1.5 Limits at Infinity and Continuity

Limits as x Approaches Infinity and the Concept of Continuity

Differential Calculus

Outline

What is a Limit at Infinity?

- So far, we've studied $\lim_{x \to a} f(x)$ as x approaches a finite value a.
- Now, we consider what happens as x becomes extremely large (positive or negative).
- This is important for understanding long-term behavior of functions.

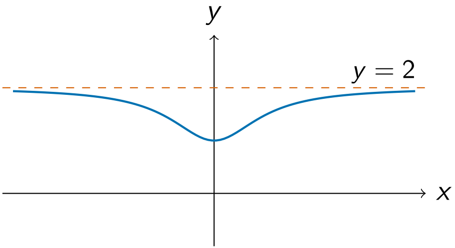
Definition: Limit at Infinity (Informal)

Definition 1.5.1

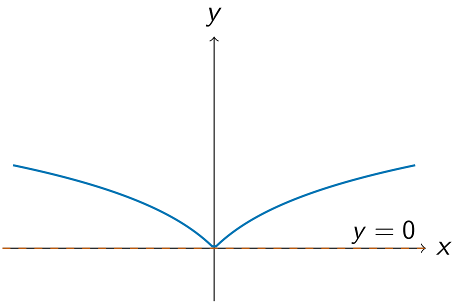
We write $\lim_{x\to\infty} f(x) = L$ if $f(x)$ gets closer and closer to L as x becomes very large and positive.
Similarly, $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ gets closer and closer to L as x becomes very large and negative.

Example: Limits at Infinity

Function with a Limit at $+\infty$ and $-\infty$



Function with No Limit at $-\infty$



Basic Limits at Infinity

Theorem 1.5.3

Let $c \in \mathbb{R}$:

$$\lim_{x\to\infty} c = c \qquad \lim_{x\to-\infty} c = c$$

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

Arithmetic of Limits at Infinity

Theorem 1.5.4

If $\lim_{x\to\infty} f(x) = F$ and $\lim_{x\to\infty} g(x) = G$ exist, then:

- $\lim_{x\to\infty} f(x) \pm g(x) = F \pm G$
- $\lim_{x\to\infty} f(x)g(x) = FG$
- $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{F}{G}$, provided $G \neq 0$
- $\lim_{x\to\infty} f(x)^p = F^p$ (if defined for all x)

Powers and Roots at Infinity

- For all rational $r > 0$, $\lim_{x \to \infty} \frac{1}{x^r} = 0$
- $\lim_{x \to -\infty} \frac{1}{x^r} = 0$ only if denominator of r is not even
- Example: $\lim_{x \to \infty} \frac{1}{x^{1/2}} = 0$, but $\lim_{x \to -\infty} \frac{1}{x^{1/2}}$ does not exist

Example: Rational Function at Infinity

Example 1.5.5

Compute $\lim_{x\to\infty} \frac{x^2-3x+4}{3x^2+8x+1}$

$$\begin{aligned} \frac{x^2-3x+4}{3x^2+8x+1} &= \frac{x^2(1-3/x+4/x^2)}{x^2(3+8/x+1/x^2)} \\ &= \frac{1-3/x+4/x^2}{3+8/x+1/x^2} \\ \lim_{x\to\infty} \frac{x^2-3x+4}{3x^2+8x+1} &= \frac{1}{3} \end{aligned}$$

Example 1.5.6

Compute $\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{5x-1}$

$$\sqrt{4x^2+1} = x\sqrt{4+1/x^2}$$

$$\frac{\sqrt{4x^2+1}}{5x-1} = \frac{x\sqrt{4+1/x^2}}{x(5-1/x)} = \frac{\sqrt{4+1/x^2}}{5-1/x}$$

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{5x-1} = \frac{2}{5}$$

Example: Root Function at $-\infty$

Example 1.5.7

Compute $\lim_{x \to -\infty} \frac{\sqrt{4x^2+1}}{5x-1}$

$$\sqrt{4x^2+1} = |x|\sqrt{4+1/x^2} = -x\sqrt{4+1/x^2} \text{ for } x < 0$$

$$\frac{\sqrt{4x^2+1}}{5x-1} = \frac{-x\sqrt{4+1/x^2}}{x(5-1/x)} = -\frac{\sqrt{4+1/x^2}}{5-1/x}$$

$$\lim_{x \to -\infty} \frac{\sqrt{4x^2+1}}{5x-1} = -\frac{2}{5}$$

Example: Dominant Power at Infinity

Example 1.5.8

Compute $\lim_{x\to\infty} x^{7/5} - x$

$$x^{7/5} - x = x^{7/5}\left(1 - \frac{1}{x^{2/5}}\right)$$

$$\lim_{x\to\infty} x^{7/5} = +\infty$$

$$\lim_{x\to\infty} 1 - 1/x^{2/5} = 1$$

$$\text{So, } \lim_{x\to\infty} x^{7/5} - x = +\infty$$

Arithmetic of Infinite Limits (1/2)

Theorem 1.5.9

Let $f(x), g(x), h(x)$ be functions with $\lim_{x\to a} f(x) = +\infty$, $\lim_{x\to a} g(x) = +\infty$, $\lim_{x\to a} h(x) = H$.

- $\lim_{x\to a} f(x) + g(x) = +\infty$
- $\lim_{x\to a} f(x) + h(x) = +\infty$
- $\lim_{x\to a} f(x) - g(x)$ is undetermined
- $\lim_{x\to a} f(x) - h(x) = +\infty$
- $\lim_{x\to a} cf(x) = +\infty$ if $c > 0$, $-\infty$ if $c < 0$, 0 if $c = 0$
- $\lim_{x\to a} f(x)g(x) = +\infty$

Arithmetic of Infinite Limits (2/2)

Theorem 1.5.9 (cont'd)

- $\lim_{x\to a} f(x)h(x) = +\infty$ if $H > 0$, $-\infty$ if $H < 0$, undetermined if $H = 0$
- $\lim_{x\to a} \frac{f(x)}{g(x)}$ is undetermined
- $\lim_{x\to a} \frac{f(x)}{h(x)} = +\infty$ if $H > 0$, $-\infty$ if $H < 0$, undetermined if $H = 0$
- $\lim_{x\to a} \frac{h(x)}{f(x)} = 0$
- $\lim_{x\to a} f(x)^p = +\infty$ if $p > 0$, 0 if $p < 0$, 1 if $p = 0$

Example: Undetermined Forms

Example 1.5.10

Let $f(x) = x^{-2}$, $g(x) = 2x^{-2}$, $h(x) = x^{-2} - 1$. As $x \to 0$:

$$\lim_{x \to 0} f(x) = +\infty, \quad \lim_{x \to 0} g(x) = +\infty, \quad \lim_{x \to 0} h(x) = +\infty$$

- $\lim_{x \to 0} f(x) - g(x) = \lim_{x \to 0} -x^{-2} = -\infty$
- $\lim_{x \to 0} f(x) - h(x) = \lim_{x \to 0} 1 = 1$
- $\lim_{x \to 0} g(x) - h(x) = \lim_{x \to 0} x^{-2} + 1 = +\infty$

What is Continuity?

Definition 1.6.1

A function $f(x)$ is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$.

- If f is not continuous at a, it is **discontinuous** at a.
- f is **continuous** if it is continuous at every $a \in \mathbb{R}$.

Continuity on Intervals

Definition 1.6.3

A function $f(x)$ is continuous on $[a, b]$ if:

- $f(x)$ is continuous on (a, b)
- $f(x)$ is continuous from the right at a
- $f(x)$ is continuous from the left at b

Types of Discontinuity

- **Jump Discontinuity**: function jumps from one value to another
- **Infinite Discontinuity**: function goes to $+\infty$ or $-\infty$
- **Removable Discontinuity**: function could be made continuous by redefining a single point

Examples: Discontinuity

- $f(x) = \begin{cases} x & x < 1 \\ x + 2 & x \geq 1 \end{cases}$ (jump at $x = 1$)
- $g(x) = \begin{cases} 1/x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$ (infinite at $x = 0$)
- $h(x) = \begin{cases} \frac{x^3 - x^2}{x - 1} & x \neq 1 \\ 0 & x = 1 \end{cases}$ (removable at $x = 1$)

Arithmetic of Continuity

Theorem 1.6.5

If $f(x)$ and $g(x)$ are continuous at a, then so are:

- $f(x) + g(x)$, $f(x) - g(x)$
- $cf(x)$, $f(x)g(x)$
- $\frac{f(x)}{g(x)}$ (if $g(a) \neq 0$)

Continuity of Polynomials and Rational Functions

Theorem 1.6.7

Every polynomial is continuous everywhere. Every rational function is continuous except where its denominator is zero.

Continuity of Common Functions

Theorem 1.6.8

The following are continuous everywhere in their domains:

- Polynomials, rational functions
- Roots and powers
- Trig functions and their inverses
- Exponential and logarithm

Example: Where is $\sin(x)/(2+\cos(x))$ Continuous?

- Numerator $\sin(x)$ is continuous everywhere
- Denominator $2+\cos(x)$ is continuous and never zero
- So $\sin(x)/(2+\cos(x))$ is continuous everywhere

Example: Where is $\sin(x)/(x^2 - 5x + 6)$ Continuous?

- Numerator and denominator are continuous
- Denominator is zero at $x = 2, 3$
- So function is continuous everywhere except $x = 2, 3$

Compositions and Continuity

Theorem 1.6.10

If g is continuous at a and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

Example: Compositions

- $f(x) = \sin(x^2 + \cos(x))$ is continuous everywhere
- $g(x) = \sqrt{\sin(x)}$ is continuous where $\sin(x) \geq 0$

Intermediate Value Theorem (IVT)

Theorem 1.6.12

Let f be continuous on $[a, b]$. If Y is between $f(a)$ and $f(b)$, then there is $c \in [a, b]$ with $f(c) = Y$.

IVT: What Does It Mean?

- If f is continuous on $[a, b]$, then f takes every value between $f(a)$ and $f(b)$ at least once
- The IVT does not say how many such c exist, just that at least one does
- If f is not continuous, IVT may fail

IVT: Real-World Example

- If you start a hike at the bottom and end at the top, you must pass every height in between
- If you and a friend start at different times, you must meet somewhere in between

IVT: Locating Zeros

- If f is continuous and $f(a) < 0$, $f(b) > 0$, then there is $c \in [a, b]$ with $f(c) = 0$
- The bisection method repeatedly halves the interval to locate the zero more precisely

Example: IVT and Bisection

Example 1.6.14

Show $f(x) = x - 1 + \sin(\pi x/2)$ has a zero in $[0, 1]$.

- $f(0) = -1 < 0$, $f(1) = 1 > 0$
- f is continuous (sum of continuous functions)
- By IVT, there is $c \in [0, 1]$ with $f(c) = 0$

Example 1.6.15

Use bisection to find a zero of $f(x) = x - 1 + \sin(\pi x/2)$ in $[0, 1]$.

- $f(0) = -1$, $f(1) = 1$
- $f(0.5) = 0.207 > 0 \rightarrow$ new interval $[0, 0.5]$
- $f(0.25) = -0.367 < 0 \rightarrow$ new interval $[0.25, 0.5]$
- $f(0.375) = -0.069 < 0 \rightarrow$ new interval $[0.375, 0.5]$
- $f(0.4375) = 0.072 > 0 \rightarrow$ new interval $[0.375, 0.4375]$

Practice: 1 and 2

Practice 1:

$$\lim_{x\to\infty} \frac{2x^2-5}{x^2+1}$$

Practice 2:

$$\lim_{x\to-\infty} \frac{3x^3+4x}{2x^3-7}$$

Practice: 3 and 4

Practice 3:

$$\lim_{x \to \infty} \frac{5x - 1}{\sqrt{x^2 + 2}}$$

Practice 4:

$$\lim_{x \to -\infty} \frac{\sqrt{9x^2 + 1}}{2x + 5}$$

Practice 5:

$$\lim_{x\to\infty} \frac{x^3 - 2x}{4x^3 + 1}$$

Practice 6:

Where is $f(x) = \dfrac{x^2 - 4}{x^2 + 1}$ continuous?

Solutions to Practice 1 and 2

Practice 1:

$$\lim_{x \to \infty} \frac{2x^2 - 5}{x^2 + 1}$$

Solution: Divide numerator and denominator by x^2:

$$\frac{2 - 5/x^2}{1 + 1/x^2} \to \frac{2}{1} = 2$$

Practice 2:

$$\lim_{x \to -\infty} \frac{3x^3 + 4x}{2x^3 - 7}$$

Solution: Divide by x^3:

$$\frac{3 + 4/x^2}{2 - 7/x^3} \to \frac{3}{2}$$

Solutions to Practice 3 and 4

Practice 3:

$$\lim_{x\to\infty} \frac{5x-1}{\sqrt{x^2+2}}$$

Solution: For large x, $\sqrt{x^2+2} \sim x$, so $\frac{5x-1}{x} \to 5$. **Practice 4:**

$$\lim_{x\to-\infty} \frac{\sqrt{9x^2+1}}{2x+5}$$

Solution: $\sqrt{9x^2+1} \sim |3x| = -3x$ for $x \to -\infty$, so $\frac{-3x}{2x} \to \frac{-3}{2}$.

Solutions to Practice 5 and 6

Practice 5:

$$\lim_{x\to\infty} \frac{x^3 - 2x}{4x^3 + 1}$$

Solution: Divide by x^3:

$$\frac{1 - 2/x^2}{4 + 1/x^3} \to \frac{1}{4}$$

Practice 6:

$$\text{Where is } f(x) = \frac{x^2 - 4}{x^2 + 1} \text{ continuous?}$$

Solution: Numerator and denominator are continuous everywhere; denominator is never zero, so $f(x)$ is continuous for all x.