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Indefinite Stochastic LQ Control with Jumps*

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Abstract

This paper studies a stochastic linear quadratic (LQ) problem in the infinite time horizon with Markovian jumps in parameter values. In contrast to the deterministic case, the cost weighting matrices of the state and control are allowed to be indinifite here. When the generator matrix of the jump process – which is assumed to be a Markov chain – is known and time-invariant, the well-posedness of the indefinite stochastic LQ problem is shown to be equivalent to the solvability of a system of coupled generalized algebraic Riccati equations (CGAREs) that involves equality and inequality constraints. To analyze the CGAREs, linear matrix inequalities (LMIs) are utilized, and the equivalence between the feasibility of the LMIs and the solvability of the CGAREs is established. Finally, an LMI-based algorithm is devised to slove the CGAREs via a semidefinite programming, and numerical results are presented to illustrate the proposed algorithm.

Keywords: Stochastic LQ control, coupled generalized algebraic Riccati equations, linear matrix inequality, semidefinite programming, mean-square stability.

1 Introduction

In this paper, we consider indefinite stochastic linear quadratic (LQ) control with jumps in the following form:

$$\min E\left\{\int_0^{+\infty}\begin{bmatrix}x(t)\\u(t)\end{bmatrix}'\begin{bmatrix}Q(r_t) & L(r_t)\\L(r_t)' & R(r_t)\end{bmatrix}\begin{bmatrix}x(t)\\u(t)\end{bmatrix}dt\,\middle|\, r_0=i\right\},$$

$$\text{s.t.}\left\{\begin{array}{rl} dx(t)= & [A(r_t)x(t)+B(r_t)u(t)]dt\\ & +[C(r_t)x(t)+D(r_t)u(t)]dW(t),\\ x(0)= & x_0\in\mathbf{R}^n,\end{array}\right.$$

where r_t is a Markov chain taking values in $\{1,\cdots,l\}$, $W(t)$ is a Brownian motion independent of r_t, and $A(r_t)=A_i$, $B(r_t)=B_i$, $C(r_t)=C_i$, $D(r_t)=D_i$ $Q(r_t)=Q_i$, $R(r_t)=R_i$ and $L(r_t)=L_i$ when $r_t=i$ $(i=1,\cdots,l)$. Here the matrices A_i, etc. are given with appropriate dimensions. The Markov chain r_t has the transition probabilities given by:

$$\mathbf{P}\{r_{t+\Delta t}=j|r_t=i\}=\left\{\begin{array}{ll}\pi_{ij}\Delta t+o(\Delta t), & \text{if } i\neq j,\\ 1+\pi_{ii}\Delta t+o(\Delta t), & \text{else,}\end{array}\right. \quad (1)$$

where $\pi_{ij}\geq 0$ for $i\neq j$ and $\pi_{ii}=-\sum_{j\neq i}\pi_{ij}$.

The stochastic LQ control problem is one of the most fundamental tools in modern engineering. In most literature, it is a common assumption that the cost weighting matrix of control be positive definite (see [7, 4]). However, this assumption has been challenged by some recent works ([6, 3, 2]) that a class of stochastic LQ problems with indefinite control weights may still be sensible and well-posed. Note that this phenomenon may occur only when the diffusion coefficient of the system dynamics depends on the control, meaning that controls could or would influence the uncertainty scale in the system.

During the last decade, LQ control problems with jumps have been extensively studied (see [1, 13, 10, 11, 20]). However, the existing works usually set the diffusion coefficients as either 0 or $\sigma(r_t)(\neq 0)$ independent of the state or control. As a result, they have to assume, again, that the control cost weighting matrices be positive definite. To elaborate, take the special case when the diffusion term is absent. Assuming that the state weighting matrix in the cost is non-negative definite and the control weighting matrix is positive definite, the LQ control problem is automatically well-posed and can be solved via the system of coupled algebraic Riccati equations (CAREs):

$$\begin{array}{l}A_i'P_i+P_iA_i-(P_iB_i+L_i)R_i^{-1}(P_iB_i+L_i)'+Q_i\\ \quad+\sum_{j=1}^l\pi_{ij}P_j=0,\qquad i=1,\cdots,l.\end{array} \quad (2)$$

Moreover, it can be shown that this system has a solution $(P_1^*,\cdots,P_l^*)$ based on which the optimal control is represented as

$$u^*(t)=-\sum_{i=1}^l R_i^{-1}(P_i^*B_i+L_i)'x^*(t)\chi_{\{r_t=i\}}(t), \quad (3)$$

where $\chi(t)$ is the indicator function. However, in many real problems the analytical solutions to the CAREs (2) are very hard to obtain due to the large size of the problem. Several numerical algorithms have been therefore proposed for solving the coupled Riccati equations (see [19, 14]). Recently, an algorithm based on convex op-

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timization over linear matrix inequalities (LMIs):

$$\begin{cases} \left[\begin{array}{c|c} A_i'P_i+P_iA_i+Q_i+\sum_{j=1}^{l}\pi_{ij}P_j & P_iB_i+L_i \\ \hline B_i'P_i+L_i' & R_i \end{array}\right] \geq 0, \\ i=1,\cdots,l, \end{cases} \tag{4}$$

put forward by Ait Rami and El Ghaoui [1], successfully solves the CAREs (2) in polynomial time, using currently available software [9].

Now, if we extend the above special case to the indefinite LQ case to be studied in this paper, we must consider the following system of coupled generalized algebraic Riccati equations (CGAREs):

$$\begin{cases} A_i'P_i + P_iA_i + C_i'P_iC_i + Q_i + \sum_{j=1}^{l}\pi_{ij}P_j \\ \quad -(P_iB_i + C_i'P_iD_i + L_i)(R_i + D_i'P_iD_i)^{-1} \\ \quad \cdot(P_iB_i + C_i'P_iD_i + L_i)' = 0, \\ R_i + D_i'P_iD_i > 0, \qquad i = 1,\cdots,l. \end{cases} \tag{5}$$

If there exists a solution $(P_1^*,\cdots,P_l^*)$ to the above equation with $R_i + D_i'P_i^*D_i > 0$ $(i = 1,\cdots,l)$, then a possible optimal feedback control would be

$$\begin{aligned} u^*(t) = &-\textstyle\sum_{i=1}^{l}(R_i + D_i'P_i^*D_i)^{-1} \\ &\cdot(P_i^*B_i+C_i'P_i^*D_i+L_i)'x^*(t)\chi_{\{r_t=i\}}(t). \end{aligned} \tag{6}$$

However, there are some fundamental differences and difficulties with the CGAREs (5) compared to its special case CAREs (2). First, the equality constraint part of the CGAREs (5) is more complicated than its counterpart in CAREs (2) for the inverses now involve the unknown $(P_1,\cdots,P_l)$. Second, there exist l *additional* strictly positive definiteness constraints in the equations.

In this paper, we develop an analytical and computational approach to solving the CGAREs (5). The key idea is to utilize LMIs as a powerful tool to solve the problem in polynomial time based on solving a semidefinite programming (SDP) [5, 18]. Moreover, we show that, provided that the systems is stabilizing in the mean-square sense, our approach always yields the *maximal* solution to the CGAREs (5), which in turn guarantees that (6) is indeed an optimal feedback control.

This paper is an announcement of the main results in [12].

2 Problem Formulation and Preliminaries

2.1 Notation

We make use of the following basic notation in this paper:

$\mathbf{R}^n$:	n-dimensional real Euclidean space;
$\mathbf{R}^{n\times m}$:	the set of all $n \times m$ real matrices;
$\mathcal{S}^n$:	the set of all $n\times n$ symmetric matrices;
$\mathcal{S}_+^n$:	the subset of all nonnegative definite matrices of $\mathcal{S}^n$;
$\hat{\mathcal{S}}_+^n$:	the subset of all positive definite matrices of $\mathcal{S}^n$;
M' :	the transpose of any matrix M;
$M > 0$:	the symmetric matrix M is positive definite;
$M \geq 0$:	the symmetric matrix M is nonnegative definite;
$\mathbf{Tr}(M)$:	the trace of any square matrix M.

2.2 Problem Formulation

First of all, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbf{P})$ be a given filtered probability space where there live a standard one-dimensional Brownian motion $W(t)$ on $[0,+\infty)$ (with $W(0) = 0$) and a Markov chain $r_t \in \{1,2,\cdots,l\}$ with the generator $\Pi = (\pi_{ij})$, and $\mathcal{F}_t = \sigma\{W(s),r_s|0 \leq s \leq t\}$. The Brownian motion is assumed to be one dimensional only for simplicity; there is no essential difference for the multi-dimensional case. In addition, the process r_t and $W(t)$ are assumed to be independent throughout this paper. Define

$$L_2^{loc}(\mathbf{R}^k) = \left\{ \begin{array}{l} \phi(\cdot,\cdot) : [0,+\infty) \times \Omega \mapsto \mathbf{R}^k \mid \phi(\cdot,\cdot) \\ \text{is } \mathcal{F}_t\text{-adapted, Lebesgue measurable,} \\ \text{and } E\int_0^T |\phi(t,\omega)|^2dt < +\infty,\ \forall T \geq 0 \end{array} \right\}.$$

Consider the linear stochastic differential equation subject to Markovian jumps defined by

$$\begin{cases} dx(t) &= [A(r_t)x(t) + B(r_t)u(t)]dt \\ &\quad +[C(r_t)x(t) + D(r_t)u(t)]dW(t), \\ x(0) &= x_0 \in \mathbf{R}^n, \end{cases} \tag{7}$$

where $A(r_t) = A_i$, $B(r_t) = B_i$, $C(r_t) = C_i$ and $D(r_t) = D_i$ when $r_t = i$, while A_i, etc., $i = 1,2,\cdots,l$, are given matrices of suitable sizes. A process $u(\cdot)$ is called a control if $u(\cdot) \in L_2^{loc}(\mathbf{R}^k)$.

Definition 2.1 A control $u(\cdot)$ is called (mean-square) stabilizing with respect to (w.r.t.) a given initial state (x_0,i) if the corresponding state $x(\cdot)$ of (7) with $x(0) = x_0$ and $r_0 = i$ satisfies $\lim_{t\to+\infty} E[x(t)'x(t)] = 0$.

Definition 2.2 The system (7) is called (mean-square) stabilizable if there exists a feedback control $u(t) = \sum_{i=1}^{l}K_ix(t)\chi_{\{r_t=i\}}(t)$, where $K_1,\cdots,K_l$ are given matrices, which is stabilizing w.r.t. any initial state (x_0,i).

Next, for a given $(x_0, i) \in \mathbf{R}^n \times \{1, 2, \cdots, l\}$, we define the corresponding set of admissible controls:

$$\mathcal{U}(x_0, i) \triangleq \left\{ u(\cdot) \in L_2^{loc}(\mathbf{R}^{n_u}) \,\middle|\, \begin{array}{l} u(\cdot) \text{ is mean-square} \\ \text{stabilizing w.r.t. } (x_0, i) \end{array} \right\}.$$

It is easily seen that $\mathcal{U}(x_0, i)$ is a convex subset of $L_2^{loc}(\mathbf{R}^{n_u})$.

For each $(x_0, i, u(\cdot)) \in \mathbf{R}^n \times \{1, 2, \cdots, l\} \times \mathcal{U}(x_0, i)$, the optimal control problem is to find a control which minimizes the following quadratic cost associated with (7)

$$\begin{aligned} & J(x_0, i; u(\cdot)) \\ = \ & E\left\{ \int_0^{+\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(r_t) & L(r_t) \\ L(r_t)' & R(r_t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \,\middle|\, r_0 = i \right\}, \end{aligned} \tag{8}$$

where $Q(r_t) = Q_i$, $R(r_t) = R_i$ and $L(r_t) = L_i$ when $r_t = i$, while Q_i, etc., $i = 1, 2, \cdots, l$, are given matrices with suitable sizes. The value function V is defined as

$$V(x_0, i) = \inf_{u(\cdot) \in \mathcal{U}(x_0, i)} J(x_0, i; u(\cdot)). \tag{9}$$

Since the symmetric matrices $\begin{bmatrix} Q_i & L_i \\ L_i' & R_i \end{bmatrix}$, $i = 1, \cdots, l$, are allowed to be indefinite, the above optimization problem is referred to as an indefinite LQ problem. It should be noted that due to the indefiniteness the cost functional $J(x_0, i; u(\cdot))$ is not necessarily convex in $u(\cdot)$.

Definition 2.3 The LQ problem is called well-posed if

$$-\infty < V(x_0, i) < +\infty, \quad \forall x_0 \in \mathbf{R}^n, \quad \forall i = 1, \cdots, l. \tag{10}$$

A well-posed problem is called attainable (w.r.t. (x_0, i)) if there is a control $u^*(\cdot) \in \mathcal{U}(x_0, i)$ that achieves $V(x_0, i)$. In this case the control $u^*(\cdot)$ is called optimal (w.r.t. (x_0, i)).

The following two basic assumptions are imposed throughout this paper.

Assumption 2.1 The system (7) is mean-square stabilizable.

Assumption 2.2 The data appearing in the LQ problem (7) – (8) satisfy, for every i,

$$\begin{array}{l} A_i, C_i \in \mathbf{R}^{n \times n},\ B_i, D_i \in \mathbf{R}^{n \times n_u}, \\ Q_i \in \mathcal{S}^n,\ L_i \in \mathbf{R}^{n \times n_u},\ R_i \in \mathcal{S}^{n_u}. \end{array}$$

2.3 CGAREs and LMIs

Define the operator $\mathcal{R}_i$ by

$$\begin{aligned} & \mathcal{R}_i(X_1, \cdots, X_l) \\ \triangleq \ & A_i'X_i + X_iA_i + C_i'X_iC_i + Q_i + \textstyle\sum_{j=1}^l \pi_{ij}X_j \\ & -(X_iB_i + C_i'X_iD_i + L_i)(R_i + D_i'X_iD_i)^{-1} \\ & \cdot(B_i'X_i + D_i'X_iC_i + L_i'), \quad i = 1, \cdots, l. \end{aligned} \tag{11}$$

Associated with the stochastic LQ problem (7)–(8) there is a system of CGAREs:

$$\begin{cases} \mathcal{R}_i(P_1, \cdots, P_l) = 0, \\ R_i + D_i'P_iD_i > 0, \qquad i = 1, \cdots, l. \end{cases} \tag{12}$$

The key idea of this paper is to reformulate the CGAREs as LMIs, which is a powerful tool to treat the original LQ problem by using convex optimization techniques. Let us first introduce the general notion of LMIs [18].

Definition 2.4 Let symmetric matrices $F_0, F_1, \cdots, F_m \in \mathcal{S}^n$ be given. Inequalities consisting of any combination of the following relations

$$\begin{cases} F(x) \triangleq F_0 + \sum_{i=1}^m x_iF_i > 0, \\ F(x) \triangleq F_0 + \sum_{i=1}^m x_iF_i \geq 0, \end{cases} \tag{13}$$

are called LMIs with respect to the variable $x = (x_1, \ldots, x_m)' \in \mathbf{R}^m$.

The LMIs associated with the CGAREs (12) are

$$\begin{cases} \left[\begin{array}{c|c} \mathcal{M}_i & \mathcal{L}_i \\ \hline \mathcal{L}_i' & R_i + D_i'P_iD_i \end{array}\right] \geq 0, \\ R_i + D_i'P_iD_i > 0, \qquad i = 1, \cdots, l, \end{cases} \tag{14}$$

with respect to the variables $(P_1, \cdots, P_l) \in (\mathcal{S}^n)^l$, where $\mathcal{M}_i = A_i'P_i + P_iA_i + C_i'P_iC_i + Q_i + \sum_{j=1}^l \pi_{ij}P_j$ and $\mathcal{L}_i = P_iB_i + C_i'P_iD_i + L_i$.

2.4 Mean-square Stabilizability

Mean-square stabilizability is an important issue that needs to be addressed for the LQ problem in the infinite time horizon. The following lemma, originally proved in [8], relates the stabilizability of the system (7) to the feasibility of certain coupled Lyapunov inequalities that are essentially LMIs.

Lemma 2.1 ([8]) *The following properties are equivalent.*

(i) *System (7) is mean-square stabilizable.*

(ii) *There exist matrices $Y_1, \cdots, Y_l$ and symmetric matrices $X_1, \cdots, X_l$ such that*

$$\left[\begin{array}{c|c} M_i & C_iX_i + D_iY_i \\ \hline X_iC_i' + Y_i'D_i' & -X_i \end{array}\right] < 0, \quad i = 1, \cdots, l, \tag{15}$$

where $M_i = A_iX_i + X_iA_i' + B_iY_i + Y_i'B_i' + \sum_{j=1}^l \pi_{ij}X_j$. In this case the feedback $u(t) = \sum_{i=1}^l Y_iX_i^{-1}x(t)\chi_{\{r_t=i\}}(t)$ is stabilizing w.r.t. any initial (x_0, i).

The above result also gives an efficient numerical way of checking mean-square stabilizability by using LMIs.

3 Well-posedness of LQ Problem

Before looking for an optimal control of the LQ problem, we study its well-posedness via the feasibility of the associated LMIs. Now let us define a subset $\mathcal{P}$:

$$\mathcal{P} \triangleq \left\{ (P_1, \cdots, P_l) \in (\mathcal{S}^n)^l \,\middle|\, \begin{array}{l} \mathcal{R}_i(P_1, \cdots, P_l) \geq 0, \\ R_i + D_i' P_i D_i > 0, i = 1, \cdots, l \end{array} \right\}. \tag{16}$$

Theorem 3.1 *If $\mathcal{P} \neq \emptyset$, then the LQ problem (7) − (8) is well-posed. Moreover, we have*

(i) $V(x_0, i) \geq x_0' P_i x_0$, $\forall x_0 \in \mathbf{R}^n$, $\forall i = 1, \cdots, l$, $\forall (P_1, \cdots, P_l) \in \mathcal{P}$.

(ii) *If* $\mathcal{P} \cap (\mathcal{S}_+^n)^l \neq \emptyset$, *then* $V(x_0, i) \geq 0, \forall x_0 \in \mathbf{R}^n$, $\forall i = 1, \cdots, l$.

Corollary 3.1 *If the CGAREs (12) admit a solution, then the LQ problem (7) − (8) is well-posed.*

4 Solving CGAREs via LMIs

In this section, we develop analytical and computational approach to solving the CGAREs via the LMIs and the associated SDP. Set
$\mathcal{G} = \{(P_1, \cdots, P_l) \in (\mathcal{S}^n)^l | R_i + D_i' P_i D_i > 0,\ i = 1, \cdots, l\}$.

Definition 4.1 A solution $(P_1, \cdots, P_l) \in \mathcal{G}$ of the CGAREs (12) is called its maximal solution if for any $(\bar{P}_1, \cdots, \bar{P}_l) \in \mathcal{G}$ with $\mathcal{R}_i(\bar{P}_1, \cdots, \bar{P}_l) \geq 0$, it holds $P_i - \bar{P}_i \geq 0$, for $i = 1, \cdots, l$.

It is evident from the above definition that the maximal solution must be unique if it exists. We also show in this section that, provided that the system is mean-square stabilizable, our approach always yields the maximal solution to the CGAREs (12).

4.1 CGAREs vs. SDP and Its Dual

First of all, let us recall some definition and results about primal SDP problems and their duals.

Definition 4.2 Let a vector $c = (c_1, \cdots, c_m)' \in \mathbf{R}^m$ and matrices $F_0, F_1, \cdots, F_m \in \mathcal{S}^n$ be given. The following optimization problem

$$\left\{ \begin{array}{ll} \min & c'x, \\ \text{s.t.} & F(x) \equiv F_0 + \sum_{i=1}^m x_i F_i \geq 0, \end{array} \right. \tag{17}$$

is called an SDP. Moreover, the dual problem of the SDP (17) is defined as

$$\left\{ \begin{array}{ll} \max & -\mathbf{Tr}(F_0 Z), \\ \text{s.t.} & Z \in \mathcal{S}^n, \mathbf{Tr}(Z F_i) = c_i, i = 1, \cdots, m, Z \geq 0. \end{array} \right. \tag{18}$$

In this subsection, we pose an additional assumption that the interior of the set $\mathcal{P}$ is nonempty. Consider the following SDP problem

$$\begin{array}{ll} \max & \sum_{i=1}^l \mathbf{Tr}(P_i) \\ \text{s.t.} & \left\{ \begin{array}{ll} \left[\begin{array}{c|c} \mathcal{M}_i & \mathcal{L}_i \\ \hline \mathcal{L}_i' & R_i + D_i' P_i D_i \end{array} \right] \geq 0, & \\ P_i - P_i^0 \geq 0, & i = 1, \cdots, l, \end{array} \right. \end{array} \tag{19}$$

where $\mathcal{M}_i = A_i' P_i + P_i A_i + C_i' P_i C_i + Q_i + \sum_{j=1}^l \pi_{ij} P_j$ and $\mathcal{L}_i = P_i B_i + C_i' P_i D_i + L_i$.

Theorem 4.1 *The dual problem of* (19) *can be formulated as follows*

$$\begin{array}{ll} \max & -\sum_{i=1}^l [\mathbf{Tr}(Q_i S_i + L_i U_i - W_i P_i^0) + \mathbf{Tr}(U_i L_i + R_i T_i)], \\ \text{s.t.} & \left\{ \begin{array}{l} A_i S_i + S_i A_i' + C_i S_i C_i' + B_i U_i + U_i' B_i' \\ \quad + D_i U_i C_i' + C_i U_i' D_i' + D_i T_i D_i' \\ \quad + \sum_{j=1}^l \pi_{ji} S_j + W_i + I = 0, \\ \left[\begin{array}{cc} S_i & U_i' \\ U_i & T_i \end{array} \right] \geq 0, \quad W_i \geq 0, \quad i = 1, \cdots, l, \end{array} \right. \end{array} \tag{20}$$

where $(S_i, T_i, W_i, U_i) \in \mathcal{S}^n \times \mathcal{S}^{n_u} \times \mathcal{S}^n \times \mathbf{R}^{n_u \times n}$ *for every* i.

Theorem 4.2 *The dual problem* (20) *is strictly feasible if and only if the system* (7) *is mean-square stabilizable.*

We are now ready to present the existence of the solution of the CGAREs (12) via the SDP (19).

Theorem 4.3 *The optimal solution set of* (19) *is nonempty and any optimal solution* $(P_1^*, \cdots, P_l^*)$ *must satisfy the CGAREs* (12).

The following result indicates that any optimal solution of the primal SDP gives rise to a *stabilizing* control of the original LQ problem.

Theorem 4.4 *Let* $(P_1^*, \cdots, P_l^*) \in \mathcal{P}$ *be an optimal solution to the primal SDP* (19). *Then the feedback control* $u(t) = -\sum_{i=1}^l (R_i + D_i' P_i^* D_i)^{-1} (B_i' P_i^* + D_i' P_i^* C_i + L_i') x(t) \chi_{\{r_t = i\}}(t)$ *is stabilizing for the system* (7).

Theorem 4.5 *There exists a unique optimal solution to the SDP* (19), *which is also the maximal solution to the CGAREs* (12).

4.2 Regularization

In the previous subsection, we proved our main results under the assumption that the interior of $\mathcal{P}$ was nonempty. Now let us remove this assumption by a regularization argument.

For notational convenience, we rewrite the CGAREs (12) as follows

$$\left\{ \begin{array}{ll} \mathcal{R}_i(P, Q, R) = 0, & \\ R_i + D_i' P_i D_i > 0, & i = 1, \cdots, l, \end{array} \right. \tag{21}$$

where

$$P = (P_1, \cdots, P_l), Q = (Q_1, \cdots, Q_l), R = (R_1, \cdots, R_l),$$

and

$$\begin{array}{l} \mathcal{R}_i(P, Q, R) \\ \triangleq A_i' P_i + P_i A_i + C_i' P_i C_i + Q_i + \sum_{j=1}^l \pi_{ij} P_j \\ \quad -(P_i B_i + C_i' P_i D_i + L_i)(R_i + D_i' P_i D_i)^{-1} \\ \quad \cdot (B_i' P_i + D_i' P_i C_i + L_i'). \end{array}$$

Let us now present the main result of this paper.

Theorem 4.6 *Let* $Q \in (\mathcal{S}^n)^l$ *and* $R \in (\mathcal{S}^{n_u})^l$ *be given. The following are equivalent:*

(i) *There exists P^0 such that $\mathcal{R}_i(P^0, Q, R) \geq 0$ and $R_i + D_i' P_i^0 D_i > 0$, $\forall i = 1, \cdots, l$.*

(ii) *There exists a solution to the CGAREs* (21).

Moreover, when (i) *or* (ii) *holds, the CGAREs* (21) *has a maximal solution $\overline{P}$ which is the unique optimal solution to the following SDP problem*

$$\begin{array}{ll} \max & \sum_{i=1}^{l} \mathbf{Tr}(P_i) \\ \text{s.t.} & \left\{ \begin{array}{l} \left[\begin{array}{c|c} \mathcal{M}_i & \mathcal{L}_i \\ \hline \mathcal{L}_i' & R_i + D_i' P_i D_i \end{array} \right] \geq 0, \\ R_i + D_i' P_i D_i > 0, \qquad i = 1, \cdots, l, \end{array} \right. \end{array} \tag{22}$$

where $\mathcal{M}_i = A_i' P_i + P_i A_i + C_i' P_i C_i + Q_i + \sum_{j=1}^{l} \pi_{ij} P_j$ and $\mathcal{L}_i = P_i B_i + C_i' P_i D_i + L_i$.

5 Optimal LQ Control

In the previous sections we proved that the feasibility of the LMIs is necessary and sufficient for the solvability of the CGAREs. In this section, we show that the value function of the LQ problem (7)-(8) can be expressed in terms of the maximal solution to the CGAREs (21). Moreover, if there exists an optimal control of the LQ problem then it is necessarily represented as a feedback via the maximal solution to the CGAREs.

Theorem 5.1 *Assume that Theorem* 4.6-(i) *holds. Then the LQ problem* (7)-(8) *is well-posed and the value function is given by $V(x_0, i) = x_0' \overline{P}_i x_0$, $\forall x_0 \in \mathbf{R}^n$, $\forall i = 1, 2, \cdots, l$, where $\overline{P} = (\overline{P}_1, \cdots, \overline{P}_l)$ is the maximal solution to the CGAREs* (21).

Theorem 5.2 *Assume that Theorem* 4.6-(i) *holds. If there exists an optimal control of the LQ problem* (7)-(8) *then it must be unique and represented by the feedback control $u(t) = -\sum_{i=1}^{l} (R_i + D_i' \overline{P}_i D_i)^{-1} (B_i' \overline{P}_i + D_i' \overline{P}_i C_i + L_i') x(t) \chi_{\{r_t = i\}}(t)$, where $\overline{P} = (\overline{P}_1, \cdots, \overline{P}_l)$ is the maximal solution to the CGAREs* (21).

6 Numerical Examples

In this section, we report our numerical experiments for a two-mode jump linear system based on the approach developed in the previous sections. Note that the numerical algorithm we have used for checking LMIs or solving SDP [9, 16] is based on an interior-point method [18] which has a polynomial complexity [17].

The system dynamics (7) in our experiments is specified by the following matrices

$$A_1 = \begin{bmatrix} 0.211324 & 0.330327 & 0.849745 \\ 0.756043 & 0.665381 & 0.685731 \\ 0.000221 & 0.628391 & 0.878216 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.068374 & 0.726350 & 0.232074 \\ 0.560848 & 0.198514 & 0.231223 \\ 0.662356 & 0.544257 & 0.216463 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.883388 & 0.932961 \\ 0.652513 & 0.214600 \\ 0.307609 & 0.312642 \end{bmatrix}, B_2 = \begin{bmatrix} 0.361636 & 0.482647 \\ 0.292226 & 0.332171 \\ 0.566424 & 0.593509 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.501534 & 0.632574 & 0.043733 \\ 0.436858 & 0.405195 & 0.481850 \\ 0.269312 & 0.918470 & 0.263955 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.414810 & 0.778312 & 0.685689 \\ 0.280649 & 0.211903 & 0.153121 \\ 0.128005 & 0.112135 & 0.697085 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.841551 & 0.878412 \\ 0.406202 & 0.113836 \\ 0.409482 & 0.199833 \end{bmatrix}, D_2 = \begin{bmatrix} 0.561866 & 0.890622 \\ 0.589617 & 0.504221 \\ 0.685398 & 0.349361 \end{bmatrix},$$
$$\Pi = \begin{bmatrix} -0.3873779 & 0.3873779 \\ 0.9222899 & -0.9222899 \end{bmatrix}.$$

6.1 Numerical Test of Mean-square Stabilizability

We have shown in Lemma 2.1-(ii) that the controlled system under consideration is mean-square stabilizable if and only if (15) is feasible (with respect to the variables X_1, X_2, Y_1 and Y_2). Hence we may check the mean-square stabilizability by solving these LMIs. We find that the following matrices $X_1 = X_1'$, $X_2 = X_2'$, Y_1 and Y_2 satisfy (15):

$$X_1 = \begin{bmatrix} 2140.1643 & 480.89285 & 68.981119 \\ 480.89285 & 848.24038 & -241.62951 \\ 68.981119 & -241.62951 & 198.80034 \end{bmatrix},$$
$$X_2 = \begin{bmatrix} 2250.234 & 1049.5331 & 158.56745 \\ 1049.5331 & 1655.5074 & -241.96314 \\ 158.56745 & -241.96314 & 184.02253 \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} -3040.9847 & -2502.3645 & 389.80811 \\ 883.63967 & 1739.9203 & -831.8653 \end{bmatrix},$$
$$Y_2 = \begin{bmatrix} 1773.8534 & 1944.5197 & -191.23136 \\ -4966.487 & -4263.3261 & 47.977139 \end{bmatrix},$$

which give rise to the stabilizing feedback control law $u(t) = K_1 x(t)$ (while $r_t = 1$) and $u(t) = K_2 x(t)$ (while $r_t = 2$) with the following feedback gain

$$K_1 = Y_1 X_1^{-1} = \begin{bmatrix} -0.720528 & -2.924272 & -1.343456 \\ 0.279059 & 1.03009 & -3.029238 \end{bmatrix},$$
$$K_2 = Y_2 X_2^{-1} = \begin{bmatrix} 0.371952 & 0.916097 & -0.155138 \\ -1.136042 & -2.072044 & -1.484828 \end{bmatrix}.$$

6.2 Numerical Solutions of the CGAREs

Now we proceed to solve the CGAREs (21) for four cases with different (R_1, R_2) under fixed weights $(Q_1, Q_2) = (\mathbf{diag}(1,0,1), \mathbf{diag}(1,1,0)) \geq (0,0)$ and $(L_1, L_2) = (0,0)$ via solving the SDP (22).

(1) R_1 and R_2 singular

Take $(R_1, R_2) = (0,0)$. In this case, we first find that the condition of Theorem 4.6-(i) is satisfied by solving the corresponding LMIs. Hence there must be a maximal solution to the CGAREs (21). We find the following solution

$$P_1 = \begin{bmatrix} 2.8311352 & -2.5169171 & -7.7591546 \\ -2.5169171 & 3.5202908 & 9.7968223 \\ -7.7591546 & 9.7968223 & 32.708219 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 1.9601908 & 0.1772990 & -1.6708539 \\ 0.1772990 & 2.6087885 & 3.0761238 \\ -1.6708539 & 3.0761238 & 10.969290 \end{bmatrix},$$

with the residual $\|\mathcal{R}_1(P_1, P_2)\| = 8.386 \times 10^{-10}$ and $\|\mathcal{R}_2(P_1, P_2)\| = 1.070 \times 10^{-9}$.

(2) R_1 and R_2 negative definite

Setting $R_1 = -0.129782I$ and $R_2 = -0.093287I$, we have a maximal solution

$$P_1 = \begin{bmatrix} 1.0736044 & -1.068867 & -1.9944942 \\ -1.068867 & -0.0430021 & 3.0293761 \\ -1.9944942 & 3.0293761 & 8.3215428 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 0.8369850 & -0.4812074 & -0.5792971 \\ -0.4812074 & -0.1052452 & 0.6058664 \\ -0.5792971 & 0.6058664 & 1.0999906 \end{bmatrix},$$

with the residual $\|\mathcal{R}_1(P_1, P_2)\| = 1.310 \times 10^{-11}$ and $\|\mathcal{R}_2(P_1, P_2)\| = 2.740 \times 10^{-11}$.

(3) R_1 and R_2 indefinite

Choose

$$R_1 = \begin{bmatrix} -0.01 & 0.03 \\ 0.03 & 0.02 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.01 & 0.03 \\ 0.03 & -0.02 \end{bmatrix}.$$

The negative and positive eigenvalues of R_1 are -0.028541 and 0.038541 respectively, and those of R_2 are -0.038541 and 0.028541 respectively. We find the following solution

$$P_1 = \begin{bmatrix} 2.496156 & -2.3250539 & -6.7462339 \\ -2.3250539 & 3.1132333 & 8.9556074 \\ -6.7462339 & 8.9556074 & 29.363909 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 1.7063785 & -0.0240018 & -1.4628917 \\ -0.0240018 & 2.0991262 & 2.6495389 \\ -1.4628917 & 2.6495389 & 9.3117296 \end{bmatrix},$$

with the residual $\|\mathcal{R}_1(P_1, P_2)\| = 9.927 \times 10^{-11}$ and $\|\mathcal{R}_2(P_1, P_2)\| = 1.563 \times 10^{-9}$.

7 Conclusion

This paper considers a class of stochastic LQ control problems with Markovian jumps in the parameters in infinite time horizon with indefinite state and control cost weighting matrices. The associated CGAREs are extensively investigated, analytically and computationally, via LMIs.

A crucial assumption in the paper is the non-singularity of $R_i + D_i' P_i D_i$ $(i = 1, \cdots, l)$. A challenging problem is how to weaken this assumption. Another problem is to extend the LMI technique to the LQ control of jump systems in a finite time horizon where differential Riccati equations have to be involved.

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