The Chain Rule

One More Tool for Differentiation

Differential Calculus

Outline

Why We Need the Chain Rule

- We have learned derivatives of sums, products, and quotients
- We know derivatives of basic functions (polynomials, trig, exponential, etc.)
- But what about compositions like $\sin(x^2)$, e^{3x+1}, or $(x^2+1)^5$?
- The chain rule tells us how to differentiate composite functions

What is a Composite Function?

Definition

A composite function is a function formed by combining two functions:

$$f(g(x)) = f \circ g(x)$$

where:

- $g(x)$ is the "inside" function
- $f(x)$ is the "outside" function

The Chain Rule - Statement

Theorem

Let f and g be differentiable functions, then:

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

In words: Differentiate the outside function, then multiply by the derivative of the inside function.

Alternative Form of Chain Rule

Differential Notation

If $y = f(u)$ and $u = g(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Memory Aid: It looks like the *du* terms cancel out!

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1: Power of a Function

Find the derivative of $f(x) = (x^2 + 3)^4$

Solution to Example 1

Solution:

$$
\begin{aligned}
f(x) &= (x^2+3)^4 \\
f'(x) &= 4(x^2+3)^3 \cdot \frac{d}{dx}(x^2+3) \\
&= 4(x^2+3)^3 \cdot 2x \\
&= 8x(x^2+3)^3
\end{aligned}
$$

Example 2: Trigonometric Function

Find the derivative of $f(x) = \sin(3x + 2)$

Solution to Example 2

Solution:

$$\begin{aligned} f(x) &= \sin(3x+2) \\ f'(x) &= \cos(3x+2) \cdot \frac{d}{dx}(3x+2) \\ &= \cos(3x+2) \cdot 3 \\ &= 3\cos(3x+2) \end{aligned}$$

Example 3: Exponential Function

Find the derivative of $f(x) = e^{x^2+1}$

Solution to Example 3

Solution:

$$\begin{aligned}
f(x) &= e^{x^2+1} \\
f'(x) &= e^{x^2+1} \cdot \frac{d}{dx}(x^2+1) \\
&= e^{x^2+1} \cdot 2x \\
&= 2xe^{x^2+1}
\end{aligned}$$

Power Rule with Chain Rule

General Formula

For any differentiable function $g(x)$ and any power n:

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Linear Argument Rule

General Formula

For any differentiable function $f(x)$ and constants a, b:

$$\frac{d}{dx} f(ax + b) = a \cdot f'(ax + b)$$

Practice: 1 and 2

Practice 1:

$$\text{Find the derivative of } f(x) = (2x^2 + 3x + 1)^4$$

Practice 2:

$$\text{Find the derivative of } f(x) = \sin(5x - 2)$$

Practice: 3 and 4

Practice 3:

$$\text{Find the derivative of } f(x) = e^{3x^2+2x}$$

Practice 4:

$$\text{Find the derivative of } f(x) = \ln(4x^3 - 7x)$$

Practice: 5 and 6

Practice 5:

Find the derivative of $f(x) = \cos^3(2x+1)$

Practice 6:

Find the derivative of $f(x) = \sqrt{x^2+4}$

Practice: 7 and 8

Practice 7:

Find the derivative of $f(x) = \tan(3x^2 - 1)$

Practice 8:

Find the derivative of $f(x) = (x^3 + 2x)^5$

Practice: 9 and 10

Practice 9:

Find the derivative of $f(x) = \sin^2(x^2 + 1)$

Practice 10:

Find the derivative of $f(x) = e^{\sin(x)}$

Practice: 11 and 12

Practice 11:

$$\text{Find the derivative of } f(x) = \ln(\cos(x))$$

Practice 12:

$$\text{Find the derivative of } f(x) = \arctan(x^3 + 2x)$$

Practice: 13 and 14

Practice 13:

Find the derivative of $f(x) = \cos(e^x + x^2)$

Practice 14:

Find the derivative of $f(x) = \sqrt{\ln(x^2 + 1)}$

Practice: 15 and 16

Practice 15:

Find the derivative of $f(x) = \sin(\ln(x))$

Practice 16:

Find the derivative of $f(x) = e^{\arctan(x)}$

Practice: 17 and 18

Practice 17:

Find the derivative of $f(x) = \sin^3(\cos(x^2))$

Practice 18:

Find the derivative of $f(x) = \ln(\sqrt{e^x + \sin(x)})$

Practice: 19 and 20

Practice 19:

$$\text{Find the derivative of } f(x) = e^{\sin(\ln(x))}$$

Practice 20:

$$\text{Find the derivative of } f(x) = \arctan(\sqrt{x^2 + \cos(x)})$$

Practice: 21 and 22

Practice 21:

Find the derivative of $f(x) = \cos(\ln(\sin(x)))$

Practice 22:

Find the derivative of $f(x) = \sqrt{e^{\sin(x)} + \cos(x)}$

Practice: 23 and 24

Practice 23:

$$\text{Find the derivative of } f(x) = \ln(\arctan(e^x))$$

Practice 24:

$$\text{Find the derivative of } f(x) = \sin(\cos(\tan(x)))$$

Practice: 25

Practice 25:

Find the derivative of $f(x) = e^{\sin(\cos(\ln(x)))}$

Solution to Practice 1

Practice 1:
Find the derivative of $f(x) = (2x^2 + 3x + 1)^4$
Solution:

$$\begin{aligned} f'(x) &= 4(2x^2 + 3x + 1)^3 \cdot \frac{d}{dx}(2x^2 + 3x + 1) \\ &= 4(2x^2 + 3x + 1)^3 \cdot (4x + 3) \\ &= 4(2x^2 + 3x + 1)^3(4x + 3) \end{aligned}$$

Solution to Practice 2

Practice 2:
Find the derivative of $f(x) = \sin(5x - 2)$
Solution:

$$\begin{aligned} f'(x) &= \cos(5x-2) \cdot \frac{d}{dx}(5x-2) \\ &= \cos(5x-2) \cdot 5 \\ &= 5\cos(5x-2) \end{aligned}$$

Solution to Practice 3

Practice 3:
Find the derivative of $f(x) = e^{3x^2+2x}$
Solution:

$$f'(x) = e^{3x^2+2x} \cdot \frac{d}{dx}(3x^2 + 2x)$$
$$= e^{3x^2+2x} \cdot (6x + 2)$$
$$= e^{3x^2+2x}(6x + 2)$$

Solution to Practice 4

Practice 4:
Find the derivative of $f(x) = \ln(4x^3 - 7x)$
Solution:

$$\begin{aligned} f'(x) &= \frac{1}{4x^3 - 7x} \cdot \frac{d}{dx}(4x^3 - 7x) \\ &= \frac{1}{4x^3 - 7x} \cdot (12x^2 - 7) \\ &= \frac{12x^2 - 7}{4x^3 - 7x} \end{aligned}$$

Solution to Practice 5

Practice 5:
Find the derivative of $f(x) = \cos^3(2x+1)$
Solution:

$$\begin{aligned}
f'(x) &= 3\cos^2(2x+1) \cdot \frac{d}{dx}(\cos(2x+1)) \\
&= 3\cos^2(2x+1) \cdot (-\sin(2x+1)) \cdot \frac{d}{dx}(2x+1) \\
&= 3\cos^2(2x+1) \cdot (-\sin(2x+1)) \cdot 2 \\
&= -6\cos^2(2x+1)\sin(2x+1)
\end{aligned}$$

Solution to Practice 6

Practice 6:
Find the derivative of $f(x) = \sqrt{x^2 + 4}$
Solution:

$$
\begin{aligned}
f'(x) &= \frac{1}{2\sqrt{x^2+4}} \cdot \frac{d}{dx}(x^2 + 4) \\
&= \frac{1}{2\sqrt{x^2+4}} \cdot 2x \\
&= \frac{x}{\sqrt{x^2+4}}
\end{aligned}
$$

Solution to Practice 7

Practice 7:
Find the derivative of $f(x) = \tan(3x^2 - 1)$
Solution:

$$\begin{aligned} f'(x) &= \sec^2(3x^2 - 1) \cdot \frac{d}{dx}(3x^2 - 1) \\ &= \sec^2(3x^2 - 1) \cdot 6x \\ &= 6x \sec^2(3x^2 - 1) \end{aligned}$$

Solution to Practice 8

Practice 8:
Find the derivative of $f(x) = (x^3 + 2x)^5$
Solution:

$$\begin{aligned} f'(x) &= 5(x^3 + 2x)^4 \cdot \frac{d}{dx}(x^3 + 2x) \\ &= 5(x^3 + 2x)^4 \cdot (3x^2 + 2) \\ &= 5(x^3 + 2x)^4(3x^2 + 2) \end{aligned}$$

Solution to Practice 9

Practice 9:
Find the derivative of $f(x) = \sin^2(x^2 + 1)$
Solution:

$$\begin{aligned} f'(x) &= 2\sin(x^2+1) \cdot \frac{d}{dx}(\sin(x^2+1)) \\ &= 2\sin(x^2+1) \cdot \cos(x^2+1) \cdot \frac{d}{dx}(x^2+1) \\ &= 2\sin(x^2+1) \cdot \cos(x^2+1) \cdot 2x \\ &= 4x\sin(x^2+1)\cos(x^2+1) \end{aligned}$$

Solution to Practice 10

Practice 10:
Find the derivative of $f(x) = e^{\sin(x)}$
Solution:

$$\begin{aligned} f'(x) &= e^{\sin(x)} \cdot \frac{d}{dx}(\sin(x)) \\ &= e^{\sin(x)} \cdot \cos(x) \\ &= e^{\sin(x)} \cos(x) \end{aligned}$$

Solution to Practice 11

Practice 11:
Find the derivative of $f(x) = \ln(\cos(x))$
Solution:

$$\begin{aligned}
f'(x) &= \frac{1}{\cos(x)} \cdot \frac{d}{dx}(\cos(x)) \\
&= \frac{1}{\cos(x)} \cdot (-\sin(x)) \\
&= -\frac{\sin(x)}{\cos(x)} \\
&= -\tan(x)
\end{aligned}$$

Solution to Practice 12

Practice 12:
Find the derivative of $f(x) = \arctan(x^3 + 2x)$
Solution:

$$\begin{aligned} f'(x) &= \frac{1}{1 + (x^3 + 2x)^2} \cdot \frac{d}{dx}(x^3 + 2x) \\ &= \frac{1}{1 + (x^3 + 2x)^2} \cdot (3x^2 + 2) \\ &= \frac{3x^2 + 2}{1 + (x^3 + 2x)^2} \end{aligned}$$

Solution to Practice 13

Practice 13:
Find the derivative of $f(x) = \cos(e^x + x^2)$
Solution:

$$\begin{aligned} f'(x) &= -\sin(e^x + x^2) \cdot \frac{d}{dx}(e^x + x^2) \\ &= -\sin(e^x + x^2) \cdot (e^x + 2x) \\ &= -(e^x + 2x)\sin(e^x + x^2) \end{aligned}$$

Solution to Practice 14

Practice 14:
Find the derivative of $f(x) = \sqrt{\ln(x^2+1)}$
Solution:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{d}{dx}(\ln(x^2+1)) \\ &= \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) \\ &= \frac{1}{2\sqrt{\ln(x^2+1)}} \cdot \frac{1}{x^2+1} \cdot 2x \\ &= \frac{x}{(x^2+1)\sqrt{\ln(x^2+1)}} \end{aligned}$$

Solution to Practice 15

Practice 15:
Find the derivative of $f(x) = \sin(\ln(x))$
Solution:

$$\begin{aligned} f'(x) &= \cos(\ln(x)) \cdot \frac{d}{dx}(\ln(x)) \\ &= \cos(\ln(x)) \cdot \frac{1}{x} \\ &= \frac{\cos(\ln(x))}{x} \end{aligned}$$

Solution to Practice 16

Practice 16:
Find the derivative of $f(x) = e^{\arctan(x)}$
Solution:

$$\begin{aligned} f'(x) &= e^{\arctan(x)} \cdot \frac{d}{dx}(\arctan(x)) \\ &= e^{\arctan(x)} \cdot \frac{1}{1+x^2} \\ &= \frac{e^{\arctan(x)}}{1+x^2} \end{aligned}$$

Solution to Practice 17

Practice 17:
Find the derivative of $f(x) = \sin^3(\cos(x^2))$
Solution:

$$\begin{aligned}
f'(x) &= 3\sin^2(\cos(x^2)) \cdot \frac{d}{dx}(\sin(\cos(x^2))) \\
&= 3\sin^2(\cos(x^2)) \cdot \cos(\cos(x^2)) \cdot \frac{d}{dx}(\cos(x^2)) \\
&= 3\sin^2(\cos(x^2)) \cdot \cos(\cos(x^2)) \cdot (-\sin(x^2)) \cdot \frac{d}{dx}(x^2) \\
&= 3\sin^2(\cos(x^2)) \cdot \cos(\cos(x^2)) \cdot (-\sin(x^2)) \cdot 2x \\
&= -6x\sin^2(\cos(x^2))\cos(\cos(x^2))\sin(x^2)
\end{aligned}$$

Practice 18:

Find the derivative of $f(x) = \ln(\sqrt{e^x + \sin(x)})$

Solution:

$$\begin{aligned}
f'(x) &= \frac{1}{\sqrt{e^x + \sin(x)}} \cdot \frac{d}{dx}(\sqrt{e^x + \sin(x)}) \\
&= \frac{1}{\sqrt{e^x + \sin(x)}} \cdot \frac{1}{2\sqrt{e^x + \sin(x)}} \cdot \frac{d}{dx}(e^x + \sin(x)) \\
&= \frac{1}{2(e^x + \sin(x))} \cdot (e^x + \cos(x)) \\
&= \frac{e^x + \cos(x)}{2(e^x + \sin(x))}
\end{aligned}$$

Solution to Practice 19

Practice 19:

Find the derivative of $f(x) = e^{\sin(\ln(x))}$

Solution:

$$\begin{aligned} f'(x) &= e^{\sin(\ln(x))} \cdot \frac{d}{dx}(\sin(\ln(x))) \\ &= e^{\sin(\ln(x))} \cdot \cos(\ln(x)) \cdot \frac{d}{dx}(\ln(x)) \\ &= e^{\sin(\ln(x))} \cdot \cos(\ln(x)) \cdot \frac{1}{x} \\ &= \frac{e^{\sin(\ln(x))} \cos(\ln(x))}{x} \end{aligned}$$

Solution to Practice 20

Practice 20:
Find the derivative of $f(x) = \arctan(\sqrt{x^2 + \cos(x)})$
Solution:

$$
\begin{aligned}
f'(x) &= \frac{1}{1 + (\sqrt{x^2 + \cos(x)})^2} \cdot \frac{d}{dx}(\sqrt{x^2 + \cos(x)}) \\
&= \frac{1}{1 + x^2 + \cos(x)} \cdot \frac{1}{2\sqrt{x^2 + \cos(x)}} \cdot \frac{d}{dx}(x^2 + \cos(x)) \\
&= \frac{1}{1 + x^2 + \cos(x)} \cdot \frac{1}{2\sqrt{x^2 + \cos(x)}} \cdot (2x - \sin(x)) \\
&= \frac{2x - \sin(x)}{2(1 + x^2 + \cos(x))\sqrt{x^2 + \cos(x)}}
\end{aligned}
$$

Solution to Practice 21

Practice 21:
Find the derivative of $f(x) = \cos(\ln(\sin(x)))$
Solution:

$$\begin{aligned}
f'(x) &= -\sin(\ln(\sin(x))) \cdot \frac{d}{dx}(\ln(\sin(x))) \\
&= -\sin(\ln(\sin(x))) \cdot \frac{1}{\sin(x)} \cdot \frac{d}{dx}(\sin(x)) \\
&= -\sin(\ln(\sin(x))) \cdot \frac{1}{\sin(x)} \cdot \cos(x) \\
&= -\frac{\cos(x)\sin(\ln(\sin(x)))}{\sin(x)}
\end{aligned}$$

Solution to Practice 22

Practice 22:
Find the derivative of $f(x) = \sqrt{e^{\sin(x)} + \cos(x)}$
Solution:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{e^{\sin(x)} + \cos(x)}} \cdot \frac{d}{dx}(e^{\sin(x)} + \cos(x)) \\ &= \frac{1}{2\sqrt{e^{\sin(x)} + \cos(x)}} \cdot (e^{\sin(x)}\cos(x) - \sin(x)) \\ &= \frac{e^{\sin(x)}\cos(x) - \sin(x)}{2\sqrt{e^{\sin(x)} + \cos(x)}} \end{aligned}$$

Solution to Practice 23

Practice 23:
Find the derivative of $f(x) = \ln(\arctan(e^x))$
Solution:

$$\begin{aligned} f'(x) &= \frac{1}{\arctan(e^x)} \cdot \frac{d}{dx}(\arctan(e^x)) \\ &= \frac{1}{\arctan(e^x)} \cdot \frac{1}{1+(e^x)^2} \cdot \frac{d}{dx}(e^x) \\ &= \frac{1}{\arctan(e^x)} \cdot \frac{1}{1+e^{2x}} \cdot e^x \\ &= \frac{e^x}{\arctan(e^x)(1+e^{2x})} \end{aligned}$$

Solution to Practice 24

Practice 24:
Find the derivative of $f(x) = \sin(\cos(\tan(x)))$
Solution:

$$\begin{aligned} f'(x) &= \cos(\cos(\tan(x))) \cdot \frac{d}{dx}(\cos(\tan(x))) \\ &= \cos(\cos(\tan(x))) \cdot (-\sin(\tan(x))) \cdot \frac{d}{dx}(\tan(x)) \\ &= \cos(\cos(\tan(x))) \cdot (-\sin(\tan(x))) \cdot \sec^2(x) \\ &= -\sec^2(x)\cos(\cos(\tan(x)))\sin(\tan(x)) \end{aligned}$$

Solution to Practice 25

Practice 25:
Find the derivative of $f(x) = e^{\sin(\cos(\ln(x)))}$
Solution:

$$\begin{aligned}
f'(x) &= e^{\sin(\cos(\ln(x)))} \cdot \frac{d}{dx}(\sin(\cos(\ln(x)))) \\
&= e^{\sin(\cos(\ln(x)))} \cdot \cos(\cos(\ln(x))) \cdot \frac{d}{dx}(\cos(\ln(x))) \\
&= e^{\sin(\cos(\ln(x)))} \cdot \cos(\cos(\ln(x))) \cdot (-\sin(\ln(x))) \cdot \frac{d}{dx}(\ln(x)) \\
&= e^{\sin(\cos(\ln(x)))} \cdot \cos(\cos(\ln(x))) \cdot (-\sin(\ln(x))) \cdot \frac{1}{x} \\
&= -\frac{e^{\sin(\cos(\ln(x)))} \cos(\cos(\ln(x))) \sin(\ln(x))}{x}
\end{aligned}$$

Key Points - Chain Rule

- **Basic Form:** $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$
- **Alternative Form:** $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
- **Key Idea:** Differentiate outside, multiply by derivative of inside

Common Applications

- **Powers:** $\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$
- **Linear arguments:** $\frac{d}{dx}f(ax+b) = af'(ax+b)$
- **Trig functions:** $\frac{d}{dx}\sin(g(x)) = \cos(g(x)) \cdot g'(x)$
- **Exponential:** $\frac{d}{dx}e^{g(x)} = e^{g(x)} \cdot g'(x)$

The chain rule is essential for finding derivatives of complex functions and appears frequently in applications across mathematics, physics, and engineering.

Thank You!

Questions?

The Chain Rule is your final tool for differentiation!