On Feasibility and Convergence of Linear Periodic H_∞ Filters

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Abstract— In this paper, we study the feasibility and convergence properties of linear periodic discrete-time H_∞ filters. Based on quasi-lifting techniques, we extend existing results for time invariant systems to periodic time varying systems. A sufficient condition ensuring feasibility and convergence to the periodic stabilizing solution will be given.

Keywords: linear periodic discrete-time systems, H_∞ filters, periodic Riccati equation, feasibility and convergence analysis.

I. Introduction

An important and often encountered class of linear time-varying systems is the class of periodic systems where the system parameters vary periodically. Such class of systems are often encountered in many fields of engineering. For example, all forms of bauded data transmission exhibit cyclostationary characteristics. Amplitude-modulated signals can also be modeled as periodic if carrier synchronization is obtained. Also, the statistics of video signals exhibit a periodicity at the horizontal line rate. A detailed discussion of cyclostationarity in communications and signal processing can be found in [1], [2] and references cited therein. To sum up, it is quite important to study the periodic systems.

In the past decade, the state-space H_∞ design and analysis method has been extended to the study of periodic systems, and many results have been reported in the literatures (see, e.g. [3], [4] and [5]). In these results, it is well known that periodic Riccati differential (difference) equation plays an important role in H_∞ filtering and control of linear periodic systems. Thus, it is important to study the properties of the solutions of H_∞-type periodic Riccati differential (difference) equation.

The performance and the solution of periodic Riccati equation associated with H_2 filtering and control problems for linear periodic systems have been extensively investigated (see, e.g. [6], [7] and references cited therein). However, in spite of such developments, a thorough analysis of the feasibility and convergent behavior of the solution of the H_∞-type periodic Riccati equation is still lacking. Actually, to the best of authors' knowledge, only the convergence of H_∞ filters for continuous-time periodic systems was investigated in [4]. However, as we shall see later, the performance of discrete-time case is rather different from that of the continuous-time case.

Very recently, some attention has been devoted to the performance analysis of H_∞ filters for linear time-invariant systems. In particular, for discrete time-invariant systems, several interesting results on the convergence analysis of Riccati difference equation associated with H_∞ filtering has been reported (see, [8], [9] and [10]). It is noted that the non-existence of an H_∞ filter over a finite horizon is not necessarily associated with the solution of the Riccati difference equation (RDE) becoming unbounded. Rather, the existence of the filter requires the fulfillment at each step of a suitable matrix inequality (feasibility condition). This requirement is stricter than that of the continuous time case, because feasibility can be lost even when the solution of the corresponding RDE remains bounded. The problem is then to find conditions under which the feasibility of the solutions of RDE over an arbitrarily long time interval is ensured and that it converges to the stabilizing solution of algebraic Riccati equation (ARE).

Although some progresses have been made on the analysis of the H_∞-type difference Riccati equation for discrete time-invariant systems, the study of feasibility and asymptotic behavior of periodic Riccati equation in the H_∞ case is still lacking. In this paper, we will examine the performance analysis of linear periodic discrete-time H_∞ filters. We extend existing results for time invariant systems ([9]) to the feasibility and convergence analysis of periodic H_∞ filters. A sufficient condition ensuring feasibility and convergence to the periodic stabilizing solution will be given.

The remainder of this paper is organized as follows: In Section 2, we give the problem formulation and derive some preliminary results necessary for obtaining the main results. The main results on the performance analysis of periodic H_∞-difference Riccati equation are given in Section 3. One numerical examples are illustrated in Section 4. Finally, a brief conclusion is drawn in Section 5.

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Notations: Most of the notations used in this note are fairly standard. $\mathcal{Z}$ denotes the set of non-negative integers. $\Re^n$ denotes the n-dimensional Euclidean space and $||\cdot||$ refers to Euclidean vector norm. $l_2[0,N]$ stands for the space of square summable vector sequence over $[0,N]$, and $||\cdot||_2$ is the $l_2[0,N]$ norm defined by $||\cdot||_2 := \sqrt{\sum_0^N ||\cdot||^2}$. Next, we introduce the decomposition of a symmetric matrix M into its positive part and negative part, i.e. $M = M^+ + M^-$, where $M^+ \geq 0$ ($M^- \leq 0$, resp.). In addition, the nonzero eigenvalues of M^+ (M^-, resp.) are the positive (negative, resp.) eigenvalues of M. Finally, we denote $\sum_{i=m}^n X_i = X_m + X_{m+1} + X_{m+2} + \cdots + X_n$ ($n \geq m$) and $\sum_{i=m}^n X_i = 0$ ($n < m$).

II. Preliminaries

Consider the following linear periodic discrete-time system

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k \omega_k & (1) \\ y_k &= C_k x_k + D_k \omega_k & (2) \\ z_k &= L_k x_k & (3) \end{aligned}$$

where $x_k \in \Re^n$ is the system state, $\omega_k \in \Re^q$ is the noise which belongs to $l_2[0,\infty)$, $y_k \in \Re^m$ is the output measurements, $z_k \in \Re^p$ is a linear combination of the state variables to be estimated, and A_k, B_k, C_k, D_k, and L_k are known real bounded matrices of finite period $K \geq 1$ and with appropriate dimensions, i.e., they satisfy

$$\begin{aligned} A_{i+mK} &= A_i, \quad B_{i+mK} = B_i, \quad C_{i+mK} = C_i, \\ D_{i+mK} &= D_i, \quad L_{i+mK} = L_i. \\ & (\ \forall m \in \mathcal{Z},\ i = 0, 1, \cdots, K-1.) \end{aligned}$$

We shall assume that $D_k[B_k^T \ \ D_k^T] = [0 \ \ I]$. Such assumption is quite standard in H_2 and H_∞ filtering problems (see, e.g. [11]). It means that the process noise and the measurement noise are uncorrelated and that all components of the measurement vector are noisy.

Let $\hat{z}_k$ be an estimate of z_k, we then adopt the following worst-case performance measure:

Finite Horizon :

$$J_N = \sup_{0 \neq (x_0, \omega) \in \Re^n \times l_2[0,N]} \frac{||z - \hat{z}||_2^2}{x_0^T R_0 x_0 + ||\omega||_2^2}$$

Infinite Horizon :

$$J_\infty = \sup_{0 \neq \omega \in l_2[0,\infty)} \frac{||z - \hat{z}||_2^2}{||\omega||_2^2}$$

where $R_0 = R_0^T > 0$ is a given weight matrix for the unknown initial state x_0. For a given scalar γ, the H_∞ filtering problem is to design a linear filter such that $J_N < \gamma^2$ or $J_\infty < \gamma^2$. It is worth noting that an estimator of z_k is called a priori filter when $\hat{z}_k$ is obtained based on the output measurements $\{y_0, y_1, \cdots, y_{k-1}\}$.

The design of H_∞ filters for general linear time-varying systems has been extensively studied in some publications (see, e.g. [10] and references cited therein). It is well known that solutions to H_∞ filtering problems over finite-horizon and infinite-horizon are dependent on the existence of positive definite solution of associated Riccati difference equation,

$$\begin{aligned} P_{k+1} &= A_k P_k A_k^T - A_k P_k \hat{C}_k^T (\hat{C}_k P_k \hat{C}_k^T + R)^{-1} \hat{C}_k P_k A_k^T \\ &\quad + B_k B_k^T \end{aligned} \tag{4}$$

where $\hat{C}_k = \begin{bmatrix} C_k \\ \frac{1}{\gamma} L_k \end{bmatrix}$, $R = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

Note that the matrices A_k, B_k and $\hat{C}_k$ in (4) are periodic with period K, hence (4) will be referred to as a K-periodic Riccati difference equation (PRDE).

Let us introduce some definitions on linear periodic systems that will be used throughout this paper. First, we shall use the notation $\Pi_{k,l}$ to denote the transition matrix of A_k. Note that the eigenvalues of $\Pi_k = \Pi_{k+K,k}$ are independent of k and are named the characteristic multipliers of A_k ([7]). Moreover, it is worth stressing that the linear periodic system (1) or the periodic matrix A_k is asymptotically stable if and only if all the characteristic multipliers of A_k are inside the open unit disc ([2]).

Let $W_{k,l}^o$ and $W_{k,l}^r$ denote the observability Gramian matrix of (C_k, A_k) and reachability Gramian matrix of (A_k, B_k) (see, e.g. [7]). Then we adopt the following definitions of detectability, reachability and stabilizing solution for linear periodic discrete-time systems.

Definition 1: ([7]) The pair (C_k, A_k) will be called detectable if and only if the discrete-time pair $(W_{k,k+K}^o, \Pi_k)$ is detectable.

Definition 2: ([7]) An eigenvalue λ of Π_k will be called (A_k, B_k)-reachable at time k if and only if it is a reachable mode of $(\Pi_k, W_{k,k+K}^r)$.

Definition 3: ([7]) A real symmetric periodic matrix P_k^s is said to be a periodic stabilizing solution of the PRDE (4) if P_k^s is a symmetric K-periodic solution of the PRDE (4) and all the characteristic multipliers of the matrix

$$\hat{A}_k = A_k - A_k P_k^s \hat{C}_k^T (\hat{C}_k P_k^s \hat{C}_k^T + R)^{-1} \hat{C}_k \tag{5}$$

are inside the open unit disc, i.e., $\hat{A}_k$ is asymptotically stable.

Note that the periodic stabilizing solution, if it exists, is unique.

Throughout this paper, we adopt the following assumption:

Assumption 1: (a) Matrix A_k is invertible;
(b) (C_k, A_k) is detectable and (A_k, B_k) is reachable.

The following theorems, which provide solutions to the finite-horizon and infinite-horizon H_∞ filtering problems for linear discrete-time periodic system (1)-(3), can be obtained with simple extension of existing results (see, e.g. [10])

Theorem 1: (Finite-horizon H_∞ filter) Consider the system (1)-(3) and let $R_0 = R_0^T > 0$ be a given initial state weighting matrix. Then there exists a filter such that $J_N < \gamma^2$ if and only if there exists a symmetric positive definite solution P_k over $[0, N]$ satisfying the PRDE (4) with the initial state $P_0 = R_0^{-1}$, and such that $P_k^{-1} - \gamma^{-2} L_k^T L_k > 0$. Moreover, if the above conditions are satisfied, a suitable filter is given by

$$\begin{aligned} \hat{x}_{k+1} &= A_k \hat{x}_k + \hat{K}_k (y_k - C_k \hat{x}_k), \\ &\qquad \hat{x}_0 = initial\ guess \quad (6) \\ \hat{z}_k &= L_k \hat{x}_k \quad (7) \end{aligned}$$

where $\hat{K}_k = A_k U_k C_k^T (C_k U_k C_k^T + I)^{-1}$ and $U_k = P_k + \gamma^{-2} P_k L_k^T (I - \gamma^{-2} L_k P_k L_k^T)^{-1} L_k P_k$.

Theorem 2: (Infinite-horizon H_∞ filter) Consider the system (1)-(3). Given a scalar $\gamma > 0$, if Assumption 2.1 holds, then there exists an asymptotically stable periodic filter such that the filtering error dynamics is asymptotically stable and $J_\infty < \gamma^2$ if and only if there exists symmetric positive definite K-period stabilizing solution P_k^s over $[0, \infty)$ satisfying the PRDE (4) such that $(P_k^s)^{-1} - \gamma^{-2} L_k^T L_k > 0$. In such case, a suitable filter is obtained by replacing P_k with P_k^s in equalities (6)-(7).

Note that the reachability of (A_k, B_k) as assumed in the Assumption 2.1 ensures that the strict positives of P_k^s if $J_\infty < \gamma^2$. It is clear that the existence of H_∞ filter is related to the PRDE (4) and the fulfillment of a suitable matrix inequality (feasibility condition). So in this paper, we shall introduce the following definition of feasible solution of PRDE.

Definition 4: (Feasible solution of PRDE) A real positive definite solution P_k of the PRDE (4) is termed a "feasible solution" if it satisfies

$$P_k^{-1} - \gamma^{-2} L_k^T L_k > 0 \quad (8)$$

at every step k.

In the sequel, in view of Theorem 2.2, we make the following assumption.

Assumption 2: There exists a symmetric positive definite periodic stabilizing solution P_k^s to the PRDE (4) such that $(P_k^s)^{-1} - \gamma^{-2} L_k^T L_k > 0$ for the given γ.

Since $\hat{A}_k$ of (5) can be rewritten as $A_k[(P_k^s)^{-1} - \gamma^{-2} L_k^T L_k + C_k^T C_k]^{-1} (P_k^s)^{-1}$, it is clear that $\hat{A}_k$ is invertible when A_k is invertible and P_k^s is feasible.

Under Assumption 2.2, the feasibility and convergence analysis problem studied in this paper can be stated as follows: Given an arbitrarily large N, find suitable conditions on the initial state P_0, such that the solution P_k of the PRDE (4) is feasible at every step $k \in [0, N]$ and converges to the periodic stabilizing solution P_k^s as $N \to \infty$.

We end this section by giving the following lemma which will be needed in the proof of our main results. This lemma can be established via a proof similar to that found in [12].

Lemma 1: Consider the PRDE (4). Let P_k^1 and P_k^2 be two solutions of (4) with different initial conditions $P_0^2 \geq P_0^1 > 0$. Then, under Assumption 2.1, when P_k^2 is feasible, it results in $P_k^2 \geq P_k^1 > 0$ and P_k^1 is feasible too. Furthermore, if $P_0^2 > P_0^1$, then $P_k^2 > P_k^1$ for $\forall k \geq 0$.

III. Convergence & Feasibility Analysis of Periodic H_∞ Filters

In this section, we will derive a sufficient condition for the feasibility and convergence of symmetric positive definite solution of the H_∞-PRDE to the periodic stabilizing solution.

Before presenting the main results of this section, some notations are introduced as follows.

First, we define

$$\begin{aligned} G_k &= -(P_k^s)^{-1} - (P_k^s)^{-1} \Big[\gamma^{-2} L_k^T L_k - (P_k^s)^{-1} \Big]^{-1} \\ &\quad \cdot (P_k^s)^{-1} \end{aligned} \quad (9)$$

and

$$M_k = \hat{A}_k^{-T} (G_k + \hat{C}_k^T \hat{R}_k^{-1} \hat{C}_k) \hat{A}_k^{-1} - G_{k+1} \quad (10)$$

where $\hat{R}_k = \hat{C}_k P_k^s \hat{C}_k^T + R$, $\hat{A}_k$, $\hat{C}_k$, L_k and R are the same as in (5) and (4). It is clear that G_k and M_k are both known K-periodic real bounded matrices.

Next we define Θ_i $(i \in [0, K-1])$ as follows,

$$\Theta_i = \hat{A}_{K-1+i} \hat{A}_{K-2+i} \cdot \cdots \cdot \hat{A}_{1+i} \hat{A}_i \quad (11)$$

where $\hat{A}_i$ $(i \in [0, K-1])$ is defined in (5). Θ_i is the lifted system matrix of the periodic system $x_{k+1} = \hat{A}_k x_k$ at $i = 0, 1, \cdots K-1$. We denote the transition matrix as follows

$$\begin{aligned} \hat{\Phi}_{k,l} &= \hat{A}_{k-1} \hat{A}_{k-2} \cdots \hat{A}_{l+1} \hat{A}_l, \ k > l \quad (12) \\ \hat{\Phi}_{k,l} &= I, \ k = l \quad (13) \end{aligned}$$

Since $\hat{A}_i$ is invertible, $\hat{\Phi}_{k,l}$ is invertible for any $k > l$. Furthermore, as P_k^s is the stabilizing solution, the matrix Θ_i is stable.

We shall also define Ψ_i $(i \in [0, K-1])$ as follows:

$$\Psi_i = \sum_{l=1}^{K} \hat{\Phi}_{K+i,l+i}^{-T} M_{l-1+i}^{-} \hat{\Phi}_{K+i,l+i}^{-1} \quad (14)$$

where M_i $(i \in [0, K-1])$ is as defined in (10) and M_i^- is the negative part of M_i as defined in Section 1.

Next, we introduce the following K Lyapunov equations

$$\Theta_i^T Y_i \Theta_i - Y_i = -\Psi_i \ , \qquad i \in [0, K-1] \quad (15)$$

where Θ_i and Ψ_i $(i \in [0, K-1])$ are known real bounded matrices as defined in (11) and (14), respectively.

Before presenting the main results, we introduce the following lemma which will be needed in the proof of the main results.

Lemma 2: ([10]) Let P_k^1 and P_k^2 be two solutions of PRDE (4) with different initial conditions P_0^1 and P_0^2, respectively. We then have the following identity:

$$\begin{aligned} P_{k+1}^2 - P_{k+1}^1 &= \hat{\Phi}_{k+1,0} [I + (P_0^2 - P_0^1) \Omega_k]^{-1} (P_0^2 - P_0^1) \\ &\quad \cdot \hat{\Phi}_{k+1,0}^T \end{aligned} \quad (16)$$

where

$$\Omega_k = \sum_{j=0}^{k} \hat{\Phi}_{j,0}^T \hat{C}_j^T \hat{R}_j^{-1} \hat{C}_j \hat{\Phi}_{j,0} \quad (17)$$

and $\hat{\Phi}_{k,l}$ is as defined in (13).

The following Theorem establishes a relationship between the initial state $P_0 > 0$ and the feasibility of the solution of PRDE (4).

Theorem 3: Consider the periodic Riccati difference equations (4). Suppose that Assumptions 2.1 and 2.2 hold, and let

$$\Delta_i = \left[G_0 + \hat{\Phi}_{i,0}^T(\Psi_i - Y_i)\hat{\Phi}_{i,0} - \sum_{l=1}^{i} \hat{\Phi}_{l,0}^T M_{l-1}^- \hat{\Phi}_{l,0} + \epsilon I\right]^{-1} \quad \forall i \in [0, K-1] \tag{18}$$

where $\epsilon > 0$ is sufficiently small, Y_i ($\forall i \in [0, K-1]$) are solutions of Lyapunov equations (15), and G_0, M_i, $\hat{\Phi}_{k,l}$ and Ψ_i ($\forall i \in [0, K-1]$) are known real bounded matrices as defined in (9)-(14), respectively. In such case, Δ_i is positive definite.

Then, the real positive definite solution P_k of PRDE (4) is feasible over $[0, \infty)$ if the initial state satisfies

$$0 < P_0 < P_0^s + \Delta_i \ , \qquad \forall i \in [0, K-1] \tag{19}$$

Proof: Let us first show that Δ_i in (18) is positive definite. It is worth noting that $G_k \geq 0$ since P_k^s is feasible. Next, refer to Lemma 21.6 of [13], then from (15), we have

$$Y_i = \sum_{j=0}^{\infty} (\Theta_i^T)^j \Psi_i \Theta_i^j = \Psi_i + \sum_{j=1}^{\infty} (\Theta_i^T)^j \Psi_i \Theta_i^j \ , \quad i \in [0, \ K-1] \tag{20}$$

In view of (20) and the fact that $\Psi_i \leq 0$, we have $\Psi_i - Y_i \geq 0$ ($i \in [0, K-1]$). Note also that $M_l^- \leq 0$. Then we can conclude that $\Delta_i > 0$ for sufficiently small $\epsilon > 0$.

Now let us prove the main result. The proof can be divided into three subcases depending on the relation between initial state P_0 and P_0^s.

(i) The case when $P_0 < P_0^s$.

Because P_k^s is the feasible solution of (4), the feasibility of P_k follows directly from Lemma 2.1.

(ii) The case when $P_0 > P_0^s$.

At first, let's define $X_k = P_k - P_k^s$. It is worth noting that X_k is invertible when we consider Lemma 3.1. Then, applying Lemma 3.1 of [14] to (4), we readily obtain that X_k satisfies

$$\begin{aligned} X_{k+1} &= \hat{A}_k X_k \hat{A}_k^T \\ &\quad - \hat{A}_k X_k \hat{C}_k^T (\hat{C}_k X_k \hat{C}_k^T + \hat{R}_k)^{-1} \hat{C}_k X_k \hat{A}_k^T \\ &= \hat{A}_k (X_k^{-1} + \hat{C}_k^T \hat{R}_k^{-1} \hat{C}_k)^{-1} \hat{A}_k^T \end{aligned} \tag{21}$$

where the initial state $X_0 = P_0 - P_0^s$, $\hat{A}_k$ and $\hat{R}_k$ are the same as in (5) and (10).

Now letting $Z_k = X_k^{-1} - G_k$, where G_k is defined by (9). From the invertibility of $\hat{A}_k$ and (21), we have

$$Z_{k+1} = (\hat{A}_k^{-1})^T Z_k \hat{A}_k^{-1} + M_k \tag{22}$$

where M_k is defined by (10) and the initial state $Z_0 = (P_0 - P_0^s)^{-1} - G_0$.

Since P_k^s is feasible, it is clear that the feasibility of P_k is equivalent to the positive definiteness of Z_k, which follows from $Z_k = (P_k^s)^{-1}\big[\big((P_k^s)^{-1} - P_k^{-1}\big)^{-1} - \big((P_k^s)^{-1} - \gamma^{-2} L_k^T L_k\big)^{-1}\big](P_k^s)^{-1}$.

Now consider the following Lyapunov equation

$$\hat{Z}_{k+1} = \hat{A}_k^{-T} \hat{Z}_k \hat{A}_k^{-1} + M_k^- \tag{23}$$

with the initial state $\hat{Z}_0 = Z_0$. By definition, $M_k \geq M_k^-$, so that $Z_k \geq \hat{Z}_k$. Then $\hat{Z}_k > 0$ is sufficient to guarantee the positivity of Z_k.

In (23), it is clear that $\hat{A}_k$ and M_k^- are both known real bounded K-periodic matrices. After some tedious algebraic manipulations, the solution of (23) can be expressed as follows

$$\begin{aligned} \hat{Z}_{mK+i} &= (\Theta_i^T)^{-m}\Big[\hat{\Phi}_{i,0}^{-T} Z_0 \hat{\Phi}_{i,0}^{-1} + \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} \\ &\quad + \sum_{j=1}^{m} (\Theta_i^T)^j \Psi_i \Theta_i^j\Big]\Theta_i^{-m} \\ &\geq (\Theta_i^T)^{-m}\Big[\hat{\Phi}_{i,0}^{-T} Z_0 \hat{\Phi}_{i,0}^{-1} + \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} \\ &\quad + \sum_{j=1}^{\infty} (\Theta_i^T)^j \Psi_i \Theta_i^j\Big]\Theta_i^{-m} \\ &\quad i = 0, 1, 2, \cdots, K-1 \ ; \ m \in \mathcal{Z} \end{aligned} \tag{24}$$

It is worth stressing that the above set Z_{mK+i} ($i \in [0, K-1]$, $m \in \mathcal{Z}$) forms the complete solution of (23).

Comparing (24) and (20), we have

$$\begin{aligned} \hat{Z}_{mK+i} &\geq (\Theta_i^T)^{-m}\Big[\hat{\Phi}_{i,0}^{-T} Z_0 \hat{\Phi}_{i,0}^{-1} + \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} \\ &\quad + Y_i - \Psi_i\Big]\Theta_i^{-m} \end{aligned} \tag{25}$$

where $i \in [0, K-1]$ and $m \in \mathcal{Z}$.

Observe from (25) that if

$$\hat{\Phi}_{i,0}^{-T} Z_0 \hat{\Phi}_{i,0}^{-1} + \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} + Y_i - \Psi_i > 0 \ , \quad \forall i \in [0, K-1] \tag{26}$$

then $\hat{Z}_k > 0$ ($\forall k \in \mathcal{Z}$), and, in turn $Z_k > 0$, which means that P_k keeps feasibility over $[0, \infty)$.

Because $Z_0 = (P_0 - P_0^s)^{-1} - G_0$, hence (26) can be rewritten as

$$\hat{\Phi}_{i,0}^{-T}[(P_0 - P_0^s)^{-1} - G_0]\hat{\Phi}_{i,0}^{-1} + \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} + Y_i - \Psi_i > 0 \tag{27}$$

where $i \in [0, K-1]$.

In view of (20) and the fact that $\Psi_i \leq 0$, $\Psi_i - Y_i \geq 0$ ($i \in [0, K-1]$). Noting that $G_0 \geq 0$, then (27) holds if P_0 satisfies (19) for sufficiently small $\epsilon > 0$.

(iii) The case that $P_0 - P_0^s$ is not sign definite.

There always exists a $\bar{P}_0$ satisfying (19) and such that $\bar{P}_0 > P_0$ and $\bar{P}_0 > P_0^s$. Then in view of subcase (ii), the solution $\bar{P}_k$ of PRDE (4) starting from $\bar{P}_0$ is feasible. Finally, the feasibility of P_k readily comes from Lemma 2.1. $\bigtriangledown\bigtriangledown\bigtriangledown$

We shall now study the convergence of the solution of the PRDE (4). Let us introduce the following Lyapunov equation:

$$\Omega^s = \hat{\Phi}_{K,0}^T \Omega^s \hat{\Phi}_{K,0} + \Upsilon \tag{28}$$

where Υ is a constant matrix defined as follows:

$$\Upsilon = \sum_{j=0}^{K-1} \hat{\Phi}_{j,0}^T \hat{C}_j^T \hat{R}_j^{-1} \hat{C}_j \hat{\Phi}_{j,0}$$

and $\hat{\Phi}_{k,l}$ is as defined in (13).

The following theorem gives the main result of this paper. It establishes a relationship between the initial state P_0 and the convergence as well as feasibility of the solution of PRDE (4).

Theorem 4: Let P_k be symmetric positive definite solution of PRDE (4), and suppose that Assumptions 2.1 and 2.2 hold. Then, P_k is feasible over $[0, \infty)$ and converges to the stabilizing solution P_k^s, i.e. $\lim_{k\to\infty}(P_k - P_k^s) = 0$, if the initial state satisfies

(a) $0 < P_0 < P_0^s + \Delta_i$ $(\forall i \in [0, K-1])$, where Δ_i is defined as in (18).

(b) $I + (P_0 - P_0^s)\Omega^s$ is nonsingular, where Ω^s is the solution of Lyapunov equality (28).

Proof: It should be noted that P_k is feasible over $[0, \infty)$ from Theorem 3.1. So in what follows, we focus on the proof of the convergence of P_k.

Similar to the proof of Theorem 3.1, we define $X_k = P_k - P_k^s$. Then by (21), we have

$$\begin{aligned} X_{k+1} &= \hat{A}_k X_k \hat{A}_k^T \\ &\quad - \hat{A}_k X_k \hat{C}_k^T (\hat{C}_k X_k \hat{C}_k^T + \hat{R}_k)^{-1} \hat{C}_k X_k \hat{A}_k^T \\ &= \hat{A}_k \Big[I + X_k \hat{C}_k^T \hat{R}_k^{-1} \hat{C}_k \Big]^{-1} X_k \hat{A}_k^T \end{aligned} \tag{29}$$

where the initial state $X_0 = P_0 - P_0^s$, $\hat{A}_k$ and $\hat{R}_k$ are the same as in (5) and (10).

Considering Lemma 3.1, the solution of (29) can be expressed as follows:

$$X_{k+1} = \hat{\Phi}_{k+1,0} (I + X_0 \Omega_k)^{-1} X_0 \hat{\Phi}_{k+1,0}^T \tag{30}$$

where Ω_k and $\hat{\Phi}_{k,l}$ are as defined in (17) and (13), respectively.

Using the same quasi-lifting techniques as that used in the proof of Theorem 3.1, Ω_k in the equality (30) can be expressed as the solution of following Lyapunov equation with K different initial states ($m \in \mathcal{Z}$ and $m \geq 1$):

$$\begin{aligned} \Omega_{mK+i} &= \hat{\Phi}_{K,0}^T \Omega_{(m-1)K+i} \hat{\Phi}_{K,0} + \Upsilon \ , \\ &\quad (i = 0, 1, 2 \cdots K-1) \end{aligned} \tag{31}$$

where Υ is a constant matrix defined as follows:

$$\Upsilon = \sum_{j=0}^{K-1} \hat{\Phi}_{j,0}^T \hat{C}_j^T \hat{R}_j^{-1} \hat{C}_j \hat{\Phi}_{j,0}$$

and K initial states Ω_i are

$$\Omega_i = \sum_{j=0}^{i} \hat{\Phi}_{j,0}^T \hat{C}_j^T \hat{R}_j^{-1} \hat{C}_j \hat{\Phi}_{j,0} \quad , \qquad i \in [0, K-1]$$

Since $\hat{\Phi}_{K,0}$ is stable, Ω_k converges to the stable solution Ω^s as $k \to \infty$. Then it is clear that $\lim_{k\to\infty}(I + X_0\Omega_k)^{-1} = (I + X_0\Omega^s)^{-1} = [I + (P_0 - P_0^s)\Omega^s]^{-1}$.

Hence, using the above fact and considering conditions (a) and (b) and noting that $\hat{\Phi}_{k,l}$ is stable, we conclude immediately from (30) that X_k converges to 0, i.e. $\lim_{k\to\infty}(P_k - P_k^s) = 0$. $\bigtriangledown\bigtriangledown\bigtriangledown$

In Theorem 3.2, we give a sufficient condition for feasibility and convergence of periodic H_∞ filters. This condition requires the initial state P_0 satisfy K inequalities simultaneously and keeping $I + (P_0 - P_0^s)\Omega^s$ nonsingular. It can be computed by using Matlab LMI Toolbox.

It is worth noting that when period $K = 0$, then $\Psi_0 = M^-$, $P_0^s = P^s$, $\Omega^s = 0$, and the sufficient conditions of Theorem 3.2 reduces to one of associated time-invariant H_∞ filters ([9]).

In general, the feasible and convergent conditions in Theorem 3.2 are only sufficient, but it can be easily shown that they are also necessary when $K = 1$ and $\gamma \to \infty$. Since in such case, $G_k = 0$, $M_k^- = 0$, $Y_i = 0$ and $\Omega^s = 0$, so the global convergence can be guaranteed with known results on Kalman filtering.

The invertibility assumption of A_k in Assumption 2.1 is crucial in the derivation of our results and it seems rather difficult to eliminate.

IV. Numerical Examples

In this section, we shall present a numerical examples to illustrate the application of the results proposed in this paper.

Consider the periodic discrete-time system (1)-(3) with period $K = 2$ and $A_0 = 0.1$, $A_1 = 0.5$, $B = [1 \ \ 0]$, $C_0 = 10$, $C_1 = 1.5$, $D = [0 \ \ 1]$, $L = 2$.

By applying Theorem 2.2, we obtain the optimal $\gamma = 2.11$. The corresponding periodic stabilizing solution of PRDE (4) is $P_0^s = 1.106$ and $P_1^s = 1.000$.

For the finite-horizon case, according to Theorem 3.2, if the initial state P_0 satisfies

$$0 < P_0 < 1.113 \tag{32}$$

then P_k remains feasible as well as converges to the periodic stabilizing solution P_k^s as $k \to \infty$.

If we suppose that the initial state $P_0 = 1.11$ which satisfies (32), then simulation results show that P_k remains feasible and converges to the periodic stabilizing solution P_k^s over $[0, \infty)$. On the other hand, if P_0 does not satisfy (32), for example when $P_0 = 1.20$, then solution of the PRDE (4) shows that feasibility is lost at the first step where $P_0^{-1} - \gamma^{-2} L^T L = -0.0651 < 0$!

This simple example illustrates the importance of inequality (8) in ensuring the feasibility of PRDE (4) in H_∞ a posteriori filter design.

V. Conclusion

In this paper, we have studied the feasibility and convergent problem of linear periodic discrete-time H_∞ filters. Based on quasi-lifting techniques, a sufficient condition for ensuring the feasibility and convergence of such filters has been given. This condition requires the initial state P_0 to satisfy K inequalities simultaneously and the nonsingularity of a certain matrix.

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