

# **MATH 101 Study Notes**

## **Integral Calculus with**

## **Applications**

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# 1. General Principles

## 1.1. Special Cases

In this course,

- log is ln.

## 1.2. Sigma Notation

$$\sum_{i=1}^n i \tag{1}$$

means ‘the sum of  $i$  from 1 to  $n$ , where  $i$  is an integer’.

## 2. Definite Integral

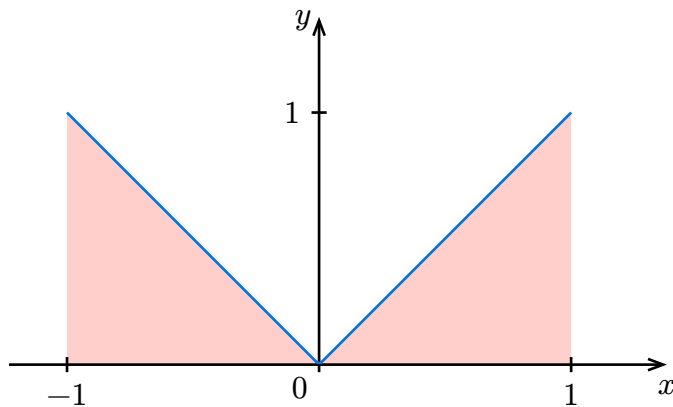
$$\int_a^b f(x) \, dx \quad (2)$$

means ‘the integral of  $f(x)$  from  $a$  to  $b$ ’.

Take  $f(x) = |x|$ , its integral from  $-1$  to  $1$  is:

$$\int_{-1}^1 |x| \, dx \quad (3)$$

looks like:



### 2.1. Estimation

Right Riemann Sum (RRS) is an estimation of the area under the curve using rectangles with the right endpoint as the height.

For example,

$$\int_0^8 \sqrt{x} \, dx. \quad (4)$$

Using RRS with 4 rectangles, we have:

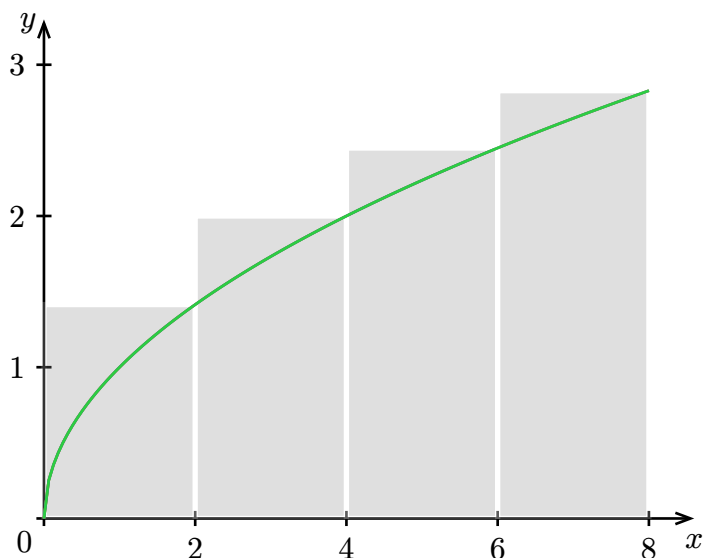
1. each rectangle has width  $\frac{8-0}{4} = 2$ ,
2. the right endpoints are 2, 4, 6, 8,
3. the heights are  $\sqrt{2}$ ,  $\sqrt{4}$ ,  $\sqrt{6}$ ,  $\sqrt{8}$ .

That gives us the estimation:

$$\sum_{i=1}^4 2\sqrt{2i} = 2\sqrt{2} + 2\sqrt{4} + 2\sqrt{6} + 2\sqrt{8} \quad (5.1)$$

$$\approx 2.83 + 4 + 5.29 + 5.66 \quad (5.2)$$

$$= 17.78. \quad (5.3)$$



Similarly, Left Riemann Sum (LRS), Midpoint Riemann Sum (MRS), and Trapezoidal Riemann Sum (TRS) exist.

The generalized formula with  $n$  rectangles/trapeziums from  $a$  to  $b$  are:

$$\text{RRS}(a, b, n) = \sum_{i=1}^n f(x_i) \Delta x \quad (6.1)$$

$$\text{LRS}(a, b, n) = \sum_{i=1}^n f(x_{i-1}) \Delta x \quad (6.2)$$

$$\text{MRS}(a, b, n) = \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x \quad (6.3)$$

$$\text{TRS}(a, b, n) = \sum_{i=1}^n (f(x_{i-1}) + f(x_i)) \Delta \frac{x}{2}, \quad (6.4)$$

where  $\Delta x = \frac{b-a}{n}$  and  $x_i = a + i\Delta x$ .

For an increasing function ( $f'(x) > 0$ ), RRS is an overestimation, and LRS is an underestimation. For a function concave up ( $f''(x) > 0$ ), TRS is an overestimation.

## 2.2. Signed Area

If the 'area under the curve' is below the  $x$ -axis, it can be called 'negative'. Hence, the integral of a function can be interpreted as the signed area of a curve, which can be positive or negative.

Say, if we have an odd function ( $\pi$  rotation symmetry about the origin), then its signed area/integral over a symmetric interval is 0.

## 2.3. Precise Calculation

Using Riemann Sums, we can see that the more rectangles we use, the closer the estimation is to the actual value.

So, let's bust up  $n$  to infinity:

$$\lim_{n \rightarrow \infty} \text{RRS}(a, b, n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \quad (7.1)$$

$$= \text{the actual signed area} \quad (7.2)$$

$$= \text{wait isn't this the definition of the integral?} \quad (7.3)$$

$$= \int_a^b f(x) \, dx. \quad (7.4)$$

### 3. Fundamental Theorem of Calculus

#### 3.1. Part 1

The Fundamental Theorem of Calculus (FTC) states that the derivative of the integral of a function is the function itself.

$$\left( \int_a^x f(t) \, dt \right)' = f(x). \quad (8)$$

#### 3.2. Integral Properties

Integral with range  $a$  to  $b$  is a linear operator:

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx \quad (9.1)$$

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx. \quad (9.2)$$

Scaling and summing are also allowed:

$$\int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx \quad (10.1)$$

$$\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx. \quad (10.2)$$

However, the integral of a product is not the product of integrals:

$$\int_a^b f(x)g(x) \, dx \neq \int_a^b f(x) \, dx \int_a^b g(x) \, dx, \quad (11)$$

nor can an integral ‘scale’ like a scalar:

$$\int_a^b f(x)g(x) \, dx \neq g(x) \int_a^b f(x) \, dx. \quad (12)$$

#### 3.3. Anti-derivative

The anti-derivative of a function  $f(x)$  is a function  $F(x)$  such that  $F'(x) = f(x)$ .

For example, the anti-derivative of  $x^n$  is  $\left(\frac{1}{n+1}\right)x^{n+1} + c$ , where  $c$  is a constant.  $c$  can be any number, since the derivative of a constant is 0.

Unfortunately, there is no systematic way to find the anti-derivative of a function, but there are some common rules to follow.

$$\int \frac{1}{x} \, dx = \ln(|x|) + c \quad (13.1)$$

$$\int e^x \, dx = e^x + c \quad (13.2)$$

$$\int \ln(x) \, dx = x \ln(x) - x + c \quad (13.3)$$

$$\int \tan(x) \, dx = -\ln(|\cos(x)|) + c. \quad (13.4)$$

Also, there are functions we can't find the anti-derivative for, like  $e^{-x^2}$ .

### 3.4. Part 2

The second part of the Fundamental Theorem of Calculus states that the integral of a function can be calculated by finding an anti-derivative of the function. For a definite integral, such operation will cancel out the constant  $c$ :

$$\int_a^b f(x) \, dx = F(b) - c - F(a) - (-c) \quad (14.1)$$

$$= F(b) - F(a). \quad (14.2)$$

### 3.5. Even and Odd Functions

**Even function**  $f(x) = f(-x)$ , symmetrical about the  $y$ -axis.

**Odd function**  $f(x) = -f(-x)$ , symmetrical about the origin.

For an even function, the integral over a symmetric interval is twice the integral over half the interval:

$$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx. \quad (15)$$

For an odd function, the integral over a symmetric interval is 0:

$$\int_{-a}^a f(x) \, dx = 0. \quad (16)$$