Pre-Calculus 11

6.4 Reciprocal Functions

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Overview

What We Will Learn

1. Understand the concept of reciprocal functions
2. Learn about asymptotes and invariant points
3. Graph reciprocal functions of linear and quadratic functions
4. Determine domain and range of reciprocal functions
5. Apply these concepts to solve problems

Key Questions:

- How do we find the reciprocal of a function?
- What happens to the graph when we take the reciprocal?
- Where do asymptotes occur and why?
- What are invariant points and why are they important?

What is a Reciprocal?

Definition

The reciprocal of a number a is $\frac{1}{a}$, where $a \neq 0$.
$a\frac{1}{a}a \neq 0$

Key Properties :

- $a \cdot \frac{1}{a} = 1$ (product of a number and its reciprocal is 1)
- The reciprocal of 1 is 1, the reciprocal of -1 is -1
- The reciprocal of 0 is undefined
- Taking the reciprocal preserves the sign

Examples of Reciprocals

Complete the table:

Number	**Reciprocal**	**Check**
2	$\frac{1}{2}$	$2 \cdot \frac{1}{2} = 1$
-5	$-\frac{1}{5}$	$-5 \cdot (-\frac{1}{5}) = 1$
0.5	2	$0.5 \cdot 2 = 1$
1	1	$1 \cdot 1 = 1$
-1	-1	$-1 \cdot (-1) = 1$
0	Undefined	Cannot divide by zero

Pattern: Small numbers have large reciprocals, large numbers have small reciprocals.

The Reciprocal of a Function

Definition

Given a function $f(x)$, its reciprocal function is $y = \frac{1}{f(x)}$.
$f(x)y = \frac{1}{f(x)}$

Examples :

- If $f(x) = x + 3$, then $y = \frac{1}{x+3}$
- If $f(x) = 2x^2 - 5$, then $y = \frac{1}{2x^2-5}$
- If $f(x) = \sqrt{x}$, then $y = \frac{1}{\sqrt{x}}$

Important Notes :

- The reciprocal function is undefined where $f(x) = 0$
- The sign of $f(x)$ is preserved in the reciprocal
- The reciprocal function has different behavior than the original function

Key Concepts

1. **Vertical Asymptotes** : Occur where $f(x) = 0$
2. **Horizontal Asymptote** : Usually $y = 0$ as $x \to \pm\infty$
3. **Invariant Points** : Where $f(x) = 1$ or $f(x) = -1$

Why These Points Are Important :

- Vertical asymptotes: Division by zero is undefined
- Invariant points: $\frac{1}{1} = 1$ and $\frac{1}{-1} = -1$
- These points help us sketch the graph accurately

Understanding Asymptotes

Vertical Asymptotes :

- Occur when $f(x) = 0$ because $\frac{1}{0}$ is undefined
- The graph approaches but never touches these lines
- Example: For $f(x) = x - 2$, the reciprocal has a vertical asymptote at $x = 2$

Horizontal Asymptotes :

- Usually $y = 0$ because as $|f(x)|$ gets large, $\frac{1}{f(x)}$ gets small
- The graph approaches this line as $x \to \pm\infty$
- Example: For $f(x) = x^2$, as $x \to \infty$, $y = \frac{1}{x^2} \to 0$

Graphing the Reciprocal of a Linear Function

Step-by-Step Process :

1. Graph the original linear function $f(x)$
2. Find where $f(x) = 0$ (vertical asymptote)
3. Find where $f(x) = 1$ and $f(x) = -1$ (invariant points)
4. Take the reciprocal of y-coordinates for other points
5. Sketch the reciprocal function

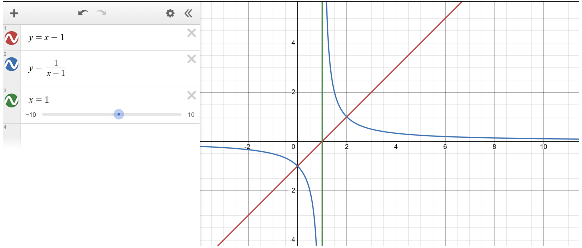
Example: $f(x) = x - 1$

- $f(x) = 0$ when $x = 1$ (vertical asymptote)
- $f(x) = 1$ when $x = 2$ (invariant point)
- $f(x) = -1$ when $x = 0$ (invariant point)

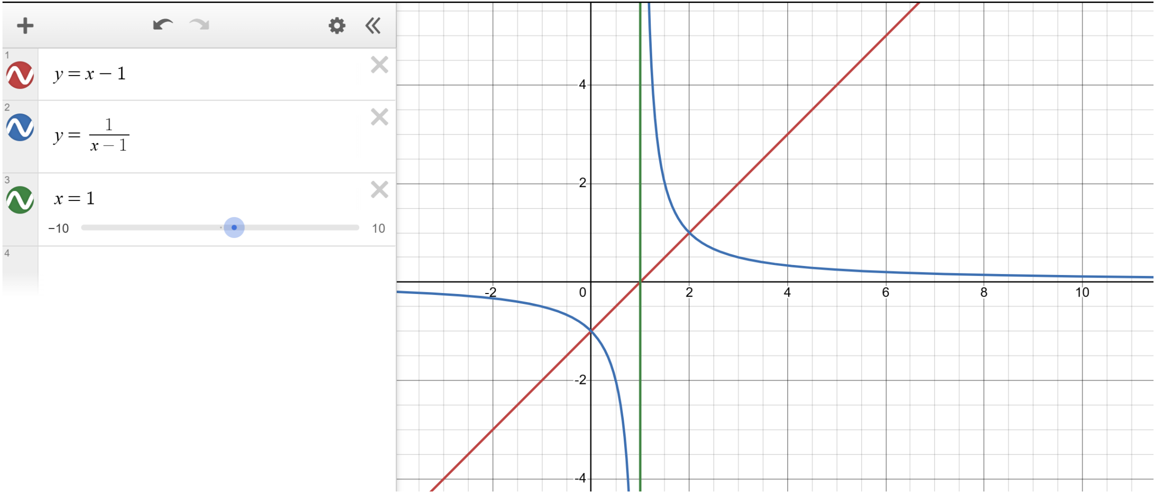
Original Function: $f(x) = x - 1$ (linear)
Reciprocal Function: $y = \frac{1}{x-1}$

Key Points:

- Vertical asymptote: $x = 1$
- Invariant points: $(2, 1)$ and $(0, -1)$
- As $x \to \infty$, $y \to 0^+$
- As $x \to -\infty$, $y \to 0^-$



Graphing Example: $f(x) = x - 1$ (Enlarged)



Graphing the Reciprocal of a Quadratic Function

Key Differences from Linear Functions :

- Quadratic functions can have 0, 1, or 2 x-intercepts
- Each x-intercept becomes a vertical asymptote
- Can have up to 4 invariant points (2 for $f(x) = 1$, 2 for $f(x) = -1$)
- The graph has more complex behavior

Example: $f(x) = x^2 - 4$

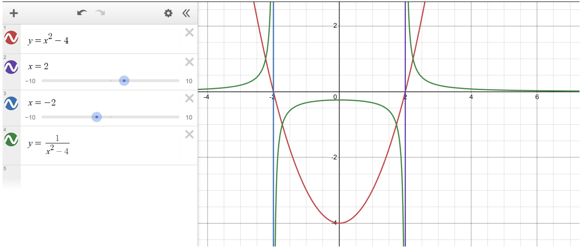
- $f(x) = 0$ at $x = 2$ and $x = -2$ (two vertical asymptotes)
- $f(x) = 1$ at $x = \pm\sqrt{5}$ (two invariant points)
- $f(x) = -1$ at $x = \pm\sqrt{3}$ (two invariant points)

Graphing Example: $f(x) = x^2 - 4$

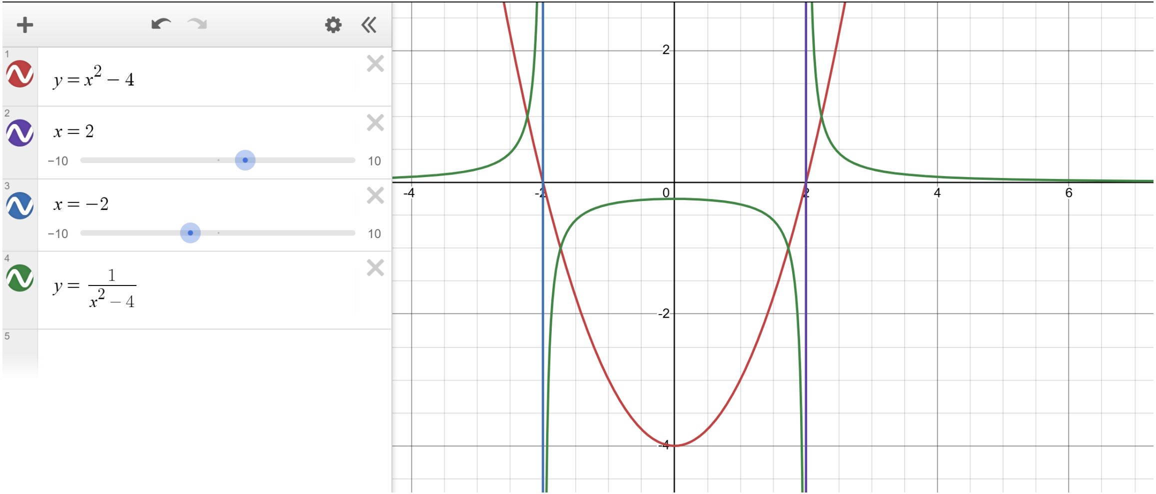
Original Function: $f(x) = x^2 - 4$ (parabola)
Reciprocal Function: $y = \frac{1}{x^2-4}$

Key Features:

- Vertical asymptotes: $x = 2$ and $x = -2$
- Invariant points: $(\pm\sqrt{5}, 1)$ and $(\pm\sqrt{3}, -1)$
- Horizontal asymptote: $y = 0$
- Three distinct regions: left, middle, right



Graphing Example: $f(x) = x^2 - 4$ (Enlarged)



Domain and Range

Domain

All real numbers x such that $f(x) \neq 0$.
$f(x) \neq 0x$

Range

All real numbers $y \neq 0$ (except possibly at invariant points).
$y \neq 0$

Examples :

- For $y = \frac{1}{x+2}$: Domain $= \{x | x \neq -2\}$, Range $= \{y | y \neq 0\}$
- For $y = \frac{1}{x^2}$: Domain $= \{x | x \neq 0\}$, Range $= \{y | y > 0\}$

Practice: Find the Reciprocal Function and Key Features

For each function $f(x)$ below, complete the following:

1. Write the reciprocal function $y = \frac{1}{f(x)}$
2. Find all vertical asymptotes
3. Find all invariant points
4. State the domain and range
5. Describe the key features of the graph

Functions to analyze:

- $f(x) = x + 2$
- $f(x) = 2x - 4$
- $f(x) = x^2 - 1$
- $f(x) = x^2 + 2x + 1$
- $f(x) = x^2 + 4$

Solution Q1: $f(x) = x + 2$

Reciprocal Function: $y = \frac{1}{x+2}$

Vertical Asymptote: $x = -2$ (where $f(x) = 0$)

Invariant Points:

- $f(x) = 1$ when $x + 2 = 1$, so $x = -1$. Point: $(-1, 1)$
- $f(x) = -1$ when $x + 2 = -1$, so $x = -3$. Point: $(-3, -1)$

Domain: $\{x | x \neq -2\}$
Range: $\{y | y \neq 0\}$

Graph Features: Hyperbola with vertical asymptote at $x = -2$, approaches $y = 0$ as $x \to \pm\infty$

Solution Q2: $f(x) = 2x - 4$

Reciprocal Function: $y = \frac{1}{2x-4}$

Vertical Asymptote: $x = 2$ (where $f(x) = 0$)

Invariant Points:

- $f(x) = 1$ when $2x - 4 = 1$, so $x = 2.5$. Point: $(2.5, 1)$
- $f(x) = -1$ when $2x - 4 = -1$, so $x = 1.5$. Point: $(1.5, -1)$

Domain: $\{x|x \neq 2\}$
Range: $\{y|y \neq 0\}$

Graph Features: Hyperbola with vertical asymptote at $x = 2$, approaches $y = 0$ as $x \to \pm\infty$

Solution Q3: $f(x) = x^2 - 1$

Reciprocal Function: $y = \frac{1}{x^2-1}$

Vertical Asymptotes: $x = 1$ and $x = -1$ (where $f(x) = 0$)

Invariant Points:

- $f(x) = 1$ when $x^2 - 1 = 1$, so $x^2 = 2$, $x = \pm\sqrt{2}$. Points: $(\sqrt{2}, 1)$, $(-\sqrt{2}, 1)$
- $f(x) = -1$ when $x^2 - 1 = -1$, so $x^2 = 0$, $x = 0$. Point: $(0, -1)$

Domain: $\{x | x \neq 1, x \neq -1\}$
Range: $\{y | y \neq 0\}$

Graph Features: Three regions separated by vertical asymptotes, approaches $y = 0$ as $x \to \pm\infty$

Solution Q4: $f(x) = x^2 + 2x + 1 = (x+1)^2$

Reciprocal Function: $y = \frac{1}{(x+1)^2}$

Vertical Asymptote: $x = -1$ (where $f(x) = 0$)

Invariant Points:

- $f(x) = 1$ when $(x+1)^2 = 1$, so $x + 1 = \pm 1$, $x = 0$ or $x = -2$. Points: $(0, 1)$, $(-2, 1)$
- $f(x) = -1$ when $(x+1)^2 = -1$ (no real solution)

Domain: $\{x | x \neq -1\}$
Range: $\{y | y > 0\}$ (since $(x+1)^2 \geq 0$)

Graph Features: Always positive, vertical asymptote at $x = -1$, approaches $y = 0$ as $x \to \pm\infty$

Solution Q5: $f(x) = x^2 + 4$

Reciprocal Function: $y = \frac{1}{x^2+4}$

Vertical Asymptotes: None (since $x^2 + 4 > 0$ for all real x)

Invariant Points:

- $f(x) = 1$ when $x^2 + 4 = 1$, so $x^2 = -3$ (no real solution)
- $f(x) = -1$ when $x^2 + 4 = -1$, so $x^2 = -5$ (no real solution)

Domain: All real numbers $\mathbb{R}$
Range: $\{y | 0 < y \leq \frac{1}{4}\}$ (maximum at $x = 0$)

Graph Features: Bell-shaped curve, maximum at $(0, \frac{1}{4})$, approaches $y = 0$ as $x \to \pm\infty$

Match each function with its reciprocal function:

- $f(x) = x - 3$
- $f(x) = x^2 - 9$
- $f(x) = x^2 + 1$
- $f(x) = 3x + 6$

Reciprocal Functions:

- $y = \frac{1}{x-3}$
- $y = \frac{1}{x^2-9}$
- $y = \frac{1}{x^2+1}$
- $y = \frac{1}{3x+6}$

Summary

Key Points

1. The reciprocal of $f(x)$ is $y = \frac{1}{f(x)}$
2. Vertical asymptotes occur where $f(x) = 0$
3. Invariant points occur where $f(x) = 1$ or $f(x) = -1$
4. Domain excludes values where $f(x) = 0$
5. Range is usually all real numbers except $y = 0$
6. The graph behavior changes significantly from the original function

Graphing Strategy :

1. Find vertical asymptotes (where $f(x) = 0$)
2. Find invariant points (where $f(x) = \pm 1$)
3. Determine behavior as $x \to \pm\infty$
4. Sketch the graph using these key features