Pre-Calculus 11

6.3 Solving Absolute Value Equations

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Solving Absolute Value Equations

Key Idea

When solving $|x| = a$, the value inside the absolute value can be both positive or negative.

For example: $|x| = 5$ means $x = 5$ or $x = -5$.

General Steps:

1. Isolate the absolute value expression.
2. Set up two cases: one for the positive, one for the negative.
3. Solve each case for x.
4. Check for extraneous roots by substituting back into the original equation.

Example: Solve $|x + 3| = 5$

Step 1: Set up two cases.

$x + 3 = 5$ or $x + 3 = -5$

Step 2: Solve each case.

$x = 2$ or $x = -8$

Step 3: Check for extraneous roots.

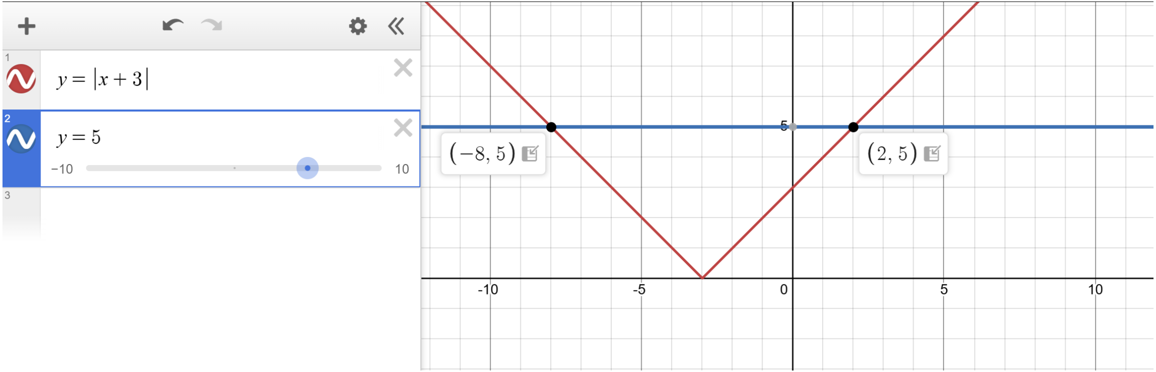
Substitute $x = 2$: $|2 + 3| = |5| = 5$ (valid)

Substitute $x = -8$: $|-8 + 3| = |-5| = 5$ (valid)

Final Answer: $x = 2$ and $x = -8$

Graphical Interpretation

If we solve $|x+3| = 5$ graphically, we are finding the x-values where $y = |x+3|$ intersects $y = 5$.



Steps for Solving Absolute Value Equations

Steps

1. **Isolate** the absolute value.
2. **Set up two cases** (positive and negative).
3. **Solve** each case for x.
4. **Check for extraneous roots** by substituting back into the original equation.

Extraneous roots occur when a solution does not satisfy the original equation (e.g., if the absolute value equals a negative number).

Which Equations Have No Solutions?

Which of the following equations will have no solutions?

- $|x - 5| = 20$
- $|x + 4| = -7$
- $|-2x + 3| = -10$
- $|-x - 7| = 13$
- $|x - 5| = -20$
- $|-x| = 5$

Hint: An absolute value cannot equal a negative number.

Practice: Solve for x and Check for Extraneous Roots

Solve for x and check for extraneous roots:

- $|x-1| = x+1$
- $|7-16x| = 3x+8$
- $|x^2-4| = 2x$
- $|2x-2| = 3x-8$
- $|2x^2-16x+8| = x^2-3x+5$
- $|3x-15| = 5x-9$

Solution Q1: $|x-1| = x+1$

Case 1: $x - 1 = x + 1$ →-1=1$(No solution)$
Case 2: $x - 1 = -(x + 1)$ →x-1 = -x-1→2x=0→x=0
Check: $|0 - 1| = |-1| = 1$, $0 + 1 = 1$ (Valid)
Final Answer: $x = 0$

Solution Q2: $|7 - 16x| = 3x + 8$

Case 1: $7 - 16x = 3x + 8 \to 7 - 16x - 3x = 8 \to -19x = 1 \to x = -\frac{1}{19}$
Case 2: $7 - 16x = -(3x + 8) \to 7 - 16x = -3x - 8 \to 7 + 8 = -3x + 16x \to 15 = 13x \to x = \frac{15}{13}$
Check: Substitute both values into the original equation to verify.
$x = -\frac{1}{19}$: $|7 - 16(-\frac{1}{19})| = 3(-\frac{1}{19}) + 8$
$|7 + \frac{16}{19}| = -\frac{3}{19} + 8$
$|\frac{149}{19}| = \frac{149}{19}$ (Valid)
$x = \frac{15}{13}$: $|7 - 16 \cdot \frac{15}{13}| = 3 \cdot \frac{15}{13} + 8$
$|7 - \frac{240}{13}| = \frac{45}{13} + 8$
$|\frac{91-240}{13}| = \frac{45+104}{13}$
$|\frac{-149}{13}| = \frac{149}{13}$ (Valid)
Final Answer: $x = -\frac{1}{19}$ and $x = \frac{15}{13}$

Practice: More Challenging Questions

Try these more challenging absolute value equations:

1. $|2x - 5| = |x + 4|$
2. $|x^2 - 6x + 8| = 2$
3. $|3x + 1| = 2x - 4$
4. $|x - 2| + |x + 2| = 6$

Solution Q7: $|2x-5| = |x+4|$

Case 1: $2x-5 = x+4 \rightarrow x = 9$
Case 2: $2x-5 = -(x+4) \rightarrow 2x-5 = -x-4 \rightarrow 3x = 1 \rightarrow x = \frac{1}{3}$
Case 3: $-(2x-5) = x+4 \rightarrow -2x+5 = x+4 \rightarrow -3x = -1 \rightarrow x = \frac{1}{3}$
Case 4: $-(2x-5) = -(x+4) \rightarrow -2x+5 = -x-4 \rightarrow -x = -9 \rightarrow x = 9$
Unique solutions: $x = 9,\ \frac{1}{3}$
Check: Both values satisfy the original equation.

Solution Q8: $|x^2 - 6x + 8| = 2$

Case 1: $x^2 - 6x + 8 = 2 \rightarrow x^2 - 6x + 6 = 0 \rightarrow x = 3 \pm \sqrt{3}$
Case 2: $x^2 - 6x + 8 = -2 \rightarrow x^2 - 6x + 10 = 0 \rightarrow x = 3 \pm i$ (no real solution)
Final Answer: $x = 3 + \sqrt{3},\ 3 - \sqrt{3}$

Solution Q9: $|3x+1| = 2x-4$

Case 1: $3x+1 = 2x-4 \to x = -5$
Case 2: $3x+1 = -(2x-4) \to 3x+1 = -2x+4 \to 5x = 3 \to x = \frac{3}{5}$
Check: $x = -5$: $|3(-5)+1| = |-15+1| = |-14| = 14$, $2(-5)-4 = -10-4 = -14$ (not valid, since $|3x+1|$ cannot be negative)
$x = \frac{3}{5}$: $|3 \cdot \frac{3}{5} + 1| = |\frac{9}{5}+1| = |\frac{14}{5}| = \frac{14}{5}$, $2 \cdot \frac{3}{5} - 4 = \frac{6}{5} - 4 = \frac{6}{5} - \frac{20}{5} = -\frac{14}{5}$ (not valid)
Final Answer: No real solution.

Solution Q10: $|x-2|+|x+2|=6$

Consider three intervals: $x \leq -2$, $-2 < x < 2$, $x \geq 2$.
Case 1: $x \leq -2$
$|x-2| = -(x-2)$, $|x+2| = -(x+2)$
$-(x-2) + -(x+2) = 6$ $-x+2-x-2 = 6$ $-2x = 6 \rightarrow x = -3$
Check: $x = -3 \leq -2$, $|-3-2| + |-3+2| = |-5| + |-1| = 5+1 = 6$ (valid)
Case 2: $-2 < x < 2$
$|x-2| = -(x-2)$, $|x+2| = x+2$
$-(x-2) + (x+2) = 6$ $-x+2+x+2 = 6$ $4 = 6$ (no solution)
Case 3: $x \geq 2$
$|x-2| = x-2$, $|x+2| = x+2$
$(x-2) + (x+2) = 6$ $2x = 6 \rightarrow x = 3$
Check: $x = 3 \geq 2$, $|3-2| + |3+2| = 1+5 = 6$ (valid)
Final Answer: $x = -3, \ x = 3$