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Parametric method for spacecraft trajectory tracking control problem with stochastic thruster fault

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Abstract: This study deals with the spacecraft trajectory tracking control problem with stochastic thruster fault of the chaser spacecraft, whose aim is to make the output of the Markov jump linear system track the output of a given reference model. A new succinct general complete parametric expression for the tracking controller is established. Considering the actuator malfunction, a set of Clohessy–Wiltshire equations with Markov jump parameters are proposed to describe the relative motion while the target spacecraft into a circular orbit. A linear matrix inequality method is presented to make the Markov jump system stochastically stable and to guarantee the input constraints. Based on the theory of the generalised Sylvester equations, a parametric method is established for the model reference tracking problem. For certain systems, the proposed algorithm contains extra degrees of freedom in parametric design, which can be used to achieve some additional performance. The final numerical simulation results show the effectiveness of the proposed method.

1 Introduction

In recent years, with the development of control theory and space technology, there has been a tremendous interest in spacecraft trajectory tracking control, which is the foundation of many space missions, such as repairing [1], rendezvous and docking [2, 3], space intercepting [4], in-orbit monitoring and satellite networking [5, 6]. The planning of a space mission often begins with the design of the spacecraft trajectory, and analysing its stability, control strategies and feasibility. In general, most of the trajectory tracking missions can be regarded as proximity relative orbital manoeuvring control problems, which can be described by autonomous non-linear differential equations whose linearised equations are known as Clohessy–Wiltshire (C–W) equations [7] while the target orbit is a circle or Tschauner–Hempel (T–H) equations [8] when the target orbit is elliptical.

The basic requirement of trajectory tracking is to make the closed-loop system stable and to have the ability to track the given command, which can be translated into a model reference tracking control problem. Model tracking control is an important research field in control theory, and it has many practical applications arising in areas such as the control of mechanical systems, aircraft and air traffic control, and radar antenna tracking control, see [9–14]. A three-stage method for solving the optimal tracking problem for switched systems is proposed in [12]. By combining model-reference mechanism with robust adaptive radial basis function shown in [13], a fault-tolerant model-following controller for non-linear systems is presented. Liu and Jia [14] investigate the adaptive consensus problem of linear multi-agent systems with partially unknown parameters and bounded disturbances, under the guidance of an active leader with a reference input signal. In [15], a model tracking control logic which consists of both the feed-forward of the reference model system and the feedback of the controlled plant states is presented, and general complete parametric expressions for the controllers formed by the control logic are given. Based on the results of eigenstructure assignment in linear systems presented in [16], the complete parametric representations of the controllers are obtained. Referring to the control logic in [15, 16], a new method and parametric expression for the controller is presented to solve the matrix equations related to the model output tracking problem.

In practice, malfunctions of spacecraft thruster are frequently encountered because of the harsh working environment, therefore designing a controller to maintain spacecraft stability and acceptable performance, despite stochastic faults of system actuator, is a critical issue for aerospace engineers [17]. The trajectory tracking system with stochastic actuator faults can be regarded as a Markov jump system, which has attracted a lot of researchers in the past several decades since many practical systems can be modelled by stochastic systems with Markov jumping parameters [18]. Moreover, many different approaches have been proposed to stabilise the Markov linear systems, see [19–21]. Li *et al.* [19] present that the optimal control problem for continuous-time linear systems subject to Markovian jumps in the parameters relies on the study of a countably infinite set of coupled algebraic Riccati equations. In [20], Boukas deals with the tracking problem for the Markov jump systems with external finite energy disturbance. In [21], the stability and robust state feedback stabilisation problems for a class of non-linear discrete-time descriptor Markov jump systems with parameter uncertainties are investigated. However, this paper on model reference tracking control of Markov jump systems is still relatively few.

In this paper, the trajectory tracking problem with stochastic actuator faults while the target spacecraft in a circular orbit is considered. With three independent continuous control accelerations (or thrusts) being used as the control signals to the C–W equations, the problem can be formulated as a model reference tracking control problem of Markov jump linear system. The aim is to make the output of the linear jump system track the output of a given reference model which can provide a desired trajectory. A model reference output tracking controller is established, and it consists of two parts: one is the feedback of the controlled plant states, the other is the feed-forward of the states of the reference model system. A linear matrix inequality (LMI) method is proposed to design the state feedback controller to maintain stability, and two additional LMI conditions are presented to guarantee the input constraints. Based on some matrix analysis theories about generalised inverse matrix and a complete parametric solution to the non-homogeneous generalised Sylvester equations (GSEs) shown in [16], a more succinct general complete parametric expression for the controller is presented in this paper. Then a new effective algorithm to solve this kind of model reference output tracking problem is established. The algorithm contains extra degrees of freedom in parametric design, and it can be used to cope with some additional performance.

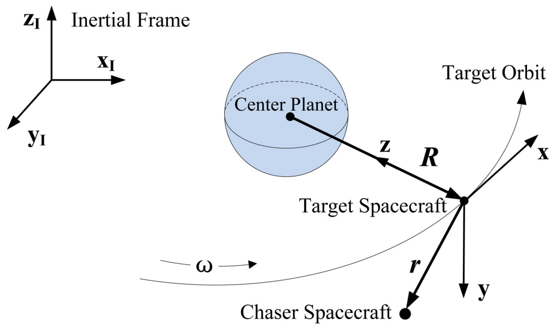

Fig. 1 *Orbit coordinate of target spacecraft and related definitions*

The rest of this paper is organised as follows. In Section 2, the proximity relative orbital tracking problem is formulated. Section 3 obtains the feedback controller, feed-forward controller and the algorithm for solving the tracking problem. In Section 4, the designed controllers are applied to the spacecraft flying around mission and the simulation results show the validity of the proposed methodology to handle the tracking problem. Concluding remarks follow in Section 5.

2 Problem formulation

The relative position of the target spacecraft and the chaser spacecraft is shown in Fig. 1, where $\boldsymbol{r}$ is the vector from the target spacecraft to the chaser spacecraft and $\boldsymbol{R}$ is the vector from the centre of gravity to the target spacecraft. Assuming that the target spacecraft are not actuated by thrusters, and the external forces (e.g. solar pressure, air drag, higher gravity terms etc.) on the chaser and the target are identical. In the case of the spacecraft engines working well, the relative motion of the chaser spacecraft in the inertial frame is [22]

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\mu\left(\frac{\boldsymbol{R}+\boldsymbol{r}}{|\boldsymbol{R}+\boldsymbol{r}|^3} - \frac{\boldsymbol{R}}{|\boldsymbol{R}|^3}\right) + \boldsymbol{a}_\mathrm{f}, \tag{1}$$

where μ is the gravity constant and $\boldsymbol{a}_\mathrm{f}$ is the acceleration vector due to thrust forces on the chaser spacecraft.

Considering the target-orbital rotating coordinate system $x-y-z$, as shown in Fig. 1, the origin is fixed at the mass centre of the target and the z-axis points in the nadir direction. The y-axis is normal to the orbital plane, opposite the angular momentum vector, and the x-axis completes the right-hand system. The symbol ω is the orbital rate of the rotating coordinate system. Assume that the target spacecraft is in a circular orbit and the distance between the chaser and the target is much smaller than the distance between the target and the centre of the gravity field (i.e. $R \gg r$). Then, with the notation $\boldsymbol{r} = [x\,y\,z]^\mathrm{T}$ and $\boldsymbol{a}_\mathrm{f} = [a_\mathrm{f}^x\,a_\mathrm{f}^y\,a_\mathrm{f}^z]^\mathrm{T}$, the kinetic equations describing the relative motion can be linearised as

$$\begin{cases} \ddot{x} = 2\omega\dot{z} + a_\mathrm{f}^x \\ \ddot{y} = -\omega^2 y + a_\mathrm{f}^y \\ \ddot{z} = -2\omega\dot{x} + 3\omega^2 z + a_\mathrm{f}^z \end{cases}, \tag{2}$$

which are known as the C–W equations [7].

Choosing the state vector, the input vector and the output vector as

$$\xi(t) = [x(t)\,y(t)\,z(t)\,\dot{x}(t)\,\dot{y}(t)\,\dot{z}(t)]^\mathrm{T},$$
$$u(t) = [a_\mathrm{f}^x(t)\,a_\mathrm{f}^y(t)\,a_\mathrm{f}^z(t)]^\mathrm{T},\ y(t) = [x(t)\,y(t)\,z(t)]^\mathrm{T},$$

then the C–W equations (2) can be written as the following constant system:

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + Bu(t) \\ y(t) = C\xi(t) \end{cases}, \tag{3}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2\omega \\ 0 & -\omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\omega^2 & -2\omega & 0 & 0 \end{bmatrix},\quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Considering the case of the spacecraft thrusts failure and the random occurrence of the abrupt changes in thrust forces, this kind of stochastic fault can be described as a continuous-time discrete-state Markov process in general. Then the relative motion equations can be written as the following stochastic Markov jump system:

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + B(\gamma(t))u(t) \\ y(t) = C\xi(t) \end{cases}, \tag{4}$$

where $\{\gamma(t), t \ge 0\}$ is a Markov random process that represents the jump mode of the system and takes values in a finite set $S = \{1, 2, \ldots, N\}$, and $B(\gamma(t))$ are known input matrices which represent various types of actuator failures. The stationary mode transition probabilities of the system are given by

$$\Pr\{\gamma(t+h) = j \,|\, \gamma(t) = i\} = \begin{cases} \lambda_{ij}h + o(h), i \ne j \\ 1 + \lambda_{ij}h + o(h), i = j \end{cases},$$

where $h > 0$, $\lim_{h\to 0^+}((o(h))/h) = 0$, and the transition probability rate matrix Λ can be written as

$$\Lambda = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1N} \\ \vdots & \ddots & \vdots \\ \lambda_{N1} & \cdots & \lambda_{NN} \end{bmatrix}, \tag{5}$$

where $\lambda_{ij} \ge 0$ for $i \ne j$ and $\lambda_{ii} = -\sum_{j=1, i \ne j}^{N} \lambda_{ij}$.

For system (4), a state feedback control law $u(t) = K_\mathrm{s}(\gamma(t))\xi(t)$ is designed, which has the same transition properties with the stochastic system. For convenience, let

$$B_i = B(\gamma(t) = i),\quad K_{si} = K_\mathrm{s}(\gamma(t) = i),$$

when the system is in mode $\gamma(t) = i$ (unless otherwise specified, all subscript i below represents this meaning). Such that the closed-loop system is given by

$$\dot{\xi}(t) = A_i^*\xi(t), \tag{6}$$

where

$$A_i^* = A + B_iK_{si}.$$

Introduce a set of matrices $\{F_i\}$ to describe the failure channel of the thrust forces, in form of

$$F_i = \mathrm{diag}(f_{i1}, f_{i2}, f_{i3}), \tag{7}$$

where $0 \le f_{ij} \le 1$, $i = 1, 2, \ldots, N$. When the system is in mode $\gamma(t) = i$, define that $f_{ij} = 0$ means the actuator component in No. j-direction is in completely outage case ($j = 1, 2, 3$ denotes x, y, z-direction, respectively), $f_{ij} = 1$ means operation normally, and

$0 < f_{ij} < 1$ means in partial degradation case. Then the input matrix B_i can be written as

$$B_i = BF_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ f_{i1} & 0 & 0 \\ 0 & f_{i2} & 0 \\ 0 & 0 & f_{i3} \end{bmatrix}. \tag{8}$$

Definition 1: The continuous-time Markov jump stochastic system (6) is stochastically stable (SS) if for every initial condition $\xi(0) = \xi_0$ and $\gamma(0) = \gamma_0 \in S$, there holds

$$\int_0^\infty \mathrm{E}\{\, \| \xi(t) \|^2 \}\mathrm{d}t < \infty,$$

where the symbol 'E' represents the mathematical expectation.

This paper mainly concentrates on the trajectory tracking control problem of spacecraft. The basic requirement is to make the closed-loop system stable and have the ability to track the given command, thus it can be attributed to a model reference output tracking problem. Considering the general situation, let the given command signal generated by the following linear continuous-time reference model:

$$\begin{cases} \dot{\xi}_m(t) = A_m \xi_m(t) \\ y_m(t) = C_m \xi_m(t) \end{cases}, \tag{9}$$

where $\xi_m(t) \in \mathbb{R}^p$ and $y_m(t) \in \mathbb{R}^m$ are, respectively, the state vector and the output vector of the reference model, and A_m and C_m are known matrices of appropriate dimensions.

In view of the system (4) is a Markov jump system, the objective is to design a controller that force the output vector ($y(t)$) of the system to track the output vector ($y_m(t)$) of the reference model in mean-square sense, which is to satisfy the following tracking requirement:

$$\int_0^\infty \mathrm{E}\{\, \| y(t) - y_m(t) \|^2 \}\mathrm{d}t < \infty, \tag{10}$$

for arbitrary initial values $\xi(0)$, $\xi_m(0)$ and $\gamma(0)$.

Then the purpose of this paper is to find a solution to the problem below.

Problem 1 Given the continuous-time Markov jump system (4) and the linear constant reference model (9), design a controller in form of

$$u(t) = K_i \xi(t) + K_{mi} \xi_m(t), \tag{11}$$

where $K_{mi} = K_m(\gamma(t) = i)$, such that

i. The closed-loop system (6) is SS.
ii. The output vector ($y(t)$) of the system is able to track the output vector ($y_m(t)$) of the reference model in mean-square sense [i.e. satisfy the tracking requirement (10)].

Before closing this section, recall the following lemma which will be used in the proof of the main results.

Lemma 1 (Schur complement lemma): The LMI

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} < 0,$$

is equivalent to

$$A_{22} < 0, \quad A_{11} - A_{12} A_{22}^{-1} A_{21} < 0,$$

where $A_{11} = A_{11}^{\mathrm{T}}$, $A_{22} = A_{22}^{\mathrm{T}}$ and $A_{12} = A_{21}^{\mathrm{T}}$.

3 Controller design

According to Problem 1, two parts need to be done: the design of the feedback controller to make the system stable, and the feed-forward compensator to make the tracking requirement be satisfied. First of all, the existence conditions of the controller (11) should be given.

3.1 Existence conditions of the controller

Theorem 1: Problem 1 has a solution if system (4) is stochastically stabilisable and there exist matrices G_i and H_i ($i = 1, 2, \ldots, N$) satisfying the following matrix equations:

$$\begin{cases} AG_i + B_i H_i = G_i A_m, & \text{(12a)} \\ CG_i = C_m. & \text{(12b)} \end{cases}$$

In this case, the feed-forward gain matrices K_{mi} can be chosen as

$$K_{mi} = H_i - K_i G_i, \tag{13}$$

while K_i are any real matrices such that the closed-loop system (6) is SS.

Proof: Let

$$\begin{cases} \delta\xi(t) = \xi(t) - G_i \xi_m(t) \\ \delta u(t) = u(t) - H_i \xi_m(t), \\ \delta y(t) = y(t) - y_m(t) \end{cases} \tag{14}$$

then

$$\begin{aligned} \delta\dot{\xi}(t) &= \dot{\xi}(t) - G_i \dot{\xi}_m(t) = A\xi(t) + B_i u(t) - G_i A_m \xi_m(t) \\ &= A\,\delta\xi(t) + B_i\,\delta u(t) + (AG_i + B_i H_i - G_i A_m)\xi_m(t), \\ \delta y(t) &= C\xi(t) - C_m \xi_m(t) = C\,\delta\xi(t) + (CG_i - C_m)\xi_m(t). \end{aligned}$$

If the conditions (12a) and (12b) are satisfied, it is able to show

$$\begin{cases} \delta\dot{\xi}(t) = A\,\delta\xi(t) + B_i\,\delta u(t) \\ \delta y(t) = C\,\delta\xi(t) \end{cases}. \tag{15}$$

Notice that systems (15) and (4) have the same structure, hence, for arbitrary state feedback control law $u(t) = K_i \xi(t)$ which can stabilise the system (4), the state feedback control law

$$\delta u(t) = K_i\,\delta\xi(t) \tag{16}$$

can also stabilise system (15). That is to say the system

$$\begin{cases} \delta\dot{x}(t) = (A + B_i K_i)\,\delta\xi(t) \\ \delta y(t) = C\,\delta\xi(t) \end{cases}$$

is SS, hence

$$\int_0^\infty \mathrm{E}\{\, \| \delta y(t) \|^2 \}\mathrm{d}t = \int_0^\infty \mathrm{E}\{\, \| y(t) - y_m(t) \|^2 \}\mathrm{d}t < \infty.$$

The tracking requirement of Problem 1 is satisfied. Combining (14) and (16), the controller can be written as

$$u(t) = K_i \xi(t) + (H_i - K_i G_i)\xi_m(t).$$

Then it can be obtained that $K_{mi} = H_i - K_i G_i$. The proof is finished. □

3.2 Feedback controller design

Lemma 2 ([23]): The continuous-time Markov jump stochastic system (6) is SS if and only if there exists a set of symmetric positive-definite matrices $P = (P_1, P_2, \dots, P_N) > 0$, such that the following continuous-time stochastic Lyapunov equations hold

$$A_i^{*\mathrm{T}} P_i + P_i A_i^* + \sum_{j=1}^{N} \lambda_{ij} P_j = -Q_i, \quad i = 1, 2, \dots, N, \tag{17}$$

for arbitrarily given symmetric positive-definite matrices $(Q_1, Q_2, \dots, Q_N)$.
Further, the following corollary can be obtained.

Corollary 1: System (6) is SS if there exists a set of symmetric positive-definite matrices $P = (P_1, P_2, \dots, P_N) > 0$, such that the following relation holds for each $i \in S$:

$$A_i^{*\mathrm{T}} P_i + P_i A_i^* + \sum_{j=1}^{N} \lambda_{ij} P_j < 0\,. \tag{18}$$

Consider the control input constraints in practice, gives

$$\| u(t) \|_2 \le u_{\max}, \tag{19}$$

where $u_{\max}$ represents the maximum acceleration that the spacecraft thruster can provide.

By using the Schur complement lemma, the following theorem is proposed to show the conditions which guarantee the stability and the input constraints of the closed-loop jump system.

Theorem 2: The continuous-time Markov jump stochastic system (6) is SS and satisfies the control input constraints described in (19), if there exists a given positive number ρ, a set of matrices W_i and symmetric positive-definite matrices X_i, $i \in S$, which meet the following conditions for each $i \in S$:

$$\begin{bmatrix} \rho I & \xi^{\mathrm{T}}(0) \\ \xi(0) & X_i \end{bmatrix} \ge 0, \tag{20}$$

$$\begin{bmatrix} u_{\max}^2 I & W_i \\ W_i^{\mathrm{T}} & \dfrac{1}{\rho} X_i \end{bmatrix} \ge 0, \tag{21}$$

$$\begin{bmatrix} \Phi_i & \Gamma_i \\ \Gamma_i^{\mathrm{T}} & \Delta_i \end{bmatrix} < 0, \tag{22}$$

where

$$\Phi_i = AX_i + X_i A^{\mathrm{T}} + B_i W_i + W_i^{\mathrm{T}} B_i^{\mathrm{T}} + \lambda_{ii} X_i,$$
$$\Gamma_i = \left[\sqrt{\lambda_{i1}} X_i \cdots \sqrt{\lambda_{i,i-1}} X_i \sqrt{\lambda_{i,i+1}} X_i \cdots \sqrt{\lambda_{iN}} X_i\right],$$
$$\Delta_i = \mathrm{diag}\left[X_1 \quad \cdots \quad X_{i-1} \quad X_{i+1} \quad \cdots \quad X_N\right].$$

In this case, the feedback gain matrices are given by

$$K_i = W_i X_i^{-1}. \tag{23}$$

Proof: It follows from Corollary 1 that the system (6) is SS if there exist symmetric positive-definite matrices P_i, such that the following holds for each $i \in S$:

$$(A + B_i K_i)^{\mathrm{T}} P_i + P_i (A + B_i K_i) + \sum_{j=1}^{N} \lambda_{ij} P_j < 0\,.$$

Due to P_i is symmetric positive-definite matrix, there exists inverse matrix P_i^{-1}, satisfying $P_i^{-1} = (P_i^{-1})^{\mathrm{T}}$. Pre-multiplying and post-multiplying by P_i^{-1} both sides of the aforementioned inequality gives

$$P_i^{-1} A^{\mathrm{T}} + P_i^{-1} K_i^{\mathrm{T}} B_i^{\mathrm{T}} + A P_i^{-1} + B_i K_i P_i^{-1} + P_i^{-1} \left(\sum_{j=1}^{N} \lambda_{ij} P_j\right) P_i^{-1} < 0\,.$$

By defining $X_i = P_i^{-1}$ and $W_i = K_i P_i^{-1}$, the above inequality is turned into the following form:

$$X_i A^{\mathrm{T}} + W_i^{\mathrm{T}} B_i^{\mathrm{T}} + AX_i + B_i W_i + X_i \left(\sum_{j=1}^{N} \lambda_{ij} X_j^{-1}\right) X_i < 0\,. \tag{24}$$

By using Lemma 1, Schur complement lemma, inequality (24) can be equivalently arranged into the form of inequality (22).

There exists a positive number ρ satisfying the initial condition $\xi^{\mathrm{T}}(0) P_i \xi(0) \le \rho$ for every $i \in S$. Then while $t > 0$ it follows that:

$$\xi^{\mathrm{T}}(t) P_i \xi(t) \le \xi^{\mathrm{T}}(0) P_i \xi(0) \le \rho\,. \tag{25}$$

By using the Schur complement lemma, inequality (25) can be equivalently arranged into the form of inequality (20).

In view of (19), gives

$$u^{\mathrm{T}}(t) u(t) = \xi^{\mathrm{T}}(t) K_i^{\mathrm{T}} K_i \xi(t) \le u_{\max}^2. \tag{26}$$

Combining (25) and (26), and using the Schur complement lemma, it can be obtained that

$$\begin{bmatrix} u_{\max}^2 I & K_i \\ K_i^{\mathrm{T}} & \dfrac{1}{\rho} P_i \end{bmatrix} \ge 0. \tag{27}$$

Pre-multiplying and post-multiplying by $\mathrm{diag}(I, X_i)$ both sides of the aforementioned inequality, inequality (21) can be obtained. The proof is then completed. □

It is easy for Matlab LMI toolbox to solve the inequalities.

3.3 Feed-forward compensator design

The purpose of feed-forward compensator is to make the tracking requirement of Problem 1 be satisfied. According to Theorem 1 the existence conditions of the controller (11) have be obtained, and the next most major problem is to find a set of matrices G_i and H_i ($i = 1, 2, \dots, N$) satisfying the matrix equations (12a) and (12b).

Based on some relative theories in matrix analysis [24], the following lemma is proposed.

Lemma 3: The linear matrix equations (12b) have a solution if and only if there exists a generalised inverse matrix C^- of matrix C, satisfying

$$CC^- C_m = C_m, \tag{28}$$

then the general complete solution of (12b) can be written as

$$G_i = C^- C_m + (I - C^- C) Y_i, \tag{29}$$

where $Y_i \in \mathbb{R}^{n \times p}$ are arbitrary matrices.

For systems covered in this paper, condition (28) is obviously established. Substituting (29) into (12a), gives

$$\begin{aligned} A(C^- C_m + (I - C^- C) Y_i) + B_i H_i \\ = (C^- C_m + (I - C^- C) Y_i) A_m, \end{aligned}$$

which can be written as

$$(I - C^{-}C)Y_i A_m - A(I - C^{-}C)Y_i = B_i H_i + (AC^{-}C_m - C^{-}C_m A_m).$$

Then (12a) and (12b) can be written in the following first-order non-homogeneous GSEs form

$$EY_i A_m - \hat{A}Y_i = B_i H_i + F, \tag{30}$$

where the matrices $E, \hat{A}$ and F are the parameter matrices in the form of

$$\begin{aligned} E &= (I - C^{-}C), \quad \hat{A} = A(I - C^{-}C), \\ F &= AC^{-}C_m - C^{-}C_m A_m. \end{aligned} \tag{31}$$

By establishing the general complete parametric expressions for matrices Y_i and H_i satisfying the GSEs (30), the matrices G_i can be computed by (29), then a complete parametric solution of (12a) and (12b) is obtained.

Lemma 4 ([25]): If the matrix pair $(\hat{A}, B_i)$ is controllable, there exists a unimodular matrix $Q_i^*(s) \in \mathbb{R}^{(n+r)\times(n+r)}[s]$ satisfying the Smith form reduction

$$\begin{bmatrix} sE - \hat{A} & -B_i \end{bmatrix} Q_i^*(s) = \begin{bmatrix} I_n & 0 \end{bmatrix}. \tag{32}$$

Partition the unimodular matrix $Q_i^*(s)$ as follows:

$$Q_i^*(s) = \begin{bmatrix} U_i(s) & N_i(s) \\ T_i(s) & D_i(s) \end{bmatrix}, \tag{33}$$

where $U_i(s) \in \mathbb{R}^{n\times n}[s]$, $T_i(s) \in \mathbb{R}^{r\times n}[s]$, $N_i(s) \in \mathbb{R}^{n\times r}[s]$, and $D_i(s) \in \mathbb{R}^{r\times r}[s]$. Then let the four polynomial matrices be written in the following form:

$$\begin{cases} U_i(s) = \sum_{j=0}^{\varphi} U_{ij}s^j, \ U_{ij} \in \mathbb{R}^{n\times n} \\ T_i(s) = \sum_{j=0}^{\varphi} T_{ij}s^j, \ T_{ij} \in \mathbb{R}^{r\times n} \\ N_i(s) = \sum_{j=0}^{\varphi} N_{ij}s^j, \ N_{ij} \in \mathbb{R}^{n\times r} \\ D_i(s) = \sum_{j=0}^{\varphi} D_{ij}s^j, \ D_{ij} \in \mathbb{R}^{r\times r} \end{cases}. \tag{34}$$

Combining with Lemma 4, the following lemma is presented for obtaining the general solution to the non-homogeneous generalised Sylvester equations (GSEs) (30).

Lemma 5 ([16]): When the pair $(\hat{A}, B_i)$ is controllable, the general complete solution (Y_i, H_i) to the non-homogeneous GSEs (30) is given by

$$\begin{cases} Y_i = \sum_{j=0}^{\varphi} N_{ij}Z_i A_m^j + \sum_{j=0}^{\varphi} U_{ij}FA_m^j \\ H_i = \sum_{j=0}^{\varphi} D_{ij}Z_i A_m^j + \sum_{j=0}^{\varphi} T_{ij}FA_m^j \end{cases}, \tag{35}$$

where $Z_i \in \mathbb{R}^{r\times p}$ are arbitrary parameter matrices.

By using Lemmas 3–5, the following result about the general solution to the matrix equations (12a) and (12b) is immediately obtained.

Theorem 3: Let the matrices $E, \hat{A} \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times p}$ be given by (31), and $U_i(s) \in \mathbb{R}^{n\times n}[s]$, $T_i(s) \in \mathbb{R}^{r\times n}[s]$, $N_i(s) \in \mathbb{R}^{n\times r}[s]$, and $D_i(s) \in \mathbb{R}^{r\times r}[s]$ be in the form of (34) and given by (32) and (33). If the matrix pairs $(\hat{A}, B_i)$ are controllable, then a complete parametric solution of the matrix equations (12a) and (12b) can be obtained by (29) and (35) as follows:

$$\begin{cases} G_i = E\sum_{j=0}^{\varphi} N_{ij}Z_i A_m^j + E\sum_{j=0}^{\varphi} U_{ij}FA_m^j + C^{-}C_m \\ H_i = \sum_{j=0}^{\varphi} D_{ij}Z_i A_m^j + \sum_{j=0}^{\varphi} T_{ij}FA_m^j \end{cases}, \tag{36}$$

where $Z_i \in \mathbb{R}^{r\times p}$ are arbitrary parameter matrices which represent the degree of freedom in the solution.

Then the feed-forward compensation gain matrices can be obtained by (13). Notice that the matrices G_i and H_i contain the arbitrary parameter matrices Z_i, in some practical applications, Z_i can be utilised as the design degrees of freedom, and optimised to achieve some additional performance.

3.4 Algorithm for solving the controller

The purpose of Problem 1 is to design a controller which make the closed-loop system (6) stable in the mean-square sense and track the desired trajectory. Given the reference model coefficient matrices A_m, C_m, the transition rate matrix Λ, the input matrices B_i which represent various types of actuator failures for every $i = 1, 2, \ldots, N$, then the following algorithm for solving Problem 1 can be presented.

Algorithm 1:

Step 1:Solve the LMI equations described by (20)–(22). If a set of W_i and X_i can be found, then the feedback gain matrices are given by $K_i = W_i X_i^{-1}$; otherwise the feedback control law does not exist.

Step 2:Judge whether the matrix pair $(\hat{A}, B_i)$ [$\hat{A}$ is given by (31)] is controllable. If it is, solve the SFR described by (32) and go to next step; otherwise the feed-forward control law does not exist or the algorithm is failed.

Step 3:Compute G_i and H_i by Theorem 3, then the feed-forward compensation gain matrices are given by $K_{mi} = H_i - K_i G_i$, and the parametric form of the model reference tracking controller is obtained as follows: $u(t) = K_i\xi(t) + K_{mi}\xi_m(t), \ i = 1, 2, \ldots, N$.

Step 4:In view of the arbitrary parameter matrices Z_i contained in the parametric form, select appropriate optimising index to optimise the controller parameters, and obtain the final controller expression.

4 Simulation results

In this section, a numerical simulation will be used to demonstrate the effectiveness of the proposed approach to circular orbit model reference tracking control.

Suppose that the target spacecraft is in the geosynchronous orbit, a circular orbit on the Earth's equator plane with the same period as Earth rotation. The orbital parameters are as follows: the radius $R = 4.2241 \times 10^7\,\text{m}$, the period $T = 86164\,\text{s}$ and the angular rate $\omega = 7.2921 \times 10^{-5}\,\text{rad/s}$. Give the initial state vector (including the initial relative position and the initial relative velocity) as $\xi(0) = [1000\,\text{m}, 1000\,\text{m}, 1000\,\text{m}, 1\,\text{m/s}, 1\,\text{m/s}, 1\,\text{m/s}]^{\text{T}}$

For the matrices $\{F_i\}$ describing the failure channel of the thrust forces in form of (7), consider three mode: mode 1, $f_{11} = f_{12} = f_{13} = 1$, means all actuator operating normally; mode 2, $f_{21} = 0.5, f_{22} = f_{23} = 1$, means the actuator component in the x-direction is in partial degradation case; and mode 3, $f_{32} = 0.5, f_{31} = f_{33} = 1$, means the actuator component in the y-direction is in partial degradation case. The jumping input matrices B_i are given as follows:

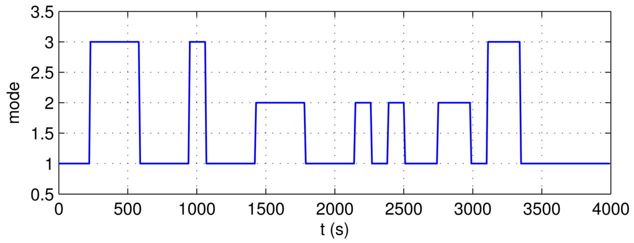

Fig. 2 *Chaser spacecraft's Markov jumping mode*

$$B_1=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad B_2=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\\0.5&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad B_3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&0\\0&0.5&0\\0&0&1\end{bmatrix},$$

and the transition probability rate matrix

$$\Lambda=\begin{bmatrix}-0.4&0.2&0.2\\0.5&-0.5&0.0\\0.5&0.0&-0.5\end{bmatrix}. \tag{37}$$

In this paper, interests are put on chaser spacecraft flying around the target spacecraft. Assuming that the trajectory of the chaser spacecraft is a circular orbit centred on the target spacecraft, and the fly-around plane is coincident with the x– y plane. The radius $R_c = 500\,\mathrm{m}$ and the period $T_c = 1000\,\mathrm{s}$, and then the angular rate $\omega_c = 2\pi/T_c = 0.002\pi\,\mathrm{rad/s}$. To achieve the target, the reference model in form of (9) with $p = 3$ are designed as follows:

$$A_m=\begin{bmatrix}0&0.002\pi&0\\-0.002\pi&0&0\\0&0&-0.01\end{bmatrix},\quad C_m=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \tag{38}$$

with the initial value $\xi_m(0) = [250\sqrt{2}\,\mathrm{m},\ 250\sqrt{2}\,\mathrm{m},\ 0]^{\mathrm{T}}$.

Assuming that the maximum acceleration which the spacecraft thruster can provide is $u_{\max} = 0.1\,\mathrm{m/s^2}$. Based on Algorithm 1, the main results of each step are presented as follows:

i. By using Theorem 2, W_i and X_i are obtained by Matlab LMI toolbox. The feedback gain matrices are given by $K_i = W_iX_i^{-1}$.
ii. Obviously, the matrix pairs ($\hat{A}$, B_i) [$\hat{A}$ is given by (31)] are controllable. Let B_i in form of (8), solve the SFR described by (32) and then the solution can be obtained as (see equation below)
iii. Regardless of the optimisation problem, the arbitrary parameter matrices Z_i can be generated by Matlab randomly. For convenience, let $Z_i = I_3, i = 1, 2, 3$. By using Theorem 3, G_i and H_i can be obtained. Then compute the feed-forward compensation gain matrices by $K_{mi} = H_i - K_iG_i$.

Finally, the model reference controller in the form of $u(t) = K_i\xi(t) + K_{mi}\xi_m(t),\ (i = 1, 2, 3)$ can be obtained.

Set the simulation time $T_{\text{simu}} = 4000\,\mathrm{s}$, the jump period $T_{\text{jump}} = 120\,\mathrm{s}$, a random mode changing curve based on the transition probability rate matrix (37) is shown in Fig. 2. This Markov jumping mode shows the failure or abrupt changes in the thrust forces of the chaser spacecraft.

By simulation, the state trajectories of the reference model with coefficient matrices (38) and the closed-loop system consisting of the jump C–W equations and the model reference tracking controller described by (11) are obtained. The simulation results of the tracking task are shown in Fig. 3. The position and velocity change relative to the target spacecraft are shown as Figs. 3*a* and *b*, respectively, the trajectories in target-orbital rotating coordinate system are depicted in Fig. 3*c*, and the tracking errors are recorded in Fig. 3*d*.

The simulation results from Figs. 3*a–d* show that: even in the case of thruster fault, the controlled system can also track the reference signal rapidly and smoothly. It follows that the closed-loop system is asymptotically stable, and it is clearly that the chaser spacecraft can track the reference model accurately. In this numerical simulation example, the tracking mission is accomplished at about $T_s = 3200\,\mathrm{s}$.

With the initial states $\xi(0)$ and $\xi_m(0)$, the magnitude of the control signals are recorded in Fig. 4, which shows the jumping characteristics of the controller. The assumed maximum acceleration thruster can provide is $u_{\max} = 0.1\,\mathrm{m/s^2}$, and it is clearly to see that the desired thrust constraint is satisfied.

The spacecraft trajectory tracking control problem mainly suffers from initial state (include position and velocity) errors and external environmental disturbance in practice. In order to better validate the performance of the proposed approach, a small initial errors case has been considered. The initial state vector is set as $\xi(0) = [100\,\mathrm{m},\ 100\,\mathrm{m},\ 100\,\mathrm{m},\ 0.1\,\mathrm{m/s},\ 0.1\,\mathrm{m/s},\ 0.1\,\mathrm{m/s}]^{\mathrm{T}}$. Similarly, solve the controller based on Algorithm 1, and give the simulation results. The dynamic curves are basically similar with those figures above. Therefore, to save space, only the figure of tracking errors shown in Fig. 5 is given.

By comparing Figs. 3*d* and 5*a*, it can be seen clearly that the transition time in this simulation is much less than that in the case of large initial error. Simulation results show that the proposed method is effective in both cases (small initial errors and large initial errors), and the smaller the initial error allows better tracking performance.

For better representing the control accuracy, Fig. 5*b* about the partial enlargement of tracking errors is given. As it can be seen from the figure, under the condition of stochastic thruster fault, closed-loop system controlled by the designed controller is still able to maintain a desirable control accuracy.

The simulation results show that: under the condition of stochastic thruster fault, the controlled system can also track the

$$U_i(s)=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\end{bmatrix},\quad N_i(s)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

$$T_i(s)=\begin{bmatrix}\frac{s}{f_{i1}}&0&-\frac{2\omega}{f_{i1}}&\frac{1}{f_{i1}}&0&0\\0&\frac{s}{f_{i2}}&0&0&\frac{1}{f_{i2}}&0\\\frac{2\omega}{f_{i3}}&0&\frac{s}{f_{i3}}&0&0&\frac{1}{f_{i3}}\end{bmatrix},\quad D_i(s)=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

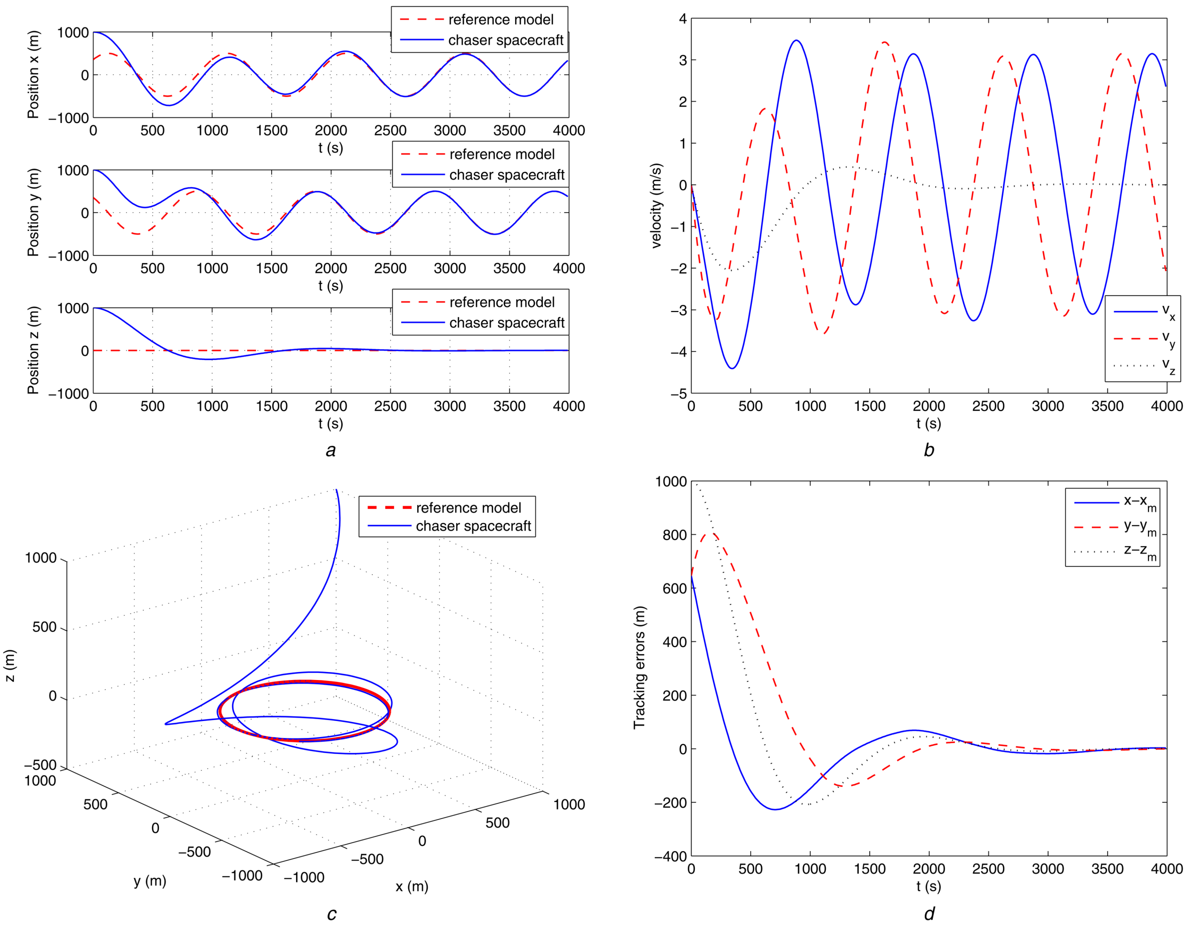

Fig. 3 *Tracking result*
(a) Relative position, ***(b)*** Relative velocity, ***(c)*** Trajectories of reference model and spacecraft, ***(d)*** Tracking errors

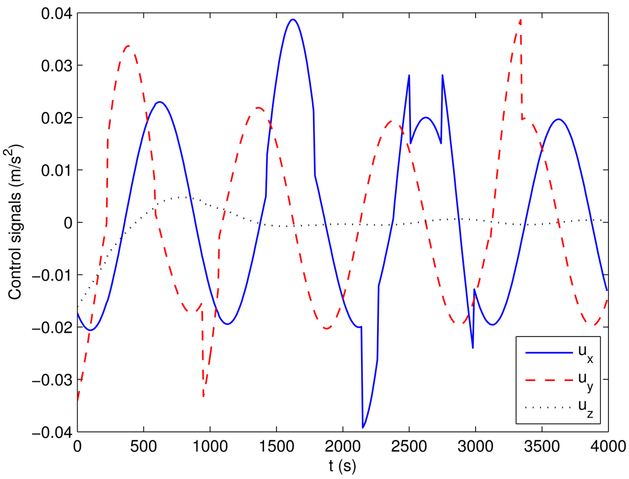

Fig. 4 *Control signals*

reference signal rapidly and smoothly, and satisfy the performance requirement of maximum thrust constraints and other indicators.

In order to indicate the better performance of the proposed method, a conventional PID method with saturation constraint is proposed. The system parameters and the initial errors are the same as the proposed approach. The calculation process is omitted, and the simulation results are given directly in Fig. 6. Compared with Fig. 5, it follows that the overshoot in Fig. 6 is much bigger, and the transition time in Fig. 6 is more than that in Fig. 5. In other words, the proposed method in this paper has better performance of smooth and fast response. In addition, the proposed parametric expression is succinct, and the approach is very convenient for programming. The free parameter matrices contained in the complete parametric solution brings an extra degree of freedom for controller design, which can be used to cope with some additional performance optimisation.

5 Conclusion

This paper studied the spacecraft trajectory tracking control problem with stochastic thruster fault of the chaser spacecraft. The main task is to make the closed-loop system stable and have the ability to track of the given command. With the target spacecraft into a circular orbit, the C–W equations with Markov jump parameters are proposed to describe the relative motion. An LMI method is presented to make the Markov jump system be asymptotically mean square stable and guarantee the input constraints. Based on the theory of the GSEs, a parametric method is established for the model reference tracking problem. The proposed algorithms have the extra degree of freedom in parametric design, and it can be used to cope with some robustness requirements. A numerical simulation of spacecraft flying around was developed to show the effectiveness of the proposed method.

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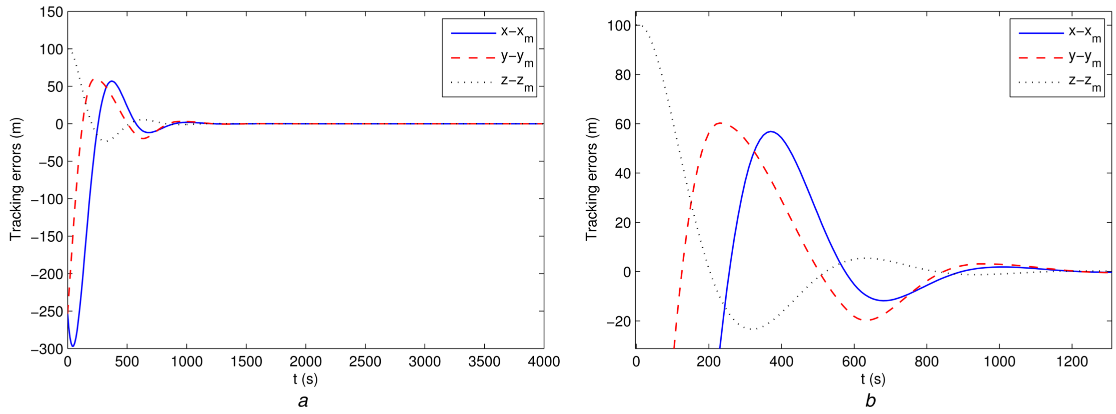

Fig. 5 *Tracking errors in the case of small initial errors*
(a) Tracking errors, ***(b)*** Partial enlargement of tracking errors

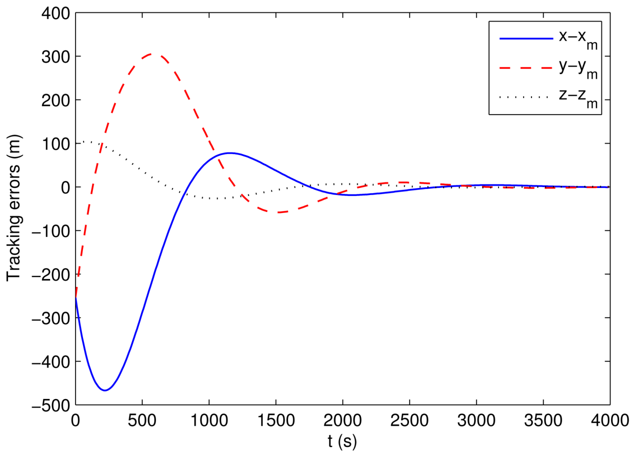

Fig. 6 *Tracking errors with PID method*

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