Curve Sketching

Applications of Derivatives: Sketching Graphs

Differential Calculus

Outline

Why Curve Sketching?

- Derivatives help us understand the shape of a function's graph.
- We use $f(x)$, $f'(x)$, and $f''(x)$ to identify key features:
  - Domain, intercepts, asymptotes
  - Increasing/decreasing, maxima/minima
  - Concavity, inflection points
- Goal: Efficiently sketch accurate graphs using calculus tools.

Domain, Intercepts, and Asymptotes

- **Domain:** Where is $f(x)$ defined? (Watch for denominators, roots, discontinuities)
- **Intercepts:** x-intercepts: solve $f(x) = 0$; y-intercept: $f(0)$
- **Vertical Asymptotes:** Where $f(x) \to \pm\infty$ (often zeros of denominator)
- **Horizontal Asymptotes:** $\lim_{x\to\pm\infty} f(x)$

Example: Domain and Asymptotes

Example: $f(x) = \frac{x+1}{(x+3)(x-2)}$

- Domain: $x \neq -3, 2$
- y-intercept: $f(0) = -\frac{1}{6}$
- x-intercept: $x = -1$
- Vertical asymptotes: $x = -3, 2$
- Horizontal asymptote: $y = 0$

How to Find the First Derivative

Step-by-Step:

1. **Start with the function:** $f(x)$
2. **Apply differentiation rules:**
   - Power rule: $\frac{d}{dx}x^n = nx^{n-1}$
   - Sum/difference rule: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
   - Product/quotient/chain rules as needed
3. **Simplify the result:** Write $f'(x)$ in simplest form

Example:

$$f(x) = x^4 - 6x^3$$
$$f'(x) = 4x^3 - 18x^2$$

How to Test for Increasing/Decreasing

Step-by-Step:

1. **Find** $f'(x)$
2. **Solve** $f'(x) = 0$ to find critical points
3. **Identify intervals** between/around critical points
4. **Test the sign of** $f'(x)$ in each interval:
   - If $f'(x) > 0$ on an interval, $f(x)$ is increasing there
   - If $f'(x) < 0$ on an interval, $f(x)$ is decreasing there

Example:

$$f'(x) = 4x^3 - 18x^2$$

$$4x^3 - 18x^2 = 0 \implies x^2(4x - 18) = 0 \implies x = 0, \frac{9}{2}$$

Test $f'(x)$ in intervals $(-\infty, 0)$, $(0, \frac{9}{2})$, $(\frac{9}{2}, \infty)$.

How to Find the Second Derivative

Step-by-Step:

1. **Start with the first derivative:** $f'(x)$
2. **Differentiate again:** $f''(x) = \frac{d}{dx} f'(x)$
3. **Simplify the result:** Write $f''(x)$ in simplest form

Example:

$$f'(x) = 4x^3 - 18x^2$$
$$f''(x) = 12x^2 - 36x$$

How to Test for Concavity

Step-by-Step:

1. **Find** $f''(x)$
2. **Solve** $f''(x) = 0$ to find possible inflection points
3. **Identify intervals** between/around these points
4. **Test the sign of** $f''(x)$ in each interval:
   - If $f''(x) > 0$ on an interval, $f(x)$ is concave up there
   - If $f''(x) < 0$ on an interval, $f(x)$ is concave down there

Example:

$$f''(x) = 12x^2 - 36x = 12x(x-3)$$

$$12x(x-3) = 0 \implies x = 0, 3$$

Test $f''(x)$ in intervals $(-\infty, 0)$, $(0, 3)$, $(3, \infty)$.

Symmetries

- **Even:** $f(-x) = f(x)$ (symmetric about y-axis)
- **Odd:** $f(-x) = -f(x)$ (symmetric about origin)
- **Periodic:** $f(x + P) = f(x)$ (repeats every P)

Curve Sketching Checklist

1. Domain, intercepts, asymptotes
2. Symmetry (even, odd, periodic)
3. Critical points, singular points
4. Increasing/decreasing intervals
5. Concavity, inflection points

Worked Example

Sketch: $f(x) = x^3 - 3x + 1$

- Domain: all real x
- y-intercept: $f(0) = 1$
- $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$
- Critical points: $x = \pm 1$
- $f''(x) = 6x$, inflection at $x = 0$

Worked Example: Analysis

- $x < -1$: $f'(x) > 0$ (increasing)
- $-1 < x < 1$: $f'(x) < 0$ (decreasing)
- $x > 1$: $f'(x) > 0$ (increasing)
- $x < 0$: $f''(x) < 0$ (concave down)
- $x > 0$: $f''(x) > 0$ (concave up)
- Inflection at $(0, 1)$

Summary

- Use $f(x)$, $f'(x)$, $f''(x)$ to find all key features
- Sketch using intercepts, asymptotes, critical points, inflection points
- Check symmetry and periodicity
- Draw smooth curves between features

Example: $f(x) = x^4 - 4x^3$ (Domain and Intercepts)

Step 1: Domain

- Domain: all real x

Step 2: Intercepts

- y-intercept: $f(0) = 0$
- x-intercepts: $f(x) = 0 \implies x^4 - 4x^3 = 0 \implies x^3(x-4) = 0 \implies x = 0, 4$
- No vertical/horizontal asymptotes (polynomial)

Example: $f(x) = x^4 - 4x^3$ (First Derivative)

Step 3: First Derivative

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

Critical points: $x = 0, 3$

- $x < 0$: $f'(x) < 0$ (decreasing)
- $0 < x < 3$: $f'(x) < 0$ (decreasing)
- $x > 3$: $f'(x) > 0$ (increasing)

Example: $f(x) = x^4 - 4x^3$ (Second Derivative)

Step 4: Second Derivative

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Inflection points: $x = 0, 2$

- $x < 0$: $f''(x) > 0$ (concave up)
- $0 < x < 2$: $f''(x) < 0$ (concave down)
- $x > 2$: $f''(x) > 0$ (concave up)

Example: $f(x) = x^4 - 4x^3$ (Summary)

Step 5: Summary

- Decreasing for $x < 3$, increasing for $x > 3$
- Inflection points at $x = 0, 2$
- Local minimum at $x = 3$

Example: $f(x) = \dfrac{x}{x^2 - 4}$ (Domain and Asymptotes)

Step 1: Domain

- Domain: $x \neq 2, -2$

Step 2: Intercepts and Asymptotes

- y-intercept: $f(0) = 0$
- x-intercept: $x = 0$
- Vertical asymptotes: $x = 2, -2$
- Horizontal asymptote: $y = 0$

Example: $f(x) = \dfrac{x}{x^2 - 4}$ (First Derivative)

Step 3: First Derivative

$$f'(x) = \frac{(x^2-4)\cdot 1 - x\cdot 2x}{(x^2-4)^2} = \frac{x^2 - 4 - 2x^2}{(x^2-4)^2} = \frac{-x^2-4}{(x^2-4)^2}$$

- Numerator always negative, denominator always positive (except at asymptotes)
- $f'(x) < 0$ for all $x \neq 2, -2$ (function always decreasing)

Example: $f(x) = \dfrac{x}{x^2 - 4}$ (Second Derivative)

Step 4: Second Derivative Let $u = -x^2 - 4$, $v = (x^2 - 4)^2$:

$$f''(x) = \frac{u'v - uv'}{v^2}$$

$$u' = -2x, \quad v' = 2(x^2 - 4) \cdot 2x = 4x(x^2 - 4)$$

$$f''(x) = \frac{(-2x)(x^2 - 4)^2 - (-x^2 - 4) \cdot 4x(x^2 - 4)}{(x^2 - 4)^4}$$

$$= \frac{-2x(x^2 - 4)^2 + 4x(x^2 + 4)(x^2 - 4)}{(x^2 - 4)^4}$$

$$= \frac{-2x(x^2 - 4) + 4x(x^2 + 4)}{(x^2 - 4)^3}$$

Example: $f(x) = \dfrac{x}{x^2 - 4}$ (Concavity and Summary)

Step 5: Concavity

- Test sign of $f''(x)$ in intervals between asymptotes.

Step 6: Summary

- Always decreasing
- Vertical asymptotes at $x = 2, -2$
- Horizontal asymptote at $y = 0$

Example: $f(x) = \sin x$ (Domain and Intercepts)

Step 1: Domain

- Domain: all real x

Step 2: Intercepts and Periodicity

- y-intercept: $f(0) = 0$
- x-intercepts: $x = n\pi,\ n \in \mathbb{Z}$
- No vertical/horizontal asymptotes
- Periodic: period 2π

Example: $f(x) = \sin x$ (First Derivative and Increasing/Decreasing)

Step 3: First Derivative

$$f'(x) = \cos x$$

Step 4: Increasing/Decreasing

- $f'(x) = 0$ at $x = \frac{\pi}{2} + n\pi$
- $f'(x) > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ (increasing)
- $f'(x) < 0$ on $(\frac{\pi}{2}, \frac{3\pi}{2})$ (decreasing)

Example: $f(x) = \sin x$ (Second Derivative and Concavity)

Step 5: Second Derivative

$$f''(x) = -\sin x$$

Step 6: Concavity and Summary

- $f''(x) = 0$ at $x = n\pi$
- $f''(x) > 0$ on $(0, \pi)$ (concave up)
- $f''(x) < 0$ on $(-\pi, 0)$ (concave down)
- Inflection points at $x = n\pi$
- Maxima at $x = \frac{\pi}{2} + 2n\pi$, minima at $x = \frac{3\pi}{2} + 2n\pi$
- Periodic, smooth wave

Thank You!

Questions?

Curve sketching is a powerful application of calculus!