PHYS 170 Study Notes
<u>Mechanics</u>

Yecheng Liang

Contents

1. General Principles

1.1. The Four Horseman of Mechanics

- Length
- Mass
- Time
- Force

So you basically take three of them and solve the 1 left.

1.2. US Customary Units

LENGTH	MASS	TIME	FORCE
meter m	kilogram kg	second s	force $\mathrm{kg\,m\,s^{-2}}$
foot ft	slug $\mathrm{lb\,s^{2}\,ft^{-1}}$	second s	pound lb

Table 1: SI and US Customary (FPS) Units for Mechanics

1.3. Gravity

$$F = G\frac{m_1 m_2}{r^2} \tag{1.1}$$

$$F = ma \tag{1.2}$$

In this course, we will use

$$g = 9.81\ \mathrm{m\,s^{-2}} \tag{2}$$

which happens to be true for Vancouver.

1.4. Vector Notation

In this course, vectors are upright bold, and vector magnitudes are italicized bold, while unit vectors are italics with an hat over.

$$\mathbf{A}\text{ has a magnitude of }\boldsymbol{A}\text{ in direction }\hat{\boldsymbol{A}}. \tag{3}$$

In manuscript, of course we cannot strike bold, so we use overhead arrow for vectors instead.

1.5. Angle Unit

In this course, angles are in degrees.

2. Force Vectors

Force, having both magnitude and direction, is a vector. Intuitively, we can apply all kinds of vector operations to forces, as you would learn in MATH 152.

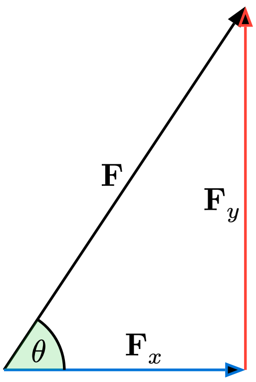
2.1. Addition

Use "tip to tail" for triangular method of addition: draw the vectors head to tail, and the resultant vector is the vector from the tail of the first vector to the head of the last vector.

2.2. Force Components

$$\mathbf{F} = \boldsymbol{x}\hat{\boldsymbol{\imath}} + \boldsymbol{y}\hat{\boldsymbol{\jmath}} \tag{4}$$

where x, y are magnitudes of the force in the $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ directions.



Force $\mathbf{F}$ can be represented as a combination of $\mathbf{F}_x$ and $\mathbf{F}_y$

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y \tag{5}$$

or as a polar coordinate of angle $\theta = \arctan\left(\frac{\boldsymbol{F}_y}{\boldsymbol{F}_x}\right)$ and magnitude $\boldsymbol{F}$

$$\mathbf{F} = \boldsymbol{F}(\cos(\theta) + \sin(\theta)). \tag{6}$$

To generalize it, we can write it as

$$\mathbf{F} = \boldsymbol{F}_x\hat{\boldsymbol{\imath}} + \boldsymbol{F}_y\hat{\boldsymbol{\jmath}} \tag{7.1}$$

$$= \boldsymbol{F}\left(\cos(\theta)\hat{\boldsymbol{\imath}} + \sin(\theta)\hat{\boldsymbol{\jmath}}\right) \tag{7.2}$$

where $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ are unit vectors in the x, y directions. This is the Cartesian form of a vector.

For a force with 2 dimensions, we call it a coplanar force.

Sometimes, non-linear equations arise from problems involving forces. Gladly use math solvers for those.

2.3. Unit Vector

To disregard magnitude and only focus on direction, we use unit vector, which we divide a vector by its magnitude, $\hat{\boldsymbol{u}} = \frac{\mathbf{A}}{\boldsymbol{A}}$.

2.4. 3D Forces

Forces in 3D are $\mathbf{F} = \boldsymbol{F}_x\hat{\boldsymbol{\imath}} + \boldsymbol{F}_y\hat{\boldsymbol{\jmath}} + \boldsymbol{F}_z\hat{\boldsymbol{k}}$, with their magnitudes being $\boldsymbol{F} = \sqrt{\boldsymbol{F}_x^2 + \boldsymbol{F}_y^2 + \boldsymbol{F}_z^2}$.

To determine orientation of the axis, we use the right-hand rule: make a thumb up using your right hand, the side of the curling fingers is x, the arm is y, and the thumb is z.

2.4.1. Direction of Cartesian Vector

The direction of a Cartesian vector is the angles between the vector and the **positive** axis. α, β, γ each corresponds to the angle from the positive x, y, z axis.

$$\cos(\alpha) = \frac{\boldsymbol{F}_x}{\boldsymbol{F}} \tag{8.1}$$

$$\cos(\beta) = \frac{\boldsymbol{F}_y}{\boldsymbol{F}} \tag{8.2}$$

$$\cos(\gamma) = \frac{\boldsymbol{F}_z}{\boldsymbol{F}} \tag{8.3}$$

Therefore,

$$\hat{\boldsymbol{u}} = \cos(\alpha)\hat{\boldsymbol{i}} + \cos(\beta)\hat{\boldsymbol{j}} + \cos(\gamma)\hat{\boldsymbol{k}} \tag{9}$$

and

$$\mathbf{F} = \boldsymbol{F}\hat{\boldsymbol{u}} \tag{10.1}$$

$$= \boldsymbol{F}\left(\cos(\alpha)\hat{\boldsymbol{i}} + \cos(\beta)\hat{\boldsymbol{j}} + \cos(\gamma)\hat{\boldsymbol{k}}\right) \tag{10.2}$$

The directions satisfy $-180° < \alpha, \beta, \gamma < 180°$ and have identity

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1. \tag{11}$$

2.4.2. Determining 3D Force Components

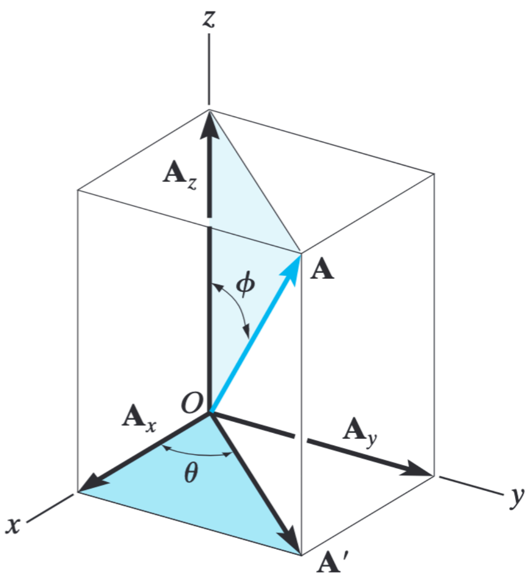


Figure 1: A Cartesian Vector

With magnitude $\boldsymbol{F}$ and angles from the positive z-axis φ and from the positive x-axis θ, we can determine the force components by first solving for $\boldsymbol{F}_z$, then $\boldsymbol{F}_{xy}$ followed by $\boldsymbol{F}_x$ and $\boldsymbol{F}_y$.

$$\boldsymbol{F}_z = \boldsymbol{F}\cos(\varphi) \tag{12.1}$$

$$\boldsymbol{F}_{xy} = \boldsymbol{F}\sin(\varphi) \tag{12.2}$$

$$\boldsymbol{F}_x = \boldsymbol{F}_{xy}\cos(\theta) \tag{12.3}$$

$$\boldsymbol{F}_y = \boldsymbol{F}_{xy}\sin(\theta) \tag{12.4}$$

Or instead, given 2 (β, γ) of the 3 Cartesian angles, we can determine the force by

$$\cos(\alpha) = \sqrt{1 - \cos^2(\beta) - \cos^2(\gamma)} \tag{13.1}$$

$$\mathbf{F} = \boldsymbol{F}\Big(\cos(\alpha)\hat{\boldsymbol{\imath}} + \cos(\beta)\hat{\boldsymbol{\jmath}} + \cos(\gamma)\hat{\boldsymbol{k}}\Big). \tag{13.2}$$

2.5. Position Vectors

Position vectors are vectors that describe the position of a point in space relative to a reference point.

As obvious, we need 3 coordinates to locate a point in 3D space. Point $P(x, y, z)$ has position vector $\mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ relative to the origin.

Note that the position vector does not always come from the origin, it can be relative to arbitrary points. Given $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$, the position vector of B relative to A is

$$\mathbf{r} = (x_B - x_A)\hat{\boldsymbol{\imath}} + (y_B - y_A)\hat{\boldsymbol{\jmath}} + (z_B - z_A)\hat{\boldsymbol{k}}. \tag{14}$$

Connecting to unit vectors, $\mathbf{u} = \frac{\mathbf{F}}{\boldsymbol{F}}$,

$$\mathbf{F} = \boldsymbol{F}\mathbf{u} = \boldsymbol{F}\frac{\mathbf{r}}{\boldsymbol{r}}. \tag{15}$$

To simplify calculation, let $X = \frac{\boldsymbol{F}}{\boldsymbol{r}}$,

$$\mathbf{F} = X\mathbf{r} \tag{16.1}$$

$$\boldsymbol{F} = X\boldsymbol{r}. \tag{16.2}$$

2.6. Vector Operations

Mostly taught in MATH 152, but here again anyways.

2.6.1. Dot Product & Angle Between Vectors

$$\mathbf{A} \cdot \mathbf{B} = \boldsymbol{AB}\cos(\theta) \tag{17.1}$$

$$= A_x B_x + A_y B_y + A_z B_z \tag{17.2}$$

2.6.2. Parallel & Perpendicular Components

Two vectors are parallel if their cross product is a zero vector, and perpendicular if their dot product is zero.

Given a vector $\mathbf{A}$, its parallel component is

$$\mathbf{A}_{\parallel} = (\mathbf{A} \cdot \hat{\boldsymbol{u}})\hat{\boldsymbol{u}} \tag{18}$$

and its perpendicular component is

$$\mathbf{A}_{\perp} = \mathbf{A} - \mathbf{A}_{\parallel}. \tag{19}$$

2.6.3. Projection

The projection of $\mathbf{A}$ onto $\mathbf{B}$ is

$$\mathbf{A}_{\text{proj on B}} = (\mathbf{A} \cdot \hat{\boldsymbol{u}}_{\boldsymbol{B}})\hat{\boldsymbol{u}}_{\boldsymbol{B}}. \tag{20}$$

Note the similarity to the parallel component formula.

3. Equilibrium of a Particle

This section only concerns static equilibrium, where the particle is at rest. In other words,

$$\sum \mathbf{F} = 0. \tag{21}$$

3.1. Free-Body Diagram

To draw a free-body diagram,

1. Draw a labeled right-hand coordinate system.
2. Draw outlined shape.
3. Show all forces.
   Since this is static equilibrium, account for *active* and *reactive* forces.
4. Identify and label each force.

3.2. Coplanar Force Systems

On a 2D plane, forces can be resolved into x, y components.

$$\sum \mathbf{F}_x = 0 \tag{22.1}$$

$$\sum \mathbf{F}_y = 0. \tag{22.2}$$

3.3. 3D Force Systems

Forces in 3D can be resolved into x, y, z components.

$$\sum \mathbf{F}_x = 0 \tag{23.1}$$

$$\sum \mathbf{F}_y = 0 \tag{23.2}$$

$$\sum \mathbf{F}_z = 0. \tag{23.3}$$

Be very careful with the signs of the forces.

3.4. Solving Equilibrium Problems

With multiple forces and their direction vectors (*not unit vectors*) $\mathbf{r}$,

$$F = \mathbf{r}X \text{ or } Y \text{ or } Z... \tag{24}$$

where $X, Y, Z... = \frac{F}{r}$. Solving for X, Y, Z gives the forces. To solve a typical X, Y, Z system, we can use the `rref` function on a matrix of the coefficients in the equations and their RHS. More in MATH 152.

Why don't we directly compute the values? Try and see for yourself, you will comeback to the X, Y, Z's.

A sanity check would be to ensure that the forces are on the same magnitude as the original forces, and are in somewhat canceling directions.

As always, check angles and signs. Forces project to the negative axis have $\alpha, \beta, \gamma > 90°$.

4. Force System Resultants

Forces can produce difference results.
Force Causes translation of a body.
Moment/Torque Causes rotation of a body.

Translation happens on the same plane as the forces, while rotation happens perpendicular to the plane.

In this course of statics, we study not the translations or rotations, but the *tendencies* for the bodies to act so.

These results combined are called the force system resultants.

4.1. Moment of a Force

Intuitively, magnitude of the moment of a force $\mathbf{F}$ about point O is

$$M_O = \boldsymbol{F}d \tag{25}$$

where d is the perpendicular distance from O to the line of action of $\mathbf{F}$.

Multiple moments? Just sum them up.

By convention (curl of fingers in the Right Hand Rule), positive moments point to the positive z axis, counterclockwise.

To understand rotation in a simple way, think of (or actually do) opening a door at different points in different directions.

Now, be reminded that the cross product of two vectors produces a vector perpendicular to the plane of the two vectors.

4.1.1. Cross Product

Samely, more in MATH 152.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{26.1}$$

$$= \left(A_y B_z - A_z B_y\right)\hat{\boldsymbol{\imath}} - \left(A_x B_z - A_z B_x\right)\hat{\boldsymbol{\jmath}} + \left(A_x B_y - A_y B_x\right)\hat{\boldsymbol{k}} \tag{26.2}$$

which is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

And it is non-commutative.

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{27.1}$$

$$-\mathbf{C} = \mathbf{B} \times \mathbf{A} \tag{27.2}$$

It is mandated that we show construction of the determinant in exams even if we use calculators.

4.2. Vector Formulation of Moment

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{28}$$

where $\mathbf{r}$ is the position vector of the point of application of $\mathbf{F}$ relative to O, in other words, *any* vector from O to the line of action of $\mathbf{F}$.

The magnitude of $\mathbf{M}_O$ is then

$$\begin{aligned}
\boldsymbol{M}_O &= \boldsymbol{r}\boldsymbol{F}\sin(\theta) && (29.1)\\
&= \boldsymbol{F}d && (29.2)\\
\mathbf{M}_{\text{axis}} &= \mathbf{u}_{a_x} \times (\mathbf{r} \times \mathbf{F}) && (29.3)\\
&= \mathbf{u}_{a_x} \times \mathbf{M}_O \text{ (this can commute).} && (29.4)
\end{aligned}$$

Using our recent knowledge of Cartesian vectors and cross products, we can write the moment as

$$\mathbf{M}_O = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \tag{30}$$