}_i$ denote $\tilde{A}(\theta_t)$, $\tilde{B}(\theta_t)$ and $\tilde{C}(\theta_t)$, respectively, when $\theta_t = i$, $i = 1, \ldots, \sigma$.

In order to put the $\mathcal{H}_\infty$ filtering problem for system $(\mathcal{S})$ in a stochastic setting, let the space $\mathcal{L}_2[(\Omega, \mathcal{F}, I\!P)]$ of $\mathcal{F}$-measurable processes $\{\tilde{z}(t)\}$ for which

$$\| \tilde{z} \|_2 := \left\{ \mathbf{E}\left[\int_0^\infty \tilde{z}^T(t)\tilde{z}(t)dt \right] \right\}^{\frac{1}{2}} < \infty.$$

For the sake of notation simplification, $||\cdot||_2$ will be used to denote the norm either in $\mathcal{L}_2[(\Omega, \mathcal{F}, I\!P)]$ or in $\mathcal{L}_2$, the later defined by

$$\|w\|_2 := \left[\int_0^\infty w^T(t)w(t)dt \right]^{\frac{1}{2}}, \quad \text{for } w \in \mathcal{L}_2.$$

Before formulating the $\mathcal{H}_\infty$ filtering problem, we recall the notion of *internal mean square stability*.

Definition 2.1: System $(\mathcal{S})$ is said to be internally mean square stable (IMSS), if the solution to the stochastic differential equation

$$\dot{x}(t) = A(\theta_t)x(t)$$

is such that $\boldsymbol{E}[\, \|x(t)\|^2 \,] \to 0$ as $t \to \infty$ for any finite initial condition $x(0) \in \mathbb{R}^n$ and $\theta_0 \in \Xi$. $\square$

This paper is concerned with the following $\mathcal{H}_\infty$ filtering problem for system $(\mathcal{S})$:

Given a scalar $\gamma > 0$, determine an asymptotically stable filter (5) which ensures that the estimation error system $(\mathcal{S}_e)$ is IMSS and its $\mathcal{L}_2$-induced gain, i.e. $\mathcal{H}_\infty$ norm, is less than γ, namely:

$$\|\mathcal{S}_e\|_\infty := \sup_{w \in \mathcal{L}_2} \left\{ \frac{\|\tilde{z}\|_2}{\|w\|_2}; \; w \not\equiv 0, \; \xi(0) = 0 \right\} < \gamma.$$

We conclude this section by recalling a version of Finsler's lemma that will be used in the derivation of the main result of this paper.

Lemma 2.1: Given matrices $\Psi_i = \Psi_i^T \in \mathbb{R}^{n \times n}$ and $H_i \in \mathbb{R}^{m \times n}$, $i = 1, \ldots, \kappa$, then

$$\begin{gathered} x_i^T \Psi_i x_i < 0, \; \forall \, x_i \in \mathbb{R}^n : H_i x_i = 0, \; x_i \neq 0; \\ i = 1, \ldots, \kappa \end{gathered} \tag{8}$$

if and only if there exist matrices $L_i \in \mathbb{R}^{n \times m}$, $i = 1, \ldots, \kappa$, such that

$$\Psi_i + L_i H_i + H_i^T L_i^T < 0, \quad i = 1, \ldots, \kappa. \tag{9}$$

Note that conditions (9) remain sufficient for (8) to hold even when arbitrary constraints are imposed to the scaling matrices L_i, including setting $L_i = L$, $i = 1, \ldots, \kappa$.

III. The $\mathcal{H}_\infty$ Filter

We first recall a version of the bounded real lemma (BRL) for MJL systems that will be used to bound the $\mathcal{H}_\infty$ norm of system $(\mathcal{S}_e)$. The following LMI based BRL can be readily derived from a result in [16], which is given in terms of a set of coupled algebraic Riccati equations.

Lemma 3.1: Consider system $(\mathcal{S}_e)$ and let $\gamma > 0$ be a given scalar. Then the following statements are equivalent:

(a) System $(\mathcal{S}_e)$ is IMSS and $\|\mathcal{S}_e\|_\infty < \gamma$.

(b) There exist matrices $P_i > 0$, $i = 1, \ldots, \sigma$, satisfying the following LMIs:

$$\begin{bmatrix} \tilde{A}_i^T P_i + P_i \tilde{A}_i + \bar{P}_i & P_i \tilde{B}_i & \tilde{C}_i^T \\ \tilde{B}_i^T P_i & -\gamma I & 0 \\ \tilde{C}_i & 0 & -\gamma I \end{bmatrix} < 0, \; i = 1, \ldots, \sigma \tag{10}$$

where

$$\bar{P}_i = \sum_{j=1}^{\sigma} \lambda_{ij} P_j, \quad i = 1, \ldots, \sigma.$$

Furthermore, $V(x(t),\theta_t)=x^T(t)P(\theta_t)x(t)$, where $P(\theta_t)=P_i$ when $\theta_t = i$, $i = 1,\dots,\sigma$ is a stochastic Lyapunov function for the unforced system of $(\mathcal{S}_1)$. □

The *mode-independent* $\mathcal{H}_\infty$ filtering method to be developed in this paper is based on Lemma 3.1 and with the Lyapunov matrix P_i parameterized as follows [1]:

$$P_i = M^T N_i^{-1} M > 0, \quad i = 1,\dots,\sigma \tag{11}$$

where N_i, $i = 1,\dots,\sigma$ are symmetric positive definite matrices and M is a nonsingular matrix. Moreover, it will be assumed that $\lambda_{ii} \neq 0$, $i = 1,\dots,\sigma$. This assumption rules out the existence of a *terminal mode* in the system, namely an operation mode i such that if it is "visited", i.e. $\lambda_{ki} \neq 0$ for some $k \in \Xi$, $k \neq i$, then the system will remain in that operation mode.

The next theorem presents the main result of this paper, which is derived from the inequalities of (10) with P_i as in (11) and using Finsler's lemma together with appropriate parameterizations of the matrix M and the filter matrices.

Theorem 3.1: Consider the system $(\mathcal{S})$ and let $\gamma > 0$ be a given scalar. There exists an n-th order filter (5) which ensures that the estimation error system $(\mathcal{S}_e)$ is IMSS and $\|\mathcal{S}_e\|_\infty < \gamma$ if there exist matrices $R \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n_y}$, $Z \in \mathbb{R}^{n_z\times n}$, and symmetric matrices $\mathcal{N}_i \in \mathbb{R}^{2n\times 2n}$, $i = 1,\dots,\sigma$, satisfying the following LMIs:

$$\begin{bmatrix} \Omega+\Omega^T & \star & \star & \star & \star \\ \mathcal{A}_i-\mathcal{N}_i & \lambda_{ii}\mathcal{N}_i & \star & \star & \star \\ 0 & \mathcal{B}_i^T & -\gamma I & \star & \star \\ \mathcal{C}_i & 0 & 0 & -\gamma I & 0 \\ \Lambda_i\Omega & 0 & 0 & 0 & -\tilde{\mathcal{N}}_i \end{bmatrix} < 0, \ i = 1,\dots,\sigma \tag{12}$$

where

$$\Omega = \begin{bmatrix} R & 0 \\ S+W & W \end{bmatrix}, \tag{13}$$

$$\mathcal{A}_i = \begin{bmatrix} RA_i & 0 \\ SA_i + X + YC_i & X \end{bmatrix}, \tag{14}$$

$$\mathcal{B}_i = \begin{bmatrix} RB_i \\ SB_i + YD_i \end{bmatrix}, \ \mathcal{C}_i = \begin{bmatrix} L_i - Z & -Z \end{bmatrix}, \tag{15}$$

$$\tilde{\mathcal{N}}_i = \mathrm{diag}\{\mathcal{N}_1, \cdots, \mathcal{N}_{i-1}, \mathcal{N}_{i+1}, \cdots, \mathcal{N}_\sigma\}, \tag{16}$$

$$\Lambda_i = \left[\sqrt{\lambda_{i1}} I \cdots \sqrt{\lambda_{i(i-1)}} I \ \ \sqrt{\lambda_{i(i+1)}} I \cdots \sqrt{\lambda_{i\sigma}} I\right]^T \tag{17}$$

Moreover, the transfer function matrix of a suitable filter is given by

$$H_{\hat{z}y}(s) = Z(sI - W^{-1}X)^{-1} W^{-1} Y. \tag{18}$$

Proof. It will be shown that if the inequalities of (12) hold, then the filter (18) ensures that conditions (10) of Lemma 3.1 are satisfied with a matrix $P_i > 0$ as in (11).

First, note that with a matrix P_i of the form (11), the inequalities of (10) are equivalent to:

$$\Phi_i^T \Upsilon_i \Phi_i < 0, \quad i = 1,\dots,\sigma \tag{19}$$

where

$$\Upsilon_i = \begin{bmatrix} 0 & \star & \star & \star & \star \\ M\tilde{A}_i & \lambda_{ii}N_i & \star & \star & \star \\ 0 & \tilde{B}_i^T M^T & -\gamma I_{n_w} & \star & \star \\ \tilde{C}_i & 0 & 0 & -\gamma I_{n_z} & \star \\ \tilde{M}\Lambda_i & 0 & 0 & 0 & -\tilde{N}_i \end{bmatrix}, \tag{20}$$

$$\Phi_i = \begin{bmatrix} I & 0 & 0 & 0 \\ N_i^{-1}M & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{21}$$

$$\tilde{N}_i = \mathrm{diag}\{N_1, \cdots, N_{i-1}, N_{i+1}, \cdots, N_\sigma\}, \tag{22}$$

$$\tilde{M} = \mathrm{diag}\{M, \cdots, M\}. \tag{23}$$

Define $n \times n$ nonsingular matrices U and V such that $UV = W$ and let the matrix M be parameterized as follows:

$$M = \begin{bmatrix} R^{-1} & R^{-1}V^T \\ -U^{-1}SR^{-1} & U^{-1}V^T - U^{-1}SR^{-1}V^T \end{bmatrix}^{-1}. \tag{24}$$

Moreover, let the transformation matrix

$$\mathcal{T} = \begin{bmatrix} R^T & S^T \\ 0 & U^T \end{bmatrix}. \tag{25}$$

Note that

$$M^{-T}\mathcal{T} = \begin{bmatrix} I & 0 \\ V & V \end{bmatrix}. \tag{26}$$

Now it will be shown that the matrices M and $\mathcal{T}$ as above are nonsingular. From (12) it follows that $\Omega+\Omega^T < 0$, which implies that the matrices R and W are nonsingular. Therefore, in view of the definition of U and V, the matrices $\mathcal{T}$ and $M^{-T}\mathcal{T}$ are nonsingular and thus M^{-T} is a nonsingular matrix as well.

Next, introduce the matrices

$$\mathcal{T}_a = \mathrm{diag}\{M^{-T}\mathcal{T}, M^{-T}\mathcal{T}, I_{n_w}, I_{n_z}, \tilde{M}^{-T}\tilde{\mathcal{T}}\} \tag{27}$$

$$\tilde{\mathcal{T}} := \mathrm{diag}\{\mathcal{T}, \cdots, \mathcal{T}\} \tag{28}$$

where $\tilde{\mathcal{T}}$ contains $(\sigma-1)$ blocks $\mathcal{T}$.

Performing the congruent transformation $\mathcal{T}_a^T(\cdot)\mathcal{T}_a$ on Υ_i, inequality (19) is equivalent to:

$$\eta^T \mathcal{T}_a^T \Upsilon_i \mathcal{T}_a \eta < 0, \ \ \eta = \mathcal{T}_a^{-1}\Phi_i\xi, \ \ \forall \xi \in \mathbb{R}^\alpha, \quad \xi \neq 0, \ \ \alpha = 2n\sigma + n_w + n_y . \tag{29}$$

Considering that

$$\left\{\eta : \eta = \mathcal{T}_a^{-1}\Phi_i\xi, \ \forall \xi \in \mathbb{R}^\alpha, \ \xi \neq 0\right\} = \left\{\eta : H_i\mathcal{T}_a\eta = 0, \ \eta \neq 0\right\}$$

where

$$H_i = \begin{bmatrix} M & -N_i & 0 & 0 & 0 \end{bmatrix}, \tag{30}$$

it also holds that (29) is equivalent to

$$\eta^T \mathcal{T}_a^T \Upsilon_i \mathcal{T}_a \eta < 0, \ \ \forall \eta \neq 0 : H_i \mathcal{T}_a \eta = 0 . \tag{31}$$

By Lemma 2.1, (31) holds if the following inequality is feasible for some matrix $\tilde{L}_i$ of appropriate dimensions:

$$\mathcal{T}_a^T \Upsilon_i \mathcal{T}_a + \tilde{L}_i H_i \mathcal{T}_a + \mathcal{T}_a^T H_i^T \tilde{L}_i^T < 0\,. \qquad (32)$$

Without loss of generality, let the matrix $\tilde{L}_i$ be rewritten as $\tilde{L}_i = \mathcal{T}_a^T L_i$. Then (32) is equivalent to:

$$\mathcal{T}_a^T \left(\Upsilon_i + L_i H_i + H_i^T L_i^T \right) \mathcal{T}_a < 0\,. \qquad (33)$$

Setting

$$L_i = \begin{bmatrix} I_{2n} & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

inequality (33) becomes

$$\mathcal{T}_a^T \tilde{\Upsilon}_i \mathcal{T}_a < 0 \qquad (34)$$

where

$$\tilde{\Upsilon}_i = \begin{bmatrix} M + M^T & \star & \star & \star & \star \\ M\tilde{A}_i - N_i^T & \lambda_{ii} N_i & \star & \star & \star \\ 0 & \tilde{B}_i^T M^T & -\gamma I_{n_w} & \star & \star \\ \tilde{C}_i & 0 & 0 & -\gamma I_{n_z} & \star \\ \tilde{M}\Lambda_i & 0 & 0 & 0 & -\tilde{N}_i \end{bmatrix}.$$

Consider the following state-space realization for the filter (18)

$$A_f = VW^{-1}XV^{-1}, \;\; B_f = VW^{-1}Y, \;\; C_f = ZV^{-1} \qquad (35)$$

and let the matrix $\mathcal{N}_i$ be defined as

$$\mathcal{N}_i := \mathcal{T}^T M^{-1} N_i M^{-T} \mathcal{T}. \qquad (36)$$

By performing straightforward matrix manipulations, it can be easily shown that

$$\mathcal{T}^T \tilde{A}_i M^{-T} \mathcal{T} = \mathcal{A}_i, \;\; \mathcal{T}^T \tilde{B}_i = \mathcal{B}_i, \;\; \tilde{C}_i M^{-T} \mathcal{T} = \mathcal{C}_i, \qquad (37)$$

$$\mathcal{T}^T M^{-T} \mathcal{T} = \Omega, \;\; \tilde{\mathcal{T}} \Lambda_i M^{-T} \mathcal{T} = \Lambda_i \Omega, \qquad (38)$$

$$\tilde{\mathcal{T}}^T \tilde{M}^{-1} \tilde{N}_i \tilde{M}^{-T} \tilde{\mathcal{T}} = \tilde{\mathcal{N}}_i. \qquad (39)$$

Next, taking into account (27), (28) and (36)-(39), it can be readily verified that (34) is identical to (12). Thus, (10) is satisfied with $P_i = M^T N_i^{-1} M$. Finally, the filter transfer matrix of (18) is readily obtained from (35). ∇∇∇

Remark 3.1: Theorem 3.1 provides an LMI method for designing a *mode-independent* $\mathcal{H}_\infty$ linear filter for continuous-time MJL systems with a non-accessible jumping parameter. This is contrast with [8] which has developed a *mode-dependent* $\mathcal{H}_\infty$ filter for MJL systems under the assumption that the jumping parameter is accessible. □

IV. Robust $\mathcal{H}_\infty$ Filters

Theorem 3.1 can be easily extended to the case where the transition rate matrix $\Lambda = [\lambda_{ij}]$ is unknown, but belongs to a given polytope, namely $\Lambda \in \mathcal{P}_\Lambda$, where $\mathcal{P}_\Lambda$ is a polytope with vertices Λ_i, $i = 1, \ldots, \mathcal{V}_\Lambda$, i.e.

$$\mathcal{P}_\Lambda := \left\{ \Lambda \mid \Lambda = \sum_{r=1}^{\mathcal{V}_\Lambda} \alpha_r \Lambda_r ; \;\; \alpha_r \geq 0, \; \sum_{r=1}^{\mathcal{V}_\Lambda} \alpha_r = 1 \right\} \qquad (40)$$

where $\Lambda_r = [\lambda_{ij}^{(r)}]$, $i, j = 1, \ldots, \sigma$, $r = 1, \ldots, \mathcal{V}_\Lambda$ are given transition rate matrices. Note that the convex hull of transition rate matrices is also a transition rate matrix.

Attention is focused on designing a mode-independent filter (5) that ensures internal mean square stability and a prescribed $\mathcal{H}_\infty$ performance for the estimation error system $(\mathcal{S}_e)$ over the polytope $\mathcal{P}_\Lambda$. In the above setting, and considering that the inequalities of (12) are affine in λ_{ij}, the following robust $\mathcal{H}_\infty$ filtering result follows readily from Theorem 3.1.

Theorem 4.1: Consider system $(\mathcal{S})$ with an uncertain transition rate matrix Λ belonging to the polytope $\mathcal{P}_\Lambda$ and let $\gamma > 0$ be a given scalar. There exists an n-th order filter (5) which ensures that the estimation error system $(\mathcal{S}_e)$ is IMSS and $\|\mathcal{S}_e\|_\infty < \gamma$ over the polytope $\mathcal{P}_\Lambda$ if there exist matrices $R \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n_y}$, $Z \in \mathbb{R}^{n_z\times n}$, and symmetric matrices $\mathcal{N}_i \in \mathbb{R}^{2n\times 2n}$, $i = 1, \ldots, \sigma$, satisfying the following LMIs:

$$\begin{bmatrix} \Omega + \Omega^T & \star & \star & \star & \star \\ \mathcal{A}_i - \mathcal{N}_i & \lambda_{ii}^{(r)} \mathcal{N}_i & \star & \star & \star \\ 0 & \mathcal{B}_i^T & -\gamma I & \star & \star \\ \mathcal{C}_i & 0 & 0 & -\gamma I & 0 \\ \Lambda_i^{(r)} \Omega & 0 & 0 & 0 & -\tilde{\mathcal{N}}_i \end{bmatrix} < 0,$$

$$r = 1, \ldots, \mathcal{V}_\Lambda, \;\; i = 1, \ldots, \sigma$$

where

$$\Lambda_i^{(r)} = \left[\sqrt{\lambda_{i1}^{(r)}} I \cdots \sqrt{\lambda_{i(i-1)}^{(r)}} I \;\; \sqrt{\lambda_{i(i+1)}^{(r)}} I \cdots \sqrt{\lambda_{i\sigma}^{(r)}} I \right]^T$$

and the other matrices are as defined in (13)-(16). In addition, the transfer function matrix of a suitable filter is given by (18). □

In the sequel, we shall consider the design of a robust $\mathcal{H}_\infty$ filter in the case where the matrices A_i, B_i, C_i, D_i and L_i, $i = 1, \ldots, \sigma$, of system $(\mathcal{S})$ are uncertain, but belong to given matrix polytopes $\mathcal{P}_i$, $i = 1, \ldots, \sigma$, described by:

$$\mathcal{P}_i := \Big\{ (A_i, B_i, C_i, D_i, L_i) \mid (A_i, B_i, C_i, D_i, L_i) = \sum_{k=1}^{\mathcal{V}_i} \alpha_{i_k} \Big(A_{i_k}, B_{i_k}, C_{i_k}, D_{i_k}, L_{i_k} \Big); \; \alpha_{i_k} \geq 0, \; \sum_{k=1}^{\mathcal{V}_i} \alpha_{i_k} = 1 \Big\} \qquad (41)$$

where $\mathcal{V}_i$ is the number of vertices of $\mathcal{P}_i$ and A_{i_k}, B_{i_k}, C_{i_k}, D_{i_k} and L_{i_k} are given matrices.

The filter (5) to be designed should guarantee internal mean square stability and a prescribed $\mathcal{H}_\infty$ performance for the estimation error system $(\mathcal{S}_e)$ over the polytopes $\mathcal{P}_i$, $i = 1, \ldots, \sigma$. To solve this robust $\mathcal{H}_\infty$ filtering problem, a parametric Lyapunov function approach is adopted, where the Lyapunov matrix P_i of (11) depends on the uncertain parameter vector $\alpha_i = [\alpha_{i_1}, \ldots, \alpha_{i_{\mathcal{V}_i}}]^T$ that characterizes the uncertainty matrix polytope $\mathcal{P}_i$. To this end, P_i is assumed to be parameterized in the form $P_i = M^T N_i^{-1}(\alpha_i) M$, with

$$N_i(\alpha_i) = \sum_{k=1}^{\mathcal{V}_i} \alpha_{i_k} N_i^{(k)}$$

where $N_i^{(k)},\ i=1,\dots,\sigma,\ k=1,\dots,\mathcal{V}_i$ are symmetric positive definite matrices.

As the LMIs in (12) are affine in the matrices A_i, B_i, C_i, D_i and L_i, the following robust $\mathcal{H}_\infty$ filtering result is obtained from Theorem 3.1.

Theorem 4.2: Consider system $(\mathcal{S})$ with the matrices A_i, B_i, C_i, D_i and L_i belonging to the polytope $\mathcal{P}_i$, $i=1,\dots,\sigma$ and let $\gamma>0$ be a given scalar. There exists an n-th order filter (5) which ensures that the estimation error system $(\mathcal{S}_e)$ is IMSS and $\|\mathcal{S}_e\|_\infty<\gamma$ over the polytopes $\mathcal{P}_i$, $i=1,\dots,\sigma$, if there exist matrices $R\in\mathbb{R}^{n\times n}$, $S\in\mathbb{R}^{n\times n}$, $W\in\mathbb{R}^{n\times n}$, $X\in\mathbb{R}^{n\times n}$, $Y\in\mathbb{R}^{n\times n_y}$, $Z\in\mathbb{R}^{n_z\times n}$, and symmetric matrices $\mathcal{N}_i^{(k)}\in\mathbb{R}^{2n\times 2n}$, $i=1,\dots,\sigma$, $k=1,\dots,\mathcal{V}_i$, satisfying the following LMIs:

$$\begin{bmatrix} \Omega+\Omega^T & \star & \star & \star & \star \\ \mathcal{A}_i^{(k)}-\mathcal{N}_i^{(k)} & \lambda_{ii}\mathcal{N}_i^{(k)} & \star & \star & \star \\ 0 & (\mathcal{B}_i^{(k)})^T & -\gamma I & \star & \star \\ \mathcal{C}_i^{(k)} & 0 & 0 & -\gamma I & 0 \\ \Lambda_i\Omega & 0 & 0 & 0 & -\tilde{\mathcal{N}}_i^{[\pi_i]} \end{bmatrix}<0,$$

$$k=1,\dots,\mathcal{V}_i,\ \ \forall\pi_i\in J_i,\ \ i=1,\dots,\sigma$$

where Ω and Λ_i are as defined in (13) and (17), respectively,

$$\mathcal{A}_i^{(k)}=\begin{bmatrix} RA_{i_k} & 0 \\ SA_{i_k}+X+YC_{i_k} & X\end{bmatrix},$$

$$\mathcal{B}_i^{(k)}=\begin{bmatrix} RB_{i_k} \\ SB_{i_k}+YD_{i_k}\end{bmatrix},\ \ \mathcal{C}_i^{(k)}=\begin{bmatrix} L_{i_k}-Z & -Z\end{bmatrix},$$

and J_i denotes the set of all $(\sigma-1)$-tuples

$$\pi_i=[p_1,\dots,\ p_{i-1},\ p_{i+1},\dots,\ p_\sigma],\ \forall p_k\in\{1,\dots,\mathcal{V}_k\}$$

where to each π_i is associated the matrix

$$\tilde{\mathcal{N}}_i^{[\pi_i]}=\mathrm{diag}\left\{\mathcal{N}_1^{(p_1)},\cdots,\mathcal{N}_{i-1}^{(p_{i-1})},\mathcal{N}_{i+1}^{(p_{i+1})},\cdots,\mathcal{N}_\sigma^{(p_\sigma)}\right\}$$

Moreover, the transfer function matrix of a suitable filter is given by (18). □

Remark 4.1: It should be noted that the robust $\mathcal{H}_\infty$ filter design method of Theorem 4.2 is based on a Lyapunov function that depends on both the jumping parameter θ_t and the uncertain parameters in the system matrices for each operation mode. This is in contrast with existing robust $\mathcal{H}_\infty$ filtering results for MJL systems, such as those of [2], [9] and [21], which are based on Lyapunov functions that are independent of the system uncertain parameters and require the jumping parameter to be accessible.

Observe that Theorem 4.2 reduces to a filtering method based on an uncertainty-independent Lyapunov function by imposing the constraints $\mathcal{N}_i^{(k)}=\mathcal{N}_i,\ k=1,\dots,\mathcal{V}_i$. Similarly to the case of systems without jumps, it turns out that the result of Theorem 4.2 is less conservative than that based on an uncertainty-independent Lyapunov function. This fact will be illustrated by Example 2 in the next section.

V. EXAMPLES

Two examples are presented to illustrate the applicability of the $\mathcal{H}_\infty$ filter designs of Theorems 3.1, 4.1 and 4.2.

Example 1: Consider the example in [21] without time-delay, namely let the system $(\mathcal{S})$ of (1)-(3) with two operating modes described by:

$$A_1=\begin{bmatrix} -3 & 1 & 0 \\ 0.3 & -2.5 & 1 \\ -0.1 & 0.3 & -3.8\end{bmatrix},\quad B_1=\begin{bmatrix}1\\0\\1\end{bmatrix} \tag{42}$$

$$C_1=\begin{bmatrix}0.8 & 0.3 & 0\end{bmatrix} \tag{43}$$

$$D_1=0.2,\quad L_1=\begin{bmatrix}0.5 & -0.1 & 1\end{bmatrix} \tag{44}$$

$$A_2=\begin{bmatrix} -2.5 & 0.5 & -0.1 \\ 0.1 & -3.5 & 0.3 \\ -0.1 & 1 & -2\end{bmatrix},\quad B_2=\begin{bmatrix}-0.6\\0.5\\0\end{bmatrix} \tag{45}$$

$$C_2=\begin{bmatrix}-0.5 & 0.2 & 0.3\end{bmatrix} \tag{46}$$

$$D_2=0.5,\quad L_2=\begin{bmatrix}0 & 1 & 0.6\end{bmatrix}. \tag{47}$$

Regarding the transition rate matrix Λ, two cases will be considered:

Case 1: Λ is exactly known and given by:

$$\Lambda=\begin{bmatrix}-0.5 & 0.5\\ 0.3 & -0.3\end{bmatrix}.$$

The minimum upper-bound γ on the $\mathcal{H}_\infty$ norm of the estimation error system obtained with the filter design of Theorem 3.1 is $\gamma=0.74038$ and the corresponding filter matrices are

$$A_f=\begin{bmatrix}-0.8413 & -0.2774 & 0.4500\\ 1.4460 & -4.3347 & -1.2151\\ -0.7942 & 1.5064 & -0.7335\end{bmatrix}$$

$$B_f=\begin{bmatrix}0.4219 & 0.8664 & 0.2159\end{bmatrix}^T$$

$$C_f=\begin{bmatrix}0.0789 & 0.3777 & 0.9254\end{bmatrix}.$$

Case 2: Λ is unknown, but belongs to a polytope $\mathcal{P}_\Lambda$, as defined in (40), with 2 vertices given by:

$$\Lambda_1=\begin{bmatrix}-0.35 & 0.35\\ 0.2 & -0.2\end{bmatrix}$$

$$\Lambda_2=\begin{bmatrix}-0.65 & 0.65\\ 0.4 & -0.4\end{bmatrix}.$$

The minimum γ obtained with the robust $\mathcal{H}_\infty$ filter design of Theorem 4.1 is $\gamma=0.92641$ and the corresponding filter matrices are

$$A_f=\begin{bmatrix}-0.5102 & -1.0052 & 0.4784\\ 2.2254 & -5.5181 & -1.4752\\ -1.1438 & 2.1146 & -0.3834\end{bmatrix}$$

$$B_f=\begin{bmatrix}0.4845 & 0.9460 & 0.0772\end{bmatrix}^T$$

$$C_f=\begin{bmatrix}0.0472 & 0.3850 & 0.9535\end{bmatrix}.$$

Example 2: Let the system of Example 1/ Case 1, except that now the matrices in (42)-(47) become the nominal system matrices $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ and $\bar{D}_i$, $i=1,2$, which are subject to uncertainties, namely the matrices A_i, B_i, C_i and D_i are now uncertain and given by:

$$\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} \bar{A}_i & \bar{B}_i \\ \bar{C}_i & \bar{D}_i \end{bmatrix} + \delta_i \begin{bmatrix} L_{1i} \\ L_{2i} \end{bmatrix} \begin{bmatrix} N_{1i} & N_{2i} \end{bmatrix}, \quad i=1,2 \tag{48}$$

where δ_1 and δ_2 are independent unknown constant scalars in the interval $[-1,1]$ and

$$L_{11} = \begin{bmatrix} 0.1 & 0 & 0.2 \end{bmatrix}^T, \quad L_{21} = 0.2$$

$$N_{11} = \begin{bmatrix} 0.2 & 0 & 0.1 \end{bmatrix}, \quad N_{21} = 0.2$$

$$L_{12} = \begin{bmatrix} 0.1 & 0.1 & 0 \end{bmatrix}^T, \quad L_{22} = 0.1$$

$$N_{12} = \begin{bmatrix} 0.1 & 0.1 & 0 \end{bmatrix}, \quad N_{22} = 0.1$$

The above uncertain matrices A_i, B_i, C_i and D_i are of polytopic-type as in (41) with the polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ having two vertices, which are readily obtained from (48) by setting δ_i equal to -1 and 1.

The minimum γ obtained with the robust $\mathcal{H}_\infty$ filter design of Theorem 4.2 is $\gamma = 0.94947$ and the corresponding filter matrices are

$$A_f = \begin{bmatrix} -2.2939 & -0.2702 & -0.3352 \\ -0.3440 & -1.8175 & 0.1267 \\ 0.3881 & 0.1321 & -1.6095 \end{bmatrix}$$

$$B_f = \begin{bmatrix} 1.2222 & 0.6085 & 0.3318 \end{bmatrix}^T$$

$$C_f = \begin{bmatrix} 0.1806 & 0.2499 & 0.7426 \end{bmatrix}.$$

Note that if the Lyapunov matrix P_i of (11) is constrained to be uncertainty-independent, i.e. by setting $\mathcal{N}_i^{(k)} = \mathcal{N}_i$, $k=1,2$ in Theorem 4.2, the minimum achievable γ in this case is $\gamma = 1.32633$. This result demonstrates the superiority of the uncertainty-dependent Lyapunov function approach in terms of achieved performance.

VI. CONCLUSIONS

This paper has addressed the problem of $\mathcal{H}_\infty$ filtering for a class of continuous-time MJL systems where the jumping parameter is not accessible. An LMI method is proposed for designing a *mode-independent* filter that ensures the mean square stability of the estimation error dynamics and a prescribed upper-bound on the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error. Robust $\mathcal{H}_\infty$ filters have also been proposed for MJL systems subject to polytopic-type parameter uncertainty in either the transition rate matrix of the Markov process, or in the matrices of the state-space model for the possible modes of operation of the system. In the later case, we have developed a robust $\mathcal{H}_\infty$ filtering method based on a Lyapunov function that depends on the uncertain parameters.

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