Lesson 6: Completing the Square Converting to APQ Form

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I) Perfect Trinomials - Part 1

Key Concepts

- **When a perfect trinomial is factored, both binomials will be equal**
  - $x^2 + 12x + 36 = (x+6)(x+6) = (x+6)^2$
  - $x^2 - 14x + 49 = (x-7)(x-7) = (x-7)^2$
  - $x^2 + 4x + 4 = (x+2)(x+2) = (x+2)^2$

I) Perfect Trinomials - Part 2

Key Concepts (Cont.)

- **The third term in a perfect trinomial is equal to the second term divided by 2 and then squared**
  - $\left(\frac{12}{2}\right)^2 = 6^2 = 36$
  - $\left(\frac{-14}{2}\right)^2 = (-7)^2 = 49$
  - $\left(\frac{4}{2}\right)^2 = 2^2 = 4$
- **The term in the binomial is equal to the second term divided by 2**
  - $\frac{12}{2} = 6$
- **When we CTS, we change the trinomial into a perfect trinomial**

II) What is Completing the Square?

Key Concept

Completing the Square is a process that changes a quadratic function from:

$$\textbf{Standard Form: } y = Ax^2 + Bx + C$$

to

$$\textbf{Vertex Form: } y = a(x-p)^2 + q$$

Example: Completing the Square (Part 1)

Example: y

Starting with $y = x^2 - 8x + 10$:

- **Bracket the first two terms!**

$$y = (x^2 - 8x) + 10$$

- **Divide the second term by 2 and square it!**
- The purpose is to make the expression in the bracket into a perfect square!

$$y = (x^2 - 8x + 16) + 10 - 16$$

- **Take the negative square outside of the brackets!**

$$y = (x^2 - 8x + 16) + (10 - 16)$$

Example: Completing the Square (Part 2)

Example: y

Continuing from $y = (x^2 - 8x + 16) + (10 - 16)$:

- **The trinomial becomes two equal binomials**

$$y = (x - 4)(x - 4) - 6$$

- **Simplified to Vertex Form:**

$$y = (x - 4)^2 - 6$$

- Now the equation is in vertex form: $a = 1$, $p = 4$, $q = -6$.
- Vertex: $(4, -6)$

Practice: Complete the Square and Find the Vertex

Problem

Complete the square and find the vertex for:

$$y = x^2 + 10x + 15$$

Practice: Complete the Square and Find the Vertex (Solution) - Part 1

Solution - Part 1

For $y = x^2 + 10x + 15$:

- **Bracket the first two terms!**

$$y = (x^2 + 10x) + 15$$

- **Divide the second term by 2 and square it!**
- Purpose: Make the expression in the bracket into a perfect square!

$$y = (x^2 + 10x + 25) + 15 - 25$$

Practice: Complete the Square and Find the Vertex (Solution) - Part 2

Solution - Part 2 (Cont.)

Continuing for $y = (x^2 + 10x + 25) + 15 - 25$:

- **Take the negative square outside of the brackets!**

$$y = (x^2 + 10x + 25) + (15 - 25)$$

- **The trinomial becomes two equal binomials**

$$y = (x + 5)(x + 5) - 10$$

- **Now the equation is in vertex form:**

$$y = (x + 5)^2 - 10$$

- Here, $a = 1$, $p = -5$, $q = -10$.
- **Vertex:** $(-5, -10)$

III) CTS with a Leading Coefficient

Key Steps

- **Factor out any coefficient for x^2**

$$y = 3x^2 - 12x + 15$$
$$y = 3(x^2 - 4x) + 15$$

- **Divide the second term by 2 and square it!**
- This makes the expression in the bracket into a perfect square!

$$y = 3\left(x^2 - 4x + \left(\frac{-4}{2}\right)^2\right) + 15 - 3\left(\frac{-4}{2}\right)^2$$

- **Take the negative square outside of the brackets and multiply with coefficient in front!**

$$y = 3\left(x^2 - 4x + 4\right) + 15 - 3(4)$$
$$y = 3(x^2 - 4x + 4) + 15 - 12$$

III) CTS with a Leading Coefficient (Cont.)

Key Steps (Cont.)

Continuing for $y = 3(x^2 - 4x + 4) + 15 - 12$:

- **The trinomial becomes two equal binomials**

$$y = 3(x-2)(x-2) + 3$$

- **Now the equation is in vertex form:**

$$y = 3(x-2)^2 + 3$$

- Here, $a = 3,\ p = 2,\ q = 3.$
- **Vertex:** $(2, 3)$

Practice: Convert to APQ Form (Problem 1)

Problem 1

Convert the following equation to APQ Form:

$$y = 4x^2 + 8x - 5$$

Practice: Convert to APQ Form (Solution 1) - Part 1

Solution 1 - Part 1

For $y = 4x^2 + 8x - 5$:

- **Factor out any coefficient for x^2**

$$y = 4(x^2 + 2x) - 5$$

- **Divide the second term by 2 and square it!**

$$y = 4\left(x^2 + 2x + \left(\frac{2}{2}\right)^2\right) - 5 - 4\left(\frac{2}{2}\right)^2$$
$$y = 4(x^2 + 2x + 1) - 5 - 4(1)$$

Practice: Convert to APQ Form (Solution 1) - Part 2

Solution 1 - Part 2 (Cont.)

Continuing for $y = 4(x^2 + 2x + 1) - 5 - 4$:

- **The trinomial becomes two equal binomials**

$$y = 4(x+1)(x+1) - 9$$

- **Vertex Form:**

$$y = 4(x+1)^2 - 9$$

Practice: Convert to APQ Form (Problem 2)

Problem 2

Convert the following equation to APQ Form:

$$y = \frac{1}{3}x^2 - 6x + 20$$

Practice: Convert to APQ Form (Solution 2) - Part 1

Solution 2 - Part 1

For $y = \frac{1}{3}x^2 - 6x + 20$:

- **Factor out any coefficient for x^2**

$$y = \frac{1}{3}(x^2 - 18x) + 20$$

- **Divide the second term by 2 and square it!**

$$y = \frac{1}{3}(x^2 - 18x + (-9)^2) + 20 - \frac{1}{3}(-9)^2$$
$$y = \frac{1}{3}(x^2 - 18x + 81) + 20 - \frac{1}{3}(81)$$

Practice: Convert to APQ Form (Solution 2) - Part 2

Solution 2 - Part 2 (Cont.)

Continuing for $y = \frac{1}{3}(x^2 - 18x + 81) + 20 - \frac{1}{3}(81)$:

- **Simplify constants**

$$y = \frac{1}{3}(x^2 - 18x + 81) + 20 - 27$$
$$y = \frac{1}{3}(x^2 - 18x + 81) - 7$$

- **The trinomial becomes two equal binomials**

$$y = \frac{1}{3}(x - 9)^2 - 7$$

Practice: Convert to APQ Form (Problem 3)

Problem 3

Convert the following equation to APQ Form:

$$y = -\frac{1}{4}x^2 + 2x - 1$$

Practice: Convert to APQ Form (Solution 3) - Part 1

Solution 3 - Part 1

For $y = -\frac{1}{4}x^2 + 2x - 1$:

- **Factor out any coefficient for x^2**

$$y = -\frac{1}{4}(x^2 - 8x) - 1$$

- **Divide the second term by 2 and square it!**

$$y = -\frac{1}{4}(x^2 - 8x + (-4)^2) - 1 - \left(-\frac{1}{4}\right)(-4)^2$$
$$y = -\frac{1}{4}(x^2 - 8x + 16) - 1 - \left(-\frac{1}{4}\right)(16)$$

Practice: Convert to APQ Form (Solution 3) - Part 2

Solution 3 - Part 2 (Cont.)

Continuing for $y = -\frac{1}{4}(x^2 - 8x + 16) - 1 - \left(-\frac{1}{4}\right)(16)$:

- **Simplify constants**

$$y = -\frac{1}{4}(x^2 - 8x + 16) - 1 + 4$$

- **The trinomial becomes two equal binomials**

$$y = -\frac{1}{4}(x - 4)^2 + 3$$

- **Vertex Form:**

$$y = -\frac{1}{4}(x - 4)^2 + 3$$

Key Steps - Part 1

When an equation is in APQ form, solving for the x-intercepts requires very little algebra.

1. **Step 1:** The "y" coordinate is zero at the x-intercept, so the "Y variable should be zero."

$$y = 2(x + 3)^2 - 18$$
$$0 = 2(x + 3)^2 - 18$$

2. **Step 2:** Isolate the brackets with the exponent.

$$18 = 2(x + 3)^2$$
$$9 = (x + 3)^2$$

IV) Solving Equations in APQ Form - Part 2

Key Steps - Part 2 (Cont.)

Continuing from $(x+3)^2 = 9$:

1. **Step 3:** Square root both sides to get rid of the exponent. Remember: There are two answers! Positive & negative!

$$\begin{aligned} \pm\sqrt{9} &= x+3 \\ \pm 3 &= x+3 \end{aligned}$$

2. **Step 4:** Solve for "x" by adding/subtracting the constant.

$$x = -3 \pm 3$$

3. You now have two answers:
   - $x_1 = -3 + 3 = 0$
   - $x_2 = -3 - 3 = -6$

Practice: Solve for "x" (Problem 1)

Problem 1

Solve for "x":

$$(x+8)^2 - 25 = 0$$

Practice: Solve for "x" (Solution 1) - Part 1

Solution 1 - Part 1

For $(x+8)^2 - 25 = 0$:

$$\begin{aligned}(x+8)^2 &= 25\\ x+8 &= \pm\sqrt{25}\\ x+8 &= \pm 5\end{aligned}$$

Practice: Solve for "x" (Solution 1) - Part 2

Solution 1 - Part 2 (Cont.)

Continuing for $x + 8 = \pm 5$:

$$x = -8 \pm 5$$

So, $x_1 = -8 + 5 = -3$ and $x_2 = -8 - 5 = -13$.

Practice: Solve for "x" (Problem 2)

Problem 2

Solve for "x":

$$5(x-4)^2 - 45 = 0$$

Practice: Solve for "x" (Solution 2)

Solution 2

For $5(x-4)^2 - 45 = 0$:

$$\begin{aligned} 5(x-4)^2 &= 45 \\ (x-4)^2 &= 9 \\ x-4 &= \pm\sqrt{9} \\ x-4 &= \pm 3 \\ x &= 4 \pm 3 \end{aligned}$$

So, $x_1 = 4 + 3 = 7$ and $x_2 = 4 - 3 = 1$.

Practice: Solve for "x" (Problem 3)

Problem 3

Solve for "x":

$$3x^2 + 7 = 0$$

Practice: Solve for "x" (Solution 3)

Solution 3

For $3x^2 + 7 = 0$:

$$3x^2 = -7$$
$$x^2 = -\frac{7}{3}$$

Since a square cannot be negative, there are **No Real Solutions**.

Practice: Solve for "x" by CTS (Problem 4)

Problem 4

Solve for "x" by completing the square:

$$0 = x^2 + 4x + 6$$

Practice: Solve for "x" by CTS (Solution 4) - Part 1

Solution 4 - Part 1

For $0 = x^2 + 4x + 6$:

- **Bracket the first two terms!**

$$0 = (x^2 + 4x) + 6$$

- **Divide the second term by 2 and square it!**

$$0 = (x^2 + 4x + 4) + 6 - 4$$

Solution 4 - Part 2 (Cont.)

Continuing for $0 = (x^2 + 4x + 4) + 6 - 4$:

- **Take the negative square outside of the brackets!**

$$0 = (x^2 + 4x + 4) + 2$$

- **The trinomial becomes two equal binomials**

$$0 = (x + 2)^2 + 2$$

- **Solve for "x" by square rooting both sides:**

$$-2 = (x + 2)^2$$

- Since a square cannot be negative, there are **No Real Solutions**.

Practice: Solve for "x" by CTS (Problem 5)

Problem 5

Solve for "x" by completing the square:

$$0 = 3x^2 + 9x - 6$$

Practice: Solve for "x" by CTS (Solution 5) - Part 1

Solution 5 - Part 1

For $0 = 3x^2 + 9x - 6$:

- **Factor out any coefficient for x^2**

$$0 = 3(x^2 + 3x) - 6$$

- **Divide the second term by 2 and square it!**

$$0 = 3\left(x^2 + 3x + (1.5)^2\right) - 6 - 3(1.5)^2$$
$$0 = 3(x^2 + 3x + 2.25) - 6 - 6.75$$

- **Take the negative square outside of the brackets and multiply with coefficient in front!**

$$0 = 3(x^2 + 3x + 2.25) - 12.75$$

Practice: Solve for "x" by CTS (Solution 5) - Part 2

Solution 5 - Part 2 (Cont.)

Continuing for $0 = 3(x^2 + 3x + 2.25) - 12.75$:

- **The trinomial becomes two equal binomials**

$$0 = 3(x + 1.5)^2 - 12.75$$

- **Solve for "x" by square rooting both sides:**

$$\begin{aligned} 12.75 &= 3(x + 1.5)^2 \\ 4.25 &= (x + 1.5)^2 \\ \pm\sqrt{4.25} &= x + 1.5 \\ x &= -1.5 \pm \sqrt{4.25} \end{aligned}$$

- So, $x_1 = -1.5 + \sqrt{4.25}$ and $x_2 = -1.5 - \sqrt{4.25}$.

Practice: Solve for "x" (Problem 6)

Problem 6

Solve for "x":

$$2x^2 + 16x + 30 = 0$$

Practice: Solve for "x" (Solution 6) - Part 1

Solution 6 - Part 1

For $2x^2 + 16x + 30 = 0$:

- **Factor out any coefficient for x^2**

$$0 = 2(x^2 + 8x) + 30$$

- **Divide the second term by 2 and square it!**

$$0 = 2(x^2 + 8x + (-4)^2) + 30 - 2(-4)^2$$
$$0 = 2(x^2 + 8x + 16) + 30 - 2(16)$$

- **Simplify constants**

$$0 = 2(x^2 + 8x + 16) + 30 - 32$$
$$0 = 2(x^2 + 8x + 16) - 2$$

Practice: Solve for "x" (Solution 6) - Part 2

Solution 6 - Part 2 (Cont.)

Continuing for $0 = 2(x^2 + 8x + 16) - 2$:

- **The trinomial becomes two equal binomials**

$$0 = 2(x+4)^2 - 2$$

- **Solve for "x" by square rooting both sides:**

$$\begin{aligned} 2 &= 2(x+4)^2 \\ 1 &= (x+4)^2 \\ \pm\sqrt{1} &= x+4 \\ x &= -4 \pm 1 \end{aligned}$$

- So, $x_1 = -4 + 1 = -3$ and $x_2 = -4 - 1 = -5$.

Practice: Solve for "x" (Problem 7)

Problem 7

Solve for "x":

$$0.25x^2 - 2x + 1 = 0$$

Practice: Solve for "x" (Solution 7) - Part 1

Solution 7 - Part 1

For $0.25x^2 - 2x + 1 = 0$:

- **Factor out any coefficient for x^2**

$$0 = 0.25(x^2 - 8x) + 1$$

- **Divide the second term by 2 and square it!**

$$0 = 0.25(x^2 - 8x + (-4)^2) + 1 - 0.25(-4)^2$$
$$0 = 0.25(x^2 - 8x + 16) + 1 - 0.25(16)$$

- **Simplify constants**

$$0 = 0.25(x^2 - 8x + 16) + 1 - 4$$

Practice: Solve for "x" (Solution 7) - Part 2

Solution 7 - Part 2 (Cont.)

Continuing for $0 = 0.25(x^2 - 8x + 16) + 1 - 4$:

- **The trinomial becomes two equal binomials**

$$0 = 0.25(x-4)^2 - 3$$

- **Solve for "x" by square rooting both sides:**

$$\begin{aligned} 3 &= 0.25(x-4)^2 \\ 12 &= (x-4)^2 \\ \pm\sqrt{12} &= x - 4 \\ \pm 2\sqrt{3} &= x - 4 \\ x &= 4 \pm 2\sqrt{3} \end{aligned}$$

- So, $x_1 = 4 + 2\sqrt{3}$ and $x_2 = 4 - 2\sqrt{3}$.

IV) CTS with Algebra Tiles

Concept: y

The tiles can be organized together to become a square

- Side 1: $x + 5$
- Side 2: $x + 5$
- So, $(x+5)(x+5) = (x+5)^2$

IV) CTS with Algebra Tiles (Cont.)

Concept: y

The equation is now in vertex form. Create a bunch of zero pairs to complete the square.

- Side 1: $x - 3$
- Side 2: $x - 3$
- Additional constant: -7
- So, $(x-3)^2 - 7$

Word Problem: Rock Thrown into Air (Problem)

Problem

A rock is thrown into the air. The height of the rock is given by the formula:

$$h(t) = -3.5t^2 + 18t + 4$$

Where "H" is the height in meters and "T" is the time after the rock is thrown in seconds.

1. Convert the following equation to APQ form.
2. When will the rock be at the maximum height?
3. What is the maximum height?

Solution Part A: Convert to APQ Form - Part 1

For $h(t) = -3.5t^2 + 18t + 4$:

- **Factor out any coefficient for t^2**

$$h(t) = -3.5\left(t^2 - \frac{18}{3.5}t\right) + 4$$

$$h(t) = -3.5(t^2 - 5.142857t) + 4 \quad \text{(approx.)}$$

- **Divide the second term by 2 and square it!**

$$h(t) = -3.5\left(t^2 - 5.142857t + \left(\frac{-5.142857}{2}\right)^2\right) + 4 - (-3.5)\left(\frac{-5.142857}{2}\right)^2$$

$$h(t) = -3.5(t^2 - 5.142857t + (2.5714285)^2) + 4 + 3.5(2.5714285)^2$$

Word Problem: Rock Thrown into Air (Solution Part A) - Part 2

Solution Part A: Convert to APQ Form - Part 2 (Cont.)

Continuing for $h(t) = -3.5(t^2 - 5.142857t + (2.5714285)^2) + 4 + 3.5(2.5714285)^2$:

- **Simplify and combine constants**

$$h(t) = -3.5(t^2 - 5.142857t + 6.612244) + 4 + 23.142854$$

- **The trinomial becomes two equal binomials**

$$h(t) = -3.5(t - 2.5714285)^2 + 27.142854$$

- **APQ Form:**

$$h(t) = -3.5(t - 2.57)^2 + 27.14 \quad \text{(approx.)}$$

Word Problem: Rock Thrown into Air (Solution Part B C)

Solution Part B C: Max Height

For $h(t) = -3.5(t - 2.57)^2 + 27.14$:

- The rock will be at maximum height at the vertex of the equation.
- From the APQ form, the vertex is $(p, q) = (2.57, 27.14)$.
- **When will the rock be at the maximum height?**
  - **Answer:** Approximately 2.57 seconds.
- **What is the maximum height?**
  - **Answer:** Approximately 27.14 meters.