Linear Approximation & Taylor Polynomials

Applications of Derivatives: Approximating Functions

Differential Calculus

Outline

Why Linear Approximation?

- Sometimes we need to approximate function values that are difficult to compute exactly
- Linear approximation uses the tangent line to estimate function values near a known point
- It's based on the idea that near a point, a function looks approximately linear
- Applications include:
    - Estimating square roots, exponentials, logarithms
    - Error analysis in measurements
    - Quick mental calculations
    - Understanding function behavior locally

Zeroth Approximation

- The simplest approximation: use a constant function
- $F(x) = f(a)$ for all x
- This is just the function value at the point a
- Very crude approximation - only good at the point itself
- Example: $f(x) = e^x$ at $a = 0$ gives $F(x) = 1$

Linear Approximation Concept

- Improve on zeroth approximation by using a linear function
- Allow $F(x) = A + Bx$ for some constants A and B
- Requirements:
  - $F(a) = f(a)$ (same value at $x = a$)
  - $F'(a) = f'(a)$ (same slope at $x = a$)
- This means $F(x)$ is the tangent line to $f(x)$ at $x = a$

Deriving the Linear Approximation

Step-by-step derivation:

- Let $F(x) = A + Bx$
- Then $F(a) = A + Ba = f(a)$
- And $F'(x) = B$, so $F'(a) = B = f'(a)$
- Therefore $B = f'(a)$
- Substitute: $A + a \cdot f'(a) = f(a)$
- So $A = f(a) - a \cdot f'(a)$

Linear Approximation Formula

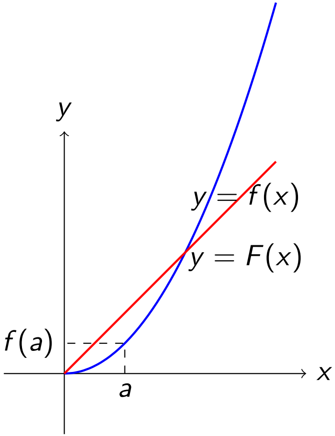
The Linear Approximation:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Key Points:

- This is exactly the equation of the tangent line at $x = a$
- Good approximation for x close to a
- Requires knowing $f(a)$ and $f'(a)$
- The approximation improves as x gets closer to a

Geometric Interpretation



Visual:

- Blue curve: $y = f(x)$
- Red line: $y = F(x)$ (tangent line)

Example 1: Estimating $e^{0.1}$

Problem: Use linear approximation to estimate $e^{0.1}$

Think about:

- What function should you use?
- What point a should you choose?
- What are $f(a)$ and $f'(a)$?
- How close is 0.1 to your chosen point?

Example 1: Estimating $e^{0.1}$ - Solution

Solution:

- Let $f(x) = e^x$ and $a = 0$
- $f(0) = e^0 = 1$
- $f'(x) = e^x$, so $f'(0) = e^0 = 1$
- Linear approximation: $f(x) \approx f(0) + f'(0)(x - 0) = 1 + x$
- For $x = 0.1$: $e^{0.1} \approx 1 + 0.1 = 1.1$
- Actual value: $e^{0.1} = 1.105170918...$
- Our approximation is accurate to about 3 decimal places!

Example 2: Estimating $\sqrt{4.1}$

Problem: Use linear approximation to estimate $\sqrt{4.1}$
Think about:

- What function should you use?
- What point a should you choose? (Consider what's easy to compute)
- What are $f(a)$ and $f'(a)$?
- How close is 4.1 to your chosen point?

Example 2: Estimating $\sqrt{4.1}$ - Solution

Solution:

- Let $f(x) = \sqrt{x}$ and $a = 4$
- $f(4) = \sqrt{4} = 2$
- $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$
- Linear approximation: $f(x) \approx f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4)$
- For $x = 4.1$: $\sqrt{4.1} \approx 2 + \frac{1}{4}(0.1) = 2 + 0.025 = 2.025$
- Actual value: $\sqrt{4.1} = 2.024845673...$
- Our approximation is very accurate!

Beyond Linear: Quadratic Approximation

- Linear approximation uses a straight line (degree 1 polynomial)
- We can improve by using a quadratic function (degree 2 polynomial)
- Let $F(x) = A + Bx + Cx^2$
- Requirements:
  - $F(a) = f(a)$ (same value)
  - $F'(a) = f'(a)$ (same first derivative)
  - $F''(a) = f''(a)$ (same second derivative)

Quadratic Approximation Formula

The Quadratic Approximation:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Key Points:

- This is a parabola that matches the function's value, slope, and curvature at $x = a$
- Better approximation than linear for x further from a
- Requires knowing $f(a)$, $f'(a)$, and $f''(a)$
- The $\frac{1}{2}$ factor comes from the second derivative

Example: Quadratic Approximation of e^x

Problem: Find the quadratic approximation of $f(x) = e^x$ at $a = 0$
Solution:

- $f(0) = e^0 = 1$
- $f'(x) = e^x$, so $f'(0) = 1$
- $f''(x) = e^x$, so $f''(0) = 1$
- Quadratic approximation: $f(x) \approx 1 + x + \frac{x^2}{2}$
- For $x = 0.1$: $e^{0.1} \approx 1 + 0.1 + \frac{0.01}{2} = 1.105$
- This is even more accurate than the linear approximation!

Taylor Polynomials - General Form

- We can continue this process to get higher degree approximations
- The nth degree Taylor polynomial about $x = a$:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

- This polynomial matches the function's value and first n derivatives at $x = a$
- Higher degree = better approximation (for smooth functions)

Deriving Taylor Polynomials

Step-by-step derivation:

- Let $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$
- We want $T_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, 2, \ldots, n$
- Evaluating at $x = a$: $T_n(a) = c_0 = f(a)$
- First derivative: $T_n'(a) = c_1 = f'(a)$
- Second derivative: $T_n''(a) = 2c_2 = f''(a) \implies c_2 = \frac{f''(a)}{2}$
- In general: $c_k = \frac{f^{(k)}(a)}{k!}$

Taylor Polynomial Formula

The nth Order Taylor Polynomial:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Key Points:

- Each term involves a higher derivative divided by factorial
- The $(x-a)^k$ terms ensure good approximation near $x = a$
- Special case $a = 0$: called Maclaurin polynomial
- Can extend $T_n(x)$ to $T_{n+1}(x)$ by adding one more term

Common Taylor Series

Important Taylor series about $x = 0$ (Maclaurin series):

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ (for $|x| < 1$)
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ (for $|x| < 1$)
- $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$ (binomial series)

Example 1: Taylor Polynomial for e^x

Problem: Find the 3rd order Taylor polynomial for $f(x) = e^x$ about $a = 0$
Solution:

- $f(0) = e^0 = 1$
- $f'(x) = e^x$, so $f'(0) = 1$
- $f''(x) = e^x$, so $f''(0) = 1$
- $f'''(x) = e^x$, so $f'''(0) = 1$
- $T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
- For $x = 0.1$: $e^{0.1} \approx 1 + 0.1 + 0.005 + 0.000167 = 1.105167$
- Very accurate approximation!

Example 2: Taylor Polynomial for $\sin x$

Problem: Find the 5th order Taylor polynomial for $f(x) = \sin x$ about $a = 0$
Solution:

- $f(0) = \sin(0) = 0$
- $f'(x) = \cos(x)$, so $f'(0) = 1$
- $f''(x) = -\sin(x)$, so $f''(0) = 0$
- $f'''(x) = -\cos(x)$, so $f'''(0) = -1$
- $f^{(4)}(x) = \sin(x)$, so $f^{(4)}(0) = 0$
- $f^{(5)}(x) = \cos(x)$, so $f^{(5)}(0) = 1$
- $T_5(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}$

Example 3: Taylor Polynomial for $\ln(1+x)$

Problem: Find the 4th order Taylor polynomial for $f(x) = \ln(1+x)$ about $a = 0$
Solution:

- $f(0) = \ln(1) = 0$
- $f'(x) = \frac{1}{1+x}$, so $f'(0) = 1$
- $f''(x) = -\frac{1}{(1+x)^2}$, so $f''(0) = -1$
- $f'''(x) = \frac{2}{(1+x)^3}$, so $f'''(0) = 2$
- $f^{(4)}(x) = -\frac{6}{(1+x)^4}$, so $f^{(4)}(0) = -6$
- $T_4(x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

Approximation Error

- The error in our approximation is $|f(x) - T_n(x)|$
- For linear approximation: error is approximately $\frac{f''(c)}{2}(x-a)^2$ for some c between a and x
- For quadratic approximation: error is approximately $\frac{f'''(c)}{6}(x-a)^3$
- In general, the error decreases as:
  - x gets closer to a
  - The degree of the polynomial increases
  - The function becomes smoother

Remainder Term

Taylor's Remainder Theorem: The error in the nth order Taylor approximation is:

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is some point between a and x.

Key Points:

- This gives us a bound on the error
- The error decreases as $(x-a)^{n+1}$ when x is close to a
- Higher derivatives control the error size

Practice Problem 1

Problem: Use linear approximation to estimate $\ln(1.1)$
Hint: Use $f(x) = \ln(x)$ and choose a point where $\ln(x)$ is easy to compute.

Practice Problem 2

Problem: Use linear approximation to estimate $\sin(0.1)$

Hint: Use $f(x) = \sin(x)$ and $a = 0$ (since $\sin(0) = 0$ and $\cos(0) = 1$).

Practice Problem 3

Problem: Use quadratic approximation to estimate $\cos(0.1)$
Hint: Use $f(x) = \cos(x)$ and $a = 0$, then find $f(0)$, $f'(0)$, and $f''(0)$.

Practice Problem 4

Problem: Use linear approximation to estimate $\sqrt{9.1}$
Hint: Use $f(x) = \sqrt{x}$ and choose $a = 9$ (since $\sqrt{9} = 3$).

Practice Problem 5

Problem: Find the quadratic approximation of $f(x) = \frac{1}{1-x}$ at $a = 0$
Hint: Find $f(0)$, $f'(0)$, and $f''(0)$, then use the quadratic formula.

Practice Problem 6

Problem: Find the 3rd order Taylor polynomial for $f(x) = \cos x$ about $a = 0$
Hint: Calculate $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

Practice Problem 7

Problem: Use the 2nd order Taylor polynomial to estimate $e^{0.2}$
Hint: Use $f(x) = e^x$ and $a = 0$, then find the quadratic approximation.

Practice Problem 8

Problem: Find the 4th order Taylor polynomial for $f(x) = \frac{1}{1+x}$ about $a = 0$
Hint: Calculate the first 4 derivatives at $x = 0$.

Practice Problem 1 - Solution

Solution: Estimate $\ln(1.1)$

- Let $f(x) = \ln(x)$ and $a = 1$
- $f(1) = \ln(1) = 0$
- $f'(x) = \frac{1}{x}$, so $f'(1) = 1$
- Linear approximation: $f(x) \approx f(1) + f'(1)(x-1) = 0 + 1(x-1) = x - 1$
- For $x = 1.1$: $\ln(1.1) \approx 1.1 - 1 = 0.1$
- Actual value: $\ln(1.1) \approx 0.0953$
- Our approximation is quite good!

Practice Problem 2 - Solution

Solution: Estimate $\sin(0.1)$

- Let $f(x) = \sin(x)$ and $a = 0$
- $f(0) = \sin(0) = 0$
- $f'(x) = \cos(x)$, so $f'(0) = \cos(0) = 1$
- Linear approximation: $f(x) \approx f(0) + f'(0)(x - 0) = 0 + 1 \cdot x = x$
- For $x = 0.1$: $\sin(0.1) \approx 0.1$
- Actual value: $\sin(0.1) \approx 0.0998$
- Very accurate approximation!

Practice Problem 3 - Solution

Solution: Estimate $\cos(0.1)$ using quadratic approximation

- Let $f(x) = \cos(x)$ and $a = 0$
- $f(0) = \cos(0) = 1$
- $f'(x) = -\sin(x)$, so $f'(0) = 0$
- $f''(x) = -\cos(x)$, so $f''(0) = -1$
- Quadratic approximation: $f(x) \approx 1 + 0 \cdot x + \frac{-1}{2}x^2 = 1 - \frac{x^2}{2}$
- For $x = 0.1$: $\cos(0.1) \approx 1 - \frac{0.01}{2} = 0.995$
- Actual value: $\cos(0.1) \approx 0.9950$
- Excellent approximation!

Practice Problem 4 - Solution

Solution: Estimate $\sqrt{9.1}$

- Let $f(x) = \sqrt{x}$ and $a = 9$
- $f(9) = \sqrt{9} = 3$
- $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$
- Linear approximation: $f(x) \approx 3 + \frac{1}{6}(x - 9)$
- For $x = 9.1$: $\sqrt{9.1} \approx 3 + \frac{1}{6}(0.1) = 3 + 0.0167 = 3.0167$
- Actual value: $\sqrt{9.1} \approx 3.0166$
- Very accurate!

Practice Problem 5 - Solution

Solution: Quadratic approximation of $f(x) = \frac{1}{1-x}$ at $a = 0$

- $f(0) = \frac{1}{1-0} = 1$
- $f'(x) = \frac{1}{(1-x)^2}$, so $f'(0) = 1$
- $f''(x) = \frac{2}{(1-x)^3}$, so $f''(0) = 2$
- Quadratic approximation: $f(x) \approx 1 + 1 \cdot x + \frac{2}{2}x^2 = 1 + x + x^2$
- This gives us the first three terms of the geometric series!

Practice Problem 6 - Solution

Solution: 3rd order Taylor polynomial for $f(x) = \cos x$ about $a = 0$

- $f(0) = \cos(0) = 1$
- $f'(x) = -\sin(x)$, so $f'(0) = 0$
- $f''(x) = -\cos(x)$, so $f''(0) = -1$
- $f'''(x) = \sin(x)$, so $f'''(0) = 0$
- $T_3(x) = 1 + 0 \cdot x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 = 1 - \frac{x^2}{2}$
- Note: The 3rd order polynomial is actually quadratic because $f'''(0) = 0$

Practice Problem 7 - Solution

Solution: Estimate $e^{0.2}$ using 2nd order Taylor polynomial

- Let $f(x) = e^x$ and $a = 0$
- $f(0) = 1,\ f'(0) = 1,\ f''(0) = 1$
- $T_2(x) = 1 + x + \frac{x^2}{2}$
- For $x = 0.2$: $e^{0.2} \approx 1 + 0.2 + \frac{0.04}{2} = 1 + 0.2 + 0.02 = 1.22$
- Actual value: $e^{0.2} \approx 1.2214$
- Very good approximation!

Practice Problem 8 - Solution

Solution: 4th order Taylor polynomial for $f(x) = \frac{1}{1+x}$ about $a = 0$

- $f(0) = 1$
- $f'(x) = -\frac{1}{(1+x)^2}$, so $f'(0) = -1$
- $f''(x) = \frac{2}{(1+x)^3}$, so $f''(0) = 2$
- $f'''(x) = -\frac{6}{(1+x)^4}$, so $f'''(0) = -6$
- $f^{(4)}(x) = \frac{24}{(1+x)^5}$, so $f^{(4)}(0) = 24$
- $T_4(x) = 1 - x + \frac{2}{2!}x^2 - \frac{6}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x + x^2 - x^3 + x^4$

Summary

- Linear approximation: $f(x) \approx f(a) + f'(a)(x-a)$
- Quadratic approximation: $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$
- Taylor polynomial: $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$
- Choose a close to the point you want to approximate
- Choose a where $f(a)$ and derivatives are easy to compute
- Higher degree polynomials give better approximations
- Taylor series provide systematic way to find polynomial approximations

Thank You!

Questions?

Taylor polynomials are powerful tools for approximating functions!