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Exponential Stability for Discrete-Time Infinite Markov Jump Systems

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***Abstract*—This paper addresses a class of discrete-time stochastic systems with infinite Markov jump parameter. First of all, spectral criterion and Lyapunov type criteria are presented for the exponential stability of considered systems. Further, under the condition of strong detectability, a Barbashin-Krasovskii stability theorem is shown in terms of a generalized Lyapunov equation with a positive semi-definite term. As an application of the proposed Lyapunov stability criterion, a sufficient condition of ℓ_2 input-state stability is derived for the infinite Markov jump systems disturbed by finite-energy random disturbance.**

***Index Terms*—exponential stability, strong detectability, infinite Markov jump systems, ℓ_2 input-state stability.**

I. Introduction

MARKOV jump systems have found applications in a wide range of fields, such as robot manipulation [15], fault-tolerant detection [20] and portfolio optimization [23]. Therefore, the related control theory has been extensively studied [10], [16], [19]. Particularly, fruitful results have been devoted to stability analysis [3], [6], [21], [22]. Note that the existing works are mostly carried out on condition that the Markov process has a finite state space. In recent years, infinite Markov jump systems have attracted considerable research attention [1], [2], [7], [11], [12]. It has been recognized that there is essential difference between the performance of finite and infinite Markov jump systems. For example, it is well known that stochastic stability is equivalent to asymptotic mean square stability for stationary finite Markov jump systems, while this is not case when infinite Markov jump systems are concerned [2], [7]. By allowing the involved Markov chain to take infinite possible values, infinite Markov jump systems can be utilized to characterize more real plants in practice. At the same time, some technical challenges arise from the analysis on this subject. An evident difficulty lies in that the causal and anticausal Lyapunov operators associated with infinite Markov jump systems are no more adjoint (see later Remark 2.1).

In this paper, we focus on the exponential stability of discrete-time linear stochastic systems with infinite Markov jump parameter and multiplicative noises. Spectral criterion and Lyapunov type criteria will be established respectively for verifying exponential stability. Further, based on the obtained Lyapunov stability criterion, a sufficient condition of ℓ_2 input-state stability is provided, which guarantees the system state to remain stochastically stable while infinite Markov jump systems are disturbed by random signals with finite energy. It is expected that this result will play a key role to the development of related H_∞ control theory; see [17]. The main contributions of this work are twofold: firstly, by means of the proposed stability criteria, the connections will be clarified among asymptotic stability, stochastic stability and exponential stability. More specifically, if an infinite Markov jump system is exponentially stable, then it must be stochastically stable. However, the converse implication is not true. Secondly, some peculiar properties about infinite Markov jump systems are revealed: it is only a sufficient but not necessary condition of stochastic stability that the spectral radius of causal Lyapunov operator is less than one. Besides, for any finite-energy random disturbance, stochastic stability of the autonomous infinite Markov jump system can not guarantee the disturbed dynamics to be ℓ_2 input-state stable. These results are helpful to better understand the distinctive properties of infinite Markov jump systems.

The outline of this paper is organized as follows. Section 2 provides some technical preliminaries and basic concepts. Section 3 contains the main results. In Subsection 3.1, we are concentrated on the analysis of exponential stability and stabilization. Further, Subsection 3.2 deals with the ℓ_2 input-state stability of disturbed infinite Markov jump systems. Finally, this paper is ended in Section 4 with a brief concluding remark.

Notations. $\mathcal{C}$: the set of all complex numbers; $\mathcal{R}^n$: n-dimensional real Euclidean space; $\mathcal{R}^{m\times n}$: the linear space of all m by n real matrices; $\|\cdot\|$: the Euclidean norm of $\mathcal{R}^n$ or the operator norm of $\mathcal{R}^{m\times n}$; A': the transpose of a matrix (or vector) A; S_n: the set of all $n\times n$ symmetric matrices; $A>0$ (≥ 0): A is positive (semi-)definite; I_n: the $n\times n$ identity matrix; $\mathbf{1}_{(\cdot)}$: the indicator function; $r(\cdot)$: the spectral radius of an operator; $\mathbf{Z}_+=\{0,1,2,\cdots\}$; $\mathcal{D}=\{1,2,\cdots\}$.

II. Preliminaries

On a complete probability space $(\Omega,\mathcal{F},\mathcal{P})$, we consider the following discrete-time linear system with infinite Markov jump parameter and multiplicative noises:

$$\begin{cases} x_{t+1}=A_0(\eta_t)x_t+G_0(\eta_t)u_t+\sum\limits_{k=1}^{d}[A_k(\eta_t)x_t \\ \qquad\quad +G_k(\eta_t)u_t]w_t^k, \\ z_t=C_0(\eta_t)x_t+\sum\limits_{k=1}^{d}C_k(\eta_t)x_tw_t^k,\ t\in\mathbf{Z}_+, \end{cases} \tag{1}$$

where $x_t\in\mathcal{R}^n$, $u_t\in\mathcal{R}^{n_u}$, and $z_t\in\mathcal{R}^{n_z}$ represent the system state, control input, and controlled output, respectively.

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Let $w_t = \{w_t = (w_t^1, \cdots, w_t^d)\}$ be a sequence of wide sense stationary, second-order processes which satisfy $E(w_t) = 0$ and $E(w_t w_s') = I_d \delta_{(t-s)}$ with $\delta_{(\cdot)}$ being a Kronecker function. In (1), the coefficients are all real matrix-valued sequences with suitable dimension, and the Markov jump parameter $\{\eta_t\}_{t\in\mathbf{Z}_+}$ has a countably infinite state space $\mathcal{D}$ with switchings governed by a stationary transition probability matrix $\mathbf{P} = [p(i,j)]$, where $p(i,j) = P(\eta_{t+1} = j|\eta_t = i)$. Assume that $\mathbf{P}$ is nondegenerate. Moreover, the stochastic processes $\{w_t\}_{t\in\mathbf{Z}_+}$ and $\{\eta_t\}_{t\in\mathbf{Z}_+}$ are mutually independent. Let $\mathcal{F}_k$ be the σ-algebra generated by $\{\eta_t, w_s|0 \le t \le k, 0 \le s \le k-1\}$. When $k = 0$, we set $\mathcal{F}_0 = \sigma\{\eta_0\}$. Denote by $l^2(0,\infty;\mathcal{R}^m)$ the space of $\mathcal{R}^m$-valued stochastic processes $\{y_{(t,\omega)} : \mathbf{Z}_+ \times \Omega \to \mathcal{R}^m\}$, which are $\mathcal{F}_t$-measurable and $\sum_{t=0}^{\infty} E\|y_t\|^2 < \infty$. Thus, $l^2(0,\infty;\mathcal{R}^m)$ is a real Hilbert space with the norm induced by the usual inner product: $\|y\|_{l^2(0,\infty;\mathcal{R}^m)} = (\sum_{t=0}^{\infty} E\|y_t\|^2)^{\frac{1}{2}} < \infty$.

Let $\mathbb{H}_1^{m\times n}$ represent the set $\{H|H = (H(1), H(2), \cdots),\ H(i) \in \mathcal{R}^{m\times n}\}$ where H satisfies $\sum_{i=1}^{\infty} \|H(i)\| < \infty$. It can be verified that $\mathbb{H}_1^{m\times n}$ is a Banach space with the norm $\|H\|_1 = \sum_{i=1}^{\infty} \|H(i)\|$. Similarly, we can define another Banach space $\mathbb{H}_\infty^{m\times n}$, where the norm is given by $\|H\|_\infty = \sup_{i\in\mathbf{Z}_+} \|H(i)\|$. In the sequel, all coefficients of the considered systems are set to have a finite norm $\|\cdot\|_\infty$. When $m = n$, $\mathbb{H}_1^{m\times n}$ is simplified as $\mathbb{H}_1^n$ and so is $\mathbb{H}_\infty^{m\times n}$. If $H(i) \in S_n$ and $H(i) \ge 0$ for $i \in \mathcal{D}$, $\mathbb{H}_1^n$ ($\mathbb{H}_\infty^n$) will be denoted by $\mathbb{H}_1^{n+}$ (resp., $\mathbb{H}_\infty^{n+}$). For $M, N \in \mathbb{H}_1^{n+}$, $M \le N$ means that $M(i) \le N(i)$ for all $i \in \mathbf{Z}_+$. In this case, we have $\|M\|_1 \le \|N\|_1$. Moreover, given a Banach space $\mathbb{X}$, we denote by $\mathscr{B}(\mathbb{X})$ the Banach space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{X}$. For $\Gamma \in \mathscr{B}(\mathbb{X})$, its uniform induced norm is represented as $\|\Gamma\|_{\mathbb{X}}$.

For $U \in \mathbb{H}_1^n$ and $V \in \mathbb{H}_\infty^n$, we can introduce the following operators $\mathcal{L}(U)$, $\mathcal{T}(V)$ and $\mathcal{E}(V)$:

$$\begin{cases} \mathcal{L}_i(U) = \sum_{k=0}^{d} \sum_{j=1}^{\infty} p(j,i) A_k(j) U(j) A_k(j)', \\ \mathcal{T}_i(V) = \sum_{k=0}^{d} A_k(i)' [\sum_{j=1}^{\infty} p(i,j) V(j)] A_k(i), \\ \mathcal{E}_i(V) = \sum_{j=1}^{\infty} p(i,j) V(j). \end{cases} \tag{2}$$

Remark 2.1: It is easy to show that $\mathcal{L}$, $\mathcal{T}$ and $\mathcal{E}$ are all linear positive operators. Moreover, $\mathcal{L} \in \mathscr{B}(\mathbb{H}_1^n)$ and $\mathcal{T}, \mathcal{E} \in \mathscr{B}(\mathbb{H}_\infty^n)$. For infinite Markov jump systems, $\mathcal{T}$ is not the adjoint operator of $\mathcal{L}$, as is different from the case of finite Markov jump systems [5]. In fact, $\mathcal{L}$ and $\mathcal{T}$ are defined on two different spaces.

Next, let us recall some basic concepts of infinite Markov jump systems ([2], [14], [18]).

Definition 2.1: The zero state equilibrium of discrete-time linear infinite Markov jump system

$$x_{t+1} = A_0(\eta_t) x_t + \sum_{k=1}^{d} A_k(\eta_t) x_t w_t^k,\ t \in \mathbf{Z}_+, \tag{3}$$

or $(\mathbf{A}; \mathbf{P})$ is called:
(i) asymptotically mean square stable (AMSS) if for any $(x_0, \eta_0) \in \mathcal{R}^n \times \mathcal{D}$, $\lim_{t\to\infty} E(\|x_t\|^2) = 0$;
(ii) stochastically stable if for any $(x_0, \eta_0) \in \mathcal{R}^n \times \mathcal{D}$, $\sum_{t=0}^{\infty} E(\|x_t\|^2) < \infty$;
(iii) exponentially mean square stable with conditioning (EMSS-C) if there exist $\beta \ge 1$ and $\alpha \in (0,1)$ such that $E[\|x_t\|^2 \big| \eta_0 = i] \le \beta\alpha^t \|x_0\|^2$ for all $t \in \mathbf{Z}_+$, $(x_0, i) \in \mathcal{R}^n \times \mathcal{D}_0$ and all $(\{\eta_t\}_{t\in\mathbf{Z}_+}, \{w_t\}_{t\in\mathbf{Z}_+})$ satisfying the previous descriptions. Here, $\mathcal{D}_0 = \{i|P(\eta_0 = i) > 0, i \in \mathcal{D}\}$.

If there exists a sequence $F \in \mathbb{H}_\infty^{n_u \times n}$ such that the zero state equilibrium of the closed-loop system

$$\begin{aligned} x_{t+1} = [A_0(\eta_t) + G_0(\eta_t) F(\eta_t)] x_t + \sum_{k=1}^{d} [A_k(\eta_t) \\ + G_k(\eta_t) F(\eta_t)] x_t w_t^k,\ t \in \mathbf{Z}_+ \end{aligned} \tag{4}$$

is stochastically stable, then system (1) or $(\mathbf{A}, \mathbf{G}; \mathbf{P})$ is called stochastically stabilizable. Here, $u_t = F(\eta_t) x_t$ is called a stabilizing feedback. Further, if system (4) is EMSS-C, then $(\mathbf{A}, \mathbf{G}; \mathbf{P})$ is called exponentially stabilizable and $u_t = F(\eta_t) x_t$ is called an exponentially stabilizing feedback.

Definition 2.2: The discrete-time infinite Markov jump system (1) with $u_t \equiv 0$ or $(\mathbf{A}, \mathbf{C}; \mathbf{P})$ is called:
(i) stochastically detectable, if there exists a sequence $H \in \mathbb{H}_\infty^{n\times n_z}$ such that $(\mathbf{A} + \mathbf{HC}; \mathbf{P})$ is stochastically stable;
(ii) strongly detectable, if $(\mathbf{A} + \mathbf{HC}; \mathbf{P})$ is EMSS-C.

The following formula can be shown by reasoning as Theorem 2.1 of [8].

Proposition 2.1: Let $X_t(i) = E[x_t x_t' \mathbf{1}_{(\eta_t = i)}]$ $(i \in \mathcal{D})$ with x_t being the state of system (3), then $X_t \in \mathbb{H}_1^{n+}$ and $X_{t+1}(i) = \mathcal{L}_i(X_t)$.

III. Main Results

A. Stability and Stabilization

In this subsection, some criteria will be provided for the exponential stability and stabilization of system (3). Above all, let us recall that for any $\Gamma \in \mathscr{B}(\mathbb{X})$, the spectral radius of Γ is given by $r(\Gamma) = \max\{|\lambda| \big| \lambda \in \Lambda(\Gamma)\}$, where $\Lambda(\Gamma)$ is the spectral set of Γ, i.e., the set of all $\lambda \in \mathcal{C}$ such that $\lambda - \Gamma$ is not invertible on $\mathbb{X}$. Next, we give a spectral criterion for exponential stability of system (3).

Lemma 3.1: If $r(\mathcal{L}) < 1$, then $(\mathbf{A}; \mathbf{P})$ is EMSS-C. Conversely, if $(\mathbf{A}; \mathbf{P})$ is EMSS-C, then $r(\mathcal{L}) < 1$.

Proof: "⇒": Let x_t be the state of $(\mathbf{A}; \mathbf{P})$ starting from $(x_0, i) \in \mathcal{R}^n \times \mathcal{D}_0$, then it follows from Proposition 2.1 that $X_t = \mathcal{L}^t(X_0)$, where $X_t(i) = E[x_t x_t' \mathbf{1}_{(\eta_t = i)}]$ and $X_0(j) = \mathbf{1}_{(\eta_0 = j)} x_0 x_0'$. Similar to Corollary 5.1 of [4], we can show that

$$E(\|x_t\|^2 | \eta_0 = i) = x_0' [\mathcal{T}^t(I)](i) x_0, \tag{5}$$

where $I = \{I_n, I_n, \cdots\} \in \mathbb{H}_\infty^n$. By Theorem 6 of [18], there holds

$$x_0' [\mathcal{T}^t(I)](i) x_0 = \|\mathcal{L}^t(X_0)\|_1. \tag{6}$$

If $r(\mathcal{L}) < 1$, then there exist $\beta \ge 1$ and $0 < \alpha < 1$ such that $\|\mathcal{L}^t(X_0)\|_1 \le \beta\alpha^t \|X_0\|_1$, which together with (5) leads to

$$E(\|x_t\|^2 | \eta_0 = i) \le \beta\alpha^t \|X_0\|_1 \le \beta\alpha^t \|x_0\|^2. \tag{7}$$

"$\Leftarrow$": If $(\mathbf{A};\mathbf{P})$ is EMSS-C, by Theorem 3.4 of [14], the operator $\mathcal{T}$ defines an exponentially stable anticausal evolution on $\mathbb{H}_{\infty}^{n}$. From Theorem 7 of [18], we can deduce that $\mathcal{L}$ defines an exponentially stable causal evolution on $\mathbb{H}_1^n$. That is, there exist $\beta \geq 1$ and $\alpha \in (0,1)$ such that $\|\mathcal{L}^t\|_{\mathbb{H}_1^n} \leq \beta\alpha^t$ for all $t \in \mathbf{Z}_+$. By use of Gelfand's formula, we have $r(\mathcal{L}) = \lim_{t\to\infty} \|\mathcal{L}^t\|_{\mathbb{H}_1^n}^{\frac{1}{t}} \leq \alpha < 1$, which completes the proof. ■

From Definition 2.1, it is easy to deduce that if $(\mathbf{A};\mathbf{P})$ is EMSS-C, then it is stochastically stable. Moreover, if $(\mathbf{A};\mathbf{P})$ is stochastically stable, then it must be AMSS. Hence, the following result is trivial.

Corollary 3.1: If $r(\mathcal{L}) < 1$, then $(\mathbf{A};\mathbf{P})$ is stochastically stable and hence AMSS.

In what follows, let us investigate whether $r(\mathcal{L}) < 1$ under the assumption that $(\mathbf{A};\mathbf{P})$ is stochastically stable. The next example will show that this implication does not necessarily hold.

Example 3.1: Consider system (3) with $n = 1$, $d = 1$, $A_0(\eta_t) = A_1(\eta_t) = \frac{\eta_t}{\sqrt{2}(\eta_t+1)}$ ($t \in \mathbf{Z}_+$). The transition probability of η_t is given by $p_{i,i+1} = 1$ and $p_{i,j} = 0$ if $j \neq i+1$. Let $P(\eta_0 = 1) = 1$ (i.e., $\mathcal{D}_0 = \{1\}$). For any $x_0 \in \mathcal{R}^n$, the state of $x_{t+1} = A_0(\eta_t)x_t + A_1(\eta_t)x_t w_t^1$ satisfies

$$E(\|x_t\|^2) = \frac{1}{(1+t)^2}\|x_0\|^2, \ t \in \mathbf{Z}_+, \tag{8}$$

so $(\mathbf{A};\mathbf{P})$ is stochastically stable. However, it is clear that system (3) is not EMSS-C in this case. In fact, we can derive that $r(\mathcal{L}) \geq 1$. In $\mathbb{H}_1^1$, by taking an orthogonal basis $\{e_j\}_{j\in\mathcal{D}}$ with $e_j(i) = \delta_{(j-i)}$, it can be computed that

$$\mathcal{L}^t(e_1) = \frac{1}{(1+t)^2}e_{t+1}, \ \forall t = 1,2,\cdots,$$

which implies

$$\frac{1}{(1+t)^2} = \frac{\|\mathcal{L}^t(e_1)\|_1}{\|e_1\|_1} \leq \sup_{\substack{x\neq 0\\ x\in\mathbb{H}_1^1}} \frac{\|\mathcal{L}^t(x)\|_1}{\|x\|_1} = \|\mathcal{L}^t\|_{\mathbb{H}_1^1}.$$

Thus, we obtain the following inequality:

$$\frac{1}{(1+t)^{\frac{2}{t}}} \leq \|\mathcal{L}^t\|_{\mathbb{H}_1^1}^{\frac{1}{t}}. \tag{9}$$

By letting $t \to \infty$ in (9), we arrive at $r(\mathcal{L}) \geq 1$.

Remark 3.1: The above analysis indicates that when $(\mathbf{A};\mathbf{P})$ is stochastically stable, it may be not EMSS-C. Besides, by Remark 6 of [2], we know that even if $(\mathbf{A};\mathbf{P})$ is AMSS, it is not necessarily stochastically stable. These implications among three stability concepts are summarized in Fig. 1, where "SS" stands for the stochastic stability.

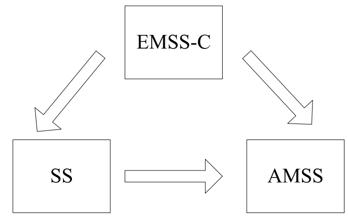

Fig. 1. The relationships among three stability concepts of system (3).

Next, a Lyapunov stability theorem will be presented for exponential stability of system (3). For simplicity, we denote $\tilde{\mathbb{H}}_{\infty}^{n+} = \{H | H \in \mathbb{H}_{\infty}^{n+}, H(i) > \varepsilon I_n, \varepsilon > 0, i \in \mathcal{D}\}$.

Theorem 3.1: $(\mathbf{A};\mathbf{P})$ is EMSS-C if and only if one of the following conditions holds:

(i) there exists a $Y \in \tilde{\mathbb{H}}_{\infty}^{n+}$ such that

$$X(i) - \sum_{k=0}^{d} A_k(i)'\mathcal{E}_i(X)A_k(i) = Y(i), \ i \in \mathcal{D} \tag{10}$$

has a solution $X \in \tilde{\mathbb{H}}_{\infty}^{n+}$;

(ii) there exists a $X \in \tilde{\mathbb{H}}_{\infty}^{n+}$ such that

$$X(i) - \sum_{k=0}^{d} A_k(i)'\mathcal{E}_i(X)A_k(i) > \alpha I_n, \ i \in \mathcal{D}, \tag{11}$$

where $\alpha > 0$ is independent of i;

(iii) for any $Y \in \tilde{\mathbb{H}}_{\infty}^{n+}$, there exists a $X \in \tilde{\mathbb{H}}_{\infty}^{n+}$ such that

$$X(i) - \sum_{k=0}^{d} A_k(i)'\mathcal{E}_i(X)A_k(i) = Y(i), \ i \in \mathcal{D}. \tag{12}$$

Proof: Let x_t be the state of system (3) corresponding to the initial state $(x,i) \in \mathcal{R}^n \times \mathcal{D}_0$. From Definition 2.1 and the formula (5), it follows that $(\mathbf{A};\mathbf{P})$ is EMSS-C if and only if the operator $\mathcal{T}$ generates an exponentially stable anticausal evolution. By use of Theorem 5.2 of [4], the necessity and sufficiency of assertions (i)-(iii) are proven. ■

By use of Theorem 3.1(ii) and the uniform Schur's complement (cf. Theorem 2.2, [17]), we can get the following LMI criteria for exponential stabilization and strong detectability.

Corollary 3.2: (i) $(\mathbf{A},\mathbf{G};\mathbf{P})$ is exponentially stabilizable if there exists a positive definite matrix $X \in S_n$ and $W \in \mathbb{H}_{\infty}^{n_u\times n}$ such that for some $\epsilon > 0$ independent of i,

$$\begin{bmatrix} X & \Phi_i(X,W) \\ \Phi_i(X,W)' & \mathcal{X} \end{bmatrix} > \epsilon I_{n(d+2)}, \ i \in \mathcal{D}, \tag{13}$$

where

$$\begin{aligned} \Phi_i(X,W) &= [A_0(i)X + G_0(i)W(i) \ \cdots \\ &\qquad \cdots \ A_d(i)X + G_d(i)W(i)], \\ \mathcal{X} &= \mathrm{diag}\{\underbrace{X,\cdots,X}_{d+1}\}. \end{aligned}$$

Moreover, if the LMIs (13) are solvable, then an exponentially stabilizing feedback gain is given by $F(i) = W(i)X^{-1}$ ($i \in \mathcal{D}$);

(ii) $(\mathbf{A},\mathbf{C};\mathbf{P})$ is strongly detectable if there is a positive definite matrix $Y \in S_n$ and $Z \in \mathbb{H}_{\infty}^{n\times n_z}$ such that for some $\epsilon' > 0$ independent of i,

$$\begin{bmatrix} Y & \Psi_i(Y,Z) \\ \Psi_i(Y,Z)' & \mathcal{Y} \end{bmatrix} > \epsilon' I_{n(d+2)}, \ i \in \mathcal{D}, \tag{14}$$

where

$$\begin{aligned} \Psi_i(Y,Z) &= [YA_0(i) + Z(i)C_0(i) \ \cdots \\ &\qquad \cdots \ YA_d(i) + Z(i)C_d(i)], \\ \mathcal{Y} &= \mathrm{diag}\{\underbrace{Y,\cdots,Y}_{d+1}\}. \end{aligned}$$

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Moreover, if the LMIs (14) are feasible, then a stabilizing injection is given by $H(i) = Y^{-1}Z(i)$ $(i \in \mathcal{D})$.

Remark 3.2: By Definition 2.2, if $(\mathbf{A}, \mathbf{C}; \mathbf{P})$ is strongly detectable, then it must be stochastically detectable. Hence, Corollary 3.2(ii) also supplies a sufficient condition for checking stochastic detectability of $(\mathbf{A}, \mathbf{C}; \mathbf{P})$. Similarly, Corollary 3.2(i) can ensure $(\mathbf{A}, \mathbf{G}; \mathbf{P})$ to be stochastically stabilizable.

As an immediate application of Theorem 3.1(i), it is easy to obtain a Lyapunov-equation criterion of exponential stabilization and strong detectability, which is of theoretical value.

Corollary 3.3: (i) $(\mathbf{A}, \mathbf{G}; \mathbf{P})$ is exponentially stabilizable if and only if there exists a pair $(X, Y) \in \tilde{\mathbb{H}}_\infty^{n+} \times \tilde{\mathbb{H}}_\infty^{n+}$ such that

$$X(i) = \sum_{k=0}^{d} [A_k(i) + G_k(i)F(i)]' \mathcal{E}_i(X)[A_k(i) + G_k(i)F(i)] + Y(i), \qquad i \in \mathcal{D} \tag{15}$$

admits a solution $F \in \mathbb{H}_\infty^{n_u \times n}$;
(ii) $(\mathbf{A}, \mathbf{C}; \mathbf{P})$ is strongly detectable if and only if there exists a pair $(Z, W) \in \tilde{\mathbb{H}}_\infty^{n+} \times \tilde{\mathbb{H}}_\infty^{n+}$ such that

$$W(i) = \sum_{k=0}^{d} [A_k(i) + H(i)C_k(i)]' \mathcal{E}_i(W)[A_k(i) + H(i)C_k(i)] + Z(i), \qquad i \in \mathcal{D} \tag{16}$$

has a solution $H \in \mathbb{H}_\infty^{n \times n_z}$.

Example 3.2: Suppose that the state space of infinite Markov chain $\{\eta_t\}_{t \in \mathbf{Z}_+}$ can be divided into several clusters $\mathcal{D} = \bigcup_{j=1}^{N} \mathcal{D}_j$, where $\mathcal{D}_j = \{j_1, j_2, \cdots\}$. Moreover, the transition probability among different clusters satisfies $P(\eta_{t+1} \in \mathcal{D}_j | \eta_t \in \mathcal{D}_i) = 0$ if $i \neq j$. Then, $(\mathbf{A}, \mathbf{G}; \mathbf{P})$ is exponentially stabilizable if and only if $(\mathbf{A}_j, \mathbf{G}_j; \mathbf{P}_j)$ is exponentially stabilizable. Here, $\mathbf{A}_j$ denotes $\{A_0(i), \cdots, A_d(i)\}$ while $i \in \mathcal{D}_j$, and so does $\{\mathbf{G}_j, \mathbf{P}_j\}$. More specifically, according to the division of $\mathcal{D}$, the equation (15) can be decoupled as follows:

$$\begin{aligned} X(j_s) = Y(j_s) &+ \sum_{k=0}^{d} [A_k(j_s) + G_k(j_s)F(j_s)]' \\ &\cdot [\sum_{j_k \in \mathcal{D}_j} p(j_s, j_k)X(j_k)][A_k(j_s) + G_k(j_s)F(j_s)], \\ &j_s \in \mathcal{D}_j, \ j = 1, 2, \cdots, N. \end{aligned} \tag{17}$$

Clearly, (15) is feasible if and only if the Lyapunov equations (17) are all feasible. By Corollary 3.2(i), the previous conclusion is in order. By the same arguments, $(\mathbf{A}, \mathbf{C}; \mathbf{P})$ is strongly detectable if and only if $(\mathbf{A}_j, \mathbf{C}_j; \mathbf{P}_j)$ is strongly detectable.

Now, we are prepared to prove a Barbashin-Krasovskii stability theorem under the condition of strong detectability.

Theorem 3.2: If $(\mathbf{A}, \mathbf{C}; \mathbf{P})$ is strongly detectable, then $(\mathbf{A}; \mathbf{P})$ is EMSS-C if and only if there exists a $X \in \mathbb{H}_\infty^{n+}$ such that

$$X(i) - \sum_{k=0}^{d} A_k(i)' \mathcal{E}_i(X) A_k(i) = Q(i), \ i \in \mathcal{D}, \tag{18}$$

where $Q(i) = \sum_{k=0}^{d} C_k(i)' C_k(i)$.

Proof: Let us first show the "only if" assertion. To this end, we construct the following backward difference equations:

$$\begin{cases} X(t,i) - \sum_{k=0}^{d} A_k(i)' \mathcal{E}_i(X(t+1)) A_k(i) = Q(i), \\ X(T+1, i) = 0 \in S_n, \ i \in \mathcal{D}, \ t = 0, 1, \cdots, T. \end{cases} \tag{19}$$

It is obvious that there exits a unique solution $X_T(t) \in \mathbb{H}_\infty^{n+}$ satisfying (19). Thus, we have

$$\begin{aligned} &E[x'_{t+1} X_T(t+1, \eta_{t+1}) x_{t+1} - x'_t X_T(t, \eta_t) x_t | \eta_0] \\ &= E\left\{ x'_t \left[\sum_{k=0}^{d} A_k(\eta_t)' \mathcal{E}_{\eta_t}(X(t+1)) A_k(\eta_t) \right] x_t \middle| \eta_0 \right\} \\ &\quad - E[x'_t X_T(t, \eta_t) x_t | \eta_0] \\ &= -E[x'_t Q(\eta_t) x_t | \eta_0], \end{aligned} \tag{20}$$

where x_t is the state of $(\mathbf{A}; \mathbf{P})$. Summing up (20) from 0 to T and using (19), we get

$$0 \leq x'_0 X_T(0, i) x_0 = \sum_{t=0}^{T} E[x'_t Q(\eta_t) x_t | \eta_0 = i]. \tag{21}$$

On one hand, (21) implies that for any $x_0 \in \mathcal{R}^n$, $i \in \mathcal{D}_0$, and $T \in \mathbf{Z}_+$,

$$\begin{aligned} x'_0 X_T(0, i) x_0 &= \sum_{t=0}^{T} E[x'_t Q(\eta_t) x_t | \eta_0 = i] \\ &\leq \sum_{t=0}^{T+1} E[x'_t Q(\eta_t) x_t | \eta_0 = i] = x'_0 X_{T+1}(0, i) x_0. \end{aligned} \tag{22}$$

On the other hand, since $(\mathbf{A}; \mathbf{P})$ is EMSS-C, we can select an infinite Markov chain $\{\eta_t\}_{t \in \mathbf{Z}_+}$ with $\mathcal{D}_0 = \mathcal{D}$ (see Definition 2.1). Because $Q \in \mathbb{H}_\infty^{n+}$, it follows from (21) that

$$\begin{aligned} x'_0 X_T(0, i) x_0 &= \sum_{t=0}^{T} E[x'_t Q(\eta_t) x_t | \eta_0 = i] \\ &\leq \|Q\|_\infty \sum_{t=0}^{\infty} E[\|x_t\|^2 | \eta_0 = i] < \infty. \end{aligned} \tag{23}$$

Due to the arbitrariness of x_0, (22) and (23) guarantee that $X_T(0, i) \to X(i) \geq 0$ $(i \in \mathcal{D})$ as $T \to \infty$. Moreover, because the coefficients of (19) and the transition probability are time-invariant, there stands $X_T(1, i) = X_{T-1}(0, i)$. Now, taking $t = 0$ in (19) and letting $T \to \infty$, we conclude that (18) admits a solution $X \in \mathbb{H}_\infty^{n+}$.

Next, we will show the converse assertion. To this end, let us check two operators $\mathcal{M} : \mathbb{H}_\infty^{n_z} \to \mathbb{H}_\infty^{n}$ and $\Xi : \mathbb{H}_\infty^{n+n_z} \to \mathbb{H}_\infty^{n+n_z}$:

$$\mathcal{M}_i(X) = \sum_{k=0}^{d} C_k(i)' X(i) C_k(i), \tag{24}$$

$$\Xi_i(Y) = \sum_{k=0}^{d} \begin{bmatrix} A_k(i) Y(i) A_k(i)' & A_k(i) Y(i) C_k(i)' \\ C_k(i) Y(i) A_k(i)' & C_k(i) Y(i) C_k(i)' \end{bmatrix}. \tag{25}$$

Clearly, Ξ is a positive Lyapunov operator. Moreover, since $(\mathbf{A}, \mathbf{C}; \mathbf{P})$ is strongly detectable, by Definition 2.2 and Lemma

3.1, there exists $H \in \mathbb{H}_\infty^{n\times n_z}$ such that $r(\Upsilon^H) < 1$, where the operator Υ^H is given by

$$\Upsilon_i^H(U) = \sum_{k=0}^{d}\sum_{j=1}^{\infty} p(j,i)[A_k(j) + H(j)C_k(j)]U(j)[A_k(j) + H(j)C_k(j)]', \ \forall U \in \mathbb{H}_1^n.$$

According to Definition 9 of [18], the pair $(\mathcal{T}, \mathcal{M})$ is weakly detectable. Further, it is easy to compute that $\mathcal{M}_i(I) = Q(i)$ where $I = \{I_{n_z}, I_{n_z}, \cdots\} \in \mathbb{H}_\infty^{n_z}$. From Corollary 19 of [18], we deduce that if the equation (18) admits a solution $X \in \mathbb{H}_\infty^{n+}$, then $\mathcal{T}$ defines an exponentially stable anticausal evolution on $\mathbb{H}_\infty^{n+}$. By Theorem 3.4 of [14], $(\mathbf{A};\mathbf{P})$ is EMSS-C. The proof is completed. ■

Remark 3.3: We should point out that in the above theorem, X and Q are only required to be positive semi-definite; while in Theorem 3.1(i), both X and Y must be uniformly positive. It can be further shown that, the positive semi-definite solution of (18), if it exists, must be unique. By Theorem 3.2, the structural property is incorporated into the stability analysis. Hence, this result has potential applications to treat the infinite horizon control problems of (1), such as LQ [2] and mixed H_2/H_∞ control [13].

B. ℓ_2 Input-State Stability

In this subsection, we focus on the following discrete-time infinite Markov jump system:

$$x_{t+1} = A_0(\eta_t)x_t + B_0(\eta_t)v_t + \sum_{k=1}^{d}[A_k(\eta_t)x_t + B_k(\eta_t)v_t]w_t^k, \qquad t \in \mathbf{Z}_+, \tag{26}$$

where $v_t \in l^2(0,\infty;\mathcal{R}^{n_v})$ is a random disturbance signal. In the study of disturbance attenuation performance and H_∞ control of stochastic systems, a fundamental prerequisite to evaluate the ℓ_2-gain (i.e., H_∞ norm) is that the disturbed system is stochastically stable if the exogenous disturbance $v \in l^2(0,\infty;\mathcal{R}^{n_v})$. From a physical viewpoint, this means that if the disturbance signal has finite energy, then the resulting system state retains finite energy. However, this property does not always stand. Therefore, we aim to investigate under what condition the aforementioned input-state stability holds true.

Definition 3.1: System (26) or $(\mathbf{A},\mathbf{B};\mathbf{P})$ is called ℓ_2 input-state stable if for any $x_0 \in \mathcal{R}^n$ and $\eta_0 \in \mathcal{D}$, $x \in l^2(0,\infty;\mathcal{R}^n)$ whenever $v \in l^2(0,\infty;\mathcal{R}^{n_v})$.

Below, we show the main result of this subsection.

Theorem 3.3: If $(\mathbf{A};\mathbf{P})$ is EMSS-C, then $(\mathbf{A},\mathbf{B};\mathbf{P})$ is ℓ_2 input-state stable.

Proof: If $(\mathbf{A};\mathbf{P})$ is EMSS-C, by Theorem 3.1(iii), there is a $X \in \tilde{\mathbb{H}}_\infty^{n+}$ $(I \le X < \gamma I, \ \gamma > 1)$ satisfying

$$X(i) - \sum_{k=0}^{d} A_k(i)'\mathcal{E}_i(X)A_k(i) = I_n, \ i \in \mathcal{D}. \tag{27}$$

Without loss of generality, take an infinite Markov chain satisfying $\mathcal{D}_0 = \mathcal{D}$. For any $(x_0,\eta_0) \in \mathcal{R}^n \times \mathcal{D}$ and $v \in l^2(0,\infty;\mathcal{R}^{n_v})$, let x_t be the corresponding state of $(\mathbf{A},\mathbf{B};\mathbf{P})$, then we have

$$\begin{aligned} &E[x_{t+1}'X(\eta_{t+1})x_{t+1}|\eta_0] - E[x_t'X(\eta_t)x_t|\eta_0] \\ &= -E\left\{\|x_t\|^2 + \sum_{k=0}^{d}[2x_t'A_k(\eta_t)'\mathcal{E}_{\eta_t}(X)B_k(\eta_t)v_t + v_t'B_k(\eta_t)'\mathcal{E}_{\eta_t}(X)B_k(\eta_t)v_t]\Big|\eta_0\right\}. \end{aligned} \tag{28}$$

By completing the squares, (28) can be rewritten as follows:

$$\begin{aligned} &E[x_{t+1}'X(\eta_{t+1})x_{t+1}|\eta_0] - E[x_t'X(\eta_t)x_t|\eta_0] \\ &= -\frac{1}{2}E(\|x_t\|^2|\eta_0) - \frac{1}{2}\sum_{k=0}^{d}E[\|\frac{1}{\sqrt{d+1}}x_t \\ &\quad -2\sqrt{d+1}A_k(\eta_t)'\mathcal{E}_{\eta_t}(X)B_k(\eta_t)v_t\|^2|\eta_0] \\ &\quad +\sum_{k=0}^{d}E\{v_t'B_k(\eta_t)'[\mathcal{E}_{\eta_t}(X) + 2(d+1)\mathcal{E}_{\eta_t}(X) \\ &\quad \cdot A_k(\eta_t)A_k(\eta_t)'\mathcal{E}_{\eta_t}(X)]B_k(\eta_t)v_t|\eta_0\}. \end{aligned} \tag{29}$$

Since $X \in \tilde{\mathbb{H}}_\infty^{n+}$ and $\max\{\|A\|_\infty, \|B\|_\infty\} < \infty$, there is $\mu > 0$ such that

$$\begin{aligned} &\|B_k(\eta_t)'[\mathcal{E}_{\eta_t}(X) + 2(d+1)\mathcal{E}_{\eta_t}(X) \\ &\quad \cdot A_k(\eta_t)A_k(\eta_t)'\mathcal{E}_{\eta_t}(X)]B_k(\eta_t)\|_\infty \le \mu, \end{aligned}$$

which, together with (29), leads to

$$\begin{aligned} &E[x_{t+1}'X(\eta_{t+1})x_{t+1}|\eta_0] - E[x_t'X(\eta_t)x_t|\eta_0] \\ &\le -\frac{1}{2}E(\|x_t\|^2|\eta_0) + \mu(d+1)E(\|v_t\|^2|\eta_0). \end{aligned} \tag{30}$$

Recalling $I \le X < \gamma I$, we have $-E(\|x_t\|^2|\eta_0) < -\gamma^{-1}E[x_t'X(\eta_t)x_t|\eta_0]$. Thus, it follows from (30) that

$$\begin{aligned} &E[x_{t+1}'X(\eta_{t+1})x_{t+1}|\eta_0] < \varrho E[x_t'X(\eta_t)x_t|\eta_0] \\ &\quad +\mu(d+1)E(\|v_t\|^2|\eta_0), \ \varrho = 1 - \frac{1}{2\gamma}. \end{aligned} \tag{31}$$

By induction on (31), we can obtain

$$\begin{aligned} &E[x_t'X(\eta_t)x_t|\eta_0] < \varrho^t E[x_0'X(\eta_0)x_0|\eta_0] \\ &\quad +\mu(d+1)\sum_{s=0}^{t-1}\varrho^{t-s-1}E(\|v_s\|^2|\eta_0). \end{aligned} \tag{32}$$

Now, via taking sum over $[0,\infty)$ and noting $I \le X$, (32) gives

$$\begin{aligned} &\sum_{t=0}^{\infty}E(\|x_t\|^2|\eta_0) \le \sum_{t=0}^{\infty}E[x_t'X(\eta_t)x_t|\eta_0] \\ &< \mu(d+1)\sum_{t=0}^{\infty}\sum_{s=0}^{t-1}\varrho^{t-s-1}E(\|v_s\|^2|\eta_0) \\ &\quad +\sum_{t=0}^{\infty}\varrho^t E[x_0'X(\eta_0)x_0|\eta_0]. \end{aligned} \tag{33}$$

By rearranging the order of summation in (33), it is got that

$$\begin{aligned} \sum_{t=0}^{\infty}E(\|x_t\|^2|\eta_0) &< \frac{1}{1-\varrho}\{E[x_0'X(\eta_0)x_0|\eta_0 \\ &\quad +\mu(d+1)\sum_{t=0}^{\infty}E(\|v_t\|^2|\eta_0)\}, \end{aligned} \tag{34}$$

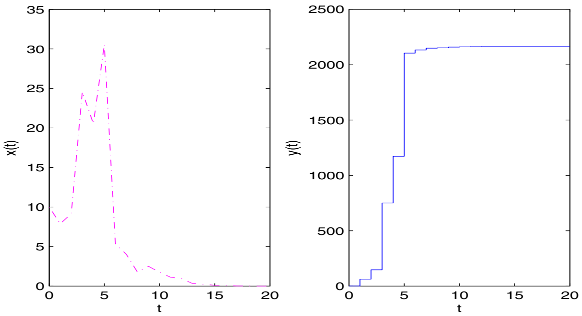

Fig. 2. A sampled state trajectory of (26) and the tendency of $y(t) = \sum_{k=0}^{t} E\|x_k\|^2$.

which yields that for any $(x_0, \eta_0) \in \mathcal{R}^n \times \mathcal{D}$, if $\sum_{t=0}^{\infty} E(\|v_t\|^2) < \infty$, then $\sum_{t=0}^{\infty} E(\|x_t\|^2) < \infty$. The proof is completed. ■

Example 3.3: In (26), we take the coefficients $A_0(\eta_t) = \frac{\eta_t}{\sqrt{2}(\eta_t+1)}$, $A_1(\eta_t) = \frac{\eta_t}{\sqrt{3}(\eta_t+1)}$, and $B_0(\eta_t) = B_1(\eta_t) \equiv 1$, then $(\mathbf{A}; \mathbf{P})$ becomes EMSS-C. When $v_t = 8e^{-t} \sin 10t$, according to Theorem 3.3, the disturbed system is ℓ_2 input-state stable. Simulations have been demonstrated in Fig.2, where the initial condition is $(x_0, \eta_0) = (10, 1)$. Clearly, Fig.2 shows that $\sum_{k=0}^{\infty} E\|x_k\|^2$ is finite.

As shown in [9], if $(\mathbf{A}, \mathbf{B}; \mathbf{P})$ is ℓ_2 input-state stable, then $(\mathbf{A}; \mathbf{P})$ is stochastically stable. However, the following example shows that the stochastic stability of $(\mathbf{A}; \mathbf{P})$ can not guarantee $(\mathbf{A}, \mathbf{B}; \mathbf{P})$ to be ℓ_2 input-state stable.

Example 3.4: Let $B_0(\eta_t) = B_1(\eta_t) \equiv 1$ and the other descriptions of system (26) be the same as in Example 3.1, then it has been verified that $(\mathbf{A}; \mathbf{P})$ is stochastically stable. For the random signal $v_t = \sqrt{\frac{t+1}{2(t+2)}} \cdot x_t \in l^2(0, \infty; \mathcal{R}^1)$, the resulting state of system (26) satisfies

$$\begin{aligned} E\|x_t\|^2 &= \left[\frac{t}{t+1} + (\frac{t}{t+1})^{\frac{1}{2}}\right]^2 E\|x_{t-1}\|^2 \\ &\geq \frac{t}{t+1} E\|x_{t-1}\|^2, \end{aligned} \tag{35}$$

which leads to $E\|x_t\|^2 \geq \frac{1}{t+1}\|x_0\|^2$ for any $x_0 \in \mathcal{R}^1$. So, $(\mathbf{A}, \mathbf{B}; \mathbf{P})$ is not ℓ_2 input-state stable in this case.

IV. Conclusion

In this paper, we have studied the exponential stability and ℓ_2 input-state stability of discrete-time infinite Markov jump systems with multiplicative noises. For exponential stability, spectral criterion and Lyapunov type criteria have been established respectively. By combing with strong detectability, a Barbashin-Krasovskii stability theorem has also been provided, which will be useful in the infinite horizon control designs of the considered systems. Moreover, the ℓ_2 input-state stability has also been discussed for the disturbed infinite Markov jump systems. This research makes it possible to explore the H_∞ theory about system (26) in future.

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