1.3 The Limit of a Function

Introduction to Limits

Differential Calculus

Outline

Limit Notation

We write:

$$\lim_{x \to a} f(x) = L$$

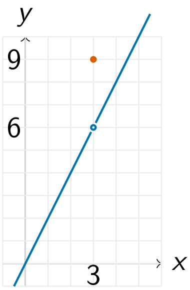
This should be read as:
The limit of $f(x)$ as x approaches a is L.

- This is shorthand notation to avoid writing long sentences
- Mathematically precise and language-independent
- Can also be written as: $f(x) \to L$ as $x \to a$
- Both notations mean exactly the same thing

Understanding Limits with a Simple Example

Consider the function:

$$f(x) = \begin{cases} 2x & \text{if } x < 3 \\ 9 & \text{if } x = 3 \\ 2x & \text{if } x > 3 \end{cases}$$



What happens as x approaches 3?

Let's plug in values close to 3:

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	5.8	5.98	5.998	6.002	6.02	6.2

- As x gets closer to 3, $f(x)$ gets closer to 6
- We write: $\lim_{x \to 3} f(x) = 6$
- Note: $f(3) = 9$, but the limit is 6
- The limit does NOT depend on the value at $x = 3$

Definition 1.3.3

Informal Definition

We write $\lim_{x \to a} f(x) = L$ if the value of the function $f(x)$ is sure to be arbitrarily close to L whenever the value of x is close enough to a, **without being exactly** a.

- This is an informal definition sufficient for most purposes
- The condition "without being exactly a" is important
- We'll see why this matters when we study derivatives
- For now, this gives us a working understanding of limits

Computing a Limit

Consider:

$$\lim_{x \to 2} \frac{x - 2}{x^2 + x - 6}$$

- If we try to compute $f(2)$, we get $\frac{0}{0}$ which is **undefined**
- This is exactly why we need limits!
- We must "sneak up" on points where functions are not defined

Important:

$\frac{0}{0}$ is **not** ∞ and it is **not** 1. It is **undefined**.

Let's plug in values close to 2

Numerical Analysis:

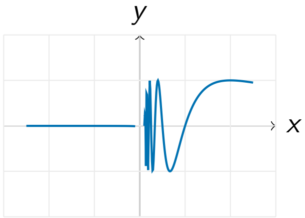
x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.20408	0.20040	0.20004	0.19996	0.19960	0.19608

- As x approaches 2, $f(x)$ approaches 0.2
- Therefore: $\lim_{x\to 2} \frac{x-2}{x^2+x-6} = 0.2$
- The limit exists even though the function is not defined at $x = 2$

When Limits Don't Exist - Case 1

Consider:

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$$



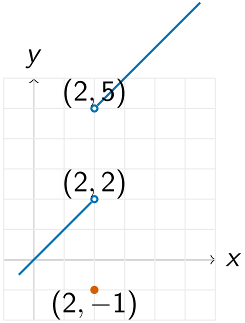
Oscillates faster and faster as $x \to 0$

- As $x \to 0$, $\frac{\pi}{x}$ becomes larger and larger
- sin oscillates faster and faster
- Function doesn't approach a single number
- Therefore: $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = \text{DNE}$

When Limits Don't Exist - Case 2

Consider:

$$f(x) = \begin{cases} x & \text{if } x < 2 \\ -1 & \text{if } x = 2 \\ x + 3 & \text{if } x > 2 \end{cases}$$



Approaching from Different Sides

Let's plug in values close to 2:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	1.9	1.99	1.999	5.001	5.01	5.1

- From below: $f(x) \to 2$
- From above: $f(x) \to 5$
- Since we get different values, the limit does not exist
- $\lim_{x\to 2} f(x) = \text{DNE}$

Definition 1.3.7

Left-Hand Limit:

$$\lim_{x \to a^-} f(x) = K$$

When $f(x)$ gets closer to K as $x < a$ approaches a from below.

Right-Hand Limit:

$$\lim_{x \to a^+} f(x) = L$$

When $f(x)$ gets closer to L as $x > a$ approaches a from above.

- Also called "left-hand" and "right-hand" limits
- Be careful to include the superscript $+$ and $-$
- Alternative notations exist but we'll use the standard ones

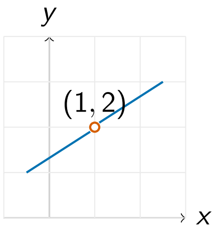
Theorem 1.3.8

Important Theorem:

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L$$

- The two-sided limit exists only if both one-sided limits exist and are equal
- If either one-sided limit doesn't exist, or if they're different, then the two-sided limit doesn't exist
- This gives us a systematic way to check if limits exist

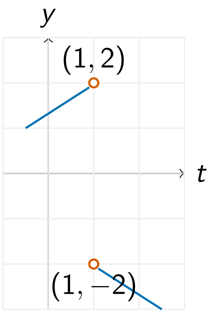
Function $f(x)$



Function $f(x)$**:**

$$\lim_{x \to 1^-} f(x) = 2$$
$$\lim_{x \to 1^+} f(x) = 2$$
$$\therefore \lim_{x \to 1} f(x) = 2$$

Function $g(t)$



Function $g(t)$:

$$\lim_{t \to 1^-} g(t) = 2$$

$$\lim_{t \to 1^+} g(t) = -2$$

$$\therefore \lim_{t \to 1} g(t) = \text{DNE}$$

Definition 1.3.10: Positive Infinity

Positive Infinity:

$$\lim_{x \to a} f(x) = +\infty$$

When $f(x)$ becomes arbitrarily large and positive as x approaches a.

Definition 1.3.10: Negative Infinity

Negative Infinity:

$$\lim_{x \to a} f(x) = -\infty$$

When $f(x)$ becomes arbitrarily large and negative as x approaches a.

Graph: Infinite Limits

Definition 1.3.11

Right-Hand Infinite Limits:

$$\lim_{x \to a^+} f(x) = +\infty \text{ or } \lim_{x \to a^+} f(x) = -\infty$$

Left-Hand Infinite Limits:

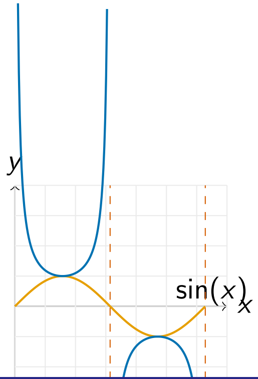
$$\lim_{x \to a^-} f(x) = +\infty \text{ or } \lim_{x \to a^-} f(x) = -\infty$$

- These occur when functions approach infinity from only one side
- Very common in rational functions and trigonometric functions
- Important for understanding vertical asymptotes

Consider:

$$g(x) = \frac{1}{\sin(x)}$$

Find the one-sided limits as $x \to \pi$.



Understanding the Behavior: Analysis

Analysis:

- As $x \to \pi^-$: $\sin(x) \to 0^+$ (small positive numbers)
- Therefore: $\dfrac{1}{\sin(x)} \to +\infty$
- As $x \to \pi^+$: $\sin(x) \to 0^-$ (small negative numbers)
- Therefore: $\dfrac{1}{\sin(x)} \to -\infty$

Understanding the Behavior: Result

Result:

$$\lim_{x \to \pi^-} \frac{1}{\sin(x)} = +\infty$$

$\lim_{x \to \pi^+} \frac{1}{\sin(x)} = -\infty$

- Since the one-sided limits are different, $\lim_{x \to \pi} \frac{1}{\sin(x)} = \text{DNE}$
- But the one-sided infinite limits give us more information than just "DNE"

What We've Learned

Limits That Exist:

- $\lim_{x \to a} f(x) = L$ where L is a finite number
- Both one-sided limits exist and are equal

Limits That Don't Exist:

1. **Oscillation**: Function oscillates wildly (like $\sin(\frac{\pi}{x})$)
2. **Jump**: One-sided limits are different
3. **Infinite**: Function approaches $\pm\infty$

One-Sided Limits:

- $\lim_{x \to a^-} f(x) = L$ (left-hand limit)
- $\lim_{x \to a^+} f(x) = L$ (right-hand limit)
- Can be finite numbers or $\pm\infty$

Important Concepts

Notation:

$$\lim_{x \to a} f(x) = L \quad \text{vs} \quad f(a) = L$$

Critical Points:

- Limits describe behavior **near** a point, not **at** the point
- A function can have a limit at $x = a$ even if $f(a)$ is undefined
- Infinity is not a number - it's a description of behavior
- One-sided limits help us understand discontinuities

Next Steps:

We'll develop systematic methods for computing limits using limit laws and algebraic techniques.