Pre-Calculus 11
Chapter 6: Absolute Values and Reciprocal Functions

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Chapter Overview

This chapter covers absolute values and reciprocal functions, including:

- Understanding and evaluating absolute values

- Graphing absolute value functions and transformations

- Solving absolute value equations and inequalities

- Understanding reciprocal functions and their properties

- Solving equations with multiple absolute values

1 6.1 Basics with Absolute Values

Key Concepts

> **Absolute Value Definition**
>
> - $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$
>
> - $|x| = \sqrt{x^2}$
>
> - Absolute value represents distance from zero on the number line
>
> - Always non-negative: $|x| \geq 0$ for all real numbers x

Properties

- $|a \cdot b| = |a| \cdot |b|$

- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ (where $b \neq 0$)

- $|a + b| \leq |a| + |b|$ (Triangle Inequality)

- $|x| = |-x|$

Examples

1. $|-15| = 15$

2. $|25 - 17| = |8| = 8$

3. $|x + 5| = 12$ has solutions $x = 7$ and $x = -17$

2 6.2 Absolute Value Functions

Key Concepts

> **Graphing Absolute Value Functions**
>
> - $y = |f(x)|$ reflects any negative parts of $f(x)$ above the x-axis
>
> - Find x-intercepts of $f(x)$ to determine reflection points
>
> - Piecewise definition: $y = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$

Common Transformations

- $y = |x|$: V-shape with vertex at $(0, 0)$

- $y = |x - h| + k$: V-shape with vertex at (h, k)

- $y = a|x - h| + k$: V-shape with vertex at (h, k), slope $\pm a$

Examples

1. $y = |x^2 - 4|$: Reflects the parabola below x-axis upward

2. $y = |x + 2|$: V-shape with vertex at $(-2, 0)$

3. $y = |x^3 - x|$: Reflects negative parts of cubic function

3 6.3 Solving Absolute Value Equations

Key Concepts

> **Solving Absolute Value Equations**
>
> 1. For $|x| = a$, set up two cases: $x = a$ and $x = -a$
>
> 2. For $|f(x)| = a$, solve $f(x) = a$ and $f(x) = -a$
>
> 3. Always check for extraneous solutions
>
> 4. $|x| = -a$ has no solution if $a > 0$

Solution Process

1. Isolate the absolute value expression
2. Set up two equations (positive and negative cases)
3. Solve each equation separately
4. Check all solutions in the original equation

Examples

1. $|x - 3| = 7$: $x - 3 = 7$ or $x - 3 = -7$; $x = 10$ or $x = -4$
2. $|2x - 5| = |x + 4|$: Consider all sign combinations; $x = 9$ or $x = \frac{1}{3}$
3. $|x - 1| = x + 1$: Only $x = 0$ is valid (check for extraneous roots)

4 6.4 Reciprocal Functions

Key Concepts

> **Reciprocal Functions**
>
> - $y = \frac{1}{f(x)}$ is the reciprocal of $f(x)$
> - Vertical asymptotes occur where $f(x) = 0$
> - Invariant points occur where $f(x) = 1$ or $f(x) = -1$
> - Horizontal asymptote is usually $y = 0$

Graphing Process

1. Graph the original function $f(x)$
2. Mark vertical asymptotes where $f(x) = 0$
3. Mark invariant points where $f(x) = \pm 1$
4. Take reciprocals of other y-values
5. Sketch the reciprocal function

Examples

1. $y = \frac{1}{x-2}$: Vertical asymptote at $x = 2$, invariant point at $(3, 1)$
2. $y = \frac{1}{x^2 - 4}$: Vertical asymptotes at $x = \pm 2$, horizontal asymptote at $y = 0$
3. $y = \frac{1}{\sqrt{x-3}}$: Domain $x > 3$, vertical asymptote at $x = 3$

5 6.5 Solving Equations with Two Absolute Values

Key Concepts

> **Multiple Absolute Values**
>
> - Consider all possible sign combinations for each absolute value
>
> - Use number line to determine valid intervals
>
> - Check all solutions for extraneous roots
>
> - For $|x - a| + |x - b| = c$, consider 3 cases based on relative positions

Solution Strategy

1. Identify critical points (where each absolute value equals zero)

2. Divide number line into intervals based on critical points

3. For each interval, determine signs of absolute value expressions

4. Set up and solve equations for each valid case

5. Check all solutions in original equation

Examples

1. $|x - 2| + |x + 6| = 11$: Solutions $x = 3.5$ and $x = -7.5$

2. $|x - 1| + |x + 5| = 8$: Solutions $x = 2$ and $x = -6$

3. $|x + 3| - |x - 2| = 4$: Solution $x = 1.5$

Practice Problems

Basic Absolute Values

1. Evaluate: $|-23| + |17| - |8|$

2. Solve: $|3x + 7| = 13$

3. Order from least to greatest: $|-5.2|, |-3.8|, |-7.1|$

Absolute Value Functions

1. Graph $y = |x^2 - 9|$

2. Write piecewise form for $y = |2x - 6|$

3. Find domain and range of $y = |x^3 - 4x|$

Solving Equations

1. Solve $|x - 4| = 2x + 1$

2. Solve $|x^2 - 5x + 6| = 2$

3. Solve $|x + 2| + |x - 3| = 7$

Reciprocal Functions

1. Find asymptotes of $y = \frac{1}{x^2 - 16}$

2. Find invariant points of $y = \frac{1}{x - 5}$

3. Graph $y = \frac{1}{x^2 + 1}$

Chapter Summary

Key Takeaways

- Master the definition and properties of absolute values

- Understand how to graph absolute value functions and transformations

- Practice solving absolute value equations with proper case analysis

- Learn to identify asymptotes and invariant points in reciprocal functions

- Develop systematic approach for equations with multiple absolute values

- Always check for extraneous solutions and state domains appropriately

Common Mistakes to Avoid

Watch Out For

- Forgetting to check for extraneous solutions

- Not considering all cases when solving absolute value equations

- Confusing absolute value functions with reciprocal functions

- Missing vertical asymptotes in reciprocal functions

- Incorrectly applying absolute value properties