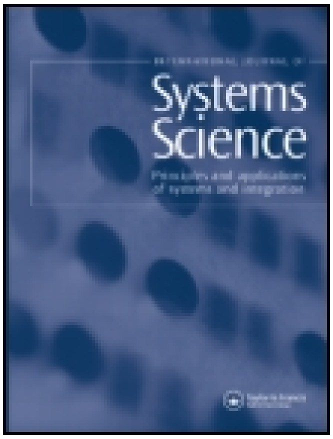

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H∞ filtering for Markovian jump linear systems

C. E. de Souza and M. D. Fragoso*

The problem of $\mathcal{H}_\infty$ filtering for continuous-time linear systems with Markovian jump is investigated. It was assumed that the jumping parameter was available. This paper develops necessary and sufficient conditions for designing a Markovian jump linear filter that ensures a prescribed bound on the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error. The main result is tailored via linear matrix inequalities.

1. Introduction

It is a well-known fact that in a great variety of stochastic modelling problems, it is very difficult to know *precisely* the *statistics* of the additive noise actuating in the system. This is a particularly important issue when we are dealing with what is known in the specialized literature as the *filtering problem*. One way to deal with this issue is to use a very popular measure of performance, the $\mathcal{H}_\infty$-norm, which has been introduced in the robust control setting (e.g. Doyle *et al.* 1989, and references therein). In this context, the filtering problem is known in the literature as the $\mathcal{H}_\infty$ *filtering problem*, and its success can be confirmed, in part, by the amount of available literature on this subject (e.g. Grimble 1988, Bernstein and Haddad 1989, Yaesh and Shaked 1989, Shaked 1990, Banavar and Speyer 1991, Nagpal and Khargonekar 1991, Shaked and Theodor 1992, Xie and de Souza 1993, Geromel and de Oliveira 2001 and references therein). Roughly speaking, in the $\mathcal{H}_\infty$ filtering approach, the noise sources one considers are arbitrary signals with bounded energy, or bounded average power, and the estimator is designed to guarantee that the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error is less than a certain prescribed level.

The subject-matter of this paper is to study the problem of $\mathcal{H}_\infty$ filtering for a class of linear continuous-time systems whose structures are subject to abrupt parameters changes (jumps), modelled here via a continuous-time finite-state Markov chain (it is also known in the literature as the class of Markovian jump linear systems). These changes may be a consequence of random component failures or repairs, abrupt environmental disturbances, changes in the operating point of a nonlinear plant, etc. This can be found, for instance, in control of solar thermal central receivers, robotic manipulator systems, aircraft control systems, large flexible structures for space stations (such as antenna, solar arrays), etc. Several authors have analysed different aspects of such a class and some successful applications have, in part, spurred a considerable interest on it (e.g. Sworder 1969, Wonham 1970, Tugnait 1982, Mariton 1987, Blom and Bar-Shalom 1988, Fragoso 1988, Ji and Chizeck 1990, Yaz 1991, de Souza and Fragoso 1993, Costa 1994 and references therein). In particular, with regard to the filtering problem, minimum variance filtering schemes for discrete-time systems have been studied in, for instance, Tugnait (1982), Blom and Bar-Shalom (1988), Yaz (1991) and Costa (1994). To the best of the authors' knowledge, to date the problem of $\mathcal{H}_\infty$ filtering for this class of systems has not yet been addressed.

The problem addressed here is the design of a Markovian jump linear filter for the above class of Markovian jump linear systems, which provides a mean-square stable estimation error dynamics and a prescribed bound on the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error. Necessary and sufficient conditions in terms of linear matrix inequalities (LMIs) are developed for solving this $\mathcal{H}_\infty$ filtering problem. In the case where there is no jump in the system, the proposed result provides an LMI-based methodology for designing an $\mathcal{H}_\infty$ filter for linear systems with nonzero initial conditions.

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Department of Systems and Control, Laboratório Nacional de Computação Cientifica – LNCC/MCT, Av. Getúlio Vargas 333, 25651-070, Petrópolis, RJ, Brazil.

*To whom correspondence should be addressed. email: frag@lncc.br

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1.1. *Notation*

Throughout this paper, a superscript 'T' stands for a matrix transposition, $\Re^n$ denotes the n-dimensional Euclidean space, $\Re^{n\times m}$ is the set of all $n \times m$ real matrices, $\|\cdot\|$ is the Euclidean vector norm, and $\mathcal{L}_2$ is the space of square integrable vector functions over $[0, \infty)$. For a real matrix P, $P > 0$ (respectively, $P \geq 0$), means that P is symmetric and positive definite (respectively, positive semidefinite) and $\|x\|_P^2$ denotes $x^{\mathrm{T}}Px$ for a real vector x and a real matrix $P > 0$.

2. Problem formulation

Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and consider the following class of stochastic systems:

$$(\Sigma): \quad \dot{x}(t) = A(\theta_t)x(t) + B(\theta_t)w(t);$$

$$x(0) = x_0, \quad \theta_0 = i_0 \tag{1}$$

$$y(t) = C(\theta_t)x(t) + D(\theta_t)w(t) \tag{2}$$

$$z(t) = L(\theta_t)x(t), \tag{3}$$

where $x(t) \in \Re^n$ is the state, $x_0 \in \Re^n$ is an unknown initial state, $w(t) \in \Re^q$ is the noise signal, which is assumed to be an arbitrary signal in $\mathcal{L}_2$, $y(t) \in \Re^m$ is the measurement, and $z(t) \in \Re^p$ is the signal to be estimated. $\{\theta_t\}$ is a homogeneous Markov process with right continuous trajectories and taking values on the finite set $\phi = \{1, 2, \ldots, N\}$ with stationary transition probabilities:

$$\mathrm{Prob}\{\theta_{t+h} = j | \theta_t = i\} = \begin{cases} \lambda_{ij}h + o(h), & i \neq j \\ 1 + \lambda_{ii}h + o(h), & i = j \end{cases},$$

where $h > 0$ and $\lambda_{ij} \geq 0$ is the transition rate from the state i to j, $i \neq j$, and

$$\lambda_{ii} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \lambda_{ij}. \tag{4}$$

The set ϕ comprises the various operation modes of system (Σ) and for each possible value of $\theta_t = i$, $i \in \phi$, the matrices associated with the 'i-th mode' will be denoted by

$$A_i := A(\theta_t = i),\ B_i := B(\theta_t = i),\ C_i := C(\theta_t = i),$$
$$D_i := D(\theta_t = i),\ L_i := L(\theta_t = i),$$

where A_i, B_i, C_i, D_i and L_i are constant matrices for any $i \in \phi$.

It is assumed that the jumping process, $\{\theta_t\}$, is accessible, i.e. the operation mode of system (Σ) is known for every $t \geq 0$.

This paper is concerned with obtaining an estimate, $\hat{z}(t)$, of $z(t)$, via a causal Markovian jump linear filter using the measurement y and which provides a uniformly small estimation error, $z - \hat{z}$, for all $w \in \mathcal{L}_2$.

To put the $\mathcal{H}_\infty$ filtering problem for system (Σ) in a stochastic setting, introduce the space $\mathcal{L}_1[\Omega, \mathcal{F}, \mathcal{P}]$ of $\mathcal{F}$-measurable processes, $z(t) - \hat{z}(t)$, for which

$$\|z - \hat{z}\|_2 := \left\{ E\left[\int_0^\infty [z(t) - \hat{z}(t)]^{\mathrm{T}}[z(t) - \hat{z}(t)\,\mathrm{d}t\right]\right\}^{1/2} < \infty,$$

where $E[\cdot]$ stands for the mathematical expectation. For the sake of notation simplification, $\|\cdot\|_2$ will be used indistinctly to denote the norm either in $\mathcal{L}_2[\Omega, \mathcal{F}, \mathcal{P}]$ or in $\mathcal{L}_2$, defined by

$$\|w\|_2 := \left[\int_0^\infty w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t\right]^{1/2}, \quad \text{for } w \in \mathcal{L}_2.$$

Before formulating the $\mathcal{H}_\infty$ filtering problem, we recall the notion of internal mean square stability and state an important auxiliary result.

Definition 2.1: System (1) is said to be *internally mean square stable*, if the solution of

$$\dot{x}(t) = A(\theta_t)x(t)$$

is such that $E(\|x(t)\|^2) \to 0$, as $t \to \infty$ for arbitrary initial condition (x_0, θ_0). □

Lemma 2.1: *The system* (1) *is internally mean square stable if and only if* $x(t) \in \mathcal{L}_2[\Omega, \mathcal{F}, \mathcal{P}]$ *for every* $w(t) \in \mathcal{L}_2$, *i.e.*

$$\int_0^\infty E[\|x(t)\|^2]\,\mathrm{d}t = c < \infty, \quad \textit{for any } w(t) \in \mathcal{L}_2 \tag{5}$$

with c a positive real number.

Proof: An adaptation of Theorem 5.2 in Fragoso and Costa (2000).

The filtering problem we address in this paper is as follows:

Given an a priori estimate, $\hat{x}_0$, *of the initial state,* x_0, *design a Markovian jump linear filter that provides an estimate,* $\hat{z}(t)$, *of* $z(t)$ *based on* $\{y(\tau),\ 0 \leq \tau \leq t\}$ *and* $\{\theta_\tau,\ 0 \leq \tau \leq t\}$ *such that the estimation error system is internally mean square stable and*

$$\mathcal{J}(\hat{x}_0, R) := \sup_{x_0 \in \Re^n, w \in \mathcal{L}_2} \left\{ \left[\frac{\|z - \hat{z}\|_2^2}{\|w\|_2^2 + \|x_0 - \hat{x}_0\|_R^2}\right]^{1/2} : \|w\|_2^2 + \|x_0 - \hat{x}_0\|_R^2 \neq 0 \right\} < \gamma, \tag{6}$$

where $R > 0$ *is a given weighting matrix for the initial state estimation error,* $\gamma > 0$ *is a given scalar which specifies the level of 'noise' attenuation in the estimation error.*

The weighting matrix R is a measure of the degree of confidence in the estimate $\hat{x}_0$ relative to the uncertainty in w. A 'large' value of R indicates that $\hat{x}_0$ is very close to x_0.

In the case of filtering problems where the effect of the initial state is ignored, without loss of generality, x_0 and $\hat{x}_0$ can be set to zero and (6) is replaced by

$$\mathcal{J}_0 := \sup_{w \in \mathcal{L}_2} \left\{ \frac{\|z - \hat{z}\|_2}{\|w\|_2} : x_0 = 0,\ \hat{x}_0 = 0,\ w \not\equiv 0 \right\} < \gamma. \tag{7}$$

Attention will be focused on the design of an n-th-order filter. Since the matrices in system (Σ) are known at time t (as θ_t is available) and it is required that in the absence of w,

$$E[\|x(t) - \hat{x}(t)\|^2] \to 0 \quad \text{and} \quad E[\|z(t) - \hat{z}(t)\|^2] \to 0, \quad \text{as} \quad t \to \infty, \tag{8}$$

where $\hat{x}(t)$ is the state of the filter, irrespective of the internal mean square stability of (1), without loss of generality, the following structure for the Markovian jump linear filter will be adopted

$$(\Sigma_f): \quad \dot{\hat{x}}(t) = A(\theta_t)\hat{x}(t) + K(\theta_t)[y(t) - C(\theta_t)\hat{x}(t)], \quad \hat{x}(0) = \hat{x}_0 \tag{9}$$

$$\hat{z}(t) = L(\theta_t)\hat{x}(t), \tag{10}$$

where the filter gain matrix $K(\theta_t)$ is to be determined. The reason for that is as follows. Let a Markovian jump linear filter in the following general form

$$(\tilde{\Sigma}_f): \quad \dot{\hat{x}}(t) = A_f(\theta_t)\hat{x}(t) + B_f(\theta_t)y(t) \tag{11}$$

$$\hat{z}(t) = C_f(\theta_t)\hat{x}(t). \tag{12}$$

Hence, in view of (1)–(3) the corresponding estimation error system can be described by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) - \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ A - B_f C - A_f & A_f \end{bmatrix} \begin{bmatrix} x(t) \\ x(t) - \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B \\ B - B_f D \end{bmatrix} w(t)$$

$$z(t) - \hat{z}(t) = [L - C_f \quad C_f] \begin{bmatrix} x(t) \\ x(t) - \hat{x}(t) \end{bmatrix},$$

where the matrices dependence on θ_t has been omitted. The above implies that in order for (8) to hold in the absence of w and irrespective of the internal mean square stability of the system (1), it is required that $A_f(\theta_t) = A(\theta_t) - B_f(\theta_t)C(\theta_t)$ and $C_f(\theta_t) = L(\theta_t)$.

Remark 2.1: Observe that no 'non-singularity assumption', namely

$$D(\theta_t)D^{\mathrm{T}}(\theta_t) > 0, \quad \forall \theta_t \in \phi, \quad \text{and} \quad \forall t \geq 0$$

is imposed to the filtering problem treated in this paper. This is in contrast with the $\mathcal{H}_\infty$ filtering approaches in Nagpal and Khargonekar (1991), Shaked (1990) and Yaesh and Shaked (1989) for linear systems without jumps. □

3. $\mathcal{H}_\infty$ Markovian jump filter

Before presenting our filter design, the motivation for the approach adopted here will be discussed. First, recall that since θ_t is accessible, the system (Σ) could be seen as a linear time-varying system defined by matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$ and $L(t)$ which can be computed online based on the state of the process $\{\theta_t\}$ at time t. Indeed

$$A(t) = A_i,\ B(t) = B_i,\ C(t) = C_i,\ D(t) = D_i,\ L(t) = L_i,$$

when $\theta(t) = i$. This implies that well known results on $\mathcal{H}_\infty$ filtering for linear time-varying systems, such as those in Nagpal and Khargonekar (1991), could be used to solve the $\mathcal{H}_\infty$ filtering problem for system (Σ). Subject to the assumption that

$$S(t) := D(t)D^{\mathrm{T}}(t) > 0, \quad \forall t \geq 0,$$

it follows from the results in Nagpal and Khargonekar (1991) that this $\mathcal{H}_\infty$ filtering problem is solvable via a strictly proper linear filter, if and only if, there exists a bounded symmetric positive definite solution $P(t)$ over $[0, \infty)$ to the Riccati differential equation

$$\dot{P} = (A - BD^{\mathrm{T}}S^{-1}C)P + P(A - BD^{\mathrm{T}}S^{-1}C)^{\mathrm{T}} + P(\gamma^{-2}L^{\mathrm{T}}L - C^{\mathrm{T}}S^{-1}C)P + B(I - D^{\mathrm{T}}S^{-1}D)B^{\mathrm{T}}; \quad P(0) = R^{-1}, \tag{13}$$

such that the time-varying system

$$\dot{\eta}(t) = [A - (PC^{\mathrm{T}} + BD^{\mathrm{T}})S^{-1}C + \gamma^{-2}PL^{\mathrm{T}}L]\eta(t) \tag{14}$$

is exponentially stable. Note that, for simplicity of notation, the time-dependence in the matrices of (13) and (14) has been omitted.

Further, a suitable filter is of the form

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + K(t)[y(t) - C(t)\hat{x}(t)]; \quad \hat{x}(0) = \hat{x}_0 \tag{15}$$

$$\hat{z}(t) = L(t)\hat{x}(t), \tag{16}$$

where

$$K(t) = [P(t)C(t) + B(t)D^{\mathrm{T}}(t)]S^{-1}(t). \tag{17}$$

Note that in view of the above, the filter gain, $K(t)$, of (15) will not be constant while system (Σ) remains in a certain operation mode, i.e. the jumping process remains in a certain state, $j \in \phi$. Thus, the filter of (15) and (16) is not a Markovian jump linear system, which is highly undesirable, as the underlying system (Σ) is a Markovian jump linear system. Another undesirable feature of the above filter is that since the matrices of system (Σ) are not available *a priori*, the existence of a bounded solution to the Riccati differential equation (15) with the required stability property, i.e. the existence of a filter, cannot be ascertained offline. Also note that the computation of the filter gain, $K(t)$, if it exists, can only be carried out online and would involve the solution of a quadratic matrix differential equation with time-varying matrix coefficients, which is numerically unattractive. The filter design proposed in this paper will not exhibit the above undesirable features.

The approach developed here to solve the $\mathcal{H}_\infty$ filtering problem for system (Σ) has the following features:

- Design leads to a Markovian jump linear filter.
- Existence of a filter can be ascertained offline.
- Filter gain, $K(\theta_t)$, can assume N possible values, $K_i = K(\theta_t)$, when $\theta_t = i$, and the matrices K_i, $\forall i \in \phi$, can be calculated offline.

The following theorem presents a solution to the $\mathcal{H}_\infty$ filtering problem for the Markovian jumping linear system (Σ).

Theorem 3.1: *Consider the system (Σ) and let $\gamma > 0$ be a given scalar. Let $\hat{x}_0$ be an a-priori estimate of the initial state and $R > 0$ a given initial state error weighting matrix. Then there exists a Markovian jump filter of the form of (9) and (10) such that the estimation error system is internally mean square stable and $\mathcal{J}(\hat{x}_0, R) < \gamma$ if and only if for all $i \in \phi$ there exist matrices $X_i > 0$ and Y_i satisfying the following LMIs:*

$$\begin{bmatrix} A_i^{\mathrm{T}} X_i + X_i A_i - C_i^{\mathrm{T}} Y_i^{\mathrm{T}} - Y_i C_i + \sum_{j=1}^{N} \lambda_{ij} X_j + L_i^{\mathrm{T}} L_i & X_i B_i - Y_i D_i \\ B_i^{\mathrm{T}} X_i - D_i^{\mathrm{T}} Y_i^{\mathrm{T}} & -\gamma^2 I \end{bmatrix} < 0, \quad \forall i \in \phi \tag{18}$$

$$X_{i_0} - \gamma^2 R \le 0, \tag{19}$$

where i_0 is the state assumed by $\{\theta_t\}$ at $t = 0$. Moreover, a suitable filter is given by

$$\dot{\hat{x}}(t) = A_i \hat{x}(t) + K_i[y(t) - C_i \hat{x}(t)]; \quad \hat{x}(0) = \hat{x}_0 \tag{20}$$

$$\hat{z}(t) = L_i \hat{x}(t) \tag{21}$$

for $\theta_t = i$, $i \in \phi$, where

$$K_i = K(\theta_i = i) = X_i^{-1} Y_i. \tag{22}$$

Proof: *Sufficiency*: First note that by considering (1)–(3) and (20)–(21), and defining $\tilde{x} := x - \hat{x}$, it follows that an estimation error, $\tilde{z} := z - \hat{z}$, can be described by

$$(\Sigma_e): \quad \dot{\tilde{x}}(t) = \tilde{A}(\theta_t)\tilde{x}(t) + \tilde{B}(\theta_t) w(t);$$

$$\tilde{x}(0) = x_0 - \hat{x}_0 \tag{23}$$

$$\tilde{z}(t) = L(\theta_t)\tilde{x}(t), \tag{24}$$

where

$$\tilde{A}(\theta_t) = A(\theta_t) - K(\theta_t) C(\theta_t),$$
$$\tilde{B}(\theta_t) = B(\theta_t) - K(\theta_t) D(\theta_t). \tag{25}$$

In view of (18) and taking into account (22), we have that

$$(A_i - K_i C_i)^{\mathrm{T}} X_i + X_i (A_i - K_i C_i) + \sum_{i=1}^{N} \lambda_{ij} X_j < 0, \ \forall i \in \phi.$$

By Theorem 3.1 and Proposition 3.5 in Feng *et al.* (1992), this implies that the error system of (23) is *internally mean square stable*.

Now for $T > 0$, define the following cost function

$$J(T) := E\left\{ \int_0^{\mathrm{T}} \left[[z(t) - \hat{z}(t)]^{\mathrm{T}} [z(t) - \hat{z}(t)] - \gamma^2 w^{\mathrm{T}}(t) w(t) \right] \mathrm{d}t \right\}. \tag{26}$$

Note that $\{\tilde{x}(t), \theta_t\}$ is a Markov process and the associated one-parameter semigroup of linear Markov transition operators $\{S_h : h \in [0, T]\}$ which characterizes $\{\tilde{x}(t), \theta(t)\}_{t \in [s,T]}$, is given by

$$S_h \cdot g(t, \tilde{x}(t), \theta(t)) := E_{\tilde{z}(t),\theta(t)}[g(t+h, \tilde{x}(t+h), \theta(t+h))]. \tag{27}$$

By the infinitesimal generator of the Markov process $\{\tilde{x}(t)(t), \theta(t)\}_{t \in [s,T]}$, we mean the operator $\mathcal{T}$, defined by

$$\mathcal{T} \cdot g(t, x(t), \theta(t)) = \lim_{h \downarrow 0} \frac{S_h g(t, \tilde{x}(t)(t), \theta(t)) - S_0 g(t, x(t), \theta(t))}{h}, \tag{28}$$

where $g(\cdot)$ is a real continuous, bounded, functional in $\mathcal{D}(\mathcal{T})$ for which the above limit exists (the domain of definition of the operator $\mathcal{T}$). The limit required is the uniform limit (see Fragoso and Baczynski 2001, for a detailed discussion on this subject).

In addition, using (28), (23) and bearing in mind the transition probability for $\{\theta_t\}$, we get easily that the infinitesimal generator in our scenario is given then by

$$\mathcal{T} \cdot g(t, x(t), \theta(t)) = [\tilde{A}(\theta_t)\tilde{x} + \tilde{B}(\theta_t) w(t)]^{\mathrm{T}} g_{\tilde{x}}(\tilde{x}(t), \theta_t) + \sum_{j=1}^{N} \lambda_{\theta_t j} g(\tilde{x}(t), j), \tag{29}$$

where $g(\cdot)$ is a real continuous, bounded, functional in $\mathcal{D}(\mathcal{T})$ with partial derivatives

$$g_{\tilde{x}} := \left[\frac{\partial g}{\partial \tilde{x}_1}, \frac{\partial g}{\partial \tilde{x}_2}, \ldots, \frac{\partial g}{\partial \tilde{x}_n}\right]^{\mathrm{T}},$$

where $\tilde{x}_j$ denotes the j-th component of $\tilde{x}$.

Now, for $g(t, x(t), \theta(t)) = \tilde{x}^{\mathrm{T}}(t)X(\theta_t)\tilde{x}(t)$, adding and subtracting $\mathcal{T} \cdot (\tilde{x}^{\mathrm{T}}(t)X(\theta_t)\tilde{x}(t))$ to (26), where $X(\theta_t) = X_i$ when $\theta_t = i$, $i \in \phi$, and considering (24), (25) and (29), we have:

$$\begin{aligned} J(T) = E\Bigg\{\int_0^{\mathrm{T}} \Bigg\{ & \tilde{x}^{\mathrm{T}}(t)\Bigg[(A(\theta_t) - K(\theta_t)C(\theta_t))^{\mathrm{T}} X(\theta_t) \\ & + X(\theta_t)(A(\theta_t) - K(\theta_t)C(\theta_t)) + L^{\mathrm{T}}(\theta_t)L(\theta_t) \\ & + \sum_{j=1}^{N} \lambda_{\theta_t j} X_j\Bigg]\tilde{x}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) \\ & + \tilde{x}^{\mathrm{T}}(t)X(\theta_t)[B(\theta_t) - K(\theta_t)D(\theta_t)w(t) \\ & + w^{\mathrm{T}}(t)[B(\theta_t) - K(\theta_t)D(\theta_t)]^{\mathrm{T}} X(\theta_t)\tilde{x}(t) \\ & - \mathcal{T} \cdot (\tilde{x}^{\mathrm{T}}(t)X(\theta_t)\tilde{x}(t))\Bigg\}\, \mathrm{d}t\Bigg\}. \end{aligned} \tag{30}$$

Now, since the system of (23) is internally mean square stable and $w \in \mathcal{L}_2[0, \infty)$, it follows from Lemma 2.1 that

$$\lim_{T\to\infty} E[\tilde{x}^{\mathrm{T}}(T)X(\theta_T)\tilde{x}(T)] = 0, \tag{31}$$

as well as that $J(T)$, given by (26), is well defined as $T \to \infty$.

Using Dynkin's formula (Ito and McKean 1965) together with (31) and defining

$$Y(\theta_t) := X(\theta_t)K(\theta_t), \quad \forall \theta_t \in \phi \tag{32}$$

it results from (30) that

$$\begin{aligned} \|z - \hat{z}\|_2^2 - \gamma^2[\|w\|_2^2 + \tilde{x}^{\mathrm{T}}(0)R\tilde{x}(0)] \\ = E\left\{\int_0^{\infty} [\tilde{x}^{\mathrm{T}} \quad \omega^{\mathrm{T}}]\Psi(\theta_t)\begin{bmatrix}\tilde{x} \\ w\end{bmatrix} \mathrm{d}t\right\} \\ + \tilde{x}^{\mathrm{T}}(0)(X_{i_0} - \gamma^2 R)\tilde{x}(0), \end{aligned} \tag{33}$$

where

$$\Psi(\theta_t) = \begin{bmatrix} M(\theta_t) & X(\theta_t)B(\theta_t) - Y(\theta_t)D(\theta_t) \\ B^{\mathrm{T}}(\theta_t)X(\theta_t) - D^{\mathrm{T}}(\theta_t)Y^{\mathrm{T}}(\theta_t) & -\gamma^2 I \end{bmatrix}$$

$$\begin{aligned} M(\theta_t) = {} & A^{\mathrm{T}}(\theta_t)X(\theta_t) + X(\theta_t)A(\theta_t) - C^{\mathrm{T}}(\theta_t)Y^{\mathrm{T}}(\theta_t) \\ & - Y(\theta_t)C(\theta_t) + \sum_{j=1}^{N} \lambda_{\theta_t j} X_j + L^{\mathrm{T}}(\theta_t)L(\theta_t). \end{aligned}$$

Finally, considering the inequalities (18) and (19), the result follows from (33).

Necessity: Suppose that there exists a filter of the form of (9) and (10) such that the estimation error system (Σ_e) given by (23) and (24) is internally mean-square stable and $\mathcal{J}(\hat{x}_0, R) < \gamma$. Consider the Markovian jumping system obtained from (23) and (24) by adding the output $\varepsilon x(t)$, where ε is a positive real number, i.e. let the system

$$\Sigma_{ea}: \quad \dot{\tilde{x}}(t) = \tilde{A}(\theta_t)\tilde{x}(t) + \tilde{B}(\theta_t)w(t); \quad \tilde{x}(0) = x_0 - \hat{x}_0 \tag{34}$$

$$\tilde{z}_a(t) = \begin{bmatrix} L(\theta_t) \\ \varepsilon I \end{bmatrix}\tilde{x}(t). \tag{35}$$

Note that the system (34) is internally mean-square stable.

Considering that $\mathcal{J}(\hat{x}_0, R) < \gamma$ and $\|\tilde{z}_a\|_2^2 = \|\tilde{z}\|_2^2 + \varepsilon^2\|x\|_2^2$, where $\tilde{z}$ is the output of the system (Σ_e), it follows that there exists a sufficiently small real number $\varepsilon > 0$ such that

$$\mathcal{J}(\Sigma_{ea}, \hat{x}_0, R) := \sup_{x_0\in\Re^n, w\in\mathcal{L}_2}\left\{\left[\frac{\|\tilde{z}_a\|_2^2}{\|w\|_2^2 + \|x_0 - \hat{x}_0\|_R^2}\right]^{1/2} : \|w\|_2^2 + \|x_0 - \hat{x}_0\|_R^2 \not\equiv 0\right\} < \gamma.$$

This implies that for the system (Σ_{ea}),

$$\sup_{w\in\mathcal{L}_2}\left\{\frac{\|\tilde{z}_a\|_2}{\|w\|_2} : \tilde{x}(0) = 0,\ w \neq 0\right\} < \gamma.$$

Hence, it follows from Theorem 3.2 in Pan and Basar (1996) that there exist matrices $X_i > 0$, $\forall i \in \phi$, satisfying the following coupled algebraic Riccati equations

$$\begin{aligned} \tilde{A}_i^{\mathrm{T}} X_i + X_i\tilde{A}_i + \gamma^{-2}X_i\tilde{B}_i\tilde{B}_i^{\mathrm{T}} X_i + \sum_{j=1}^{N} \lambda_{ij}X_j \\ + L_i^{\mathrm{T}} L_i + \varepsilon^2 I = 0, \quad \forall i \in \phi. \end{aligned} \tag{36}$$

Moreover, the Markovian jump linear system

$$\dot{\xi}(t) = [\tilde{A}(\theta_t) + \gamma^{-2}\tilde{B}(\theta_t)\tilde{B}^{\mathrm{T}}(\theta_t)X(\theta_t)]\xi(t) \tag{37}$$

is mean-square stable, where $X(\theta_t) = X_i$ when $\theta_t = i$, $i \in \phi$. Note that $X(\theta_t)$ satisfies

$$\begin{aligned} \tilde{A}^{\mathrm{T}}(\theta_t)X(\theta_t) + X(\theta_t)\tilde{A}(\theta_t) + \gamma^{-2}X(\theta_t)\tilde{B}(\theta_t)\tilde{B}^{\mathrm{T}}(\theta_t)X(\theta_t) \\ + \sum_{j=1}^{N} \lambda_{\theta_t j}X(\theta_t) + L^{\mathrm{T}}(\theta_t)L(\theta_t) + \varepsilon^2 I = 0. \end{aligned} \tag{38}$$

By defining $Y_i = X_iK_i$, (36) implies that the matrices X_i and Y_i satisfy the inequalities

$$A_i^{\mathrm{T}} X_i + X_i A_i - C_i^{\mathrm{T}} Y_i^{\mathrm{T}} - Y_i C_i + \gamma^{-2}(X_i B_i - Y_i D_i)^{\mathrm{T}}(X_i B_i - Y_i D_i) + \sum_{j=1}^{N} \lambda_{ij} X_j + L_i^{\mathrm{T}} L_i < 0, \ \forall i \in \phi,$$

which are equivalent to (18). It remains to be shown that $X_{i_0} - \gamma^2 R \le 0$. Suppose that this is not true. Then, there exists $\eta \in \Re^n$, $\eta \ne \hat{x}_0$ such that $(\eta - \hat{x}_0)^{\mathrm{T}}(X_{i_0} - \gamma^2 R)(\eta - \hat{x}_0) > 0$. Similarly to the sufficiency proof, it can be readily established that for the system (Σ_{ea}) and $X(\theta_t)$ as above,

$$\begin{aligned}\mathcal{T} \cdot [\tilde{x}^{\mathrm{T}}(t) X(\theta_t)\tilde{x}(t)] &= \tilde{x}^{\mathrm{T}}(t)\Big[\tilde{A}^{\mathrm{T}}(\theta_t)X(\theta_t) + X(\theta_t)\tilde{A}(\theta_t) \\ &\quad + X(\theta_t)\tilde{A}(\theta_t) + \sum_{j=1}^{N}\lambda_{\theta_t j}X(\theta_t)\Big]\tilde{x}^{\mathrm{T}}(t) \\ &\quad + \tilde{x}^{\mathrm{T}}(t)X(\theta_t)\tilde{B}(\theta_t)w(t) \\ &\quad + w^{\mathrm{T}}(t)\tilde{B}^{\mathrm{T}}(\theta_t)X(\theta_t)\tilde{x} \\ &= \gamma^2 w^{\mathrm{T}}(t)w(t) - \tilde{z}_a^{\mathrm{T}}(t)\tilde{z}_a(t) \\ &\quad - \gamma^2[w(t) - \hat{w}(t)]^{\mathrm{T}}[w(t) - \hat{w}(t)],\end{aligned} \tag{39}$$

where $\hat{w}(t) = \gamma^{-2}\tilde{B}^{\mathrm{T}}(\theta_t)X(\theta_t)\tilde{x}(t)$ and where the second equality has been obtained by considering (38) and the definition of $\tilde{z}_a(t)$ given by (35).

Now choose $x_0 = \eta$ and $w(t) = \hat{w}(t)$, $t \ge 0$, for the system (Σ_{ea}) Under these conditions and since the system (37) is mean square stable, it follows that the system (Σ_{ea}) satisfies

$$\lim_{t\to\infty} E[\tilde{x}^{\mathrm{T}}(t)X(\theta_t)\tilde{x}(t)] = 0.$$

Moreover, integrating (39) from 0 to ∞, taking expectation and using Dynkin's formula, one obtains that

$$-(\eta - \hat{x}_0)^{\mathrm{T}} X_{i_0}(\eta - \hat{x}_0) = \gamma^2\|w\|_2^2 - \|\tilde{z}_a\|_2^2$$

or yet

$$\begin{aligned}&\gamma^2[\|w\|_2^2 + (\eta - \hat{x}_0)^{\mathrm{T}} R(\eta - \hat{x}_0)] - \|\tilde{z}_a\|_2^2 \\ &\quad = (\eta - \hat{x}_0)^{\mathrm{T}}(\gamma^2 R - X_{i_0})(\eta - \hat{x}_0) < 0\end{aligned}$$

which is a contradiction as $\mathcal{J}(\Sigma_{ea}, \hat{x}_0, R) < \gamma$. □

Remark 3.1: Theorem 3.1 provides a method for designing $\mathcal{H}_\infty$ Markovian jump linear filters for linear systems subject to Markovian jumping parameters. The proposed design is given in terms of linear matrix inequalities, which has the advantage that it can be solved numerically very efficiently using recently developed algorithms for solving LMIs.

It should be observed that the problem of designing an optimal $\mathcal{H}_\infty$ Markovian jump linear filter, i.e. for the smallest possible $\gamma > 0$, can be easily solved via the following convex programming problem in $\kappa := \gamma^2$, X_i and Y_i, $\forall i \in \phi$:

$$\begin{aligned}&\text{minimize} \quad \kappa \\ &\text{subject to} \quad \kappa > 0, X_i > 0, (18) \text{ and } (19) \text{ with } \gamma^2 = \kappa.\end{aligned}$$

□

When the effect of the initial state is ignored, without loss of generality, x_0 and $\hat{x}_0$ can be set to zero. Thus, the inequality of (19) will no longer be required as this case corresponds to choosing a sufficiently large R (in the sense that its smallest eigenvalue approaches infinity). In such a situation, Theorem 3.1 specializes as follows:

Corollary 3.1: *Consider the system* (Σ) *and let* $\gamma > 0$ *be a given scalar. Then there exists a Markovian jump filter of the form of* (9) *and* (10) *such that the estimation error system is internally mean square stable and* $\mathcal{J}_0 < \gamma$, *if and only if for all* $i \in \phi$ *there exist matrices* $X_i > 0$ *and* Y_i *such that the LMIs* (18) *are satisfied. Moreover, a suitable filter is given by* (20)–(22) *with* $\hat{x}_0 = 0$.

In the case of one mode operation, i.e. there are no jumps in system (Σ), we have $N = 1$, $\phi = \{1\}$ and $\lambda_{11} = 0$. Denoting the matrices of system (Σ) by A, B, C, D and L, Theorem 3.1 reduces to the following result.

Corollary 3.2: *Consider the system* (Σ) *with no jumps and let* $\gamma > 0$ *be a given scalar. Let* $\hat{x}_0$ *be an a-priori initial state estimate and* $R > 0$ *a given initial state error weighting matrix. Then there exists an n-th order strictly proper linear filter such that the estimation error system is asymptotically stable and* $\mathcal{J}(\hat{x}_0, R) < \gamma$, *if and only if there exist matrices* $X > 0$ *and* Y *satisfying the following LMIs:*

$$\begin{bmatrix} A^{\mathrm{T}}X + XA - C^{\mathrm{T}}Y^{\mathrm{T}} - YC & XB - YD \\ B^{\mathrm{T}}X - D^{\mathrm{T}}Y^{\mathrm{T}} & -\gamma^2 I \end{bmatrix} < 0 \tag{40}$$

$$X - \gamma^2 R \le 0. \tag{41}$$

Moreover, a suitable filter is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K[y(t) - C\hat{x}(t)]; \quad \hat{x}(0) = \hat{x}_0$$
$$\hat{z}(t) = L\hat{x}(t),$$

where $K = X^{-1}Y$.

Remark 3.2: Corollary 3.2 provides LMI based necessary and sufficient conditions for designing an $\mathcal{H}_\infty$ filter for linear systems with non-zero initial conditions. This is in contrast with the approach developed in Nagpal and Khargonekar (1991), which is in terms of an algebraic Riccati equation and is also restricted to "nonsingular" $\mathcal{H}_\infty$ filtering problems. When the effect of the initial state is ignored, Corollary 3.2 reduces to an LMI approach for $\mathcal{H}_\infty$ filtering which has been recently proposed in Geromel and de Oliveira (2001).

4. Conclusions

This paper has addressed the problem of $\mathcal{H}_\infty$ filtering for a class of continuous-time Markovian jump linear systems under the assumption that the jumping parameter is accessible. We have developed necessary and sufficient conditions for the existence of a Markovian jump linear filter such that the resulting estimation error system is internally mean-square stable and the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error is smaller than a prescribed value. The proposed conditions, as well as the filter design, are given in terms of the feasibility of linear matrix inequalities and IM-AGIMB.

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