MATH 152 Study Notes

Linear Systems

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Contents

1. Scalars

A scalar x has

$$x \in \mathbb{R}. \tag{1}$$

In this course,

$$x \in \mathbb{C} \text{ (complex numbers)} \tag{2}$$

is also a scalar.

2. Vectors

A vector is 2 or more scalars arranged in a predetermined order.

When written, an arrow is placed above the variable to indicate that it is a vector.

$$x \text{ is a normal variable,} \tag{3.1}$$
$$\vec{x} \text{ is a vector.} \tag{3.2}$$

In printed media, vectors are often written in boldface.

$$\boldsymbol{x} \text{ is a vector.} \tag{4}$$

2.1. Vector Dimensions

The number of scalars in a vector is called the dimension of the vector. For example,

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ is a } \mathbb{R}^2 \tag{5.1}$$
$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ is a } \mathbb{R}^4 \tag{5.2}$$

2.2. Linear Combinations

A linear combination of vectors is the sum of the vectors multiplied by scalars. Each vector is a orthogonal basis vector.

For example, $\boldsymbol{a} \in \mathbb{R}^3$ can be written as

$$\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{6}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are the orthogonal basis vectors for a 3 dimensional space.

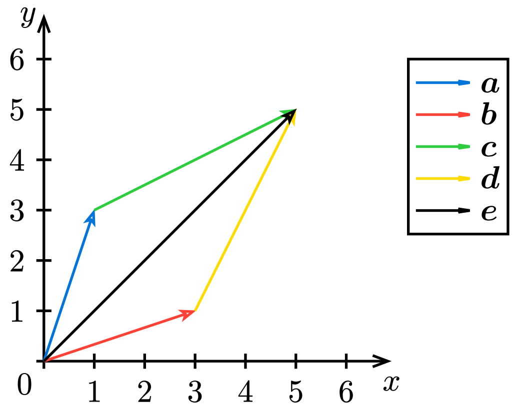
2.3. Vector Operations

2.3.1. Vector & Scalar

Yeah just do it.

2.4. Addition & Subtraction

For addition and subtraction, simply add or subtract the corresponding scalars. Commutate, associate and distribute them.



$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c} + \boldsymbol{d} = \boldsymbol{e} \tag{7.1}$$

$$\boldsymbol{e} - \boldsymbol{a} = \boldsymbol{b} \tag{7.2}$$

$$\boldsymbol{e} - \boldsymbol{c} = \boldsymbol{d} \tag{7.3}$$

By subtracting vectors, we can find the vector that connects the two vectors. The length of the connecting vector is the distance between the two vectors.

2.4.1. Dot Product

The two vectors must have the same size/dimension. The dot product of two vectors is the sum of the products of the corresponding scalars.

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{8}$$

Note that the result is a scalar.

Dot product is commutative (even with scalars) and distributive.

2.4.2. Angle Between Vectors

The angle between two vectors can be found using the dot product.

$$\cos(\theta) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\| \ \|\boldsymbol{b}\|} \tag{9}$$

where θ is the smallest angle between the two vectors.
(This can be proven using Pythagorean theorem.)

If neither of the vectors is the zero vector, and the dot product is 0, then the vectors are orthogonal (perpendicular) to each other.

Given $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, the angle between them is

$$\cos(\theta) = \frac{1 \times 1 + 1 \times -1 + 1 \times 1}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2 + -1^2 + 1^2}} = \frac{1}{3} \tag{10.1}$$

$$\theta \approx 1.23\,\mathrm{rad} = 70.53°. \tag{10.2}$$

Note: if the dot product of two vectors $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, then vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perbendicular**.

2.4.3. Vector Length

The length of a vector is the square root of the sum of the squares of the scalars in the vector, which is also the square root of the dot product of the vector with itself. It can be notated as $\|\boldsymbol{a}\|$.

$$\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{11}$$

For example, $\|\boldsymbol{x} - \boldsymbol{c}\| = 1$ means all points which are 1 unit away from $\boldsymbol{c}$, a circle.

2.4.4. Projection

The projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is the vector that is parallel to $\boldsymbol{b}$ and has the same length as the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

$$\begin{aligned} \operatorname{proj}_{\boldsymbol{b}} \boldsymbol{a} &= \text{'shadow' length of } \boldsymbol{a} \text{ on } \boldsymbol{b} \times \text{direction of } \boldsymbol{b} && (12.1) \\ &= (\boldsymbol{a} \cdot \boldsymbol{b}) \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|^2} && (12.2) \\ &= \left(\boldsymbol{a} \cdot \hat{b}\right) \hat{b} && (12.3) \end{aligned}$$

Here’s a proof:

The projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ can be represented as a scalar multipled by $\boldsymbol{b}$, where $\boldsymbol{b} \neq 0$.

$$\operatorname{proj}_{\boldsymbol{b}} \boldsymbol{a} = s\boldsymbol{b} \tag{13}$$

We know that the perpendicular vector, connecting point on $\boldsymbol{b}$ to the end of $\boldsymbol{a}$, can be denoted as $\boldsymbol{a} - s\boldsymbol{b}$.

Hence, we know that

$$(\boldsymbol{a} - s\boldsymbol{b}) \cdot \boldsymbol{b} = 0 \tag{14}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} - s(\boldsymbol{b} \cdot \boldsymbol{b}) = 0 \tag{15}$$

$$s = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}} \tag{16}$$

Plugging back into Equation 13, we hence get:

$$\operatorname{proj}_{\boldsymbol{b}} \boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{b}\|^2} \boldsymbol{b} \tag{17}$$

Further, a vector $\boldsymbol{a}$ has $\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \hat{a}$, so

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{b}\|^2} \boldsymbol{b} = \left(\boldsymbol{a} \cdot \hat{b}\right) \hat{b}. \tag{18}$$

2.4.5. Parallelogram

A parallelogram is a quadrilateral with opposite sides parallel. Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the area of the parallelogram formed by them is

$$
\begin{aligned}
A &= \text{base} \times \text{height} && (19.1)\\
&= \|\boldsymbol{a}\| \; \|\boldsymbol{b}\| \sin(\theta) && (19.2)\\
&= \|\boldsymbol{a}\| \; \|\boldsymbol{b}\| \cos\left(\frac{\pi}{2} - \theta\right) && (19.3)\\
&= \boldsymbol{a}_{\perp} \cdot \boldsymbol{b} && (19.4)
\end{aligned}
$$

where $\boldsymbol{a}_{\perp}$ is the vector perpendicular to $\boldsymbol{a}$.

2.5. Matrices

A matrix is a rectangular array of scalars. It can also be thought of as rows of vectors.

$$
\begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{20}
$$

2.5.1. Determinant of 2x2 Matrix

Given a matrix $\boldsymbol{a} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of a 2x2 matrix is

$$
\det \boldsymbol{a} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{21}
$$

2.5.2. Determinant of 3x3 Matrix

Let a matrix be defined as:

$$
\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{22}
$$

The minor matrix for an element is the 2x2 matrix obtained from deleting the row and column that element is in.
e.g.

$$
\text{minor matrix of } f = \begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix} \tag{23}
$$

One way to find the determinant is by row expansion, which is taking each of the elements in a row OR column and multiplying it by the determinant of the minor matrix. Take note of the alternating signs. (This is knkown as "Laplace Expansion" for anyone curious).

The signs for computing the sum of each element x its minor is:

$$
\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \tag{24}
$$

If you pick row 1:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{25}$$

If you pick row 2:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}. \tag{26}$$

If you pick column 2:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}. \tag{27}$$

Tip pick any row / column with the **MOST ZEROES**.

The determinant of a 3x3 matrix is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh \tag{28.1}$$

$$\text{(try find a pattern here!)} \tag{28.2}$$

This can be visualised as putting copying the first two columns onto the right, then for each 3 number diagonal in the shape of a backslash, take the product of the three numbers and sum them, and for each 3 number diagonal in the shape, subtract the each product from the previous sum.

For a parallelogram formed by two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the area is the absolute value of the determinant of the matrix formed by the vectors:

$$A_{\text{parallelogram}} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \tag{29}$$

For example, for $\boldsymbol{a} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, the area of the parallelogram is

$$A_{\text{parallelogram}} = \begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} = 11. \tag{30}$$

2.5.3. Linear Dependence

2.5.4. Cross Product

The cross product of two vectors is a vector that is orthogonal to the two vectors,
a.k.a. the normal vector of the plane formed by the two vectors,
a.k.a. the vector that is perpendicular to the two vectors.

Given $\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, the cross product is

$$
\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} && (31.1) \\
&= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}. && (31.2)
\end{aligned}
$$

In a Right Hand (RH) coordinate system, the cross product is the vector that points in the direction of the thumb when the fingers of the right hand curl from $\boldsymbol{a}$ to $\boldsymbol{b}$. In other words, the cross product is the vector that is orthogonal/perpendicular to the two vectors.

$$
\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0 \qquad (32)
$$

2.5.5. Volume of Parallelepiped

The volume of a parallelepiped formed by three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ is the absolute value of the determinant of the matrix formed by the vectors.

$$
\begin{aligned}
V_{\text{parallelepiped}} &= \text{base area} \times \text{height} && (33.1) \\
&= A_{\text{parallelogram}} \times \cos(\theta)\ \|\boldsymbol{c}\| && (33.2) \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} && (33.3) \\
&= |\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})|. && (33.4)
\end{aligned}
$$

3. Lines, Curves and Planes in Vector Form

3.1. Lines in 2D Space

There is a line L. Take a point on the line, $\boldsymbol{p} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, then take a vector that is parallel to the line, $\boldsymbol{l} = \begin{bmatrix} i \\ j \end{bmatrix}$. The line can be represented as

$$\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{l} \tag{34}$$

where $t \in \mathbb{R}$.

Alternatively,

$$\begin{cases} x = it + x_0 \\ y = jt + y_0 \end{cases} \tag{35}$$

The directional vector $\boldsymbol{l}$ is can be compared to other vectors to determine if they are parallel or perpendicular, or neither.

- If $\boldsymbol{l}_1 \cdot \boldsymbol{l}_2 = 0$, then the two vectors are perpendicular.
- Else, if $\boldsymbol{l}_1 = c\boldsymbol{l}_2$ where c is a scalar constant, then the two vectors are parallel.

Where there is a line, there is a normal vector to the line, $\boldsymbol{n}$. Thus the line can also be represented as

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{p}) = 0 \tag{36.1}$$

$$\boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}. \tag{36.2}$$

If we have a line $x - 2y + 3z = 0$, we know that the LHS is not 0, while the RHS $\boldsymbol{n} \cdot \boldsymbol{p} = 0$, $\boldsymbol{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so the line must pass through the origin. Additionally, the normal vector is $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

3.2. Planes in 3D Space

There is a plane S in $\mathbb{R}^3$. Take a point on the plane, $\boldsymbol{p} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$. If $\boldsymbol{x}$ is a point on the plane, $\boldsymbol{x} - \boldsymbol{p}$ must be perpendicular to the normal vector of the plane from origin, $\boldsymbol{n}$. Thus the plane can be represented as

$$\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{p}) = 0 \quad \text{or} \quad \boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}. \tag{37}$$

Similarly, if we have a plane $x - 2y + 3z = 0$, we know that the LHS is not 0, while the RHS $\boldsymbol{n} \cdot \boldsymbol{p} = 0, \boldsymbol{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so the plane must pass through the origin. Additionally, the normal vector is $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

The plane can also be seen as:

$$ax + by + cz = d. \tag{38}$$

where $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boldsymbol{n}$ and $d = \boldsymbol{n} \cdot \boldsymbol{p}$. We can see that this is an expansion of the $\boldsymbol{n} \cdot \boldsymbol{x} = \boldsymbol{n} \cdot \boldsymbol{p}$ form.

Alternatively, we can use the parametric form:

$$\boldsymbol{x} - \boldsymbol{p} = s\boldsymbol{u} + t\boldsymbol{v} \tag{39}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors on the plane and $s, t \in \mathbb{R}$.

For example, a plane S has parametric form

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \tag{40}$$

To find the normal representation, we can do a cross product of the two vectors on the plane to get the normal vector.

$$\begin{aligned} \boldsymbol{n} &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} && (41.1) \\ &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} && (41.2) \\ &= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}. && (41.3) \end{aligned}$$

Another example, given plane S's equation $x + 2y + 3z = 6$, we can find its parametric form by locating three points on the plane, then finding two vectors connecting the dots.

However, a better way is to restrict two of the variables to 0 and the third to 1.

$$\begin{aligned} x + 2y + 3z &= 6 && (42.1) \\ y &= s && (42.2) \\ z &= t && (42.3) \\ x &= 6 - 2y - 3z && (42.4) \\ &= 6 - 2s - 3t. && (42.5) \end{aligned}$$

$$\boldsymbol{x} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}. \tag{43}$$

3.2.1. Distance from Point to Plane

The distance from a point $P(x_0, y_0, z_0)$ to a plane $ax + by + cz = d$ is

$$d_{\text{distance}} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{44}$$

3.3. Lines in 3D Space

3D line L can be written as

$$
\boldsymbol{x} = \boldsymbol{p} + t\boldsymbol{l}, \tag{45}
$$

the same as in 2D space.

The line can also be represented as the intersection of two planes.

$$
\begin{cases} \boldsymbol{n}_1 \cdot \boldsymbol{x} = \boldsymbol{n}_1 \cdot \boldsymbol{p}_1 \\ \boldsymbol{n}_2 \cdot \boldsymbol{x} = \boldsymbol{n}_2 \cdot \boldsymbol{p}_2 \end{cases} \tag{46.1}
$$

$$
\boldsymbol{l} = \boldsymbol{n}_1 \times \boldsymbol{n}_2. \tag{46.2}
$$

Example: given a line L with

$$
\begin{cases} x + y + z = 3 \\ x - y + 2z = -7 \end{cases} \tag{47}
$$

Firstly, the planes' normals are obviously

$$
\boldsymbol{n}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{48.1}
$$

$$
\boldsymbol{n}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}. \tag{48.2}
$$

So we can have $\boldsymbol{l}$

$$
\boldsymbol{l} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}. \tag{49}
$$

Let $z = 0$,

$$
\begin{cases} x + y = 3 \\ x - y = -7 \end{cases} \tag{50.1}
$$

$$
\begin{cases} x = -2 \\ y = 5 \end{cases}. \tag{50.2}
$$

Thus, $\boldsymbol{p} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$ is on L.

$$
\boldsymbol{x} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}. \tag{51}
$$

Inversely, given a line L with

$$
\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \tag{52}
$$

we can find two planes that intersect at the line. Look for two combinations of $t = x, y, z$ that when plugged in, will remove t from the equation. That will give us two planes.

3.3.1. Projection of a Line onto A Plane

Let $\boldsymbol{a}$ be a line and S be a plane in 3D space.

In case that S is one of the coordinate planes, the projection is simple enough (taught in PHYS 170).

$$\boldsymbol{a}_{\text{proj on xy-plane}} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \tag{53}$$

and so for the other two planes.

In general, the projection in 3D space resembles the projection in 2D space.

$$\boldsymbol{a}_{\parallel} = \boldsymbol{a} - \boldsymbol{a}_{\perp} \tag{54.1}$$

$$= \boldsymbol{a}_{\text{proj}} \tag{54.2}$$

4. Systems of Linear Equations

A system of linear equations is a set of equations that can be written in the parametric form

$$\begin{cases} a_{11}x + b_{12}y + c_{13}z = d_1 \\ a_{21}x + b_{22}y + c_{23}z = d_2 \\ a_{31}x + b_{32}y + c_{33}z = d_3 \end{cases} \tag{55}$$

where $a_{ij}, b_i, (i, j \in (1, 2, 3)) \in \mathbb{R}$, which i is for row and j is for column.

Notice that each equation defines a plane in 3D space. The planes can intersect at a point, form a line, or (have 2 or more of them) be parallel.

4.1. Linear Dependence and Independence

Linear Dependence (LD) Non-zero vectors which are parallel to each other, or can be expressed as a scalar multiple of each other, or most generally, $s_1\boldsymbol{a}_1 + s_2\boldsymbol{a}_2 + \ldots = \mathbf{0}$ can be true when *not all* s's are 0.

In 2D geometry, the parallelogram formed by two LD vectors would have 0 area.

Linear Independence (LI) Non-zero vectors which satisfy none of the LD requirements.

Linear Combination A vector expressed as a sum of scalar multiples of other vectors.

Basis A set of LI vectors that can be used to express any vector in the space. In an $\mathbb{R}^n$ space, the basis has n vectors. Hence, a set of vectors are definitely LD if there are more than n vectors in the set.

If a set of vectors are LD, any subset of them *is possible to be* LI.

For example,

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix} \text{ are LD} \tag{56.1}$$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ are LI.} \tag{56.2}$$

To check if a set of vectors are LI, we can form a matrix with the vectors as *rows* and find the determinant. If the determinant is not 0, then the vectors are LI.

For example,

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0, \tag{57}$$

so $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are LI.

For 3 3D vectors, the other way is to check the volume of the parallelepiped formed by the vectors, $|\boldsymbol{a} \times (\boldsymbol{b} \cdot \boldsymbol{c})|$. If the volume is not 0, then the vectors are LI.

4.2. Solving Systems of Linear Equations

Previously, we solve systems of linear equations by substitution or elimination. (No way I'm explaining elimination here.)

Two equations and two unknowns are good, how about three for each? A hundred for each? This is when matrices come in handy.

Using elimination, we are essentially multiplying the equations by a scalar and adding them together. Recall that this is a linear combination of the equations. If the system is solvable, we must be able to express one equation as a linear combination of the others.

To solve the system, look at the following example:

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_1 + 2x_2 + 3x_3 = 9 \\ 2x_1 + 3x_2 + x_3 = 7 \end{cases} \tag{58.1}$$

$$\begin{cases} (2) = (2) - (1) \\ (3) = (3) - 2(1) \end{cases} \quad \begin{cases} x_1 + x_2 + x_3 = 4 \\ x_2 + 2x_3 = 5 \\ x_2 - x_3 = -1 \end{cases} \tag{58.2}$$

$$\dots \tag{58.3}$$

How about we stop copying the x's and write the coefficients and right-hand-side in a matrix? Put coefficients in each row into rows to form a matrix, and the right-hand-side into an augmented column.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 5 & 7 & 1 \\ 2 & -5 & 5 & 7 \end{array}\right] \xrightarrow[(3)=(3)-(1)]{(2)=(2)-2(1)} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & -6 & 2 & -8 \end{array}\right] \tag{59.1}$$

$$\xrightarrow[\text{"row echelon form"}]{(3)=(3)+2(2)} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 4 & 2 \end{array}\right] \tag{59.2}$$

$$\xrightarrow{(3)=(3)/4} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \tag{59.3}$$

$$\xrightarrow[x_3=1/2]{\text{substitute back}} \begin{cases} x_1 = -1 \\ x_2 = \frac{3}{2} \\ x_3 = \frac{1}{2} \end{cases} \tag{59.4}$$

Voila! This is very similar to the Gaussian elimination we learned probably in high school, but there are reasons why we do it this way.

Also, the matrix with the weird lines is called the augmented matrix.

Row Echelon Form (REF) To write a matrix in row echelon form, we need to make sure that:
- The all-zero rows are at the bottom.
- There are no identical LHS rows.
- There are increasing number of zeroes from upper to lower rows.

When the last row is all zeros, we cannot give a unique solution to the system. Instead, we assign a parameter, like t, to the last variable and solve for the others.

For example,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & 4 \end{array}\right] \xrightarrow[(3)=(3)-2(1)]{(2)=(2)-(1)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \tag{60.1}$$

$$\xrightarrow{(3)=(3)-(2)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \tag{60.2}$$

$$0x_1 + 0x_2 + 0x_3 = 1 \text{ (impossible)} \tag{60.3}$$

$$\text{No solution.} \tag{60.4}$$

Reduced Row Echelon Form (RREF) Now, we can answer why the function we use on calculators in PHYS 170 is called “rref”.

4.2.1. Checking for Linear Dependence by REF

To check if a set of vectors are LD, we can form a matrix with the vectors as ***columns*** and find the REF.

If there is a row of all zeros, then the vectors are LD.

4.3. Distance from Line to Line

Given two lines L_1 and L_2 in 3D space. The distance between the two lines is the length of the vector that connects the two lines and is perpendicular to both lines.

Thus, a cross product of the two directional vectors of the lines gives the direction of the connecting vector.

For example, given $L_1 : \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $L_2 : \boldsymbol{x} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, the direction of the connecting vector is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{61.1}$$

$$\ldots \tag{61.2}$$

Alternatively, we can find two arbitrary points on each line, connect them, and find the projection of the connecting vector onto the direction of the perpendicular line from one of the points.

4.4. Rank and Solution Structure

Rank The number of LI (non-zero) rows in the REF of the non-augmented part of matrix.