MATH 101 Study Notes

Integral Calculus with Applications

Yecheng Liang

Contents

1. General Principles 3
   1.1. Special Cases 3
   1.2. Sigma Notation 3
2. Definite Integral 4
   2.1. Estimation 4
   2.2. Signed Area 5
   2.3. Precise Calculation 5
3. Fundamental Theorem of Calculus 7
   3.1. Part 1 7
   3.2. Integral Properties 7
   3.3. Anti-derivative 7
   3.4. Part 2 8
   3.5. Even and Odd Functions 8

1. General Principles

1.1. Special Cases

In this course,

- $\log$ is $\ln$.

1.2. Sigma Notation

$$\sum_{i=1}^{n} i \tag{1}$$

means ‘the sum of i from 1 to n, where i is an integer’.

2. Definite Integral

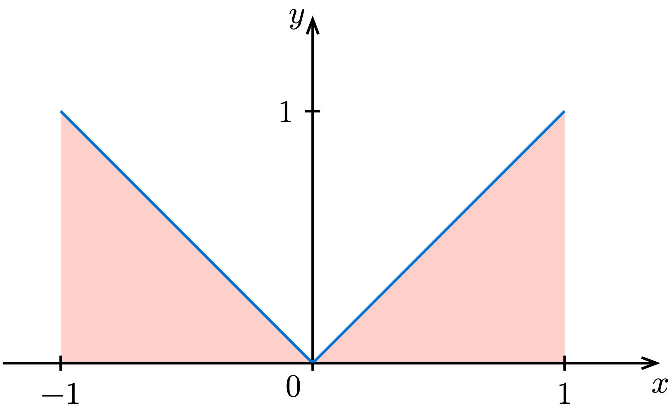
$$\int_a^b f(x) \, \mathrm{d}x \tag{2}$$

means 'the integral of $f(x)$ from a to b'.

Take $f(x) = |x|$, its integral from -1 to 1 is:

$$\int_{-1}^{1} |x| \, \mathrm{d}x \tag{3}$$

looks like:



2.1. Estimation

Right Riemann Sum (RRS) is an estimation of the area under the curve using rectangles with the right endpoint as the height.

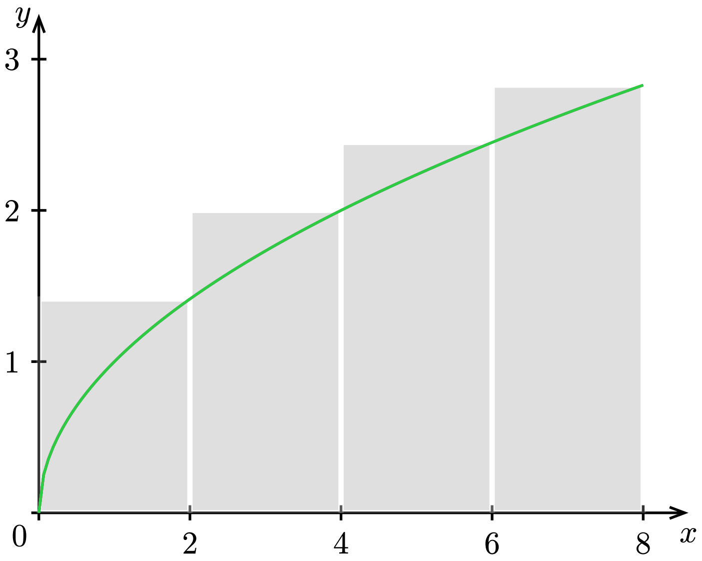
For example,

$$\int_0^8 \sqrt{x} \, \mathrm{d}x \, . \tag{4}$$

Using RRS with 4 rectangles, we have:
1. each rectangle has width $\frac{8-0}{4} = 2$,
2. the right endpoints are $2, 4, 6, 8$,
3. the heights are $\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}$.

That gives us the estimation:

$$\begin{aligned} \sum_{i=1}^{4} 2\sqrt{2i} &= 2\sqrt{2} + 2\sqrt{4} + 2\sqrt{6} + 2\sqrt{8} && (5.1) \\ &\approx 2.83 + 4 + 5.29 + 5.66 && (5.2) \\ &= 17.78. && (5.3) \end{aligned}$$



Similarly, Left Riemann Sum (LRS), Midpoint Riemann Sum (MRS), and Trapezoidal Riemann Sum (TRS) exist.

The generalized formula with n rectangles/trapeziums from a to b are:

$$\mathrm{RRS}(a, b, n) = \sum_{i=1}^{n} f(x_i)\Delta x \tag{6.1}$$

$$\mathrm{LRS}(a, b, n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x \tag{6.2}$$

$$\mathrm{MRS}(a, b, n) = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x \tag{6.3}$$

$$\mathrm{TRS}(a, b, n) = \sum_{i=1}^{n} (f(x_{i-1}) + f(x_i))\Delta \frac{x}{2}, \tag{6.4}$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

For an increasing function ($f'(x) > 0$), RRS is an overestimation, and LRS is an underestimation. For a function concave up ($f''(x) > 0$), TRS is an overestimation.

2.2. Signed Area

If the 'area under the curve' is below the x-axis, it can be called 'negative'. Hence, the integral of a function can be interpreted as the signed area of a curve, which can be positive or negative.

Say, if we have an odd function (π rotation symmetry about the origin), then its signed area/integral over a symmetric interval is 0.

2.3. Precise Calculation

Using Riemann Sums, we can see that the more rectangles we use, the closer the estimation is to the actual value.

So, let's bust up n to infinity:

$$
\begin{aligned}
\lim_{n\to\infty} \mathrm{RRS}(a,b,n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x && (7.1)\\
&= \text{the actual signed area} && (7.2)\\
&= \text{wait isn't this the definition of the integral?} && (7.3)\\
&= \int_a^b f(x)\,\mathrm{d}x\,. && (7.4)
\end{aligned}
$$

3. Fundamental Theorem of Calculus

3.1. Part 1

The Fundamental Theorem of Calculus (FTC) states that the derivative of the integral of a function is the function itself.

$$\left(\int_a^x f(t)\,\mathrm{d}t\right)' = f(x). \tag{8}$$

3.2. Integral Properties

Integral with range a to b is a linear operator:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x \tag{9.1}$$

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x\,. \tag{9.2}$$

Scaling and summing are also allowed:

$$\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x \tag{10.1}$$

$$\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x\,. \tag{10.2}$$

However, the integral of a product is not the product of integrals:

$$\int_a^b f(x)g(x)\,\mathrm{d}x \neq \int_a^b f(x)\,\mathrm{d}x \int_a^b g(x)\,\mathrm{d}x, \tag{11}$$

nor can an integral 'scale' like a scalar:

$$\int_a^b f(x)g(x)\,\mathrm{d}x \neq g(x)\int_a^b f(x)\,\mathrm{d}x\,. \tag{12}$$

3.3. Anti-derivative

The anti-derivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

For example, the anti-derivative of x^n is $\left(\frac{1}{n+1}\right)x^{n+1} + c$, where c is a constant. c can be any number, since the derivative of a constant is 0.

Unfortunately, there is no systematic way to find the anti-derivative of a function, but there are some common rules to follow.

Function	Anti-derivative
$\frac{1}{x}$	$\ln(\lvert x\rvert) + c$
e^x	$e^x + c$
$\ln(x)$	$x\ln(x) - x + c$
$\tan(x)$	$-\ln(\lvert\cos(x)\rvert) + c$
$\sin(x)$	$-\cos(x) + c$
$\cos(x)$	$\sin(x) + c$
$\sec^2(x)$	$\tan(x) + c$
$\csc^2(x)$	$-\cot(x) + c$
$\sec(x)\tan(x)$	$\sec(x) + c$
$\csc(x)\cot(x)$	$-\csc(x) + c$

Also, there are functions we can't find the anti-derivative for, like e^{-x^2}.

3.4. Part 2

The second part of the Fundamental Theorem of Calculus states that the integral of a function can be calculated by finding an anti-derivative of the function. For a definite integral, such operation will cancel out the constant c:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - c - F(a) - (-c) \tag{13.1}$$

$$= F(b) - F(a). \tag{13.2}$$

To put it in other words, once c is fixed, the integral has bounds, and is definite.

3.5. Even and Odd Functions

Even function $f(x) = f(-x)$, symmetrical about the y-axis.
Odd function $f(x) = -f(-x)$, symmetrical about the origin.

For an even function, the integral over a symmetric interval is twice the integral over half the interval:

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x\,. \tag{14}$$

For an odd function, the integral over a symmetric interval is 0:

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0. \tag{15}$$