1.4 Calculating Limits with Limit Laws

Limit Laws and Their Applications

Differential Calculus

Outline

Building Blocks of Functions

- Constants: c
- Monomials: x^n
- Trigonometric functions: $\sin(x)$, $\cos(x)$, $\tan(x)$
- (Soon: exponentials, inverses, etc.)

Functions are constructed from these using:

- Addition, subtraction, multiplication, division
- Substitution (composition)

Limits of Building Blocks

- We want to compute limits of these basic pieces
- Then use arithmetic to compute limits of more complicated functions
- This avoids plugging in numbers or ϵ-δ arguments

Limits of Polynomials

Key Fact

For any polynomial $P(x)$ and any real number a:

$$\lim_{x \to a} P(x) = P(a)$$

- To evaluate the limit, just plug in the number
- We will build up to this result step by step

Easiest Limits

Theorem 1.4.1

Let $a, c \in \mathbb{R}$. Then:

$$\lim_{x \to a} c = c$$

$$\lim_{x \to a} x = a$$

Understanding Theorem 1.4.1

- a, c are real numbers
- $\lim_{x \to a} c = c$: The limit of a constant function is just that constant
- $\lim_{x \to a} x = a$: The limit of $f(x) = x$ as x approaches a is a

Arithmetic of Limits

Theorem 1.4.2

Let $a, c \in \mathbb{R}$, and $f(x), g(x)$ defined near a with $\lim_{x \to a} f(x) = F$, $\lim_{x \to a} g(x) = G$ (both real). Then:

- $\lim_{x \to a}(f(x) + g(x)) = F + G$
- $\lim_{x \to a}(f(x) - g(x)) = F - G$
- $\lim_{x \to a} cf(x) = cF$
- $\lim_{x \to a}(f(x)g(x)) = FG$
- If $G \neq 0$, $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$

Applying Arithmetic of Limits

Example 1.4.3

Suppose $\lim_{x\to 1} f(x) = 3$ and $\lim_{x\to 1} g(x) = 2$.

$$\lim_{x\to 1} 3f(x) = 3 \times 3 = 9$$

$$\lim_{x\to 1} 3f(x) - g(x) = 3 \times 3 - 2 = 7$$

$$\lim_{x\to 1} f(x)g(x) = 3 \times 2 = 6$$

$$\lim_{x\to 1} \frac{f(x)}{f(x) - g(x)} = \frac{3}{3-2} = 3$$

Step-by-Step Example

Example 1.4.4

Find $\lim_{x\to 3} 4x^2 - 1$

$$
\begin{aligned}
\lim_{x\to 3} 4x^2 - 1 &= \lim_{x\to 3} 4x^2 - \lim_{x\to 3} 1 \\
&= 4 \lim_{x\to 3} x^2 - 1 \\
&= 4(\lim_{x\to 3} x)^2 - 1 \\
&= 4 \times 3 \times 3 - 1 \\
&= 36 - 1 = 35
\end{aligned}
$$

Another Example

Example 1.4.5

Compute $\lim_{x\to 2} \frac{x}{x-1}$

$$\lim_{x\to 2} x = 2$$

$$\lim_{x\to 2} x - 1 = 2 - 1 = 1$$

$$\lim_{x\to 2} \frac{x}{x-1} = \frac{2}{1} = 2$$

When the Denominator is Zero

Example 1.4.6

Be careful: If $\lim_{x\to a} g(x) = 0$, the limit law for quotients does not apply!

- If $\lim_{x\to a} f(x) \neq 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \text{DNE}$
- If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, more analysis is needed:
  - $\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} = \text{DNE}$
  - $\lim_{x\to 0} \frac{x^2}{x} = 0$
  - $\lim_{x\to 0} \frac{x^2}{x^4} = \lim_{x\to 0} \frac{1}{x^2} = +\infty$
  - $\lim_{x\to 0} \frac{x}{x} = 1$

Rational Function Example

Example 1.4.7

Let $h(x) = \frac{2x-3}{x^2+5x-6}$. Find $\lim_{x\to 2} h(x)$.

$$\lim_{x\to 2} 2x - 3 = 2 \times 2 - 3 = 1$$

$$\lim_{x\to 2} x^2 + 5x - 6 = 2^2 + 5 \times 2 - 6 = 4 + 10 - 6 = 8$$

$$\lim_{x\to 2} h(x) = \frac{1}{8}$$

When the Limit Does Not Exist

Example 1.4.7 (continued)

Find $\lim_{x\to 1} \frac{2x-3}{x^2+5x-6}$

$$\lim_{x\to 1} 2x-3 = 2\times 1-3 = -1$$

$$\lim_{x\to 1} x^2+5x-6 = 1^2+5\times 1-6 = 1$$

Since denominator $\to 0$ and numerator $\to -1 \neq 0$, the limit does not exist.

Powers and Roots

Theorem 1.4.8

Let n be a positive integer, $a \in \mathbb{R}$, and $\lim_{x\to a} f(x) = F$ (real). Then:

- $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n = F^n$
- If n even and $F > 0$, or n odd, then $\lim_{x\to a}(f(x))^{1/n} = (\lim_{x\to a} f(x))^{1/n} = F^{1/n}$
- More generally, if $F > 0$ and p is real, $\lim_{x\to a}(f(x))^p = (\lim_{x\to a} f(x))^p = F^p$

Example: Roots and Powers

Example 1.4.9

$\lim_{x\to 2}(4x^2-3)^{1/3} = (4\times 2^2-3)^{1/3} = (16-3)^{1/3} = 13^{1/3}$

Limits of Polynomials and Rational Functions

Theorem 1.4.10

Let $a \in \mathbb{R}$, $P(x)$ a polynomial, $R(x)$ a rational function. Then:

- $\lim_{x \to a} P(x) = P(a)$
- If $R(x)$ is defined at $x = a$, $\lim_{x \to a} R(x) = R(a)$

Quick Examples

$$\lim_{x\to 2}\frac{2x-3}{x^2+5x-6}=\frac{4-3}{4+10-6}=\frac{1}{8}$$

$$\lim_{x\to 2}4x^2-1=16-1=15$$

$$\lim_{x\to 2}\frac{x}{x-1}=\frac{2}{2-1}=2$$

When Denominator is Zero: Factor and Cancel

Example 1.4.11

Compute $\lim_{x \to 1} \frac{x^3 - x^2}{x - 1}$

- Both numerator and denominator $\to 0$ as $x \to 1$
- Factor: $x^3 - x^2 = x^2(x - 1)$
- $\frac{x^3 - x^2}{x - 1} = x^2$ for $x \neq 1$
- So $\lim_{x \to 1} \frac{x^3 - x^2}{x - 1} = \lim_{x \to 1} x^2 = 1$

General Principle

Theorem 1.4.12

If $f(x) = g(x)$ except at $x = a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ (if the latter exists).

Example: Factor and Cancel

Example 1.4.13

Compute $\lim_{h\to 0} \frac{(1+h)^2-1}{h}$

$$(1+h)^2 - 1 = 1 + 2h + h^2 - 1 = 2h + h^2$$
$$\frac{2h+h^2}{h} = 2 + h$$

So $\lim_{h\to 0} \frac{(1+h)^2-1}{h} = 2$

Example: Factor and Cancel (Short Version)

Example 1.4.14

$\lim_{h\to 0} \frac{(1+h)^2-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0} 2 + h = 2$

Example: Factor and Cancel (Terse)

Example 1.4.15

$\lim_{h\rightarrow 0} \frac{(1+h)^2-1}{h} = \lim_{h\rightarrow 0} 2+h = 2$

Radical Example

Example 1.4.16

Compute $\lim_{x\to 0} \frac{x}{\sqrt{1+x}-1}$

- Both numerator and denominator $\to 0$ as $x \to 0$
- Multiply numerator and denominator by $\sqrt{1+x}+1$
- $\frac{x}{\sqrt{1+x}-1} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \frac{x(\sqrt{1+x}+1)}{x}$
- $= \sqrt{1+x}+1$ for $x \neq 0$
- So $\lim_{x\to 0} \frac{x}{\sqrt{1+x}-1} = 2$

Squeeze Theorem

Theorem 1.4.17 (Squeeze/Sandwich/Pinch Theorem)

If $f(x) \leq g(x) \leq h(x)$ for all x near a (except possibly at a), and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Squeeze Theorem Example

Example 1.4.18

Compute $\lim_{x\to 0} x^2 \sin(\pi/x)$

- $-1 \leq \sin(\pi/x) \leq 1$ for all $x \neq 0$
- $-x^2 \leq x^2 \sin(\pi/x) \leq x^2$
- $\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0$
- By the squeeze theorem, $\lim_{x\to 0} x^2 \sin(\pi/x) = 0$

Squeeze Theorem Example 2

Example 1.4.19

Let $1 \leq f(x) \leq x^2 - 2x + 2$. Find $\lim_{x\to 1} f(x)$

- $\lim_{x\to 1} 1 = 1$
- $\lim_{x\to 1} x^2 - 2x + 2 = 1 - 2 + 2 = 1$
- By the squeeze theorem, $\lim_{x\to 1} f(x) = 1$

Why the Squeeze Theorem Works (Intuition)

If $f(x) \leq g(x) \leq h(x)$ and $f(x), h(x) \to L$ as $x \to a$, then $g(x)$ is trapped between two functions that both get arbitrarily close to L.
For any $\epsilon > 0$, we can make $f(x)$ and $h(x)$ within ϵ of L by taking x close enough to a. Then $g(x)$ is also within ϵ of L.
This is the essence of the squeeze theorem.

Practice: 1 and 2

Practice 1:

$$\lim_{x \to 3} \left(\frac{4x - 2}{x + 2} \right)^4$$

Practice 2:

$$\lim_{t \to -3} \frac{1 - t}{\cos(t)}$$

Practice: 3 and 4

Practice 3:

$$\lim_{h\to 0} \frac{(2+h)^2-4}{2h}$$

Practice 4:

$$\lim_{t\to -2} \frac{t-5}{t+4}$$

Practice: 5 and 6

Practice 5:

$$\lim_{x \to 1} 5x^3 + 4$$

Practice 6:

$$\lim_{t \to -1} \frac{t-2}{t+3}$$

Practice: 7 and 8

Practice 7:

$$\lim_{x\to 1} \frac{\log(1+x) - x}{x^2}$$

Practice 8:

$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

Practice: 9 and 10

Practice 9:

$$\lim_{x\to 4} \frac{x^2 - 4x}{x^2 - 16}$$

Practice 10:

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2}$$

Practice: 11 and 12

Practice 11:

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$$

Practice 12:

$$\lim_{t \to 2} \frac{1}{2t^4 - 3t^3 + t}$$

Practice: 13 and 14

Practice 13:

$$\lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$$

Practice 14:

$$\lim_{x \to 2} \frac{\sqrt{x + 7} - \sqrt{11 - x}}{2x - 4}$$

Practice: 15 and 16

Practice 15:

$$\lim_{x\to 1} \frac{\sqrt{x+2}-\sqrt{4-x}}{x-1}$$

Practice 16:

$$\lim_{x\to 3} \frac{\sqrt{x-2}-\sqrt{4-x}}{x-3}$$

Solution to Practice 1

Practice 1:

$$\lim_{x \to 3} \left(\frac{4x-2}{x+2} \right)^4$$

Solution:

$$\lim_{x \to 3} \frac{4x-2}{x+2} = \frac{4 \times 3 - 2}{3+2} = \frac{12-2}{5} = 2$$

$$\text{So,} \quad \lim_{x \to 3} \left(\frac{4x-2}{x+2} \right)^4 = 2^4 = 16$$

Solution to Practice 2

Practice 2:

$$\lim_{t \to -3} \frac{1-t}{\cos(t)}$$

Solution:

$$\lim_{t \to -3} (1-t) = 1-(-3) = 4$$

$$\lim_{t \to -3} \cos(t) = \cos(-3) = \cos(3) \approx -0.9900$$

$$\text{So,} \quad \lim_{t \to -3} \frac{1-t}{\cos(t)} \approx \frac{4}{-0.9900} \approx -4.04$$

Solution to Practice 3

Practice 3:

$$\lim_{h \to 0} \frac{(2+h)^2 - 4}{2h}$$

Solution:

$$\begin{aligned}
(2+h)^2 - 4 &= 4 + 4h + h^2 - 4 = 4h + h^2 \\
\frac{4h + h^2}{2h} &= \frac{h(4+h)}{2h} = \frac{4+h}{2} \quad (h \neq 0) \\
\lim_{h \to 0} \frac{4+h}{2} &= 2
\end{aligned}$$

Solution to Practice 4

Practice 4:

$$\lim_{t \to -2} \frac{t-5}{t+4}$$

Solution:

$$\lim_{t \to -2} (t-5) = -2-5 = -7$$

$$\lim_{t \to -2} (t+4) = -2+4 = 2$$

$$\text{So,} \quad \lim_{t \to -2} \frac{t-5}{t+4} = \frac{-7}{2}$$

Solution to Practice 5

Practice 5:

$$\lim_{x \to 1} 5x^3 + 4$$

Solution:

$$\lim_{x \to 1} 5x^3 + 4 = 5 \times 1^3 + 4 = 5 + 4 = 9$$

Solution to Practice 6

Practice 6:

$$\lim_{t \to -1} \frac{t-2}{t+3}$$

Solution:

$$\lim_{t \to -1} (t-2) = -1-2 = -3$$

$$\lim_{t \to -1} (t+3) = -1+3 = 2$$

So, $$\lim_{t \to -1} \frac{t-2}{t+3} = \frac{-3}{2}$$

Solution to Practice 7

Practice 7:

$$\lim_{x \to 1} \frac{\log(1+x) - x}{x^2}$$

Solution:

Direct substitution: $\dfrac{\log(2) - 1}{1^2} = \log(2) - 1 \approx -0.3069$

Solution to Practice 8

Practice 8:

$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

Solution:

$$x^2 - 4 = (x-2)(x+2)$$

$$\frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2} \quad (x \neq 2)$$

$$\lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$$

Solution to Practice 9

Practice 9:

$$\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 16}$$

Solution:

$$x^2 - 4x = x(x-4)$$

$$x^2 - 16 = (x-4)(x+4)$$

$$\frac{x^2 - 4x}{x^2 - 16} = \frac{x(x-4)}{(x-4)(x+4)} = \frac{x}{x+4} \quad (x \neq 4)$$

$$\lim_{x \to 4} \frac{x}{x+4} = \frac{4}{8} = \frac{1}{2}$$

Solution to Practice 10

Practice 10:

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$$

Solution:

$$x^2 + x - 6 = (x-3)(x+2)$$

$$\frac{x^2 + x - 6}{x - 2} = \frac{(x-3)(x+2)}{x-2} \text{ (cannot cancel, so sub}$$

$$\lim_{x \to 2} \frac{2^2 + 2 - 6}{2 - 2} = \frac{4 + 2 - 6}{0} = \frac{0}{0} \text{ (indeterminate)}$$

Try factoring numerator: $x^2 + x - 6 = (x-2)(x+3)$

$$\frac{x^2 + x - 6}{x - 2} = \frac{(x-2)(x+3)}{x-2} = x + 3 \quad (x \neq 2)$$

$$\lim_{x \to 2} x + 3 = 5$$

Solution to Practice 11

Practice 11:

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$$

Solution:

$$x^2 - 9 = (x - 3)(x + 3)$$

$$\frac{x^2 - 9}{x + 3} = \frac{(x - 3)(x + 3)}{x + 3} = x - 3 \quad (x \neq -3)$$

$$\lim_{x \to -3} x - 3 = -3 - 3 = -6$$

Solution to Practice 12

Practice 12:

$$\lim_{t\to 2} \frac{1}{2t^4 - 3t^3 + t}$$

Solution:

$$2t^4 - 3t^3 + t\Big|_{t=2} = 2 \times 16 - 3 \times 8 + 2 = 32 - 24 + 2 = 10$$

$$\lim_{t\to 2} \frac{1}{2t^4 - 3t^3 + t} = \frac{1}{10}$$

Solution to Practice 13

Practice 13:

$$\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1}$$

Solution:

$$\text{Substitute:} \quad x=-1 \implies \sqrt{(-1)^2+8}-3=\sqrt{9}-3=3-3=0, \; x+1=0$$

Indeterminate, so rationalize:

$$\frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3} = \frac{x^2+8-9}{(x+1)(\sqrt{x^2+8}+3)} = \frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)}$$

$$= \frac{(x-1)(x+1)}{(x+1)(\sqrt{x^2+8}+3)} = \frac{x-1}{\sqrt{x^2+8}+3} \quad (x \neq -1)$$

$$\lim_{x \to -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$$

Solution to Practice 14

Practice 14:

$$\lim_{x\to 2} \frac{\sqrt{x+7}-\sqrt{11-x}}{2x-4}$$

Solution:

$$\text{Substitute:} \quad x=2 \implies \sqrt{2+7}-\sqrt{11-2}=3-3=0,\ 2x-4=0$$

Indeterminate, so rationalize:

$$\frac{\sqrt{x+7}-\sqrt{11-x}}{2x-4}\cdot\frac{\sqrt{x+7}+\sqrt{11-x}}{\sqrt{x+7}+\sqrt{11-x}}=\frac{(x+7)-(11-x)}{(2x-4)(\sqrt{x+7}+\sqrt{11-x})}$$

$$=\frac{2x-4}{(2x-4)(\sqrt{x+7}+\sqrt{11-x})}=\frac{1}{\sqrt{x+7}+\sqrt{11-x}} \quad (x\neq 2)$$

$$\lim_{x\to 2}\frac{1}{\sqrt{2+7}+\sqrt{11-2}}=\frac{1}{3+3}=\frac{1}{6}$$

Solution to Practice 15

Practice 15:

$$\lim_{x \to 1} \frac{\sqrt{x+2} - \sqrt{4-x}}{x-1}$$

Solution:

$$\text{Substitute:} \quad x = 1 \implies \sqrt{1+2} - \sqrt{4-1} = \sqrt{3} - \sqrt{3} = 0, \;\; x - 1 = 0$$

Indeterminate, so rationalize:

$$\frac{\sqrt{x+2} - \sqrt{4-x}}{x-1} \cdot \frac{\sqrt{x+2} + \sqrt{4-x}}{\sqrt{x+2} + \sqrt{4-x}} = \frac{(x+2) - (4-x)}{(x-1)(\sqrt{x+2} + \sqrt{4-x})}$$

$$= \frac{2x-2}{(x-1)(\sqrt{x+2} + \sqrt{4-x})} = \frac{2(x-1)}{(x-1)(\sqrt{x+2} + \sqrt{4-x})} = \frac{2}{\sqrt{x+2} + \sqrt{4-x}} \quad (x \neq 1)$$

$$\lim_{x \to 1} \frac{2}{\sqrt{1+2} + \sqrt{4-1}} = \frac{2}{\sqrt{3} + \sqrt{3}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Solution to Practice 16

Practice 16:

$$\lim_{x\to 3} \frac{\sqrt{x-2}-\sqrt{4-x}}{x-3}$$

Solution:

$$\text{Substitute:} \quad x=3 \implies \sqrt{3-2}-\sqrt{4-3}=1-1=0, \;\; x-3=0$$

Indeterminate, so rationalize:

$$\frac{\sqrt{x-2}-\sqrt{4-x}}{x-3}\cdot\frac{\sqrt{x-2}+\sqrt{4-x}}{\sqrt{x-2}+\sqrt{4-x}}=\frac{(x-2)-(4-x)}{(x-3)(\sqrt{x-2}+\sqrt{4-x})}$$

$$=\frac{2x-6}{(x-3)(\sqrt{x-2}+\sqrt{4-x})}=\frac{2(x-3)}{(x-3)(\sqrt{x-2}+\sqrt{4-x})}=\frac{2}{\sqrt{x-2}+\sqrt{4-x}} \quad (x\neq 3)$$

$$\lim_{x\to 3}\frac{2}{\sqrt{3-2}+\sqrt{4-3}}=\frac{2}{1+1}=1$$