The Derivative and the Tangent Line

Definition, Geometric Meaning, and Applications

Differential Calculus

Outline

What is the Derivative?

- The derivative measures how a function changes as its input changes.
- It is the "instantaneous rate of change" or the "slope of the tangent line" at a point.

Definition of the Derivative

Limit Definition

The derivative of $f(x)$ at $x = a$ is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- If this limit exists, f is said to be differentiable at a.
- $f'(a)$ is also called the "slope of the tangent line" to f at $x = a$.

Alternate Notation

- $f'(x)$ (prime notation)
- $\frac{df}{dx}$ or $\frac{dy}{dx}$ (Leibniz notation)
- $D_x f(x)$ (operator notation)

Example 1: Derivative of $f(x) = |x|$ at $x = 0$

Question: Find the derivative of $f(x) = |x|$ at $x = 0$ using the definition.

Solution to Example 1

Solution:

$$f'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

$$\text{If } h > 0,\ \frac{|h|}{h} = 1; \quad \text{If } h < 0,\ \frac{|h|}{h} = -1$$

$$\text{Left-hand limit: } \lim_{h \to 0^-} \frac{|h|}{h} = -1$$

$$\text{Right-hand limit: } \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

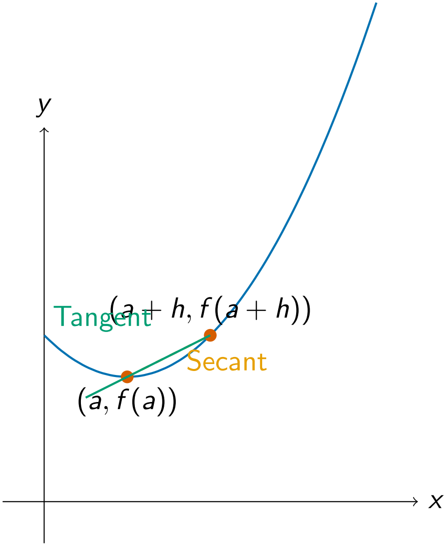
Since left and right limits are not equal, the derivative does not exist at $x = 0$.

$f(x) = |x| \quad x = 0$

Secant Line and Tangent Line

- The secant line through $(a, f(a))$ and $(a + h, f(a + h))$ has slope $\frac{f(a+h)-f(a)}{h}$.
- As $h \to 0$, the secant line approaches the tangent line at $x = a$.

Tangent Line Visualization



Example 2: Derivative and Tangent Line for $f(x) = x^3$ at $x = 1$

Question: Find the derivative of $f(x) = x^3$ at $x = 1$ and the equation of the tangent line.

Solution to Example 2

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2 \\
f'(1) &= 3(1)^2 = 3 \\
f(1) &= 1^3 = 1 \\
\text{Tangent line: } y &= f(1) + f'(1)(x-1) = 1 + 3(x-1)
\end{aligned}
$$

Summary: Derivative and Tangent Line

- The derivative $f'(a)$ is the slope of the tangent line to f at $x = a$.
- The tangent line at $x = a$ has equation:

$$y = f(a) + f'(a)(x - a)$$

- The process of finding the derivative is called "differentiation".

Practice: 1 and 2

Practice 1:

Find the derivative of $f(x) = x^2$

Practice 2:

Find the equation of the tangent line to $f(x) = x^2$ at $x = 1$

Practice: 3 and 4

Practice 3:

Find the derivative of $f(x) = \sqrt{x}$ at $x = 4$

Practice 4:

Find the derivative of $f(x) = |x - 2|$ at $x = 2$

Practice: 5

Practice 5:

A particle moves along a line so that its position at time t is $s(t) = t^2 - 4t + 5$.

Find the instantaneous velocity at t=3.

Solution to Practice 1

Practice 1:
Find the derivative of $f(x) = x^2$
Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) = 2x \end{aligned}$$

Solution to Practice 2

Practice 2:
Find the equation of the tangent line to $f(x) = x^2$ at $x = 1$
Solution:

$$f(1) = 1^2 = 1$$
$$f'(x) = 2x \implies f'(1) = 2$$
$$\text{Tangent line: } y = f(1) + f'(1)(x-1) = 1 + 2(x-1)$$

Solution to Practice 3 (Part 1)

Practice 3:
Find the derivative of $f(x) = \sqrt{x}$ at $x = 4$
Solution:

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\text{Let } x = 4:$$

$$f'(4) = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$$

Solution to Practice 3 (Part 2)

Solution (continued):

Multiply numerator and denominator by $\sqrt{4+h}+2$:

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}
\end{aligned}
$$

Solution to Practice 4 (Part 1)

Practice 4:

Find the derivative of $f(x) = |x-2|$ at $x = 2$

Solution:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{|2+h-2| - |2-2|}{h} \\ &= \lim_{h\to 0} \frac{|h|}{h} \end{aligned}$$

Solution to Practice 4 (Part 2)

Solution (continued):

$$\text{If } h > 0,\ \frac{|h|}{h} = 1$$

$$\text{If } h < 0,\ \frac{|h|}{h} = -1$$

$$\text{Left-hand limit: } \lim_{h \to 0^-} \frac{|h|}{h} = -1$$

$$\text{Right-hand limit: } \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

Solution to Practice 4 (Part 3)

Solution (continued):

Since left and right limits are not equal,
the derivative does not exist at $x = 2$.

$f(x) = |x - 2| \quad x = 2$

Solution to Practice 5 (Part 1)

Practice 5:
A particle moves along a line so that its position at time t is $s(t) = t^2 - 4t + 5$. Find the instantaneous velocity at $t = 3$.
Solution:

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

$$s'(t) = \lim_{h \to 0} \frac{(t+h)^2 - 4(t+h) + 5 - (t^2 - 4t + 5)}{h}$$

Solution to Practice 5 (Part 2)

Solution (continued):

$$
\begin{aligned}
&= \lim_{h \to 0} \frac{t^2 + 2th + h^2 - 4t - 4h + 5 - t^2 + 4t - 5}{h} \\
&= \lim_{h \to 0} \frac{2th + h^2 - 4h}{h} \\
&= \lim_{h \to 0} (2t - 4 + h) = 2t - 4
\end{aligned}
$$

Solution to Practice 5 (Part 3)

Solution (continued):

$$v(3) = 2 \times 3 - 4 = 2$$

Answer: The instantaneous velocity at $t = 3$ is 2 units/time.