Lesson 4: The Quadratic Formula

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I) Quadratic Functions in Standard Form: (A, B, C)

Key Concepts

- Most quadratic functions are written in **standard form**:

$$y = ax^2 + bx + c$$

- We know it's quadratic because the largest exponent of "x" is 2.
- The letters "a", "b", and "c" are **coefficients** (real numbers).
- The constant "c" is the **Y-intercept**.
  - If "x" is equal to zero, then $y = c$.
- The coefficient "a" indicates which way the graph opens:
  - If $a > 0$, the parabola opens **up**.
  - If $a < 0$, the parabola opens **down**.

Practice: Find Coefficients and Properties (Problem 1)

Problem 1

For the given quadratic function, identify the coefficients "a", "b", "c", the Y-intercept, and state which way the graph opens:

$$y = 2x^2 - 5x + 7$$

Practice: Find Coefficients and Properties (Solution 1)

Solution 1

For $y = 2x^2 - 5x + 7$:

- $a = 2$
- $b = -5$
- $c = 7$
- **Y-intercept:** $(0, 7)$ (Since $x = 0 \implies y = 7$)
- **Opens:** Up (Since $a = 2 > 0$)

Practice: Find Coefficients and Properties (Problem 2)

Problem 2

For the given quadratic function, identify the coefficients "a", "b", "c", the Y-intercept, and state which way the graph opens:

$$y = -3x^2 + x - 9$$

Practice: Find Coefficients and Properties (Solution 2)

Solution 2

For $y = -3x^2 + x - 9$:

- $a = -3$
- $b = 1$
- $c = -9$
- **Y-intercept:** $(0, -9)$ (Since $x = 0 \implies y = -9$)
- **Opens:** Down (Since $a = -3 < 0$)

Practice: Find Coefficients and Properties (Problem 3)

Problem 3

For the given quadratic function, identify the coefficients "a", "b", "c", the Y-intercept, and state which way the graph opens:

$$y = 5(x-2)^2 + 1$$

(Hint: Expand and simplify to standard form first.)

Practice: Find Coefficients and Properties (Solution 3)

Solution 3

First, expand and simplify $y = 5(x-2)^2 + 1$:

$$y = 5(x^2 - 4x + 4) + 1$$
$$y = 5x^2 - 20x + 20 + 1$$
$$y = 5x^2 - 20x + 21$$

Now, for $y = 5x^2 - 20x + 21$:

- $a = 5$
- $b = -20$
- $c = 21$
- **Y-intercept:** $(0, 21)$
- **Opens:** Up (Since $a = 5 > 0$)

II) The Quadratic Formula

Solving Quadratic Equations

- When solving a quadratic equation in standard form ($ax^2 + bx + c = 0$), we can use the Quadratic Formula to solve for the "x" variable:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The letters "a", "b", and "c" are the coefficients from the standard form.
- **Important Conditions:**
  - $a \neq 0$ (Cannot divide by zero!)
  - $b^2 - 4ac \geq 0$ (Cannot square root a negative number in real numbers)
- The Quadratic Formula can be used to find the "roots" (x-intercepts) without a graphing calculator.
- It is particularly useful for equations that **cannot be factored**.

Conditions for Using the Quadratic Formula

Key Rules

- **One side of the equation must be zero!** (Move all terms to one side).
  - Example: If $3x^2 + 7 = 2x$, rearrange to $3x^2 - 2x + 7 = 0$.
- **Equation must be a Quadratic Function and in "Standard Form":** $ax^2 + bx + c = 0$.
- **Discriminant Condition:** If $(b^2 - 4ac)$ is negative, then you will have "NO Real Solutions"! (No real answer for x).
  - Example: $x^2 + x + 1 = 0$ has $b^2 - 4ac = 1^2 - 4(1)(1) = 1 - 4 = -3 < 0$. So, no real solutions.

Ex: Solve for "x" (Problem 1)

Problem 1

Solve for "x":

$$x^2 + 3x - 10 = 0$$

Ex: Solve for "x" (Solution 1) - Part 1

Solution 1: Identify Coefficients

For $x^2 + 3x - 10 = 0$:

- First, identify coefficients: $a = 1$, $b = 3$, $c = -10$.

Ex: Solve for "x" (Solution 1) - Part 2

Solution 1: Plug into Formula

Plug coefficients into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-10)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{9 + 40}}{2}$$

$$x = \frac{-3 \pm \sqrt{49}}{2}$$

$$x = \frac{-3 \pm 7}{2}$$

Ex: Solve for "x" (Solution 1) - Part 3

Solution 1: Final Answers

You get two answers:

$$x_1 = \frac{-3+7}{2} = \frac{4}{2} = 2$$
$$x_2 = \frac{-3-7}{2} = \frac{-10}{2} = -5$$

Ex: Solve for "x" to 2 decimal places (Problem 2)

Problem 2

Solve for "x" to 2 decimal places:

$$2x^2 - 6x - 5 = 0$$

Ex: Solve for "x" to 2 decimal places (Solution 2) - Part 1

Solution 2: Identify Coefficients

For $2x^2 - 6x - 5 = 0$:

- First, identify coefficients: $a = 2$, $b = -6$, $c = -5$.

Solution 2: Plug into Formula

Plug coefficients into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(-5)}}{2(2)}$$

$$x = \frac{6 \pm \sqrt{36 + 40}}{4}$$

$$x = \frac{6 \pm \sqrt{76}}{4}$$

Ex: Solve for "x" to 2 decimal places (Solution 2) - Part 3

Solution 2: Final Answers

Continue solving for x:

$$x = \frac{6 \pm \sqrt{76}}{4}$$
$$x = \frac{6 \pm 8.7178}{4} \quad \text{(approx.)}$$

You get two answers:

$$x_1 = \frac{6 + 8.7178}{4} = \frac{14.7178}{4} \approx 3.68$$
$$x_2 = \frac{6 - 8.7178}{4} = \frac{-2.7178}{4} \approx -0.68$$

Ex: Solve for "x" (Problem 3 - No Real Solutions)

Problem 3

Solve for "x":

$$x^2 - 4x + 7 = 0$$

Ex: Solve for "x" (Solution 3 - No Real Solutions)

Solution 3

For $x^2 - 4x + 7 = 0$:

- First, identify coefficients: $a = 1$, $b = -4$, $c = 7$.
- Plug coefficients into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16 - 28}}{2}$$

$$x = \frac{4 \pm \sqrt{-12}}{2}$$

- Since we are taking the square root of a negative number (-12), there are **No Real Solutions**.

Using the Quadratic Formula to Find the Vertex - Part 1

Key Ideas

- The quadratic formula gives you the two x-intercepts (if they exist).
- The vertex is exactly in the middle of the two x-intercepts.
- We can find the x-coordinate of the vertex by averaging the two x-intercepts from the quadratic formula. Let's see how this works:

Using the Quadratic Formula to Find the Vertex - Part 2 (Derivation)

Derivation of Vertex Formula

Given the two x-intercepts $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$:

$$x_{\text{vertex}} = \frac{x_1 + x_2}{2}$$

$$x_{\text{vertex}} = \frac{\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) + \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)}{2}$$

Using the Quadratic Formula to Find the Vertex - Part 3 (Derivation)

Derivation of Vertex Formula (Cont.)

Continuing the derivation:

$$x_{\text{vertex}} = \frac{\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}}{2}$$

$$x_{\text{vertex}} = \frac{\frac{-2b}{2a}}{2}$$

$$x_{\text{vertex}} = \frac{-b}{2a}$$

So, the x-coordinate of the vertex is given by:

$$x_{\text{vertex}} = \frac{-b}{2a}$$

Using the Quadratic Formula to Find the Vertex - Part 4

Finding the Y-coordinate

- To find the y-coordinate of the vertex, plug this x_{vertex} value ($\frac{-b}{2a}$) back into the original quadratic equation $y = ax^2 + bx + c$.
- This formula $x_{\text{vertex}} = \frac{-b}{2a}$ is also known as the equation of the **Axis of Symmetry**.

Ex: Find Axis of Symmetry and Vertex (Problem 1)

Problem 1

Given the equation below, find the equation of the Axis of Symmetry (AOS) and the coordinates of the Vertex:

$$y = x^2 - 8x + 15$$

Ex: Find Axis of Symmetry and Vertex (Solution 1) - Part 1

Solution 1: Identify Coefficients

For $y = x^2 - 8x + 15$:

- Identify coefficients: $a = 1$, $b = -8$, $c = 15$.

Ex: Find Axis of Symmetry and Vertex (Solution 1) - Part 2

Solution 1: Axis of Symmetry (AOS)

Axis of Symmetry (AOS):

$$x = \frac{-b}{2a}$$

$$x = \frac{-(-8)}{2(1)}$$

$$x = \frac{8}{2}$$

$$x = 4$$

Solution 1: Vertex Coordinates

Vertex:

- Plug $x = 4$ into the original equation to find the y-coordinate:

$$y = (4)^2 - 8(4) + 15$$
$$y = 16 - 32 + 15$$
$$y = -1$$

- The coordinates of the Vertex are $(4, -1)$.

Ex: Find Axis of Symmetry and Vertex (Problem 2)

Problem 2

Given the equation below, find the equation of the Axis of Symmetry (AOS) and the coordinates of the Vertex:

$$y = -2x^2 - 12x - 10$$

Ex: Find Axis of Symmetry and Vertex (Solution 2)

Solution 2

For $y = -2x^2 - 12x - 10$:

- Identify coefficients: $a = -2$, $b = -12$, $c = -10$.
- **Axis of Symmetry (AOS):**

$$x = \frac{-b}{2a}$$
$$x = \frac{-(-12)}{2(-2)}$$
$$x = \frac{12}{-4}$$
$$x = -3$$

- **Vertex:**
  - Plug $x = -3$ into the original equation to find the y-coordinate:

Word Problem: Ball Toss (Problem)

Problem

A ball is thrown upward from a 1.5-meter platform with an initial velocity of 14 meters per second. The height h (in meters) of the ball after t seconds is given by the formula:

$$h(t) = -4.9t^2 + 14t + 1.5$$

1. When does the ball reach its maximum height?
2. What is the maximum height that the ball reaches?
3. When does the ball hit the ground after it is thrown? (Round to 2 decimal places)

Word Problem: Ball Toss (Solution Part 1A)

Solution Part 1A: Max Height Time

For $h(t) = -4.9t^2 + 14t + 1.5$, we have $a = -4.9$, $b = 14$, $c = 1.5$.

1. **When does the ball reach its maximum height?**
2. To find the time to max height, use $t = \frac{-b}{2a}$:

$$t = \frac{-(14)}{2(-4.9)}$$
$$t = \frac{-14}{-9.8}$$
$$t \approx 1.43 \text{ seconds}$$

Word Problem: Ball Toss (Solution Part 1B)

Solution Part 1B: Max Height Value

For $h(t) = -4.9t^2 + 14t + 1.5$, we have $a = -4.9$, $b = 14$, $c = 1.5$.

2. **What is the maximum height that the ball reaches?**
3. Plug $t \approx 1.43$ seconds into the height formula:

$$\begin{aligned} h(1.43) &= -4.9(1.43)^2 + 14(1.43) + 1.5 \\ h(1.43) &= -4.9(2.0449) + 20.02 + 1.5 \\ h(1.43) &= -10.02 + 20.02 + 1.5 \\ h(1.43) &\approx 11.5 \text{ meters} \end{aligned}$$

Word Problem: Ball Toss (Solution Part 2A.1)

Solution Part 2A.1: Set up Equation

For $h(t) = -4.9t^2 + 14t + 1.5$, we have $a = -4.9$, $b = 14$, $c = 1.5$.

1. **When does the ball hit the ground after it is thrown?**
2. When the ball hits the ground, the height $h(t) = 0$. So, we solve:

$$0 = -4.9t^2 + 14t + 1.5$$

Word Problem: Ball Toss (Solution Part 2A.2A)

Solution 2A.2A: Apply Quadratic Formula

For $0 = -4.9t^2 + 14t + 1.5$, we have $a = -4.9$, $b = 14$, $c = 1.5$.

- Use the Quadratic Formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(14) \pm \sqrt{(14)^2 - 4(-4.9)(1.5)}}{2(-4.9)}$$

Word Problem: Ball Toss (Solution Part 2A.2B.1A)

Solution 2A.2B.1A: Simplify Square Root

For $0 = -4.9t^2 + 14t + 1.5$, we have $a = -4.9$, $b = 14$, $c = 1.5$.

- Continuing the simplification of the Quadratic Formula:

$$t = \frac{-14 \pm \sqrt{196 + 29.4}}{-9.8}$$

Word Problem: Ball Toss (Solution Part 2A.2B.1B)

Solution 2A.2B.1B: Intermediate Result

For $0 = -4.9t^2 + 14t + 1.5$, we have $a = -4.9$, $b = 14$, $c = 1.5$.

- Continuing the simplification:

$$t = \frac{-14 \pm \sqrt{225.4}}{-9.8}$$

Solution Part 2B.1: Approximate Value

For $h(t) = -4.9t^2 + 14t + 1.5$, we have $a = -4.9$, $b = 14$, $c = 1.5$.

- Calculating the approximate value:

$$t = \frac{-14 \pm \sqrt{225.4}}{-9.8}$$
$$t = \frac{-14 \pm 15.013}{-9.8} \quad \text{(approx.)}$$

Word Problem: Ball Toss (Solution Part 2B.2)

Solution Part 2B.2: Final Answers and Conclusion

For $h(t) = -4.9t^2 + 14t + 1.5$, we have $a = -4.9,\ b = 14,\ c = 1.5$.

- Two possible values for t:

$$t_1 = \frac{-14 + 15.013}{-9.8} = \frac{1.013}{-9.8} \approx -0.10 \text{ seconds (extraneous)}$$
$$t_2 = \frac{-14 - 15.013}{-9.8} = \frac{-29.013}{-9.8} \approx 2.96 \text{ seconds}$$

- Since time cannot be negative, the ball hits the ground after approximately **2.96 seconds**.

Word Problem: Rocket Launch (Problem)

Problem

A small rocket is launched from a height of 5 meters above the ground. Its height h (in meters) above the ground t seconds after launch is modeled by the equation:

$$h(t) = -3t^2 + 18t + 5$$

The rocketś tracking device is designed to activate when the rocket is at a height of 20 meters. After how many seconds should the tracking device activate on its way down? (Round to 2 decimal places)

Word Problem: Rocket Launch (Solution Part 1)

Solution Part 1: Set up Equation and Identify Coefficients

We want to find t when $h(t) = 20$. So, set the equation:

$$\begin{aligned} 20 &= -3t^2 + 18t + 5 \\ 0 &= -3t^2 + 18t + 5 - 20 \\ 0 &= -3t^2 + 18t - 15 \end{aligned}$$

For $0 = -3t^2 + 18t - 15$, we have $a = -3$, $b = 18$, $c = -15$.

Word Problem: Rocket Launch (Solution Part 2A.1)

Solution Part 2A.1: Apply Quadratic Formula

For $0 = -3t^2 + 18t - 15$, we have $a = -3$, $b = 18$, $c = -15$.

- Use the Quadratic Formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(18) \pm \sqrt{(18)^2 - 4(-3)(-15)}}{2(-3)}$$

Word Problem: Rocket Launch (Solution Part 2A.2)

Solution Part 2A.2: Simplify Quadratic Formula

For $0 = -3t^2 + 18t - 15$, we have $a = -3$, $b = 18$, $c = -15$.

- Continuing the simplification:

$$t = \frac{-18 \pm \sqrt{324 - 180}}{-6}$$

$$t = \frac{-18 \pm \sqrt{144}}{-6}$$

Solution Part 2B.1: Calculate Values

For $0 = -3t^2 + 18t - 15$, we have $a = -3$, $b = 18$, $c = -15$.

- Continuing the calculation:

$$t = \frac{-18 \pm 12}{-6}$$

Word Problem: Rocket Launch (Solution Part 2B.2)

Solution Part 2B.2: Final Conclusion

For $0 = -3t^2 + 18t - 15$, we have $a = -3$, $b = 18$, $c = -15$.

- Two possible values for t:

$$t_1 = \frac{-18 + 12}{-6} = \frac{-6}{-6} = 1 \text{ second (on the way up)}$$
$$t_2 = \frac{-18 - 12}{-6} = \frac{-30}{-6} = 5 \text{ seconds (on the way down)}$$

- The tracking device should activate on its way down after approximately **5 seconds**.

III) Where Does the Quadratic Formula Come From?

Derivation using Completing the Square

We will derive the Quadratic Formula by taking the standard quadratic equation and applying the method of **Completing the Square**. Then, we will isolate "x".
Starting with the standard form:

$$ax^2 + bx + c = 0$$

Follow the next slides for step-by-step derivation.

Derivation Step 1: Isolate Constant Term

Step 1

Move the constant term "c" to the right side of the equation:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ ax^2 + bx &= -c \end{aligned}$$

Derivation Step 2: Divide by "a"

Step 2

Divide the entire equation by the coefficient "a" (since $a \neq 0$):

$$\frac{ax^2}{a} + \frac{bx}{a} = \frac{-c}{a}$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Derivation Step 3: Complete the Square

Step 3

To complete the square on the left side, take half of the coefficient of "x" (which is $\frac{b}{a}$), square it, and add it to both sides. Half of $\frac{b}{a}$ is $\frac{b}{2a}$. Squaring it gives $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Derivation Step 4: Factor and Combine Terms

Step 4

Factor the perfect square trinomial on the left side and combine the terms on the right side by finding a common denominator ($4a^2$):

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} \cdot \frac{4a}{4a} + \frac{b^2}{4a^2}$$
$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Derivation Step 5: Take the Square Root

Step 5

Take the square root of both sides. Remember to include the $\pm$ sign on the right side:

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2|a|}$$

Note: We can use $2a$ instead of $2|a|$ because the $\pm$ already accounts for both positive and negative cases.

Derivation Step 6: Isolate "x"

Step 6

Subtract $\frac{b}{2a}$ from both sides to isolate "x":

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is the Quadratic Formula.