Optimisation

Applications of Derivatives: Finding Maximum and Minimum Values

Differential Calculus

Outline

Why Optimisation?

- One important application of differential calculus is to find the maximum (or minimum) value of a function.
- This often finds real world applications in problems such as:
  - Maximizing area with limited materials
  - Minimizing cost while maintaining quality
  - Finding optimal dimensions for containers
  - Determining shortest paths or distances
- Goal: Use derivatives to find where functions achieve their extreme values.

Types of Extrema

- **Global (Absolute) Maximum:** The largest value of $f(x)$ on an interval
- **Global (Absolute) Minimum:** The smallest value of $f(x)$ on an interval
- **Local Maximum:** $f(x) \leq f(c)$ for all x near c
- **Local Minimum:** $f(x) \geq f(c)$ for all x near c

Critical Points and Singular Points

- **Critical Point:** Where $f'(c) = 0$ (derivative exists and equals zero)
- **Singular Point:** Where $f'(c)$ does not exist
- **Important:** Local extrema can occur at:
    - Critical points
    - Singular points
    - Endpoints of the interval

First Derivative Test

Theorem: If $f(x)$ has a local maximum or minimum at $x = c$ and $f'(c)$ exists, then $f'(c) = 0$.
Note: The converse is NOT true! A critical point doesn't guarantee a local extremum.
Example: $f(x) = x^3$ has $f'(0) = 0$ but no local extremum at $x = 0$.

Second Derivative Test

Theorem: Let $f(x)$ be defined on interval I and $c \in I$ with $f'(c) = 0$.

- If $f''(c) < 0$, then $f(x)$ has a local maximum at c
- If $f''(c) > 0$, then $f(x)$ has a local minimum at c
- If $f''(c) = 0$, the test is inconclusive

Note: This test only works when $f'(c) = 0$ and $f''(c)$ exists.

Method for Finding Global Extrema

Corollary: To find the global maximum and minimum of $f(x)$ on $[a, b]$:

1. Find all critical points: solve $f'(x) = 0$
2. Find all singular points: where $f'(x)$ doesn't exist
3. Evaluate $f(x)$ at all critical points, singular points, and endpoints
4. The largest value is the global maximum, smallest is the global minimum

Example: Finding Global Extrema

Find the global extrema of $f(x) = 2x^{5/3} + 3x^{2/3}$ **on** $[-1, 1]$
Solution:

- Endpoints: $f(-1) = 1$, $f(1) = 5$
- $f'(x) = \frac{10x+6}{3x^{1/3}}$
- Singular point: $x = 0$ (denominator zero), $f(0) = 0$
- Critical point: $f'(x) = 0$ when $x = -3/5$, $f(-3/5) \approx 1.28$
- Global maximum: 5 at $x = 1$
- Global minimum: 0 at $x = 0$

General Problem-Solving Strategy

1. **Read** the problem carefully
2. **Draw** a diagram
3. **Define** variables with units
4. **Find** relations between variables
5. **Reduce** to a function of one variable
6. **Find** critical points and evaluate
7. **Check** that answers make sense
8. **Answer** the question asked

Example 1: Fencing Problem

Problem: A farmer has 400m of fencing. What is the largest rectangular paddock that can be enclosed?

Think about:

- What variables should you define?
- What is the constraint equation?
- How can you express area as a function of one variable?
- What is the domain of your function?

Example 1: Fencing Problem - Solution

Solution:

- Let dimensions be x by y metres
- Area: $A = xy$
- Constraint: $2x + 2y = 400$ (use all fencing)
- So $y = 200 - x$
- Area function: $A(x) = x(200 - x) = 200x - x^2$
- Domain: $0 \leq x \leq 200$

Example 1: Fencing Problem - Solution (Continued)

Find maximum area:

- $A'(x) = 200 - 2x$
- Critical point: $200 - 2x = 0 \implies x = 100$
- Evaluate: $A(0) = 0$, $A(100) = 10,000$, $A(200) = 0$
- Maximum area: $10,000\ \mathrm{m}^2$ when $x = 100\mathrm{m}$, $y = 100\mathrm{m}$
- Answer: A square paddock of $100\mathrm{m} \times 100\mathrm{m}$

Example 2: Box Volume

Problem: A rectangular sheet of cardboard is 6" by 9". Four identical squares are cut from the corners and the remaining piece is folded into an open box. What size squares maximize the volume?

Think about:

- What will be the dimensions of the box after folding?
- How can you express volume as a function of the square size?
- What are the constraints on the square size?
- What is the domain of your volume function?

Example 2: Box Volume - Solution

Solution:

- Let square side length be x inches
- Box dimensions: $(9 - 2x) \times (6 - 2x) \times x$
- Volume: $V(x) = x(9 - 2x)(6 - 2x) = 54x - 30x^2 + 4x^3$
- Domain: $0 \leq x \leq 3$ (since $6 - 2x \geq 0$)

Example 2: Box Volume - Solution (Continued)

Find maximum volume:

- $V'(x) = 54 - 60x + 12x^2 = 6(9 - 10x + 2x^2)$
- Critical points: $x = \frac{5\pm\sqrt{7}}{2}$
- Only $x = \frac{5-\sqrt{7}}{2} \approx 1.18$ is in domain
- Evaluate: $V(0) = 0$, $V(1.18) \approx 24.5$, $V(3) = 0$
- Maximum volume: approximately 24.5 cubic inches

Example 3: Distance Problem

Problem: Find the point on the line $y = 6 - 3x$ that is closest to the point $(7, 5)$.
Think about:

- How do you calculate distance between two points?
- How can you express distance as a function of x?
- Is it easier to minimize distance or distance squared?
- What is the domain of your function?

Example 3: Distance Problem - Solution

Solution:

- Let (x, y) be a point on the line
- Distance: $d = \sqrt{(x-7)^2 + (y-5)^2}$
- Since $y = 6 - 3x$: $d = \sqrt{(x-7)^2 + (1-3x)^2}$
- Minimize $d^2 = (x-7)^2 + (1-3x)^2 = 10x^2 - 20x + 50$

Example 3: Distance Problem - Solution (Continued)

Find minimum distance:

- $\frac{d}{dx}(d^2) = 20x - 20$
- Critical point: $20x - 20 = 0 \implies x = 1$
- When $x = 1$: $y = 6 - 3(1) = 3$
- Distance: $d = \sqrt{10(1)^2 - 20(1) + 50} = \sqrt{40} = 2\sqrt{10}$
- Answer: The point $(1, 3)$ is closest to $(7, 5)$

Example 4: Snell's Law

Problem: Use Fermat's principle (light takes the path of least time) to derive Snell's law: $\frac{\sin\theta_i}{\sin\theta_r} = \frac{c_a}{c_w}$

Think about:

- How do you calculate time for light to travel a distance?
- What variables can you use to describe the path?
- How can you express total time as a function of one variable?
- What happens when you minimize this function?

Example 4: Snell's Law - Solution

Solution:

- Let c_a = speed in air, c_w = speed in water
- Total time: $T = \frac{\ell_P}{c_a} + \frac{\ell_Q}{c_w}$
- Minimize T with respect to x (position on interface)
- $\frac{dT}{dx} = -\frac{\sin\theta_i}{c_a} + \frac{\sin\theta_r}{c_w}$
- At minimum: $\frac{\sin\theta_i}{c_a} = \frac{\sin\theta_r}{c_w}$
- Therefore: $\frac{\sin\theta_i}{\sin\theta_r} = \frac{c_a}{c_w}$

Unbounded Domains

Theorem: If $f(x)$ is defined for all real x and:

- $\lim_{x \to \pm\infty} f(x) = +\infty$, then global minimum occurs at a critical or singular point
- $\lim_{x \to \pm\infty} f(x) = -\infty$, then global maximum occurs at a critical or singular point

Example: $f(x) = x^2 - 4x + 5$ has $\lim_{x \to \pm\infty} f(x) = +\infty$, so minimum occurs at critical point $x = 2$.

Practice Problem 1

Problem: Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius 5.
Hint: Use the fact that if a rectangle is inscribed in a circle, its diagonal is the diameter of the circle.

Practice Problem 2

Problem: A cylindrical can is to hold 1000 cm^3. Find the radius and height that minimize the surface area.
Hint: Use the volume constraint to express height in terms of radius.

Practice Problem 3

Problem: Find the point on the parabola $y = x^2$ that is closest to the point $(0, 3)$.
Hint: Minimize the distance squared between (x, x^2) and $(0, 3)$.

Practice Problem 4

Problem: A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.
Hint: Express cost as a function of width, then minimize.

Practice Problem 5

Problem: Find the global maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on the interval $[-1, 4]$.
Hint: Find critical points, evaluate at endpoints, and compare.

Practice Problem 6

Problem: A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is a minimum?
Hint: Let x be the length used for the square, then $(10 - x)$ is used for the triangle.

Practice Problem 7

Problem: Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side length 6 if one side of the rectangle lies on the base of the triangle.
Hint: Use similar triangles to relate the rectangle's height to its width.

Practice Problem 8

Problem: A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, find the dimensions that will allow the maximum amount of light to pass through.
Hint: Express area as a function of the rectangle's width, using the perimeter constraint.

Solution: Rectangle inscribed in circle of radius 5

- Let rectangle have dimensions $2x$ by $2y$
- Diagonal constraint: $(2x)^2 + (2y)^2 = (10)^2$ (diameter $= 10$)
- So $x^2 + y^2 = 25$
- Area: $A = 4xy = 4x\sqrt{25 - x^2}$
- $A'(x) = 4\sqrt{25 - x^2} + 4x \cdot \frac{-x}{\sqrt{25-x^2}} = \frac{4(25-x^2-x^2)}{\sqrt{25-x^2}}$
- Critical point: $25 - 2x^2 = 0 \implies x = \frac{5}{\sqrt{2}}$
- Then $y = \frac{5}{\sqrt{2}}$ (square!)
- Maximum area: $4 \cdot \frac{5}{\sqrt{2}} \cdot \frac{5}{\sqrt{2}} = 50$ square units

Practice Problem 2 - Solution

Solution: Cylindrical can with volume 1000 cm³

- Volume: $V = \pi r^2 h = 1000$
- So $h = \frac{1000}{\pi r^2}$
- Surface area: $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$
- $S(r) = 2\pi r^2 + \frac{2000}{r}$
- $S'(r) = 4\pi r - \frac{2000}{r^2}$
- Critical point: $4\pi r = \frac{2000}{r^2} \implies r^3 = \frac{500}{\pi} \implies r = \sqrt[3]{\frac{500}{\pi}} \approx 5.4$ cm
- $h = \frac{1000}{\pi(5.4)^2} \approx 10.8$ cm

Practice Problem 3 - Solution

Solution: Point on $y = x^2$ closest to $(0, 3)$

- Distance squared: $d^2 = (x-0)^2 + (x^2-3)^2 = x^2 + x^4 - 6x^2 + 9 = x^4 - 5x^2 + 9$
- $\frac{d}{dx}(d^2) = 4x^3 - 10x = 2x(2x^2 - 5)$
- Critical points: $x = 0$ or $x = \pm\sqrt{\frac{5}{2}}$
- Test: $d^2(0) = 9$, $d^2(\sqrt{\frac{5}{2}}) = \frac{25}{4} - \frac{25}{2} + 9 = \frac{11}{4}$
- Minimum at $x = \sqrt{\frac{5}{2}} \approx 1.58$
- Point: $(\sqrt{\frac{5}{2}}, \frac{5}{2})$

Practice Problem 4 - Solution

Solution: Storage container cost optimization

- Let width $= x$, length $= 2x$, height $= h$
- Volume: $2x^2h = 10 \implies h = \frac{5}{x^2}$
- Cost: $C = 10 \cdot 2x^2 + 6 \cdot (2xh + 2 \cdot 2xh) = 20x^2 + 6 \cdot 6xh = 20x^2 + 36xh$
- $C(x) = 20x^2 + 36x \cdot \frac{5}{x^2} = 20x^2 + \frac{180}{x}$
- $C'(x) = 40x - \frac{180}{x^2}$
- Critical point: $40x = \frac{180}{x^2} \implies x^3 = 4.5 \implies x = \sqrt[3]{4.5} \approx 1.65$ m
- Cost: $C(1.65) \approx \$163.50$

Practice Problem 5 - Solution

Solution: Global extrema of $f(x) = x^3 - 3x^2 + 1$ on $[-1, 4]$

- $f'(x) = 3x^2 - 6x = 3x(x - 2)$
- Critical points: $x = 0, 2$
- Evaluate: $f(-1) = -1 - 3 + 1 = -3$
- $f(0) = 1$, $f(2) = 8 - 12 + 1 = -3$, $f(4) = 64 - 48 + 1 = 17$
- Global maximum: 17 at $x = 4$
- Global minimum: -3 at $x = -1$ and $x = 2$

Practice Problem 6 - Solution

Solution: Wire cutting for minimum total area

- Let x = length for square, $(10 - x)$ = length for triangle
- Square side length: $\frac{x}{4}$, area: $(\frac{x}{4})^2 = \frac{x^2}{16}$
- Triangle side length: $\frac{10-x}{3}$, area: $\frac{\sqrt{3}}{4}(\frac{10-x}{3})^2 = \frac{\sqrt{3}}{36}(10-x)^2$
- Total area: $A(x) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(10-x)^2$
- $A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(10-x)$
- Critical point: $\frac{x}{8} = \frac{\sqrt{3}}{18}(10-x) \implies x \approx 4.35$ m
- Use about 4.35 m for square, 5.65 m for triangle

Practice Problem 7 - Solution

Solution: Rectangle in equilateral triangle

- Let rectangle width $= 2x$, height $= y$
- Using similar triangles: $\frac{y}{\sqrt{3}} = \frac{3-x}{3}$
- So $y = \sqrt{3}(1 - \frac{x}{3}) = \sqrt{3} - \frac{x}{\sqrt{3}}$
- Area: $A(x) = 2x \cdot (\sqrt{3} - \frac{x}{\sqrt{3}}) = 2\sqrt{3}x - \frac{2x^2}{\sqrt{3}}$
- $A'(x) = 2\sqrt{3} - \frac{4x}{\sqrt{3}}$
- Critical point: $2\sqrt{3} = \frac{4x}{\sqrt{3}} \implies x = \frac{3}{2}$
- Dimensions: width $= 3$, height $= \frac{\sqrt{3}}{2}$

Practice Problem 8 - Solution

Solution: Norman window with perimeter 30 ft

- Let rectangle width $= 2r$, height $= h$, semicircle radius $= r$
- Perimeter: $2h + 2r + \pi r = 30 \implies h = 15 - r - \frac{\pi r}{2}$
- Area: $A = 2rh + \frac{\pi r^2}{2} = 2r(15 - r - \frac{\pi r}{2}) + \frac{\pi r^2}{2}$
- $A(r) = 30r - 2r^2 - \pi r^2 + \frac{\pi r^2}{2} = 30r - 2r^2 - \frac{\pi r^2}{2}$
- $A'(r) = 30 - 4r - \pi r$
- Critical point: $30 = r(4 + \pi) \implies r = \frac{30}{4+\pi} \approx 4.2$ ft
- $h = 15 - 4.2 - \frac{\pi \cdot 4.2}{2} \approx 4.2$ ft

Summary

- Use derivatives to find critical points ($f'(x) = 0$)
- Check singular points (where $f'(x)$ doesn't exist)
- Evaluate function at critical points, singular points, and endpoints
- Compare values to find global extrema
- Always verify that answers make physical sense
- Optimization problems require careful setup and constraint handling

Questions?

Optimization is a powerful application of calculus in the real world!