Chapter 5.1: Rational Functions

Simplifying Rational Expressions and NPV's

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June 16, 2025

What are Rational Expressions?

Definition

- A rational expression is a fraction where both the numerator and denominator are polynomials
- In this section, the numerator and denominator are binomials and trinomials that can be factored
- "x" can't be an exponent or inside a radical, exponents of 'x' must be integers [not fractions]

Example

$$\begin{aligned}\frac{x^2-3x-4}{x^2-16} &= \frac{(x-4)(x+1)}{(x+4)(x-4)} \\ &= \frac{x+1}{x+4}\end{aligned}$$

Are These Rational Expressions?

Practice

Indicate whether the following expressions are rational expressions:

1. $\frac{x^2-4x-9}{x^3-3}$
2. $\frac{x^3-8}{x^4-5}$
3. $\frac{x^2-7}{x^3-10}$
4. $\frac{x^3-9x^2+4}{x^2-12}$
5. $\frac{x^4-1}{x^3+3}$
6. $\frac{x^2-3}{x^4+5}$
7. $\frac{x^4-7x^2+2}{x^3-8}$
8. $\frac{x^3-5}{x^2+3}$
9. $\frac{x^4-6}{x^3+x}$

Are These Rational Expressions? - Solutions Part 1

Detailed Solutions

1. $\frac{x^2-4x-9}{x^3-3}$
   Yes - Both numerator and denominator are polynomials with integer exponents
2. $\frac{x^3-8}{x^4-5}$
   Yes - Both numerator and denominator are polynomials with integer exponents
3. $\frac{x^2-7}{x^3-10}$
   Yes - Both numerator and denominator are polynomials with integer exponents
4. $\frac{x^3-9x^2+4}{x^2-12}$
   Yes - Both numerator and denominator are polynomials with integer exponents

Detailed Solutions

5. $\frac{x^4-1}{x^3+3}$
   Yes - Both numerator and denominator are polynomials with integer exponents
6. $\frac{x^2-3}{x^4+5}$
   Yes - Both numerator and denominator are polynomials with integer exponents
7. $\frac{x^4-7x^2+2}{x^3-8}$
   Yes - Both numerator and denominator are polynomials with integer exponents
8. $\frac{x^3-5}{x^2+3}$
   Yes - Both numerator and denominator are polynomials with integer exponents
9. $\frac{x^4-6}{x^3+x}$
   Yes - Both numerator and denominator are polynomials with integer exponents

Are These Rational Expressions? - Explanation

Key Points

- All expressions are rational expressions because:
  - Both numerator and denominator are polynomials
  - All exponents are integers
  - No radicals in the expressions
  - No fractional exponents
- Examples of expressions that would NOT be rational:
  - $\frac{\sqrt{x}}{x^2}$ (contains radical)
  - $\frac{x^{1/2}}{x^3}$ (contains fractional exponent)
  - $\frac{\sin x}{x^2}$ (contains trigonometric function)

More Examples of Non-Rational Expressions (1/2)

Examples: Radicals Fractional Exponents

1. Expressions with Radicals:
   - $\frac{\sqrt{x+1}}{x^2}$ (square root in numerator)
   - $\frac{x^2}{\sqrt[3]{x-2}}$ (cube root in denominator)
   - $\frac{\sqrt{x^2+1}}{x+1}$ (square root of polynomial)
2. Expressions with Fractional Exponents:
   - $\frac{x^{3/2}}{x^2}$ (fractional exponent in numerator)
   - $\frac{x^2}{x^{-1/3}}$ (negative fractional exponent)
   - $\frac{(x+1)^{1/4}}{x^2}$ (fractional exponent of binomial)

More Examples of Non-Rational Expressions (2/2)

Examples: Trig

3. Expressions with Trigonometric Functions:
   - $\frac{\sin x}{x^2}$ (sine function)
   - $\frac{x^2}{\cos x}$ (cosine function)
   - $\frac{\tan x}{x+1}$ (tangent function)
4. Expressions with Logarithms:
   - $\frac{\ln x}{x^2}$ (natural logarithm)
   - $\frac{x^2}{\log x}$ (common logarithm)
   - $\frac{\log_2(x+1)}{x}$ (logarithm with base 2)
5. Mixed Non-Rational Expressions:
   - $\frac{\sqrt{x}+\sin x}{x^2}$ (combination of radical and trig)
   - $\frac{x^{1/2}+\ln x}{x+1}$ (combination of fractional exponent and log)
   - $\frac{\sqrt{x^2+1}+\cos x}{x^3}$ (combination of radical and trig)

Why These Are Not Rational Expressions (1/2)

Explanation: Radicals Fractional Exponents

- **Radicals**:
  - Cannot be written as polynomials
  - Involve non-integer exponents
  - Example: $\sqrt{x} = x^{1/2}$
- **Fractional Exponents**:
  - Not allowed in polynomials
  - Cannot be simplified to integer exponents
  - Example: $x^{3/2} = \sqrt{x^3}$

Why These Are Not Rational Expressions (2/2)

Explanation: Trig Functions Logarithms

- **Trigonometric Functions**:
  - Not polynomials
  - Cannot be expressed as finite sums of terms
  - Example: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
- **Logarithms**:
  - Not polynomials
  - Cannot be expressed as finite sums of terms
  - Example: $\ln x$ cannot be written as a polynomial

Simplifying Rational Expressions

Rules

- You can only simplify fractions when you have a common factor in both the numerator & denominator
- When simplifying binomials, factor out the common factor first, then simplify
- You can only cancel out a binomial when it's a common factor in both the Numerator & Denominator

Common Mistakes

- You CANNOT cancel a common term from the top and bottom if it is added or subtracted!
- Example: $\frac{x+3}{x+5} \neq \frac{3}{5}$
- Example: $\frac{x+5}{x-5} \neq 1$
- Example: $\frac{2x+10}{x+5} \neq 4$

Practice: Factor and Simplify

Problems

Simplify each expression:

1. $\frac{a^2-6a-3}{a-3}$
2. $\frac{a^2-2ab-b^2}{a^2+ab}$
3. $\frac{3xy^2-18y}{y^2-2xy}$
4. $\frac{3a^2+3ab-60b^2}{2a^2+4ab-48b^2}$

Practice: Solutions Part 1

Detailed Solutions

1. $\frac{a^2-6a-3}{a-3}$

$$\begin{aligned} &= \frac{(a-3)(a+1)}{a-3} \\ &= a+1 \end{aligned}$$

2. $\frac{a^2-2ab-b^2}{a^2+ab}$

$$\begin{aligned} &= \frac{(a-b)(a+b)}{a(a+b)} \\ &= \frac{a-b}{a} \end{aligned}$$

Practice: Solutions Part 2

Detailed Solutions

3. $\frac{3xy^2-18y}{y^2-2xy}$

$$= \frac{3y(xy-6)}{y(y-2x)}$$
$$= \frac{3(xy-6)}{y-2x}$$

4. $\frac{3a^2+3ab-60b^2}{2a^2+4ab-48b^2}$

$$= \frac{3(a^2+ab-20b^2)}{2(a^2+2ab-24b^2)}$$
$$= \frac{3(a+5b)(a-4b)}{2(a+6b)(a-4b)}$$
$$= \frac{3(a+5b)}{2(a+6b)}$$

Non-Permissible Values (NPV)

Definition

- Permissible $\rightarrow$ Allowed, Non-Permissible $\rightarrow$ Not Allowed
- When evaluating rational expressions, the denominator is not allowed to be Zero
- Can't divide by zero $\rightarrow$ Undefined!
- Any value of "x" that makes the denominator equal to zero is called a NPV

Steps to Find NPV

1. Take the entire denominator
2. Make it equal to zero
3. Solve for "x"
4. Factor the denominator
5. These are values that "x" cannot be

Find the NPV

Practice

Find the non-permissible values for each expression:

1. $\frac{3x-6}{6x+2}$
2. $\frac{2x^2-2x-8}{4x^2-81}$
3. $\frac{5x^2-10x+12}{x^2-11x+30}$
4. $\frac{2x^2-2x-8}{x^2-7xy+10y^2}$

NPV Solutions Part 1

Detailed Solutions

1. $\frac{3x-6}{6x+2}$

$$\begin{aligned} 6x + 2 &= 0 \\ 6x &= -2 \\ x &= -\frac{1}{3} \end{aligned}$$

NPV: $x \neq -\frac{1}{3}$

2. $\frac{2x^2-2x-8}{4x^2-81}$

$$\begin{aligned} 4x^2 - 81 &= 0 \\ (2x+9)(2x-9) &= 0 \\ x &= -\frac{9}{2} \text{ or } x = \frac{9}{2} \end{aligned}$$

NPV: $x \neq \pm\frac{9}{2}$

NPV Solutions Part 2

Detailed Solutions

3. $\frac{5x^2-10x+12}{x^2-11x+30}$

$$\begin{aligned} x^2 - 11x + 30 &= 0 \\ (x-5)(x-6) &= 0 \\ x &= 5 \text{ or } x = 6 \end{aligned}$$

NPV: $x \neq 5, 6$

4. $\frac{2x^2-2x-8}{x^2-7xy+10y^2}$

$$\begin{aligned} x^2 - 7xy + 10y^2 &= 0 \\ (x-5y)(x-2y) &= 0 \\ x &= 5y \text{ or } x = 2y \end{aligned}$$

NPV: $x \neq 5y, 2y$

Additional Challenging NPV Problems

Advanced Practice

Find the non-permissible values for each expression:

1. $\frac{x^3-8}{x^4-16}$
2. $\frac{x^2-9}{x^3-27}$
3. $\frac{x^4-1}{x^3+x^2-x-1}$
4. $\frac{x^3-2x^2-5x+6}{x^4-5x^2+4}$

Challenging NPV Solutions Part 1

Detailed Solutions

1. $\frac{x^3-8}{x^4-16}$

$$\begin{aligned} x^4 - 16 &= 0 \\ (x^2+4)(x^2-4) &= 0 \\ (x^2+4)(x+2)(x-2) &= 0 \\ x &= \pm 2 \end{aligned}$$

NPV: $x \neq \pm 2$

2. $\frac{x^2-9}{x^3-27}$

$$\begin{aligned} x^3 - 27 &= 0 \\ (x-3)(x^2+3x+9) &= 0 \\ x &= 3 \end{aligned}$$

NPV: $x \neq 3$

Challenging NPV Solutions Part 2

Detailed Solutions

3. $\frac{x^4-1}{x^3+x^2-x-1}$

$$\begin{aligned} x^3+x^2-x-1 &= 0 \\ (x+1)(x^2-1) &= 0 \\ (x+1)(x+1)(x-1) &= 0 \\ x &= \pm 1 \end{aligned}$$

NPV: $x \neq \pm 1$

4. $\frac{x^3-2x^2-5x+6}{x^4-5x^2+4}$

$$\begin{aligned} x^4-5x^2+4 &= 0 \\ (x^2-4)(x^2-1) &= 0 \\ (x+2)(x-2)(x+1)(x-1) &= 0 \\ x &= \pm 2, \pm 1 \end{aligned}$$

Summary

Key Points

- Rational expressions are fractions with polynomials
- Simplify by factoring and canceling common factors
- Watch out for common mistakes in simplification
- Always find and state non-permissible values
- Remember: denominator cannot be zero