Arithmetic of Derivatives

A Differentiation Toolbox

Differential Calculus

Outline

Why We Need Differentiation Rules

- Computing derivatives using the limit definition becomes unwieldy for complex functions
- We need efficient tools to break down complicated derivatives into simple ones
- Similar to how we used "arithmetic of limits" for computing limits

Basic Derivatives We Know

Simple Functions

$$\frac{d}{dx}1 = 0$$
$$\frac{d}{dx}x = 1$$
$$\frac{d}{dx}x^2 = 2x$$
$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

Derivative of Sum and Difference

Lemma 2.4.1

Let $f(x)$, $g(x)$ be differentiable functions and let $c \in \mathbb{R}$ be a constant. Then:

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$
$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$
$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

Linearity of Differentiation

Theorem 2.4.2

Let $f(x)$, $g(x)$ be differentiable functions, let $\alpha, \beta \in \mathbb{R}$ be constants and define:

$$S(x) = \alpha f(x) + \beta g(x)$$

Then:

$$\frac{dS}{dx} = S'(x) = \alpha f'(x) + \beta g'(x)$$

The Product Rule

Theorem 2.4.3

Let $f(x)$, $g(x)$ be differentiable functions, then:

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

- The derivative of a product is NOT the product of derivatives
- Remember: "First times derivative of second, plus second times derivative of first"

Special Case: Derivative of a Square

Corollary 2.4.4

Let $f(x)$ be a differentiable function, then:

$$\frac{d}{dx}[f(x)]^2 = 2f(x) \cdot f'(x)$$

The Quotient Rule

Theorem 2.4.5

Let $f(x)$, $g(x)$ be differentiable functions, then:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

- This derivative exists except at points where $g(x) = 0$
- Remember: "Low d-high minus high d-low, over low squared"

Special Case: Derivative of a Reciprocal

Corollary 2.4.6

Let $g(x)$ be a differentiable function, then:

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{g'(x)}{[g(x)]^2}$$

Example 1: Using Linearity

Find the derivative of $f(x) = 3x^2 + 5x - 2$

Solution to Example 1

Solution:

$$\begin{aligned}
f(x) &= 3x^2 + 5x - 2 \\
f'(x) &= \frac{d}{dx}(3x^2) + \frac{d}{dx}(5x) + \frac{d}{dx}(-2) \\
&= 3 \cdot \frac{d}{dx}(x^2) + 5 \cdot \frac{d}{dx}(x) + 0 \\
&= 3 \cdot 2x + 5 \cdot 1 \\
&= 6x + 5
\end{aligned}$$

Example 2: Using Product Rule

Find the derivative of $f(x) = x^2 \cdot \sqrt{x}$

Solution:

$$\begin{aligned}
f(x) &= x^2 \cdot \sqrt{x} \\
f'(x) &= \frac{d}{dx}(x^2) \cdot \sqrt{x} + x^2 \cdot \frac{d}{dx}(\sqrt{x}) \\
&= 2x \cdot \sqrt{x} + x^2 \cdot \frac{1}{2\sqrt{x}} \\
&= 2x\sqrt{x} + \frac{x^2}{2\sqrt{x}} \\
&= 2x\sqrt{x} + \frac{x\sqrt{x}}{2} \\
&= \frac{5x\sqrt{x}}{2}
\end{aligned}$$

Example 3: Using Quotient Rule

Find the derivative of $f(x) = \frac{x^2+1}{x-1}$

Solution to Example 3

Solution:

$$
\begin{aligned}
f(x) &= \frac{x^2+1}{x-1} \\
f'(x) &= \frac{\frac{d}{dx}(x^2+1)\cdot(x-1)-(x^2+1)\cdot\frac{d}{dx}(x-1)}{(x-1)^2} \\
&= \frac{2x\cdot(x-1)-(x^2+1)\cdot 1}{(x-1)^2} \\
&= \frac{2x^2-2x-x^2-1}{(x-1)^2} \\
&= \frac{x^2-2x-1}{(x-1)^2}
\end{aligned}
$$

Practice: 1 and 2

Practice 1:

$$\text{Find the derivative of } f(x) = 4x^3 - 2x^2 + 7x - 3$$

Practice 2:

$$\text{Find the derivative of } f(x) = x^3 \cdot (x^2 + 1)$$

Practice: 3 and 4

Practice 3:

$$\text{Find the derivative of } f(x) = \frac{x^2 - 4}{x + 2}$$

Practice 4:

$$\text{Find the derivative of } f(x) = \frac{1}{x^2 + 1}$$

Practice 5:

Find the derivative of $f(x) = (x^2 + 3x)(x^3 - 2)$

Practice: 6 and 7

Practice 6:

$$\text{Find the derivative of } f(x) = \frac{x^3 + 2x}{x^2 - 1}$$

Practice 7:

$$\text{Find the derivative of } f(x) = (x^2 + 1)^2$$

Practice: 8 and 9

Practice 8:

$$\text{Find the derivative of } f(x) = \frac{1}{x^3 + 3x}$$

Practice 9:

$$\text{Find the derivative of } f(x) = x^2 \cdot \sqrt{x^2 + 1}$$

Practice: 10 and 11

Practice 10:

$$\text{Find the derivative of } f(x) = \frac{\sqrt{x}}{x^2+1}$$

Practice 11:

$$\text{Find the derivative of } f(x) = (x^3 - 2x)(x^2+3)^2$$

Practice: 12 and 13

Practice 12:

$$\text{Find the derivative of } f(x) = \frac{x^2 - 4}{x^2 + 4}$$

Practice 13:

$$\text{Find the derivative of } f(x) = \frac{1}{(x^2 + 1)^2}$$

Practice: 14 and 15

Practice 14:

$$\text{Find the derivative of } f(x) = x \cdot \sqrt{x^2 - 1}$$

Practice 15:

$$\text{Find the derivative of } f(x) = \frac{x^3 + x}{x^2 - x + 1}$$

Practice: 16 and 17

Practice 16:

Find the derivative of $f(x) = x^2 \sin(x)$

Practice 17:

Find the derivative of $f(x) = e^x \cos(x)$

Practice: 18 and 19

Practice 18:

$$\text{Find the derivative of } f(x) = \ln(x^2 + 1)$$

Practice 19:

$$\text{Find the derivative of } f(x) = \frac{\sin(x)}{x}$$

Practice: 20 and 21

Practice 20:

$$\text{Find the derivative of } f(x) = \arctan(x^2)$$

Practice 21:

$$\text{Find the derivative of } f(x) = x \cdot e^x$$

Practice: 22 and 23

Practice 22:

$$\text{Find the derivative of } f(x) = \frac{\ln(x)}{x}$$

Practice 23:

$$\text{Find the derivative of } f(x) = \arcsin(\sqrt{x})$$

Practice: 24 and 25

Practice 24:

$$\text{Find the derivative of } f(x) = \sin(x)\cos(x)$$

Practice 25:

$$\text{Find the derivative of } f(x) = \frac{e^x}{x^2+1}$$

Solution to Practice 1

Practice 1:
Find the derivative of $f(x) = 4x^3 - 2x^2 + 7x - 3$
Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(4x^3) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}(3) \\ &= 4 \cdot 3x^2 - 2 \cdot 2x + 7 \cdot 1 - 0 \\ &= 12x^2 - 4x + 7 \end{aligned}$$

Solution to Practice 2 (Part 1)

Practice 2:
Find the derivative of $f(x) = x^3 \cdot (x^2 + 1)$
Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3) \cdot (x^2 + 1) + x^3 \cdot \frac{d}{dx}(x^2 + 1) \\ &= 3x^2 \cdot (x^2 + 1) + x^3 \cdot 2x \end{aligned}$$

Solution to Practice 2 (Part 2)

Solution (continued):

$$
\begin{aligned}
&= 3x^4 + 3x^2 + 2x^4 \\
&= 5x^4 + 3x^2
\end{aligned}
$$

Solution to Practice 3 (Part 1)

Practice 3:
Find the derivative of $f(x) = \frac{x^2-4}{x+2}$
Solution:

$$f'(x) = \frac{\frac{d}{dx}(x^2-4)\cdot(x+2)-(x^2-4)\cdot\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$= \frac{2x\cdot(x+2)-(x^2-4)\cdot 1}{(x+2)^2}$$

Solution to Practice 3 (Part 2)

Solution (continued):

$$
\begin{aligned}
&= \frac{2x^2 + 4x - x^2 + 4}{(x+2)^2} \\
&= \frac{x^2 + 4x + 4}{(x+2)^2} \\
&= \frac{(x+2)^2}{(x+2)^2} = 1
\end{aligned}
$$

Solution to Practice 4

Practice 4:
Find the derivative of $f(x) = \frac{1}{x^2+1}$
Solution:

$$
\begin{aligned}
f'(x) &= -\frac{\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
&= -\frac{2x}{(x^2+1)^2}
\end{aligned}
$$

Solution to Practice 5 (Part 1)

Practice 5:
Find the derivative of $f(x) = (x^2 + 3x)(x^3 - 2)$
Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2 + 3x) \cdot (x^3 - 2) + (x^2 + 3x) \cdot \frac{d}{dx}(x^3 - 2) \\ &= (2x + 3) \cdot (x^3 - 2) + (x^2 + 3x) \cdot 3x^2 \end{aligned}$$

Solution to Practice 5 (Part 2)

Solution (continued):

$$\begin{aligned} &= 2x^4 - 4x + 3x^3 - 6 + 3x^4 + 9x^3 \\ &= 5x^4 + 12x^3 - 4x - 6 \end{aligned}$$

Solution to Practice 6 (Part 1)

Practice 6:
Find the derivative of $f(x) = \frac{x^3+2x}{x^2-1}$
Solution:

$$
\begin{aligned}
f'(x) &= \frac{\frac{d}{dx}(x^3+2x)\cdot(x^2-1) - (x^3+2x)\cdot\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \\
&= \frac{(3x^2+2)\cdot(x^2-1) - (x^3+2x)\cdot 2x}{(x^2-1)^2}
\end{aligned}
$$

Solution to Practice 6 (Part 2)

Solution (continued):

$$= \frac{3x^4 - 3x^2 + 2x^2 - 2 - 2x^4 - 4x^2}{(x^2-1)^2}$$

$$= \frac{x^4 - 5x^2 - 2}{(x^2-1)^2}$$

Solution to Practice 7

Practice 7:
Find the derivative of $f(x) = (x^2 + 1)^2$
Solution:

$$\begin{aligned} f'(x) &= 2(x^2+1) \cdot \frac{d}{dx}(x^2+1) \\ &= 2(x^2+1) \cdot 2x \\ &= 4x(x^2+1) \end{aligned}$$

Solution to Practice 8

Practice 8:
Find the derivative of $f(x) = \frac{1}{x^3+3x}$
Solution:

$$\begin{aligned} f'(x) &= -\frac{\frac{d}{dx}(x^3+3x)}{(x^3+3x)^2} \\ &= -\frac{3x^2+3}{(x^3+3x)^2} \\ &= -\frac{3(x^2+1)}{(x^3+3x)^2} \end{aligned}$$

Solution to Practice 9 (Part 1)

Practice 9:
Find the derivative of $f(x) = x^2 \cdot \sqrt{x^2+1}$
Solution:

$$f'(x) = \frac{d}{dx}(x^2) \cdot \sqrt{x^2+1} + x^2 \cdot \frac{d}{dx}(\sqrt{x^2+1})$$
$$= 2x \cdot \sqrt{x^2+1} + x^2 \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

Solution (continued):

$$
\begin{aligned}
&= 2x\sqrt{x^2+1} + \frac{x^3}{\sqrt{x^2+1}} \\
&= \frac{2x(x^2+1) + x^3}{\sqrt{x^2+1}} \\
&= \frac{2x^3 + 2x + x^3}{\sqrt{x^2+1}} \\
&= \frac{3x^3 + 2x}{\sqrt{x^2+1}}
\end{aligned}
$$

Practice 10:
Find the derivative of $f(x) = \frac{\sqrt{x}}{x^2+1}$
Solution:

$$
\begin{aligned}
f'(x) &= \frac{\frac{d}{dx}(\sqrt{x}) \cdot (x^2+1) - \sqrt{x} \cdot \frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
&= \frac{\frac{1}{2\sqrt{x}} \cdot (x^2+1) - \sqrt{x} \cdot 2x}{(x^2+1)^2}
\end{aligned}
$$

Solution (continued):

$$= \frac{\frac{x^2+1}{2\sqrt{x}} - 2x\sqrt{x}}{(x^2+1)^2}$$

$$= \frac{x^2+1-4x^2}{2\sqrt{x}(x^2+1)^2}$$

$$= \frac{1-3x^2}{2\sqrt{x}(x^2+1)^2}$$

Solution to Practice 11 (Part 1)

Practice 11:
Find the derivative of $f(x) = (x^3 - 2x)(x^2 + 3)^2$
Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3 - 2x) \cdot (x^2 + 3)^2 + (x^3 - 2x) \cdot \frac{d}{dx}[(x^2 + 3)^2] \\ &= (3x^2 - 2) \cdot (x^2 + 3)^2 + (x^3 - 2x) \cdot 2(x^2 + 3) \cdot 2x \end{aligned}$$

Solution (continued):

$$
\begin{aligned}
&= (3x^2 - 2)(x^4 + 6x^2 + 9) + 4x(x^5 - 2x^3 + 3x^3 - 6x) \\
&= 3x^6 + 18x^4 + 27x^2 - 2x^4 - 12x^2 - 18 + 4x^6 - 8x^4 + 12x^4 - 24x^2 \\
&= 7x^6 + 26x^4 - 9x^2 - 18
\end{aligned}
$$

Solution to Practice 12

Practice 12:
Find the derivative of $f(x) = \frac{x^2-4}{x^2+4}$
Solution:

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(x^2-4)\cdot(x^2+4) - (x^2-4)\cdot\frac{d}{dx}(x^2+4)}{(x^2+4)^2} \\ &= \frac{2x\cdot(x^2+4) - (x^2-4)\cdot 2x}{(x^2+4)^2} \\ &= \frac{2x^3+8x-2x^3+8x}{(x^2+4)^2} \\ &= \frac{16x}{(x^2+4)^2} \end{aligned}$$

Solution to Practice 13

Practice 13:
Find the derivative of $f(x) = \frac{1}{(x^2+1)^2}$
Solution:

$$
\begin{aligned}
f'(x) &= -\frac{\frac{d}{dx}[(x^2+1)^2]}{[(x^2+1)^2]^2} \\
&= -\frac{2(x^2+1)\cdot 2x}{(x^2+1)^4} \\
&= -\frac{4x(x^2+1)}{(x^2+1)^4} \\
&= -\frac{4x}{(x^2+1)^3}
\end{aligned}
$$

Solution to Practice 14 (Part 1)

Practice 14:

Find the derivative of $f(x) = x \cdot \sqrt{x^2 - 1}$

Solution:

$$f'(x) = \frac{d}{dx}(x) \cdot \sqrt{x^2-1} + x \cdot \frac{d}{dx}(\sqrt{x^2-1})$$
$$= 1 \cdot \sqrt{x^2-1} + x \cdot \frac{1}{2\sqrt{x^2-1}} \cdot 2x$$

Solution to Practice 14 (Part 2)

Solution (continued):

$$
\begin{aligned}
&= \sqrt{x^2 - 1} + \frac{x^2}{\sqrt{x^2 - 1}} \\
&= \frac{x^2 - 1 + x^2}{\sqrt{x^2 - 1}} \\
&= \frac{2x^2 - 1}{\sqrt{x^2 - 1}}
\end{aligned}
$$

Solution to Practice 15 (Part 1)

Practice 15:
Find the derivative of $f(x) = \frac{x^3+x}{x^2-x+1}$
Solution:

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(x^3+x)\cdot(x^2-x+1)-(x^3+x)\cdot\frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2} \\ &= \frac{(3x^2+1)\cdot(x^2-x+1)-(x^3+x)\cdot(2x-1)}{(x^2-x+1)^2} \end{aligned}$$

Solution (continued):

$$= \frac{3x^4 - 3x^3 + 3x^2 + x^2 - x + 1 - 2x^4 + x^3 - 2x^2 + x}{(x^2 - x + 1)^2}$$
$$= \frac{x^4 - 2x^3 + 2x^2 + 1}{(x^2 - x + 1)^2}$$

Derivatives of Common Functions

Power Functions

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$
$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

Exponential and Log

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \ln(a)$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

Trigonometric Functions

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$
$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

Derivatives of Inverse Trigonometric Functions

Inverse Trigonometric Functions

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Solution to Practice 16

Practice 16:
Find the derivative of $f(x) = x^2 \sin(x)$
Solution:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(x^2) \cdot \sin(x) + x^2 \cdot \frac{d}{dx}(\sin(x)) \\
&= 2x \cdot \sin(x) + x^2 \cdot \cos(x) \\
&= 2x \sin(x) + x^2 \cos(x)
\end{aligned}
$$

Solution to Practice 17

Practice 17:
Find the derivative of $f(x) = e^x \cos(x)$
Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(e^x) \cdot \cos(x) + e^x \cdot \frac{d}{dx}(\cos(x)) \\ &= e^x \cdot \cos(x) + e^x \cdot (-\sin(x)) \\ &= e^x \cos(x) - e^x \sin(x) \\ &= e^x(\cos(x) - \sin(x)) \end{aligned}$$

Solution to Practice 18

Practice 18:
Find the derivative of $f(x) = \ln(x^2 + 1)$
Solution:

$$\begin{aligned} f'(x) &= \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) \\ &= \frac{1}{x^2+1} \cdot 2x \\ &= \frac{2x}{x^2+1} \end{aligned}$$

Solution to Practice 19

Practice 19:
Find the derivative of $f(x) = \frac{\sin(x)}{x}$
Solution:

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(\sin(x)) \cdot x - \sin(x) \cdot \frac{d}{dx}(x)}{x^2} \\ &= \frac{\cos(x) \cdot x - \sin(x) \cdot 1}{x^2} \\ &= \frac{x\cos(x) - \sin(x)}{x^2} \end{aligned}$$

Solution to Practice 20

Practice 20:
Find the derivative of $f(x) = \arctan(x^2)$
Solution:

$$\begin{aligned} f'(x) &= \frac{1}{1+(x^2)^2} \cdot \frac{d}{dx}(x^2) \\ &= \frac{1}{1+x^4} \cdot 2x \\ &= \frac{2x}{1+x^4} \end{aligned}$$

Solution to Practice 21

Practice 21:
Find the derivative of $f(x) = x \cdot e^x$
Solution:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(x) \cdot e^x + x \cdot \frac{d}{dx}(e^x) \\
&= 1 \cdot e^x + x \cdot e^x \\
&= e^x + xe^x \\
&= e^x(1 + x)
\end{aligned}
$$

Solution to Practice 22

Practice 22:
Find the derivative of $f(x) = \frac{\ln(x)}{x}$
Solution:

$$f'(x) = \frac{\frac{d}{dx}(\ln(x)) \cdot x - \ln(x) \cdot \frac{d}{dx}(x)}{x^2}$$
$$= \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x^2}$$
$$= \frac{1 - \ln(x)}{x^2}$$

Solution to Practice 23 (Part 1)

Practice 23:
Find the derivative of $f(x) = \arcsin(\sqrt{x})$
Solution:

$$f'(x) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$
$$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

Solution to Practice 23 (Part 2)

Solution (continued):

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}}$$
$$= \frac{1}{2\sqrt{x(1-x)}}$$

Solution to Practice 24

Practice 24:
Find the derivative of $f(x) = \sin(x)\cos(x)$
Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\sin(x)) \cdot \cos(x) + \sin(x) \cdot \frac{d}{dx}(\cos(x)) \\ &= \cos(x) \cdot \cos(x) + \sin(x) \cdot (-\sin(x)) \\ &= \cos^2(x) - \sin^2(x) \\ &= \cos(2x) \end{aligned}$$

Solution to Practice 25

Practice 25:
Find the derivative of $f(x) = \frac{e^x}{x^2+1}$
Solution:

$$
\begin{aligned}
f'(x) &= \frac{\frac{d}{dx}(e^x)\cdot(x^2+1) - e^x\cdot\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
&= \frac{e^x\cdot(x^2+1) - e^x\cdot 2x}{(x^2+1)^2} \\
&= \frac{e^x(x^2+1-2x)}{(x^2+1)^2} \\
&= \frac{e^x(x^2-2x+1)}{(x^2+1)^2}
\end{aligned}
$$