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Asymptotic Behavior of Periodic H_∞ *a Posteriori* Filters*

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Abstract. The asymptotic behavior of linear periodic discrete-time H_∞ *a posteriori* filters is discussed in this paper. We extend existing results for time-invariant H_∞ filters to study the problems arising from periodic discrete-time systems. Based on quasi-lifting techniques, a sufficient condition for ensuring feasibility and convergence of H_∞ *a posteriori* filters is given.
Key words: Periodic discrete-time systems, H_∞ *a posteriori* filters, Riccati equations, feasibility and convergence analysis.

1. Introduction

In many practical engineering systems, such as those in the fields of electronics, biology, economy management, and chemistry, periodic characteristics are frequently encountered. This has motivated the study of control and filtering problems of periodic systems, and a number of results have been reported in the literature (see, e.g., [1] and references cited therein). In the early 1990s, by extending the state-space H_∞ design and analysis method, some progress in the H_∞ optimal estimation and control of periodic systems had already been made (see, e.g., [9]–[11]). In these results, the periodic Riccati difference equation plays a key role in H_∞ filtering and control for periodic discrete-time systems. It is therefore important to study the asymptotic behavior of the H_∞-type periodic Riccati difference equation.

The performance of the periodic difference Riccati equation associated with the H_2 filtering and control problem for linear periodic systems has been extensively investigated (see, e.g., [2] and references cited therein). A few pioneering works

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on the performance analysis of the H_∞ Riccati difference equation for time-invariant systems have also been reported (see, e.g., [3] and [4]). However, in spite of such developments, an analysis of the asymptotic behavior of the H_∞-type periodic Riccati equation is still lacking.

First, the H_∞ filtering is quite different from the H_2 case because the nonexistence of an H_∞ filter over a finite horizon is not necessarily associated with the solution of the periodic Riccati difference equation (PRDE) becoming unbounded. Rather, the existence of the filter requires the fulfillment at each step of a suitable matrix inequality (feasibility condition). We stress that feasibility may be lost even when the solution of the corresponding PRDE remains bounded. In such a case, we wish to find conditions under which the solutions of the PRDE over an arbitrarily long time interval are feasible, and for which there is convergence toward the periodic stabilizing solution of PRDE. Second, although the feasible and convergent problems of H_∞ filters were investigated in [3] and [4], only the time-invariant version of the difference Riccati equation arising from H_∞ filtering was addressed.

This paper is concerned with asymptotic behavior analysis of linear periodic discrete-time H_∞ *a posteriori* filters. Existing results for time-invariant systems ([3] and [4]) are extended to the feasibility and convergence analysis of periodic H_∞ filters. Based on quasi-lifting techniques, a sufficient condition ensuring feasibility and convergence to the periodic stabilizing solution will be given in terms of certain matrix inequalities.

The remainder of this paper is organized as follows. In Section 2, we give the problem formulation and derive some preliminary results. The main results on the analysis of the periodic H_∞-difference Riccati equation are given in Section 3. Two numerical examples are illustrated in Section 4. Section 5 concludes the paper.

Notation

The notation used in this paper is fairly standard. The superscript T denotes matrix transposition. $\mathcal{Z}$ denotes the set of nonnegative integers. $\Re^n$ denotes the n-dimensional Euclidean space, and $\|\cdot\|$ refers to a Euclidean vector norm. The term $l_2[0, N]$ stands for the space of square summable vector sequences over $[0, N]$, and $\|\cdot\|_2$ is the $l_2[0, N]$ norm defined by $\|\cdot\|_2 := \sqrt{\sum_0^N \|\cdot\|^2}$. Next, we introduce the decomposition of a symmetric matrix M into its positive part and negative part, i.e., $M = M^+ + M^-$, where $M^+ \geq 0\,(M^- \leq 0)$. In addition, the nonzero eigenvalues of $M^+ (M^-$, resp.) are the positive (negative, resp.) eigenvalues of M. Finally, we denote $\sum_{i=m}^n X_i = X_m + X_{m+1} + X_{m+2} + \cdots + X_n$ $(n \geq m)$ and $\sum_{i=m}^n X_i = 0$ $(n < m)$.

2. Problem formulation and preliminaries

Consider the following linear periodic discrete-time system:

$$x_{k+1} = A_k x_k + B_k \omega_k \tag{2.1}$$

$$y_k = C_k x_k + D_k \omega_k \tag{2.2}$$

$$z_k = L_k x_k, \tag{2.3}$$

where $x_k \in \Re^n$ is the system state, $\omega_k \in \Re^q$ is the noise, $y_k \in \Re^m$ is the output measurements, $z_k \in \Re^p$ is a linear combination of the state variables to be estimated, and A_k, B_k, C_k, D_k, and L_k are known real bounded matrices of finite period $K \geq 1$ and with appropriate dimensions, i.e., they satisfy

$$\begin{aligned} &A_{i+mK} = A_i, \qquad B_{i+mK} = B_i, \qquad C_{i+mK} = C_i, \\ &D_{i+mK} = D_i, \qquad L_{i+mK} = L_i \qquad (\forall m \in \mathcal{Z}, i = 0, 1, \ldots, K-1). \end{aligned}$$

In this paper, it will be tacitly assumed that $D_k[B_k^T \;\; D_k^T] = [0 \;\; I]$. Such an assumption corresponds to the usual hypothesis of uncorrelated process and measurement noise and normalized measurement noise.

Let $\hat{z}_k$ denote the estimate of z_k and $\hat{z} = \{\hat{z}_k, k = 0, 1, 2, \ldots\}$. Then we shall adopt the following worst-case performance measure:

$$\text{Finite horizon:} \qquad J_N = \sup_{0 \neq (x_0, \omega) \in \Re^n \times l_2[0,N]} \frac{\|z_k - \hat{z}_k\|_2^2}{x_0^T R_0 x_0 + \|\omega_k\|_2^2}.$$

$$\text{Infinite horizon:} \qquad J_\infty = \sup_{0 \neq \omega \in l_2[0,\infty)} \frac{\|z_k - \hat{z}_k\|_2^2}{\|\omega_k\|_2^2},$$

where $R_0 = R_0^T > 0$ is a given weight matrix for the unknown initial state x_0. For a given scalar γ, the H_∞ filtering problem is to design a linear filter such that $J_N < \gamma^2$ or $J_\infty < \gamma^2$. Note that an estimator of z_k is called an *a posteriori* filter if $\hat{z}_k$ is obtained based on the output measurements $\{y_0, y_1, \ldots, y_k\}$, whereas $\hat{z}_k$ is referred to as an *a priori* filter if it is obtained by using the measurements $\{y_0, y_1, \ldots, y_{k-1}\}$.

The design of H_∞ *a posteriori* filters for general linear time-varying systems has been previously studied in some publications (see, e.g., [7], [8] and references cited therein). It is well known that solutions to H_∞ *a posteriori* filtering problems over the finite horizon and infinite horizon are dependent on the existence of positive definite solutions of the associated Riccati difference equation,

$$P_{k+1} = A_k P_k A_k^T - A_k P_k \hat{C}_k^T (\hat{C}_k P_k \hat{C}_k^T + R)^{-1} \hat{C}_k P_k A_k^T + B_k B_k^T, \tag{2.4}$$

where $\hat{C}_k = \begin{bmatrix} C_k \\ \frac{1}{\gamma} L_k \end{bmatrix}$, $R = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

In this paper, the matrix coefficients in (2.4) are periodic functions; hence, (2.4) will be referred to as the periodic Riccati difference equation (PRDE). Note that the difference between the H_∞-type PRDE and the Kalman-type PRDE is that R in PRDE (2.4) is sign indefinite in the H_∞ type.

We shall introduce several definitions for linear periodic systems that will be used throughout this paper. First, we shall use the notation $\Pi_{k,l}$ to denote the transition matrix of A_k. Note that the eigenvalues of $\Pi_k = \Pi_{k+K,k}$ are independent of k and are called the characteristic multipliers of A_k ([6]). Moreover, we stress that the linear periodic system (2.1) or the periodic matrix A_k is asymptotically stable if and only if all the characteristic multipliers of A_k are inside the open unit disc ([1]).

Let $W^o_{k,l}$ and $W^r_{k,l}$ be the observability Gramian matrix of (C_k, A_k) and the reachability Gramian matrix of (A_k, B_k) (see, e.g., [6]). Then we adopt the following definitions of detectability and reachability for linear periodic discrete-time systems.

Definition 2.1 [6]. The pair (C_k, A_k) will be called detectable if and only if the discrete-time pair $(W^o_{k,k+K}, \Pi_k)$ is detectable.

Definition 2.2 [6]. An eigenvalue λ of Π_k will be called (A_k, B_k)-reachable at time k if and only if it is a reachable mode of $(\Pi_k, W^r_{k,k+K})$.

The notion of stabilizing solutions to PRDE (2.4) is defined as follows.

Definition 2.3 [6]. A real symmetric periodic matrix P^s_k is said to be a periodic stabilizing solution of the PRDE (2.4) if P^s_k is a symmetric K-periodic solution of the PRDE (2.4) and all the characteristic multipliers of the matrix

$$\hat{A}_k = A_k - A_k P^s_k \hat{C}^T_k (\hat{C}_k P^s_k \hat{C}^T_k + R)^{-1} \hat{C}_k \tag{2.5}$$

are inside the open unit disc, i.e., $\hat{A}_k$ is asymptotically stable.

Throughout this paper, we make the following assumption.

Assumption 2.1.

(a) *Matrix A_k is invertible.*
(b) *(C_k, A_k) is detectable, and (A_k, B_k) is reachable.*

By simple extension of existing results (see, e.g., [7] and [8]), we have the following theorems, which give solutions to the finite-horizon and infinite-horizon H_∞ *a posteriori* filtering problems for linear discrete-time periodic system (2.1)–(2.3).

Theorem 2.1 Finite-horizon H_∞ *a posteriori* filter. *Consider the system (2.1)–(2.3) and let $R_0 = R_0^T > 0$ be a given initial state weighting matrix. Then there exists an H_∞* a posteriori *filter such that $J_N < \gamma^2$ if and only if there exist symmetric positive definite solutions P_k over $[0, N]$ satisfying* PRDE (2.4) *with $P_0 = R_0^{-1}$ and such that $P_k^{-1} + C_k^T C_k - \gamma^{-2} L_k^T L_k > 0$. Moreover, if the above conditions are satisfied, a suitable filter is given by*

$$\hat{x}_{k+1} = A_k \hat{x}_k + \hat{K}_{k+1}(y_{k+1} - C_{k+1} A_k \hat{x}_k), \qquad \hat{x}_0 = \textit{initial guess} \tag{2.6}$$

$$\hat{z}_{k+1} = L_{k+1} \hat{x}_{k+1}, \tag{2.7}$$

where $\hat{K}_{k+1} = P_{k+1}C_{k+1}^T(C_{k+1}P_{k+1}C_{k+1}^T + I)^{-1}$.

Theorem 2.2 Infinite-horizon H_∞ *a posteriori* filter. *Consider the system* (2.1)–(2.3). *If Assumption* 2.1 *holds, then there exists an asymptotically stable periodic* H_∞ a posteriori *filter such that the filtering error dynamics is asymptotically stable and* $J_\infty < \gamma^2$ *if and only if there exist symmetric positive definite* K*-periodic stabilizing solutions* P_k^s *over* $[0, \infty)$ *satisfying* PRDE (2.4) *and such that* $(P_k^s)^{-1} + C_k^T C_k - \gamma^{-2}L_k^T L_k > 0$. *In such a case, a suitable periodic filter is obtained by replacing* P_k *with* P_k^s *in equalities* (2.6) *and* (2.7).

It is clear that the existence of an H_∞ filter is related to the PRDE (2.4) and the fulfillment of a suitable matrix inequality (feasibility condition). So, we shall introduce another definition that will play a key role in the remainder of the paper.

Definition 2.4 Feasible solution of PRDE. A real positive definite solution P_k of the PRDE (2.4) is termed a "feasible solution" if it satisfies

$$P_k^{-1} + C_k^T C_k - \gamma^{-2}L_k^T L_k > 0 \tag{2.8}$$

at every step k.

It is worth indicating that the feasible condition of an H_∞ *a posteriori* filter is different from that of an H_∞ *a priori* filter (see, e.g., [8]). Next, in view of Theorem 2.2, we shall make the following assumption in this paper.

Assumption 2.2. *There exists a symmetric positive definite periodic stabilizing solution* P_k^s *to the* PRDE (2.4) *such that* $(P_k^s)^{-1} + C_k^T C_k - \gamma^{-2}L_k^T L_k > 0$ *for the given* γ.

Note that the periodic stabilizing solution, if it exists, is unique. Besides, because $\hat{A}_k$ of (2.5) can be rewritten as $A_k[(P_k^s)^{-1} + C_k^T C_k - \gamma^{-2}L_k^T L_k]^{-1}(P_k^s)^{-1}$, it is clear that $\hat{A}_k$ is invertible when A_k is invertible and P_k^s is feasible.

The feasibility and convergence analysis problem studied in this paper can be stated as follows: *Given an arbitrarily large N, find suitable conditions on the initial state* P_0 *such that the solution* P_k *of* PRDE (2.4) *is feasible at every step* $k \in [0, N]$, *and converges to the periodic stabilizing solution* P_k^s *as* $N \to \infty$.

We end this section by establishing the following two technical lemmas, which will be needed in the proof of our main results.

Lemma 2.1. *Consider the* PRDE (2.4). *Let* P_k^1 *and* P_k^2 *be two solutions of* (2.4) *with different initial conditions* $P_0^2 \geq P_0^1 > 0$. *Then, under Assumption* 2.1, *when* P_k^2 *is feasible, it results in* $P_k^2 \geq P_k^1 > 0$ *and* P_k^1 *is feasible too. Furthermore, if* $P_0^2 > P_0^1$, *then* $P_k^2 > P_k^1$.

Proof. The proof can be carried out similarly way to the proof of a result in [3]. □

The following lemma provides an alternative test for the feasibility condition (2.8).

Lemma 2.2. *Let P_k be symmetric positive definite matrices. Then following two statements are equivalent:*

(i) $P_k^{-1} + C_k^T C_k - \gamma^{-2} L_k^T L_k = P_k^{-1} + \hat{C}_k^T R \hat{C}_k > 0$
(ii) $\eta(\hat{C}_k P_k \hat{C}_k^T + R) < 0$,

where $\hat{C}_k$ and R are the same as in (2.4), and $\eta\left(\begin{bmatrix} \mathrm{H}_1 & \mathrm{H}_3 \\ \mathrm{H}_3^{\mathrm{T}} & \mathrm{H}_2 \end{bmatrix}\right) = \mathrm{H}_2 - \mathrm{H}_3^{\mathrm{T}} \mathrm{H}_1^{-1} \mathrm{H}_3$.

Proof. The result readily follows from Corollary 1 in [7]. □

3. Asymptotic analysis of periodic H_∞ filters

In this section, a sufficient condition for the feasibility and convergence of the symmetric positive definite solution of the H_∞-PRDE to the periodic stabilizing solution will be derived. Before presenting the main result of this section, some notation is introduced as follows.

First, we define

$$M_k = \hat{C}_k^T \hat{R}_k^{-1} \hat{C}_k, \tag{3.1}$$

where $\hat{R}_k = \hat{C}_k P_k^s \hat{C}_k^T + R$, $\hat{C}_k$ and R are the same as in (2.4), and P_k^s is the periodic stabilizing solution of (2.4). It is clear that M_k is a known real-bounded K-periodic matrix.

Next we define Θ_i $(i \in [0, K-1])$ as follows:

$$\Theta_i = \hat{A}_{K-1+i} \hat{A}_{K-2+i} \cdots \hat{A}_{1+i} \hat{A}_i, \tag{3.2}$$

where $\hat{A}_i (i \in [0, K-1])$ is defined in (2.5). Θ_i is the lifted system matrix of the periodic system $x_{k+1} = \hat{A}_k x_k$ at $i = 0, 1, \ldots, K-1$. We denote the transition matrix as follows:

$$\hat{\Phi}_{k,l} = \begin{cases} \hat{A}_{k-1} \hat{A}_{k-2} \ldots \hat{A}_{l+1} \hat{A}_l, & k > l \\ I, & k = l. \end{cases} \tag{3.3}$$

Because $\hat{A}_i$ is invertible, $\hat{\Phi}_{k,l}$ is invertible for any $k > l$. And because P_k^s is the stabilizing solution, the matrix Θ_i is stable.

We shall also define Ψ_i $(i \in [0, K-1])$ as follows:

$$\Psi_i = \sum_{l=1}^{K} \hat{\Phi}_{K+i,l+i}^{-T} M_{l-1+i}^{-} \hat{\Phi}_{K+i,l+i}^{-1}, \tag{3.4}$$

where M_i $(i \in [0, K-1])$ is as defined in (3.1), and M_i^- is the negative part of M_i.

Finally, we introduce the following K Lyapunov equations:

$$\Theta_i^T Y_i \Theta_i - Y_i = -\Psi_i, \qquad i \in [0, K-1], \tag{3.5}$$

where Θ_i and Ψ_i ($i \in [0, K-1]$) are known real-bounded matrices as defined in (3.2) and (3.4), respectively.

The following theorem gives a sufficient condition for ensuring the feasibility of the solutions of PRDE (2.4).

Theorem 3.1. *Consider the* PRDE (2.4). *Suppose that Assumptions* 2.1 *and* 2.2 *hold, and let*

$$\Delta_i = \left\{\left[\hat{\Phi}_{i,0}^T(\Psi_i - Y_i)\hat{\Phi}_{i,0} - M_0 - \sum_{l=1}^{i} \hat{\Phi}_{l,0}^T M_{l-1}^- \hat{\Phi}_{l,0}\right]^+ + \epsilon I\right\}^{-1}, \forall i \in [0, K-1], \tag{3.6}$$

where $\epsilon > 0$ is sufficiently small, $Y_i (\forall i \in [0, K-1])$ are solutions of Lyapunov equations (3.5), *M_i, $\hat{\Phi}_{k,l}$ and $\Psi_i (\forall i \in [0, K-1])$ are known real bounded matrices as defined in* (3.1), (3.3) *and* (3.4), *respectively, and the notation $[\cdot]^+$ is as defined in Section* 1.

Then, the solution P_k of PRDE (2.4) *is feasible over* $[0, \infty)$ *if the initial state satisfies*

$$0 < P_0 < P_0^s + \Delta_i, \qquad i \in [0, K-1]. \tag{3.7}$$

Proof. The proof of feasibility can be divided into three subcases depending on the relation between P_0 and P_0^s.

(i) *The case when* $P_0 < P_0^s$. Because P_k^s is the feasible solution of (2.4), the feasibility of P_k follows from Lemma 2.1 directly.

(ii) *The case when* $P_0 > P_0^s$.

Let's define $X_k = P_k - P_k^s$. In light of Lemma 2.1, $X_k > 0$. Then, applying Lemma 3.1 of [5] to (2.4), we readily obtain that X_k satisfies

$$\begin{aligned} X_{k+1} &= \hat{A}_k X_k \hat{A}_k^T - \hat{A}_k X_k \hat{C}_k^T (\hat{C}_k X_k \hat{C}_k^T + \hat{R}_k)^{-1} \hat{C}_k X_k \hat{A}_k^T \\ &= \hat{A}_k (X_k^{-1} + \hat{C}_k^T \hat{R}_k^{-1} \hat{C}_k)^{-1} \hat{A}_k^T, \end{aligned} \tag{3.8}$$

where $X_0 = P_0 - P_0^s$, and $\hat{A}_k$ and $\hat{R}_k$ are the same as in (2.5) and (3.1).

According to Lemma 2.2, the feasible condition (2.8) is equivalent to $\eta(\hat{C}_k P_k \hat{C}_k^T + R) < 0$, and noting that $\eta(\hat{C}_k P_k \hat{C}_k^T + R) = \eta(\hat{C}_k X_k \hat{C}_k^T + \hat{R})$, the feasible condition is thus equivalent to $\eta(\hat{C}_k X_k \hat{C}_k^T + \hat{R}) < 0$, which is equivalent to $X_k^{-1} + \hat{C}_k^T \hat{R}_k^{-1} \hat{C}_k > 0$. That is, P_k is feasible if $X_k^{-1} + \hat{C}_k^T \hat{R}_k^{-1} \hat{C}_k > 0$.

Now, let $Z_k = X_k^{-1} + \hat{C}_k^T \hat{R}_k^{-1} \hat{C}_k$. Because $\hat{A}_k$ is invertible, it follows from (3.8) that

$$Z_{k+1} = (\hat{A}_k^{-1})^T Z_k \hat{A}_k^{-1} + M_k, \tag{3.9}$$

where M_k is defined by (3.1) and $Z_0 = (P_0 - P_0^s)^{-1} + M_0$.

It is obvious that the feasibility of P_k is equivalent to the positive definiteness of Z_k. Hence, the proof is reduced to showing that if the initial state P_0 satisfies (3.7), then $Z_k > 0$.

Now consider the following Lyapunov equation:

$$\hat{Z}_{k+1} = \hat{A}_k^{-T} \hat{Z}_k \hat{A}_k^{-1} + M_k^-, \tag{3.10}$$

with the initial state $\hat{Z}_0 = Z_0$. By definition, $M_k \geq M_k^-$, so that $Z_k \geq \hat{Z}_k$. Then $\hat{Z}_k > 0$ is sufficient to guarantee the positivity of Z_k.

Let us now examine (3.10). It is clear that $\hat{A}_k$ and M_k^- are both known real-bounded matrices of finite period K. After some tedious algebraic manipulations, the solution of (3.10) can be expressed as follows:

$$\hat{Z}_{mK+i} = (\Theta_i^T)^{-m}\left[\hat{\Phi}_{i,0}^{-T} Z_0 \hat{\Phi}_{i,0}^{-1} + \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} + \sum_{j=1}^{m} (\Theta_i^T)^j \Psi_i \Theta_i^j\right]\Theta_i^{-m}$$

$$\geq (\Theta_i^T)^{-m}\left[\hat{\Phi}_{i,0}^{-T} Z_0 \hat{\Phi}_{i,0}^{-1} + \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} + \sum_{j=1}^{\infty} (\Theta_i^T)^j \Psi_i \Theta_i^j\right]\Theta_i^{-m},$$

$$i = 0, 1, 2, \ldots, K-1;\ m \in \mathcal{Z}. \tag{3.11}$$

Note that the above set Z_{mK+i} ($i \in [0, K-1]$, $m \in \mathcal{Z}$) forms the complete solution of (3.10).

Referring to Lemma 21.6 of [12], then from (3.5), we have

$$Y_i = \sum_{j=0}^{\infty} (\Theta_i^T)^j \Psi_i \Theta_i^j = \Psi_i + \sum_{j=1}^{\infty} (\Theta_i^T)^j \Psi_i \Theta_i^j, \qquad i \in [0,\ K-1]. \tag{3.12}$$

Comparing (3.11) and (3.12), we have

$$\hat{Z}_{mK+i} \geq (\Theta_i^T)^{-m}\left[\hat{\Phi}_{i,0}^{-T} Z_0 \hat{\Phi}_{i,0}^{-1} + \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} + Y_i - \Psi_i\right]\Theta_i^{-m}, \tag{3.13}$$

where $i \in [0, K-1]$ and $m \in \mathcal{Z}$.

In view of (3.13), if

$$\hat{\Phi}_{i,0}^{-T} Z_0 \hat{\Phi}_{i,0}^{-1} + \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} + Y_i - \Psi_i > 0, \qquad \forall i \in [0, K-1], \tag{3.14}$$

then $\hat{Z}_k > 0$ ($\forall k \in \mathcal{Z}$), and in turn $Z_k > 0$, which means that P_k remains feasibility over $[0, \infty)$. Because $Z_0 = (P_0 - P_0^s)^{-1} + M_0$, (3.14) can be rewritten as

$$\hat{\Phi}_{i,0}^{-T}[(P_0 - P_0^s)^{-1} + M_0]\hat{\Phi}_{i,0}^{-1}$$
$$+ \sum_{l=1}^{i} \hat{\Phi}_{i,l}^{-T} M_{l-1}^- \hat{\Phi}_{i,l}^{-1} + Y_i - \Psi_i > 0, \quad \forall i \in [0, K-1], \tag{3.15}$$

i.e.,

$$(P_0 - P_0^s)^{-1} > \hat{\Phi}_{i,0}^T(\Psi_i - Y_i)\hat{\Phi}_{i,0} - M_0 - \sum_{l=1}^{i} \hat{\Phi}_{l,0}^T M_{l-1}^- \hat{\Phi}_{l,0}, \qquad \forall i \in [0, K-1]. \tag{3.16}$$

On the other hand, (3.7) implies that

$$(P_0 - P_0^s)^{-1} > \left[\hat{\Phi}_{i,0}^T(\Psi_i - Y_i)\hat{\Phi}_{i,0} - M_0 - \sum_{l=1}^{i}\hat{\Phi}_{l,0}^T M_{l-1}^- \hat{\Phi}_{l,0}\right]^+ + \epsilon I, \qquad \forall i \in [0, K-1]. \tag{3.17}$$

Hence, (3.16) holds if P_0 satisfies (3.7) for sufficiently small $\epsilon > 0$. So, (3.14) holds and $\hat{Z}_k > 0$, and, in turn $Z_k > 0$, which means that P_k remains feasible over $[0, \infty)$.

(iii) *The case that* $P_0 - P_0^s$ *is not sign definite.*

There always exists a $\bar{P}_0$ satisfying (3.7) and such that $\bar{P}_0 > P_0$ and $\bar{P}_0 > P_0^s$. Then in view of case (ii), the solution $\bar{P}_k$ of PRDE (2.4) starting from $\bar{P}_0$ is feasible. Finally, the feasibility of P_k readily comes from Lemma 2.1. □

In what follows, we shall study the convergent behavior of the solution of the PRDE (2.4). Let us first introduce the following Lyapunov equation:

$$\Omega^s = \hat{\Phi}_{K,0}^T \Omega^s \hat{\Phi}_{K,0} + \Upsilon, \tag{3.18}$$

where Υ is a constant matrix defined as $\Upsilon = \sum_{j=0}^{K-1} \hat{\Phi}_{j,0}^T \hat{C}_j^T \hat{R}_j^{-1} \hat{C}_j \hat{\Phi}_{j,0}$ and $\hat{\Phi}_{k,l}$ is as defined in (3.3).

The following theorem gives the main result of this paper. It establishes a relationship between the initial state P_0 and the convergence as well as the feasibility of the solution of the PRDE (2.4).

Theorem 3.2. *Let* P_k *be symmetric positive definite solutions of the* PRDE (2.4), *and suppose that Assumptions* 2.1 *and* 2.2 *hold. Then,* P_k *is feasible over* $[0, \infty)$ *and it converges to the stabilizing solution* P_k^s, *i.e.,* $\lim_{k\to\infty}(P_k - P_k^s) = 0$, *if the initial state satisfies*

(i) $0 < P_0 < P_0^s + \Delta_i$ $(\forall i \in [0, K-1])$, *where* Δ_i *is defined as in* (3.6).

(ii) $I + (P_0 - P_0^s)\Omega^s$ *is nonsingular, where* Ω^s *is the solution of Lyapunov equality* (3.18).

Proof. Note that P_k is feasible over $[0, \infty)$ from Theorem 3.1. So in what follows, we focus on the proof of the convergence of P_k.

Similarly to the proof of Theorem 3.1, we define $X_k = P_k - P_k^s$. Then by (3.8), we have

$$\begin{aligned} X_{k+1} &= \hat{A}_k X_k \hat{A}_k^T - \hat{A}_k X_k \hat{C}_k^T (\hat{C}_k X_k \hat{C}_k^T + \hat{R}_k)^{-1} \hat{C}_k X_k \hat{A}_k^T \\ &= \hat{A}_k [I + X_k \hat{C}_k^T \hat{R}_k^{-1} \hat{C}_k]^{-1} X_k \hat{A}_k^T, \end{aligned} \tag{3.19}$$

where the initial state $X_0 = P_0 - P_0^s$, and $\hat{A}_k$ and $\hat{R}_k$ are the same as in (2.5) and (3.1).

After some tedious algebraic manipulations, the solution of (3.19) can be expressed as follows:

$$X_{k+1} = \hat{\Phi}_{k+1,0}(I + X_0\Omega_k)^{-1} X_0 \hat{\Phi}_{k+1,0}^T, \tag{3.20}$$

where

$$\Omega_k = \sum_{j=0}^{k} \hat{\Phi}_{j,0}^T \hat{C}_j^T \hat{R}_j^{-1} \hat{C}_j \hat{\Phi}_{j,0} \tag{3.21}$$

and $\hat{\Phi}_{k,l}$ is as defined in (3.3).

Using the same quasi-lifting techniques as used in the proof of Theorem 3.1, (3.21) can be expressed as the solution of the following Lyapunov equation with K different initial states ($m \in \mathcal{Z}$ and $m \geq 1$):

$$\Omega_{mK+i} = \hat{\Phi}_{K,0}^T \Omega_{(m-1)K+i} \hat{\Phi}_{K,0} + \Upsilon, \qquad (i = 0, 1, 2, \ldots, K-1), \tag{3.22}$$

where Υ is a constant matrix defined as $\Upsilon = \sum_{j=0}^{K-1} \hat{\Phi}_{j,0}^T \hat{C}_j^T \hat{R}_j^{-1} \hat{C}_j \hat{\Phi}_{j,0}$ and K initial states are $\Omega_i = \sum_{j=0}^{i} \hat{\Phi}_{j,0}^T \hat{C}_j^T \hat{R}_j^{-1} \hat{C}_j \hat{\Phi}_{j,0}$ ($i = 0, 1, 2, \ldots, K-1$). Because $\hat{\Phi}_{K,0}$ is stable, Ω_k converges to the stable solution Ω^s as $k \to \infty$. Then it is clear that $\lim_{k\to\infty}(I + X_0\Omega_k)^{-1} = (I + X_0\Omega^s)^{-1} = [I + (P_0 - P_0^s)\Omega^s]^{-1}$.

Hence, considering conditions (i), and (ii), and noting that $\hat{\Phi}_{k,l}$ is stable, we conclude immediately from (3.20) that X_k converges to 0, i.e., $\lim_{k\to\infty}(P_k - P_k^s) = 0$. □

In Theorems 3.1 and 3.2, we have studied the asymptotic behavior of periodic H_∞ *a posteriori* filters and we have given a sufficient condition that guarantees the feasibility and convergence of such filters. Our main idea is based on quasi-lifting techniques. The sufficient condition requires that the initial state P_0 satisfy K inequalities simultaneously and the nonsingularity of a certain matrix. It can be computed by using the MATLAB LMI Toolbox.

Note that when period $K = 0$, i.e., the time-invariant system case, then $\Psi_0 = M_0^-$ and $\Omega^s = 0$, so the sufficient conditions of Theorem 3.2 reduce to those of associated time-invariant H_∞ *a posteriori* filters ([3]).

In general, the conditions of Theorem 3.2 are only sufficient, but it can be easily shown that they are also necessary when $K = 0$ and $\gamma \to \infty$. In such a case, $M_k^- = 0$, $Y_i = 0$, and $\Omega^s = 0$, so the global convergence can be guaranteed with known results on Kalman filtering.

4. Numerical examples

We shall now present two examples to illustrate the results developed in this paper.

Example 4.1. Consider a simple periodic discrete-time system (2.1)–(2.3) with period $K = 2$ and $A_0 = -0.1$, $A_1 = 0.4$, $B_0 = [0.1 \;\; 0.3]$, $B_1 = [-0.4 - 0.2]$, $C = 1$, $D = [0 \;\; 1]$, $L = 1$.

By applying Theorem 2.2, we obtain optimal $\gamma = 0.46$. The corresponding periodic stabilizing solution of PRDE (2.4) is $P_0^s = 0.118$ and $P_1^s = 0.234$.

For the finite-horizon case, according to Theorem 3.2, if the initial state P_0

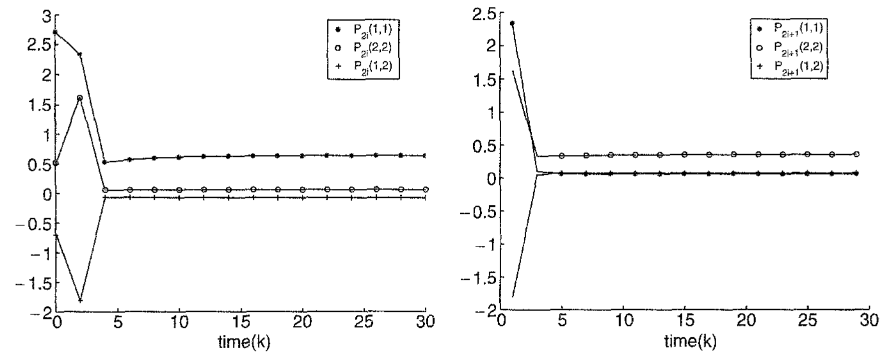

Figure 1. Time-course of P_k of Example 4.2.

satisfies

$$0 < P_0 < 0.169, \tag{4.1}$$

then P_k remains feasible and it converges to the periodic stabilizing solution P_k^s. However, P_0 does not satisfy (4.1), for example, when we let $P_0 = 0.255$, then the solution of the PRDE (2.4) shows that feasibility is lost at the eighth step where $P_8^{-1} - \gamma^{-2}L^T L + C^T C = -0.5907 < 0$.

This simple example illustrates the importance of inequality (2.8) in ensuring the feasibility of PRDE (2.4) in H_∞ *a posteriori* filter design.

Example 4.2. Consider the following second-order periodic discrete-time system with period $K = 2$ and

$$A_0 = \begin{bmatrix} 0.4 & -0.7 \\ -0.2 & 0.1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -0.1 & 0.2 \\ 0.4 & 0.1 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} -0.4 & 0 \\ 0.1 & -0.2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0.3 \end{bmatrix},$$
$$C = [\,1 \quad 0\,], \qquad D = [\,0 \quad 1\,], \qquad L = [\,1 \quad 1\,].$$

Also applying Theorem 2.2, we obtain optimal $\gamma = 0.85$, and the corresponding periodic stabilizing solution of PRDE (2.4) is

$$P_0^s = \begin{bmatrix} 0.6296 & -0.0801 \\ -0.0801 & 0.0548 \end{bmatrix}, \qquad P_1^s = \begin{bmatrix} 0.0608 & 0.0704 \\ 0.0704 & 0.3559 \end{bmatrix}.$$

For the finite-horizon case, by using Theorem 3.2, we could find that if the initial state P_0 satisfies

$$0 < P_0 < P_0^s + \begin{bmatrix} 2.0833 & -0.6032 \\ -0.6032 & 0.4252 \end{bmatrix} = \begin{bmatrix} 2.7134 & -0.6833 \\ -0.6833 & 0.4800 \end{bmatrix} = \bar{P}_0,$$

then P_k remains feasible as well as converges to the periodic stabilizing solution P_k^s, i.e., $\lim_{k\to\infty}(P_k - P_k^s) = 0$. Figure 1 shows the convergence of P_k from the initial state $\bar{P}_0$.

5. Conclusion

In this paper, we have studied the feasible and convergent problem of linear periodic discrete time H_∞ *a posteriori* filters. Based on quasi-lifting techniques, a sufficient condition for ensuring the feasibility and convergence of such filters has been given. The results are supported with numerical examples. The techniques used may also be applied to study the dual problem arising from H_∞ control.

References

[1] S. Bittani and P. Colaneri, Analysis of discrete time linear periodic systems, in *Control and Dynamics Systems*, vol. 78, Academic Press, New York, pp. 313–339, 1996.

[2] P. Bolzern, P. Colaneri, and G. de Nicolao, The periodic Riccati equation, in *The Riccati Equation*, Springer, Heidelberg, pp. 127–162, 1991.

[3] P. Bolzern, P. Colaneri, and G. de Nicolao, Transient and asymptotic analysis of discrete time H_∞-filters, *Eur. J. Control*, 3, 317–324, 1997.

[4] P. Bolzern and M. Maroni, New conditions for the convergence of H_∞ filters and predictors, *IEEE Trans. Autom. Control*, 44, 1564–1568, 1999.

[5] C. E. de Souza, On stabilizing properties of solutions of the Riccati difference equation, *IEEE Trans. Autom. Control*, 34, 1113–1116, 1989.

[6] C. E. de Souza, Periodic strong solution for the optimal filtering problem of linear discrete time periodic systems, *IEEE Trans. Autom. Control*, 36, 333–338, 1991.

[7] B. Hassibi, A. H. Sayed, and T. Kailath, Linear estimation in Krein spaces—part II: application, *IEEE Trans. Autom. Control*, 41, 34–49, 1996.

[8] B. Hassibi, A. H. Sayed, and T. Kailath, *Indefinite-quadratic Estimation and Control*, SIAM Press, Philadelphia, PA, 1999.

[9] L. Xie and C. E. de Souza, H_∞ state estimation for linear periodic systems, *IEEE Trans. Autom. Control*, 38, 1704–1707, 1993.

[10] L. Xie, C. E. de Souza, and M. Fragoso, Robust H_∞ filtering for a class of uncertain periodic systems, in *Proceedings of the 30th IEEE Conference on Decision and Control*, Brighton, England, pp. 527–532, 1991.

[11] B. Xie and V. Syrmos, Robust stabilization for discrete time periodic systems with an H_∞-bounded uncertainty, in *Proceedings of the 36th IEEE Conference on Decision and Control*, San Diego, CA, pp. 2175–2180, 1997.

[12] K. Zhou, J. Doyle, and P. Khargonekar, *Robust and Optimal Control*, Prentice-Hall, Englewood Cliffs, NJ, 1996.