Multivariable Functions

Partial Derivatives and Second Order Partials

Differential Calculus

Outline

What are Multivariable Functions?

- Functions that depend on more than one variable
- Examples: $f(x, y) = x^2 + y^2$, $g(x, y, z) = xyz + \sin(x)$
- Input: multiple variables (e.g., (x, y) or (x, y, z))
- Output: single real number
- Visualized as surfaces in 3D space

Key Differences from Single Variable Functions:

- Domain: subset of $\mathbb{R}^n$ (n-dimensional space)
- Range: subset of $\mathbb{R}$ (real numbers)
- More complex behavior and visualization
- Multiple ways to approach a point

Examples of Multivariable Functions - Part 1

Common Examples:

- **Linear function:** $f(x, y) = 2x + 3y - 1$
- **Quadratic function:** $f(x, y) = x^2 + y^2$
- **Exponential function:** $f(x, y) = e^{x+y}$
- **Trigonometric function:** $f(x, y) = \sin(x)\cos(y)$
- **Rational function:** $f(x, y) = \frac{x^2+y^2}{x+y}$

Examples of Multivariable Functions - Part 2

Real-world Applications:

- Temperature distribution: $T(x, y, t)$ (position and time)
- Pressure in a fluid: $P(x, y, z)$ (3D position)
- Economic models: $C(x, y)$ (cost as function of labor and materials)
- Physics: $E(x, y, z, t)$ (energy field)
- Population growth: $P(x, y, t)$ (spatial and temporal)
- Chemical concentration: $C(x, y, z, t)$ (diffusion processes)

Domain and Range - Part 1

Domain: Set of all valid input points (x, y) or (x, y, z)

Examples:

- $f(x, y) = \sqrt{x^2 + y^2}$: Domain $= \mathbb{R}^2$ (all real pairs)
- $f(x, y) = \frac{1}{x+y}$: Domain $= \{(x, y) : x + y \neq 0\}$
- $f(x, y) = \ln(xy)$: Domain $= \{(x, y) : xy > 0\}$

Domain and Range - Part 2

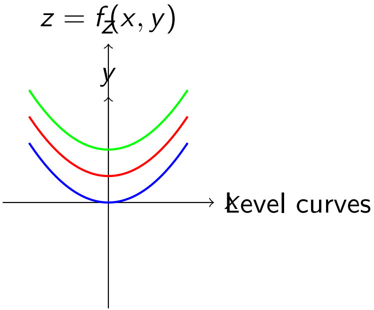
More Domain Examples:

- $f(x, y) = \sqrt{1 - x^2 - y^2}$: Domain $= \{(x, y) : x^2 + y^2 \leq 1\}$
- $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$: Domain $= \{(x, y) : (x, y) \neq (0, 0)\}$
- $f(x, y, z) = \ln(xyz)$: Domain $= \{(x, y, z) : xyz > 0\}$

Range Examples:

- $f(x, y) = x^2 + y^2$: Range $= [0, \infty)$
- $f(x, y) = \sin(x)\cos(y)$: Range $= [-1, 1]$
- $f(x, y) = e^{-(x^2+y^2)}$: Range $= (0, 1]$

Visualizing Multivariable Functions



Visualization Methods:

- **3D surfaces:** Plot $z = f(x, y)$ in 3D space
- **Level curves:** Curves where $f(x, y) = c$ (constant)
- **Contour plots:** 2D representation of level curves
- **Cross-sections:** Fix one variable and plot the result

What are Partial Derivatives?

Definition: Rate of change of a function with respect to one variable while holding others constant

Notation:

- $\frac{\partial f}{\partial x}$ or f_x: partial derivative with respect to x
- $\frac{\partial f}{\partial y}$ or f_y: partial derivative with respect to y
- $\frac{\partial f}{\partial z}$ or f_z: partial derivative with respect to z

Geometric Interpretation:

- $\frac{\partial f}{\partial x}$: slope of tangent line in x-direction
- $\frac{\partial f}{\partial y}$: slope of tangent line in y-direction
- Each partial derivative gives the rate of change along one axis

Computing Partial Derivatives - Method

Method: Treat all other variables as constants and differentiate with respect to the variable of interest

Key Rules:

- When differentiating with respect to x, treat y and z as constants
- When differentiating with respect to y, treat x and z as constants
- Use all standard differentiation rules (product rule, chain rule, etc.)
- The order of partial differentiation matters for mixed partials

Example: For $f(x, y) = x^2 + 3xy + y^2$

$$\frac{\partial f}{\partial x} = 2x + 3y \quad \text{(treat } y \text{ as constant)}$$
$$\frac{\partial f}{\partial y} = 3x + 2y \quad \text{(treat } x \text{ as constant)}$$

Partial Derivatives - Example 1

Example: $f(x, y) = e^{xy}\sin(x)$
Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= ye^{xy}\sin(x) + e^{xy}\cos(x)\\ &= e^{xy}(y\sin(x) + \cos(x))\end{aligned}$$

$$\frac{\partial f}{\partial y} = xe^{xy}\sin(x)$$

Explanation:

- For $\frac{\partial f}{\partial x}$: Use product rule on $e^{xy}\sin(x)$
- For $\frac{\partial f}{\partial y}$: Only e^{xy} depends on y, so $\sin(x)$ is constant

Partial Derivatives - Example 2

Example: $f(x, y, z) = x^2y + yz^2 + xz$
Solution:

$$\frac{\partial f}{\partial x} = 2xy + z$$
$$\frac{\partial f}{\partial y} = x^2 + z^2$$
$$\frac{\partial f}{\partial z} = 2yz + x$$

Explanation:

- $\frac{\partial f}{\partial x}$: x^2y gives $2xy$, yz^2 gives 0, xz gives z
- $\frac{\partial f}{\partial y}$: x^2y gives x^2, yz^2 gives z^2, xz gives 0
- $\frac{\partial f}{\partial z}$: x^2y gives 0, yz^2 gives $2yz$, xz gives x

Partial Derivatives - Example 3

Example: $f(x,y) = \frac{x^2+y^2}{x+y}$
Solution for $\frac{\partial f}{\partial x}$:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(2x)(x+y) - (x^2+y^2)(1)}{(x+y)^2} \\
&= \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2} \\
&= \frac{x^2 + 2xy - y^2}{(x+y)^2}
\end{aligned}$$

Explanation:

- Used quotient rule: $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v \cdot u' - u \cdot v'}{v^2}$
- $u = x^2 + y^2$, so $u' = 2x$
- $v = x + y$, so $v' = 1$

Partial Derivatives - Example 3 (Continued)

Solution for $\frac{\partial f}{\partial y}$:

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{(2y)(x+y)-(x^2+y^2)(1)}{(x+y)^2} \\
&= \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} \\
&= \frac{2xy+y^2-x^2}{(x+y)^2}
\end{aligned}$$

Explanation:

- Used quotient rule again
- $u = x^2 + y^2$, so $u' = 2y$ (treating x as constant)
- $v = x + y$, so $v' = 1$
- Notice the symmetry between the two partial derivatives

Partial Derivatives - Example 4

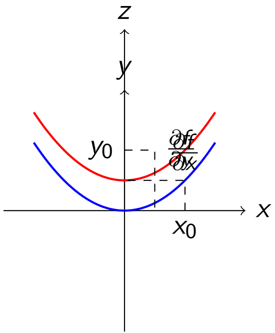
Example: $f(x, y) = \ln(x^2 + y^2)$
Solution:

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}$$
$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$$

Explanation:

- Use chain rule: $\frac{d}{dx}[\ln(u)] = \frac{1}{u} \cdot \frac{du}{dx}$
- For $\frac{\partial f}{\partial x}$: $u = x^2 + y^2$, so $\frac{du}{dx} = 2x$
- For $\frac{\partial f}{\partial y}$: $u = x^2 + y^2$, so $\frac{du}{dy} = 2y$

Geometric Interpretation of Partial Derivatives



Key Points:

- $\frac{\partial f}{\partial x}$: slope of tangent line parallel to x-axis
- $\frac{\partial f}{\partial y}$: slope of tangent line parallel to y-axis
- Both partials exist at a point if the function is differentiable there
- Partial derivatives can exist even if the function is not continuous

Second Order Partial Derivatives - Introduction

Definition: Partial derivatives of partial derivatives

Notation:

- $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}: second partial with respect to x
- $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}: second partial with respect to y
- $\frac{\partial^2 f}{\partial x \partial y}$ or f_{xy}: mixed partial (first x, then y)
- $\frac{\partial^2 f}{\partial y \partial x}$ or f_{yx}: mixed partial (first y, then x)

Clairaut's Theorem: If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$

Example: For $f(x, y) = x^2 y + xy^2$

$$f_x = 2xy + y^2$$
$$f_y = x^2 + 2xy$$

Second Order Partial Derivatives - Example 1

Example: $f(x, y) = x^2y + xy^2$
First order partials:

$$f_x = 2xy + y^2$$
$$f_y = x^2 + 2xy$$

Second order partials:

$$f_{xx} = 2y$$
$$f_{yy} = 2x$$
$$f_{xy} = 2x + 2y$$
$$f_{yx} = 2x + 2y$$

Note: $f_{xy} = f_{yx}$ as expected by Clairaut's theorem.

Second Order Partial Derivatives - Example 2

Example: $f(x, y) = e^{xy} + x^2y$
First order partials:

$$
\begin{aligned}
f_x &= ye^{xy} + 2xy \\
f_y &= xe^{xy} + x^2
\end{aligned}
$$

Second order partials:

$$
\begin{aligned}
f_{xx} &= y^2e^{xy} + 2y \\
f_{yy} &= x^2e^{xy} \\
f_{xy} &= e^{xy} + xye^{xy} + 2x \\
f_{yx} &= e^{xy} + xye^{xy} + 2x
\end{aligned}
$$

Note: $f_{xy} = f_{yx}$ as expected.

Second Order Partial Derivatives - Example 3

Example: $f(x, y) = \sin(xy) + x^3y^2$

First order partials:

$$f_x = y\cos(xy) + 3x^2y^2$$
$$f_y = x\cos(xy) + 2x^3y$$

Second order partials:

$$f_{xx} = -y^2\sin(xy) + 6xy^2$$
$$f_{yy} = -x^2\sin(xy) + 2x^3$$
$$f_{xy} = \cos(xy) - xy\sin(xy) + 6x^2y$$
$$f_{yx} = \cos(xy) - xy\sin(xy) + 6x^2y$$

Note: $f_{xy} = f_{yx}$ as expected.

The Hessian Matrix - Introduction

Definition: Matrix of second order partial derivatives

For $f(x, y)$:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Properties:

- Symmetric matrix (if Clairaut's theorem applies)
- Used to classify critical points
- Determinant helps determine nature of extrema
- Important in optimization problems

The Hessian Matrix - Example

Example: For $f(x, y) = x^2 + 3xy + y^2$

First order partials:

$$f_x = 2x + 3y$$
$$f_y = 3x + 2y$$

Second order partials:

$$f_{xx} = 2$$
$$f_{yy} = 2$$
$$f_{xy} = f_{yx} = 3$$

Hessian matrix:

$$H = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

Determinant:

$$\det(H) = (2)(2) - (3)(3) = 4 - 9 = -5$$

The Hessian Matrix - Applications

Applications:

- **Classifying critical points:**
  - If $\det(H) > 0$ and $f_{xx} > 0$: local minimum
  - If $\det(H) > 0$ and $f_{xx} < 0$: local maximum
  - If $\det(H) < 0$: saddle point
  - If $\det(H) = 0$: inconclusive (degenerate case)
- **Optimization problems:** Finding extrema of functions
- **Taylor series expansions:** Second order approximation
- **Convexity/concavity:** Determining function behavior

Practice Problem 1

Problem: Find all first and second order partial derivatives of $f(x, y) = x^3y^2 + e^{xy}$

Steps to follow:

1. Find f_x and f_y (first order partials)
2. Find f_{xx}, f_{yy}, f_{xy}, and f_{yx} (second order partials)
3. Verify that $f_{xy} = f_{yx}$ (Clairaut's theorem)

Hint: Remember to use the product rule and chain rule where needed.

Practice Problem 2

Problem: Find all first and second order partial derivatives of $f(x, y) = \ln(x^2 + y^2)$

Steps to follow:

1. Find f_x and f_y using the chain rule
2. Find f_{xx}, f_{yy}, f_{xy}, and f_{yx}
3. Simplify your answers

Hint: For the second derivatives, you'll need to use the quotient rule.

Practice Problem 3

Problem: Find all first and second order partial derivatives of $f(x, y, z) = x^2y + yz^2 + xz$
Steps to follow:

1. Find f_x, f_y, and f_z (first order partials)
2. Find all second order partials: f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{xz}, f_{yx}, f_{yz}, f_{zx}, f_{zy}
3. Verify that mixed partials are equal where applicable

Hint: This is a relatively simple function - each term depends on at most two variables.

Practice Problem 4

Problem: Find the Hessian matrix for $f(x, y) = x^2 + 2xy + y^2$
Steps to follow:

1. Find all second order partial derivatives
2. Construct the Hessian matrix
3. Calculate the determinant of the Hessian

Hint: This is a quadratic function, so the second derivatives will be constants.

Practice Problem 5

Problem: Show that $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ satisfies the equation $xf_x + yf_y = 0$

Steps to follow:

1. Find f_x and f_y using the quotient rule
2. Compute $xf_x + yf_y$
3. Simplify to show it equals zero

Hint: This is a homogeneous function of degree 0, which explains why this equation holds.

Practice Problem 6

Problem: Find all first and second order partial derivatives of $f(x, y) = \sin(xy) + \cos(x + y)$
Steps to follow:

1. Find f_x and f_y using trigonometric differentiation rules
2. Find all second order partials
3. Verify Clairaut's theorem

Hint: Remember that $\frac{d}{dx}[\sin(u)] = \cos(u) \cdot \frac{du}{dx}$ and $\frac{d}{dx}[\cos(u)] = -\sin(u) \cdot \frac{du}{dx}$.

Practice Problem 7

Problem: Find all first and second order partial derivatives of $f(x, y) = e^{x^2+y^2}$

Steps to follow:

1. Find f_x and f_y using the chain rule
2. Find all second order partials
3. Verify Clairaut's theorem

Hint: This is a radial function, so it has special symmetry properties.

Practice Problem 8

Problem: Find the Hessian matrix for $f(x, y) = x^3 + y^3 - 3xy$

Steps to follow:

1. Find all first and second order partial derivatives
2. Construct the Hessian matrix
3. Calculate the determinant
4. What does this tell you about the critical points?

Hint: This function has interesting critical point behavior.

Practice Problem 1 - Solution

Solution: $f(x, y) = x^3y^2 + e^{xy}$
First order partials:

$$f_x = 3x^2y^2 + ye^{xy}$$
$$f_y = 2x^3y + xe^{xy}$$

Second order partials:

$$f_{xx} = 6xy^2 + y^2e^{xy}$$
$$f_{yy} = 2x^3 + x^2e^{xy}$$
$$f_{xy} = 6x^2y + e^{xy} + xye^{xy}$$
$$f_{yx} = 6x^2y + e^{xy} + xye^{xy}$$

Note: $f_{xy} = f_{yx}$ as expected by Clairaut's theorem.

Practice Problem 2 - Solution

Solution: $f(x, y) = \ln(x^2 + y^2)$
First order partials:

$$f_x = \frac{2x}{x^2 + y^2}$$
$$f_y = \frac{2y}{x^2 + y^2}$$

Second order partials:

$$f_{xx} = \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$
$$f_{yy} = \frac{2(x^2 + y^2) - 2y(2y)}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

Practice Problem 2 - Solution (Continued)

Second order partials (continued):

$$f_{xy} = \frac{-2x(2y)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2}$$
$$f_{yx} = \frac{-2y(2x)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2}$$

Note: $f_{xy} = f_{yx}$ and $f_{xx} = -f_{yy}$.
Explanation:

- Used quotient rule for second derivatives
- Mixed partials are equal as expected by Clairaut's theorem
- Pure second derivatives have opposite signs due to symmetry

Practice Problem 3 - Solution

Solution: $f(x, y, z) = x^2y + yz^2 + xz$
First order partials:

$$f_x = 2xy + z$$
$$f_y = x^2 + z^2$$
$$f_z = 2yz + x$$

Explanation:

- $\frac{\partial f}{\partial x}$: x^2y gives $2xy$, yz^2 gives 0, xz gives z
- $\frac{\partial f}{\partial y}$: x^2y gives x^2, yz^2 gives z^2, xz gives 0
- $\frac{\partial f}{\partial z}$: x^2y gives 0, yz^2 gives $2yz$, xz gives x

Practice Problem 3 - Solution (Continued)

Second order partials:

$$f_{xx} = 2y$$
$$f_{yy} = 0$$
$$f_{zz} = 2y$$

Mixed partials:

$$f_{xy} = 2x = f_{yx}$$
$$f_{xz} = 1 = f_{zx}$$
$$f_{yz} = 2z = f_{zy}$$

Note: All mixed partials are equal as expected by Clairaut's theorem.

Key observations:

- Pure second derivatives with respect to x and z are equal ($f_{xx} = f_{zz} = 2y$)
- Pure second derivative with respect to y is zero ($f_{yy} = 0$)
- All mixed partials are constants

Practice Problem 4 - Solution

Solution: $f(x, y) = x^2 + 2xy + y^2$
Second order partials:

$$f_{xx} = 2$$
$$f_{yy} = 2$$
$$f_{xy} = 2 = f_{yx}$$

Hessian matrix:

$$H = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Determinant:

$$\det(H) = (2)(2) - (2)(2) = 4 - 4 = 0$$

Interpretation: This indicates that the function has a degenerate critical point structure.

Practice Problem 5 - Solution

Solution: $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$

First order partial - $\frac{\partial f}{\partial x}$**:**

$$
\begin{aligned}
f_x &= \frac{(2x)(x^2+y^2) - (x^2-y^2)(2x)}{(x^2+y^2)^2} \\
&= \frac{2x^3 + 2xy^2 - 2x^3 + 2xy^2}{(x^2+y^2)^2} \\
&= \frac{4xy^2}{(x^2+y^2)^2}
\end{aligned}
$$

Explanation:

- Used quotient rule: $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v \cdot u' - u \cdot v'}{v^2}$
- $u = x^2 - y^2$, so $u' = 2x$
- $v = x^2 + y^2$, so $v' = 2x$
- Notice the cancellation: $2x^3 - 2x^3 = 0$

Practice Problem 5 - Solution (Continued)

First order partial - $\frac{\partial f}{\partial y}$:

$$\begin{aligned} f_y &= \frac{(-2y)(x^2+y^2) - (x^2-y^2)(2y)}{(x^2+y^2)^2} \\ &= \frac{-2x^2y - 2y^3 - 2x^2y + 2y^3}{(x^2+y^2)^2} \\ &= \frac{-4x^2y}{(x^2+y^2)^2} \end{aligned}$$

Explanation:

- Used quotient rule again
- $u = x^2 - y^2$, so $u' = -2y$
- $v = x^2 + y^2$, so $v' = 2y$
- Notice the cancellation: $-2y^3 + 2y^3 = 0$

Practice Problem 5 - Solution (Verification)

Verification: Show that $xf_x + yf_y = 0$

$$
\begin{aligned}
xf_x + yf_y &= x \cdot \frac{4xy^2}{(x^2+y^2)^2} + y \cdot \frac{-4x^2y}{(x^2+y^2)^2} \\
&= \frac{4x^2y^2 - 4x^2y^2}{(x^2+y^2)^2} = 0
\end{aligned}
$$

Explanation: This function is homogeneous of degree 0, meaning $f(tx, ty) = f(x, y)$ for all $t \neq 0$.
Euler's theorem for homogeneous functions states that $xf_x + yf_y = nf(x, y)$ where n is the degree. Since $n = 0$, we get $xf_x + yf_y = 0$.
Physical interpretation: This function represents the ratio of the difference to the sum of squares, which has rotational symmetry.

Practice Problem 6 - Solution

Solution: $f(x, y) = \sin(xy) + \cos(x + y)$
First order partials:

$$f_x = y\cos(xy) - \sin(x + y)$$
$$f_y = x\cos(xy) - \sin(x + y)$$

Second order partials:

$$f_{xx} = -y^2\sin(xy) - \cos(x + y)$$
$$f_{yy} = -x^2\sin(xy) - \cos(x + y)$$
$$f_{xy} = \cos(xy) - xy\sin(xy) - \cos(x + y)$$
$$f_{yx} = \cos(xy) - xy\sin(xy) - \cos(x + y)$$

Note: $f_{xy} = f_{yx}$ as expected by Clairaut's theorem.

Practice Problem 7 - Solution

Solution: $f(x, y) = e^{x^2+y^2}$
First order partials:

$$f_x = 2xe^{x^2+y^2}$$
$$f_y = 2ye^{x^2+y^2}$$

Second order partials:

$$f_{xx} = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2} = 2e^{x^2+y^2}(1 + 2x^2)$$
$$f_{yy} = 2e^{x^2+y^2} + 4y^2e^{x^2+y^2} = 2e^{x^2+y^2}(1 + 2y^2)$$
$$f_{xy} = 4xye^{x^2+y^2} = f_{yx}$$

Note: This is a radial function, so it has special symmetry properties.

Practice Problem 8 - Solution

Solution: $f(x, y) = x^3 + y^3 - 3xy$

First order partials:

$$f_x = 3x^2 - 3y$$
$$f_y = 3y^2 - 3x$$

Second order partials:

$$f_{xx} = 6x$$
$$f_{yy} = 6y$$
$$f_{xy} = -3 = f_{yx}$$

Hessian matrix:

$$H = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

Determinant:

$$\det(H) = (6x)(6y) - (-3)(-3) = 36xy - 9$$

Practice Problem 8 - Solution (Continued)

Critical points: Set $f_x = 0$ and $f_y = 0$

$$\begin{aligned} 3x^2 - 3y = 0 \quad &\Rightarrow \quad y = x^2 \\ 3y^2 - 3x = 0 \quad &\Rightarrow \quad x = y^2 \end{aligned}$$

Substituting: $x = (x^2)^2 = x^4$, so $x(x^3 - 1) = 0$
This gives $x = 0$ or $x = 1$
Critical points: $(0, 0)$ and $(1, 1)$
At $(0, 0)$**:** $\det(H) = 36(0)(0) - 9 = -9 < 0$ (saddle point)
At $(1, 1)$**:** $\det(H) = 36(1)(1) - 9 = 27 > 0$ and $f_{xx} = 6 > 0$ (local minimum)

Physics Applications:

- **Temperature gradients:** $\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)$
- **Force fields:** $\vec{F} = \nabla\phi$ (gradient of potential)
- **Wave equations:** $\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$
- **Heat equation:** $\frac{\partial u}{\partial t} = \alpha\nabla^2 u$
- **Maxwell's equations:** Electromagnetic field theory

Examples:

- Temperature distribution in a room: $T(x, y, z, t)$
- Electric potential: $V(x, y, z)$
- Wave propagation: $u(x, y, z, t)$

Economics Applications:

- **Marginal products:** $\frac{\partial Q}{\partial L}$ (marginal product of labor)
- **Elasticity:** $\frac{\partial Q}{\partial P} \cdot \frac{P}{Q}$
- **Utility functions:** $\frac{\partial U}{\partial x_i}$ (marginal utility)
- **Cost functions:** $\frac{\partial C}{\partial q_i}$ (marginal cost)
- **Production functions:** $Q = f(K, L)$ (output as function of capital and labor)

Examples:

- Cobb-Douglas production: $Q = AK^\alpha L^\beta$
- Utility maximization: $U(x, y) = x^a y^b$
- Cost minimization: $C(x, y) = px + qy$

Applications of Partial Derivatives - Engineering

Engineering Applications:

- **Stress analysis:** Second derivatives in elasticity
- **Fluid dynamics:** Navier-Stokes equations
- **Heat transfer:** Fourier's law
- **Structural analysis:** Beam deflection equations
- **Control systems:** State space representation

Examples:

- Heat conduction: $\frac{\partial T}{\partial t} = \alpha \nabla^2 T$
- Beam bending: $\frac{\partial^2 w}{\partial x^2} = \frac{M}{EI}$
- Fluid flow: $\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p$

The Gradient Vector - Introduction

Definition: Vector of first order partial derivatives

For $f(x, y)$: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

For $f(x, y, z)$: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

Properties:

- Points in direction of steepest increase
- Magnitude gives rate of steepest increase
- Perpendicular to level curves/surfaces
- Used in optimization algorithms

The Gradient Vector - Examples

Example 1: For $f(x, y) = x^2 + y^2$

$$\nabla f = (2x, 2y)$$

At point $(1, 2)$: $\nabla f = (2, 4)$
Example 2: For $f(x, y) = e^{xy}$

$$\nabla f = (ye^{xy}, xe^{xy})$$

At point $(0, 1)$: $\nabla f = (e^0, 0) = (1, 0)$
Example 3: For $f(x, y, z) = x^2 + y^2 + z^2$

$$\nabla f = (2x, 2y, 2z)$$

At point $(1, 1, 1)$: $\nabla f = (2, 2, 2)$

The Laplacian Operator - Introduction

Definition: Sum of second order partial derivatives
For $f(x, y)$: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$
For $f(x, y, z)$: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
Physical Interpretation:

- Measures the "curvature" of a function
- Positive Laplacian: function is "concave up" in all directions
- Negative Laplacian: function is "concave down" in all directions
- Zero Laplacian: harmonic function (important in physics)

The Laplacian Operator - Applications

Applications:

- **Heat equation:** $\frac{\partial u}{\partial t} = \alpha \nabla^2 u$
- **Wave equation:** $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$
- **Poisson's equation:** $\nabla^2 \phi = \rho$
- **Laplace's equation:** $\nabla^2 \phi = 0$
- **Schrödinger equation:** $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$

Examples:

- For $f(x, y) = x^2 + y^2$: $\nabla^2 f = 2 + 2 = 4$
- For $f(x, y) = \sin(x)\cos(y)$: $\nabla^2 f = -\sin(x)\cos(y) - \sin(x)\cos(y) = -2\sin(x)\cos(y)$
- For $f(x, y) = \ln(x^2 + y^2)$: $\nabla^2 f = 0$ (harmonic function)

Summary - Key Concepts

- **Multivariable functions** depend on multiple variables
- **Partial derivatives** measure rate of change with respect to one variable
- **Second order partials** include pure and mixed derivatives
- **Clairaut's theorem** states that mixed partials are equal under continuity
- **Hessian matrix** organizes second order partials
- **Gradient vector** points in direction of steepest increase
- **Laplacian operator** is sum of pure second order partials

Summary - Applications

- **Physics:** Temperature gradients, force fields, wave equations
- **Economics:** Marginal products, elasticity, utility functions
- **Engineering:** Stress analysis, fluid dynamics, heat transfer
- **Optimization:** Finding extrema, gradient descent algorithms
- **Computer Science:** Machine learning, computer graphics
- **Biology:** Population dynamics, diffusion processes

Thank You!

Questions?

Multivariable calculus opens the door to understanding complex systems in multiple dimensions!

Next: Optimization, Lagrange multipliers, and multiple integrals