Numerical Methods

Newton's Method and Euler's Method

Differential Calculus

Outline

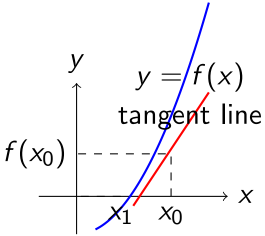
Why Numerical Methods?

- Many mathematical problems cannot be solved exactly using analytical methods
- Numerical methods provide approximate solutions using iterative procedures
- These methods are essential in:
  - Engineering and physics simulations
  - Financial modeling and optimization
  - Computer graphics and animation
  - Scientific computing and data analysis
- Two fundamental methods: Newton's Method and Euler's Method

Newton's Method - Introduction

- Newton's Method is used to find roots of equations: $f(x) = 0$
- It's an iterative method that uses linear approximation
- Based on the idea: if we have a good guess x_n, we can get a better guess x_{n+1}
- Uses the tangent line to approximate where the function crosses the x-axis
- Very fast convergence for most functions

Newton's Method - Geometric Intuition



Key Elements:

- Blue curve: $y = f(x)$ (the function we want to find roots of)
- Red line: tangent line at point $(x_0, f(x_0))$
- x_0: our initial guess
- x_1: where tangent line crosses x-axis (next guess)
- $f(x_0)$: function value at our initial guess

Newton's Method - Step-by-Step Process

The Newton's Method Process:

1. Start with initial guess x_0
2. Draw tangent line at point $(x_0, f(x_0))$
3. Find where tangent line crosses x-axis: x_1
4. Use x_1 as new guess and repeat process
5. Continue until convergence

Mathematical Insight:

- Tangent line equation: $y = f(x_0) + f'(x_0)(x - x_0)$
- Set $y = 0$ to find where it crosses x-axis
- Solve: $0 = f(x_0) + f'(x_0)(x_1 - x_0)$
- Result: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Newton's Method - Derivation

Step-by-step derivation:

- We have a guess x_n and want to find a better guess x_{n+1}
- The tangent line at $(x_n, f(x_n))$ has equation:

$$y = f(x_n) + f'(x_n)(x - x_n)$$

- This line crosses the x-axis when $y = 0$:

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n)$$

- Solving for x_{n+1}:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton's Method - Formula

Newton's Method Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Key Points:

- Requires both $f(x)$ and $f'(x)$
- Need a good initial guess x_0
- Method fails if $f'(x_n) = 0$ (horizontal tangent)
- Usually converges very quickly (quadratic convergence)
- Stop when $|x_{n+1} - x_n| < \epsilon$ or $|f(x_n)| < \epsilon$

Newton's Method - Detailed Derivation

Complete Mathematical Derivation:

- Start with initial guess x_1 for solution of $f(x) = 0$
- Tangent line at $x = x_1$: $y = F(x) = f(x_1) + f'(x_1)(x - x_1)$
- Solve $F(x) = 0$ for x_2:

$$0 = f(x_1) + f'(x_1)(x_2 - x_1)$$

$$x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

- Repeat: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Newton's Method - Algorithm

Newton's Method Algorithm:

1. Make a preliminary guess x_1
2. Define $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
3. Iterate: for each natural number n, once you have computed x_n, define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

4. Stop when $|x_{n+1} - x_n| < \epsilon$ or $|f(x_n)| < \epsilon$

Example 1: Newton's Method for Square Root

Problem: Use Newton's Method to approximate $\sqrt{5}$
Think about:

- What equation has $\sqrt{5}$ as a solution?
- What function should you use?
- What is a good initial guess?
- What is the derivative of your function?

Example 1: Newton's Method for Square Root - Solution

Solution:

- We want to solve $x^2 = 5$, so $f(x) = x^2 - 5$
- $f'(x) = 2x$
- Choose initial guess: $x_0 = 2$ (close to $\sqrt{5} \approx 2.236$)
- Newton's formula: $x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{x_n + \frac{5}{x_n}}{2}$

Example 1: Newton's Method for Square Root - Solution (Continued)

Iterations:

- $x_0 = 2$
- $x_1 = \frac{2+\frac{5}{2}}{2} = \frac{2+2.5}{2} = 2.25$
- $x_2 = \frac{2.25+\frac{5}{2.25}}{2} = \frac{2.25+2.222...}{2} = 2.2361$
- $x_3 = \frac{2.2361+\frac{5}{2.2361}}{2} = 2.236068$
- Actual value: $\sqrt{5} = 2.236068...$
- Very fast convergence!

Example 2: Newton's Method for Equation Solving

Problem: Use Newton's Method to find a root of $f(x) = x^3 - 2x - 5$

Think about:

- What is the derivative of this function?
- What would be a good initial guess?
- How can you check if your answer is reasonable?

Example 2: Newton's Method for Equation Solving - Solution

Solution:

- $f(x) = x^3 - 2x - 5$
- $f'(x) = 3x^2 - 2$
- Newton's formula: $x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$
- Choose initial guess: $x_0 = 2$ (since $f(2) = 8 - 4 - 5 = -1$ and $f(3) = 27 - 6 - 5 = 16$)

Example 2: Newton's Method for Equation Solving - Solution (Continued)

Iterations:

- $x_0 = 2$
- $x_1 = 2 - \frac{8-4-5}{12-2} = 2 - \frac{-1}{10} = 2.1$
- $x_2 = 2.1 - \frac{9.261-4.2-5}{13.23-2} = 2.1 - \frac{0.061}{11.23} = 2.0946$
- $x_3 = 2.0946 - \frac{9.19-4.189-5}{13.16-2} = 2.0946 - \frac{0.001}{11.16} = 2.0945$
- Check: $f(2.0945) = 9.19 - 4.189 - 5 = 0.001$ (very close to 0)

Example 3: Approximating $\sqrt{2}$

Problem: Use Newton's Method to approximate $\sqrt{2}$
Solution:

- Solve $f(x) = x^2 - 2 = 0$
- $f'(x) = 2x$
- Newton's formula: $x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$
- Start with $x_1 = 1.5$ (between 1 and 2)

Example 3: Approximating $\sqrt{2}$ - Iterations

Detailed Iterations:

- $x_1 = 1.5$
- $x_2 = \frac{1.5}{2} + \frac{1}{1.5} = 0.75 + 0.6667 = 1.416666667$
- $x_3 = \frac{1.416666667}{2} + \frac{1}{1.416666667} = 0.7083 + 0.7059 = 1.414215686$
- $x_4 = \frac{1.414215686}{2} + \frac{1}{1.414215686} = 0.7071 + 0.7071 = 1.414213562$
- $x_5 = \frac{1.414213562}{2} + \frac{1}{1.414213562} = 1.414213562$
- $\sqrt{2} \approx 1.414213562$ (9 decimal places accuracy!)

Example 4: Approximating π

Problem: Use Newton's Method to approximate π
Solution:

- Solve $f(x) = \sin(x) = 0$ (since $\sin(\pi) = 0$)
- $f'(x) = \cos(x)$
- Newton's formula: $x_{n+1} = x_n - \frac{\sin(x_n)}{\cos(x_n)} = x_n - \tan(x_n)$
- Start with $x_1 = 3$ (close to π)

Example 4: Approximating π - Iterations

Detailed Iterations:

- $x_1 = 3$
- $x_2 = 3 - \tan(3) = 3.142546543$
- $x_3 = 3.142546543 - \tan(3.142546543) = 3.141592653$
- $x_4 = 3.141592653 - \tan(3.141592653) = 3.141592654$
- $x_5 = 3.141592654 - \tan(3.141592654) = 3.141592654$
- $\pi \approx 3.141592654$ (9 decimal places accuracy!)

Example 5: When Newton's Method Fails

Problem: Try to solve $f(x) = \arctan(x) = 0$ starting with $x_1 = 1.5$
What happens:

- $f'(x) = \frac{1}{1+x^2}$
- Newton's formula: $x_{n+1} = x_n - (1 + x_n^2)\arctan(x_n)$
- $x_1 = 1.5$
- $x_2 = -1.69$
- $x_3 = 2.32$
- $x_4 = -5.11$
- The sequence diverges wildly!

Example 5: Why Newton's Method Failed

Analysis of Failure:

- The initial guess $x_1 = 1.5$ was too far from the root $x = 0$
- Tangent line at $x = 1.5$ was a poor approximation near $x = 0$
- Each iteration moved further from the solution
- With better initial guess $x_1 = 0.5$:
  - $x_1 = 0.5$
  - $x_2 = -0.0796$
  - $x_3 = 0.000335$
  - $x_4 = -2.51 \times 10^{-11}$
- Success with good initial guess!

Example 6: Interest Rate Calculation

Problem: A car dealer sells a car for \$23,520 with payments of \$420/month for 5 years. What interest rate is being charged?

Solution:

- Present value of 60 payments: $PV = 420\frac{1-(1+r)^{-60}}{r}$
- Set equal to \$23,520: $23520 = 420\frac{1-(1+r)^{-60}}{r}$
- Simplify: $56 = \frac{1-(1+r)^{-60}}{r}$
- Define: $f(r) = (1 - 56r)(1 + r)^{60} - 1$
- Find root of $f(r) = 0$

Example 6: Interest Rate Calculation - Iterations

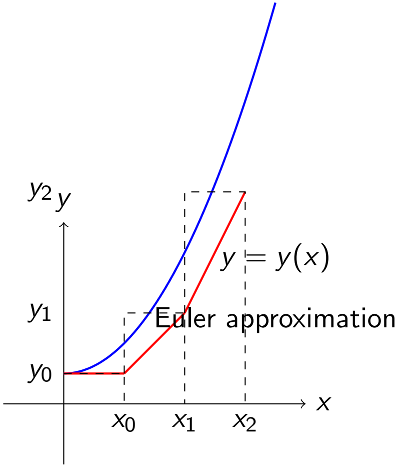
Newton's Method Application:

- $f(r) = (1 - 56r)(1 + r)^{60} - 1$
- $f'(r) = (4 - 3416r)(1 + r)^{59}$
- Start with $r_1 = 0.002$ (0.2% per month)
- $r_2 = 0.002344$
- $r_3 = 0.002292$
- $r_4 = 0.002290$
- $r_5 = 0.002290$
- Interest rate: 0.229% per month = 2.75% per year

Euler's Method - Introduction

- Euler's Method is used to solve differential equations numerically
- Given $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$
- We want to approximate the solution $y(x)$ at various points
- Uses linear approximation to step from one point to the next
- Simple but fundamental method for numerical integration

Euler's Method - Geometric Intuition



Visual:

- Start at (x_0, y_0)

Euler's Method - Derivation

Step-by-step derivation:

- We have $\frac{dy}{dx} = f(x, y)$ and want to approximate $y(x)$
- At point (x_n, y_n), the slope is $f(x_n, y_n)$
- Using linear approximation: $y(x_{n+1}) \approx y_n + f(x_n, y_n)(x_{n+1} - x_n)$
- Let $h = x_{n+1} - x_n$ (step size)
- Then: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$

Euler's Method - Formula

Euler's Method Formula:

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

where h is the step size and $x_{n+1} = x_n + h$

Key Points:

- Requires the differential equation $\frac{dy}{dx} = f(x, y)$
- Need initial condition $y(x_0) = y_0$
- Choose step size h (smaller = more accurate but more steps)
- Error accumulates with each step
- Simple but can be inaccurate for large step sizes

Example 7: Euler's Method for Simple ODE

Problem: Use Euler's Method to approximate the solution of $\frac{dy}{dx} = x + y$ with $y(0) = 1$ at $x = 0.5$ using step size $h = 0.1$

Think about:

- What is your initial condition?
- How many steps do you need?
- What is $f(x, y)$ in this case?
- How do you calculate each step?

Example 7: Euler's Method for Simple ODE - Solution

Solution:

- $\frac{dy}{dx} = x + y$, so $f(x, y) = x + y$
- Initial condition: $x_0 = 0$, $y_0 = 1$
- Step size: $h = 0.1$
- Need 5 steps to reach $x = 0.5$
- Euler's formula: $y_{n+1} = y_n + 0.1(x_n + y_n)$

Example 7: Euler's Method for Simple ODE - Solution (Continued)

Iterations:

- $x_0 = 0,\ y_0 = 1$
- $x_1 = 0.1,\ y_1 = 1 + 0.1(0 + 1) = 1.1$
- $x_2 = 0.2,\ y_2 = 1.1 + 0.1(0.1 + 1.1) = 1.1 + 0.12 = 1.22$
- $x_3 = 0.3,\ y_3 = 1.22 + 0.1(0.2 + 1.22) = 1.22 + 0.142 = 1.362$
- $x_4 = 0.4,\ y_4 = 1.362 + 0.1(0.3 + 1.362) = 1.362 + 0.1662 = 1.5282$
- $x_5 = 0.5,\ y_5 = 1.5282 + 0.1(0.4 + 1.5282) = 1.5282 + 0.19282 = 1.72102$
- Approximation: $y(0.5) \approx 1.721$

Example 8: Euler's Method for Exponential Growth

Problem: Use Euler's Method to approximate the solution of $\frac{dy}{dx} = y$ with $y(0) = 1$ at $x = 1$ using step size $h = 0.25$

Think about:

- What is the exact solution to this differential equation?
- How many steps do you need?
- How accurate do you expect the approximation to be?

Example 8: Euler's Method for Exponential Growth - Solution

Solution:

- $\frac{dy}{dx} = y$, so $f(x, y) = y$
- Initial condition: $x_0 = 0$, $y_0 = 1$
- Step size: $h = 0.25$
- Need 4 steps to reach $x = 1$
- Euler's formula: $y_{n+1} = y_n + 0.25y_n = 1.25y_n$

Example 8: Euler's Method for Exponential Growth - Solution (Continued)

Iterations:

- $x_0 = 0, \ y_0 = 1$
- $x_1 = 0.25, \ y_1 = 1.25 \times 1 = 1.25$
- $x_2 = 0.5, \ y_2 = 1.25 \times 1.25 = 1.5625$
- $x_3 = 0.75, \ y_3 = 1.25 \times 1.5625 = 1.953125$
- $x_4 = 1.0, \ y_4 = 1.25 \times 1.953125 = 2.441406$
- Approximation: $y(1) \approx 2.441$
- Exact solution: $y(x) = e^x$, so $y(1) = e \approx 2.718$
- Error: $|2.718 - 2.441| = 0.277$ (about 10% error)

Newton's Method - Error Analysis

- Newton's Method has quadratic convergence
- If r is the exact root, then $|x_{n+1} - r| \approx C|x_n - r|^2$
- This means the number of correct digits roughly doubles each iteration
- Method fails if:
  - $f'(x_n) = 0$ (horizontal tangent)
  - Initial guess is too far from root
  - Function has multiple roots close together

Euler's Method - Error Analysis

- Euler's Method has linear convergence
- Local truncation error: $O(h^2)$ per step
- Global truncation error: $O(h)$ over the entire interval
- Error accumulates with each step
- Can be improved by:
    - Using smaller step size
    - Using more sophisticated methods (Runge-Kutta)
    - Using adaptive step sizes

Practice Problem 1

Problem: Use Newton's Method to approximate $\sqrt{7}$ starting with $x_0 = 2.5$
Hint: Solve $x^2 = 7$ using $f(x) = x^2 - 7$.

Steps to follow:

1. Write down the function $f(x)$ and its derivative $f'(x)$
2. Write the Newton's Method formula
3. Perform the iterations step by step
4. Check your answer

Practice Problem 2

Problem: Use Newton's Method to find a root of $f(x) = x^3 - x - 1$ starting with $x_0 = 1$
Hint: Calculate $f'(x) = 3x^2 - 1$ and use the Newton formula.

Steps to follow:

1. Write down the function $f(x)$ and its derivative $f'(x)$
2. Write the Newton's Method formula
3. Perform the iterations step by step
4. Check your answer by evaluating $f(x)$ at your final result

Practice Problem 3

Problem: Use Euler's Method to approximate the solution of $\frac{dy}{dx} = 2x$ with $y(0) = 0$ at $x = 1$ using $h = 0.2$

Hint: This is a simple integration problem. What should the exact solution be?

Steps to follow:

1. Identify $f(x, y)$ from the differential equation
2. Determine how many steps you need
3. Use Euler's formula: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$
4. Calculate each step carefully
5. Compare with the exact solution

Practice Problem 4

Problem: Use Euler's Method to approximate the solution of $\frac{dy}{dx} = -y$ with $y(0) = 1$ at $x = 1$ using $h = 0.25$
Hint: This is exponential decay. The exact solution is $y(x) = e^{-x}$.

Steps to follow:

1. Identify $f(x, y)$ from the differential equation
2. Determine how many steps you need
3. Use Euler's formula: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$
4. Calculate each step carefully
5. Compare with the exact solution $y(x) = e^{-x}$

Practice Problem 5

Problem: Use Newton's Method to find a root of $f(x) = \cos(x) - x$ starting with $x_0 = 0.5$
Hint: You'll need to calculate $f'(x) = -\sin(x) - 1$.

Steps to follow:

1. Write down the function $f(x)$ and its derivative $f'(x)$
2. Write the Newton's Method formula
3. Perform the iterations step by step
4. Check your answer by evaluating $f(x)$ at your final result

Practice Problem 6

Problem: Use Euler's Method to approximate the solution of $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$ at $x = 0.5$ using $h = 0.1$
Hint: This is a more complex ODE. Use the Euler formula carefully.

Steps to follow:

1. Identify $f(x, y)$ from the differential equation
2. Determine how many steps you need
3. Use Euler's formula: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$
4. Calculate each step carefully, paying attention to both x_n^2 and y_n terms

Practice Problem 7

Problem: Use Newton's Method to find a root of $f(x) = e^x - 2x - 1$ starting with $x_0 = 1$
Hint: Calculate $f'(x) = e^x - 2$ and use the Newton formula.

Steps to follow:

1. Write down the function $f(x)$ and its derivative $f'(x)$
2. Write the Newton's Method formula
3. Perform the iterations step by step
4. Check your answer by evaluating $f(x)$ at your final result

Practice Problem 8

Problem: Use Newton's Method to approximate $\sqrt[3]{10}$ starting with $x_0 = 2$
Hint: Solve $x^3 = 10$ using $f(x) = x^3 - 10$.

Steps to follow:

1. Write down the function $f(x)$ and its derivative $f'(x)$
2. Write the Newton's Method formula
3. Perform the iterations step by step
4. Check your answer by cubing your final result

Practice Problem 1 - Solution

Solution: Approximate $\sqrt{7}$ using Newton's Method

- $f(x) = x^2 - 7,\ f'(x) = 2x$
- Newton's formula: $x_{n+1} = x_n - \frac{x_n^2 - 7}{2x_n} = \frac{x_n + \frac{7}{x_n}}{2}$
- $x_0 = 2.5$
- $x_1 = \frac{2.5 + \frac{7}{2.5}}{2} = \frac{2.5 + 2.8}{2} = 2.65$
- $x_2 = \frac{2.65 + \frac{7}{2.65}}{2} = \frac{2.65 + 2.6415}{2} = 2.64575$
- $x_3 = \frac{2.64575 + \frac{7}{2.64575}}{2} = 2.645751$
- $\sqrt{7} \approx 2.645751$ (very accurate!)

Practice Problem 2 - Solution

Solution: Find root of $f(x) = x^3 - x - 1$ using Newton's Method

- $f(x) = x^3 - x - 1$, $f'(x) = 3x^2 - 1$
- Newton's formula: $x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$
- $x_0 = 1$
- $x_1 = 1 - \frac{1-1-1}{3-1} = 1 - \frac{-1}{2} = 1.5$
- $x_2 = 1.5 - \frac{3.375-1.5-1}{6.75-1} = 1.5 - \frac{0.875}{5.75} = 1.3478$
- $x_3 = 1.3478 - \frac{2.448-1.3478-1}{5.45-1} = 1.3478 - \frac{0.1002}{4.45} = 1.3252$
- Root ≈ 1.3252

Practice Problem 3 - Solution

Solution: Euler's Method for $\frac{dy}{dx} = 2x$ with $y(0) = 0$

- $f(x, y) = 2x$, $h = 0.2$, need 5 steps to reach $x = 1$
- $x_0 = 0$, $y_0 = 0$
- $x_1 = 0.2$, $y_1 = 0 + 0.2(2 \times 0) = 0$
- $x_2 = 0.4$, $y_2 = 0 + 0.2(2 \times 0.2) = 0.08$
- $x_3 = 0.6$, $y_3 = 0.08 + 0.2(2 \times 0.4) = 0.08 + 0.16 = 0.24$
- $x_4 = 0.8$, $y_4 = 0.24 + 0.2(2 \times 0.6) = 0.24 + 0.24 = 0.48$
- $x_5 = 1.0$, $y_5 = 0.48 + 0.2(2 \times 0.8) = 0.48 + 0.32 = 0.80$
- Approximation: $y(1) \approx 0.80$
- Exact solution: $y(x) = x^2$, so $y(1) = 1$
- Error: $|1 - 0.80| = 0.20$ (20% error)

Practice Problem 4 - Solution

Solution: Euler's Method for $\frac{dy}{dx} = -y$ with $y(0) = 1$

- $f(x, y) = -y$, $h = 0.25$, need 4 steps to reach $x = 1$
- $x_0 = 0$, $y_0 = 1$
- $x_1 = 0.25$, $y_1 = 1 + 0.25(-1) = 0.75$
- $x_2 = 0.5$, $y_2 = 0.75 + 0.25(-0.75) = 0.75 - 0.1875 = 0.5625$
- $x_3 = 0.75$, $y_3 = 0.5625 + 0.25(-0.5625) = 0.5625 - 0.1406 = 0.4219$
- $x_4 = 1.0$, $y_4 = 0.4219 + 0.25(-0.4219) = 0.4219 - 0.1055 = 0.3164$
- Approximation: $y(1) \approx 0.3164$
- Exact solution: $y(x) = e^{-x}$, so $y(1) = e^{-1} \approx 0.3679$
- Error: $|0.3679 - 0.3164| = 0.0515$ (about 14% error)

Practice Problem 5 - Solution

Solution: Find root of $f(x) = \cos(x) - x$ using Newton's Method

- $f(x) = \cos(x) - x$, $f'(x) = -\sin(x) - 1$
- Newton's formula: $x_{n+1} = x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1}$
- $x_0 = 0.5$
- $x_1 = 0.5 - \frac{\cos(0.5) - 0.5}{-\sin(0.5) - 1} = 0.5 - \frac{0.8776 - 0.5}{-0.4794 - 1} = 0.5 - \frac{0.3776}{-1.4794} = 0.7552$
- $x_2 = 0.7552 - \frac{\cos(0.7552) - 0.7552}{-\sin(0.7552) - 1} = 0.7552 - \frac{0.7281 - 0.7552}{-0.6845 - 1} = 0.7552 - \frac{-0.0271}{-1.6845} = 0.7391$
- $x_3 = 0.7391 - \frac{\cos(0.7391) - 0.7391}{-\sin(0.7391) - 1} = 0.7391 - \frac{0.7396 - 0.7391}{-0.6736 - 1} = 0.7391 - \frac{0.0005}{-1.6736} = 0.7391$
- Root ≈ 0.7391 (this is a fixed point of $\cos(x)$)

Practice Problem 6 - Solution

Solution: Euler's Method for $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$

- $f(x, y) = x^2 + y$, $h = 0.1$, need 5 steps to reach $x = 0.5$
- $x_0 = 0$, $y_0 = 1$
- $x_1 = 0.1$, $y_1 = 1 + 0.1(0^2 + 1) = 1 + 0.1 = 1.1$
- $x_2 = 0.2$, $y_2 = 1.1 + 0.1(0.1^2 + 1.1) = 1.1 + 0.1(0.01 + 1.1) = 1.1 + 0.111 = 1.211$
- $x_3 = 0.3$,
  $y_3 = 1.211 + 0.1(0.2^2 + 1.211) = 1.211 + 0.1(0.04 + 1.211) = 1.211 + 0.1251 = 1.3361$
- $x_4 = 0.4$, $y_4 = 1.3361 + 0.1(0.3^2 + 1.3361) = 1.3361 + 0.1(0.09 + 1.3361) = 1.3361 + 0.14261 = 1.47871$
- $x_5 = 0.5$, $y_5 = 1.47871 + 0.1(0.4^2 + 1.47871) = 1.47871 + 0.1(0.16 + 1.47871) = 1.47871 + 0.163871 = 1.642581$
- Approximation: $y(0.5) \approx 1.643$

Practice Problem 7 - Solution

Solution: Find root of $f(x) = e^x - 2x - 1$ using Newton's Method

- $f(x) = e^x - 2x - 1$, $f'(x) = e^x - 2$
- Newton's formula: $x_{n+1} = x_n - \frac{e^{x_n} - 2x_n - 1}{e^{x_n} - 2}$
- $x_0 = 1$
- $x_1 = 1 - \frac{e^1 - 2(1) - 1}{e^1 - 2} = 1 - \frac{2.718 - 2 - 1}{2.718 - 2} = 1 - \frac{-0.282}{0.718} = 1.393$
- $x_2 = 1.393 - \frac{e^{1.393} - 2(1.393) - 1}{e^{1.393} - 2} = 1.393 - \frac{4.027 - 2.786 - 1}{4.027 - 2} = 1.393 - \frac{0.241}{2.027} = 1.274$
- $x_3 = 1.274 - \frac{e^{1.274} - 2(1.274) - 1}{e^{1.274} - 2} = 1.274 - \frac{3.575 - 2.548 - 1}{3.575 - 2} = 1.274 - \frac{0.027}{1.575} = 1.257$
- Root ≈ 1.257

Practice Problem 8 - Solution

Solution: Approximate $\sqrt[3]{10}$ using Newton's Method

- $f(x) = x^3 - 10,\ f'(x) = 3x^2$
- Newton's formula: $x_{n+1} = x_n - \frac{x_n^3 - 10}{3x_n^2} = x_n - \frac{x_n^3 - 10}{3x_n^2}$
- $x_0 = 2$
- $x_1 = 2 - \frac{8-10}{12} = 2 - \frac{-2}{12} = 2 + 0.167 = 2.167$
- $x_2 = 2.167 - \frac{10.18-10}{14.08} = 2.167 - \frac{0.18}{14.08} = 2.167 - 0.013 = 2.154$
- $x_3 = 2.154 - \frac{10.00-10}{13.92} = 2.154 - \frac{0.00}{13.92} = 2.154$
- $\sqrt[3]{10} \approx 2.154$ (very accurate!)

Summary

- Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for finding roots
- Euler's Method: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$ for solving ODEs
- Newton's Method has quadratic convergence (very fast)
- Euler's Method has linear convergence (slower, accumulates error)
- Both methods require good initial conditions
- Error analysis helps understand accuracy and limitations
- Newton's Method can fail with poor initial guesses
- Euler's Method accuracy improves with smaller step sizes

Thank You!

Questions?

Numerical methods are essential tools in modern mathematics and science!