have: $1 = \sum_{k=0}^{\infty} \tilde{\pi}_k =_{(\text{by}(39)\text{and}(41))} \tilde{\pi}_0 + \sum_{k=1}^{I} \pi'_k \tilde{\pi}_0/\varphi = \tilde{\pi}_0(1 + \sum_{k=1}^{I} \pi'_k/\varphi) =_{(\text{by}(20))} \tilde{\pi}_0[1 + (1-\varphi)/\varphi] = \tilde{\pi}_0/\varphi$. Therefore, $\tilde{\pi}_0 = \varphi$ and thus from (41) we have $\tilde{\pi}_k = \pi'_k$ for all $k \in \{1, 2, \ldots, I\}$.

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Finite Horizon Quadratic Optimal Control and a Separation Principle for Markovian Jump Linear Systems

O. L. V. Costa and E. F. Tuesta

***Abstract*—In this note, we consider the finite-horizon quadratic optimal control problem of discrete-time Markovian jump linear systems driven by a wide sense white noise sequence. We assume that the output variable and the jump parameters are available to the controller. It is desired to design a dynamic Markovian jump controller such that the closed-loop system minimizes the quadratic functional cost of the system over a finite horizon period of time. As in the case with no jumps, we show that an optimal controller can be obtained from two coupled Riccati difference equations, one associated to the optimal control problem when the state variable is available, and the other one associated to the optimal filtering problem. This is a principle of separation for the finite horizon quadratic optimal control problem for discrete-time Markovian jump linear systems. When there is only one mode of operation our results coincide with the traditional separation principle for the linear quadratic Gaussian control of discrete-time linear systems.**

***Index Terms*—Discrete time, finite horizon, Markovian jump systems, quadratic cost, separation principle.**

I. INTRODUCTION

In this note, we analyze a family of multiple model linear systems known as Markovian jump linear systems (MJLSs). In this family, it is assumed that the real system can be represented by a finite number of possible discrete-time linear models, and transitions among them follow a Markov chain $\theta(k) \in \{1, \ldots, N\}$. The possible values of the Markov parameter characterizes the "operation modes" of the system. This class can model dynamic systems which are inherently vulnerable to abrupt changes in their structures due, for instance, to component and or interconnections failures, abrupt environment changes, etc. For this class of systems the problems of stability, filtering, and optimal control have been analyzed in the recent literature, for example, in [1], [3]–[8], and [11]–[17].

In this note, we consider the situation in which only an output and the jump parameters of the system are available to the controller for the finite horizon linear quadratic problem of MJLS. As pointed out in [12], the optimal x-state estimator for this case is a Kalman filter (see Remark 3) for a time-varying system, and therefore the gains will be sample path dependent (see the optimal controller presented in [4], which is based on this filter). In order to get around with the sample path dependence, the authors in [12] propose a Markovian filter (that is, a filter that depends just on the present value of the Markovian parameter), based on a posterior estimate of the jump parameters. Notice however that no proof of optimality for this class of Markovian filters is presented in [12], since the authors are mainly interested in the steady-state convergence properties of the filter. In this note, we restrict ourselves to a class of dynamic Markovian jump controllers, and it is desired to minimize the quadratic cost over a finite horizon among

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all other dynamic Markovian jump controllers. As in the case when there is only one mode of operation, we show that the optimum controller (among the class of Markovian jump controllers) can be obtained from two coupled Riccati difference equations, one of them associated to the optimum controller when the state variables are available, and the other one associated to the optimum filtering problem. The proof of this property, known as a "separation principle" for discrete time Markovian jump linear systems, is the main goal of the present note. Steady-state gains are discussed in Remarks 2, 4, and 6.

We follow an approach similar to the standard theory for Kalman filter and linear quadratic Gaussian control (see, for instance, [2] and [9]) to develop a separation principle for MJLS. However, a key point in our formulation is the introduction of the indicator function for the Markov parameter in the orthogonality between the estimation error and the state estimation, presented in Theorem 2. This generalizes the standard case, in which there is only one mode of operation, so that the indicator function in this case is always equal to one. The introduction of the indicator function for the Markov parameter in Theorem 2 is essential for obtaining the principle of separation, presented in Section V.

This note is organized in the following way. In Section II, we present some notations, definitions and preliminary results, and in Section III, we recall some basic facts for the finite horizon linear quadratic optimal control problem for discrete-time Markovian jump linear systems. In Section IV, we solve the filtering problem for this class of problems. The separation principle is presented in Section V. All proofs are presented in the Appendix.

II. Notations, Definitions, and Problem Statement

A. Notations and Definitions

Let $\mathcal{H}^n$ be the linear space made up of all sequence of complex n by n matrices $(H_1, H_2, \ldots, H_N)$, and $\mathcal{H}^{n+} = \{(H_1, H_2, \ldots, H_N) \in \mathcal{H}^n; H_i \geq 0, i = 1, 2, \ldots, N\}$, where $L \geq 0 (L > 0)$ indicates that a self-adjoint matrix L is positive semi-definite (definite respectively). We set * for the transpose conjugate of a complex matrix, and will denote by $tr(.)$ the trace of a matrix (recall that for matrices A, B of appropriate dimensions, $tr(AB) = tr(BA)$). Define the sets $\mathbb{N} = \{1, \ldots, N\}$ and $\mathbb{T} = \{0, 1, \ldots T\}$ where N and T are positive integer. On a probabilistic space $(\Omega, \mathcal{F}, \mathcal{P})$, let $\{\theta(k) \in \mathbb{N}\}$ be a time-varying Markov chain taking values in $\mathbb{N}$ with transition probability $p_{ij}(k)$, $k \in \mathbb{T}$, where $p_{ij}(k) \geq 0$ for $i, j \in \mathbb{N}$, $\sum_{j=1}^{N} p_{ij}(k) = 1$ for each $i \in \mathbb{N}$, and for each $k \in \mathbb{T}$. We set $\pi_i(k) = \mathcal{P}(\theta(k) = i)$. For a set $A \in \mathcal{F}$ the indicator function 1_A is defined in the usual way, that is, for any $\omega \in \Omega$

$$1_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Finally, we define for $k \in \mathbb{T}$, $i \in \mathbb{N}$, and $V = (V_1, \ldots, V_N) \in \mathcal{H}^n$,

$$\mathcal{E}_i(V, k) = \sum_{j=1}^{N} p_{ij}(k) V_j. \tag{2}$$

B. Problem Statement

On the probabilistic space $(\Omega, \mathcal{F}, \mathcal{P})$, consider the following MJLS $\mathcal{G}_G$:

$$\mathcal{G}_G = \begin{cases} x(k+1) = A_{\theta(k)}(k)x(k) + B_{\theta(k)}(k)u(k) \\ \qquad\qquad + G_{\theta(k)}(k)w(k) \\ y(k) = L_{\theta(k)}(k)x(k) + H_{\theta(k)}(k)w(k) \\ z(k) = C_{\theta(k)}(k)x(k) + D_{\theta(k)}(k)u(k) \end{cases} \tag{3}$$

where $\{x(k); k \in \mathbb{T}\}$ represents the n-dimensional state vector, $\{u(k); k \in \mathbb{T}\}$ the m-dimensional control sequence, $\{w(k); k \in \mathbb{T}\}$ a q-dimensional wide sense white noise sequence, (that is, $E(w(k)) = 0$, $E(w(k)w(k)^*) = I$, $E(w(k)w(l)^*) = 0$ for $k \neq l$), $\{y(k); k \in \mathbb{T}\}$ the p-dimensional sequence of measurable variables, $\{z(k); k \in \mathbb{T}\}$ the r-dimensional output sequence. We assume that $C_i(k)^* D_i(k) = 0$, $D_i(k)^* D_i(k) > 0$, $G_i(k) H_i(k)^* = 0$, $H_i(k) H_i(k)^* > 0$ for $i \in \mathbb{N}, k \in \mathbb{T}$, and that the output and "operation modes" ($y(k)$, $\theta(k)$, respectively) are known at each time k. The noise $\{w(k); k \in \mathbb{T}\}$ and the Markov chain $\{\theta(k); k \in \mathbb{T}\}$ are independent sequences, and the initial condition $(x(0), \theta(0))$ are independent random variables with $E(x(0)) = \mu$, $E(x(0)x(0)^*) = S$.

Remark 1: The orthogonality assumptions $C_i(k)^* D_i(k) = 0$ and $G_i(k) H_i(k)^* = 0$ are considered here just for simplicity and could have been eliminated. Indeed if they do not hold then the control and filtering coupled Riccati difference equations (14) and (24) would have the extra terms $C_i(k)^* D_i(k)$ and $G_i(k) H_i(k)^*$ respectively as, for instance, in [9, pp. 118 and 258], for the case with no jumps.

We represent by $\mathcal{F}_k$ the σ-field generated by the random variables $\{y(t), \theta(t); t = 0, \ldots, k\}$. Thus, $\{\mathcal{F}_k\}_{k \in \mathbb{T}}$ is an increasing family of filtering with $\mathcal{F}_k \subset \mathcal{F}_{k+1} \subset \mathcal{F}$ and $\mathcal{P}$ is a probability measure such that

$$\mathcal{P}(\theta(k+1) = j | \mathcal{F}_k) = \mathcal{P}(\theta(k+1) = j | \theta(k)) = p_{\theta(k)j}(k). \tag{4}$$

We consider dynamic Markovian jump controllers $\mathcal{G}_K$ given by

$$\mathcal{G}_K = \begin{cases} \widehat{x}(k+1) = \widehat{A}_{\theta(k)}(k)\widehat{x}(k) + \widehat{B}_{\theta(k)}(k)y(k) \\ u(k) = \widehat{C}_{\theta(k)}(k)\widehat{x}(k). \end{cases} \tag{5}$$

$\hat{y}(k) = C_{\theta(k)} \hat{x}(k)$

The reason for choosing this kind of controllers is that they depend just on $\theta(k)$ (rather than on the entire past history of modes $\theta(0), \ldots, \theta(k)$), so that the closed-loop system is again a Markovian jump system. In particular, as it will be seen in see Remarks 2, 4, and 6, time-invariant parameters can be considered as an approximation for the optimal solution.

The quadratic cost of the closed-loop system $\mathcal{G}_{cl}$ and with control law $u = (u(0), \ldots, u(T-1))$ given by (5) is denoted by $\mathcal{J}(u)$, and is given by

$$\mathcal{J}(u) = \sum_{k=0}^{T-1} E\left(\|z(k)\|^2\right) + E\left(x(T)^* \mathcal{V}_{\theta(T)} x(T)\right) \tag{6}$$

and $(\mathcal{V}_1, \ldots, \mathcal{V}_N) \in \mathcal{H}^{n+}$. Therefore, the finite horizon optimal quadratic control (OC) problem we want to study is: find $(\widehat{A}_1(k), \ldots, \widehat{A}_N(k))$, $(\widehat{B}_1(k), \ldots, \widehat{B}_N(k))$, $(\widehat{C}_1(k), \ldots, \widehat{C}_N(k))$ in (5), such that the control law $u = (u(0), \ldots, u(T-1))$ induced by (5) has minimal cost $\mathcal{J}(u)$. This minimal cost will be denoted by $\mathcal{J}^{op}$.

III. Quadratic Optimal Control Problem When the State Variable Is Available

For the case in which the state variable $x(k)$ is available to the controller the solution of the quadratic optimal control problem has already been solved in the literature (see, for instance, [10]). In this section, we will adapt the results presented in [10] that will be useful for the partial observation case. On the probabilistic space $(\Omega, \mathcal{F}, \mathcal{P})$, consider the following MJLS $\mathcal{G}_G$:

$$\mathcal{G}_G = \begin{cases} x(k+1) = A_{\theta(k)}(k)x(k) + B_{\theta(k)}(k)u(k) \\ \qquad\qquad + M_{\theta(k)}(k)\nu(k) \\ y(k) = x(k) \\ z(k) = C_{\theta(k)}(k)x(k) + D_{\theta(k)}(k)u(k) \end{cases} \tag{7}$$

where $\{\nu(k); k \in \mathbb{T}\}$ represents a noise sequence satisfying

$$E\left(\nu(k)\nu(k)^* 1_{\{\theta(k)=i\}}\right) = Q_i(k) \tag{8}$$
$$E\left(\nu(0)x(0)^* 1_{\{\theta(0)=i\}}\right) = 0 \tag{9}$$

for $k \in \mathbb{T}$ [recall the definition of the indicator function $1_{\{\theta(k)=i\}}$ in (1)]. We assume that the Markov property (4) holds and moreover for any measurable functions f and g

$$E\left(f(\nu(k))\, g(\theta(k+1))\, |\mathcal{F}_k\right) = E\left(f(\nu(k))\, |\mathcal{F}_k\right) \times \sum_{j=1}^{N} p_{\theta(k)j}(k) g(j). \tag{10}$$

We want to find a sequence of control law $u = \{u(0), \ldots, u(T-1)\}$ such that for each k, $u(k)$ is $\mathcal{F}_k$-measurable and satisfies

$$E\left(\nu(k)x(k)^* 1_{\{\theta(k)=i\}}\right) = 0 \tag{11}$$

and

$$E\left(\nu(k)u(k)^* 1_{\{\theta(k)=i\}}\right) = 0. \tag{12}$$

We denote this set of admissible controllers by $\mathcal{U}_c$. For a fixed integer $\kappa \in \mathbb{T}$, $\kappa \neq T$ we consider the "intermediate" problem of minimizing

$$\mathcal{J}(x(\kappa), \theta(\kappa), \kappa, u_\kappa) = \sum_{k=\kappa}^{T-1} E\left(\|z(k)\|^2 \, |\mathcal{F}_\kappa\right) + E\left(x(T)^* \mathcal{V}_{\theta(T)} x(T) | \mathcal{F}_\kappa\right) \tag{13}$$

where the control law $u_\kappa = (u(\kappa), \ldots, u(T-1))$ satisfies, for each $\kappa \le k \le T-1$, (11), (12), and $u(k)$ is $\mathcal{F}_k$-measurable. We denote this set of admissible controllers by $\mathcal{U}_c(\kappa)$, and the optimal cost by $\mathcal{J}^{op}(x, i, \kappa)$. The intermediate problem is, thus, to optimize the performance of the system over the last $T - \kappa$ stages starting at the point $x(\kappa) = x$ and mode $\theta(\kappa) = i$. As in the stochastic linear regulator problem (see, for instance, [9]) we apply dynamic programming to obtain, by induction on the intermediate problems, the solution of minimizing $\mathcal{J}(u)$. Define the following recursive coupled Riccati difference equations $(X_1(k), \ldots, X_N(k)) \in \mathcal{H}^{n+}$, for $i \in \mathbb{N}$ and $k = T-1, \ldots, 0$:

$$\begin{aligned} X_i(k) = {} & A_i(k)^* \mathcal{E}_i(X(k+1), k) A_i(k) \\ & + C_i(k)^* C_i(k) - A_i(k)^* \mathcal{E}_i(X(k+1), k) B_i(k) \\ & \times \left(D_i(k)^* D_i(k) + B_i(k)^* \mathcal{E}_i(X(k+1), k) B_i(k)\right)^{-1} \\ & \times B_i(k)^* \mathcal{E}_i(X(k+1), k) A_i(k) \end{aligned} \tag{14}$$

where $X_i(T) = \mathcal{V}_i$. Set also

$$R_i(k) = D_i(k)^* D_i(k) + B_i(k)^* \mathcal{E}_i(X(k+1), k) B_i(k) > 0$$

and

$$F_i(k) = R_i(k)^{-1} B_i(k)^* \mathcal{E}_i(X(k+1), k) A_i(k). \tag{15}$$

Define now for $i \in \mathbb{N}$ and $k = T-1, \ldots, 0$

$$W(x, i, k) = x^* X_i(k) x + \alpha(k)$$

where

$$\alpha(k) = \sum_{t=k}^{T-1} \delta(t) \quad \alpha(T) = 0$$

$$\delta(t) = \sum_{i=1}^{N} tr\left(M_i(t) Q_i(t) M_i(t)^* \mathcal{E}_i(X(t+1), t)\right).$$

The following theorem, which is based on the results in [10] taking into account (8), (9), (11), (12), and Markov property (4) and (10), shows that $\mathcal{J}^{op}(x, i, k) = W(x, i, k)$. The proof can be found in the Appendix.

Theorem 1: For the intermediate stochastic control problems described in (13), the optimal control law $u_\kappa^{op} = (u^{op}(\kappa), \ldots, u^{op}(T-1))$ is given by

$$u^{op}(k) = -F_{\theta(k)}(k) x(k) \tag{16}$$

and the optimal cost $\mathcal{J}^{op}(x, i, k) = W(x, i, k)$. In particular, we have that the optimal control law u^{op} for the problem described in (6) is $u^{op} = u_0^{op} = (u^{op}(0), \ldots, u^{op}(T-1))$ with optimal cost $\mathcal{J}^{op} = E(\mathcal{J}^{op}(x(0), \theta(0), 0)) = E(W(x(0), \theta(0), 0))$, that is

$$\mathcal{J}^{op} = \sum_{i=1}^{N} \left[tr\left(\pi_i(0) S X_i(0)\right) + \pi_i(0) \mu^* X_i(0) \mu + \sum_{k=0}^{T-1} \pi_i(k) tr\left(M_i(k) Q_i(k) M_i(k)^* \mathcal{E}_i(X(k+1), k)\right) \right]. \tag{17}$$

Remark 2: For the case in which A_i, B_i, C_i, D_i, p_{ij} in (3) are time invariant, the control coupled Riccati difference equations (14) lead to the following control coupled algebraic Riccati equations and respective gains:

$$X_i = A_i^* \mathcal{E}_i(X) A_i + C_i^* C_i - A_i^* \mathcal{E}_i(X) B_i \times \left(D_i^* D_i + B_i^* \mathcal{E}_i(X) B_i\right)^{-1} B_i^* \mathcal{E}_i(X) A_i \tag{18}$$

$$F_i = \left(D_i^* D_i + B_i^* \mathcal{E}_i(X) B_i\right)^{-1} B_i^* \mathcal{E}_i(X) A_i. \tag{19}$$

In [5], it was presented a sufficient condition for the existence of a unique solution $(X_1, \ldots, X_N) \in \mathcal{H}^{n+}$ for (18), and convergence of $X_i(k)$ to X_i. Thus a time-invariant approximation for the optimal control problem would be to replace $F_i(k)$ in (16) by F_i as in (19), thus just requiring to keep in memory the gains $(F_1, \ldots, F_N)$.

IV. Filtering Problem

Suppose that in model (3) it is desired to find an estimator for $x(k)$ when the output and the "operation modes" ($y(k)$, $\theta(k)$ respectively) are available to the controller. In the optimal filtering (OF) problem we want to find $\widehat{A}_i(k)$, $\widehat{B}_i(k)$, $\widehat{C}_i(k)$ in (5) with $\widehat{x}(0)$ deterministic, such that minimizes $E(\|v(k)\|^2)$ for each $k = 0, 1, \ldots, T$, where $v(k) = x(k) - \widehat{x}(k)$. It will be shown that the solution to this problem is associated to a set of filtering coupled Riccati difference equations given by (24).

Remark 3: It is well known that for the case in which $(y(k), \theta(k))$, are available the best linear estimator of $x(k)$ is derived from the Kalman filter for time varying systems (see [4] and [9]), since all the values of the mode of operation are known at time k. Indeed, the recursive equation for the covariance error matrix $Z(k)$ and the gain of the filter $K(k)$ would be as follows:

$$\begin{aligned} Z(k+1) = {} & A_{\theta(k)}(k) Z(k) A_{\theta(k)}(k)^* + G_{\theta(k)}(k) G_{\theta(k)}(k)^* \\ & - A_{\theta(k)}(k) Z(k) L_{\theta(k)}(k)^* \\ & \times \left(H_{\theta(k)}(k) H_{\theta(k)}(k)^* + L_{\theta(k)}(k) Z(k) L_{\theta(k)}(k)^*\right)^{-1} \\ & \times L_{\theta(k)}(k) Z(k) A_{\theta(k)}(k)^* \\ K(k) = {} & A_{\theta(k)}(k) Z(k) L_{\theta(k)}(k)^* \\ & \times \left(H_{\theta(k)}(k) H_{\theta(k)}(k)^* + L_{\theta(k)}(k) Z(k) L_{\theta(k)}(k)^*\right)^{-1}. \end{aligned} \tag{20}$$

As pointed out by [12], offline computation of the filter (20) is inadvisable since $Z(k)$ and $K(k)$ are sample path dependent, and the number of sample paths grows exponentially in time. Indeed, on the time interval [0, T], it would be necessary to pre-compute $N + N^2 + \ldots + N^T = N((N^T - 1)/(N-1))$ gains. On the other hand, the optimal filter in the form of (5) requires much less precomputed gains (NT instead of $N((N^T-1)/(N-1))$) and depends on just $\theta(k)$ at time k, which allows to consider, as pointed out in Remark 4 below, an approximation by a Markovian time-invariant filter.

More specifically, we want to study now the following OF problem. On the probabilistic space $(\Omega, \{\mathcal{F}_k\}_{k\in\mathbb{T}}, \mathcal{F}, \mathcal{P})$, consider the following MJLS:

$$\mathcal{G}_v = \begin{cases} x(k+1) = A_{\theta(k)}(k)x(k) + B_{\theta(k)}(k)u(k) \\ \qquad\qquad + G_{\theta(k)}(k)w(k) \\ y(k) = L_{\theta(k)}(k)x(k) + H_{\theta(k)}(k)w(k) \\ v(k) = x(k) - \widehat{x}(k). \end{cases} \tag{21}$$

It is desired to minimize $E(\|v(k)\|^2)$ for each $k = 0, 1, \ldots, T$ by considering Markovian jump linear filters of the following forms:

$$\mathcal{G}_K = \begin{cases} \widehat{x}(k+1) = \widehat{A}_{\theta(k)}(k)\widehat{x}(k) + \widehat{B}_{\theta(k)}(k)y(k) \\ u(k) = \widehat{C}_{\theta(k)}(k)\widehat{x}(k) \end{cases} \tag{22}$$

with $\widehat{x}(0)$ being a deterministic value. Define $\Gamma(k) = \{i \in \mathbb{N}; \pi_i(k) \neq 0\}$, and

$$\begin{cases} \widehat{x}_e(k+1) = A_{\theta(k)}(k)\widehat{x}_e(k) + B_{\theta(k)}(k)u(k) \\ \qquad\qquad + M_{\theta(k)}(k)\left(y(k) - L_{\theta(k)}(k)\widehat{x}_e(k)\right) \\ \widehat{x}_e(0) = E(x_0) = \mu \end{cases} \tag{23}$$

where $u(k)$ is given by (22), and $M_i(k)$ is defined from the filtering coupled Riccati difference equations shown in (24)–(26) at the bottom of page. The associated error related with the estimator given in (23) is defined by

$$\widetilde{x}_e(k) = x(k) - \widehat{x}_e(k) \tag{27}$$

and, from (21)–(23), we have that

$$\begin{aligned} \widetilde{x}_e(k+1) &= \left[A_{\theta(k)}(k) - M_{\theta(k)}(k)L_{\theta(k)}(k)\right]\widetilde{x}_e(k) \\ &\quad + \left[G_{\theta(k)}(k) - M_{\theta(k)}(k)H_{\theta(k)}(k)\right]w(k) \\ \widetilde{x}_e(0) &= x(0) - E(x_0) = x(0) - \mu. \end{aligned} \tag{28}$$

It is easy to check by induction on k that $Y_i(k)$ in (24) is

$$Y_i(k) = E\left(\widetilde{x}_e(k)\widetilde{x}_e(k)^* 1_{\{\theta(k)=i\}}\right). \tag{29}$$

We have the following Theorems, proved in the Appendix.

Theorem 2: For $\widehat{x}(k)$, $\widehat{x}_e(k)$, $\widetilde{x}_e(k)$ given by (22), (23) and (28), respectively, and $i = 1, \ldots, N$, $k = 0, 1, \ldots$, we have that

$$\begin{aligned} E\left(\widetilde{x}_e(k)\widehat{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) &= 0 \\ E\left(\widetilde{x}_e(k)\widehat{x}(k)^* 1_{\{\theta(k)=i\}}\right) &= 0. \end{aligned}$$

Theorem 3: Let $v(k)$ and $(Y_1(k), \ldots, Y_N(k))$ be as in (21) and (29), respectively. Then, for every $k = 0, 1, \ldots$, $E(\|v(k)\|^2) \geq \sum_{i=1}^N tr(Y_i(k))$.

The next Theorem is straightforward from Theorem 3, and shows that the solution for the optimum filtering problem can be obtained from the filtering Riccati recursive equations $(Y_1(k), \ldots, Y_N(k))$ as in (24) and gains $(M_1(k), \ldots, M_N(k))$ as in (26).

Theorem 4: An optimal solution for the OF problem posed above is: $\widehat{C}_i(k)$ is arbitrary and

$$\begin{aligned} \widehat{A}_i^{op}(k) &= A_i(k) - M_i(k)L_i(k) + B_i(k)\widehat{C}_i(k) \\ \widehat{B}_i^{op}(k) &= M_i(k) \end{aligned} \tag{30}$$

and the optimal cost is $\sum_{k=0}^T \sum_{i=1}^N tr(Y_i(k))$.

Remark 4: For the case in which A_i, G_i, L_i, H_i, p_{ij} in (3) are time invariant and $\pi_i(k)$ converges to π_i as k goes to infinity, the filtering coupled Riccati difference equations (24) lead to the following filtering coupled algebraic Riccati equations and respective gains:

$$\begin{aligned} Y_j = \sum_{i\in\Gamma} p_{ij}\big[&A_iY_iA_i^* + \pi_iG_iG_i^* - A_iY_iL_i^* \\ &\times \left(H_iH_i^*\pi_i + L_iY_iL_i^*\right)^{-1} L_iY_iA_i^*\big] \end{aligned} \tag{31}$$

$$M_i = \begin{cases} A_iY_iL_i^* \\ \quad \times \left(H_iH_i^*\pi_i + L_iY_iL_i^*\right)^{-1}, & \text{for } \pi_i \neq 0 \\ 0, & \text{for } \pi_i = 0 \end{cases} \tag{32}$$

where $\Gamma = \{i \in \mathbb{N}; \pi_i \neq 0\}$. In [5], it was presented a sufficient condition for the existence of a unique solution $(Y_1, \ldots, Y_N) \in \mathcal{H}^{n+}$ for (31), and convergence of $Y_i(k)$ to Y_i. Convergence to the stationary state is often rapid, so that the optimal filter (30) could be approximated by the time-invariant Markovian filter

$$\widehat{x}(k+1) = \left(A_{\theta(k)} - M_{\theta(k)}L_{\theta(k)}\right)\widehat{x}(k) + M_{\theta(k)}y(k)$$

which just requires to keep in memory the gains $(M_1, \ldots, M_N)$.

V. Separation Principle for MJLS

We return now to the OC problem posed in Section II-B. First of all, we notice that

$$\begin{aligned} E\left(\|z(k)\|^2\right) &= E\left(x(k)^*C_{\theta(k)}(k)^*C_{\theta(k)}(k)x(k)\right) \\ &\quad + E\left(\left\|D_{\theta(k)}(k)u(k)\right\|^2\right) \\ &= \sum_{i=1}^N tr\left(C_i(k)^*C_i(k)E\left(x(k)x(k)^*1_{\theta(k)=i}\right)\right) \\ &\quad + E\left(\left\|D_{\theta(k)}(k)u(k)\right\|^2\right) \end{aligned} \tag{33}$$

$$\begin{aligned} Y_j(k+1) = \sum_{i\in\Gamma(k)} p_{ij}(k)\Big[&A_i(k)Y_i(k)A_i(k)^* + \pi_i(k)G_i(k)G_i(k)^* - A_i(k)Y_i(k)L_i(k)^* \\ &\times \left(H_i(k)H_i(k)^*\pi_i(k) + L_i(k)Y_i(k)L_i(k)^*\right)^{-1} L_i(k)Y_i(k)A_i(k)^*\Big] \end{aligned} \tag{24}$$

$$Y_i(0) = \pi_i(0)(S - \mu\mu^*) \tag{25}$$

$$M_i(k) = \begin{cases} A_i(k)Y_i(k)L_i(k)^*\left(H_i(k)H_i(k)^*\pi_i(k) + L_i(k)Y_i(k)L_i(k)^*\right)^{-1}, & \text{for } \pi_i(k) \neq 0 \\ 0, & \text{for } \pi_i(k) = 0 \end{cases}. \tag{26}$$

and for any control law $u = (u(0), \dots, u(T-1))$ given by (5), we have from (27) that $x(k) = \widetilde{x}_e(k) + \widehat{x}_e(k)$ and from Theorem 2

$$\begin{aligned} E\left(x(k)x(k)^* 1_{\theta(k)=i}\right) &= E\left(\widetilde{x}_e(k)\widetilde{x}_e(k)^* 1_{\theta(k)=i}\right) \\ &\quad + E\left(\widehat{x}_e(k)\widehat{x}_e(k)^* 1_{\theta(k)=i}\right) \\ &= Y_i(k) + E\left(\widehat{x}_e(k)\widehat{x}_e(k)^* 1_{\theta(k)=i}\right) \end{aligned}$$

so that (33) can be rewritten as

$$E\left(\|z(k)\|^2\right) = E\left(\|\widehat{z}_e(k)\|^2\right) + \sum_{i=1}^{N} tr\left(C_i(k)Y_i(k)C_i(k)^*\right)$$

where $\widehat{z}_e(k) = C_{\theta(k)}(k)\widehat{x}_e(k) + D_{\theta(k)}(k)u(k)$. Similarly, we have that

$$E\left(x(T)^* \mathcal{V}_{\theta(T)} x(T)\right) = E\left(\widehat{x}_e(T)^* \mathcal{V}_{\theta(T)} \widehat{x}_e(T)\right) + \sum_{i=1}^{N} tr\left(\mathcal{V}_i Y_i(T)\right)$$

and, thus

$$\begin{aligned} \mathcal{J}(u) &= \sum_{k=0}^{T-1} E\left(\|\widehat{z}_e(k)\|^2\right) + E\left(\widehat{x}_e(T)^* \mathcal{V}_{\theta(T)} \widehat{x}_e(T)\right) \\ &\quad + \sum_{k=0}^{T-1}\left[\sum_{i=1}^{N}\left[tr\left(C_i(k)Y_i(k)C_i(k)^*\right) + tr\left(\mathcal{V}_i Y_i(T)\right)\right]\right] \end{aligned} \tag{34}$$

where the terms with $Y_i(k)$ do not depend on the control variable u. Therefore, minimize (34) is equivalent to minimize

$$\mathcal{J}_e(u) = \sum_{k=0}^{T-1} E\left(\|\widehat{z}_e(k)\|^2\right) + E\left(\widehat{x}_e(T)^* \mathcal{V}_{\theta(T)} \widehat{x}_e(T)\right)$$

subject to

$$\begin{cases} \widehat{x}_e(k+1) = A_{\theta(k)}(k)\widehat{x}_e(k) + B_{\theta(k)}(k)u(k) + M_{\theta(k)}(k)\nu(k) \\ \quad \widehat{x}_e(0) = E(x_0) = \mu \end{cases}$$

where

$$\nu(k) = y(k) - L_{\theta(k)}(k)\widehat{x}_e(k) = L_{\theta(k)}(k)\widetilde{x}_e(k) + H_{\theta(k)}(k)w(k)$$

and $u(k)$ is given by (5). Let us show now that $\{\nu(k); k \in \mathbb{T}\}$ satisfies (8)–(12). Set $Q_i(k) = L_i(k)Y_i(k)L_i(k)^* + H_i(k)H_i(k)^*$. Indeed, we have from Theorem 2 and the fact that $\{w(k); k = 0, \dots, T\}$ is a wide sense white noise sequence independent of the Markov chain $\{\theta(k); k = 0, \dots, T\}$ and initial condition $x(0)$, that

$$\begin{aligned} E\left(\nu(k)\nu(k)^* 1_{\{\theta(k)=i\}}\right) &= E\left((L_i(k)\widetilde{x}_e(k) + H_i(k)w(k))\right. \\ &\quad \times (L_i(k)\widetilde{x}_e(k) \\ &\quad \left. + H_i(k)w(k))^* 1_{\{\theta(k)=i\}}\right) \\ &= L_i(k)Y_i(k)L_i(k)^* + H_i(k)H_i(k)^* \\ &= Q_i(k) \\ E\left(\nu(k)\widehat{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) &= L_i(k)E\left(\widetilde{x}_e(k)\widehat{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) \\ &\quad + H_i(k)E\left(w(k)\widehat{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) \\ &= 0 \\ E\left(\nu(k)u(k)^* 1_{\{\theta(k)=i\}}\right) &= L_i(k)E\left(\widetilde{x}_e(k)\widehat{x}(k)^* 1_{\{\theta(k)=i\}}\right) \\ &\quad \times \widehat{C}_i(k)^* + H_i(k) \\ &\quad \times E\left(w(k)\widehat{x}(k)^* 1_{\{\theta(k)=i\}}\right)\widehat{C}_i(k)^* \\ &= 0 \end{aligned}$$

for $k \in \mathbb{T}$. Moreover, for any measurable functions f and g, we have from independence between $w(k)$ and $\theta(k)$ that

$$\begin{aligned} &E\left(f(\nu(k))\, g(\theta(k+1)) \,|\mathcal{F}_k\right) \\ &\quad = E\left(f\left(L_{\theta(k)}(k)\widetilde{x}_e(k) + H_{\theta(k)}(k)w(k)\right)\right) \sum_{j=1}^{N} p_{\theta(k)j}(k)g(j) \end{aligned}$$

where the expected value is over the variable $w(k)$. Thus, the results of Section III can be applied and we have the following theorem.

Theorem 5: (A Separation Principle for MJLSs) An optimal solution for the problem posed in Section II-B is obtained from (14) and (24). The gains (15) and (26) lead to the following optimal solution:

$$\begin{aligned} \widehat{A}_i^{op}(k) &= A_i(k) - M_i(k)L_i(k) - B_i(k)F_i(k) \\ \widehat{B}_i^{op}(k) &= M_i(k) \\ \widehat{C}_i^{op}(k) &= -F_i(k) \end{aligned}$$

and the optimal cost is

$$\begin{aligned} \mathcal{J}^{op} = \sum_{i=1}^{N} \Bigg[& \pi_i(0)\mu^* X_i(0)\mu + \sum_{k=0}^{T-1} tr\left(C_i(k)Y_i(k)C_i(k)^*\right) \\ & + tr\left(\mathcal{V}_i Y_i(T)\right) + \sum_{k=0}^{T-1} \pi_i(k) \\ & \times tr\left(M_i(k)Q_i(k)M_i(k)^* \mathcal{E}_i\left(X(k+1), k\right)\right) \Bigg]. \end{aligned}$$

Remark 5: Notice that the choice of the Markovian structure for the filter as in (22) was crucial to obtain the orthogonality derived in Theorem 2, and the separation principle presented here. Other choices for the structure of the filter, with more information on the Markov chain, would lead to other notions of orthogonality and "separation principles." Therefore, the separation principle presented here is a direct consequence of the choice of the Markovian structure for the filter.

Remark 6: From Remarks 2 and 4, we have that a time-invariant Markovian controller approximation for the optimal filter in Theorem 5 would be given by the steady-state solutions (18) and (31), with gains (19) and (32), provided that that convergence conditions presented in [5] would be satisfied.

APPENDIX

We present in this appendix the proofs of Theorems 1–3.

Proof of Theorem 1: Notice that for any $u_\kappa \in \mathcal{U}_c(\kappa)$ we have from the Markov property (4) and (10), and the definitions (14) and (15) that

$$\begin{aligned} E&\left(W\left(x(k+1),\theta(k+1),k+1\right)|\mathcal{F}_K\right)-W\left(x(k),\theta(k),k\right)\\ &=-\|z(k)\|^2+\left\|R^{\frac{1}{2}}_{\theta(k)}(k)\left(u(k)+F_{\theta k}(k)x(k)\right)\right\|^2\\ &\quad+tr\left(M_{\theta(k)}(k)E\left(\nu(k)\nu(k)^*|\mathcal{F}_K\right)\right.\\ &\qquad\left.\times M_{\theta(k)}(k)^*\mathcal{E}_{\theta(k)}\left(X(k+1),k\right)\right)\\ &\quad+2tr\left(M_{\theta(k)}(k)E\left(\nu(k)x(k)^*|\mathcal{F}_K\right)\right.\\ &\qquad\left.\times A_{\theta(k)}(k)^*\mathcal{E}_{\theta(k)}\left(X(k+1),k\right)\right)\\ &\quad+2tr\left(M_{\theta(k)}(k)E\left(\nu(k)u(k)^*|\mathcal{F}_K\right)\right.\\ &\qquad\left.\times B_{\theta(k)}(k)^*\mathcal{E}_{\theta(k)}\left(X(k+1),k\right)\right)-\delta(k). \end{aligned} \tag{35}$$

From (8) and (9)

$$\begin{aligned} E&\left(tr\left(M_{\theta(k)}(k)E\left(\nu(k)\nu(k)^*|\mathcal{F}_K\right)\right.\right.\\ &\qquad\left.\left.\times M_{\theta(k)}(k)^*\mathcal{E}_{\theta(k)}\left(X(k+1),k\right)\right)\right)\\ &=E\left(tr\left(M_{\theta(k)}(k)\nu(k)\nu(k)^*M_{\theta(k)}(k)^*\right.\right.\\ &\qquad\left.\left.\times\mathcal{E}_{\theta(k)}\left(X(k+1),k\right)\right)\right)\\ &=\sum_{i=1}^{N}tr\left(M_i(k)E\left(\nu(k)\nu(k)^*1_{\theta(k)=i}\right)\right.\\ &\qquad\left.\times M_i(k)^*\mathcal{E}_i\left(X(k+1),k\right)\right)\\ &=\sum_{i=1}^{N}tr\left(M_i(k)Q_i(k)M_i(k)^*\mathcal{E}_i\left(X(k+1),k\right)\right)\\ &=\delta(k) \end{aligned} \tag{36}$$

and, similarly, from (11)

$$\begin{aligned} E&\left(tr\left(M_{\theta(k)}(k)E\left(\nu(k)x(k)^*|\mathcal{F}_K\right)\right.\right.\\ &\qquad\left.\left.\times A_{\theta(k)}(k)^*\mathcal{E}_{\theta(k)}\left(X(k+1),k\right)\right)\right)\\ &=E\left(tr\left(M_{\theta(k)}(k)\nu(k)x(k)^*A_{\theta(k)}(k)^*\right.\right.\\ &\qquad\left.\left.\times\mathcal{E}_{\theta(k)}\left(X(k+1),k\right)\right)\right)\\ &=\sum_{i=1}^{N}tr\left(M_i(k)E\left(\nu(k)x(k)^*1_{\theta(k)=i}\right)\right.\\ &\qquad\left.\times A_i(k)^*\mathcal{E}_i\left(X(k+1),k\right)\right)\\ &=0. \end{aligned} \tag{37}$$

The same arguments now using (12) yield

$$\begin{aligned} E&\left(tr\left(M_{\theta(k)}(k)E\left(\nu(k)u(k)^*|\mathcal{F}_K\right)\right.\right.\\ &\qquad\left.\left.\times B_{\theta(k)}(k)^*\mathcal{E}_{\theta(k)}\left(X(k+1),k\right)\right)\right)\\ &=E\left(tr\left(M_{\theta(k)}(k)\nu(k)u(k)^*\right.\right.\\ &\qquad\left.\left.\times B_{\theta(k)}(k)^*\mathcal{E}_{\theta(k)}\left(X(k+1),k\right)\right)\right)\\ &=\sum_{i=1}^{N}tr\left(M_i(k)E\left(\nu(k)u(k)^*1_{\theta(k)=i}\right)\right.\\ &\qquad\left.\times A_i(k)^*\mathcal{E}_i\left(X(k+1),k\right)\right)\\ &=0. \end{aligned} \tag{38}$$

From (35)–(38), we have that

$$\begin{aligned} E&\left(W\left(x(k+1),\theta(k+1),k+1\right)|\mathcal{F}_K\right)-W\left(x(k),\theta(k),k\right)\\ &=-\|z(k)\|^2+\left\|R^{\frac{1}{2}}_{\theta(k)}(k)\left(u(k)+F_{\theta k}(k)x(k)\right)\right\|^2. \end{aligned}$$

Taking the sum from $k=\kappa$ to $T-1$, and recalling that $W(x(T),\theta(T),T)=x(T)^*\mathcal{V}_{\theta(T)}x(T)$ we get that

$$\begin{aligned} E&\left(x(T)^*\mathcal{V}_{\theta(T)}x(T)|\mathcal{F}_\kappa\right)-W\left(x(\kappa),\theta(\kappa),\kappa\right)\\ &=E\left(W\left(x(T),\theta(T),T\right)|\mathcal{F}_\kappa\right)-W\left(x(\kappa),\theta(\kappa),\kappa\right)\\ &=\sum_{k=\kappa}^{T-1}E\left(E\left(W\left(x(k+1),\theta(k+1),k+1\right)|\mathcal{F}_K\right)\right.\\ &\qquad\left.-W\left(x(k),\theta(k),k\right)|\mathcal{F}_\kappa\right)\\ &=\sum_{k=\kappa}^{T-1}E\left(-\|z(k)\|^2\right.\\ &\qquad\left.+\left\|R^{\frac{1}{2}}_{\theta(k)}(k)\left(u(k)+F_{\theta k}(k)x(k)\right)\right\|^2|\mathcal{F}_\kappa\right) \end{aligned}$$

that is

$$\begin{aligned} \mathcal{J}&\left(x(\kappa),\theta(\kappa),\kappa,u_\kappa\right)\\ &=E\left(\sum_{k=\kappa}^{T-1}\|z(k)\|^2+x(T)^*\mathcal{V}_{\theta(T)}x(T)|\mathcal{F}_\kappa\right)\\ &=W\left(x(\kappa),\theta(\kappa),\kappa\right)\\ &\quad+E\left(\sum_{k=\kappa}^{T-1}\left\|R^{\frac{1}{2}}_{\theta(k)}(k)(u(k)+F_{\theta k}(k)x(k))\right\|^2|\mathcal{F}_\kappa\right) \end{aligned} \tag{39}$$

and taking the minimum over $u_\kappa \in \mathcal{U}_c(\kappa)$ in (39) we obtain that the optimal control law is given by (16), so that the second term on the right-hand side of (39) equals to zero and $\mathcal{J}(x(\kappa),\theta(\kappa),\kappa,u_\kappa)=W(x(\kappa),\theta(\kappa),\kappa)$. In particular for $\kappa=0$ we obtain that $\mathcal{J}^{op}=E(x(0)^*X_{\theta(0)}(0)x(0))+\alpha(0)$ which, after some manipulation, leads to (17), proving the desired result. □

Proof of Theorem 2: Let us prove it by induction on k. For $k=0$ it is clearly true since from independence of $x(0)$ and $\theta(0)$, and the fact that $\widehat{x}(0)$ is deterministic, $E(\widetilde{x}_e(0))=0$, we have that $E(\widetilde{x}_e(0)\widehat{x}(0)^*1_{\{\theta(0)=i\}})=\pi_i(0)E(\widetilde{x}_e(0))\widehat{x}(0)^*=0$, and similarly $E(\widetilde{x}_e(0)\widehat{x}_e(0)^*1_{\{\theta(0)=i\}})=0$. Suppose it holds for k. Then, from (22) and (28), the induction hypothesis for k, and since $w(k)$ has null mean and is not correlated with $\widehat{x}(k)$ and independent from $\theta(k)$, we have

$$\begin{aligned} E&\left(\widetilde{x}_e(k+1)\widehat{x}(k+1)^*1_{\{\theta(k+1)=j\}}\right)\\ &=\sum_{i\in\Gamma(k)}p_{ij}(k)\\ &\quad\times\left[\left(A_i(k)-M_i(k)L_i(k)\right)E\left(\widetilde{x}_e(k)\widehat{x}(k)^*1_{\{\theta(k)=i\}}\right)\right.\\ &\qquad\times\widehat{A}_i(k)+\left(A_i(k)-M_i(k)L_i(k)\right)\\ &\qquad\times E\left(\widetilde{x}_e(k)y(k)^*1_{\{\theta(k)=i\}}\right)\widehat{B}_i(k)\\ &\qquad+\left(G_i(k)-M_i(k)H_i(k)\right)E\left(w(k)\widehat{x}(k)^*1_{\{\theta(k)=i\}}\right)\\ &\qquad\times\widehat{A}_i(k)+\left(G_i(k)-M_i(k)H_i(k)\right)\\ &\qquad\left.\times E\left(w(k)y(k)^*1_{\{\theta(k)=i\}}\right)\widehat{B}_i\right]\\ &=\sum_{i\in\Gamma(k)}p_{ij}(k)\\ &\quad\times\left[\left(A_i(k)-M_i(k)L_i(k)\right)E\left(\widetilde{x}_e(k)y(k)^*1_{\{\theta(k)=i\}}\right)\right.\\ &\qquad+\left(G_i(k)-M_i(k)H_i(k)\right)\\ &\qquad\left.\times E\left(w(k)y(k)^*1_{\{\theta(k)=i\}}\right)\right]\widehat{B}_i(k). \end{aligned}$$

Notice that

$$\begin{aligned} y(k)&=L_{\theta(k)}(k)x(k)+H_{\theta(k)}(k)w(k)\\ &=L_{\theta(k)}(k)\left(\widetilde{x}_e(k)+\widehat{x}_e(k)\right)+H_{\theta(k)}(k)w(k) \end{aligned}$$

and from the induction hypothesis on k, we have that for $i \in \Gamma(k)$

$$\begin{aligned} E\big(\widetilde{x}_e(k)y(k)^* 1_{\{\theta(k)=i\}}\big) =& E\left(\widetilde{x}_e(k)\widetilde{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) L_i(k)^* \\ &+ E\left(\widetilde{x}_e(k)\widehat{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) L_i(k)^* \\ &+ E\left(\widetilde{x}_e(k)w(k)^* 1_{\{\theta(k)=i\}}\right) H_i(k)^* \\ =& Y_i(k)L_i(k)^*. \end{aligned}$$

We also have that

$$\begin{aligned} E\big(w(k)y(k)^* 1_{\{\theta(k)=i\}}\big) =& E\left(w(k)\widetilde{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) L_i(k)^* \\ &+ E\left(w(k)\widehat{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) L_i(k)^* \\ &+ E\left(w(k)w(k)^* 1_{\{\theta(k)=i\}}\right) H_i(k)^* \\ =& E(w(k)w(k)^*)\, P(\theta(k)=i) H_i(k)^* \\ =& \pi_i(k)H_i(k)^*. \end{aligned}$$

Thus, recalling that by hypothesis $G_i(k)H_i(k)^* = 0$, we have that

$$\begin{aligned} &E\left(\widetilde{x}_e(k+1)\widehat{x}_e(k+1)^* 1_{\{\theta(k+1)=j\}}\right) \\ &\quad = \sum_{i\in\Gamma(k)} p_{ij}(k) \\ &\qquad \times \left[(A_i(k) - M_i(k)L_i(k))\, Y_i L_i(k)^* \right. \\ &\qquad\quad \left. + (G_i(k) - M_i(k)H_i(k))\, H_i(k)^* \pi_i(k)\right] \widehat{B}_i(k) \\ &\quad = \sum_{i\in\Gamma(k)} p_{ij}(k) \\ &\qquad \times \left[A_i(k)Y_i(k)L_i(k)^* - M_i(k)\left(L_i(k)Y_i(k)L_i(k)^*\right.\right. \\ &\qquad\quad \left.\left. + \pi_i(k)H_i(k)H_i(k)^*\right)\right] \widehat{B}_i(k) \\ &\quad = \sum_{i=1}^{N} p_{ij}(k) \\ &\qquad \times \left[A_i(k)Y_i(k)L_i(k)^* - A_i(k)Y_i(k)L_i(k)^*\right] \widehat{B}_i(k) \\ &\quad = 0. \end{aligned}$$

Similarly, $E(\widetilde{x}_e(k+1)\widehat{x}_e(k+1)^* 1_{\{\theta(k+1)=j\}}) = 0$, proving the result. □

Proof of Theorem 3: We have from the previous theorem that

$$\begin{aligned} &E\left(\|v(k)\|^2\right) \\ &\quad = E\left(\|x(k) - \widehat{x}_e(k) + \widehat{x}_e(k) - \widehat{x}(k)\|^2\right) \\ &\quad = \sum_{i=1}^{N} E\left((\|\widetilde{x}_e(k) + (\widehat{x}_e(k) - \widehat{x}(k)))\, 1_{\{\theta(k)=i\}}\|^2\right) \\ &\quad = \sum_{i=1}^{N} tr\left(E\left((\widetilde{x}_e(k) + (\widehat{x}_e(k) - \widehat{x}(k)))\right.\right. \\ &\qquad\qquad \left.\left. \times\ (\widetilde{x}_e(k) + (\widehat{x}_e(k) - \widehat{x}(k)))^* 1_{\{\theta(k)=i\}}\right)\right) \\ &\quad = \sum_{i=1}^{N} tr\left(Y_i(k)\right) + E\left(\|\widehat{x}_e(k) - \widehat{x}(k)\|^2\right) \\ &\qquad + \sum_{i=1}^{N} tr\left(E\left(\widetilde{x}_e(k)\widehat{x}_e(k)^* 1_{\{\theta(k)=i\}}\right)\right. \\ &\qquad\qquad \left. - E\left(\widetilde{x}_e(k)\widehat{x}(k)^* 1_{\{\theta(k)=i\}}\right)\right) \\ &\quad = \sum_{i=1}^{N} tr\left(Y_i(k)\right) + E\left(\|\widehat{x}_e(k) - \widehat{x}(k)\|^2\right) \\ &\quad \geq \sum_{i=1}^{N} tr\left(Y_i(k)\right) \end{aligned}$$

since

$$\begin{aligned} E\left(\widetilde{x}_e(k)\widetilde{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) &= Y_i(k) \\ E\left(\widetilde{x}_e(k)\widehat{x}_e(k)^* 1_{\{\theta(k)=i\}}\right) &= 0 \\ E\left(\widetilde{x}_e(k)\widehat{x}(k)^* 1_{\{\theta(k)=i\}}\right) &= 0 \end{aligned}$$

and

$$E\left(\|\widehat{x}_e(k) - \widehat{x}(k)\|^2\right) \geq 0$$

completing the proof of the Theorem. □

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