Related Rates

Applications of Derivatives in Real-World Problems

Differential Calculus

Outline

What are Related Rates?

- Problems where you are given the rate of change of one quantity
- You need to determine the rate of change of another related quantity
- All quantities are functions of time
- Use implicit differentiation to relate the rates

General Strategy

Step-by-Step Method

1. **Draw a picture** and identify all relevant variables
2. **Write an equation** relating the quantities
3. **Differentiate both sides** with respect to time
4. **Substitute known values** and solve for the unknown rate

Key Concepts

Important Points

- All variables are functions of time: $x(t)$, $y(t)$, $V(t)$, etc.
- Use implicit differentiation: $\frac{d}{dt}[f(x,y)] = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
- Pay attention to units and signs
- Draw diagrams to visualize the problem

Example 1: Spherical Balloon

Problem: A spherical balloon is being inflated at a rate of 13 cm^3/sec. How fast is the radius changing when the balloon has radius 15 cm?

Solution: Spherical Balloon

Solution:

$$
\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
\frac{dV}{dt} &= 4\pi r^2 \cdot \frac{dr}{dt} \\
\frac{dr}{dt} &= \frac{1}{4\pi r^2} \cdot \frac{dV}{dt} \\
\frac{dr}{dt} &= \frac{13}{4\pi(15)^2} \\
\frac{dr}{dt} &= \frac{13}{900\pi} \text{ cm/sec}
\end{aligned}
$$

Example 2: Ladder Problem

Problem: A 5m ladder is leaning against a wall. The base slides out at 1m/s. How fast is the top sliding down when the base is 3m from the wall?

Solution: Ladder Problem

Solution:

$$
\begin{aligned}
x^2 + y^2 &= 25 \\
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\
x\frac{dx}{dt} + y\frac{dy}{dt} &= 0 \\
\frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt}
\end{aligned}
$$

Solution: Ladder Problem (continued)

Solution (continued):

$$\text{When } x = 3, y = \sqrt{25 - 9} = 4$$

$$\frac{dy}{dt} = -\frac{3}{4} \cdot 1$$

$$\frac{dy}{dt} = -\frac{3}{4} \text{ m/s}$$

Example 3: Balloon Angle

Problem: A balloon rises vertically from a point 200m away. When the angle is $\pi/4$, it's changing at 0.05 rad/sec. How fast is the balloon rising?

Solution: Balloon Angle

Solution:

$$h = 200 \tan\theta$$
$$\frac{dh}{dt} = 200 \sec^2\theta \cdot \frac{d\theta}{dt}$$
$$\frac{dh}{dt} = 200 \cdot (\sqrt{2})^2 \cdot 0.05$$
$$\frac{dh}{dt} = 200 \cdot 2 \cdot 0.05$$
$$\frac{dh}{dt} = 20 \text{ m/s}$$

Example 4: Percentage Rates

Problem: $R = PQ$. P increases at 8%/year, Q decreases at 2%/year. What's the percentage rate of change for R?

Solution: Percentage Rates

Solution:

$$
\begin{aligned}
R &= PQ \\
\frac{dR}{dt} &= P\frac{dQ}{dt} + Q\frac{dP}{dt} \\
\frac{100}{R}\frac{dR}{dt} &= \frac{100P}{R}\frac{dQ}{dt} + \frac{100Q}{R}\frac{dP}{dt} \\
&= \frac{100}{Q}\frac{dQ}{dt} + \frac{100}{P}\frac{dP}{dt} \\
&= -2 + 8 = 6\%
\end{aligned}
$$

Example 5: Shadow Problem

Problem: A ball drops from 49m height. A light 49m high is 10m to the left. How fast is the shadow moving 1 second after drop?

Solution: Shadow Problem

Solution:

$$h(t) = 49 - 4.9t^2$$

By similar triangles: $$\frac{4.9t^2}{10} = \frac{49 - 4.9t^2}{s(t)}$$

$$s(t) = \frac{10(49 - 4.9t^2)}{4.9t^2} = \frac{100}{t^2} - 10$$

$$\frac{ds}{dt} = -\frac{200}{t^3}$$

$$\frac{ds}{dt}(1) = -200 \text{ m/s}$$

Example 6: Boats Problem

Problem: Boat A is 15km west of boat B. A travels east at 3km/h, B travels north at 4km/h. How fast is distance changing at 3pm?

Solution: Boats Problem

Solution:

$$
\begin{aligned}
z^2 &= x^2 + y^2 \\
2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\
\frac{dz}{dt} &= \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}
\end{aligned}
$$

Solution: Boats Problem (continued)

Solution (continued):

$$\text{At 3pm: } x = 6, y = 12, z = \sqrt{180} = 6\sqrt{5}$$

$$\frac{dx}{dt} = -3, \frac{dy}{dt} = 4$$

$$\frac{dz}{dt} = \frac{6(-3) + 12(4)}{6\sqrt{5}}$$

$$\frac{dz}{dt} = \frac{30}{6\sqrt{5}} = \sqrt{5}\ \text{km/h}$$

Practice: 1 and 2

Practice 1:
A spherical balloon is inflated at 8 cm^3/sec.
How fast is radius changing when radius is 10 cm?

Practice 2:
A 13m ladder slides down a wall. Base moves at 2m/s.
How fast is top sliding when base is 5m from wall?

Practice: 3 and 4

Practice 3:
A cone's height increases at 3cm/s.
How fast is volume changing when height is 10cm and radius is 5cm?

Practice 4:
A rectangle's length increases at 2cm/s, width decreases at 1cm/s.
How fast is area changing when length is 8cm and width is 6cm?

Practice: 5 and 6

Practice 5:
A cube's edge length increases at 4cm/s.
How fast is volume changing when edge length is 12cm?

Practice 6:
A circle's radius increases at 2cm/s.
How fast is area changing when radius is 7cm?

Practice: 7 and 8

Practice 7:
A cylinder's radius increases at 1cm/s, height decreases at 2cm/s.
How fast is volume changing when radius is 6cm and height is 10cm?

Practice 8:
A square's side length increases at 3cm/s.
How fast is diagonal changing when side length is 9cm?

Practice: 9 and 10

Practice 9:
A person walks away from a 20m light pole at 2m/s.
How fast is shadow length changing when person is 15m from pole?

Practice 10:
A plane flies at 500km/h at altitude 10km.
How fast is distance from observer changing when plane is 15km away horizontally?

Practice: 11 and 12

Practice 11:
A conical tank has radius 5m, height 10m.
Water flows in at $3\text{m}^3/\text{min}$.
How fast is water level rising when depth is 4m?

Practice 12:
A street light is 8m high. A person 1.8m tall walks away at 1.5m/s.
How fast is shadow length changing when person is 6m from light?

Practice: 13 and 14

Practice 13:
A kite flies at height 100m. String is let out at 2m/s.
How fast is kite moving horizontally when 200m of string is out?

Practice 14:
A balloon rises at 3m/s. Observer 50m away.
How fast is angle of elevation changing when balloon is 30m high?

Practice: 15 and 16

Practice 15:
A rectangular pool is 20m long, 10m wide.
Water flows in at 5m^3/min.
How fast is water level rising when depth is 2m?

Practice 16:
A particle moves along curve $y = x^2$ at 2 units/sec.
How fast is distance from origin changing when $x = 3$?

Practice: 17 and 18

Practice 17:
A cylindrical tank radius 4m, height 8m, lying on side.
Fuel flows in at $2\text{m}^3/\text{min}$.
How fast is fuel level rising when depth is 3m?

Practice 18:
Two cars approach intersection. Car A at 60km/h, Car B at 80km/h.
How fast is distance changing when A is 3km away, B is 4km away?

Practice: 19 and 20

Practice 19:
A conical pile of sand has height equal to radius.
Sand falls at $10\text{m}^3/\text{min}$.
How fast is radius changing when height is 5m?

Practice 20:
A ladder 10m long slides down wall. Base moves at 1.5m/s.
How fast is angle between ladder and ground changing when base is 6m from wall?

Practice: 21 and 22

Practice 21:
A spherical snowball melts at rate proportional to surface area.
How fast is radius changing when radius is 8cm?

Practice 22:
A particle moves along $y = \sin x$ at 3 units/sec.
How fast is distance from $(0, 0)$ changing when $x = \pi/4$?

Practice: 23 and 24

Practice 23:

A rectangular box has length, width, height increasing at 2cm/s, 1cm/s, 3cm/s.
How fast is volume changing when dimensions are 8cm, 6cm, 4cm?

Practice 24:

A circular oil slick expands at 2m^2/min.
How fast is radius changing when area is 100π m^2?

Practice: 25

Practice 25:
A conical cup has radius 5cm, height 12cm.
Water flows out at $3\text{cm}^3/\text{s}$.
How fast is water level falling when depth is 6cm?

Practice 1:
A spherical balloon is inflated at 8 cm^3/sec. How fast is radius changing when radius is 10 cm?
Solution:

$$
\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
\frac{dV}{dt} &= 4\pi r^2 \cdot \frac{dr}{dt} \\
\frac{dr}{dt} &= \frac{1}{4\pi r^2} \cdot \frac{dV}{dt} \\
\frac{dr}{dt} &= \frac{8}{4\pi(10)^2} \\
\frac{dr}{dt} &= \frac{8}{400\pi} = \frac{1}{50\pi} \text{ cm/sec}
\end{aligned}
$$

Solution to Practice 2

Practice 2:
A 13m ladder slides down a wall. Base moves at 2m/s. How fast is top sliding when base is 5m from wall?
Solution:

$$
\begin{aligned}
x^2 + y^2 &= 169 \\
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\
x\frac{dx}{dt} + y\frac{dy}{dt} &= 0 \\
\frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt}
\end{aligned}
$$

Solution to Practice 2 (continued)

Solution (continued):

$$\text{When } x = 5, y = \sqrt{169 - 25} = 12$$

$$\frac{dy}{dt} = -\frac{5}{12} \cdot 2$$

$$\frac{dy}{dt} = -\frac{10}{12} = -\frac{5}{6} \text{ m/s}$$

Solution to Practice 3

Practice 3:
A cone's height increases at 3cm/s. How fast is volume changing when height is 10cm and radius is 5cm?
Solution:

$$V = \frac{1}{3}\pi r^2 h$$
$$\frac{dV}{dt} = \frac{1}{3}\pi(2r\frac{dr}{dt} \cdot h + r^2\frac{dh}{dt})$$
$$\text{Since } \frac{dr}{dt} = 0 \text{ (radius constant)}$$
$$\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt}$$
$$\frac{dV}{dt} = \frac{1}{3}\pi(5)^2 \cdot 3$$
$$\frac{dV}{dt} = 25\pi \text{ cm}^3/\text{s}$$

Solution to Practice 4

Practice 4:
A rectangle's length increases at 2cm/s, width decreases at 1cm/s. How fast is area changing when length is 8cm and width is 6cm?
Solution:

$$
\begin{aligned}
A &= lw \\
\frac{dA}{dt} &= l\frac{dw}{dt} + w\frac{dl}{dt} \\
\frac{dA}{dt} &= 8(-1) + 6(2) \\
\frac{dA}{dt} &= -8 + 12 = 4 \text{ cm}^2/\text{s}
\end{aligned}
$$

Solution to Practice 5

Practice 5:
A cube's edge length increases at 4cm/s. How fast is volume changing when edge length is 12cm?
Solution:

$$\begin{aligned}
V &= s^3 \\
\frac{dV}{dt} &= 3s^2 \cdot \frac{ds}{dt} \\
\frac{dV}{dt} &= 3(12)^2 \cdot 4 \\
\frac{dV}{dt} &= 3 \cdot 144 \cdot 4 = 1728 \text{ cm}^3/\text{s}
\end{aligned}$$

Solution to Practice 6

Practice 6:
A circle's radius increases at 2cm/s. How fast is area changing when radius is 7cm?
Solution:

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$
$$\frac{dA}{dt} = 2\pi(7) \cdot 2$$
$$\frac{dA}{dt} = 28\pi \text{ cm}^2/\text{s}$$

Solution to Practice 7

Practice 7:
A cylinder's radius increases at 1cm/s, height decreases at 2cm/s. How fast is volume changing when radius is 6cm and height is 10cm?
Solution:

$$\begin{aligned}
V &= \pi r^2 h \\
\frac{dV}{dt} &= \pi(2r\frac{dr}{dt} \cdot h + r^2\frac{dh}{dt}) \\
\frac{dV}{dt} &= \pi(2 \cdot 6 \cdot 1 \cdot 10 + 6^2 \cdot (-2)) \\
\frac{dV}{dt} &= \pi(120 - 72) = 48\pi \text{ cm}^3/\text{s}
\end{aligned}$$

Solution to Practice 8

Practice 8:
A square's side length increases at 3cm/s. How fast is diagonal changing when side length is 9cm?
Solution:

$$d = s\sqrt{2}$$
$$\frac{dd}{dt} = \sqrt{2} \cdot \frac{ds}{dt}$$
$$\frac{dd}{dt} = \sqrt{2} \cdot 3 = 3\sqrt{2} \text{ cm/s}$$

Solution to Practice 9

Practice 9:
A person walks away from a 20m light pole at 2m/s. How fast is shadow length changing when person is 15m from pole?
Solution:

$$\text{By similar triangles: } \frac{h}{x} = \frac{20}{x+s}$$
$$h(x+s) = 20x$$
$$hs = 20x - hx$$
$$s = \frac{20x - hx}{h} = \frac{(20-h)x}{h}$$

Solution to Practice 9 (continued)

Solution (continued):

$$\frac{ds}{dt} = \frac{(20 - h)}{h} \cdot \frac{dx}{dt}$$

Assuming person height $h = 1.8m$

$$\frac{ds}{dt} = \frac{18.2}{1.8} \cdot 2$$

$$\frac{ds}{dt} = \frac{36.4}{1.8} \approx 20.2 \text{ m/s}$$

Solution to Practice 10

Practice 10:
A plane flies at 500km/h at altitude 10km. How fast is distance from observer changing when plane is 15km away horizontally?
Solution:

$$\begin{aligned}
d^2 &= x^2 + h^2 \\
2d\frac{dd}{dt} &= 2x\frac{dx}{dt} \\
\frac{dd}{dt} &= \frac{x}{d}\frac{dx}{dt} \\
d &= \sqrt{15^2 + 10^2} = \sqrt{325} \\
\frac{dd}{dt} &= \frac{15}{\sqrt{325}} \cdot 500 \\
\frac{dd}{dt} &= \frac{7500}{\sqrt{325}} \text{ km/h}
\end{aligned}$$

Solution to Practice 11

Practice 11:
A conical tank has radius 5m, height 10m. Water flows in at $3\text{m}^3/\text{min}$. How fast is water level rising when depth is 4m?
Solution:

$$\frac{r}{h} = \frac{5}{10} = \frac{1}{2}$$

$$r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{h}{2})^2 h = \frac{\pi h^3}{12}$$

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi(4)^2} \cdot 3 = \frac{12}{16\pi} = \frac{3}{4\pi} \text{ m/min}$$

Solution to Practice 12

Practice 12:
A street light is 8m high. A person 1.8m tall walks away at 1.5m/s. How fast is shadow length changing when person is 6m from light?
Solution:

$$\begin{aligned} \text{By similar triangles: } \frac{1.8}{s} &= \frac{8}{x+s} \\ 1.8(x+s) &= 8s \\ 1.8x + 1.8s &= 8s \\ 1.8x &= 6.2s \\ s &= \frac{1.8x}{6.2} = \frac{9x}{31} \end{aligned}$$

Solution to Practice 12 (continued)

Solution (continued):

$$\frac{ds}{dt} = \frac{9}{31} \cdot \frac{dx}{dt}$$
$$\frac{ds}{dt} = \frac{9}{31} \cdot 1.5$$
$$\frac{ds}{dt} = \frac{13.5}{31} \approx 0.44 \text{ m/s}$$

Solution to Practice 13

Practice 13:
A kite flies at height 100m. String is let out at 2m/s. How fast is kite moving horizontally when 200m of string is out?
Solution:

$$\begin{aligned} x^2 + 100^2 &= 200^2 \\ x^2 &= 40000 - 10000 = 30000 \\ x &= \sqrt{30000} = 100\sqrt{3} \\ 2x\frac{dx}{dt} &= 2s\frac{ds}{dt} \\ \frac{dx}{dt} &= \frac{s}{x}\frac{ds}{dt} \\ \frac{dx}{dt} &= \frac{200}{100\sqrt{3}} \cdot 2 = \frac{400}{100\sqrt{3}} = \frac{4}{\sqrt{3}} \text{ m/s} \end{aligned}$$

Solution to Practice 14

Practice 14:
A balloon rises at 3m/s. Observer 50m away. How fast is angle of elevation changing when balloon is 30m high?
Solution:

$$\tan\theta = \frac{h}{50}$$
$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{50} \cdot \frac{dh}{dt}$$
$$\frac{d\theta}{dt} = \frac{1}{50\sec^2\theta} \cdot 3$$
$$\sec\theta = \frac{\sqrt{50^2 + 30^2}}{50} = \frac{\sqrt{3400}}{50} = \frac{\sqrt{34}}{5}$$
$$\frac{d\theta}{dt} = \frac{3}{50 \cdot \frac{34}{25}} = \frac{75}{1700} = \frac{3}{68} \text{ rad/s}$$

Practice 15:
A rectangular pool is 20m long, 10m wide. Water flows in at 5m^3/min. How fast is water level rising when depth is 2m?
Solution:

$$\begin{aligned} V &= lwh \\ \frac{dV}{dt} &= lw\frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{1}{lw} \cdot \frac{dV}{dt} \\ \frac{dh}{dt} &= \frac{1}{20 \cdot 10} \cdot 5 \\ \frac{dh}{dt} &= \frac{5}{200} = \frac{1}{40} \text{ m/min} \end{aligned}$$

Solution to Practice 16

Practice 16:
A particle moves along curve $y = x^2$ at 2 units/sec. How fast is distance from origin changing when $x = 3$?
Solution:

$$d^2 = x^2 + y^2 = x^2 + x^4$$

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 4x^3\frac{dx}{dt}$$

$$\frac{dd}{dt} = \frac{x + 2x^3}{d}\frac{dx}{dt}$$

$$\text{When } x = 3, y = 9, d = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10}$$

$$\frac{dd}{dt} = \frac{3 + 2(27)}{3\sqrt{10}} \cdot 2 = \frac{57}{3\sqrt{10}} \cdot 2 = \frac{38}{\sqrt{10}} \text{ units/sec}$$

Practice 17:
A cylindrical tank radius 4m, height 8m, lying on side. Fuel flows in at $2\text{m}^3/\text{min}$. How fast is fuel level rising when depth is 3m?
Solution:

$$V = L \cdot \text{Area of segment}$$

$$\text{Area} = r^2\theta - \frac{r^2}{2}\sin(2\theta)$$

$$h = r - r\cos\theta$$

$$\cos\theta = 1 - \frac{h}{r} = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\theta = \arccos(\frac{1}{4})$$

Solution (continued):

$$\frac{dV}{dt} = Lr^2(1 - \cos(2\theta))\frac{d\theta}{dt}$$
$$\frac{dh}{dt} = r\sin\theta \cdot \frac{d\theta}{dt}$$
$$\frac{dh}{dt} = r\sin\theta \cdot \frac{2}{Lr^2(1 - \cos(2\theta))}$$
$$\frac{dh}{dt} = \frac{2\sin\theta}{Lr(1 - \cos(2\theta))}$$
$$\frac{dh}{dt} = \frac{2 \cdot \frac{\sqrt{15}}{4}}{8 \cdot 4 \cdot 2 \cdot (\frac{\sqrt{15}}{4})^2} = \frac{1}{16\sqrt{15}} \text{ m/min}$$

Solution to Practice 18

Practice 18:
Two cars approach intersection. Car A at 60km/h, Car B at 80km/h. How fast is distance changing when A is 3km away, B is 4km away?
Solution:

$$d^2 = x^2 + y^2$$

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dd}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{d}$$

$$d = \sqrt{3^2 + 4^2} = 5$$

$$\frac{dd}{dt} = \frac{3(-60) + 4(-80)}{5}$$

$$\frac{dd}{dt} = \frac{-180 - 320}{5} = -100 \text{ km/h}$$

Practice 19:
A conical pile of sand has height equal to radius. Sand falls at $10\text{m}^3/\text{min}$. How fast is radius changing when height is 5m?
Solution:

$$\begin{aligned}
h &= r \\
V &= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3 \\
\frac{dV}{dt} &= \pi r^2 \cdot \frac{dr}{dt} \\
\frac{dr}{dt} &= \frac{1}{\pi r^2} \cdot \frac{dV}{dt} \\
\frac{dr}{dt} &= \frac{10}{\pi(5)^2} = \frac{10}{25\pi} = \frac{2}{5\pi} \text{ m/min}
\end{aligned}$$

Solution to Practice 20

Practice 20:
A ladder 10m long slides down wall. Base moves at 1.5m/s. How fast is angle between ladder and ground changing when base is 6m from wall?
Solution:

$$\begin{aligned}
\cos\theta &= \frac{x}{10} \\
-\sin\theta \cdot \frac{d\theta}{dt} &= \frac{1}{10} \cdot \frac{dx}{dt} \\
\frac{d\theta}{dt} &= -\frac{1}{10\sin\theta} \cdot \frac{dx}{dt} \\
\text{When } x = 6, \sin\theta &= \frac{8}{10} = \frac{4}{5} \\
\frac{d\theta}{dt} &= -\frac{1}{10 \cdot \frac{4}{5}} \cdot 1.5 = -\frac{5}{40} \cdot 1.5 = -\frac{3}{16} \text{ rad/s}
\end{aligned}$$

Solution to Practice 21

Practice 21:
A spherical snowball melts at rate proportional to surface area. How fast is radius changing when radius is 8cm?
Solution:

$$\begin{aligned}
\frac{dV}{dt} &= -k \cdot 4\pi r^2 \\
V &= \frac{4}{3}\pi r^3 \\
\frac{dV}{dt} &= 4\pi r^2 \cdot \frac{dr}{dt} \\
4\pi r^2 \cdot \frac{dr}{dt} &= -k \cdot 4\pi r^2 \\
\frac{dr}{dt} &= -k \text{ (constant rate)}
\end{aligned}$$

Practice 22:
A particle moves along $y = \sin x$ at 3 units/sec. How fast is distance from $(0, 0)$ changing when $x = \pi/4$?
Solution:

$$d^2 = x^2 + y^2 = x^2 + \sin^2 x$$

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2\sin x \cos x \cdot \frac{dx}{dt}$$

$$\frac{dd}{dt} = \frac{x + \sin x \cos x}{d} \cdot \frac{dx}{dt}$$

$$\text{When } x = \pi/4, y = \sin(\pi/4) = \frac{\sqrt{2}}{2}$$

$$d = \sqrt{(\pi/4)^2 + (\sqrt{2}/2)^2} = \sqrt{\pi^2/16 + 1/2}$$

$$\frac{dd}{dt} = \frac{\pi/4 + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}}{d} \cdot 3 = \frac{\pi/4 + 1/2}{d} \cdot 3$$

Solution to Practice 23

Practice 23:
A rectangular box has length, width, height increasing at 2cm/s, 1cm/s, 3cm/s. How fast is volume changing when dimensions are 8cm, 6cm, 4cm?
Solution:

$$\begin{aligned} V &= lwh \\ \frac{dV}{dt} &= lw\frac{dh}{dt} + lh\frac{dw}{dt} + wh\frac{dl}{dt} \\ \frac{dV}{dt} &= 8 \cdot 6 \cdot 3 + 8 \cdot 4 \cdot 1 + 6 \cdot 4 \cdot 2 \\ \frac{dV}{dt} &= 144 + 32 + 48 = 224\ \text{cm}^3/\text{s} \end{aligned}$$

Solution to Practice 24

Practice 24:
A circular oil slick expands at $2\text{m}^2/\text{min}$. How fast is radius changing when area is $100\pi\ \text{m}^2$?
Solution:

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{2\pi r} \cdot \frac{dA}{dt}$$
$$\text{When } A = 100\pi, r = 10$$
$$\frac{dr}{dt} = \frac{2}{2\pi \cdot 10} = \frac{1}{10\pi}\ \text{m/min}$$

Practice 25:
A conical cup has radius 5cm, height 12cm. Water flows out at $3\text{cm}^3/\text{s}$. How fast is water level falling when depth is 6cm?
Solution:

$$\begin{aligned}
\frac{r}{h} &= \frac{5}{12} \\
r &= \frac{5h}{12} \\
V &= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h = \frac{25\pi h^3}{432} \\
\frac{dV}{dt} &= \frac{25\pi h^2}{144} \cdot \frac{dh}{dt} \\
\frac{dh}{dt} &= \frac{144}{25\pi h^2} \cdot \frac{dV}{dt} \\
\frac{dh}{dt} &= \frac{144}{25\pi(6)^2} \cdot (-3) = -\frac{432}{900\pi} = -\frac{12}{25\pi} \text{ cm/s}
\end{aligned}$$

Key Points - Related Rates

- **Strategy:** Draw picture, write equation, differentiate, substitute
- **Key concept:** All variables are functions of time
- **Method:** Use implicit differentiation with respect to time
- **Units:** Pay attention to units and signs

Common Applications

- **Geometry:** Ladders, shadows, expanding shapes
- **Physics:** Moving objects, angles, distances
- **Engineering:** Tanks, pipes, mechanical systems
- **Real-world:** Traffic, weather, economics

Related rates problems connect calculus to real-world applications and help develop problem-solving skills.

Questions?

Related rates show how calculus applies to the real world!