Automatica 46 (2010) 1696–1702

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Stabilization of Markov jump linear systems using quantized state feedback☆

Nan Xiao[a], Lihua Xie[a,*], Minyue Fu[b]

[a] School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798, Singapore

[b] School of Electrical Engineering and Computer Science, University of Newcastle, Callaghan, NSW 2308, Australia

ARTICLE INFO

Article history:
Received 24 March 2009
Received in revised form
17 March 2010
Accepted 28 May 2010
Available online 8 July 2010

Keywords:
Markov jump linear systems
Mean square quadratic stability
Networked control systems
Quantized stabilization

ABSTRACT

This paper addresses the stabilization problem for single-input Markov jump linear systems via mode-dependent quantized state feedback. Given a measure of quantization coarseness, a mode-dependent logarithmic quantizer and a mode-dependent linear state feedback law can achieve optimal coarseness for mean square quadratic stabilization of a Markov jump linear system, similar to existing results for linear time-invariant systems. The sector bound approach is shown to be non-conservative in investigating the corresponding quantized state feedback problem, and then a method of optimal quantizer/controller design in terms of linear matrix inequalities is presented. Moreover, when the mode process is not observed by the controller and quantizer, a mode estimation algorithm obtained by maximizing a certain probability criterion is given. Finally, an application to networked control systems further demonstrates the usefulness of the results.

1. Introduction

Quantization of measurement and/or input signals has been known to have an undesirable effect on system performance or even stability, and therefore a lot of work has been carried out to mitigate the effect. For systems engaging digital channels for signal transmission, especially in the case where bandwidth and energy are limited, quantization becomes indispensable. Elia and Mitter (2001) first pointed out that quantization is "useful, if not essential, instead of undesirable", and also indicated that the coarsest quantizer is logarithmic in quadratic stabilization of single-input linear time-invariant (LTI) systems. A relationship between the optimal quantization density and unstable eigenvalues of the plant under consideration is established. Fu and Xie (2005) showed that, under quadratic stability, quantized stabilization is equivalent to robust stabilization of an associated system with sector-bounded uncertainty, and extended the results of Elia and Mitter (2001) to multiple-input–multiple-output (MIMO) systems and output feedback control. Based on the result in Fu and Xie (2005), quantized stabilization is considered in Gao and Chen (2008), where a quantization error-dependent Lyapunov function is adopted which offers less conservative design.

The packet-drop behavior of a typical communication channel is another important issue in networked control systems (NCSs), as it induces information loss and consequently affects the performance or even stability of the closed-loop system. There have been many interesting studies on the packet-loss issue; see, e.g. Elia (2005); Hu and Yan (2007) for networked control, Huang and Dey (2007); Sinopoli, Schenato, Franceschetti, Poolla, Jordan, and Sastry (2004) for networked estimation, and Schenato, Sinopoli, Franceschetti, Poolla, and Sastry (2007) for a survey of recent results on estimation and control over lossy channels. In Hu and Yan (2007), the stability robustness of NCSs is addressed, where the packet losses are modeled according to an i.i.d. Bernoulli distribution and the control input becomes zero when the data are lost (so-called zero-control strategy). Elia (2005) considered the mean square stabilization over a fading channel in the framework of robust control for deterministic systems with stochastic model uncertainties. One of the interesting discoveries in Elia (2005) is that the supremum of allowable packet-loss rate (probability of erasure) can be given in terms of the unstable poles of the single-input plant under investigation.

As quantization and packet drops coexist in an NCS, it is natural and reasonable to take them into consideration simultaneously. The stabilization problem over a channel containing both quantization and packet losses was first addressed in Hoshina, Tsumura, and Ishii (2007), where the packet-loss phenomenon is modeled as a binary i.i.d. process. It is shown that the upper bound of the

☆ This work was supported in part by the National Natural Science Foundation of China under Grant 60828006. The material in this paper was partially presented at the American Control Conference, June 10-12, 2009, St. Louis, Missouri, USA. This paper was recommended for publication in revised form by Associate Editor Fabrizio Dabbene under the direction of Editor Roberto Tempo.

* Corresponding author. Tel.: +65 6790 4524; fax: +65 6793 3318.
E-mail addresses: xiao0023@ntu.edu.sg (N. Xiao), elhxie@ntu.edu.sg (L. Xie), minyue.fu@newcastle.edu.au (M. Fu).

0005-1098/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2010.06.018

quantization coarseness can be given in terms of the packet-loss rate and the unstable eigenvalues of the plant. However, the results of Hoshina et al. (2007) are not applicable for the case of binary Markovian losses.

It is well known that NCSs with packet losses are related to Markov jump linear systems (MJLSs), for which there have been many existing results on stability, optimal control and robust control; see Costa, Fragoso, and Marques (2005) and references therein. The MJLS theory is applied to the H_∞ control of NCSs with binary stochastic packet losses in Seiler and Sengupta (2005), and the stabilization of NCSs undergoing bounded consecutive Markovian packet losses in Xiong and Lam (2007). Note that the so-called *current mode observation* (CMO) or no mode observation at the controller side is commonly assumed in studying the control problem of MJLSs (deSouza, 2006). Recently, for the linear quadratic regulation of MJLSs with *one-step-delayed mode observation* (OSDMO), it is shown that the optimal state feedback gain can be indexed by the one-step-delayed mode (Matei, Martins, and Baras, 2008), which inspires our study on the OSDMO case. Based on the hidden Markov models (Elliott, Aggoun, & Moore, 1995; Rabiner, 1989; Viterbi, 1967), the mode and/or state estimation with no mode observation is also considered in Elliott, Dufour, and Malcolm (2005); Ho and Chen (2006), where the mode estimation is not used to generate the control signal, and thus this is different from the situation considered in Section 3.2 of the present paper.

The rest of this paper is organized as follows. The problem under consideration is formulated in Section 2. Section 3.1 answers the following questions: (a) Is logarithmic quantization still optimal for MJLSs under the notion of mean square quadratic stability? (b) Is the sector bound approach still non-conservative in dealing with quantized stabilization of MJLSs in the mean square quadratic stability sense? (c) How does one design the optimal quantizer and controller jointly? We reveal that under the mean square quadratic stability, the smallest overall coarseness for MJLSs can be approached by adopting a mode-dependent logarithmic law operating on a mode-dependent linear state feedback similar to that of LTI systems (Elia & Mitter, 2001; Fu & Xie, 2005). Again, the sector bound approach is shown to be non-conservative in investigating the quantized feedback stabilization problem under the mean square quadratic stability. A linear matrix inequality approach is then presented to compute the optimal quantizer and the set of suitable state feedback gains. When there is no mode observation at the controller and quantizer, a mode estimation method is proposed in Section 3.2, which is further demonstrated by a numerical example. We conclude the paper in Section 5 after applying the results to the NCSs in Section 4.

Notation. $\equiv$ means "defined as". The superscript $'$ denotes the transpose of a vector or matrix. $\mathcal{R}^n$, $\mathcal{R}_+$ and $\mathcal{Z}_+$ stand for the n-dimensional Euclidean space, the set of nonnegative real numbers and integers, respectively. I is the identity matrix, and 0 denotes the zero matrix or zero vector. Furthermore, let $\Pr(\cdot)$ and $\mathrm{E}(\cdot)$ stand for the probability and the mathematical expectation operators, respectively. $\|\cdot\|$ represents the Euclidean norm for vectors. $y_{t_1}^{t_2}$ is the set $\{y_{t_1}, y_{t_1+1}, \ldots, y_{t_2}\}$ for $t_1 \le t_2$, otherwise an empty set by convention.

2. Problem formulation

As we can see from Fig. 1, a quantized feedback control system comprises three parts: a system to be controlled ($\mathbb{G}$), a controller ($\mathbb{K}$) and a quantizer ($\mathbb{Q}$).

We consider a discrete-time single-input MJLS as follows:

$$\mathbb{G}: \; x_{t+1} = A_{\theta_t} x_t + B_{\theta_t} u_t + w_t, \quad t \ge 0, \tag{1}$$

where $x_t \in \mathcal{R}^n$ is the state with x_0 being a second-order random variable, $u_t \in \mathcal{R}$ is the control input, $w_t \in \mathcal{R}^n$ is a second-order

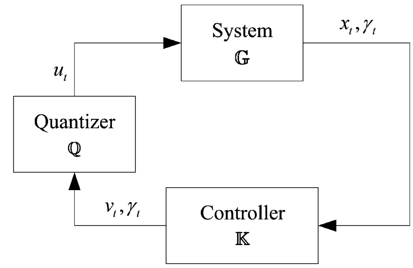


Fig. 1. Typical quantized state feedback control system.

process noise with zero mean and covariance matrix $\Sigma_{\theta_t} > 0$, and $\theta_t \in \Theta \equiv \{0, 1, \ldots, N\}$ is the system mode governed by a time-homogeneous Markov chain with initial distribution $\pi = [\pi_0 \;\; \pi_1 \cdots \pi_N]$ and transition probability matrix $\Pi = (\pi_{ij})_{i,j\in\Theta}$, where

$$\pi_i \equiv \Pr(\theta_0 = i), \qquad \pi_{ij} \equiv \Pr(\theta_{t+1} = j|\theta_t = i). \tag{2}$$

Moreover, x_0, θ_0^t, w_0^t are independent of each other for all $t \ge 0$. Suppose x_t is available at the controller, and the static quantized state feedback is denoted by

$$\mathbb{K}: \; v_t = g(x_t, \gamma_t), \tag{3}$$

$$\mathbb{Q}: \; u_t = f(v_t, \gamma_t), \tag{4}$$

where $\gamma_t \in \Theta$ is a direct observation or an estimate of system mode θ_{t-d} at the controller/quantizer side at time step t with $d \in \mathcal{Z}_+$ the constant mode observation/estimation delay. In this paper, the initial γ_k, $0 \le k \le d-1$, are chosen arbitrarily from Θ.

The closed-loop system of (1), (3) and (4) is described by

$$x_{t+1} = A_{\theta_t} x_t + B_{\theta_t} f(g(x_t, \gamma_t), \gamma_t) + w_t. \tag{5}$$

It is worth mentioning that (5) is generally nonlinear, since the control signal u_t can be a nonlinear function of v_t in (4). We adopt the following definitions of mean square stability and mean square quadratic stability.

Definition 1. For $w_t = 0$ and every initial condition x_0, θ_0, γ_0, the equilibrium point at the origin of (5) is *mean square (MS) stable* if $\lim_{t\to+\infty} \mathrm{E}[\|x_t\|^2|x_0, \theta_0, \gamma_0] = 0$; it is *mean square quadratically (MSQ) stable* if, $\forall \gamma_t \in \Theta$, there exist a positive-definite function

$$V(x_t, \gamma_t) \equiv x_t' P_{\gamma_t} x_t \tag{6}$$

and a positive-definite matrix Q_{γ_t} such that, $\forall t \ge d$,

$$\begin{aligned} \nabla V(x_t, \gamma_t) &\equiv \mathrm{E}[V(x_{t+1}, \gamma_{t+1}) - V(x_t, \gamma_t)|x_0^t, \gamma_0^t] \\ &= \mathrm{E}[V(x_{t+1}, \gamma_{t+1})|x_0^t, \gamma_0^t] - V(x_t, \gamma_t) \\ &< -x_t' Q_{\gamma_t} x_t, \quad \forall x_t \in \mathcal{R}^n, \; x_t \ne 0. \end{aligned} \tag{7}$$

Note that the MSQ stability of the equilibrium point at the origin of system (5) implies the MS stability by following a similar line of arguments as in the proof of Theorem 1 in Boukas and Liu (2001). By setting $u_t = 0, \forall t < d$, when $d \ge 1$, it is easy to see that x_d is still a second-order random variable, and thus we consider $t = d$ as the starting point in (7) to simplify the treatment.

Remark 2. Imposing condition (7) in every system mode introduces some degree of conservativeness but has the following advantages: (1) under the notion of MSQ stability, we can prove the optimality of the logarithmic quantizer defined in the next section; (2) it makes existing well-established results in robust control of MJLSs applicable in quantized feedback control.

Note that $f(\cdot,\cdot)$ in (4) is assumed to be an odd function of v_t; i.e., $f(-v_t, \gamma_t) = -f(v_t, \gamma_t)$. We define the *mode quantization density* with respect to mode i, $i \in \Theta$, similarly to that of the LTI case (Elia & Mitter, 2001) as $\eta_f(i) \equiv \limsup_{\epsilon\to 0} \frac{\#I[\epsilon, i]}{-\ln \epsilon}$, where

$\#l[\epsilon, i]$ is the number of quantization levels in the interval $[\epsilon, 1/\epsilon]$ with the quantizer $f(\cdot, i)$. Evidently, the mode quantization density is reduced to the quantization density defined in Elia and Mitter (2001) when $N = 0$. For $N \neq 0$, there is a set of mode quantization densities $\eta_f(i)$, $i = 0, 1, \ldots, N$, and we introduce the overall coarseness for an observed/estimated-mode-dependent quantizer as follows.

Definition 3. The overall coarseness of a mode-dependent quantizer $\mathbb{Q}$ is defined as

$$\mathrm{C}_f \equiv e(\eta_f(0), \eta_f(1), \ldots, \eta_f(N)), \tag{8}$$

where e is a scalar-valued function of $\eta_f(i)$, $i = 0, 1, \ldots, N$, satisfying the following property: if $\eta_{f1}(i) \le \eta_{f2}(i)$, $\forall i \in \Theta$, then

$$e(\eta_{f1}(0), \eta_{f1}(1), \ldots, \eta_{f1}(N)) \le e(\eta_{f2}(0), \eta_{f2}(1), \ldots, \eta_{f2}(N)). \tag{9}$$

The property (9) reveals that from a physical point of view the overall coarseness should always be nondecreasing when any one of the mode quantization densities is increasing and all the others are fixed. Note that the smaller the value of C_f, the coarser the quantizer. The form of e in (8) can be chosen according to physical constraints or performance requirements of the quantizer. It is easy to see that the set of $\eta_f(i)$, $i = 0, 1, \ldots, N$, corresponding to the globally optimal C_f may not be unique.

The main purpose of this paper is to find one possible combination of $\mathbb{K}$ and $\mathbb{Q}$ with the optimal C_f such that the closed-loop system is MSQ stable.

3. Main results

A mode-dependent quantizer ($\mathbb{Q}$) is said to be logarithmic if, for any $\gamma_t \in \Theta$, the corresponding set of quantization levels $\mathcal{U}_{\gamma_t}$ has the following form:

$$\mathcal{U}_{\gamma_t} = \{\pm u_l(\gamma_t) : u_l(\gamma_t) = \rho^l(\gamma_t)u_0,\ u_0 > 0,\ \text{for } l \in \pm 1, \pm 2, \ldots\} \cup \{\pm u_0\} \cup \{0\}, \tag{10}$$

where

$$\rho(\gamma_t) = \frac{1 - \delta(\gamma_t)}{1 + \delta(\gamma_t)}. \tag{11}$$

Specifically, the associated logarithmic quantizer is defined as follows.

For the given γ_t:

- if $\delta(\gamma_t) = 0$, then

$$f(v_t, \gamma_t) = v_t; \tag{12}$$

- if $0 < \delta(\gamma_t) < 1$, then

$$f(v_t, \gamma_t) = \begin{cases} u_l(\gamma_t), & \text{if } \frac{1}{1+\delta(\gamma_t)} u_l(\gamma_t) < v_t \le \frac{1}{1-\delta(\gamma_t)} u_l(\gamma_t), \\ 0, & \text{if } v_t = 0, \\ -f(-v_t, \gamma_t), & \text{if } v_t < 0; \end{cases} \tag{13}$$

- if $\delta(\gamma_t) = 1$, then

$$f(v_t, \gamma_t) = \begin{cases} u_0, & \text{if } v_t > \frac{1}{2}u_0, \\ 0, & \text{if } 0 \le v_t \le \frac{1}{2}u_0, \\ -f(-v_t, \gamma_t), & \text{if } v_t < 0. \end{cases} \tag{14}$$

There is no loss of generality by choosing the same u_0 for every $\gamma_t \in \Theta$; see Lemma 2.1 in Elia and Mitter (2001). For a logarithmic quantizer, it is easy to verify that $\eta_f(i) = -2/\ln \rho(i), \forall i \in \Theta$. Thus, the coarser the quantizer for mode i, the smaller the $\eta_f(i) \in \mathcal{R}_+ \cup \{+\infty\}$ and $\rho(i) \in [0, 1]$, or equivalently the larger the sector bound $\delta(i) \in [0, 1]$.

3.1. Quantized stabilization

Since in this subsection we only focus on global stabilization, we let $w_t = 0$ without loss of generality. The next assumption is essential to the existence of an optimal memoryless quantization strategy in the MSQ stability sense.

Assumption 1. (a). System (1) is not MS stable with $u_t = 0$ but can be MS stabilized via a linear state feedback law:

$$u_t = \bar{K}_{\gamma_t} x_t. \tag{15}$$

(b). $\forall i_1, i_2, i_3 \in \Theta$, and $\forall t \ge d$,

$$\begin{aligned} &\Pr\{\theta_t = i_1, \gamma_{t+1} = i_2 | x_0^t, \gamma_0^{t-1}, \gamma_t = i_3\} \\ &\quad = \Pr\{\theta_t = i_1, \gamma_{t+1} = i_2 | \gamma_t = i_3\}. \end{aligned} \tag{16}$$

Moreover, the conditional probability (16) denoted by $q_{i_1 i_2 i_3}$ is constant over time and known to the controller/quantizer.

Remark 4. Assumption 1(a) clearly avoids triviality and imposes a necessary restriction for ensuring the solvability of the stabilization problem. A systemic way to find a stabilizing state feedback law for an MJLS can be found in Costa et al. (2005). Assumption 1(b) facilitates an explicit evaluation of (7), and can be justified by several practical situations, as follows.

- Scheme I (CMO): $\gamma_t = \theta_t$. In this situation,

$$q_{i_1 i_2 i_3} = \begin{cases} \pi_{i_3 i_2}, & \text{if } i_1 = i_3, \\ 0, & \text{otherwise}. \end{cases}$$

- Scheme II (OSDMO): $\gamma_t = \theta_{t-1}$. In this situation,

$$q_{i_1 i_2 i_3} = \begin{cases} \pi_{i_3 i_2}, & \text{if } i_1 = i_2, \\ 0, & \text{otherwise}. \end{cases}$$

- Scheme III (Mode-independent manner): $\gamma_t = \phi$ with ϕ representing a void signal. Assumption 1(b) is reduced to "$\forall i_1 \in \Theta, \Pr\{\theta_t = i_1 | x_0^t\} = \Pr\{\theta_t = i_1\}$ is constant over time and known to the controller/quantizer", which is true if the underlying Markov chain is an i.i.d. process, i.e., $\pi_{ij} = \bar{\pi}_j, \forall i, j \in \Theta$ (Xiao, Xie, & Fu, 2009), or the Markov chain is ergodic and the initial distribution π is equal to its limiting distribution.

As the first result of this section, it will be shown that, for a fixed set of $P_i > 0$, $Q_i > 0$, $i = 0, 1, \ldots, N$, the coarsest quantization in the sense of MSQ stability can be approached by a linear state feedback law and a logarithmic quantizer. To this end, let us first define $\forall i \in \Theta$, the row vector $\mathbf{a}_i \equiv \sum_{i_1 \in \Theta, i_2 \in \Theta}[q_{i_1 i_2 i} B'_{i_1} P_{i_2} A_{i_1}]$, the matrix $F_i \equiv \sum_{i_1 \in \Theta, i_2 \in \Theta}[q_{i_1 i_2 i} A'_{i_1} P_{i_2} A_{i_1}] \ge 0$, and two scalars $b_i \equiv \sum_{i_1 \in \Theta, i_2 \in \Theta}[q_{i_1 i_2 i} B'_{i_1} P_{i_2} B_{i_1}] \ge 0$,

$$\delta_m(i) \equiv \begin{cases} +\infty, & \text{if } b_i = 0, \\ \dfrac{1}{\sqrt{K_{mi} M_i^{-1} K'_{mi}}}, & \text{otherwise}, \end{cases}$$

where

$$K_{mi} \equiv -\frac{\mathbf{a}_i}{b_i}, \qquad M_i \equiv \frac{\mathbf{a}'_i \mathbf{a}_i}{b_i^2} - \frac{F_i - P_i + Q_i}{b_i}. \tag{17}$$

Theorem 5. *Consider the MSQ stabilization with a given set of $P_i > 0$, $Q_i > 0$, $i = 0, 1, \ldots, N$ in (7) for system (1) using quantized state feedback (3) and (4). Then, under Assumption 1, the smallest C_f defined in (8) can be approached by a linear state feedback law $v_t = K_{\gamma_t} x_t$ and a logarithmic quantizer (12)–(14) with controller and quantizer parameters chosen below:*

$$K_i = \begin{cases} 0, & \text{if } \delta_m(i) > 1, \\ K_{mi}, & \text{otherwise}, \end{cases} \qquad \delta(i) = \begin{cases} 1, & \text{if } \delta_m(i) > 1, \\ \delta_m(i), & \text{otherwise}. \end{cases}$$

Proof. Suppose that $\gamma_t = i, \forall i \in \Theta$, and drop the time index $t \ge d$ when no confusion is caused. Then, for (1) with $w_t = 0$, we have

$$\nabla V(x,i) = \sum_{i_1\in\Theta,i_2\in\Theta}\left[q_{i_1i_2i}(A_{i_1}x+B_{i_1}u)'P_{i_2}(A_{i_1}x+B_{i_1}u)\right]-x'P_ix$$
$$= b_iu^2+2\mathbf{a}_ixu+x'(F_i-P_i)x. \tag{18}$$

For Case 1: $b_i = 0$. Based on the definition of b_i, it is direct to get $q_{i_1i_2i}B'_{i_1} = 0, \forall i_1, i_2 \in \Theta$ since $P_{i_2} > 0$, which further implies that $\mathbf{a}_i = 0$. The MSQ stabilization guarantees that $F_i - P_i + Q_i < 0$, and thus $K_i = 0$, i.e., $u = 0$ can be adopted, which renders $\eta_f(i) = 0$. In this situation, we may set $\delta_m(i) = +\infty$ without loss of generality.

For Case 2: $b_i \neq 0$. From (18), it is easy to get

$$\nabla V(x,i)+x'Q_ix = \{-x'M_ix+(u-K_{mi}x)^2\}b_i,$$

and therefore the MSQ stabilization ensures that $M_i > 0$. Then $\nabla V(x,i) < -x'Q_ix, \forall x \neq 0$ if and only if $u = f(v,i) \in (u_1(i), u_2(i))$, where $u_1(i) = K_{mi}x - \sqrt{x'M_ix}$, $u_2(i) = K_{mi}x + \sqrt{x'M_ix}$. By applying the orthogonal decomposition method, $M_i^{1/2}x$ can be decomposed into

$$M_i^{1/2}x = \alpha(i)M_i^{-1/2}K'_{mi}+z(i), \tag{19}$$

where $\alpha(i)$ is a scalar and vector $z(i)$ is orthogonal to $M_i^{-1/2}K'_{mi}$. Therefore, $u_1(i)$, $u_2(i)$ can be rewritten with respect to the new coordinate system (19) as

$$u_1(i) = \frac{\alpha(i)}{\delta_m(i)^2}-\sqrt{\frac{\alpha(i)^2}{\delta_m(i)^2}+z'(i)z(i)},$$
$$u_2(i) = \frac{\alpha(i)}{\delta_m(i)^2}+\sqrt{\frac{\alpha(i)^2}{\delta_m(i)^2}+z'(i)z(i)}.$$

Moreover, if $\delta_m(i) > 1$, then we can again choose $u = 0$ similarly to Case 1, since $u = 0$ belongs to the interval $(u_1(i), u_2(i))$; if $\delta_m(i) \leq 1$, then it can be proved that the optimal quantization strategy with the smallest $\eta_f(i)$ for mode i is logarithmic, as shown in (13) and (14), with $\delta(i) = \delta_m(i)$ (Elia & Mitter, 2001).

By combining the above two cases and taking note of the property (9), we can conclude that the logarithmic quantizer stated in this theorem can achieve the smallest C_f for a given set of $P_i > 0, Q_i > 0, i = 0, 1, \ldots, N$. The technique in the proof of Lemma 2.1 in Fu and Xie (2005) can still be used to prove that a linear state feedback law $v_t = K_{\gamma_t}x_t$ is sufficient to obtain the coarsest quantization for Case 2 with $\delta_m(i) \leq 1$, while, for Case 2 with $\delta_m(i) > 1$ and Case 1, the argument is trivial, since $K_i = 0$ is adopted. This completes the proof. □

The quantization error of a logarithmic quantizer is $e_t \equiv u_t - v_t = f(v_t,\gamma_t) - v_t = \Delta(v_t,\gamma_t)v_t$, where $\Delta(v_t,\gamma_t) \in [-\delta(\gamma_t), \delta(\gamma_t)]$. The closed-loop quantized feedback system with $v_t = K_{\gamma_t}x_t$ becomes the following uncertain MJLS:

$$x_{t+1} = A_{\theta_t}x_t + B_{\theta_t}(1+\Delta(K_{\gamma_t}x_t,\gamma_t))K_{\gamma_t}x_t. \tag{20}$$

Before optimizing the overall coarseness with respect to all possible $P_i > 0,\ Q_i > 0,\ i \in \Theta$ such that (20) is MSQ stable in part (b) of the theorem that follows, we note that the uncertainty in (20) is a nonlinear function of $v_t = K_{\gamma_t}x_t$, which cannot be handled directly. The validity of the sector bound approach proved in part (a) shows that quantized stabilization is equivalent to robust MSQ stabilization of an uncertain system with time-varying uncertainties.

Theorem 6. *(a). Given a logarithmic quantizer (12)–(14) with a set of fixed $\delta(i) \in [0,1]$, $i = 0, 1, \ldots, N$, system (1) under Assumption 1 is MSQ stabilizable via quantized linear state feedback if and only if the following uncertain system,*

$$x_{t+1} = A_{\theta_t}x_t + B_{\theta_t}(1+\Delta(\gamma_t))v_t, \tag{21}$$

is robustly MSQ stabilizable for uncertainty $\Delta(\gamma_t) \in [-\delta(\gamma_t), \delta(\gamma_t)]$ via a linear state feedback law $v_t = K_{\gamma_t}x_t$.

(b). Under Assumption 1, the optimal overall coarseness for system (1) to be MSQ stabilizable via quantized linear state feedback can be obtained by the following optimization:

$$\underline{C}_f \equiv \min_{S_i>0, W_i>0, Y_i, \tau(i)>0,\ \forall i\in\Theta} C_f$$

over the constraint

$$\begin{bmatrix} -S_i & S_i & Y'_i & \Phi_{0i} & \Phi_{1i} & \cdots & \Phi_{Ni} \\ * & -W_i & 0 & 0 & 0 & \cdots & 0 \\ * & * & -\tau(i) & 0 & 0 & \cdots & 0 \\ * & * & * & \Xi_{0i} & 0 & \cdots & 0 \\ * & * & * & * & \Xi_{1i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & * & * & \Xi_{Ni} \end{bmatrix} < 0, \tag{22}$$

where Φ_{ji}, Ξ_{ji} are given as (23) and (24) in Box I. Moreover, a logarithmic quantizer (12)–(14) and a linear state feedback law $v_t = K_{\gamma_t}x_t$ are sufficient to achieve the $\underline{C}_f$, and a set of suitable state feedback gains is given by $K_i = Y_iS_i^{-1}$, $i = 0, 1, \ldots, N$.

Proof. (a). Again, suppose that $\gamma_t = i, i \in \Theta$; then, for (20), we have

$$\nabla V(x,i) = \sum_{i_1\in\Theta,i_2\in\Theta}\left[q_{i_1i_2i}(A_{i_1}x+B_{i_1}(1+\Delta(K_ix,i))K_ix)'P_{i_2}\right.$$
$$\left.\times\ (A_{i_1}x+B_{i_1}(1+\Delta(K_ix,i))K_ix)\right]-x'P_ix. \tag{25}$$

Following a similar proof as that of Lemma 2.2 in Fu and Xie (2005), it can be shown that $\nabla V(x,i) < -x'Q_ix,\ \forall x \neq 0$ is equivalent to

$$\sum_{i_1\in\Theta,i_2\in\Theta}\left[q_{i_1i_2i}(A_{i_1}x+B_{i_1}(1+\Delta(i))K_ix)'P_{i_2}\right.$$
$$\left.\times\ (A_{i_1}x+B_{i_1}(1+\Delta(i))K_ix)\right]-x'P_ix < -x'Q_ix \tag{26}$$

for $x \neq 0$, where $\Delta(i)$ is defined as in (21) for $\gamma_t = i$. This kind of equivalence is true for any $i \in \Theta$, and thus, by Definition 1, inequality (26) is the condition for robust MSQ stabilization of system (21).

(b). The constraint (22) is obtained by using the Schur complement over inequality (26) and taking $S_i = P_i^{-1}$, $W_i = Q_i^{-1}$, $Y_i = K_iS_i$, where $\tau(i) > 0$ is the scaling variable. From the proof in part (a), we see that the quantized stabilization for (20) and the robust stabilization for (21) can share the same set of P_i, Q_i, $i = 0, 1, \ldots, N$, as well as the same set of feedback gains. Then the result follows directly from Theorem 5. □

For a logarithmic quantizer, the overall coarseness C_f can also be defined in terms of the set of $\delta(i)$ or $\rho(i)$, $i = 0, 1, \ldots, N$. For example, one possible choice is $C_{f1} \equiv -\min_{i\in\Theta}\{\delta(i)\}$, which captures the worst-case mode with the smallest sector bound (equivalently the largest mode quantization density) among all system modes. In this case, the optimization in part (b) of Theorem 6 becomes $\max_{S_i,W_i,Y_i,\tau(i)} \delta$ over (22) with $\delta(i) = \delta, \forall i \in \Theta$. Moreover, suppose that θ_t is driven by an ergodic Markov chain which admits a limiting probability distribution $\{\bar{\pi}_i; \bar{\pi}_i > 0, i \in \Theta\}$; then another choice could be $C_{f2} \equiv -\sqrt{\sum_{i=0}^{N}\bar{\pi}_i\delta(i)^2}$, which characterizes the weighted average quantization performance. Since, for any fixed set of $\delta(i)$, (22) is convex in S_i, W_i, Y_i and $\tau(i)$, $\underline{C}_f$ can be obtained by searching the space of $\delta(i)$, $i = 0, 1, \ldots, N$. Note that such a method may be time-consuming especially when the number of system modes N is large.

3.2. Mode estimation

When the system mode is not observed at the quantizer and controller, one may form $\mathbb{K}$ and $\mathbb{Q}$ in a mode-independent manner

$$\Phi_{ji} = \left[\sqrt{q_{j0i}}(S_iA_j' + Y_i'B_j')\ \sqrt{q_{j1i}}(S_iA_j' + Y_i'B_j')\ \cdots\ \sqrt{q_{jNi}}(S_iA_j' + Y_i'B_j')\right], \quad j = 0, 1, \ldots, N. \tag{23}$$

$$\Xi_{ji} = \begin{bmatrix} -S_0 + \tau(i)\delta(i)^2 q_{j0i}B_jB_j' & \tau(i)\delta(i)^2\sqrt{q_{j0i}q_{j1i}}B_jB_j' & \cdots & \tau(i)\delta(i)^2\sqrt{q_{j0i}q_{jNi}}B_jB_j' \\ * & -S_1 + \tau(i)\delta(i)^2 q_{j1i}B_jB_j' & \cdots & \tau(i)\delta(i)^2\sqrt{q_{j1i}q_{jNi}}B_jB_j' \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & -S_N + \tau(i)\delta(i)^2 q_{jNi}B_jB_j' \end{bmatrix}, \quad j = 0, 1, \ldots, N. \tag{24}$$

Box I.

as in Scheme III of Section 3.1, which, however, could be conservative. More generally, we can try to estimate the mode process. First of all, a special case of mode estimation is given below.

- Scheme IV (Mode estimation without process noise): $w_t = 0$ and $\forall x \neq 0, i_1, i_2, i_3 \in \Theta, i_1 \neq i_2$,

  $$A_{i_1}x + B_{i_1}f(g(x, i_3), i_3) \neq A_{i_2}x + B_{i_2}f(g(x, i_3), i_3). \tag{27}$$

  In this situation, the next estimation,

  $$\gamma_t = \hat{\theta}_{t-1} = \operatorname{argmin}_{i\in\Theta}\|x_t - A_ix_{t-1} - B_if(g(x_{t-1}, \gamma_{t-1}), \gamma_{t-1})\|^2,$$

  with arbitrary $\gamma_0 \in \Theta$, can ensure that $\gamma_t = \theta_{t-1}, \forall t \geq 1$. Thus, the result on OSDMO (Scheme II) can be applied directly.

With nonzero process noise w_t, one can still estimate the previous mode θ_{t-1} at time t based on x_0^t, γ_0^{t-1} and closed-loop system model (5). Assume that x_0 is white Gaussian and that w_t is zero-mean white Gaussian. Denote $\Omega(x, \mu, \Sigma)$ as the vector-valued Gaussian probability density function with mean vector μ and covariance matrix Σ. Suppose the initial distribution π and the transition probability matrix Π of the underlying Markov process as well as the set of covariance matrices $\Sigma_i, i = 0, 1, \ldots, N$, of process noise w_t are exactly known to the controller. The next algorithm gives an estimate of θ_{t-1} by maximizing the probability

$$L(\theta_{t-1}) \equiv \Pr\{\theta_{t-1}|x_0^t, \gamma_0^{t-1}\} \tag{28}$$

with respect to $\theta_{t-1} \in \Theta$.

Algorithm 1. A recursive procedure for finding $\gamma_t = \hat{\theta}_{t-1}$ at time $t \geq 1$ for quantized system (5), such that L defined in (28) is maximized, is stated as follows.

(a). Choose γ_0 as an arbitrary element in Θ and set $u_0 = 0$.

(b). For $t = 1$, $\gamma_1 = argmax_{i\in\Theta}[a_1(i)]$ with $a_1(i) = \pi_i\Omega(x_1, A_ix_0, \Sigma_i)$.

(c). For $t \geq 2$, $\gamma_t = argmax_{i\in\Theta}[a_t(i)]$, where $a_t(i)$ can be computed iteratively as

$$a_t(i) = \sum_{j\in\Theta} a_{t-1}(j)\pi_{ji} \times \Omega(x_t, A_ix_{t-1} + B_if(g(x_{t-1}, \gamma_{t-1}), \gamma_{t-1}), \Sigma_i).$$

The above algorithm is modified from the well-known Viterbi algorithm (Rabiner, 1989; Viterbi, 1967), where the optimality criterion, different from (28), is to find the single best mode sequence. Moreover, the maximum likelihood estimation can be used to iteratively update the parameters such as π, Π, Σ_i, if some or all of them are unknown to the controller. For more complicated cases, e.g., partial state observation with corrupted noise, approaches for mode estimation based on more sophisticated hidden Markov model may be constructed; see Elliott et al. (1995, 2005).

Remark 7. Note that, for direct mode observation $\gamma_t = \theta_{t-d}$ with $d \geq 2$, and general cases of Algorithm 1, the probability on the left-hand side of Eq. (16) becomes a function of state x and thus

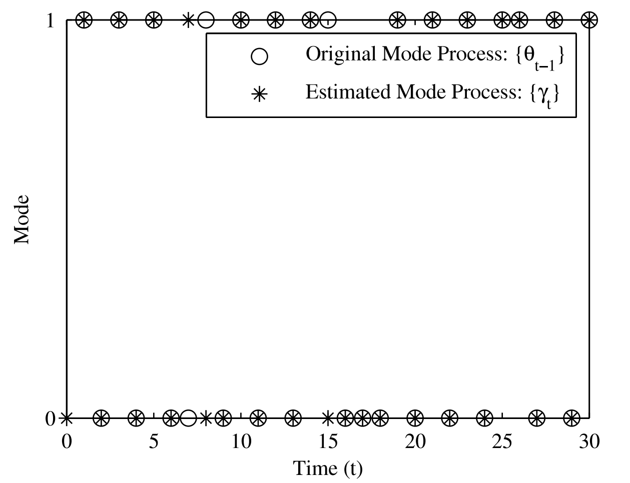


Fig. 2. Original and estimated mode processes for one sample of simulation using Algorithm 1.

dynamic, which renders an optimal memoryless quantization strategy impossible. In this situation, some dynamic or state-dependent quantization strategy would be an interesting research topic.

The next numerical example demonstrates the usefulness of Algorithm 1.

Example 8. Consider an MJLS (1) with $A_0 = 1.2$, $A_1 = -1.2$, $B_0 = B_1 = 1$ and transition probability matrix $\Pi = [0.1\ \ 0.9;\ 0.9\ \ 0.1]$.

First, suppose that direct mode observation is available at $\mathbb{K}$ and $\mathbb{Q}$. Then, for CMO (Scheme I), the smallest allowable $C_{f1} \equiv -\min_{i\in\Theta}\{\delta(i)\}$ is -0.8333 with $K_0 = -1.2$, $K_1 = 1.2$; for OSDMO (Scheme II), the smallest achievable C_{f1} is -0.7229 with $K_0 = 0.9600$, $K_1 = -0.9600$.

Second, if the system mode is not observed at $\mathbb{K}$ and $\mathbb{Q}$, then we can easily verify that the mode-independent strategy (Scheme III) cannot stabilize the system. Furthermore, assume that the covariance of w_t is given by $W_0 = W_1 = 1$ and the initial state x_0 is Gaussian distributed with mean 20 and variance 10; then the first 30 mode estimates for one sample of simulation using Algorithm 1 are shown in Fig. 2. The parameters of the controller and quantizer are chosen as in OSDMO: $K_0 = 0.9600$, $K_1 = -0.9600$, $\delta(0) = \delta(1) = 0.7229$. Fig. 3 further gives the empirical norm of state by averaging 10,000 Monte Carlo simulations. As we can see from Figs. 2 and 3, there exist some mode estimation errors, but the error rate is low, and the empirical norm of state by applying Algorithm 1 is convergent.

4. Application to NCSs

Next, we apply the results presented in Section 3 to a quantized feedback NCS as shown in Fig. 4, where an LTI plant ($\mathbb{P}$) is described in discrete-time form as

$$\mathbb{P} : x_{t+1} = Ax_t + Bz_t, \tag{29}$$

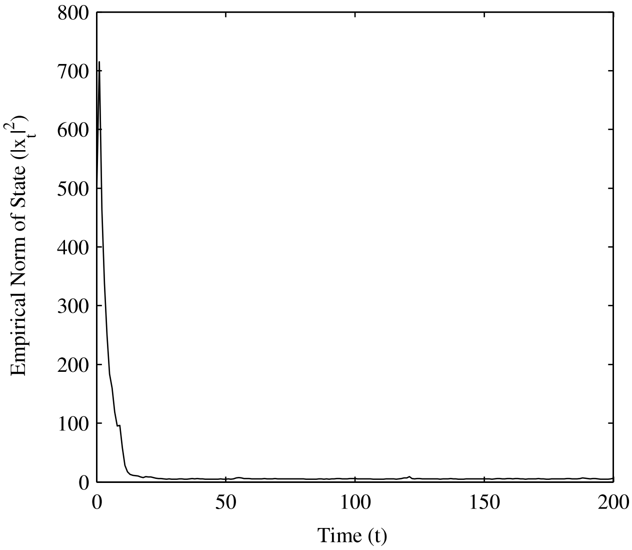


Fig. 3. The empirical norm of state by averaging 10,000 Monte Carlo simulations using Algorithm 1.

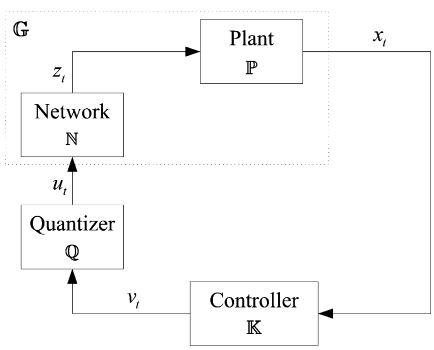


Fig. 4. Quantized control over a lossy network.

which may be obtained through discretization of a continuous-time system. Suppose a zero-control strategy is adopted in dealing with the binary dropouts over the network (ℕ):

$$\mathbb{N} : z_t = \theta_t u_t, \quad \theta_t \in \Theta = \{0, 1\}. \tag{30}$$

Then, the system (𝔾) as a combination of the network and the discrete plant can be modeled as a jump system (1) with $A_0 = A_1 = A$, $B_0 = 0$, $B_1 = B$. For a TCP-like channel (see Imer, Yüksel, and Başar (2006); Schenato et al. (2007) for more details on the TCP-like and UDP-like protocols), $\gamma_t = \theta_{t-1}$, and θ_t is driven by a Markov chain with transition probability matrix $\Pi = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}$. In this situation, the OSDMO result (Scheme II) is applicable. Note that the CMO result (Scheme I) is of theoretical importance in the quantization of MJLSs but may not be practical in the NCS depicted in Fig. 2, since it is unrealistic for the quantizer to know whether the current packet will be lost or not before the packet is sent over the network.

For the UDP-like protocol, we can easily verify that inequality (27) is true, and thus Scheme IV can be used directly when $w_t = 0$. If θ_t is assumed to be an i.i.d. random variable:

$$\Pr(\theta_t = 0) = \alpha, \qquad Pr(\theta_t = 1) = 1 - \alpha, \tag{31}$$

i.e., the system adopts an unreliable network with packet-dropout rate α, then Scheme III is applicable, and the inequality (22) is reduced to the following modified Riccati inequality:

$$A'PA - P + Q - (1-\alpha)(1-\delta^2)A'PB(B'PB)^{-1}B'PA < 0. \tag{32}$$

Based on Lemma 5.4 in Schenato et al. (2007) for modified algebraic Riccati equation, $(1-\alpha)(1-\delta^2) > 1 - \Pi_i|\lambda_i^u(A)|^{-2}$ can ensure the existence of $P > 0$ to (32), where $\lambda_i^u(A)$ denotes the i-th unstable pole of A. It is easy to check that the above result is consistent with Theorem 2.1 of Hoshina et al. (2007), which can be seen as a special case of Theorem 6(b) in this paper.

5. Conclusions

This paper has shown that, for linear systems with Markovian jump parameters, a mode-dependent logarithmic quantizer is still optimal in the MSQ stability sense, and the sector bound approach again provides a non-conservative way for studying the corresponding quantized state feedback stabilization problem. In addition, a mode estimation algorithm is presented to deal with the unknown mode process at the controller and quantizer side. Possible future work includes mode-dependent quantized feedback stabilization via a switching system approach, quantized output feedback stabilization, quantized performance control, generalization to the MIMO system case, and dynamic quantization.

Acknowledgements

The valuable comments and suggestions from the anonymous reviewers and the associate editor are very much appreciated.

References

Boukas, E., & Liu, Z. (2001). Robust H_∞ control of discrete-time Markovian jump linear systems with mode-dependent time-delays. *IEEE Transactions on Automatic Control*, *46*(12), 1918–1924.

Costa, O., Fragoso, M., & Marques, R. (2005). *Discrete-time Markov jump linear systems*. London: Springer.

de Souza, C. (2006). Robust stability and stabilization of uncertain discrete-time Markovian jump linear systems. *IEEE Transactions on Automatic Control*, *51*(5), 836–841.

Elia, N. (2005). Remote stabilization over fading channels. *Systems and Control Letters*, *54*(3), 237–249.

Elia, N., & Mitter, S. (2001). Stabilization of linear systems with limited information. *IEEE Transactions on Automatic Control*, *46*(9), 1384–1400.

Elliott, R., Aggoun, L., & Moore, J. (1995). *Hidden Markov models estimation and control*. New York: Springer.

Elliott, R., Dufour, F., & Malcolm, W. (2005). State and mode estimation for discrete-time jump Markov systems. *SIAM Journal on Control and Optimization*, *44*, 1081–1104.

Fu, M., & Xie, L. (2005). The sector bound approach to quantized feedback control. *IEEE Transactions on Automatic Control*, *50*(11), 1698–1711.

Gao, H., & Chen, T. (2008). A new approach to quantized feedback control systems. *Automatica*, *44*(2), 534–542.

Ho, T., & Chen, B. (2006). Novel extended Viterbi-based multiple-model algorithms for state estimation of discrete-time systems with Markov jump parameters. *IEEE Transactions on Signal Processing*, *54*(2), 393–404.

Hoshina, H., Tsumura, K., & Ishii, H. (2007). The coarsest logarithmic quantizers for stabilization of linear systems with packet losses. In *Proceedings of the 46th IEEE conference on decision and control*. (pp. 2235–2240) New Orleans, USA.

Huang, M., & Dey, S. (2007). Stability of Kalman filtering with Markovian packet losses. *Automatica*, *43*(4), 598–607.

Hu, S., & Yan, W. (2007). Stability robustness of networked control systems with respect to packet loss. *Automatica*, *43*(7), 1243–1248.

Imer, O., Yüksel, S., & Başar, T. (2006). Optimal control of LTI systems over unreliable communication links. *Automatica*, *42*(9), 1429–1439.

Matei, I., Martins, N., & Baras, J. (2008). Optimal linear quadratic regulator for Markovian jump linear systems, in the presence of one time-step delayed mode observations, In *Proceedings of the 17th IFAC world congress*. (pp. 8056–8061) Seoul, Korea.

Rabiner, L. (1989). A tutorial on hidden Markov models and selected applications in speech recognition. *Proceedings of the IEEE*, *77*(2), 257–286.

Schenato, L., Sinopoli, B., Franceschetti, M., Poolla, K., & Sastry, S. (2007). Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, *95*(1), 163–187.

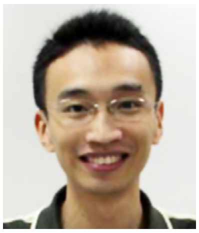
Seiler, P., & Sengupta, R. (2005). An H_∞ approach to networked control. *IEEE Transactions on Automatic Control*, *50*(3), 356–364.

Sinopoli, B., Schenato, L., Franceschetti, M., Poolla, K., Jordan, M., & Sastry, S. (2004). Kalman filtering with intermittent observations. *IEEE Transactions on Automatic Control*, *49*(9), 1453–1464.

Viterbi, A. (1967). Error bounds for convolutional codes and an asymptotically optimum decoding algorithm. *IEEE Transactions on Information Theory*, *13*(2), 260–269.

Xiao, N., Xie, L., & Fu, M. (2009). Quantized stabilization of Markov jump linear systems via state feedback. In *Proceedings of American control conference*. (pp. 4020– 4025), St. Louis, USA.

Xiong, J., & Lam, J. (2007). Stabilization of linear systems over networks with bounded packet loss. *Automatica*, *43*(1), 80–87.



Nan Xiao received his B.E. and M.E. degrees in Electrical Engineering from Tianjin University, Tianjin, China, in 2005 and 2007, respectively. He is currently pursuing a Ph.D. degree in the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. His current research interests include robust and stochastic control theory, and networked control systems.

Lihua Xie received his B.E. and M.E. degrees in Electrical Engineering from Nanjing University of Science and Technology in 1983 and 1986, respectively, and his Ph.D. degree in Electrical Engineering from the University of Newcastle, Australia, in 1992. Since 1992, he has been with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, where he is currently a professor and the Director, Centre for Intelligent Machines. He is a Changjiang visiting professor with South China University of Technology. He held teaching appointments in the Department of Automatic Control, Nanjing University of Science and Technology from 1986 to 1989.

Dr. Xie's research interests include robust control and estimation, networked control systems, sensor networks, time delay systems, and control of hard disk drive systems. In these areas, he has published many journal papers and co-authored two patents and four books. He is an editor of IET Book Series on Control and has served as an Associate Editor of several journals, including IEEE Transactions on Automatic Control, Automatica, IEEE Transactions on Control Systems Technology, IEEE Transactions on Circuits and Systems-II, and IET Proceedings on Control Theory and Applications. Dr. Xie is a Fellow of IEEE.

Minyue Fu received his Bachelor's Degree in Electrical Engineering from the University of Science and Technology of China, Hefei, China, in 1982, and M.S. and Ph.D. degrees in Electrical Engineering from the University of Wisconsin-Madison in 1983 and 1987, respectively.

From 1983 to 1987, he held a teaching assistantship and a research assistantship at the University of Wisconsin-Madison. He worked as a Computer Engineering Consultant at Nicolet Instruments, Inc., Madison, Wisconsin, during 1987. From 1987 to 1989, he served as an Assistant Professor in the Department of Electrical and Computer Engineering, Wayne State University, Detroit, Michigan. For the summer of 1989, he was employed by the Université Catholique de Louvain, Belgium. He joined the Department of Electrical and Computer Engineering, the University of Newcastle, Australia, in 1989. Currently, he is a Chair Professor in Electrical Engineering. In addition, he was a Visiting Associate Professor at University of Iowa in 1995–1996, and a Senior Fellow/Visiting Professor at Nanyang Technological University, Singapore, 2002. He holds a ChangJiang Visiting Professorship at Shandong University and visiting positions at South China University of Technology and Zhejiang University in China.

He is a Fellow of IEEE. His main research interests include control systems, signal processing and communications. He has been an Associate Editor for the IEEE Transactions on Automatic Control, Automatica and Journal of Optimization and Engineering.