Jump LQR Systems With Unknown Transition Probabilities

Ioannis Tzortzis, Charalambos D. Charalambous, and Christoforos N. Hadjicostis, *Fellow, IEEE*

***Abstract*—This article develops a robust linear quadratic regulator (LQR) approach applicable to nonhomogeneous Markov jump linear systems with uncertain transition probability distributions. The stochastic control problem is investigated under two equivalent formulations, using i) minimax optimization theory, and ii) a total variation distance metric as a tool for codifying the level of uncertainty of the jump process. By following a dynamic programming approach, a robust optimal controller is derived, which in addition to minimizing the quadratic cost, it also restricts the influence of uncertainty. A solution procedure for the LQR problem is also proposed, and an illustrative example is presented. Numerical results indicate the applicability and effectiveness of the proposed approach.**

***Index Terms*—Dynamic programming, minimax optimization, nonhomogeneous Markov jump linear systems, robust linear quadratic regulator, uncertain/ambiguous transition probabilities.**

I. Introduction

THE design of optimal controllers for linear dynamical systems, which are subject to abrupt changes in their operating modes, also known as Markov jump linear systems (MJLS), is a fundamental problem, which has received increased attention in recent years due to its wide variety of engineering applications. A popular method for designing optimal controllers for MJLS is via linear quadratic optimization theory [1], [2]. In that case, classical linear quadratic optimization methods are developed under the assumption that the Markov chain transition probabilities, used to model the jumps/transitions between the system's different operating modes, are available and accurate, however, in practice, this might not always be the case. In real applications, the designer must cope with the problem of poor, uncertain, or even, incomplete transition probability distributions, to ensure that the optimality of the controller is not affected, and moreover that, the performance of the linear quadratic regulator (LQR) controller is not compromised.

In order to tackle this problem of poor, uncertain and incomplete transition probability distributions, and to design optimal controllers, robust control theory has been developed. Good overviews of existing approaches can be found for example in [3]–[10]. Robust control approaches with incomplete and partially known transition probabilities have also received attention in [11] and [12]. Note that, while most of the aforementioned work deals with homogeneous MJLS, in many practical applications, the transition probability distribution of the jump process is time varying. Recent works, which deal with nonhomogeneous MJLS can be found in [13]–[15].

The main objective of this article is to develop a new robust linear quadratic optimization approach, applicable to MJLS with time-varying uncertain transition probability distributions, and capable of capturing and restricting the influence of uncertainty on the performance of the LQR controller. Motivated by the abovementioned discussion, and by the fact that, in practice, the transition probability distribution of the jump process is only partially revealed to the designer via a prespecified *nominal transition probability distribution*, in this article, we investigate robust linear quadratic optimization with respect to any variation in the transition probability distribution of the MJLS. To study the effect of uncertain transition probability distributions of the jump process, we employ a total variation distance metric, and we consider some level of perturbation or deviation from the nominal distribution. In particular, we assume that the true transition probability distribution of the underlying Markov chain is not completely known, but instead it is contained in a prespecified uncertainty set, of nominal distributions.

Total variation distance is selected as a tool to codify the level of uncertainty or ambiguity, by defining uncertainty sets based on the nominal and true transition probability distributions of the jump process. The emphasis on the total variation distance to model uncertainty is motivated by its natural and intuitive application to the problem at hand. More specifically, small values of total variation distance (values close to zero) imply that the true and the nominal transition probability distributions are close to each other, whereas, as the value of the total variation distance increases, the uncertainty becomes larger, which in turns implies that highly uncertain scenarios are taken into account. An important feature of the proposed uncertainty model is that we are allowed to assign different levels of ambiguity between the nominal and true transition probability distributions for different operating modes of the system. As it turns out, the proposed robust LQR approach 1) leads to an optimal controller

Manuscript received October 4, 2019; revised October 6, 2019 and March 16, 2020; accepted July 23, 2020. Date of publication August 4, 2020; date of current version May 27, 2021. This work was supported by the European Regional Development Fund and the Republic of Cyprus through the Cyprus Research and Innovation Foundation under Project POST-DOC/0916/0139. Recommended by Associate Editor R. M. Jungers. *(Corresponding author: Ioannis Tzortzis.)*

The authors are with the Department of Electrical and Computer Engineering, University of Cyprus, Nicosia 1678, Cyprus (e-mail: tzortzis.ioannis@ucy.ac.cy; chadcha@ucy.ac.cy; chadjic@ucy.ac.cy).

Color versions of one or more of the figures in this article are available online at https://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2020.3013844

with some desired robustness properties and 2) ensures the optimal performance of the LQR controller.

The stochastic control problem under investigation presupposes the following:

1) a discrete time, stochastic control system with deterministic strategies;
2) a finite state, nonhomogeneous Markov process, which is used to describe the transitions/jumps between the system's different operating modes;
3) an uncertainty or ambiguity set of the class of true, time varying, transition probability distributions of the Markov process, described by a ball with respect to the total variation distance metric;
4) a quadratic cost to be optimized.

The stochastic control problem is formulated using minimax optimization theory, in which the control aims to minimize the cost, while the transition probability distributions, from the total variation uncertainty set, aim to maximize it. A dynamic programming approach is used to derive a robust optimal control policy, a maximizing, time varying, transition probability distribution, and an explicit expression for the optimal cost. Furthermore, an equivalent formulation of the robust LQR problem is derived. This equivalent formulation is rather informative, since it includes terms that are related to the level of uncertainty in distribution, and codify the impact of incorrect distributions on the performance of the LQR controller.

The remainder of this article is organized as follows. The robust LQR problem is formulated in Section II. The solution of the robust LQR problem is given in Section III, where an LQR procedure is also proposed. In Section IV, an equivalent formulation of the robust LQR problem is derived. In Section V, a numerical example is presented to illustrate the behavior of the proposed solution. Finally, Section VI concludes this article.

II. Problem Formulation

A. Uncertain Jump Linear System

Consider a discrete-time stochastic control system, defined on filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k\geq 0}, \mathbb{P})$, with linear state dynamics

$$\begin{cases} x_{k+1} = A_k(\theta_k)x_k + B_k(\theta_k)u_k + M_k(\theta_k)w_k \\ \quad x_0 = x, \quad \theta_0 = i \end{cases} \tag{1}$$

modulated by a finite state, time varying, Markov chain $\{\theta_k, k \geq 0\} \in \Theta = \{1, 2, \ldots, n_\Theta\}$. Here, $x_k \in \mathcal{X}_k \triangleq \mathbb{R}^n$, $u_k \in \mathcal{U} \triangleq \mathbb{R}^m$ are the state and control processes, respectively, and $w_k \in \mathcal{W} \triangleq \mathbb{R}^n$ is a sequence of independent identically distributed (i.i.d.) random vectors with known probability distribution having zero mean and covariance $\Sigma_w \geq 0$. As usual, we impose the following assumption on the basic random variables.

Assumption 2.1: The basic random variables are as follows:

1) $x_0, w_0, w_1, \ldots$ are mutually independent;
2) the sequence $\{\theta_0, \theta_1, \ldots\}$ is independent with the sequence $\{x_0, w_0, w_1, \ldots\}$.

For each $\theta_k = \theta \in \Theta$, the real-valued matrices $A_k(\theta) \in \mathbb{R}^{n\times n}$, $B_k(\theta) \in \mathbb{R}^{n\times m}$, and $M_k(\theta) \in \mathbb{R}^{n\times n}$ are assumed to have bounded and measurable entries for all k. The joint distribution of the Markov chain $\{\theta_k, k \geq 0\}$ is generated from a nonhomogeneous, transition probability distribution

$$\begin{aligned} p_{ij}(k) &\triangleq P_k(\theta_{k+1} = j|\theta_0, \ldots, \theta_k = i) \\ &= P_k(\theta_{k+1} = j|\theta_k = i) \end{aligned} \tag{2}$$

where $p_{ij}(k) \geq 0$, $\forall i, j \in \Theta$, and $\sum_{j\in\Theta} p_{ij}(k) = 1$, $\forall i$. The jump process $\{\theta_k, k \geq 0\}$ that modulates the dynamical system (1), is generated from the transition probability distribution (2), and will be used to describe the jumps/transitions between the system's different operating modes.

The set of control policies, denoted by G, is the set of measurable Markov feedback control policies $g \in G$. For the construction of u_k, we suppose that at any time k the controller has available information about x_k and θ_k. It should be mentioned that, while $\{x_k\}$ is not a Markov process, the pair $\{x_k, \theta_k\}$ is a Markov process conditional on the control process. Then, for any $g \in G$, the closed-loop stochastic control system is given by

$$x^g_{k+1} = A_k(\theta_k)x^g_k + B_k(\theta_k)g_k(x^g_k, \theta_k) + M_k(\theta_k)w_k$$

with the control law $g \in G$ and the associated control process related by $u_k = g_k(x_k, \theta_k)$, where $g_k(\cdot, \cdot)$ is measurable, and $u \in \mathcal{U}_{[0,N-1]}$ with $\mathcal{U}_{[0,N-1]} \triangleq \{u_k = g_k(x_k, \theta_k) \in \mathcal{U} : \mathbb{E}\sum_{k=0}^{N-1} |u_k|^2 < \infty\}$. Note that, for the rest of this article the super index g will be omitted.

B. Uncertainty or Ambiguity Set of Markov Chain

In this article, we consider the realistic scenario in which the transition probability distribution $p_{ij}(k)$ of the Markov chain $\{\theta_k, k \geq 0\}$ is not known exactly. We model the uncertainty set of all possible, time varying, transition probability distributions $p_{ij}(k)$, by a ball centered around a nominal transition probability distribution p^0_{ij}, with respect to a total variation distance metric.[1]

Toward this end, we define the set of transition probability distributions on Θ by

$$\mathbb{P}_k(\Theta|i) \triangleq \left\{ p_{i\bullet}(k) : p_{ij}(k) \geq 0, j = 1, \ldots, n_\Theta, \right.$$
$$\left. \sum_{j\in\Theta} p_{ij}(k) = 1 \right\}, \; i \in \Theta, \; k = 0, 1, \ldots, N-1.$$

The uncertainty or ambiguity set of all possible, time varying, transition probability distributions is defined by

$$\mathbb{B}_k(i) \triangleq \left\{ p_{i\bullet}(k) \in \mathbb{P}_k(\Theta|i) : \sum_{j\in\Theta} |p_{ij}(k) - p^0_{ij}| \leq R_{TV}(i) \right\}$$
$$R_{TV}(i) \in [0, 2], \; \forall i \in \Theta, \; k = 0, 1, \ldots, N-1 \tag{3}$$

where p^0_{ij} is the nominal, time invariant, transition probability distribution, and $R_{TV}(i)$, $i \in \Theta$, is the total variation distance parameter. In general, the total variation distance parameter

[1] In the proposed approach, time varying, nominal transition probability distributions can also be used.

can also depend on time, i.e., $R_{TV}(i,k)$, $i \in \Theta$, $k = 0, 1, \ldots, N-1$.

The uncertainty or ambiguity set (3) also accounts for scenarios in which $R_{TV}(i) \neq R_{TV}(j)$, $i, j \in \Theta$. This means that it is possible to assign different uncertainty levels, between the nominal transition probability distribution and the true time-varying transition probability distribution of the jump process $\{\theta_k, k \geq 0\}$, for different operating modes of the system. This feature of (3) is of practical importance and can find applications in cases in which we are faced with the challenging problem of specifying the transition probability distribution of the system under all its different operating modes. To appreciate this, as an example, consider a control system with two operating modes, i.e., a fault-free operating mode and a faulty operating mode. While one can argue that specifying the transition probability distribution of the system when it operates on its fault-free mode is relatively easy, this is not the case when the system operates on the faulty mode.

Moreover, since the total variation distance metric is a true distance metric between probability distributions, which are not absolutely continuous, it can be defined on spaces of different dimensionality. This property of the total variation distance metric is useful in cases where the nominal model is a simplified version of the true model, and hence, defined on a lower dimensional space. This issue is discussed in more detail in the following example.

Example 2.2: Consider a true stochastic control system with three operating modes defined on Θ, and a corresponding nominal stochastic control system with two operating modes defined on $\Theta^0 \subseteq \Theta$. It is clear that, the true stochastic control system will have a time-varying transition probability distribution

$$p_{ij}(k) = \begin{pmatrix} p_{11}(k) & p_{12}(k) & p_{13}(k) \\ p_{21}(k) & p_{22}(k) & p_{23}(k) \\ p_{31}(k) & p_{32}(k) & p_{33}(k) \end{pmatrix}, \text{ with } n_\Theta = 3$$

while the corresponding nominal transition probability distribution of the Markov chain for (1), will be given by

$$p^0_{ij} = \begin{pmatrix} p^0_{11} & p^0_{12} \\ p^0_{21} & p^0_{22} \end{pmatrix}, \text{ with } n_{\Theta^0} = 2.$$

By applying function extension, one may lift the nominal transition probability distribution p^0_{ij} to Θ by simply relabeling the extended distribution as the nominal distribution with the property that $\widetilde{p}_{ij} = p^0_{ij}$, $\forall (i,j) \in \Theta^0 \times \Theta^0$, and $\widetilde{p}_{ij} = 0$, $\forall (i,j) \in \Theta \times \Theta \setminus \Theta^0 \times \Theta^0$, i.e.,

$$\widetilde{p}_{ij} = \begin{pmatrix} p^0_{11} & p^0_{12} & 0 \\ p^0_{21} & p^0_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, it follows that the uncertainty set $\mathbb{B}_k(i)$, defined by (3), now includes transition probability distributions with the same cardinality, both defined on Θ. The natural question, which arises here is on how to select the values of $\widetilde{p}_{3\bullet}$. Since no knowledge exists, one can choose the values of $\widetilde{p}_{3\bullet}$ arbitrarily so that $\widetilde{p}_{3\bullet}(k) \geq 0$, and $\sum_{j\in\Theta} \widetilde{p}_{3j} = 1$. However, in this case, the designer must also set $R_{TV}(3)$ in (3) to its maximum allowable value.

Even though, in this article, we deal with the case $n_{\Theta^0} = n_\Theta$, in general, the uncertainty model based on total variation distance metric can be applied when $n_{\Theta^0} \leq n_\Theta$. Such cases, together with the fact that when a system operates under a new unmodeled mode, under which the dynamics, inputs, and noise matrices are unknown to the designer, will be subject to future investigation.

In the literature of homogeneous MJLS, the transition probability distribution of the jump process is commonly assumed to be time invariant. The difference between the results from the literature and those presented here is that we consider the situation where the (true) transition probability distribution of the jump process depends also on time (i.e., we deal with time-varying distributions), a property which arises naturally due to the behavior of the proposed solution.

The uncertainty or ambiguity model (3) relies on the availability of a prespecified "nominal" transition probability distribution based on the planner's prior information. In particular, such nominal distributions are often our best guess and contain valuable information about the transitions of the Markov chain. However, since dynamic programming as a solution method for stochastic optimal control problems involves conditional expectation with respect to the collection of Markov chain transition probability distributions, any uncertainty in these distributions affects the optimality of the policies. Thus, the following minimax stochastic control problem is formulated.

C. Minimax Formulation

For fixed initial data $x_0 = x$ and $\theta_0 = i$, define the N-stage expected cost by

$$J_N(g, p(k) : k = 0, \ldots, N-1) \triangleq \mathbb{E}^{g,p}_{x,i}$$
$$\left[\sum_{k=0}^{N-1} (x_k^T Q_k(\theta_k) x_k + u_k^T R_k(\theta_k) u_k) + x_N^T Q_N(\theta_N) x_N\right] \quad (4)$$

where $\mathbb{E}^{g,p}_{x,i}[\cdot]$ indicates the dependence of the expectation operation on policy $g \in G$, and induced by a transition probability $p_{i\bullet}(k) \in \mathbb{B}_k(i)$, $i \in \Theta$, $k = 0, 1, \ldots, N-1$. Here, $Q_k(\theta_k)$ $(k = 0, 1, \ldots, N)$ are $n \times n$ symmetric, positive semidefinite matrices, and $R_k(\theta_k)$ $(k = 0, \ldots, N-1)$ are $m \times m$ symmetric, positive definite matrices.

Our minimax stochastic control problem is to derive an optimal control policy $g^* \in G$, and a transition probability distribution $p^*_{i\bullet}(k) \in \mathbb{B}_k(i)$ for all k that belongs to the total variation constraint set, so as to solve

$$J^* \triangleq J_N(g^*, p^*(k) : k = 0, \ldots, N-1)$$
$$= \min_{g\in G} \max_{\substack{p(k), k=0,\ldots,N-1: \\ \text{eqn (3) holds}}} J_N(g, p(k) : k = 0, \ldots, N-1). \quad (5)$$

In the following section, the solution of the robust LQR problem (5) is provided, and an LQR procedure is proposed.

III. Solution of the Robust LQR Problem

In this section, we follow a dynamic programming approach to solve the minimax stochastic control problem (5). In particular, we provide the solution of the inner optimization, and subsequently, we compute the optimal cost and the optimal policy for the outer optimization. Finally, we derive an LQR procedure for the numerical solution of (5).

A. Minimax Dynamic Programming

For $(k,x,i) \in \{0,1,\ldots,N\} \times \mathcal{X} \times \Theta$ let $V_k(x,i)$ denote the minimal cost-to-go or value function on the time horizon $\{k, k+1,\ldots,N\}$, given an optimal policy $g_t^*(\cdot)$, $t=0,\ldots,k-1$, and optimal policy $p^*(t)$, $t=0,\ldots,k-1$, such that (3) holds, defined by

$$V_k(x,i) = \min_{u\in\mathcal{U}_{[k,N-1]}} \max_{\substack{p(j),j=k+1,\ldots,N-1:\\ \text{eqn (3) holds}}} \mathbb{E}^{g,p}_{x,i} \left[\sum_{t=k}^{N-1}(x_t^TQ_t(\theta_t)x_t + u_t^TR_t(\theta_t)u_t) + x_N^TQ_N(\theta_N)x_N\right] \quad (6)$$

where $\mathbb{E}^{g,p}_{x,i}[\cdot]$ denotes conditional expectation given that $x_k^g = x$ and $\theta_k = i$ with x, i fixed. The dynamic programming algorithm gives [16]

$$V_N(x_N,\theta_N) = x_N^TQ_N(\theta_N)x_N \quad (7a)$$

$$V_k(x,i) = \min_{u_k\in\mathcal{U}} \max_{p_{i\bullet}(k)\in\mathbb{B}_k(i)} \mathbb{E}^{g,p}_{x,i}\left[x^TQ_k(i)x + u_k^TR_k(i)u_k + V_{k+1}(x_{k+1},\theta_{k+1})\right], \quad x\in\mathcal{X},\ k=N-1,\ldots,0. \quad (7b)$$

Notice that (7) relates the value function $V_k(\cdot)$ and $V_{k+1}(\cdot)$ for all $k=N-1,\ldots,0$, and generates $V_N(\cdot), V_{N-1}(\cdot),\ldots,V_0(\cdot)$ by backward recursion.

By deriving the basic steps of the proof of the maximizing and minimizing elements in (7), we will show by backward induction that the solution is of the following form:

$$V_k(x,i) = x^TP_k(i)x + r_k(i) \quad (8)$$

for some matrices $P_k(i) \succeq 0$, and scalars $r_k(i)$, $i\in\Theta$.

1) Preliminary Step 1: Clearly, the induction hypothesis in (8) is true for $k=N$ with $P_N(i) = Q_N(i)$ and $r_N(i) = 0$, $i\in\Theta$. Then, $P_N(i) = P_N(i)^T \succeq 0$, $i\in\Theta$ and $V_N(x,i) = x^TP_N(i)x + r_N(i)$. Suppose that for $t=k+1,k+2,\ldots,N$, $P_t(i) = P_t(i)^T \succeq 0$, and $V_t(x,i) = x^TP_t(i)x + r_t(i)$. It will be shown that then $V_k(x,i) = x^TP_k(i)x + r_k(i)$, where $P_k(i) = P_k(i)^T \succeq 0$ and $r_k(i)$ satisfies certain recursions. Toward this end, we rewrite (7b) as follows:

$$V_k(x,i) = \min_{u_k\in\mathcal{U}}\left\{x^TQ_k(i)x + u_k^TR_k(i)u_k + \max_{p_{i\bullet}(k)\in\mathbb{B}_k(i)} \mathbb{E}^{g,p}_{x,i}[V_{k+1}(x_{k+1},\theta_{k+1})]\right\}. \quad (9)$$

Furthermore, (9) may be expressed as follows:

$$V_k(x,i) = \min_{u_k\in\mathcal{U}}\left\{x^TQ_k(i)x + u_k^TR_k(i)u_k + \max_{p_{i\bullet}(k)\in\mathbb{B}_k(i)} \sum_{\theta_{k+1}\in\Theta}\left(\int_{\mathcal{X}_{k+1}} V_{k+1}(x_{k+1},\theta_{k+1})P^g(x_{k+1}|x,i)\right)p_{ij}(k)\right\} \quad (10)$$

where $p_{ij}(k) \triangleq P(\theta_{k+1}=j|\theta_k=i)$. Next, let us define the sequence

$$\begin{aligned}
\ell_k(x_k,\theta_k,\theta_{k+1},u_k) &\triangleq \int_{\mathcal{X}_{k+1}} V_{k+1}(x_{k+1},\theta_{k+1})P^g(x_{k+1}|x_k,\theta_k)\\
&\overset{(a)}{=} \int_{\mathcal{X}_{k+1}} \{x_{k+1}^TP_{k+1}(\theta_{k+1})x_{k+1} + r_{k+1}(\theta_{k+1})\}\, P^g(x_{k+1}|x_k,\theta_k)\\
&\overset{(b)}{=} \mathbb{E}_{w_k}\big[(A_k(\theta_k)x_k + B_k(\theta_k)u_k + M_k(\theta_k)w_k)^TP_{k+1}(\theta_{k+1})\\
&\quad (A_k(\theta_k)x_k + B_k(\theta_k)u_k + M_k(\theta_k)w_k) + r_{k+1}(\theta_{k+1})\big]\\
&\overset{(c)}{=} x_k^TA_k^T(\theta_k)P_{k+1}(\theta_{k+1})A_k(\theta_k)x_k\\
&\quad + u_k^TB_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)u_k\\
&\quad + 2x_k^TA_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)u_k\\
&\quad + \mathbb{E}_{w_k}[w_k^TM_k^T(\theta_k)P_{k+1}(\theta_{k+1})M_k(\theta_k)w_k] + r_{k+1}(\theta_{k+1})
\end{aligned} \quad (11)$$

where (a) follows by the induction hypothesis, (b) by (1) and by Assumption 2.1, and (c) using the fact that $\mathbb{E}[w_k]=0$, $k=1,2\ldots$. Note that, in (11) the term $\mathbb{E}_{w_k}[\cdot]$ denotes expectation with respect to the probability distribution of w_k. At this point, let us assume that the solution of the maximization in (10) is denoted by $p^*_{i\bullet}(k)\in\mathbb{B}_k(i)$. Then, (10) becomes

$$V_k(x,i) = \min_{u_k\in\mathcal{U}}\left\{x^TQ_k(i)x + u_k^TR_k(i)u_k + \sum_{\theta_{k+1}\in\Theta}\ell_k(x,i,\theta_{k+1},u_k)p^*_{ij}(k)\right\}. \quad (12)$$

Next, we characterize the solution of the inner optimization in (10).

2) Preliminary Step 2: First, let us define the maximum and minimum values of (11) with respect to $\theta_{k+1}\in\Theta$, by

$$\ell_{\max,k}(x_k,\theta_k,u_k) \triangleq \max_{\theta_{k+1}\in\Theta} \ell_k(x_k,\theta_k,\theta_{k+1},u_k)$$

$$\ell_{\min,k}(x_k,\theta_k,u_k) \triangleq \min_{\theta_{k+1}\in\Theta} \ell_k(x_k,\theta_k,\theta_{k+1},u_k)$$

and its corresponding sets of states that achieve the maximum and minimum values of $\ell_k(x_k,\theta_k,\theta_{k+1},u_k)$ by

$$\Theta^0(k,\theta_k) \triangleq \{\theta_{k+1}\in\Theta : \ell_k(x_k,\theta_k,\theta_{k+1},u_k) = \ell_{\max,k}(x_k,\theta_k,u_k)\} \quad (13)$$

$$\Theta_0(k,\theta_k) \triangleq \{\theta_{k+1}\in\Theta : \ell_k(x_k,\theta_k,\theta_{k+1},u_k) = \ell_{\min,k}(x_k,\theta_k,u_k)\}. \quad (14)$$

For all remaining sequences, such that $\Theta^0(k,\theta_k)\cup\Theta_0(k,\theta_k)\subset\Theta$, and for $1\le r\le |\Theta\setminus\{\Theta^0(k,\theta_k)\cup\Theta_0(k,\theta_k)\}|$, define recursively the set of states for which (11) achieves its $(j+1)$st smallest value by

$$\Theta_j(k,\theta_k) \triangleq \{\theta_{k+1} \in \Theta : \ell_k(x_k,\theta_k,\theta_{k+1},u_k) = \min \big\{\ell_k(x_k,\theta_k,\alpha_k,u_k) : \alpha_k \in \Theta\backslash(\Theta^0(k,\theta_k)\cup \big\{(\cup_{i=1}^{j}\Theta_{j-1}(k,\theta_k))\}\big)\big\}\big\},\ j \in \{1,2,\ldots,r\} \tag{15}$$

till all the elements of Θ are exhausted. Furthermore, we define the corresponding values of the sequence on these sets by

$$\ell_{\Theta_j,k}(x_k,\theta_k,u_k) \triangleq \min_{\theta_{k+1}\in\Theta\backslash(\Theta^0(k,\theta_k)\cup(\cup_{i=1}^{j}\Theta_{j-1}(k,\theta_k)))} \ell_k(x_k,\theta_k,\theta_{k+1},u_k).$$

Notice that $\Theta^0(k,\theta_k)$ is the set of $\theta_{k+1}\in\Theta$, for which $\ell_k(x_k,\theta_k,\theta_{k+1},u_k)$ achieves the function's maximum value. The set $\Theta^0(k,\theta_k)$ may contain a single element, i.e., its cardinality $|\Theta^0(k,\theta_k)|=1$, or may contain multiple elements, i.e., $|\Theta^0(k,\theta_k)|>1$. Similarly, $\Theta_{j-1}(k,\theta_k)$, $j\in\{1,2,\ldots,r+1\}$, is the set of $\theta_{k+1}\in\Theta$, for which $\ell_k(x_k,\theta_k,\theta_{k+1},u_k)$ achieves the function's (j)th minimum value. Again, $|\Theta_{j-1}(k,\theta_k)|=1$, or $|\Theta_{j-1}(k,\theta_k)|>1$. In this article, we provide the solution for the general case, i.e., $|\Theta^0|\geq 1$, and $|\Theta_{j-1}|\geq 1$ for all $j\in\{1,2,\ldots,r+1\}$. The solution for the case $|\Theta^0|=1$, and $|\Theta_{j-1}|=1$ for all $j\in\{1,2,\ldots,r+1\}$, can be found in [17].

Let $\mathcal{P}(k,\theta_k)$ denote the identified partition of Θ, in the sense of (13)–(15). For example, $\mathcal{P}(k,\theta_k)\triangleq\{\Theta^0(k,\theta_k),\Theta_0(k,\theta_k),\Theta_1(k,\theta_k),\ldots,\Theta_r(k,\theta_k)\}$. For notational convenience, also let

$$\ell_k(x_k,\theta_k,\theta_{k+1}\in\Theta^0(k,\theta_k),u_k)\triangleq\ell_{\max,k}(x_k,\theta_k,u_k)$$
$$\ell_k(x_k,\theta_k,\theta_{k+1}\in\Theta_0(k,\theta_k),u_k)\triangleq\ell_{\min,k}(x_k,\theta_k,u_k)$$
$$\ell_k(x_k,\theta_k,\theta_{k+1}\in\Theta_j(k,\theta_k),u_k)\triangleq\ell_{\Theta_j,k}(x_k,\theta_k,u_k)$$

for all $j=1,\ldots,r$. In what follows, the notation $\mathbb{E}_p^{\mathcal{P}}[\cdot]$ will be used to denote expectation with respect to the transition probability distribution $p_{ij}(k)$, over the identified partition $\mathcal{P}(k,\theta_k)$, i.e., for any function f_k of $\theta_{k+1}\in\Theta$

$$\mathbb{E}_p^{\mathcal{P}}[f_k(\theta_{k+1})]\triangleq f_k(\theta_{k+1}\in\Theta^0(k,\theta_k))\sum_{j\in\Theta^0(k,\theta_k)}p_{ij}(k) + f_k(\theta_{k+1}\in\Theta_0(k,\theta_k))\sum_{j\in\Theta_0(k,\theta_k)}p_{ij}(k) + \sum_{s=1}^{r}f_k(\theta_{k+1}\in\Theta_s(k,\theta_k))\sum_{j\in\Theta_s(k,\theta_k)}p_{ij}(k).$$

Intuitively, the solution of the maximization in (10) is obtained by identifying the partition of the state-space of the jump process $\{\theta_k,k\geq 0\}$ into disjoint sets $\{\Theta^0(k,\theta_k),\Theta_0(k,\theta_k),\Theta_1(k,\theta_k),\ldots,\Theta_r(k,\theta_k)\}$, and the measures on this partition. As the next Theorem 3.1 shows, the solution is based on finding upper and lower bounds, which are achievable and closed-form expressions of the measures, which achieve those bounds. The next result characterizes the solution of the maximization in (10).

Theorem 3.1: Define the sequence $\ell_k(x_k,\theta_k,\theta_{k+1},u_k)$ as in (11). The solution of the inner optimization in (10) is given by

$$\sum_{j\in\Theta}\ell_k(x_k=x,\theta_k=i,\theta_{k+1}=j,u_k)p^*_{ij}(k) = \ell_k(x,i,\theta_{k+1}\in\Theta^0(k,i),u_k)\sum_{j\in\Theta^0(k,i)}p^*_{ij}(k) + \ell_k(x,i,\theta_{k+1}\in\Theta_0(k,i),u_k)\sum_{j\in\Theta_0(k,i)}p^*_{ij}(k) + \sum_{s=1}^{r}\ell_k(x,i,\theta_{k+1}\in\Theta_s(k,i),u_k)\sum_{j\in\Theta_s(k,i)}p^*_{ij}(k). \tag{16}$$

The maximizing, time varying, transition probability distribution $p^*_{i\bullet}(k)\in\mathbb{B}_k(i)$, for any $\theta_k=i\in\Theta$ and $k=0,1,\ldots,N-1$ is given by

$$\sum_{j\in\Theta^0(k,i)}p^*_{ij}(k)=\sum_{j\in\Theta^0(k,i)}p^0_{ij}+\frac{\alpha_i}{2} \tag{17a}$$

$$\sum_{j\in\Theta_0(k,i)}p^*_{ij}(k)=\left(\sum_{j\in\Theta_0(k,i)}p^0_{ij}-\frac{\alpha_i}{2}\right)^+ \tag{17b}$$

$$\sum_{j\in\Theta_s(k,i)}p^*_{ij}(k)=\left(\sum_{j\in\Theta_s(k,i)}p^0_{ij} - \left(\frac{\alpha_i}{2}-\sum_{z=1}^{s}\sum_{j\in\Theta_{z-1}(k,i)}p^0_{ij}\right)^+\right)^+ \quad s=1,2,\ldots,r \tag{17c}$$

$$\alpha_i=\min\left(R_{TV}(i),2\left(1-\sum_{j\in\Theta^0(k,i)}p^0_{ij}\right)\right) \tag{17d}$$

where for convenience we denote $(x)^+\triangleq\max\{0,x\}$.

Proof: By preliminary steps 1 and 2, then it is verified that the abovementioned statement follows from [18], [19], by repeating the derivation. ■

Remark 3.2: The main feature of (17) is its explicit characterization via a water-filling solution with respect to the nominal transition probability distribution of the jump process $\{\theta_k,k\geq 0\}$ and the total variation distance design parameter, $R_{TV}(i)$, $i\in\Theta$. At this point, it should also be emphasized that between different operating modes of the MJLS, the partition of the state space Θ may vary with time. Consequently, the maximizing transition probability distribution of the jump process also depends on time, as shown in Theorem 3.1. In any case, the optimal partition of the state space of the jump process is obtained as part of the water-filling solution.

Next, we provide the solution of the outer optimization in (10).

B. Solution of the Outer Optimization Problem

Substituting (8) into (12), we obtain

$$x^T P_k(i)x + r_k(i) = \min_{u_k \in \mathcal{U}} \Bigg\{ x^T Q_k(i)x + u_k^T R_k(i)u_k + \sum_{\theta_{k+1}\in\Theta} \ell_k(x,i,\theta_{k+1},u_k)p^*_{ij}(k) \Bigg\}. \quad (18)$$

Going a step further, using (11), and also the fact that

$$\begin{aligned}
&\ell_k(x_k,\theta_k,\theta_{k+1}\in\Theta^0(k,\theta_k),u_k)\\
&= x_k^T A_k^T(\theta_k)P_{k+1}(\theta_{k+1}\in\Theta^0(k,\theta_k))A_k(\theta_k)x_k\\
&\quad + u_k^T B_k^T(\theta_k)P_{k+1}(\theta_{k+1}\in\Theta^0(k,\theta_k))B_k(\theta_k)u_k\\
&\quad + 2x_k^T A_k^T(\theta_k)P_{k+1}(\theta_{k+1}\in\Theta^0(k,\theta_k))B_k(\theta_k)u_k\\
&\quad + \mathbb{E}_{w_k}[w_k^T M_k^T(\theta_k)P_{k+1}(\theta_{k+1}\in\Theta^0(k,\theta_k))M_k(\theta_k)w_k]\\
&\quad + r_{k+1}(\theta_{k+1}\in\Theta^0(k,\theta_k)) \qquad (19)
\end{aligned}$$

and, similarly for $\ell_k(x_k,\theta_k,\theta_{k+1}\in\Theta_s(k,\theta_k),u_k)$ for all $s = 0,1,\ldots,r$, then we can rewrite (18) as follows:

$$\begin{aligned}
x^T P_k(i)x + r_k(i) = \min_{u_k\in\mathcal{U}} \Big\{ & x^T Q_k(i)x + u_k^T R_k(i)u_k\\
&+ x^T A_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]A_k(i)x\\
&+ u_k^T B_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]B_k(i)u_k\\
&+ 2x^T A_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]B_k(i)u_k\\
&+ \mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[\mathbb{E}_{w_k}[w_k^T M_k^T(i)P_{k+1}(\theta_{k+1})M_k(i)w_k]]\\
&+ \mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[r_{k+1}(\theta_{k+1})]\Big\} \qquad (20)
\end{aligned}$$

where $\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[\cdot]$ denotes expectation with respect to the maximizing transition probability distribution $p^*_{i\bullet}(k) \in \mathbb{B}_k(i)$, $\forall i \in \Theta$, over the identified partition $\mathcal{P}(k,\theta_k)$, as given in (17). Differentiating the right side of (20) with respect to u_k, and setting the derivative equal to zero, we obtain

$$\begin{aligned}
&2R_k(i)u_k + 2B_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]B_k(i)u_k\\
&\quad + 2B_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]A_k(i)x = 0.
\end{aligned}$$

Hence,

$$u_k^* = -L_k(i)x_k, \text{ for } \theta_k = i \qquad (21)$$

with

$$\begin{aligned}
L_k(i) = &\Big(R_k(i) + B_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]B_k(i)\Big)^{-1}\\
&\times B_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]A_k(i). \qquad (22)
\end{aligned}$$

In (22), the existence of the inverse follows by our assumptions on $P_k(\theta_k)$ and $R_k(\theta_k)$ (namely, the facts that $\{P_k(\theta_k),\theta_k \in \Theta\}$ are $n\times n$ symmetric, positive semidefinite matrices, and $\{R_k(\theta_k),\theta_k\in\Theta\}$ are $m\times m$ symmetric, positive definite matrices [2]). Substituting (21) into (20), we have

$$\begin{aligned}
x^T P_k(i)x + r_k(i) = x^T \Big(&Q_k(i) + L_k^T(i)R_k(i)L_k(i)\\
&+ A_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]A_k(i)\\
&+ L_k^T(i)B_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]B_k(i)L_k(i)\\
&-2A_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]B_k(i)L_k(i)\Big)x\\
&+ \mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}\big[\mathbb{E}_{w_k}[w_k^T M_k^T(i)P_{k+1}(\theta_{k+1})M_k(i)w_k]\\
&+ r_{k+1}(\theta_{k+1})\big].
\end{aligned}$$

Hence, it follows that $V_k(x,i) = x^T P_k(i)x + r_k(i)$ with

$$\begin{aligned}
P_k(i) &= Q_k(i) + L_k^T(i)\Big(R_k(i)\\
&\qquad + B_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]B_k(i)\Big)L_k(i)\\
&\qquad + A_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]A_k(i)\\
&\qquad - 2A_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]B_k(i)L_k(i)\\
&= Q_k(i) + A_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]A_k(i)\\
&\qquad - A_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]B_k(i)L_k(i) \qquad (23)
\end{aligned}$$

and

$$\begin{aligned}
r_k(i) &= \mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}\big[\mathbb{E}_{w_k}[w_k^T M_k^T(i)P_{k+1}(\theta_{k+1})M_k(i)w_k]\\
&\qquad + r_{k+1}(\theta_{k+1})\big]\\
&= \mathrm{tr}(M_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]M_k(i)\Sigma_w)\\
&\qquad + \mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[r_{k+1}(\theta_{k+1})] \qquad (24)
\end{aligned}$$

where $r_N(i) = 0$. In (23), the second equality follows by (22), while in (24), $\mathrm{tr}(\cdot)$ denotes the trace of a matrix, and $\Sigma_w \triangleq \mathbb{E}[w_k w_k^T]$ is the covariance matrix of w_k. Finally, from (8) it follows that $V_0(x,i) = x^T P_0(i)x + r_0(i)$, with

$$r_0(i) = \sum_{k=0}^{N-1} \mathrm{tr}\Big(M_k^T(i)\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]M_k(i)\Sigma_w\Big). \qquad (25)$$

Remark 3.3: A special property of the proposed solution is that the feedback gain matrices (22), and the Riccati equations (23), (24), in contrast to the classical LQR results (e.g., see [4]), are calculated using the maximizing transition probability distribution, based on uncertainty model (3) (i.e., they depend on the specified total variation distance between the nominal and the true transition probability distribution of the jump process $\{\theta_k, k\geq 0\}$). Notice that, for $R_{TV}(i) = 0, \forall i\in\Theta$, the nominal transition probability distribution p^0 is equal to the maximizing transition probability distribution $p^*_{i\bullet}(k), \forall k = 0,1,\ldots,N-1$. Consequently, the minimax stochastic control problem reduces to the classical one with a known solution given by (21), (23), and (24), calculated using the nominal transition probability distribution. Moreover, an attractive aspect of the solution of the minimax stochastic control problem under investigation is that the optimal control law (21) preserves its linearity similarly to the classical case.

Below, we state the main theorem of this article, which summarizes the previous discussion and results.

Theorem 3.4: Assume that the assumptions of Section II hold. Then, for the solution of the minimax stochastic control problem (5), the following hold.

(a) The solution of the dynamic programming equation (7) is

$$V_k(x, i) = x^T P_k(i) x + r_k(i) \tag{26}$$

for some matrices $P_k(i) \succeq 0$, and constants $r_k(i) \geq 0$, $i \in \Theta$. The optimal control is given by

$$u_k^*(x_k, i) = -L_k(i)x_k, \quad \text{for } \theta_k = i \tag{27}$$

where the feedback gain matrices $L_k(i)$ are given by (22), and the Riccati equations are calculated backwards in time as follows:

$$\begin{aligned} P_k(i) &= Q_k(i) + A_k^T(i)\mathbb{E}_{p_{i\bullet}^*(k)}[P_{k+1}(\theta_{k+1})]A_k(i) \\ &\quad - A_k^T(i)\mathbb{E}_{p_{i\bullet}^*(k)}[P_{k+1}(\theta_{k+1})]B_k(i)L_k(i) \\ P_N(i) &= Q_N(i), \quad i \in \Theta, \quad k = N-1, N-2, \ldots, 0. \end{aligned}$$

The optimal cost for the minimax problem is given by

$$J^* = \mathbb{E}^{g^*, p^*}[V_0(x_0, \theta_0)] = \mathbb{E}^{g^*, p^*}[x_0^T P_0(\theta_0)x_0 + r_0(\theta_0)] \tag{28}$$

with $r_0(i)$ given by (25).

(b) The maximizing, time varying, transition probability distribution $p_{i\bullet}^*(k) \in \mathbb{B}_k(i)$, $i \in \Theta$, $k = 0, 1, \ldots, N-1$, is given by (17), with the corresponding support sets calculated by (13)–(15).

Proof: Part (a) of Theorem 3.4 follows from the results derived in Sections III-A1 and III-B. Part (b) follows from the results derived in Section III-A2. ■

In the following section, we derive an LQR procedure for the numerical solution of the robust LQR problem.

C. LQR Procedure

As we have already mentioned, part (a) of Theorem 3.2 is to be calculated using the maximizing, time varying, transition probability distribution of the jump process $\{\theta_k, k \geq 0\}$. The main difficulty in doing so is that, while the Riccati equations are obtained offline and moving backward in time, the maximizing transition probability distribution of the jump process is obtained online and moving forward in time. To overcome this difficulty, in Algorithm 1, we provide the necessary steps to be followed in order to obtain the solution of the robust LQR problem.

In summary, the solution is obtained by following a three-step procedure. During the initialization step, the feedback gain matrices and the Riccati equations are calculated using the nominal transition probability distribution of the jump process. Then, by utilizing the feedback gain matrices and the Riccati equations, obtained under the nominal transition probability distribution, step 1 calculates the maximizing transition probability distribution online and forward in time. In step 1(a), we utilize the fact that $\ell_k(x_k, \theta_k, \theta_{k+1}, u_k)$ is to be calculated for all possible combinations of the pair (θ_k, θ_{k+1}). Hence, by substituting $u_k = -L_k(\theta_k)x_k$ back in (11) we obtain

$$\begin{aligned} &\ell_k(x_k, \theta_k, \theta_{k+1}, u_k) \\ &= x_k^T \{A_k^T(\theta_k)P_{k+1}(\theta_{k+1})A_k(\theta_k) \\ &\quad + L_k^T(\theta_k)B_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k) \\ &\quad -2A_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k)\} x_k \\ &\quad + \mathbb{E}_{w_k}[w_k^T M_k^T P_{k+1}(\theta_{k+1})M_k w_k] + r_{k+1}(\theta_{k+1}) \\ &= x_k^T S_k(\theta_k, \theta_{k+1})x_k + \mathbb{E}_{w_k}[w_k^T M_k^T P_{k+1}(\theta_{k+1})M_k w_k] \\ &\quad + r_{k+1}(\theta_{k+1}) \\ &:= \ell_k(x_k, \theta_k, \theta_{k+1}) \end{aligned} \tag{29}$$

where

$$\begin{aligned} S_k(\theta_k, \theta_{k+1}) &= A_k^T(\theta_k)P_{k+1}(\theta_{k+1})A_k(\theta_k) \\ &\quad + L_k^T(\theta_k)B_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k) \\ &\quad - 2A_k^T(\theta_k)P_{k+1}(\theta_{k+1})B_k(\theta_k)L_k(\theta_k). \end{aligned}$$

Finally, in step 2, we apply the obtained maximizing transition probability distribution to update the solution by repeating the calculations of the feedback gain matrices, the Riccati equations, and the optimal controller.

In the following section, we give an alternative characterization of Theorem 3.1, which subsequently will be used to present an equivalent formulation of (10). This equivalent formulation of (10), will help us arrive at an intuitive interpretation of ambiguity model (3), and codify the impact of uncertain MJLS on the performance of the LQR controller.

IV. Equivalent Formulation of the Robust LQR Problem

The main result of this section is Theorem 4.1, which utilizes Theorem 3.1 to derive an equivalent solution of the robust LQR problem. For the next result, we employ the definitions of sets $\Theta^0(k, \theta_k)$, $\Theta_0(k, \theta_k)$, and $\Theta_j(k, \theta_k)$, $j \in \{1, \ldots, r\}$, given by (13)–(15), respectively.

Theorem 4.1: The solution of the inner optimization in (10) is equivalently expressed as follows:

$$\begin{aligned} &\sum_{j \in \Theta} \ell_k(x_k, \theta_k = i, \theta_{k+1} = j, u_k)p_{ij}^*(k) \\ &= \mathbb{E}_{p^0}^{\mathcal{P}}[\ell_k(x_k, \theta_k = i, \theta_{k+1}, u_k)] + \beta(\alpha_i) \\ &\quad + \frac{\alpha_i}{2}\{\ell_k(x_k, \theta_k = i, \theta_{k+1} \in \Theta^0(k, i), u_k) \\ &\quad - \ell_k(x_k, \theta_k = i, \theta_{k+1} \in \widetilde{\Theta}(k, i), u_k)\} \end{aligned} \tag{30}$$

where α_i is given by (17d), and where the set $\widetilde{\Theta}(k, i)$ and the function $\beta(\alpha_i)$ are calculated as follows.

Case 1: If

$$\frac{\alpha_i}{2} \leq \sum_{j \in \Theta_0(k,i)} p_{ij}^0 \tag{31}$$

then $\widetilde{\Theta}(k, i) = \Theta_0(k, i)$ and $\beta(\alpha_i) = 0$.

Algorithm 1: LQR Procedure.

Input data. $A_k(\theta_k) \in \mathbb{R}^{n\times n}$, $B_k(\theta_k) \in \mathbb{R}^{n\times m}$, $M_k(\theta_k) \in \mathbb{R}^{n\times n}$, $Q_k(\theta_k) \in \mathbb{R}^{n\times n}$, $R_k(\theta_k) \in \mathbb{R}^{m\times m}$, known for all $\theta_k \in \Theta$ and $k = 0,\ldots,N-1$. Choose $R_{TV}(i) \in [0,2]$, $\forall i \in \Theta$, and $p^0(k) = p^0$ (time invariant) nominal transition probability distribution. Set the initial state x_0, the initial distribution of θ_0, and $P_N(\theta_N) = Q_N(\theta_N)$, $r_N(\theta_N) = 0$, for all $\theta_N \in \Theta$.

Initialization Step. (Backward recursion). For all $k = N-1,\ldots,0$, use the nominal transition probability distribution p^0, to calculate:

$$L_k(\theta_k) = \left(R_k(\theta_k) + B_k^T(\theta_k)\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]B_k(\theta_k)\right)^{-1} B_k^T(\theta_k)\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]A_k(\theta_k)$$

$$P_k(\theta_k) = Q_k(\theta_k) + A_k^T(\theta_k)\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]A_k(\theta_k) - A_k^T(\theta_k)\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]B_k(\theta_k)L_k(\theta_k)$$

$$r_k(\theta_k) = \operatorname{tr}\left(M_k^T(\theta_k)\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]M_k(\theta_k)\Sigma_w\right) + \mathbb{E}_{p^0}[r_{k+1}(\theta_{k+1})], \quad \text{for all } \theta_k \in \Theta.$$

Step 1. (Forward recursion). For all $k = 0,\ldots,N-1$, do:

a) Calculate $\ell_k(x_k,\theta_k,\theta_{k+1})$ for all $\theta_k,\theta_{k+1} \in \Theta$ given by (29).

b) Identify the support sets $\Theta^0(k,\theta_k)$, $\Theta_0(k,\theta_k)$, and $\Theta_j(k,\theta_k)$, $\forall j = 1,2,\ldots,r$, using (13), (14), and (15), respectively.

c) Calculate the maximizing transition probability distribution $p^*_{i\bullet}(k) \in \mathbb{B}_k(i)$, $\forall i \in \Theta$, given by (17).

d) Use $p^*(k)$ to obtain the Markov chain random walk, and to calculate

$$u_k = -L_k(i)x_k,$$
$$x_{k+1} = A_k(i)x_k + B_k(i)u_k + M_k(i)w_k, \text{ for } \theta_k = i \in \Theta.$$

Step 2.

a) (Backward recursion) For all $k = N-1,\ldots,0$, use the maximizing transition probability distribution $p^*(k)$, to calculate $L_k(\theta_k)$, $P_k(\theta_k)$, and $r_k(\theta_k)$, for all $\theta_k \in \Theta$ (similarly to the initialization step, but with $p^*(k)$ replacing p^0).

b) (Forward recursion) For all $k = 0,\ldots,N-1$, using the obtained Markov chain random walk (of Step 1(d)), calculate

$$u_k^* = -L_k(i)x_k,$$
$$x_{k+1} = A_k(i)x_k + B_k(i)u_k^* + M_k(i)w_k, \text{ for } \theta_k = i \in \Theta.$$

Case 2: If

$$\sum_{j\in\cup_{\ell=0}^{z-1}\Theta_\ell(k,i)} p_{ij}^0 < \frac{\alpha_i}{2} \le \sum_{j\in\cup_{\ell=0}^{z}\Theta_\ell(k,i)} p_{ij}^0, z \in \{1,2,\ldots,r\} \tag{32}$$

then $\widetilde{\Theta}(k,i) = \Theta_z(k,i)$, and $\beta(\alpha_i)$ is given by

$$\beta(\alpha_i) = \sum_{s=0}^{z-1}(\ell_k(x_k,\theta_k = i,\theta_{k+1} \in \Theta_z(k,i),u_k) - \ell_k(x_k,\theta_k = i,\theta_{k+1} = \Theta_s(k,i),u_k)) \sum_{j\in\Theta_s(k,i)} p_{ij}^0. \tag{33}$$

Proof: See the Appendix. ■

A. Verification of the Equivalence Relation

Next, we employ Theorem 4.1 to derive the analogue of (12). By substituting (30) into (12), we obtain

$$V_k(x,i) = \min_{u_k\in\mathcal{U}}\Big(x^TQ_k(i)x + u_k^TR_k(i)u_k + \mathbb{E}_{p^0}^{\mathcal{P}}[\ell_k(x,i,\theta_{k+1},u_k)] + \beta(\alpha_i) + \frac{\alpha_i}{2}\Big\{\ell_k(x,i,\theta_{k+1}\in\Theta^0(k,i),u_k) - \ell_k(x,i,\theta_{k+1}\in\widetilde{\Theta}(k,i),u_k)\Big\}\Big). \tag{34}$$

Based on Theorem 4.1, Case 1, if (31) holds then $\widetilde{\Theta}(k,i) = \Theta_0(k,i)$ and $\beta(\alpha_i) = 0$. It follows that

$$\begin{aligned} V_k(x,i) = \min_{u_k\in\mathcal{U}}\Bigg(&\left[x^T\ u_k^T\right]\begin{bmatrix}H_{11}(k,i) & H_{12}(k,i)\\ H_{12}(k,i)^T & H_{22}(k,i)\end{bmatrix}\begin{bmatrix}x\\ u_k\end{bmatrix}\\ &+ \mathbb{E}_{p^0}^{\mathcal{P}}[\mathbb{E}_{w_k}[w_k^TM_k^T(i)P_{k+1}(\theta_{k+1})M_k(i)w_k] + r_{k+1}(\theta_{k+1})]\\ &+ \frac{\alpha_i}{2}\Bigg\{\Bigg(\left[x^T\ u_k^T\right]\begin{bmatrix}H_{11}^+(k,i) & H_{12}^+(k,i)\\ H_{12}^+(k,i)^T & H_{22}^+(k,i)\end{bmatrix}\begin{bmatrix}x\\ u_k\end{bmatrix}\\ &+ \mathbb{E}_{w_k}[w_k^TM_k^T(i)P_{k+1}(\theta_{k+1}\in\Theta^0(k,i))M_k(i)w_k]\\ &+ r_{k+1}(\theta_{k+1}\in\Theta^0(k,i))\Bigg)\\ &- \Bigg(\left[x^T\ u_k^T\right]\begin{bmatrix}H_{11}^-(k,i) & H_{12}^-(k,i)\\ H_{12}^-(k,i)^T & H_{22}^-(k,i)\end{bmatrix}\begin{bmatrix}x\\ u_k\end{bmatrix}\\ &+ \mathbb{E}_{w_k}[w_k^TM_k^T(i)P_{k+1}(\theta_{k+1}\in\Theta_0(k,i))M_k(i)w_k]\\ &+ r_{k+1}(\theta_{k+1}\in\Theta_0(k,i))\Bigg)\Bigg\}\Bigg) \end{aligned} \tag{35}$$

where

$$\begin{aligned} H_{11}(k,i) &\triangleq A_k^T(i)\mathbb{E}_{p^0}^{\mathcal{P}}[P_{k+1}(\theta_{k+1})]A_k(i) + Q_k(i)\\ H_{12}(k,i) &\triangleq A_k^T(i)\mathbb{E}_{p^0}^{\mathcal{P}}[P_{k+1}(\theta_{k+1})]B_k(i)\\ H_{22}(k,i) &\triangleq B_k^T(i)\mathbb{E}_{p^0}^{\mathcal{P}}[P_{k+1}(\theta_{k+1})]B_k(i) + R_k(i)\\ H_{11}^+(k,i) &\triangleq A_k^T(i)P_{k+1}(\theta_{k+1}\in\Theta^0(k,i))A_k(i)\\ H_{12}^+(k,i) &\triangleq A_k^T(i)P_{k+1}(\theta_{k+1}\in\Theta^0(k,i))B_k(i)\\ H_{22}^+(k,i) &\triangleq B_k^T(i)P_{k+1}(\theta_{k+1}\in\Theta^0(k,i))B_k(i)\\ H_{11}^-(k,i) &\triangleq A_k^T(i)P_{k+1}(\theta_{k+1}\in\Theta_0(k,i))A_k(i)\\ H_{12}^-(k,i) &\triangleq A_k^T(i)P_{k+1}(\theta_{k+1}\in\Theta_0(k,i))B_k(i)\\ H_{22}^-(k,i) &\triangleq B_k^T(i)P_{k+1}(\theta_{k+1}\in\Theta_0(k,i))B_k(i). \end{aligned}$$

For convenience, we denote by $D_{k+1}^P((\Theta^0,\Theta_0)(k,i))$ the difference between the solution of $P_{k+1}(\theta_{k+1}\in\Theta^0(k,i))$ and

$P_{k+1}(\theta_{k+1} \in \Theta_0(k,i))$, i.e.,

$$D^P_{k+1}((\Theta^0,\Theta_0)(k,i)) \triangleq P_{k+1}(\theta_{k+1} \in \Theta^0(k,i)) - P_{k+1}(\theta_{k+1} \in \Theta_0(k,i)). \quad (36)$$

In addition, we denote by $D^r_{k+1}((\Theta^0,\Theta_0)(k,i))$ the difference between the solution of $r_{k+1}(\theta_{k+1} \in \Theta^0(k,i))$ and $r_{k+1}(\theta_{k+1} \in \Theta_0(k,i))$, i.e.,

$$D^r_{k+1}((\Theta^0,\Theta_0)(k,i)) \triangleq r_{k+1}(\theta_{k+1} \in \Theta^0(k,i)) - r_{k+1}(\theta_{k+1} \in \Theta_0(k,i)). \quad (37)$$

Differentiating the right side of (35) with respect to u_k, and setting the derivative equal to zero, the optimal control is given by

$$u^*_k = -L_k(i)x_k, \text{ for } \theta_k = i \quad (38)$$

with

$$L_k(i) = \Big(R_k(i) + B^T_k(i)\Big(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})] + \frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_0)(k,i))\Big)B_k(i)\Big)^{-1}B^T_k(i) \Big(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})] + \frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_0)(k,i))\Big)A_k(i). \quad (39)$$

Substituting (38) into (35), we obtain

$$\begin{aligned} &x^T P_k(i)x + r_k(i) \\ &= x^T \big(H_{11}(k,i) + L^T_k(i)H_{22}(k,i)L_k(i) - 2H_{12}(k,i)L_k(i) \\ &+ \frac{\alpha_i}{2}\big\{\big(H^+_{11}(k,i) + L^T_k(i)H^+_{22}(k,i)L_k(i) - 2H^+_{12}(k,i)L_k(i)\big) \\ &- \big(H^-_{11}(k,i) + L^T_k(i)H^+_{22}(k,i)L_k(i) - 2H^-_{12}(k,i)L_k(i)\big)\big\}\big)x \\ &+ \mathbb{E}^{\mathcal{P}}_{p^0}[\mathbb{E}_{w_k}[w^T_k M^T_k(i)P_{k+1}(\theta_{k+1})M_k(i)w_k]] \\ &+ \frac{\alpha_i}{2}\mathbb{E}_{w_k}[w^T_k M^T_k(i)D^P_{k+1}((\Theta^0,\Theta_0)(k,i))M_k(i)w_k] \\ &+ \mathbb{E}^{\mathcal{P}}_{p^0}[r_{k+1}(\theta_{k+1})] + \frac{\alpha_i}{2}D^r_{k+1}((\Theta^0,\Theta_0)(k,i)). \end{aligned}$$

Hence, it follows that (8) holds, with:

$$\begin{aligned} P_k(i) &= Q_k(i) + A^T_k(i)\Big(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1}] \\ &+ \frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_0)(k,i))\Big)A_k(i) - A^T_k(i)\Big(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})] \\ &+ \frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_0)(k,i))\Big)B_k(i)L_k(i) \end{aligned} \quad (40a)$$

$$P_N(i) = Q_N(i) \quad (40b)$$

and

$$\begin{aligned} r_k(i) &= \text{tr}\Big(M^T_k(i)\Big(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})] \\ &+ \frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_0)(k,i))\Big)M_k(i)\Sigma_w\Big) \\ &+ \mathbb{E}^{\mathcal{P}}_{p^0}[r_{k+1}(\theta_{k+1})] + \frac{\alpha_i}{2}D^r_{k+1}((\Theta^0,\Theta_0)(k,i)) \end{aligned} \quad (41a)$$

$$r_N(i) = 0. \quad (41b)$$

The optimal cost J^* is given by (28), with $r_0(i)$ calculated by

$$\begin{aligned} r_0(i) = &\sum_{k=0}^{N-1}\text{tr}\Big(M^T_k(i)\Big(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})] \\ &+ \frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_0)(k,i))\Big)M_k(i)\Sigma_w\Big) \\ &+ \frac{\alpha_i}{2}\sum_{k=0}^{N-2}D^r_{k+1}((\Theta^0,\Theta_0)(k,i)). \end{aligned} \quad (42)$$

Remark 4.2: Comparing the solution obtained using the maximizing transition probability distribution of Section III-A, and the solution obtained under case 1 of Theorem 4.1, it follows:

$$\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})] = \mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})] + \frac{\alpha_i}{2} \{P_{k+1}(\theta_{k+1} \in \Theta^0(k,i)) - P_{k+1}(\theta_{k+1} \in \Theta_0(k,i))\} \quad (43)$$

$$\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[r_{k+1}(\theta_{k+1})] = \mathbb{E}^{\mathcal{P}}_{p^0}[r_{k+1}(\theta_{k+1})] + \frac{\alpha_i}{2} \{r_{k+1}(\theta_{k+1} \in \Theta^0(k,i)) - r_{k+1}(\theta_{k+1} \in \Theta_0(k,i))\}. \quad (44)$$

The abovementioned expressions bring out the intuitive interpretation of the solution and the applicability of the proposed robust LQR for nonhomogeneous MJLS based on the total variation distance uncertainty model. In particular, the gains of the optimal controller, as well as, the Riccati equations are modified relative to the standard LQR (in which only the terms $\mathbb{E}_{p^0}[P_{k+1}(\theta_{k+1})]$, and $\mathbb{E}_{p^0}[r_{k+1}(\theta_{k+1})]$ are considered) and include extra terms to measure the difference between the solution of $(P_{k+1}(\theta_{k+1} \in \Theta^0(k,i)) - P_{k+1}(\theta_{k+1} \in \Theta_0(k,i)))$, and $(r_{k+1}(\theta_{k+1} \in \Theta^0(k,i)) - r_{k+1}(\theta_{k+1} \in \Theta_0(k,i)))$, for all $\theta_{k+1} \in \Theta$, which achieve the maximum and the minimum of $\ell_k(x_k,\theta_k,\theta_{k+1},u_k)$, scaled by the total variation distance parameter α_i. These extra terms enable the optimal controller to compensate for the possibility of highly uncertain distributions of the jump process and to guarantee that the optimal performance of the LQR is not compromised.

Next, we present an example to illustrate the equivalence of (43), and how to construct the maximizing transition probability distribution, for the general case when the sets defined by (13)–(15) contain multiple elements.

Example 4.3: We consider the following choice of the nominal probability distribution of the jump process:

$$p^0 = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.6 & 0.3 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}, \text{ with } n_\Theta = 3. \quad (45)$$

For the sake of this example, we assume that the partition $\mathcal{P}(k,\theta_k), \forall \theta_k \in \Theta$, and for some $k = 0,1,\ldots,N-1$, is known and is given by

$$\begin{aligned} \mathcal{P}(k,\theta_k=1) &= \{\Theta^0(k,\theta_k{=}1){=}\{1\},\ \Theta_0(k,\theta_k{=}1){=}\{2\} \\ &\qquad \Theta_1(k,\theta_k{=}1){=}\{3\}\} \\ \mathcal{P}(k,\theta_k=2) &= \{\Theta^0(k,\theta_k{=}2){=}\{3\},\ \Theta_0(k,\theta_k{=}2){=}\{2\} \\ &\qquad \Theta_1(k,\theta_k{=}2){=}\{1\}\} \\ \mathcal{P}(k,\theta_k=3) &= \{\Theta^0(k,\theta_k{=}3){=}\{1\},\ \Theta_0(k,\theta_k{=}3){=}\{2,3\}\}. \end{aligned}$$

Let us choose $R_{TV}(1) = 0.2$, $R_{TV}(2) = 0.4$, and $R_{TV}(3) = 0.8$. Next, we show the validity of (43) for $\theta_k = 1, 3$, only.

a) By (17d), we have that for $\theta_k = 1$

$$\alpha_1 = \min\left(R_{TV}(1), 2\left(1 - \sum_{j\in\Theta^0(k,1)} p^0_{1j}\right)\right)$$
$$= \min(R_{TV}(1), 2(1 - p^0_{11})) = \min(R_{TV}(1), 1)). \quad (46)$$

By (17a) and (17b), we have

$$p^*_{11}(k) = p^0_{11} + \frac{\alpha_1}{2} = p^0_{11} + \frac{\min(R_{TV}(1), 1)}{2} = 0.6$$
$$p^*_{12}(k) = \left(p^0_{12} - \frac{\alpha_1}{2}\right)^+ = \left(p^0_{12} - \frac{\min(R_{TV}(1), 1)}{2}\right)^+ = 0.2$$
$$p^*_{13}(k) = \left(p^0_{13} - \left(\frac{\alpha_1}{2} - p^0_{12}\right)^+\right)^+$$
$$= \left(p^0_{13} - \left(\frac{\min(R_{TV}(1), 1)}{2} - p^0_{12}\right)^+\right)^+ = 0.2.$$

By the left-hand side of (43), we have that

$$\mathbb{E}^{\mathcal{P}}_{p^*_{1\bullet}(k)}[P_{k+1}(\theta_{k+1})] = P_{k+1}(\theta_{k+1} \in \Theta^0(k,1)) \sum_{j\in\Theta^0(k,1)} p^*_{1j}(k)$$
$$+ P_{k+1}(\theta_{k+1} \in \Theta_0(k,1)) \sum_{j\in\Theta_0(k,1)} p^*_{1j}(k)$$
$$+ P_{k+1}(\theta_{k+1} \in \Theta_1(k,1)) \sum_{j\in\Theta_1(k,1)} p^*_{1j}(k)$$
$$= 0.6P_{k+1}(\theta_{k+1}{=}1) + 0.2P_{k+1}(\theta_{k+1}{=}2) + 0.2P_{k+1}(\theta_{k+1}{=}3).$$

By the right-hand side of (43), we have that

$$\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})] + \frac{\alpha_i}{2}\left\{P_{k+1}(\theta_{k+1} \in \Theta^0(k,1))\right.$$
$$\left.-P_{k+1}(\theta_{k+1} \in \Theta_0(k,1))\right\}$$
$$= P_{k+1}(\theta_{k+1} \in \Theta^0(k,1)) \sum_{j\in\Theta^0(k,1)} p^0_{1j}(k)$$
$$+ P_{k+1}(\theta_{k+1} \in \Theta_0(k,1)) \sum_{j\in\Theta_0(k,1)} p^0_{1j}(k)$$
$$+ P_{k+1}(\theta_{k+1} \in \Theta_1(k,1)) \sum_{j\in\Theta_1(k,1)} p^0_{1j}(k)$$
$$+ \frac{\alpha_i}{2}\left\{P_{k+1}(\theta_{k+1} \in \Theta^0(k,1)) - P_{k+1}(\theta_{k+1} \in \Theta_0(k,1))\right\}$$
$$= 0.5P_{k+1}(\theta_{k+1} \in \Theta^0(k,1)) + 0.3P_{k+1}(\theta_{k+1} \in \Theta_0(k,1))$$
$$+ 0.2P_{k+1}(\theta_{k+1} \in \Theta_1(k,1)) + 0.1\left\{P_{k+1}(\theta_{k+1} \in \Theta^0(k,1))\right.$$
$$\left.-P_{k+1}(\theta_{k+1} \in \Theta_0(k,1))\right\}$$
$$= 0.6P_{k+1}(\theta_{k+1}{=}1) + 0.2P_{k+1}(\theta_{k+1}{=}2) + 0.2P_{k+1}(\theta_{k+1}{=}3).$$

Hence, for $\theta_k = 1$ the solution obtained based on the maximizing transition probability distribution and the solution based on the equivalent formulation are identical, as expected.

Following a similar procedure, one can show the equivalence of (43) for $\theta_k = 2$, as well. In particular, the solution for $\theta_k = 2$ is given by

$$\text{LHS of (43)} = \text{RHS of (43)} = 0.6P_{k+1}(\theta_{k+1} = 1)$$
$$+ 0.1P_{k+1}(\theta_{k+1} = 2) + 0.3P_{k+1}(\theta_{k+1} = 3).$$

b) By (17d), we have that for $\theta_k = 3$

$$\alpha_3 = \min\left(R_{TV}(3), 2\left(1 - \sum_{j\in\Theta^0(k,3)} p^0_{3j}\right)\right)$$
$$= \min(R_{TV}(3), 2(1 - p^0_{31})) = \min(R_{TV}(3), 1.2). \quad (47)$$

By (17a) and (17b), we have

$$p^*_{31}(k) = p^0_{31} + \frac{\alpha_3}{2} = p^0_{31} + \frac{\min(R_{TV}(3), 1.2)}{2} = 0.8$$
$$p^*_{32}(k) + p^*_{33}(k) = \left(p^0_{32} + p^0_{33} - \frac{\alpha_3}{2}\right)^+$$
$$= \left(p^0_{32} + p^0_{33} - \frac{\min(R_{TV}(3), 1.2)}{2}\right)^+ = 0.2.$$

By the left-hand side of (43), we have

$$\mathbb{E}^{\mathcal{P}}_{p^*_{3\bullet}(k)}[P_{k+1}(\theta_{k+1})] = P_{k+1}(\theta_{k+1} \in \Theta^0(k,3)) \sum_{j\in\Theta^0(k,3)} p^*_{3j}(k)$$
$$+ P_{k+1}(\theta_{k+1} \in \Theta_0(k,3)) \sum_{j\in\Theta_0(k,3)} p^*_{3j}(k)$$
$$= 0.8P_{k+1}(\theta_{k+1} \in \Theta^0(k,3)) + 0.2P_{k+1}(\theta_{k+1} \in \Theta_0(k,3))$$
$$= 0.8P_{k+1}(\theta_{k+1} = 1) + 0.2P_{k+1}(\theta_{k+1} \in \Theta_0(k,3)).$$

By the right-hand side of (43), we have

$$\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})] + \frac{\alpha_i}{2}\left\{P_{k+1}(\theta_{k+1} \in \Theta^0(k,3))\right.$$
$$\left.-P_{k+1}(\theta_{k+1} \in \Theta_0(k,3))\right\}$$
$$= P_{k+1}(\theta_{k+1} \in \Theta^0(k,3)) \sum_{j\in\Theta^0(k,3)} p^0_{3j}(k)$$
$$+ P_{k+1}(\theta_{k+1} \in \Theta_0(k,3)) \sum_{j\in\Theta_0(k,3)} p^0_{3j}(k)$$
$$+ \frac{\alpha_i}{2}\left\{P_{k+1}(\theta_{k+1} \in \Theta^0(k,3)) - P_{k+1}(\theta_{k+1} \in \Theta_0(k,3))\right\}$$
$$= 0.4P_{k+1}(\theta_{k+1} \in \Theta^0(k,3)) + 0.6P_{k+1}(\theta_{k+1} \in \Theta_0(k,3))$$
$$+ 0.4\left\{P_{k+1}(\theta_{k+1} \in \Theta^0(k,3)) - P_{k+1}(\theta_{k+1} \in \Theta_0(k,3))\right\}$$
$$= 0.8P_{k+1}(\theta_{k+1} = 1) + 0.2P_{k+1}(\theta_{k+1} \in \Theta_0(k,3)).$$

As expected, the solution obtained based on the maximizing transition probability distribution and the solution based on the equivalent formulation are identical. Note that, since $|\Theta_0(k,3)| = 2$, then p^*_{32} and p^*_{33} must satisfy that $p^*_{32} + p^*_{33} = 0.2$. For example, choosing $p^*_{32} = 0.2$ and $p^*_{33} = 0$, then equivalence holds with $P_{k+1}(\theta_{k+1} \in \Theta_0(k,3)) = P_{k+1}(\theta_{k+1} = 2)$. Alternatively, we may choose $p^*_{33} = 0.2$ and $p^*_{32} = 0$, in which case, equivalence holds with $P_{k+1}(\theta_{k+1} \in \Theta_0(k,3)) =$

$P_{k+1}(\theta_{k+1}=3)$. In general, any combination of values for p^*_{32} and p^*_{33} such that $p^*_{32}+p^*_{33}=0.2$ is valid. To see this, let $p^*_{32}=0.1$ and $p^*_{33}=0.1$. Then, equivalence holds with $P_{k+1}(\theta_{k+1}\in\Theta_0(k,3))=\frac{1}{2}(P_{k+1}(\theta_{k+1}=2)+P_{k+1}(\theta_{k+1}=3))$.

Next, we will show that as the uncertainty set increases (i.e., (32) becomes active) then new terms enter the right side of (43) and (44), due to the water-filling behavior of the solution. In particular, if (32) holds then $\widetilde{\Theta}(k,i)=\Theta_z(k,i)$, and $\beta(\alpha_i)$ is given by (33). By (34), we have that

$$\begin{aligned}V_k(x,i)&=\min_{u_k\in\mathcal{U}}\Bigg(x^TQ_k(i)x+u_k^TR_k(i)u_k\\&+\mathbb{E}^{\mathcal{P}}_{p^0}[\ell_k(x,i,\theta_{k+1},u_k)]+\frac{\alpha_i}{2}\left\{\ell_k(x,i,\theta_{k+1}\in\Theta^0(k,i),u_k)\right.\\&\left.-\ell_k(x,i,\theta_{k+1}\in\Theta_z(k,i),u_k)\right\}+\sum_{s=0}^{z-1}\Bigg(\sum_{j\in\Theta_s(k,i)}p^0_{ij}\\&\qquad\{\ell_k(x,i,\theta_{k+1}\in\Theta_z(k,i),u_k)\\&\qquad-\ell_k(x,i,\theta_{k+1}\in\Theta_s(k,i),u_k)\}\Bigg)\Bigg).\qquad(48)\end{aligned}$$

Following a similar procedure and calculations, as for item (1) of Corollary 4.1, one can show that the optimal controller u^*_k is given by (38) with

$$\begin{aligned}L_k(i)&=\Bigg(R_k(i)+B_k^T(i)\Bigg(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})]\\&+\frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_z)(k,i))\\&+\sum_{s=0}^{z-1}\sum_{j\in\Theta_s(k,i)}D^P_{k+1}((\Theta_z,\Theta_s)(k,i))p^0_{ij}\Bigg)B_k(i)\Bigg)^{-1}\\&\times\Bigg(B_k^T(i)\Bigg(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})]+\frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_z)(k,i))\\&+\sum_{s=0}^{z-1}\sum_{j\in\Theta_s(k,i)}D^P_{k+1}((\Theta_z,\Theta_s)(k,i))p^0_{ij}\Bigg)A_k(i)\Bigg)\qquad(49)\end{aligned}$$

where $D^P_{k+1}((\Theta^0,\Theta_z)(k,i))$ and $D^P_{k+1}((\Theta_z,\Theta_s)(k,i))$ are defined similar to (36). Also, (8) holds with

$$\begin{aligned}P_k(i)&=Q_k(i)+A_k^T(i)\\&\Bigg(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1}]+\frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_z)(k,i))\\&+\sum_{s=0}^{z-1}\sum_{j\in\Theta_s(k,i)}D^P_{k+1}((\Theta_z,\Theta_s)(k,i))p^0_{ij}\Bigg)A_k(i)\\&-A_k^T(i)\Bigg(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})]+\frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_z)(k,i))\\&+\sum_{s=0}^{z-1}\sum_{j\in\Theta_s(k,i)}D^P_{k+1}((\Theta_z,\Theta_s)(k,i))p^0_{ij}\Bigg)B_k(i)L_k(i)\end{aligned}\qquad(50)$$

and

$$\begin{aligned}r_k(i)&=\mathrm{tr}\Bigg(M_k^T(i)\\&\Bigg(\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})]+\frac{\alpha_i}{2}D^P_{k+1}((\Theta^0,\Theta_z)(k,i))\\&+\sum_{s=0}^{z-1}\sum_{j\in\Theta_s(k,i)}D^P_{k+1}((\Theta_z,\Theta_s)(k,i))p^0_{ij}\Bigg)M_k(i)\Sigma_w\Bigg)\\&+\mathbb{E}^{\mathcal{P}}_{p^0}[r_{k+1}(\theta_{k+1})]+\frac{\alpha_i}{2}D^r_{k+1}((\Theta^0,\Theta_z)(k,i))\\&+\sum_{\ell=0}^{z-1}\sum_{s\in\Theta_s(k,i)}D^r_{k+1}((\Theta_z,\Theta_s)(k,i))p^0_{ij},\ \text{with } r_N(i)=0\end{aligned}\qquad(51)$$

where $D^r_{k+1}((\Theta^0,\Theta_z)(k,i))$ and $D^r_{k+1}((\Theta_z,\Theta_s)(k,i))$ are defined similar to (37). The optimal cost J^* is given by (28), with $r_0(i)$ calculated similarly to (42).

Remark 4.4: Comparing the solution obtained using the maximizing transition probability distribution of Section III-A, and the solution obtained under case 2 of Theorem 4.1, it can be noticed that

$$\begin{aligned}&\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[P_{k+1}(\theta_{k+1})]=\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})]\\&+\frac{\alpha_i}{2}\{P_{k+1}(\theta_{k+1}\in\Theta^0(k,i))-P_{k+1}(\theta_{k+1}\in\Theta_z(k,i))\}\\&+\sum_{s=0}^{z-1}(P_{k+1}(\theta_{k+1}\in\Theta_z(k,i))\\&-P_{k+1}(\theta_{k+1}\in\Theta_s(k,i)))\sum_{j\in\Theta_s(k,i)}p^0_{ij}\end{aligned}\qquad(52)$$

and similarly for $\mathbb{E}^{\mathcal{P}}_{p^*_{i\bullet}(k)}[r_{k+1}(\theta_{k+1})]$; see, for example, (44).

Next, we illustrate the equivalence of (52) by repeating Example 4.3 for $\theta_k=2$, only. Toward this end, we let the total variation distance uncertainty set to increase so that condition (32) of case 2 of Theorem 4.1 becomes active.

Example 4.5: Consider the nominal transition probability distribution given by (45), and with the assumed partition $\mathcal{P}(k,\theta_k=2)$ (as given in Example 4.3). We choose the total variation distance parameter to be equal to $R_{TV}(2)=1$. Then, by (17d), we have that $\alpha_2=\min(R_{TV}(2),1.8)=1$, and by (17a) and (17c) we obtain $p^*_{21}(k)=0.4$, $p^*_{22}(k)=0$, and $p^*_{23}(k)=0.6$. By the left-hand side of (52), we have

$$\mathbb{E}^{\mathcal{P}}_{p^*_{2\bullet}(k)}[P_{k+1}(\theta_{k+1})]=P_{k+1}(\theta_{k+1}\in\Theta^0(k,2))\sum_{j\in\Theta^0(k,2)}p^*_{2j}(k)$$

$$+ P_{k+1}(\theta_{k+1} \in \Theta_0(k,2)) \sum_{j\in\Theta_0(k,2)} p^*_{2j}(k)$$
$$+ P_{k+1}(\theta_{k+1} \in \Theta_1(k,2)) \sum_{j\in\Theta_1(k,2)} p^*_{2j}(k)$$
$$= 0.4P_{k+1}(\theta_{k+1} = 1) + 0.6P_{k+1}(\theta_{k+1} = 3). \tag{53}$$

Next, we evaluate the right-hand side of (52). By (32), since

$$0.3 = \sum_{j\in\Theta_0(k,2)} p^0_{2j} < \frac{\alpha_2}{2} = 0.5 < \sum_{j\in\Theta_0(k,2)\cup\Theta_1(k,2)} p^0_{2j} = 0.9$$

then $z = 1$. Hence,

$$\mathbb{E}^{\mathcal{P}}_{p^0}[P_{k+1}(\theta_{k+1})] + 0.5\left\{P_{k+1}(\theta_{k+1} \in \Theta^0(k,2))\right.$$
$$\left.-P_{k+1}(\theta_{k+1} \in \Theta_1(k,2))\right\} + (P_{k+1}(\theta_{k+1} \in \Theta_1(k,2))$$
$$-P_{k+1}(\theta_{k+1} \in \Theta_0(k,2))) \sum_{j\in\Theta_0(k,2)} p^0_{2j}$$
$$= 0.6P_{k+1}(\theta_{k+1}{=}1){+}0.3P_{k+1}(\theta_{k+1}{=}2){+}0.1P_{k+1}(\theta_{k+1}{=}3)$$
$$+ 0.5\left\{P_{k+1}(\theta_{k+1} = 3) - P_{k+1}(\theta_{k+1} = 1)\right\}$$
$$+ 0.3\left\{P_{k+1}(\theta_{k+1} = 1) - P_{k+1}(\theta_{k+1} = 2)\right\}$$
$$= 0.4P_{k+1}(\theta_{k+1} = 1) + 0.6P_{k+1}(\theta_{k+1} = 3). \tag{54}$$

Note that, (53) and (54) are identical as expected.

By the equivalence of the solution of the robust LQR problem, as presented in Section III-A and Section IV, it follows that Algorithm 1 may be rewritten in terms of the equivalent solution. In particular, the gain matrices of the optimal controller may be calculated using the equivalent formulation given by (39) and (49) [instead of (22)]. Similarly, the Riccati equations may be calculated using the equivalent formulation (40), (41), and (50), (51) (instead of (23) and (24)).

In the following section, a numerical example (drawn from [10], and modified to include two operating modes) is presented to illustrate the behavior of the proposed solution.

V. Numerical Example

For presentation and comparison purposes, we consider the noiseless version (drawn from [10]) of the discrete-time stochastic control system (1). In particular, we consider (1) with two states $n = 2$, and two operating modes $n_\Theta = 2$. The dynamics and input matrices $A(i)$ and $B(i)$, for each operating mode $i = 1, 2$, are given by

$$\text{Mode1: } A(1) = \begin{pmatrix} 0.9974 & 0.0539 \\ -0.1078 & 1.1591 \end{pmatrix},\ B(1) = \begin{pmatrix} 0.0013 \\ 0.0539 \end{pmatrix}$$

$$\text{Mode2: } A(2) = \begin{pmatrix} 0.6539 & 0.0974 \\ -0.3591 & 0.1078 \end{pmatrix},\ B(2) = \begin{pmatrix} 0.0013 \\ 0.0539 \end{pmatrix}$$

with state and input cost matrices

$$Q = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.05 \end{pmatrix},\quad Q_N = Q,\quad R = 0.05.$$

The time horizon is set equal to $N = 100$, with initial conditions $x_0 = [2\ \ 1]^T$, and initial distribution $p(\theta_0) = [0\ \ 1]^T$. The nominal transition probability distribution p^0, which describes the transitions/jumps of $\{\theta_k\}$ between operating mode 1 and operating mode 2, is given by

$$p^0 = \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix}. \tag{55}$$

Also, for the sake of this example, let us assume that the true transition probability distribution of the jump process is known, and is given by

$$p^{\text{true}}(k) = \begin{pmatrix} 0.8 & 0.2 \\ 0.9 & 0.1 \end{pmatrix} \text{ for all } k. \tag{56}$$

Note that

$$\sum_{j\in\Theta} |p^{\text{true}}_{ij} - p^0_{ij}| = 1.2, \text{ for } i = 1, 2.$$

In what follows, we compare the solution obtained by solving the standard LQR problem, and the solution obtained by solving the robust LQR problem. In particular, in Fig. 1, we depict optimal results obtained by performing a total of 500 Monte Carlo realizations of the Markov chain under four possible cases as follows:

C1) solution of the standard LQR problem using the nominal transition probability distribution (55) [as shown in Fig. 1(a) and (b)];

C2) solution of the standard LQR problem using the true transition probability distribution (56) [as shown in Fig. 1(c) and (d)],

C3) solution of the standard LQR problem using the nominal transition probability distribution, but with jumps/transitions of the dynamical system between different operating modes realized in simulations using the true transition probability distribution [as shown in Fig. 1(e) and (f)];

C4) solution of the robust LQR problem by applying Algorithm 1, for five different values of the total variation distance parameter $R_{TV} = 0.4, 0.6, 0.8, 1, 1.2$. In all cases, jumps/transitions of the dynamical system between different operating modes were realized in simulations using the true transition probability distribution [as shown in Fig. 1(g) and (h)].

In Fig. 1, we depict the mean values of the state trajectories and the mean value of the optimal control history for the four cases, as mentioned previously. Also, the solution of the LQR problem for cases (C1) and (C2), is obtained by applying standard LQR results, (i.e., by applying Step 2 of Algorithm 1, with p^0 and p^{true} replacing p^*, and with the Markov chain random walk obtained using nominal and true transition probability distributions, respectively).

For comparison purposes and for illustrating the effectiveness of the proposed robust LQR approach, in case (C3), we consider

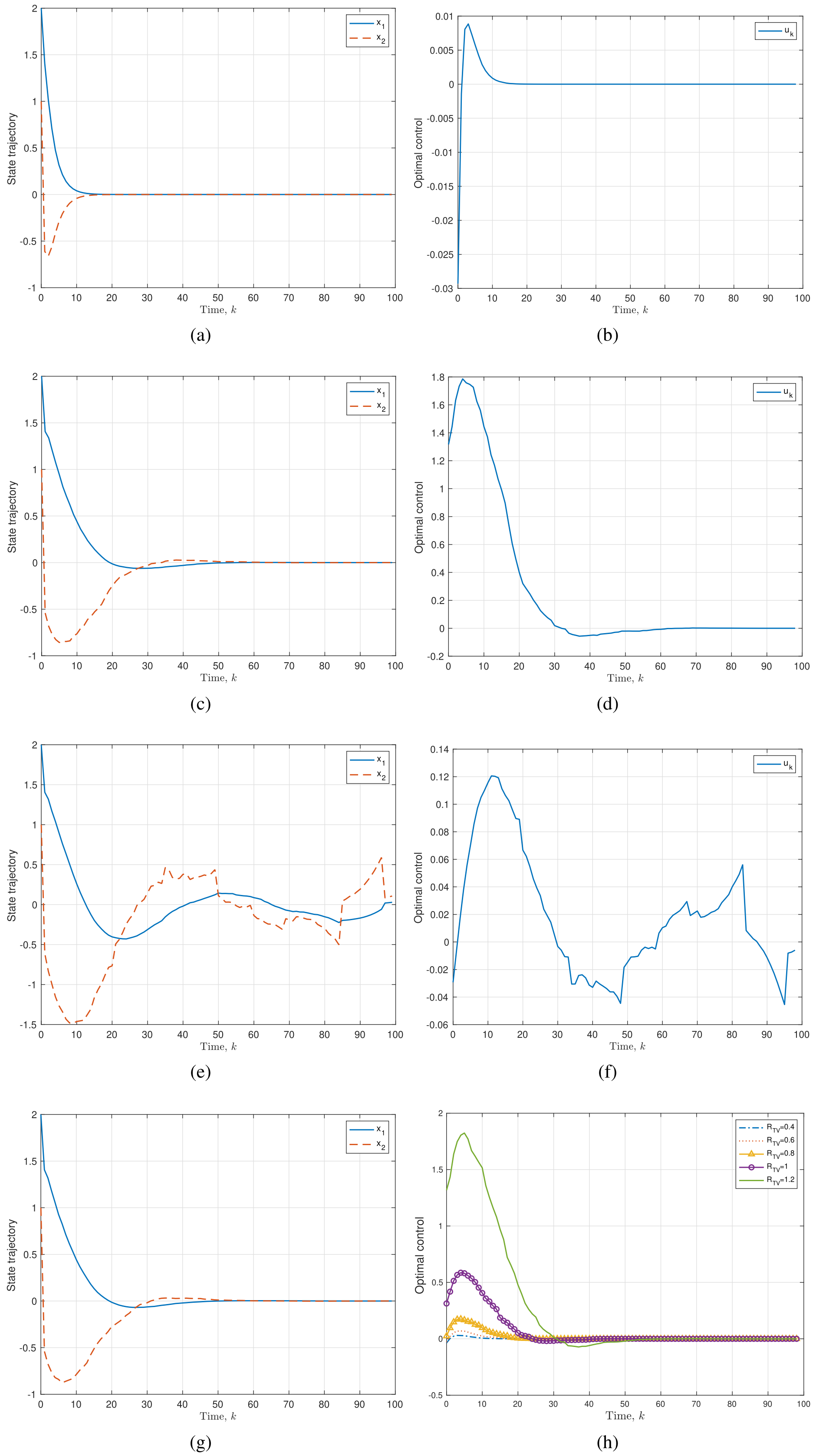

Fig. 1. Optimal control and state trajectories. (a), (b) Standard LQR using the nominal transition probability distribution, (c), (d) Standard LQR using the true transition probability distribution, (e), (f) Standard LQR using the nominal transition probability distribution, but with Markov chain random walk realized in simulations using the true transition probability distribution, and (g), (h) Robust LQR, with total variation distance parameter $R_{TV} = 0.4, 0.6, 0.8, 1, 1.2$.

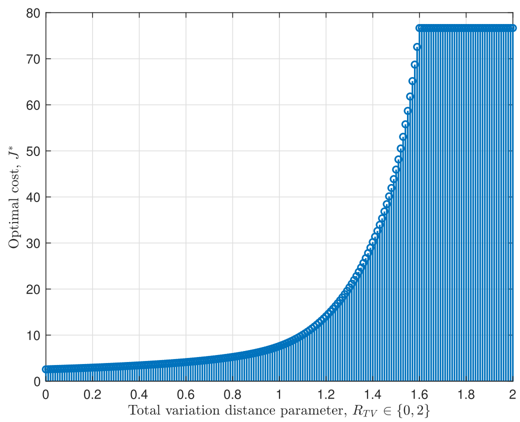

Fig. 2. Optimal cost versus total variation distance parameter.

the realistic scenario that the LQR problem is solved by following standard LQR results, i.e., using the nominal probability distribution as in (C1), and with the true transition probability distribution realizing the jumps/transitions of the dynamical system between operating mode 1 and operating mode 2. Fig. 1(e) and (f) clearly illustrates the effect of uncertainty on the performance of the linear quadratic regulator. On the contrary, Fig. 1(g) and (h) confirms that the proposed robust LQR approach restricts the influence of uncertainty on the performance of the linear quadratic regulator. In particular, Fig. 1(h) depicts the optimal control history obtained under five possible values of the total variation distance parameter, $R_{TV} = 0.4, 0.6, 0.8, 1, 1.2$, while Fig. 1(g) depicts the optimal state trajectories for total variation distance parameter $R_{TV} = 1.2$, only.

Comparing Fig. 1(h) and (d), it can be seen that as the value of the total variation distance parameter increases, i.e., from $R_{TV} = 0.4$ to $R_{TV} = 1.2$, the performance of the optimal robust controller becomes closer and closer to the performance of the optimal controller obtained by solving the standard LQR problem under the true transition probability distribution. Mainly, this is because as the value of the total variation distance parameter increases then the ambiguity set (3) increases, which in turn implies that the robust optimal controller is calculated by considering the worst-case transition probability distributions over all possible transition probability distributions within the total variation distance ambiguity set.

Fig. 2 depicts the optimal cost J^* as a function of the total variation distance parameter $R_{TV} \in [0, 2]$. It is worth noting that, as the value of R_{TV} increases then the value of J^* also increases by a small amount up to a point where thereafter any further increase of R_{TV} results in a dramatic increase of J^*. Here, there is a clear tradeoff. The designer must always choose the value of total variation distance parameter so that there is a balance between the desire for low costs and the undesirability of highly uncertain scenarios. For this particular example, it is clear that any value of $R_{TV} > 1.2$ will only serve to a limited improvement on the performance of the optimal controller. This example verifies the capability of the proposed approach, under highly uncertain MJLS, to ensure the optimality of the controller and consequently, the performance of the linear quadratic regulator.

VI. Conclusion

Presently, there is great interest in the development of effective methodologies to deal with uncertain MJLS. This interest is driven by two main factors. First, modern engineering systems are becoming exceedingly complex with an ever increased number of operating modes, making it difficult if not impossible to have complete and accurate knowledge of the system's transition probability distribution under all its different operating modes. Second, the impact of uncertainty on the performance of the optimal controller can be tremendous. Studying the effect of uncertain transition probability distributions is a key task in the analysis of MJLS as it provides the means for codifying the level of uncertainty in distribution, and ensures that the optimal performance of the controller is not compromised.

In this article, we have proposed a robust LQR approach for nonhomogeneous MJLS with respect to uncertain transition probability distributions. Total variation distance was introduced as a tool for codifying the level of uncertainty, by defining uncertainty sets based on nominal and true distributions of the jump process. The importance of the proposed uncertainty model lies in the fact that different levels of uncertainty between the system's different operating modes may be assigned. Two equivalent formulations of the robust LQR problem are derived and analyzed. Results validate the capability of the proposed LQR approach to capture and restrict the influences of uncertainty on the performance of the LQR controller.

Appendix
Proof of Theorem 4.1

We distinguish the following three steps:

(Step 1) From (17d), let $\alpha_i = 2(1 - \sum_{j\in\Theta^0(k,i)} p_{ij}^0)$. Then, by (17a)–(17c), we have $\sum_{j\in\Theta^0(k,i)} p_{ij}^*(k) = 1$, and $\sum_{j\in\Theta\setminus\Theta^0(k,i)} p_{ij}^*(k) = 0$.

Substituting $p^*(k)$ into (16), we obtain

$$\sum_{j\in\Theta} \ell_k(x_k = x, \theta_k = i, \theta_{k+1} = j, u_k) p_{ij}^*(k)$$

$$\overset{(a)}{=} \ell_k(x, i, \theta_{k+1} {\in} \Theta^0(k,i), u_k) \sum_{j\in\Theta^0(k,i)} p_{ij}^*(k)$$

$$= \ell_{\max,k}(x, i, u_k)$$

where (a) follows by the fact that $\sum_{j\in\Theta\setminus\Theta^0(k,i)} p_{ij}^*(k) = 0$ implies $p_{ij}^*(k) = 0$, $\forall j \in \Theta \setminus \Theta^0(k,i)$. Note that, for values of $\alpha_i > 2(1 - \sum_{j\in\Theta^0(k,i)} p_{ij}^0)$ the solution is equal to $\ell_{\max,k}(x_k, \theta_k = i, u_k)$, and hence, it is enough to choose $\alpha_i = 2(1 - \sum_{j\in\Theta^0(k,i)} p_{ij}^0)$.

(Step 2) From (17d), we choose $\alpha_i = R_{TV}(i) < 2(1 - \sum_{j\in\Theta^0(k,i)} p_{ij}^0)$, $\forall i \in \Theta$, and assume that the maximizing transition probability distribution given by (17a)–(17c) is such that $\sum_{j\in\Theta^0(k,i)} p_{ij}^*(k) < 1$, $\sum_{j\in\Theta_0(k,i)} p_{ij}^*(k) > 0$ and, hence, $\sum_{j\in\Theta_l(k,i)} p_{ij}^*(k) = p_{ij}^0, l = 1, 2, \ldots, r$. Substituting $p^*(k)$ into

(16), we obtain

$$\sum_{j\in\Theta}\ell_k(x_k=x,\theta_k=i,\theta_{k+1}=j,u_k)p^*_{ij}(k)$$

$$=\ell_k(x,i,\theta_{k+1}\in\Theta^0(k,i),u_k)\left(\sum_{j\in\Theta^0(k,i)}p^0_{ij}+\frac{\alpha_i}{2}\right)$$

$$+\ell_k(x,i,\theta_{k+1}\in\Theta_0(k,i),u_k)\left(\sum_{j\in\Theta_0(k,i)}p^0_{ij}-\frac{\alpha_i}{2}\right)$$

$$+\sum_{s=1}^{r}\ell_k(x,i,\theta_{k+1}\in\Theta_s(k,i),u_k)\left(\sum_{j\in\Theta_s(k,i)}p^0_{ij}\right)$$

$$=\mathbb{E}^{\mathcal{P}}_{p^0}[\ell(x,i,\theta_{k+1},u_k)]+\frac{\alpha_i}{2}\left(\ell_k(x,i,\theta_{k+1}\in\Theta^0(k,i),u_k)\right.$$

$$\left.-\ell_k(x,i,\theta_{k+1}\in\Theta_0(k,i),u_k)\right).$$

When (31) holds, then this step corresponds to (30) with $\widetilde{\Theta}(k,i)=\Theta_0(k,i)$ and $\beta(\alpha_i)=0$ (i.e., Case.1, Theorem 4.1).

(Step.3) From (17d), we choose $\alpha_i=R_{TV}(i)<2(1-\sum_{j\in\Theta^0(k,i)}p^0_{ij})$, $\forall i\in\Theta$, and assume that the maximizing transition probability distribution given by (17a)–(17c) is such that

$$\sum_{j\in\Theta^0(k,i)}p^*_{ij}(k)<1,\quad \sum_{j\in\Theta_0(k,i)}p^*_{ij}(k)=0$$

$$\sum_{j\in\Theta_1(k,i)}p^*_{ij}(k)=0$$

$$\vdots$$

$$\sum_{j\in\Theta_{z-1}(k,i)}p^*_{ij}(k)=0,\quad \sum_{j\in\Theta_z(k,i)}p^*_{ij}(k)>0$$

and

$$\sum_{j\in\Theta_l(k,i)}p^*_{ij}(k)=\sum_{j\in\Theta_l(k,i)}p^0_{ij},\quad \text{for } l=z+1,\dots,r.$$

Substituting $p^*(k)$ into (16), we obtain

$$\sum_{j\in\Theta}\ell_k(x_k=x,\theta_k=i,\theta_{k+1}=j,u_k)p^*_{ij}(k)$$

$$=\ell_k(x,i,\theta_{k+1}\in\Theta^0(k,i),u_k)\left(\sum_{j\in\Theta^0(k,i)}p^0_{ij}+\frac{\alpha_i}{2}\right)$$

$$+\ell_k(x,i,\theta_{k+1}\in\Theta_z(k,i),u_k)\Bigg($$

$$\left.\sum_{j\in\Theta_z(k,i)}p^0_{ij}+\sum_{s=1}^{z}\sum_{j\in\Theta_{s-1}(k,i)}p^0_{ij}-\frac{\alpha_i}{2}\right)$$

$$+\sum_{s=z+1}^{r}\ell_k(x,i,\theta_{k+1}\in\Theta_s(k,i),u_k)\sum_{j\in\Theta_s(k,i)}p^0_{ij}$$

$$=\mathbb{E}^{\mathcal{P}}_{p^0}[\ell(x,i,\theta_{k+1},u_k)]+\frac{\alpha_i}{2}\left(\ell_k(x,i,\theta_{k+1}\in\Theta^0(k,i),u_k)\right.$$

$$\left.-\ell_k(x,i,\theta_{k+1}\in\Theta_z(k,i),u_k)\right)$$

$$+\sum_{s=0}^{z-1}\left(\ell_k(x,i,\theta_{k+1}\in\Theta_z(k,i),u_k)\right.$$

$$\left.-\ell_k(x,i,\theta_{k+1}\in\Theta_{z-1}(k,i),u_k)\right)\sum_{j\in\Theta_{s-1}(k,i)}p^0_{ij}.$$

When (32) holds for some $z=\{1,2,\dots,r\}$, then this step corresponds to (30) with $\widetilde{\Theta}(k,i)=\Theta_z(k,i)$, and $\beta(\alpha_i)$ given by (33) (i.e., Case 2 of Theorem 4.1).

References

[1] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*. Upper Saddle River, NJ, USA: Prentice-Hall, 1990.

[2] M. Mariton, *Jump Linear Syst. Autom. Control*. New York, NY, USA: Marcel Dekker, 1990.

[3] E. Soroka and U. Shaked, “On the robustness of LQ regulators,” *IEEE Trans. Autom. Control*, vol. AC-29, no. 7, pp. 664–665, Jul. 1984.

[4] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*. Berlin, Germany: Springer Science and Business Media, 2006.

[5] M. H. Terra, J. P. Cerri, and J. Y. Ishihara, “Optimal robust linear quadratic regulator for systems subject to uncertainties,” *IEEE Trans. Autom. Control*, vol. 59, no. 9, pp. 2586–2591, Sep. 2014.

[6] M. G. Todorov and M. D. Fragoso, “New methods for mode-independent robust control of Markov jump linear systems,” *Syst. Control Lett.*, vol. 90, pp. 38–44, 2016.

[7] E.-K. Boukas, A. Swierniak, and H. Yang, “On the robustness of jump linear quadratic control,” *Int. J. Robust Nonlinear Control*, vol. 7, no. 10, pp. 899–910, 1997.

[8] A. T. Neto, J. M. Dion, and L. Dugard, “Robustness bounds for LQ regulators,” *IEEE Trans. Autom. Control*, vol. 37, no. 9, pp. 1373–1377, Sep. 1992.

[9] O. Costa, E. A. Filho, E. Boukas, and R. Marques, “Constrained quadratic state feedback control of discrete-time Markovian jump linear systems,” *Automatica*, vol. 35, no. 4, pp. 617–626, 1999.

[10] I. Tzortzis, C. D. Charalambous, T. Charalambous, C. K. Kourtellaris, and C. N. Hadjicostis, “Robust linear quadratic regulator for uncertain systems,” in *Proc. 55th IEEE Conf. Decis. Control*, Dec. 2016, pp. 1515–1520.

[11] L. Zhang, E. Boukas, and J. Lam, “Analysis and synthesis of Markov jump linear systems with time-varying delays and partially known transition probabilities,” *IEEE Trans. Autom. Control*, vol. 53, no. 10, pp. 2458–2464, Nov. 2008.

[12] L. Zhang and J. Lam, “Necessary and sufficient conditions for analysis and synthesis of Markov jump linear systems with incomplete transition descriptions,” *IEEE Trans. Autom. Control*, vol. 55, no. 7, pp. 1695–1701, Jul. 2010.

[13] J. Zhu, G. Zhang, and W. Xie, “Jump linear quadratic Gaussian problem for a class of nonhomogeneous Markov jump linear systems,” *Optimal Control Appl. Methods*, vol. 38, no. 4, pp. 668–680, 2017.

[14] Y. Zacchia Lun, A. D’Innocenzo, A. Abate, and M. D. Di Benedetto, “Optimal robust control and a separation principle for polytopic time-inhomogeneous Markov jump linear systems,” in *Proc. 56th IEEE Conf. Decis. Control*, Dec. 2017, pp. 6525–6530.

[15] Y. Zacchia Lun, A. D’Innocenzo, and M. D. Di Benedetto, “Robust stability of polytopic time-inhomogeneous Markov jump linear systems,” *Autom.*, vol. 105, pp. 286–297, 2019.

[16] I. Tzortzis, C. D. Charalambous, and T. Charalambous, “Dynamic programming subject to total variation distance ambiguity,” *SIAM J. Control Optim.*, vol. 53, no. 4, pp. 2040–2075, Jul. 2015.

[17] I. Tzortzis, C. D. Charalambous, and C. N. Hadjicostis, "Robust LQG for Markov jump linear systems," in *Proc. 58th IEEE Conf. Decis. Control*, Dec. 2019, pp. 6760–6765.

[18] I. Tzortzis, C. D. Charalambous, T. Charalambous, C. N. Hadjicostis, and M. Johansson, "Approximation of Markov processes by lower dimensional processes via total variation metrics," *IEEE Trans. Autom. Control*, vol. 62, no. 3, pp. 1030–1045, Mar. 2017.

[19] C. D. Charalambous, I. Tzortzis, S. Loyka, and T. Charalambous, "Extremum problems with total variation distance and their applications," *IEEE Trans. Autom. Control*, vol. 59, no. 9, pp. 2353–2368, Sep. 2014.

Ioannis Tzortzis received the B.Sc. degree in electrical engineering and informatics from the Budapest University of Technology and Economics, Budapest, Hungary, in 2007, and the M.Sc. and Ph.D. degrees in electrical and computer engineering from the University of Cyprus, Nicosia, Cyprus, in 2009 and 2015, respectively.

He held Postdoctoral positions with the University of Cyprus, and with KIOS Research and Innovation Center of Excellence. He is currently a Postdoctoral Researcher with the Department of Electrical and Computer Engineering, University of Cyprus. His research interests include control and optimization of systems subject to uncertainty and ambiguity, Markov decision and stochastic dynamic programming, system identification, and active fault diagnosis and control of distributed dynamical systems.

Charalambos D. Charalambous received the B.S., M.E., and Ph.D. degrees in electrical engineering from Old Dominion University, Norfolk, VA, USA, in 1987, 1988, and 1992, respectively.

In 2003, he joined the Department of Electrical and Computer Engineering, University of Cyprus. He was an Associate Professor with the University of Ottawa, School of Information Technology and Engineering, from 1999 to 2003. He was with the faculty of Department of Electrical and Computer Engineering, McGill University, as a nontenure Faculty Member, from 1995 to 1999. From 1993 to 1995, he was a Postdoctoral Fellow with Idaho State University. He is currently an Associate Editor with the Systems and Control Letters and Mathematics of Control, Signals and Systems. His research interests include stochastic dynamical decision and control systems, information theory, optimization subject to ambiguity, stochastic dynamic games, and decentralized decision systems.

Dr. Charalambous is an Associate Editor of the Journal of Nonlinear Analysis of Hybrid Systems. He was an Associate Editor for the IEEE Transactions on Automatic Control and IEEE Communications Letters.

Christoforos N. Hadjicostis (Fellow, IEEE) received the B.S. degree in electrical engineering, computer science and engineering, and in mathematics in 1993, the M.Eng. degree in electrical engineering and computer science in 1995, and the Ph.D. degree in electrical engineering and computer science in 1999 all from the Massachusetts Institute of Technology, Cambridge, Cambridge, USA.

In 1999, he joined the Faculty of the University of Illinois at Urbana–Champaign, where he was Assistant and then Associate Professor with the Department of Electrical and Computer Engineering, the Coordinated Science Laboratory, and the Information Trust Institute, Urbana, United States. Since 2007, he has been with the Department of Electrical and Computer Engineering, University of Cyprus, Nicosia, Cyprus where he is currently a Professor. His research interests include fault diagnosis and tolerance in distributed dynamic systems, error control coding, monitoring, diagnosis and control of large-scale discrete-event systems, and applications to network security, anomaly detection, energy distribution systems, medical diagnosis, biosequencing, and genetic regulatory models.

Dr. Hadjicostis is Departmental Editor of the *Journal of Discrete Event Dynamic Systems*, and an Associate Editor of *Automatica*, and the *Journal of Nonlinear Analysis of Hybrid Systems*. He was an Associate Editor for the IEEE Transactions on Automatic Control, IEEE Transactions on Automation Science and Engineering, IEEE Transactions on Control Systems Technology, and IEEE Transactions on Circuits and Systems I.