III. Concluding Remarks

A continuous-time observer, which converges in finite time, is accomplished by using the redundancy of two standard observers and a delay. What enables this result is that the individual two observers with state estimate $z_i(t)$ and state observation error $\varepsilon_i(t)$, $(i = 1, 2)$, respectively, give rise to the relations

$$\begin{aligned} z_1(t) &= x(t) + \varepsilon_1(t) \\ z_2(t) &= x(t) + \varepsilon_2(t) \\ z_1(t-D) &= x(t-D) + e^{-F_1 D} \cdot \varepsilon_1(t) \\ z_2(t-D) &= x(t-D) + e^{-F_2 D} \cdot \varepsilon_2(t) \end{aligned}$$

i.e., a set of four equations with four unknowns $x(t)$, $x(t-D)$, $\varepsilon_1(t)$, $\varepsilon_1(t)$. The state estimate $\hat{x}(t)$ is just taken to be the result, which arises from solving these equations for $x(t)$, given $z_i(t)$ and $z_i(t-D)$, $(i = 1, 2)$.

The convergence time D and the observer eigenvalues (resp. the observer gains H_i) are independent quantities to be chosen or designed. They have clearly a joint (filtering) effect on the state estimate. In particular, after the transient is over, one has from (2) that

$$\hat{x}(t) = [I_{n,n}, 0_{n,n}] \left[T, e^{FD}T\right]^{-1} \cdot \int_{t-D}^{t} e^{F(t-\tau)} \{Hy(\tau) + Gu(\tau)\} \, d\tau$$

i.e., the state estimate is generated using measurements from the finite interval $[t-D, t]$ only.

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Stochastic Stability of Jump Linear Systems

Yuguang Fang and Kenneth A. Loparo

***Abstract*—In this note, some testable conditions for mean square (i.e., second moment) stability for discrete-time jump linear systems with time-homogenous and time-inhomogenous finite state Markov chain form processes are presented.**

***Index Terms*—Jump linear systems, Kronecker product, Lyapunov equation, mean square stability, stability, stochastic systems.**

I. Introduction

In this note, second moment (mean square) stability for the jump linear system (1.1) whose form process $\{\sigma_k\}$ is a finite state time-homogenous or time-inhomogenous Markov chain is studied.

$$x_{k+1} = H(\sigma_k)x_k \tag{1.1}$$

A stochastic version of Lyapunov's second method is used to obtain a necessary and sufficient condition for second moment exponential stability if the probability transition matrix is periodic in time. This is a general result in which the results of Morozan [1] and Ji *et al.* [2] for the time-homogenous case and Krtolica *et al.* [3] can be recovered as special cases. In order to apply these results, a coupled system of Lyapunov equations needs to be solved for which Kronecker product techniques will be used and a very general sufficient condition is presented. For one-dimensional systems, this sufficient condition is also necessary.

A second moment stabilization problem for systems of type (1.1) is investigated by Ji *et al.* [2] and Feng *et al.* [4], where the equivalence between some second moment stability concepts were also proved. Mariton also studied stochastic controllability, observability, stabilizability and linear quadratic optimal control problems for continuous-time jump linear control systems, the details can be found in [6]. Krtolica *et al.* [3] applied the Kalman–Bertram decomposition to study closed-loop control systems with communication delays. The system is modeled as a jump linear system with an inhomogenous Markov chain and they obtained a necessary and sufficient condition for exponential stability. Wonham [9] systematically studied linear quadratic optimal control problems for these types of systems. Other work related to the stability of jump linear systems is summarized in [7].

Before we present the main results, some preliminaries are necessary. Suppose that $\{\sigma_k\}$ is a finite state Markov chain with state space $\underline{N}$, transition probability matrix $P = (p_{ij})_{N \times N}$ and initial distribution $p = (p_1, \ldots, p_N)$. For simplicity, assume that the initial state $x_0 \in \mathcal{R}^n$ is a (nonrandom) constant vector. Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space and let Ξ be the collection of all probabiltiy distribution on $\underline{N}$. Let $e_i \in \Xi$ be the initial distribution concentrated at the ith state, i.e., given by $P\{\sigma_0 = i\} = 1$. If properties depend on the choice of the initial distribution of the Markov form process $\{\sigma_k\}$, for each $\xi \in \Xi$, let P_ξ denote the probability measure for the Markov chain

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Y. Fang is with the Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611 USA.

K. A. Loparo is with the Department of Electrical Engineering and Computer Science, Olin Building, Case Western Reserve University, Cleveland, OH 44106-7070 USA.

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$\{\sigma_k\}$ induced by the initial distribution ξ. Also, E_ξ denotes expectation with respect to P_ξ and $\pi = (\pi_1, \ldots, \pi_N)$ is the unique invariant probability distribution for the Markov chain $\{\sigma_k\}$, when it exists. Details about Markov chains are given in [12]. Definitions of various stochastic stability concepts for jump linear systems are presented next.

Definition 1.1: Let Φ be a subset of Ξ. The jump linear system (1.1) with a Markovian form process $\{\sigma_k\}$ is said to be the following.

1) *(Asymptotically) second moment stable with respect to (w.r.t.)* Φ, if for any $x_0 \in R^n$ and any initial probability distribution $\psi \in \Phi$ of σ_k

$$\lim_{k\to\infty} E\left\{\|x_k(x_0,\omega)\|^2\right\} = 0$$

where $x_k(x_0,\omega)$ is a sample solution of (1.1) initial from $x_0 \in R^n$. If $\Phi = \Xi$, we simply say (1.1) is *asymptotically second moment stable*. Similar statements apply to the following definitions.

2) *Exponentially second moment stable w.r.t.* Φ, if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Phi$ of σ_k, there exist constants $\alpha, \beta > 0$ independent of x_0 and ψ such that

$$E\left\{\|x_k(x_0,\omega)\|^2\right\} \le \alpha \|x_0\|^2 e^{-\beta k}, \quad \forall k \ge 0.$$

3) *Stochastically second moment stable w.r.t.* Φ, if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Phi$ of σ_k,

$$\sum_{k=0}^{\infty} E\left\{\|x_k(x_0,\omega)\|^2\right\} < +\infty.$$

IV) *Almost surely (asymptotically) stable w.r.t.* Φ, if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Xi$ of σ_k

$$P\left\{\lim_{k\to\infty} \|x_k(x_0,\omega)\| = 0\right\} = 1.$$

In the case when $\{\sigma_k\}$ is an iid process with distribution $p = (p_1, \ldots, p_N)$, all the aforementioned definitions hold with $\Xi = \Phi = \{p\}$. □

The above definitions are consistent with those given in [2] and [4], and an appropriate "state" for the jump linear system is the joint process (x_k, σ_k), even though the initial distribution of the form process may be unknown. Thus, it is reasonable that the stability properties as defined are independent of the initial distribution. Of course, for a Markov chain with a single ergodic class, almost sure (sample path) stability depends only on the probability measure P_π induced by the initial distributuion π. Then, if the system is P_π-almost surely stable, it is almost surely stable (or P_ξ-almost surely stable for any $\xi \in \Xi$). However, this may not be the case for second moment stability.

II. Second Moment Stability

In this section, we study the second moment stability (or mean square stability) of the discrete-time jump linear system (1.1). As mentioned earlier, a stochastic version of Lyapunov's second method can be used to study stochastic stability. A natural candidate for a Lyapunov function is an appropriately chosen quadratic form. Morozan [5] showed that for a finite state time homogenous Markov chain $\{\sigma_k\}$ with probability transition matrix P, the system (1.1) is stochastically stable if and only if for any given positive matrices $Q(1), Q(2), \ldots, Q(N)$, there exists positive–definite matrices $P(1), P(2), \ldots, P(N)$ such that

$$\sum_{j=1}^{N} p_{ij} H^T(i) P(j) H(i) - P(i) = -Q(i), \quad i = 1, 2, \ldots, N. \tag{2.1}$$

This result is based on the choice of Lyapunov function $V(x_k, \sigma_k) = x_k^T P(\sigma_k) x_k$ where x_k is measurable with respect to the σ-algebra generated by $\sigma_{k-1}, \sigma_{k-2}, \ldots$, and the matrix $P(\sigma_k)$ depends only on σ_k. Another necessary and sufficient condition is given next.

Theorem 2.1: Suppose that $\{\sigma_k\}$ is a finite state time homogenous Markov chain with probability transition matrix P, then the system (1.1) is stochastically stable if and only if for any given positive matrices $S(1), S(2), \ldots, S(N)$, there exists positive–definite matrices $R(1), R(2), \ldots, R(N)$ such that

$$\sum_{j=1}^{N} p_{ij} H^T(j) R(j) H(j) - R(i) = -S(i), \quad i = 1, 2, \ldots, N. \tag{2.2}$$

Proof: This can be proved using the Lyapunov function $V(x_k, \sigma_k) = x_k^T R(\sigma_{k-1}) x_k$. □

Remark 1: The necessary and sufficient conditions given in (2.1) and (2.2) are equivalent.

Remark 2: From (2.2) and the theory of Lyapunov equations, the Schur stability (all eigenvalues inside the unit disk) of $\sqrt{p_{ii}}H(i)$ ($i \in \underline{N}$) is a necessary condition for second moment stability.

Equations (2.1) and (2.2) are referred to as coupled Lyapunov equations. It is not obvious which of these two necessary and sufficient conditions is better for practical applications. For the general finite state Markovian case, solving (2.1) and (2.2) requires solving N coupled matrix equations. However, for some special cases, Theorem 2.1 does provide an easier test for stochastic stability. This is summarized in the next result.

Corollary 2.2: Suppose that $\{\sigma_k\}$ is a finite state independent and identically distributed (iid) random sequence with probability distribution $\{p_1, p_2, \ldots, p_N\}$, then system (1.1) is second moment stochastically stable if and only if for some positive–definite matrix S there exists a positive–definite solution R to the following matrix equation:

$$\sum_{i=1}^{N} p_i H^T(i) R H(i) - R = -S.$$

Remark: For the iid case, (2.1) requires solving N coupled Lyapunov equations, which is more complicated than Corollary 2.2.

So far, we have only considered stochastic stability. This is not a limitation because Morozan [5] showed that second moment stability, second moment stochastic stability and exponential second moment stability of (1.1) with a time-homogenous finite state Markov chain $\{\sigma_k\}$ are equivalent. Furthermore, all of these imply almost sure (sample path) stability.

As an illustration, we apply Theorem 2.1 to the one-dimensional case.

Example 2.1: Suppose that $H(i) = a_i$ ($i \in \underline{N}$) are scalars, and define

$$A = \begin{pmatrix} p_{11}a_1^2 & p_{12}a_2^2 & \cdots & p_{1N}a_N^2 \\ p_{21}a_1^2 & p_{22}a_2^2 & \cdots & p_{2N}a_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ p_{N1}a_1^2 & p_{N2}a_2^2 & \cdots & p_{NN}a_N^2 \end{pmatrix}.$$

We want to find a necessary and sufficient condition for (1.1) to be second moment stable in the one-dimensional case. Before proceeding, Lemma A provides a result which is needed for this example: Here, the notation $A \ge_e B$ ($A \le_e B$) denotes elementwise inequalities and $\rho(A)$ denotes the spectral radius of the matrix A.

Lemma A [13, p. 493]: Given a matrix $A \ge_e 0$ and a vector $x >_e 0$ satisfying $\alpha x \le_e Ax \le_e \beta x$ for positive number α and β, then $\alpha \le \rho(A) \le \beta$. If $\alpha x <_e Ax$, then $\alpha < \rho(A)$. If $Ax <_e \beta x$, then $\rho(A) < \beta$. □

Necessary condition: Suppose that (1.1) is second moment stable, then from Theorem 2.1, for $S(1) = S(2) = \cdots = S(N) = 1$, there exist positive numbers $R(1), R(2), \ldots, R(N)$ such that

$$\sum_{j=1}^{N} p_{ij} a_j^2 R(j) - R(i) = -1 \quad (i = 1, 2, \ldots, N)$$

i.e.,

$$Ay - y = -c$$

where $y = (R(1), R(2), \ldots, R(N))^T$ and $c = (1, 1, \ldots, 1)^T$. Thus, we obtain $Ay = y - c <_e y$. Using Lemma A, we have $\rho(A) < 1$, i.e., A is Schur stable.

Sufficient condition: If A is Schur stable, i.e., $\rho(A) < 1$ with $U = (u_{ij})_{N\times N}$ and $u_{ij} = 1$, then for $\epsilon > 0$ sufficiently small, $\rho(A + \epsilon U) < 1$ and $A + \epsilon U$ is a positive matrix. From the Frobenius–Perron Theorem [13], there exists a positive vector $y >_e 0$ such that $(A + \epsilon U)y = \rho(A + \epsilon U)y$, i.e.,

$$Ay - y = \rho(A + \epsilon U)y - \epsilon Uy - y <_e y - \epsilon Uy - y = -\epsilon Uy.$$

Let $R(i) = y_i$ $(i = 1, 2, \ldots, N)$, which are positive numbers that satisfy

$$\sum_{j=1}^{N} p_{ij} a_j^2 R(j) - R(i) < 0, \quad (i = 1, 2, \ldots, N).$$

Then (2.2) is satisfied for this choice of $R(1), \ldots, R(N)$ where the positive numbers $S(1), \ldots, S(N)$ are suitably chosen. From Theorem 2.1, we conclude that (1.1) is second moment stable. Therefore (1.1) is second moment stable if and only if A is Schur stable. □

Krtolica *et al.* [3] obtained a necessary and sufficient condition for the second moment exponential stability of (1.1) with a time-inhomogenous finite state Markov chain form process $\{\sigma_k\}$. For any symmetric matrices A and B, $A \le B$ (or $A < B$) denotes that $B - A$ is a positive–semidefinite (or positive–definite) matrix. Let $\underline{m} = \{1, 2, \ldots, m\}$ for any integer m. Then, [3] showed that for a time-inhomogenous finite state Markov chain $\{\sigma_k\}$ with probability transition matrix $P = (p_{ij}(k))_{N\times N}$, the system (1.1) is exponentially second moment stable if and only if for some positive–definite matrix sequence $Q_k(1), Q_k(2), \ldots, Q_k(N)$ $(k = 0, 1, 2, \ldots)$ satisfying

$$0 < c_1 I \le Q_k(j) \le c_2 I \ (j = 1, 2, \ldots, N), \ \forall k \ge 0$$

for some positive constants c_1 and c_2, there exists positive–definite matrices $P_k(1), P_k(2), \ldots, P_k(N)$ such that

$$\sum_{j=1}^{N} p_{ij}(k+1) H^T(i) P_{k+1}(j) H(i) - P_k(i) = -Q_k(i), \ i \in \underline{N}, \quad \forall k \ge 0$$

where

$$0 < c_3 I \le P_k(i) \le c_4 I, \ \ i \in \underline{N}, \ \forall k \ge 0$$

for some positive constants c_3 and c_4. Using this result requires solving an infinite system of coupled matrix equations. If the positive–definite solutions $P_k(1), \ldots, P_k(N)$ converge as k goes to infinity, then a finite set of algebraic conditions can be obtained. If the probability transition matrix is periodic in k, the following testable condition results.

Theorem 2.3: Suppose that $\{\sigma_k\}$ is a finite state Markov chain with probability transition matrix $P = \Pi_k = (p_{ij}(k))$ satisfying $\Pi_{k+p} = \Pi_k$, then (1.1) is exponentially second moment stable if and only if for some positive–definite matrices $Q_1(j), Q_2(j), \ldots, Q_p(j)$ $(j \in \underline{N})$, there exists positive–definite matrices $P_1(j), P_2(j), \ldots, P_p(j)$ $(j \in \underline{N})$ such that

$$\begin{aligned}\sum_{j=1}^{N} p_{ij}(l) H^T(i) P_{l+1}(j) H(i) - P_l(i) &= -Q_l(i), \\ &\qquad l \in \underline{p-1}, \ i \in \underline{N} \\ \sum_{j=1}^{N} p_{ij}(p) H^T(i) P_1(j) H(i) - P_p(i) &= -Q_p(i), \\ &\qquad i \in \underline{N}.\end{aligned} \tag{2.3}$$

Theorem 2.3 provides practical testable conditions for exponential second moment stability. When the form process $\{\sigma_k\}$ is time-homogenous, that is, $p = 1$, (2.3) is equivalent to Morozan's [5] necessary and sufficient condition. Moreover, when p tends to infinity, (2.3) is equivalent to the results in [3] and may be used as an approximation to this more general case.

One approach to obtaining a solution to a Lyapuonv matrix equation is to use the Kronecker product. This also holds for coupled Lyapunov equations. Second moment stability of (1.1) with an iid form process $\{\sigma_k\}$ was first explored by Bellman [6] and generalized by many others [7]. For the basics of Kronecker products, the reader is referred to [8]. Let $A = (a_{ij})_{m\times n}$ be a real or complex matrix and define the linear operator vec$(\cdot)$ by

$$\text{vec}(A) = [a_{11}, a_{21}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn}]^T.$$

The following lemma will be used to develop the main results.

Lemma 2.4. [8]:

a) $\text{vec}(AX) = (I \otimes A)\text{vec}(X)$, $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$.

b) If $A_1 X B_1 + \cdots + A_k X B_k = C$, then

$$\left[B_1^T \otimes A_1 + \cdots + B_k^T \otimes A_k\right] \text{vec}(X) = \text{vec}(C).$$

c) $\text{vec}(AX + YB) = (I \otimes A)\text{vec}(X) + (B^T \otimes I)\text{vec}(Y)$. □

The following result is a general sufficient condition for exponential second moment stability for (1.1) with a finite state Markov chain form process.

Theorem 2.5: Suppose that $\{\sigma_k\}$ is a finite state Markov chain with probability transition matrix $\Pi_k = (p_{ij}(k))$, then (1.1) is exponentially second moment stable if the deterministic matrix product sequence $\{\prod_{i=1}^k A_i\}$ is exponentially convergent to the zero matrix, where the equation shown at the bottom of the page holds true.

Theorem 2.5 provides a test procedure for exponential second moment stability, from which testable conditions can be derived. The complexity of the stability problem is greatly reduced and the relationship between the system matrices $H(1), H(2), \ldots, H(N)$, the probability transition matrix Π_k and the second moment stability of the system is revealed. If Π_k is periodic in k, then A_k is also periodic with the same period. If $\{\sigma_k\}$ is time homogenous, A_k is a constant matrix. If Π_k can be approximated by a probability transition matrix Π, then A_k can be approximated by A. For all these three cases, simpler second moment stability criteria are obtained next.

$$A_k = \begin{pmatrix} H(1)\otimes H(1) & & & \\ & H(2)\otimes H(2) & & \\ & & \ddots & \\ & & & H(N)\otimes H(N) \end{pmatrix} \left(\Pi_{k-1}^T \otimes I\right).$$

Corollary 2.6:

a) Suppose that $\{\sigma_k\}$ is a time-homogenous finite state Markov chain with probability transition matrix $P = (p_{ij})$, then (1.1) is (stochastically, exponentially) second moment stable if the matrix

$$A = \text{diag}\{H(1) \otimes H(1), H(2) \otimes H(2), \ldots, H(N) \otimes H(N)\} \cdot \left(P^T \otimes I\right)$$

is Schur stable, i.e., all its eigenvalues are inside the unit circle in the complex plane.

b) Suppose that the probability transition matrix Π_k is periodic with period p, then the system (1.1) is exponentially second moment stable if $A_p A_{p-1} \cdots A_1$ is Schur stable.

c) Suppose that the probability transition matrix Π_k can be approximated by Π, which is also a probability transition matrix, then the system (1.1) is exponentially second moment stable if the matrix

$$A(\Pi) = \text{diag}\{H(1) \otimes H(1), H(2) \otimes H(2), \ldots, H(N) \otimes H(N)\} \cdot \left(\Pi^T \otimes I\right)$$

is Schur stable.

Remarks:

1) Part c) of Corollary 2.6 is an important result because in many practical applications, the probability transition matrix converges to a stationary matrix. In this case it is not necessary to test the stability of a deterministic time-varying system, or to test the definiteness of solutions of an infinite number of coupled Lyapunov equations as in [3]. It is only necessary to test the stability of a time-homogenous system where the probability transition matrix is replaced by its stationary limit.

2) Compared with Theorem 2.1, part a) of Corollary 2.6 provides a potential reduction in computations. Solving coupled Lyapunov equations using the Kronecker product approach requires representing matrices as expanded vectors and transforming the coupled Lyapunov equations into a linear equation with coefficient matrix B. The matrices $P(1), P(2), \ldots, P(N)$ are obtained as the solution of a linear matrix equation and the definiteness of $P(1), \ldots, P(N)$ determine the stability of the system. The matrix B is stable if and only if A is stable, so the eigenvalues of B determine the second moment exponential stability of (1.1).

3) We conjecture that the sufficient condition in a) is also necessary. However, we have not been able to give a rigorous proof for this. From Example 2.1, we know that for one dimensional jump linear systems, this conjecture is true. Corollary 2.7 establishes the result for the iid case.

□

Corollary 2.7: Suppose that $\{\sigma_k\}$ is a finite state iid form process with probability distribution $\{p_1, p_2, \ldots, p_N\}$, then (1.1) is exponentially second moment stable if and only if the matrix

$$A_0 = p_1 H(1) \otimes H(1) + p_2 H(2) \otimes H(2) + \cdots + p_N H(N) \otimes H(N)$$

is Schur stable.

III. Illustrative Examples

Example 3.1 [10]: Consider the one-dimensional jump linear system

$$x_{k+1} = a(\sigma_k) x_k \quad x_0 \text{ is given.}$$

Here, the form process is a 7-state Markov chain with the following probability transition matrix [12]:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ p_{21} & 0 & p_{23} & 0 & 0 & p_{26} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{53} & 0 & p_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & p_{72} & 0 & 0 & 0 & 0 & p_{77} \end{pmatrix}.$$

Here, {6} is an absorbing state, {3,4} is a communicating class and {1,2,5,7} are transient states. The problem is to find conditions for second moment stability.

A simple procedure is presented using Corollary 2.6 to obtain a necessary and sufficient condition for second moment stability. From Corollary 2.8, the test matrix A is given by the equation at the bottom of the page. It is easy to compute

$$\det(\lambda I - A) = \left(\lambda - p_{21} a^2(1) a^2(2)\right)\left(\lambda^2 - a^2(3) a^2(4)\right) \left(\lambda - p_{55} a^2(5)\right)\left(\lambda - a^2(6)\right)\left(\lambda - p_{77} a^2(7)\right).$$

A is Schur stable if and only if $p_{21} a^2(1) a^2(2) < 1$, $a^2(3) a^2(4) < 1$, $p_{55} a^2(5) < 1$, $a^2(6) < 1$ and $p_{77} a^2(7) < 1$, which is also a necessary and sufficient condition for (1.1) to be second moment stable. This is the same result as obtained in [10] using a different approach.

Example 3.2: Stability in each mode does not guarantee second moment stability. Consider the Schur matrices

$$H(1) = \begin{pmatrix} 0.5 & 10 \\ 0 & 0.5 \end{pmatrix} \quad H(2) = \begin{pmatrix} 0.5 & 0 \\ 10 & 0.5 \end{pmatrix}.$$

The probability transition matrix $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Choose $Q(1) = Q(2) = I$ and using this data in (2.1)

$$P(1) = \begin{pmatrix} 0.9981 & -0.0503 \\ -0.0503 & -0.0075 \end{pmatrix}$$

$$P(2) = \begin{pmatrix} -0.0075 & -0.0503 \\ -0.0503 & 0.9981 \end{pmatrix}$$

which are not positive–definite matrices. From (2.1), (1.1) is not second moment stable even though the mode matrices $H(1)$, $H(2)$ are Schur stable. The eigenvalues of the test matrix A in (a) of Corollary 2.6 are 0.25, 0.25, −0.25, −0.25, 0.0006, −0.0006, 0.4994 and −100.4994, hence, A is not Schur stable.

Assume that the form process is a two-state iid chain with the probability transition matrix $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$. Let $Q(1) = Q(2) = I$ in (2.1). The solution of the coupled Lyapunov equations is

$$P(1) = \begin{pmatrix} 0.9970 & -0.0807 \\ -0.0807 & -1.0212 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & p_{21} a^2(1) & 0 & 0 & 0 & 0 & 0 \\ a^2(2) & 0 & 0 & 0 & 0 & 0 & p_{72} a^2(2) \\ 0 & p_{23} a^2(3) & 0 & a^2(3) & p_{53} a^2(3) & 0 & 0 \\ 0 & 0 & a^2(4) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{55} a^2(5) & 0 & 0 \\ 0 & p_{26} a^2(6) & 0 & 0 & 0 & a^2(6) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^2(7) \end{pmatrix}.$$

$$P(2) = \begin{pmatrix} -1.0212 & -0.0807 \\ -0.0807 & 0.9970 \end{pmatrix}$$

which are not positive–definite, hence (1.1) is not second moment stable. The test matrix A in Corollary 2.6 has eigencalues 50.7451 and -49.75 and it is not Schur stable. Because the form process is iid, the simpler test criterion given in Corollary 2.2 can be used. Let $S = I$, the solution of the matrix equation in Corollary 2.2 is

$$R = \begin{pmatrix} -0.0121 & -0.0807 \\ -0.0807 & -0.0121 \end{pmatrix}$$

which is not positive–definite. From Corollary 2.2, (1.1) is not second moment stable. Corollary 2.7 can aslo be used to solve this problem. By direct computation, A_0 in Corollary 2.7 has eigenvalues: 0.25, -0.2451, -49.75 and 0.7451, thus A_0 is not Schur stable. From Corollary 2.7, (1.1) is not second moment stable.

Assume that the form process has the probability transition matrix $P = \begin{pmatrix} 0.2 & 0.8 \\ 0.1 & 0.9 \end{pmatrix}$. Solving the coupled Lyapunov equations in (2.1)

$$P(1) = \begin{pmatrix} 0.9787 & -0.4575 \\ -0.4573 & -9.0346 \end{pmatrix}$$
$$P(2) = \begin{pmatrix} -0.3512 & -0.0436 \\ -0.0436 & 0.9989 \end{pmatrix}$$

which are not positive–definite and (1.1) is not second moment stable. The test matrix A has eigenvalue 28.9686 and A is not Schur stable. This case is very interesting: From the probability transition matrix P, the system (1.1) stays in the stable mode 2 with greater probability. Intuitively, one might expect the system to be second moment stable. However, this is not the case as indicated by the computations. An explanation of this phenomenon is that second moment stability is an average property and even low probability events (such as switching to mode 1) can lead to instability. In fact, this can happen when $\{\sigma_k\}$ is iid. Choose $p_1 = 0.1$ and $p_2 = 0.9$, then the test matrix A_0 in Corollary 2.7 has eigenvalue 30.6422, thus A_0 is not Schur stable and (1.1) is not second moment stable. Also, for the system (1.1) with a two state iid chain having probability distribution (p_1, p_2), (1.1) is second moment stable if $0 \le p_1 \le 0.000\,03$ and (1.1) is not second moment stable if $0.000\,04 \le p_1 \le 0.999\,96$.

Example 3.3: Instability of individual modes does not imply second moment instability. Let

$$H(1) = \begin{pmatrix} 1 & -1 \\ 0 & 0.5 \end{pmatrix}, \quad H(2) = \begin{pmatrix} 0.5 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $\{\sigma_k\}$ be a two state Markov chain with probability transition matrix $P = \begin{pmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{pmatrix}$. The eigenvalues of the test matrix A in Corollary 2.6 are 0.5695, -0.2195, 0.5168, -0.2418, -0.25, 0.5, 0.5, and -0.25, thus A is Schur stable and from Corollary 2.6, (1.1) is second moment stable. Solving the coupled Lyapunov equations in (2.1) with $Q(1) = Q(2) = I$ gives

$$P(1) = \begin{pmatrix} 3.1429 & -2.2857 \\ -2.2857 & 4.6964 \end{pmatrix}$$
$$P(2) = \begin{pmatrix} 1.7143 & 0.5714 \\ 0.5714 & 5.2321 \end{pmatrix}$$

which are positive–definite. From (2.1), (1.1) is second moment stable.

Let the form process $\{\sigma_k\}$ be a time-inhomogenous two state Markov chain with the probability transition matrix

$$\Pi_k = \begin{pmatrix} 0.3 + e^{-(k+1)} & 0.7 - e^{-(k+1)} \\ 0.8 - \frac{\sin^2 k}{(k+2)^2} & 0.2 + \frac{\sin^2 k}{(k+2)^2} \end{pmatrix}.$$

Krtolica *et al.*'s [3] result requires solving an infinite system of matrix equations. Corollary 2.6 uses the steady-state probability transition matrix

$$P = \lim_{k\to\infty} \Pi_k = \begin{pmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{pmatrix}.$$

From our previous results, (1.1) with probability transition matrix P is second moment stable and from Corollary 2.6, the system (1.1) with the time-inhomogenous finite state Markov chain with probability transition matrix Π_k is exponentially second moment stable.

IV. Conclusion

This note studies the problem of mean square stability for discrete-time jump linear systems with a finite state Markov chain form process. In particular, a necessary and sufficient condition for the mean square stability of a jump linear system with a time-homogeneous finite-state Markov chain having a periodic probability transition matrix is presented and general testable sufficient conditions for mean square stability are also given. The mean square stabilization for the same class of systems is studied in a subsequent paper.

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