Feedback Control of a Class of Linear Systems with Jump Parameters

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Abstract—A class of linear systems are studied which are subject to sudden changes in parameter values. An algorithm similar in form to Kushner's stochastic maximum principle is derived and the relationship between these algorithms discussed. Systems in which the performance measure is quadratic are investigated in detail and a differential equation is derived which yields the optimal feedback gains.

I. Introduction

ONE OF THE principal reasons for introducing feedback into a control system is to make the resulting system relatively insensitive to changes in plant parameters. While the possibility that plant parameters do vary may not be included explicitly in the mathematical description of the system, the pervasive nature of such variations causes the engineer to seek to minimize their effects when and if they do occur. If the perturbations have only small influence on system behavior, classical sensitivity analysis will provide an adequate assessment of their effects. On the other hand, when the possible variations in system characteristics significantly alter the behavior of the system, it is necessary to incorporate a quantitative description of the disturbances into the system model. A possible way of doing this is to phrase one's uncertainty in terms of a stochastic model of the system. Here the plant parameters are described by stochastic processes whose statistical characterization is selected to correspond to the variation that the engineer expects in the actual plant.

For discrete systems several specific results have been obtained. When the system is described by linear difference equations and when the performance index is quadratic, the formalism of dynamic programming may be applied to advantage [1], [2].

When attention is shifted to continuous-time systems, the problem becomes significantly more complex. Several difficulties were encountered in [1] which had no direct analog in discrete time. A subtle difficulty arises if the changes in plant parameters take place suddenly. This may occur, for example, if the variations are due to component failure. Here the natural model for the parameter process might be a jump process. Since the feedback signal gives information on the parameter process, small variations in the feedback signal may indicate discrete changes in the parameter. In such a circumstance the minimum cost functional may not be differentiable with respect to the observed system variables.

The object of this paper is to extend the work on linear stochastic systems to include systems whose parameter processes are Markov jump processes with a finite number of states. To do this it will be expedient to derive an algorithm similar to the Pontryagin maximum principle. The use of the maximum principle in stochastic systems is not particularly novel, and it might be well to consider explicitly how Theorem 1 of this paper differs from results derived by other investigators. In [3, ch. 7] the maximum principle is used in the study of a tracking problem. It is important to note that the optimal control is a function of t alone. In [4] a more general stochastic problem is studied. While investigating systems with additive noise, Kushner introduced the notion of a random Hamiltonian function and derived the result that the optimal control minimizes the expectation of the Hamiltonian function conditioned on the observed system variables. It is interesting to observe that the differential equation which the adjoint variable must satisfy is identical in [3] and [4].

In this paper a result similar to [4] will be derived for a class of systems with multiplicative disturbances. For these systems it will be shown that the differential equation for the adjoint variable differs from that obtained in [4]. These results indicate that direct generalization of the results of [4] to systems with nonadditive nonwhite noise is not possible without modification.

II. System Description

It will be assumed that the system to be controlled is described by the linear vector differential equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)v(t), \quad 0 \leq t \leq T \\ x(t_0) &= x_0 \end{aligned} \tag{1}$$

where $x(t)$ is the n-dimensional state and $v(t)$ is the m-dimensional actuating signal at time t. It is intended that (1) serve as a model for a system subject to abrupt changes in parameter values perhaps because of component failures or sudden shifts in environment. If the number of possible parameter values is finite, e.g., s, it is natural to model the parameter processes with a discrete state Markov jump process. Therefore, assume that each element of the

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random matrix $[A(t), B(t)]$ is a separable Markov process and that there exists an $s \times s$ matrix $Q(t)$ such that[1]

$$\begin{aligned}\Pr\{[A(t+\Delta), B(t+\Delta)] &= [A_j, B_j] \,|\, [A(t), B(t)] = [A_i, B_i]\} \\ &= \begin{cases} q_{ij}(t)\Delta + o(\Delta), & i \neq j \\ 1 + q_{ii}(t)\Delta + o(\Delta), & i = j, i; j = 1, \cdots, s \end{cases} \end{aligned} \tag{2}$$

and a vector P such that

$$\Pr\{[A(0), B(0)] = [A_i, B_i]\} = P_i, \quad i = 1, \cdots, s.$$

To design a controller for a stochastic system, the quantity of the data fed back to the controller must be specified. For the purposes of this exposition, the feedback signal will contain both time and the instantaneous state, i.e., $(t, x(t))$. Furthermore, since the characteristics of the plant vary in a random manner, an attempt might be made to monitor these variations. Suppose that there are k sensors on the plant with outputs given by $r(t)$, a k-dimensional random process. This latter process need not be Markovian nor need it be continuous, but it will be assumed to be bounded. To avoid the complications inherent in dual control problems, it will also be required that $r(t)$ be independent of the control policy used.

With the nature of the feedback signal thus specified, the set of admissible controls may be delineated. It is at this point that the stochastic problem differs in the most striking fashion from its deterministic counterpart. Whereas, if the system characteristics are known a priori, an open-loop control is satisfactory; the performance of a stochastic system may be significantly improved by feedback. It will be supposed that at time t the controller observes the vector $(t, x(t), r(t))$. Based upon this observation the controller selects a control action $v(t)$, i.e.,

$$v(t) = \bar{u}(t,x(t),r(t)). \tag{3}$$

The bar above the u will be used to emphasize the difference between the function and its value. This distinction is important in what follows because $\bar{u}$ represents the controller and is therefore nonrandom, while $v(t)$ is the output of the controller and is thus a random process. For all practical purposes any function from the space of observations to R^m will be admissible if the solutions to

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)\bar{u}(t,x(t),r(t)) \\ x(t_0) &= x_0 \end{aligned} \tag{4}$$

are smooth enough.[2] Denote the solution to (4) by $x(t; t_0, x_0, \bar{u})$.

The engineer seeks to design a controller which will result in the minimum of a specified performance measure. The particular class of criterion functions studied here will be described by

$$J(\bar{u}; t_0, x_0, r(t_0)) = E\left\{\int_{t_0}^{T} L(\tau, x(\tau; t_0, x_0, \bar{u}), \bar{u}(\tau, x(\tau; t_0, x_0, \bar{u}), r(t_0)))d\tau \,\middle|\, t_0, x_0, r(t_0)\right\} \tag{5}$$

where $L(t, x, v)$ is non-negative and continuously differentiable with respect to x and v.

Suppose that there exists a $\bar{u}^* \epsilon \Gamma$, the set of admissible controls, with the following property: for every $t_0 \epsilon [0, T]$, every $x_0 \epsilon R^n$, and every $r(t_0)$,

$$J(\bar{u}^*; t_0, x_0, r(t_0)) = \min_{u \epsilon \Gamma} J(\bar{u}; t_0, x_0, r(t_0)). \tag{6}$$

Such a controller would clearly be labeled best in this application since it causes a minimum in the expected cost when conditioned on the information available to the controller. In addition to its existence, $\bar{u}^*$ will be assumed to have certain smoothness properties, the most important of which is that with probability one $\bar{u}_x^*(t,x,r(t))$ exists everywhere on $[0, T] \times R^n$ except perhaps on a Borel set of measure zero (see the Appendix).[3] In effect, $\bar{u}_x^*$ exists everywhere except at a "few" exceptional points which can be shown to have little effect on the design algorithm.

The basic problem is this. Given a system described by (2) and (4), find that element of Γ which minimizes the criterion functional given in (5). If a $\bar{u}^* \epsilon \Gamma$ exists which satisfies (6) and the conditions indicated in the Appendix, the next section presents an algorithm for use in evaluating $\bar{u}^*$ explicitly.

III. Stochastic Maximum Principle

A technique similar to the maximum principle is presented here which may be employed to solve the problem described in the preceding section. As mentioned earlier Theorem 1 differs in form from that presented in [4] through a difference in the equation of the "adjoint" variable. Of course, the class of problems to which each applies is distinct and the proofs are significantly different.

Fix $(t_0, x_0) \epsilon [0, T] \times R^n$. Define a vector valued random variable $p(t_0, x_0)$ as the solution to the following ordinary differential equation:

$$\begin{aligned} \frac{d}{d\tau} p(\tau, x(\tau; t_0, x_0, \bar{u}^*)) &= -[A(\tau) + B(\tau)\bar{u}_x^*(\tau, x(\tau; t_0, x_0, \bar{u}^*), r(\tau))]^T \\ &\quad \cdot p(\tau, x(\tau; t_0, x_0, \bar{u}^*)) + [L_x \\ &\quad + L_v \bar{u}_x^*(\tau, x(\tau; t_0, x_0, \bar{u}^*), r(t))]^T, \quad t_0 \leq \tau \leq T \\ p(T, x) &= 0, \quad \text{for all } x. \end{aligned} \tag{7}$$

[1] Equation (2) and numerous other relations stated here are actually only true with probability one, that is, a.s. To avoid confusion this restriction will be left implicit in the body of the paper and described in more detail in the Appendix.

[2] See the Appendix for some specific restrictions.

[3] If y is an r-dimensional vector valued function and z is s-dimensional, the matrix with elements $\partial y_i/\partial z_j$, $i = 1, \cdots, r$ and $j = 1, \cdots, s$, will be denoted by y_z.

As was noted in [4] it is not always true that $p(t) - p(t-dt) = p(t+dt) - p(t)$. However, in (7) the anomalous points have measure zero and the equation may be integrated forward in time. It can, in fact, be shown that (7) is a linear equation with bounded and measurable coefficients and forcing function [5]. Hence a solution to (7) exists and is unique. Observe that the equation for $p(t_0, x_0)$ differs from that obtained in the deterministic problem and also from that obtained by Kushner in his study of additive noise processes (see [4, eq. 10]). The fundamental reason for this variation lies in the fact that the deterministic problems require no feedback control law while in the problem posed and solved by Kushner, these additional terms do not influence the optimal control rule.

If $\bar{u}\epsilon\Gamma$, define the Hamiltonian at the point (t_0, x_0) to be

$$H(t_0, x_0, \bar{u}) = p(t_0, x_0)^T[A(t_0)x_0 + B(t_0)\bar{u}(t_0, x_0, r(t_0))] - L(t_0, x_0, \bar{u}(t_0, x_0, r(t_0))). \quad (8)$$

Then the following is true.

Theorem 1

If $\bar{u}\epsilon\Gamma$ and if $E\{p(t, x)^T[A(t), B(t)]|t, x, r(t)\}$ is measurable on $[0, T] \times R^n \times \Omega$, then the set on which $E\{H(t, x, \bar{u})|t, x, r(t)\} > E\{H(t, x, \bar{u}^*)|t, x, r(t)\}$ has measure zero.

Excluding the measure theoretic qualifiers, the theorem simply states that the optimal control maximizes the conditional expectation of the Hamiltonian under almost any feedback condition. A proof is outlined in the Appendix. It should be noted that the proof for the stochastic problem differs in a very basic manner from its deterministic counterpart. In a deterministic problem it is possible to perturb the control action $v(t)$ over an arbitrarily small time interval, and then study the resulting perturbation in the performance index. In the study of feedback controllers this is no longer possible. If the control rule $\bar{u}^*$ is perturbed over some small time interval beginning at t_1, the subsequent control action will be perturbed for all $t \geq t_1$ because of feedback.

IV. Quadratic Criterion Functionals

To illustrate the application of Theorem 1, consider the problem in which the criterion functional is quadratic and the instantaneous behavior of the system is observable. For convenience it will be assumed that $v(t)$ is a scalar and that

$$L(t, x, v) = x^TRx + v^2 \quad (9)$$

where R is a non-negative symmetric matrix. As a consequence of Theorem 1,

$$\bar{u}^*(t, x, r(t)) = \frac{1}{2}E\{p(t, x)^TB(t)|t, x, r(t)\}. \quad (10)$$

To complete the description of the controller an explicit relation between $p(t, x)$ and $(t, x, r(t))$ must be obtained. It will be supposed that the information available to the controller is sufficient to determine the instantaneous values of $A(t)$ and $B(t)$, i.e., $r(t)$ is a vector with components equal to the elements of the $[A(t), B(t)]$ matrix. While the sample functions of the elements of $A(t)$ and $B(t)$ are not continuous, they are independent of the control rule. Hence the equation for the optimal controller becomes

$$\bar{u}^*(t, x, r(t)) = \frac{1}{2}B(t)^TE\{p(t, x)|t, x, r(t)\}. \quad (11)$$

From (7) it is evident that

$$\begin{aligned}\frac{d}{d\tau}p(\tau, x(\tau; t, x, \bar{u}^*)) &= -[A(\tau) + B(\tau)\bar{u}_x^*(\tau, x(\tau; t, x, \bar{u}^*), r(\tau))]^T \\ &\quad \cdot p(\tau, x(\tau; t, x, \bar{u}^*)) + 2Rx + 2v(t) \\ &\quad \cdot \bar{u}_x^*(\tau, x(\tau; t, x, \bar{u}^*), r(\tau))^T, \quad t \leq \tau \leq T \\ p(T) &= 0.\end{aligned} \quad (12)$$

To determine the unique solution to (12), it will be supposed that $p(t, x)$ has the form

$$p(t, x) = 2K(t)x, \quad 0 \leq t \leq T, \quad x\epsilon R^n \quad (13)$$

where $K(t)$ is a random process independent of x when conditioned on $(t, x, r(t))$ and differentiable everywhere.

From (11) and (13) it follows that

$$\bar{u}^*(t, x, r(t)) = B(t)^TE\{K(t)|t, r(t)\}x$$

and hence from (12) and (13)

$$\begin{aligned}-2[A(t)^T &+ E\{K(t)^T|t, r(t)\}B(t)B(t)^T]K(t)x + 2Rx \\ &+ 2B(t)^TE\{K(t)|t,r(t)\}xB(t)^TE\{K(t)|t,r(t)\} \\ &= 2\dot{K}(t)x + 2K(t)A(t)x \\ &+ 2K(t)B(t)B^T(t)E\{K(t)|t,r(t)\}x \\ K(T) &= 0.\end{aligned}$$

Since $K(t)$ is conditionally independent of x and is symmetric with the indicated boundary conditions, it follows that

$$\begin{aligned}\dot{K}(t) = &-A(t)^TK(t) - K(t)A(t) - K(t)B(t)B(t)^T \\ &\cdot E\{K(t)|t, r(t)\} - E\{K(t)|t, r(t)\}\, B(t)B(t)^TK(t) \\ &+ E\{K(t)|t, r(t)\}B(t)B(t)^TE\{K(t)|t, r(t)\} + R \\ K(T) &= 0.\end{aligned} \quad (14)$$

It is desirable at this time to introduce some additional notation. At any time t the augmented matrix $[A(t), B(t)]$ must be in one of a finite number of different states. The event that $[A(t), B(t)] = [A_i, B_i]$ will be denoted by $(t, r(t))\epsilon[i]$. Define

$$E\{K(t)|(t, r(t))\epsilon[j]\} = \bar{K}_j(t).$$

If at time t, $(t, r(t)) \epsilon [j]$, then

$$E\{\dot{K}(t)|(t, r(t))\epsilon[j]\} = -A_j{}^T\bar{K}_j(t) - \bar{K}_j(t)A_j - \bar{K}_j(t)B_jB_j{}^T\bar{K}_j(t) + R. \quad (15)$$

From (14) it follows that $\dot{K}(t)$ is bounded on $[0, T]$ and consequently

$$E\{\dot{K}(t)|(t, r(t))\epsilon[j]\} = \lim_{\Delta\to 0} \frac{E\{K(t+\Delta)|(t, r(t))\epsilon[j]\} - E\{K(t)|(t, r(t))\epsilon[j]\}}{\Delta}.$$

The matrices $A(t)$ and $B(t)$ are Markovian, and hence

$$E\{K(t+\Delta)|(t, r(t))\epsilon[j]\} = \sum_{i=1}^{s} \bar{K}_i(t+\Delta)\Pr((t+\Delta, r(t+\Delta))\epsilon[i]|(t, r(t))\epsilon[j]).$$

Therefore, from (2)

$$E\{K(t+\Delta)|(t, r(t))\epsilon[j]\} = \bar{K}_j(t+\Delta) + \Delta \sum_{i=1}^{s} \bar{K}_i(t+\Delta)q_{ji}(t) + o(\Delta). \quad (16)$$

Finally, combining (15) and (16) and taking the indicated limit,

$$\dot{\bar{K}}_j(t) = -A_j{}^T\bar{K}_j(t) - \bar{K}_j(t)A_j - \bar{K}_j(t)B_jB_j{}^T\bar{K}_j(t) - \sum_{i=1}^{s} \bar{K}_i(t)q_{ji}(t) + R \quad (17)$$

$$\bar{K}_j(T) = 0, \quad j = 1, \cdots, s$$

and

$$\bar{u}^*(t,x,r(t)) = B_j{}^T\bar{K}_j(t)x, \quad \text{if } (t,r(t))\epsilon[j]. \quad (18)$$

Equations (17) and (18) give the solution to this stochastic control problem. The gain matrices $\bar{K}_i(t)$, $i = 1, \cdots, s$, may be obtained by direct integration of (17). The stochastic nature of this problem places itself in evidence in the intercoupling of Ricatti equations identical to those derived in stationary deterministic problems.

V. Example

Equations (17) and (18) provide the information required to design an optimal feedback controller for this system. While these equations are in a form which is convenient for integration on a digital computer, it will seldom be the case that an explicit closed-form solution of (17) can be obtained in any nontrivial problem. To illustrate the mechanics of the solution technique presented in Section IV and to indicate some of the basic differences between stochastic and deterministic problems, Theorem 1 will be applied in a formal manner to a class of problems which do admit to direct analytical solution.

Consider a system described by the following scalar equation

$$\dot{x}(t) = a(t)x(t) + v(t)$$
$$x(0) = x_0. \quad (19)$$

The function $a(t)$ is a Markov jump process with two possible states. The engineer must design a controller $\bar{u}^*$ which will observe $a(t)$ and $x(t)$ and minimize the performance measure $E\{\int_{t_0}^{\infty} [x(\tau)^2 + v(\tau)^2]d\tau | t_0, x(t_0), a(t_0)\}$.

To complete the problem formulation the probabilistic nature of the $a(t)$ process must be specified. The two possible states of $a(t)$ are 0 which is an absorbing state and 4 which is a transient state. Thus if $a(t) = 0$, it remains zero, and if $a(t) = 4$, there is a nonzero probability that $a(t+\tau) = 0$ if $\tau > 0$. Hence the Q matrix for $a(t)$ is

$$Q = \begin{bmatrix} 0 & 0 \\ q & -q \end{bmatrix}, \quad q > 0 \quad (20)$$

and the initial distribution for $a(t)$ is given by

$$P = \begin{bmatrix} p \\ 1-p \end{bmatrix}, \quad 0 \le p \le 1.$$

To solve this problem, it is observed that (17) and (18) provide a solution to similar problems in which the performance measure is the conditional expectation of $\int_{t_0}^{T} [x(\tau)^2 + v(\tau)^2]d\tau$. In this latter case the gain constants of the controller are solutions of

$$\dot{\bar{K}}_1(t) = -\bar{K}_1(t)^2 + 1$$
$$\dot{\bar{K}}_2(t) = -8\bar{K}_2(t) - \bar{K}_2{}^2(t) + qK_2(t) - qK_1(t) + 1 \quad (21)$$
$$\bar{K}_1(T) = \bar{K}_2(T) = 0.$$

For any finite value of T, (21) gives the best value for the feedback gain. As a strictly formal solution to the problem with $T = \infty$, one might employ the stationary solutions to (21) as T approaches infinity, i.e.,

$$\bar{u}^*(t,x,r(t)) = \lim_{T\to\infty} \bar{K}_j(t)x, \quad \text{if } (t,r(t))\epsilon[j].$$

If this is done, it follows immediately that

$$\lim_{T\to\infty} \bar{K}_1(t) = -1$$

and

$$\bar{u}^*(t, x, r(t)) = -x, \quad \text{if } a(t) = 0.$$

This is precisely the solution which would have been obtained in a deterministic problem if $a(t) = 0$ and eqs. (9-109) and (9-110) of [6] were used to evaluate the optimal controller. One would expect this result since if $a(t) = 0$, the problem becomes intrinsically "nonrandom."

If $a(t)$ is in its transient state, however, the situation is significantly different. Here it follows that

$$\lim_{T\to\infty} \bar{K}_2(t) = -4 + \frac{q}{2} - \sqrt{(4 - q/2)^2 + 1 + q}. \quad (22)$$

If $q = 0$, then $\lim_{T\to\infty} \bar{K}_2(t) = -4 - \sqrt{17}$. This is once again the "deterministic" solution. On the other hand, if $q = 8$, an anomaly occurs which does not appear in the deterministic problem for $\lim_{T\to\infty} \bar{K}_2(t) = -3$. Returning to (19), it now becomes clear that, if $a(t) = 4$, the closedloop system has "negative damping" and, in fact, for every real number R

$$\Pr\left\{\int_{t_0}^{\infty} [x(\tau; t_0, x_0, \bar{u}^*)^2 + \bar{u}^*(\tau, x(\tau; t_0, x_0, \bar{u}^*), r(\tau))^2]\, d\tau > R\right\} > 0.$$

In spite of this, the system is weakly asymptotically stable because

$$\lim_{t\to\infty} x(t;\, t_0,\, x_0,\, \bar{u}^*) = 0$$

for all initial conditions. Basically, the anomaly in the stochastic controller is occasioned by the fact that stabilization of the system when $a(t) = 4$ requires a lot of control "effort." Consequently, if one expects the transition in $a(t)$ to occur soon, only the "degree" of instability is maintained within bounds.

VI. Conclusions

The feedback control of a class of linear stochastic systems has been studied. It has been shown that for these systems the stochastic maximum principle derived in [4] must be modified by introducing additional terms into the equation for the adjoint variable. In [7] similar terms appeared in the context of determining the optimal strategies for a differential game. It is interesting to observe, however, that in [7] it can be shown that the equation for $p(t, x)$ reduces to that derived by Kushner if there are no constraints on the range of $v(t)$. In the problem discussed here this reduction in complexity is no longer possible. While the same results would be obtained for the systems studied in Section IV with either equation for $p(t, x)$, further investigation has provided examples of systems for which this simplification is not possible [8].

Comparing the results of this paper with those obtained in other studies of systems with random parameters reveals some apparent similarities. A related work is [9], one part of which explores the applicability of dynamic programming to systems in which the parameter processes contain terms involving the time derivative of a Poisson process. There are fundamental differences between the problem solved here and that of [9], but it might appear that some of these differences could be obviated by the simple expedient of augmenting the dimension state vector to include the parameter processes. This is not completely satisfactory since the parameter process need not be observed by the controller here while the whole state vector is observable in [9]. A more fundamental difference between (1) of this paper and eq. (1.1) of [9] relates to the probability of transition of the parameters. For example, suppose one of the parameters of (1) is described by

$$\frac{d}{dt}\, a(t) = z(t)a(t) \tag{23}$$

where (23) is to be interpreted in the sense of [9]. There are a number of deficiencies with this model in the problem posed here. The most obvious is that the mean transition rate of $a(t)$ is equal to the intensity of the $z(t)$ process and is thus independent of $a(t)$. In (2), however, the transition rate does depend upon $a(t)$.

Although the problem solved in [9] is distinct from that described here, it is interesting to observe that the differential equation for the gain coefficients in the optimal controller (see [9, eq. (3.17)]) is similar in form to (14). Due to the fact that the incremental transition probabilities were independent of the instantaneous state of the parameter process in [9], all of the expectations in eq. (3.17) are t-measurable and as a consequence, there was no need to derive equations similar to (17) of this paper.

Appendix

An outline of the proof of Theorem 1 is provided here. The proof is based upon stochastic versions of lemmas proved by Bridgeland in his investigation of feedback control of deterministic plants [10]. There are a number of measure theoretic complexities entering into the proof which will be neglected here. The interested reader will find a complete exposition in [5].

Let $(\Omega, \mathcal{A}, P)$ be the probability space on which $[A(t), B(t)]$ and $r(t)$ are defined. Denote $[0, T] \times R^n$ by B. Let μ be Lebesque measure on B. Denote by Γ the set of all functions on $B \times R^k$ to R^m such that with probability one, (4) has a unique solution which is continuous in the pair (t_0, x_0), continuable to all of $[0, T]$, and for fixed (t, t_0) satisfies a Lipschitz condition with respect to x_0 in every bounded region of R^n.

Suppose that in Γ there exists an element $\bar{u}^*$ satisfying (6). Further suppose that $\bar{u}_x{}^*$ exists everywhere except perhaps on some Borel set $M \subset B$ of μ measure zero, and that $\bar{u}^*$ satisfies a Lipschitz condition with respect to x. Then for fixed t_0 it can be shown that a unique solution to (7) exists a.s. for almost all $x_0 \epsilon R^n$.

If anomalous behavior on sets of measure zero are neglected, the proof of the theorem would proceed as follows.

Theorem 1

If $\bar{u}\epsilon\Gamma$, then $E\{H(t, x, \bar{u}^*)|t, x, r(t)\} \geq E\{H(t, x, \bar{u})|t, x, r(t)\}$, a.s.

Proof: Define the minimum cost functional as follows:

$$V(t, x, \omega) = \int_t^T L(\tau, x(\tau; t, x, \bar{u}^*), \bar{u}^*(\tau, x(\tau; t, x, \bar{u}^*), r(\tau)))d\tau, \quad \text{a.s.}$$

Suppose there exists $(t_0, x_0, r(t_0))$ and a $\bar{u}\epsilon\Gamma$ such that

$$E\{H(t_0, x_0, \bar{u}^*) - H(t_0, x_0, \bar{u})|t_0, x_0, r(t_0)\} < 0.$$

Define $\pi(t; t_0, x_0, \omega, \bar{u})$ to be

$$\pi(t; t_0, x_0, \omega, \bar{u}) = V(t, x(t; \bar{u}), \omega) + \int_{t_0}^t L(\tau, x(\tau; \bar{u}), \bar{u}(\tau, x(\tau; \bar{u}), r(\tau)))d\tau, \quad \text{a.s.}$$

where $x(t; \bar{u}) = x(t; t_0, x_0, \bar{u})$.

Then from Lemma 1 of [10]

$$\frac{d}{dt}\pi(t;t_0,x_0,\omega,\bar{u}) = \frac{d}{dt} V(t,x(t;\bar{u}),\omega) + L(t,x(t;\bar{u}),\, \bar{u}(t,x(t;u),r(t))), \quad \text{a.s.}$$

Again using Lemma 1 of [10]

$$\frac{d}{dt} V(t, x(t; \bar{u}), \omega) = -L(t, x(t; \bar{u}), \bar{u}^*(t, x(t; \bar{u}), r(t))) + \int_t^T \frac{\partial}{\partial t} L(\tau, x(\tau; t, x(t; \bar{u}), \bar{u}^*), \bar{u}^*(\tau, x(\tau; t, x(t; \bar{u}), \bar{u}^*) r(\tau))) d\tau, \quad \text{a.s.}$$

Using the chain rule,

$$\frac{\partial}{\partial t} L(\tau, x(\tau; t, x(t; \bar{u}), \bar{u}^*), \bar{u}^*(\tau, x(\tau; t, x(t; \bar{u}), \bar{u}^*), r(\tau))) = [L_x + L_v \bar{u}_x^*] \frac{\partial}{\partial t} x(\tau;t,x(t;\bar{u}),\bar{u}^*), \quad \text{a.s.}$$

Employing (4) and (7) and by interchanging orders of differentiation, one can verify directly that the right-hand side of the preceding equation is simply

$$\frac{\partial}{\partial \tau}\left[p(\tau, x(\tau; t, x(t; \bar{u}), \bar{u}^*))^T \frac{\partial}{\partial t} x(\tau; t, x(t; \bar{u}), \bar{u}^*)\right], \quad \text{a.s.}$$

Consequently,

$$\frac{d}{dt}\pi(t; t_0, x_0, \bar{u}) = H(t, x(t; \bar{u}), \bar{u}^*) - H(t, x(t; \bar{u}), \bar{u}), \quad \text{a.s.}$$

By choice of $(t_0, x_0, r(t_0))$ there exists a $\Delta > 0$ such that

$$E\{\pi(t_0 + \Delta; t_0, x_0, \bar{u}) | t_0, x_0, r(t_0)\} < E\{\pi(t_0; t_0, x_0, \bar{u}) | t_0, x_0, r(t_0)\}, \quad \text{a.s.}$$

Let

$$\bar{u}_1 = \begin{cases} \bar{u}, & \text{if } 0 \le t \le t_0 + \Delta \\ \bar{u}^*, & \text{if } t_0 + \Delta < t \le T. \end{cases}$$

Clearly $\bar{u}_1 \epsilon \Gamma$ and

$$\pi(T; t_0, x_0, \bar{u}_1) - \pi(T; t_0, x_0, \bar{u}^*) = \int_{t_0}^T \frac{d}{d\tau}\pi(\tau; t_0, x_0, \bar{u}_1) d\tau = \pi(t_0 + \Delta; t_0, x_0, \bar{u}) - \pi(t_0; t_0, x_0, \bar{u}).$$

Hence

$$E\left\{\int_{t_0}^T L(\tau, x(\tau; t_0, x_0, \bar{u}_1), \bar{u}_1(\tau, x(\tau; t_0, x_0, \bar{u}_1), r(\tau))) d\tau | t_0, x_0, r(t_0)\right\} < E\{V(t_0, x_0, \omega) | t_0, x_0, r(t_0)\}, \quad \text{a.s.}$$

This is a contradiction to (6).

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