MATH 101 Study Notes
Integral Calculus with Applications

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Contents

1. General Principles

1.1. Special Cases

In this course,

- $\log$ is $\ln$.

1.2. Sigma Notation

$$\sum_{i=1}^{n} i \tag{1}$$

means ‘the sum of i from 1 to n, where i is an integer’.

2. Definite Integral

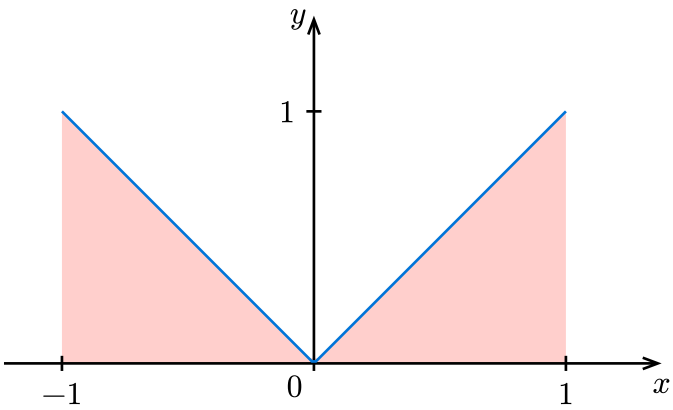
$$\int_a^b f(x)\,\mathrm{d}x \tag{2}$$

means 'the integral of $f(x)$ from a to b'.

Take $f(x) = |x|$, its integral from -1 to 1 is:

$$\int_{-1}^1 |x|\,\mathrm{d}x \tag{3}$$

looks like:



2.1. Estimation

Right Riemann Sum (RRS) is an estimation of the area under the curve using rectangles with the right endpoint as the height.

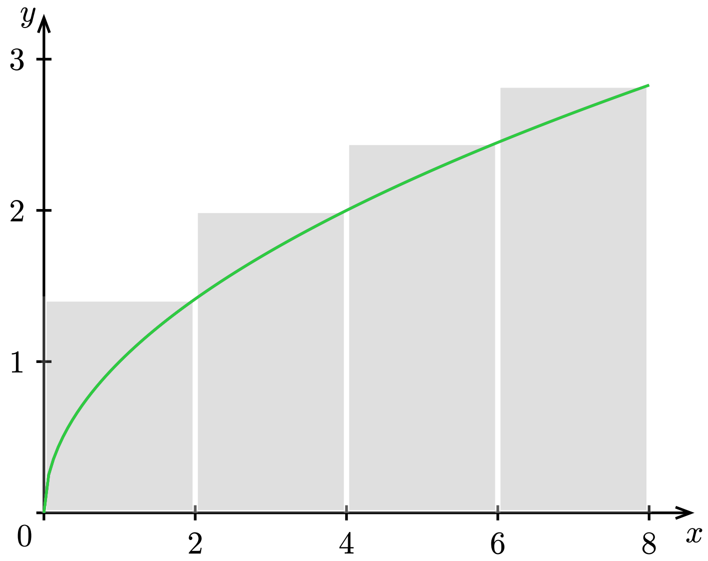
For example,

$$\int_0^8 \sqrt{x}\,\mathrm{d}x\,. \tag{4}$$

Using RRS with 4 rectangles, we have:
1. each rectangle has width $\frac{8-0}{4} = 2$,
2. the right endpoints are $2, 4, 6, 8$,
3. the heights are $\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}$.

That gives us the estimation:

$$\begin{aligned}\sum_{i=1}^{4} 2\sqrt{2i} &= 2\sqrt{2} + 2\sqrt{4} + 2\sqrt{6} + 2\sqrt{8} \\ &\approx 2.83 + 4 + 5.29 + 5.66 \\ &= 17.78.\end{aligned} \tag{5}$$



Similarly, Left Riemann Sum (LRS), Midpoint Riemann Sum (MRS), and Trapezoidal Riemann Sum (TRS) exist.

The generalized formula with n rectangles/trapeziums from a to b are:

$$\begin{aligned} \mathrm{RRS}(a, b, n) &= \sum_{i=1}^{n} f(x_i)\Delta x \\ \mathrm{LRS}(a, b, n) &= \sum_{i=1}^{n} f(x_{i-1})\Delta x \\ \mathrm{MRS}(a, b, n) &= \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x \\ \mathrm{TRS}(a, b, n) &= \sum_{i=1}^{n} (f(x_{i-1}) + f(x_i))\Delta \frac{x}{2}, \end{aligned} \tag{6}$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

For an increasing function ($f'(x) > 0$), RRS is an overestimation, and LRS is an underestimation. For a function concave up ($f''(x) > 0$), TRS is an overestimation.

2.2. Signed Area

If the 'area under the curve' is below the x-axis, it can be called 'negative'. Hence, the integral of a function can be interpreted as the signed area of a curve, which can be positive or negative.

Say, if we have an odd function (π rotation symmetry about the origin), then its signed area/integral over a symmetric interval is 0.

2.3. Precise Calculation

Using Riemann Sums, we can see that the more rectangles we use, the closer the estimation is to the actual value.

So, let's bust up n to infinity:

$$
\begin{aligned}
\lim_{n\to\infty} \mathrm{RRS}(a,b,n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \text{the actual signed area} \\
&= \text{wait isn't this the definition of the integral?} \\
&= \int_a^b f(x)\,\mathrm{d}x\,.
\end{aligned} \tag{7}
$$