MATH 101 Study Notes
Integral Calculus with Applications

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Contents

1. General Principles

In this course,

- $\log$ is $\ln$

1.1. Sigma Notation

$$\sum_{i=1}^{n} i \tag{1}$$

means “the sum of i from 1 to n, where i is an integer”.

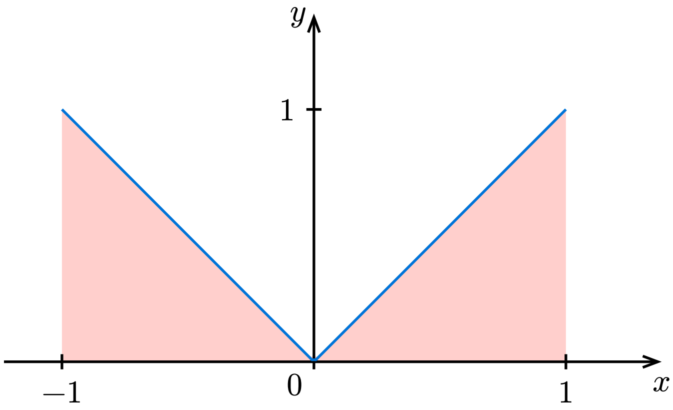
2. Definite Integral

$$\int_a^b f(x) \, \mathrm{d}x \tag{2}$$

means "the integral of $f(x)$ from a to b".

$$\int_{-1}^1 |x| \, \mathrm{d}x \tag{3}$$

and looks like:



2.1. Estimation

Right Riemann Sum (RRS) is an estimation of the area under the curve using rectangles with the right endpoint as the height.

For example,

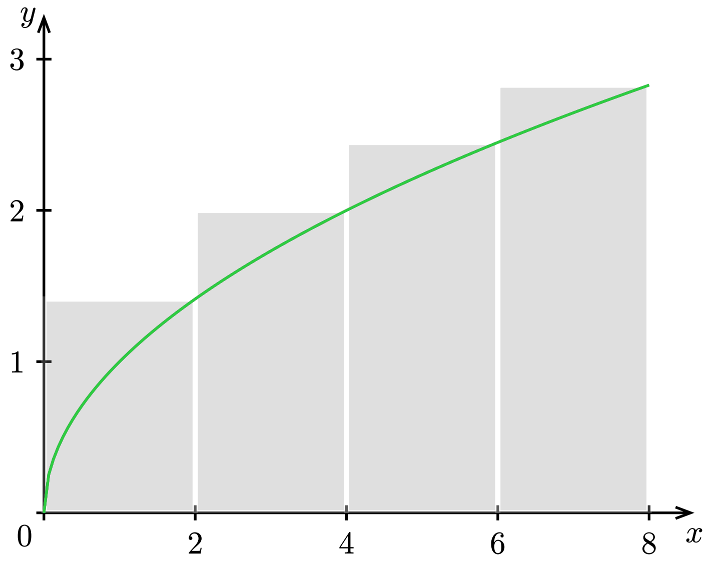
$$\int_0^8 \sqrt{x} \, \mathrm{d}x \,. \tag{4}$$

Using RRS with 4 rectangles, we have:

1. each rectangle has width $\frac{8-0}{4} = 2$,
2. the right endpoints are $2, 4, 6, 8$,
3. the heights are $\sqrt{2}$, $\sqrt{4}$, $\sqrt{6}$, $\sqrt{8}$.

That gives us the estimation:

$$\begin{aligned} \sum_{i=1}^{4} 2\sqrt{2i} &= 2\sqrt{2} + 2\sqrt{4} + 2\sqrt{6} + 2\sqrt{8} \\ &\approx 2.83 + 4 + 5.29 + 5.66 \\ &= 17.78. \end{aligned} \tag{5}$$



Similarly, Left Riemann Sum (LRS), Midpoint Riemann Sum (MRS), and Trapezoidal Riemann Sum (TRS) exist.

The generalized formula with n rectangles/trapeziums from a to b are:

$$\begin{aligned} \mathrm{RRS}(a,b,n) &= \sum_{i=1}^{n} f(x_i)\Delta x \\ \mathrm{LRS}(a,b,n) &= \sum_{i=1}^{n} f(x_{i-1})\Delta x \\ \mathrm{MRS}(a,b,n) &= \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x \\ \mathrm{TRS}(a,b,n) &= \sum_{i=1}^{n} (f(x_{i-1})+f(x_i))\Delta\frac{x}{2}, \end{aligned} \tag{6}$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

For an increasing function ($f'(x) > 0$), RRS is an overestimation, and LRS is an underestimation. For a function concave up ($f''(x) > 0$), TRS is an overestimation.

2.2. Signed Area

If the 'area under the curve' is below the x-axis, it can be called 'negative'. Hence, the integral of a function can be interpreted as the signed area of a curve, which can be positive or negative.

2.3. Precise Calculation

Using Riemann Sums, we can see that the more rectangles we use, the closer the estimation is to the actual value.