Lesson 8: Quadratic Inequalities and Discriminants

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I) Visualizing Inequalities

Key Concept

- One way to solve inequalities is to visualize them as graphs
- For linear inequalities:
  - Left side is a straight line
  - Right side (y=0) is the x-axis
  - Inequality shows where the line is above/below the x-axis
- For quadratic inequalities:
  - Left side is a parabola
  - Right side (y=0) is the x-axis
  - Inequality shows where the parabola is above/below the x-axis

I) Linear Inequality Example

Example: $3x - 4 > 0$

- Straight line: $y = 3x - 4$
- Slope: $m = 3$
- Y-intercept: $b = -4$
- Looking for where line is above x-axis
- Solution: $x > \frac{4}{3}$

I) Quadratic Inequality Example

Example: $x^2 - 3x - 4 > 0$

- Parabola: $y = x^2 - 3x - 4$
- Factored form: $(x+1)(x-4) > 0$
- X-intercepts: $x = -1$ and $x = 4$
- Looking for where parabola is above x-axis
- Solution: $x < -1$ or $x > 4$

II) Steps for Solving Inequalities

Steps

1. Move all terms to one side (make one side zero)
2. Solve for x (find intersection points)
3. Sketch the graph
4. Use inequality to determine if looking for:
   - Above the x-axis
   - Below the x-axis
   - Equal to the x-axis

II) Examples of Different Cases

Examples

- $3x - 8 + 4 \leq 0$: Line below or equal to x-axis
- $x^2 - 9 < 0$: Parabola below x-axis
- $x^2 + 2x + 6 > 0$: Parabola above x-axis

III) Practice Problem 1

Problem

Solve: $5x + 2 \geq 17$

III) Practice Problem 1: Solution

Solution

$$
\begin{aligned}
5x + 2 &\geq 17 \\
5x &\geq 15 \\
x &\geq 3
\end{aligned}
$$

III) Practice Problem 2

Problem

Solve: $2x^2 + 5x - 7 \leq 0$

III) Practice Problem 2: Solution - Step 1

Solution: Step 1 - Find X-intercepts

To solve the quadratic inequality $2x^2 + 5x - 7 \leq 0$, first, we need to find the x-intercepts of the corresponding quadratic equation $2x^2 + 5x - 7 = 0$. We use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(2)(-7)}}{2(2)}$$

$$x = \frac{-5 \pm \sqrt{25 + 56}}{4}$$

$$x = \frac{-5 \pm \sqrt{81}}{4}$$

$$x = \frac{-5 \pm 9}{4}$$

III) Practice Problem 2: Solution - Step 2

Solution: Step 2 - Use Test Values

The x-intercepts $x_1 = -3.5$ and $x_2 = 1$ divide the number line into three intervals: $(-\infty, -3.5]$, $[-3.5, 1]$, and $[1, +\infty)$.
We choose a test value from each interval and substitute it into the original inequality $2x^2 + 5x - 7 \leq 0$ to check if the inequality holds true.

Interval 1: $(-\infty, -3.5]$

- Test value: $x = -4$
- Substitute: $2(-4)^2 + 5(-4) - 7 = 2(16) - 20 - 7 = 32 - 20 - 7 = 5$
- Check inequality: $5 \leq 0$ (False)

Interval 2: $[-3.5, 1]$

- Test value: $x = 0$
- Substitute: $2(0)^2 + 5(0) - 7 = -7$
- Check inequality: $-7 \leq 0$ (True)

III) Practice Problem 2: Solution - Step 3

Solution: Step 3 - Use Test Values (Cont.) and State Solution

Interval 3: $[1, +\infty)$

- Test value: $x = 2$
- Substitute: $2(2)^2 + 5(2) - 7 = 2(4) + 10 - 7 = 8 + 10 - 7 = 11$
- Check inequality: $11 \leq 0$ (False)

Based on the test values, the inequality $2x^2 + 5x - 7 \leq 0$ is true only in the interval where the test value yielded a true statement.
Therefore, the solution to the inequality is:

$$-3.5 \leq x \leq 1$$

In interval notation, this can be written as $[-3.5, 1]$.

III) Practice Problem 3

Problem

Solve the inequality: $x^2 + 2x - 8 < 0$

III) Practice Problem 3: Solution - Step 1

Solution: Step 1 - Find X-intercepts

First, find the x-intercepts of the corresponding quadratic equation $x^2 + 2x - 8 = 0$. We can factor the quadratic expression:

$$x^2 + 2x - 8 = 0$$
$$(x + 4)(x - 2) = 0$$

So, the x-intercepts are $x = -4$ and $x = 2$. These roots divide the number line into three intervals: $(-\infty, -4)$, $(-4, 2)$, and $(2, +\infty)$.

III) Practice Problem 3: Solution - Step 2

Solution: Step 2 - Use Test Values

Now, we choose a test value from each interval and substitute it into the original inequality $x^2 + 2x - 8 < 0$ to check if the inequality holds true.

Interval 1: $(-\infty, -4)$

- Test value: $x = -5$
- Substitute: $(-5)^2 + 2(-5) - 8 = 25 - 10 - 8 = 7$
- Check inequality: $7 < 0$ (False)

Interval 2: $(-4, 2)$

- Test value: $x = 0$
- Substitute: $(0)^2 + 2(0) - 8 = -8$
- Check inequality: $-8 < 0$ (True)

III) Practice Problem 3: Solution - Step 2 (Cont.)

Solution: Step 2 - Use Test Values (Cont.)

Interval 3: $(2, +\infty)$

- Test value: $x = 3$
- Substitute: $(3)^2 + 2(3) - 8 = 9 + 6 - 8 = 7$
- Check inequality: $7 < 0$ (False)

III) Practice Problem 3: Solution - Step 3

Solution: Step 3 - State the Solution

Based on the test values, the inequality $x^2 + 2x - 8 < 0$ is true in the intervals where the test value yielded a true statement.
Therefore, the solution to the inequality is:

$$-4 < x < 2$$

In interval notation, this can be written as $(-4, 2)$.

IV) Nature of Roots

Key Concept

- The discriminant determines the nature of roots
- Formula: $D = b^2 - 4ac$
- Three possible cases:
  - $D > 0$: Two distinct real roots
  - $D = 0$: One double root
  - $D < 0$: No real roots

IV) Discriminant Examples

Examples

1. $x^2 - 4x + 7 = 8$

$$D = (-4)^2 - 4(1)(-1)$$
$$D = 16 + 4 = 20 > 0$$

   Two distinct roots

2. $3x^2 - 5x + 12 = 0$

$$D = (-5)^2 - 4(3)(12)$$
$$D = 25 - 144 = -119 < 0$$

   No real roots

V) Finding k Values

Problem Type

- Find values of k that give:
  - Two distinct roots
  - One double root
  - No real roots
- Use discriminant to solve for k

V) Example: Finding k

Example

For what values of k does $x^2 + kx + 8 = 0$ have:

1. Two distinct roots
2. One double root
3. No real roots

V) Example: Solution

Solution

1. Two distinct roots:

$$
\begin{aligned}
k^2 - 32 &> 0 \\
k^2 &> 32 \\
k &> 4\sqrt{2} \text{ or } k < -4\sqrt{2}
\end{aligned}
$$

2. One double root:

$$
\begin{aligned}
k^2 - 32 &= 0 \\
k &= \pm 4\sqrt{2}
\end{aligned}
$$

3. No real roots:

$$
\begin{aligned}
k^2 - 32 &< 0 \\
-4\sqrt{2} < k &< 4\sqrt{2}
\end{aligned}
$$

VI) Practice Problem 1

Problem

For what values of k does $9x^2 - 2kx + 4 = 0$ have:

1. Two equal roots
2. No real roots

VI) Practice Problem 1: Solution

Solution

1. Two equal roots:

$$\begin{aligned} 4k^2 - 144 &= 0 \\ k^2 &= 36 \\ k &= \pm 6 \end{aligned}$$

2. No real roots:

$$\begin{aligned} 4k^2 - 144 &< 0 \\ -6 < k &< 6 \end{aligned}$$

VI) Practice Problem 2

Problem

For what values of k does $(2k-1)x^2 - 8x + 4 = 0$ have:

1. Two different roots
2. No real roots

VI) Practice Problem 2: Solution

Solution

1. Two different roots:

$$\begin{aligned} 64 - 16(2k-1) &> 0 \\ 64 - 32k + 16 &> 0 \\ -32k &> -80 \\ k &< 2.5 \end{aligned}$$

2. No real roots:

$$\begin{aligned} 64 - 16(2k-1) &< 0 \\ k &> 2.5 \end{aligned}$$

General Practice Problems

Solve Each Inequality

Solve each of the following inequalities:

1. $3x - 10 > 5x + 4$
2. $x^2 - 7x + 10 \geq 0$
3. $2x^2 + x - 3 < 0$
4. $-x^2 + 6x - 9 \leq 0$
5. $x(x - 5) > 14$

General Practice Problems: Solution - Inequality 1

Solution: $3x - 10 > 5x + 4$

$$
\begin{aligned}
3x - 10 &> 5x + 4 \\
3x - 5x &> 4 + 10 \\
-2x &> 14 \\
x &< \frac{14}{-2} \\
x &< -7
\end{aligned}
$$

Final Solution: $x < -7$

General Practice Problems: Solution - Inequality 2 (Step 1)

Solution: $x^2 - 7x + 10 \geq 0$ (Step 1 - Find X-intercepts)

To solve the inequality $x^2 - 7x + 10 \geq 0$, first, find the x-intercepts of the corresponding equation $x^2 - 7x + 10 = 0$. Factor the quadratic expression:

$$x^2 - 7x + 10 = 0$$
$$(x-2)(x-5) = 0$$

So, the x-intercepts are $x = 2$ and $x = 5$. These roots divide the number line into three intervals: $(-\infty, 2]$, $[2, 5]$, and $[5, +\infty)$.

General Practice Problems: Solution - Inequality 2 (Step 2)

Solution: $x^2 - 7x + 10 \geq 0$ (Step 2 - Use Test Values)

Choose a test value from each interval and substitute it into the original inequality $x^2 - 7x + 10 \geq 0$ to check if the inequality holds true.

Interval 1: $(-\infty, 2]$

- Test value: $x = 0$
- Substitute: $(0)^2 - 7(0) + 10 = 10$
- Check inequality: $10 \geq 0$ (True)

Interval 2: $[2, 5]$

- Test value: $x = 3$
- Substitute: $(3)^2 - 7(3) + 10 = 9 - 21 + 10 = -2$
- Check inequality: $-2 \geq 0$ (False)

General Practice Problems: Solution - Inequality 2 (Step 3)

Solution: $x^2 - 7x + 10 \geq 0$ (Step 3 - Final Solution)

Interval 3: $[5, +\infty)$

- Test value: $x = 6$
- Substitute: $(6)^2 - 7(6) + 10 = 36 - 42 + 10 = 4$
- Check inequality: $4 \geq 0$ (True)

Based on the test values, the inequality $x^2 - 7x + 10 \geq 0$ is true in the intervals where the test value yielded a true statement.
Final Solution: $x \leq 2$ or $x \geq 5$
In interval notation: $(-\infty, 2] \cup [5, +\infty)$.

General Practice Problems: Solution - Inequality 3 (Step 1)

Solution: $2x^2 + x - 3 < 0$ (Step 1 - Find X-intercepts)

To solve the inequality $2x^2 + x - 3 < 0$, first, find the x-intercepts of the corresponding equation $2x^2 + x - 3 = 0$. Factor the quadratic expression:

$$2x^2 + x - 3 = 0$$
$$(2x + 3)(x - 1) = 0$$

So, the x-intercepts are $2x + 3 = 0 \Rightarrow x = -\frac{3}{2} = -1.5$ and $x - 1 = 0 \Rightarrow x = 1$. These roots divide the number line into three intervals: $(-\infty, -1.5)$, $(-1.5, 1)$, and $(1, +\infty)$.

Solution: $2x^2 + x - 3 < 0$ (Step 2 - Use Test Values)

Choose a test value from each interval and substitute it into the original inequality $2x^2 + x - 3 < 0$ to check if the inequality holds true.

Interval 1: $(-\infty, -1.5)$

- Test value: $x = -2$
- Substitute: $2(-2)^2 + (-2) - 3 = 2(4) - 2 - 3 = 8 - 2 - 3 = 3$
- Check inequality: $3 < 0$ (False)

Interval 2: $(-1.5, 1)$

- Test value: $x = 0$
- Substitute: $2(0)^2 + (0) - 3 = -3$
- Check inequality: $-3 < 0$ (True)

General Practice Problems: Solution - Inequality 3 (Step 3)

Solution: $2x^2 + x - 3 < 0$ (Step 3 - Final Solution)

Interval 3: $(1, +\infty)$

- Test value: $x = 2$
- Substitute: $2(2)^2 + (2) - 3 = 2(4) + 2 - 3 = 8 + 2 - 3 = 7$
- Check inequality: $7 < 0$ (False)

Based on the test values, the inequality $2x^2 + x - 3 < 0$ is true only in the interval where the test value yielded a true statement.

Final Solution: $-1.5 < x < 1$

In interval notation: $(-1.5, 1)$ or $(-\frac{3}{2}, 1)$.

General Practice Problems: Solution - Inequality 4 (Step 1)

Solution: $-x^2 + 6x - 9 \leq 0$ (Step 1 - Find X-intercepts)

To solve the inequality $-x^2 + 6x - 9 \leq 0$, first, find the x-intercepts of the corresponding equation $-x^2 + 6x - 9 = 0$. Factor the quadratic expression:

$$\begin{aligned} -(x^2 - 6x + 9) &= 0 \\ -(x-3)^2 &= 0 \end{aligned}$$

So, there is one x-intercept (a double root) at $x = 3$. This root divides the number line into two intervals: $(-\infty, 3]$ and $[3, +\infty)$.

General Practice Problems: Solution - Inequality 4 (Step 2)

Solution: $-x^2 + 6x - 9 \leq 0$ (Step 2 - Use Test Values and Final Solution)

Choose a test value from each interval and substitute it into the original inequality $-x^2 + 6x - 9 \leq 0$ to check if the inequality holds true.

Interval 1: $(-\infty, 3]$

- Test value: $x = 0$
- Substitute: $-(0)^2 + 6(0) - 9 = -9$
- Check inequality: $-9 \leq 0$ (True)

Interval 2: $[3, +\infty)$

- Test value: $x = 4$
- Substitute: $-(4)^2 + 6(4) - 9 = -16 + 24 - 9 = -1$
- Check inequality: $-1 \leq 0$ (True)

Since the inequality holds true for both intervals, and at $x = 3$ (where $-x^2 + 6x - 9 = 0$), the solution includes all real numbers.

Final Solution: All real numbers ($x \in \mathbb{R}$) In interval notation: $(-\infty, +\infty)$.

General Practice Problems: Solution - Inequality 5 (Step 1)

Solution: $x(x-5) > 14$ (Step 1 - Find X-intercepts)

To solve the inequality $x(x-5) > 14$, first, rearrange it into standard quadratic form and find the x-intercepts of the corresponding equation:

$$x(x-5) > 14$$
$$x^2 - 5x > 14$$
$$x^2 - 5x - 14 = 0$$

Factor the quadratic expression:

$$(x-7)(x+2) = 0$$

So, the x-intercepts are $x = 7$ and $x = -2$. These roots divide the number line into three intervals: $(-\infty, -2)$, $(-2, 7)$, and $(7, +\infty)$.

General Practice Problems: Solution - Inequality 5 (Step 2)

Solution: $x(x-5) > 14$ (Step 2 - Use Test Values)

Choose a test value from each interval and substitute it into the original inequality $x^2 - 5x - 14 > 0$ to check if the inequality holds true.

Interval 1: $(-\infty, -2)$

- Test value: $x = -3$
- Substitute: $(-3)^2 - 5(-3) - 14 = 9 + 15 - 14 = 10$
- Check inequality: $10 > 0$ (True)

Interval 2: $(-2, 7)$

- Test value: $x = 0$
- Substitute: $(0)^2 - 5(0) - 14 = -14$
- Check inequality: $-14 > 0$ (False)

General Practice Problems: Solution - Inequality 5 (Step 3)

Solution: $x(x-5) > 14$ (Step 3 - Final Solution)

Interval 3: $(7, +\infty)$

- Test value: $x = 8$
- Substitute: $(8)^2 - 5(8) - 14 = 64 - 40 - 14 = 10$
- Check inequality: $10 > 0$ (True)

Based on the test values, the inequality $x(x-5) > 14$ is true in the intervals where the test value yielded a true statement.

Final Solution: $x < -2$ or $x > 7$

In interval notation: $(-\infty, -2) \cup (7, +\infty)$.

General Practice Problems (Cont.)

Nature of Roots

Determine the nature of the roots for each equation (Do not solve):

1. $x^2 - 10x + 25 = 0$
2. $3x^2 + 2x + 1 = 0$
3. $2x^2 - 7x - 4 = 0$

General Practice Problems: Solution - Nature of Roots 1

Solution: $x^2 - 10x + 25$

For the equation $x^2 - 10x + 25 = 0$, we have $a = 1$, $b = -10$, and $c = 25$. Calculate the discriminant $D = b^2 - 4ac$:

$$D = (-10)^2 - 4(1)(25)$$
$$D = 100 - 100$$
$$D = 0$$

Since $D = 0$, there is **one double root (or two equal real roots)**.

General Practice Problems: Solution - Nature of Roots 2

Solution: $3x^2 + 2x + 1$

For the equation $3x^2 + 2x + 1 = 0$, we have $a = 3$, $b = 2$, and $c = 1$. Calculate the discriminant $D = b^2 - 4ac$:

$$D = (2)^2 - 4(3)(1)$$
$$D = 4 - 12$$
$$D = -8$$

Since $D < 0$, there are **no real roots**.

General Practice Problems: Solution - Nature of Roots 3

Solution: $2x^2 - 7x - 4$

For the equation $2x^2 - 7x - 4 = 0$, we have $a = 2$, $b = -7$, and $c = -4$. Calculate the discriminant $D = b^2 - 4ac$:

$$D = (-7)^2 - 4(2)(-4)$$
$$D = 49 - (-32)$$
$$D = 49 + 32$$
$$D = 81$$

Since $D > 0$, there are **two distinct real roots**.

General Practice Problems (Cont.)

Solving for k

For what values of 'k' does the equation have:

1. $x^2 + (k-2)x + 9 = 0$ have two equal roots?
2. $kx^2 - 4x + k = 0$ have no real roots?
3. $(k+1)x^2 + 5x + 2 = 0$ have two distinct real roots?

General Practice Problems: Solution - Solving for k 1

Solution: $x^2 + (k-2)x + 9$

For the equation $x^2 + (k-2)x + 9 = 0$, we have $a = 1$, $b = (k-2)$, and $c = 9$. For two equal roots, the discriminant D must be equal to zero ($D = b^2 - 4ac = 0$).

$$\begin{aligned}(k-2)^2 - 4(1)(9) &= 0 \\ (k-2)^2 - 36 &= 0 \\ (k-2)^2 &= 36 \\ k-2 &= \pm\sqrt{36} \\ k-2 &= \pm 6\end{aligned}$$

This gives two possible values for k:

- $k - 2 = 6 \Rightarrow k = 8$
- $k - 2 = -6 \Rightarrow k = -4$

Final Solution: $k = 8$ or $k = -4$.

General Practice Problems: Solution - Solving for k 2

Solution: $kx^2 - 4x + k$

For the equation $kx^2 - 4x + k = 0$, we have $a = k$, $b = -4$, and $c = k$. For no real roots, the discriminant D must be less than zero ($D = b^2 - 4ac < 0$).

$$
\begin{aligned}
(-4)^2 - 4(k)(k) &< 0 \\
16 - 4k^2 &< 0 \\
-4k^2 &< -16 \\
k^2 &> \frac{-16}{-4} \\
k^2 &> 4
\end{aligned}
$$

To solve $k^2 > 4$, we find the critical points $k = \pm 2$. Testing intervals, we find that the inequality holds true for $k < -2$ or $k > 2$. **Final Solution:** $k < -2$ or $k > 2$.
In interval notation: $(-\infty, -2) \cup (2, +\infty)$.

General Practice Problems: Solution - Solving for k 3

Solution: $(k+1)x^2 + 5x + 2$

For the equation $(k+1)x^2 + 5x + 2 = 0$, we have $a = (k+1)$, $b = 5$, and $c = 2$. For two distinct real roots, the discriminant D must be greater than zero ($D = b^2 - 4ac > 0$).

$$
\begin{aligned}
(5)^2 - 4(k+1)(2) &> 0 \\
25 - 8(k+1) &> 0 \\
25 - 8k - 8 &> 0 \\
17 - 8k &> 0 \\
-8k &> -17 \\
k &< \frac{-17}{-8} \\
k &< \frac{17}{8}
\end{aligned}
$$

General Practice Problems: Solution - Solving for k 3 (Cont.)

Solution: $(k+1)x^2 + 5x + 2$

Additionally, for the equation to be a quadratic, the coefficient of x^2 cannot be zero, so $k + 1 \neq 0 \Rightarrow k \neq -1$.
Final Solution: $k < \frac{17}{8}$ and $k \neq -1$.
In interval notation: $(-\infty, -1) \cup (-1, \frac{17}{8})$.