MATH 101 Study Notes
<u>Integral Calculus with Applications</u>

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Contents

1. General Principles

1.1. Special Cases

In this course,

- $\log$ is $\ln$.

1.2. Sigma Notation

$$\sum_{i=1}^{n} i \tag{1}$$

means ‘the sum of i from 1 to n, where i is an integer’.

2. Definite Integral

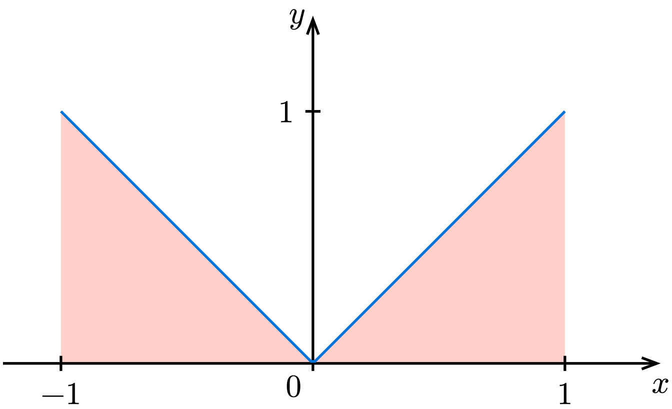
$$\int_a^b f(x)\,\mathrm{d}x \tag{2}$$

means 'the integral of $f(x)$ from a to b'.

Take $f(x) = |x|$, its integral from -1 to 1 is:

$$\int_{-1}^{1} |x|\,\mathrm{d}x \tag{3}$$

looks like:



2.1. Estimation

Right Riemann Sum (RRS) is an estimation of the area under the curve using rectangles with the right endpoint as the height.

For example,

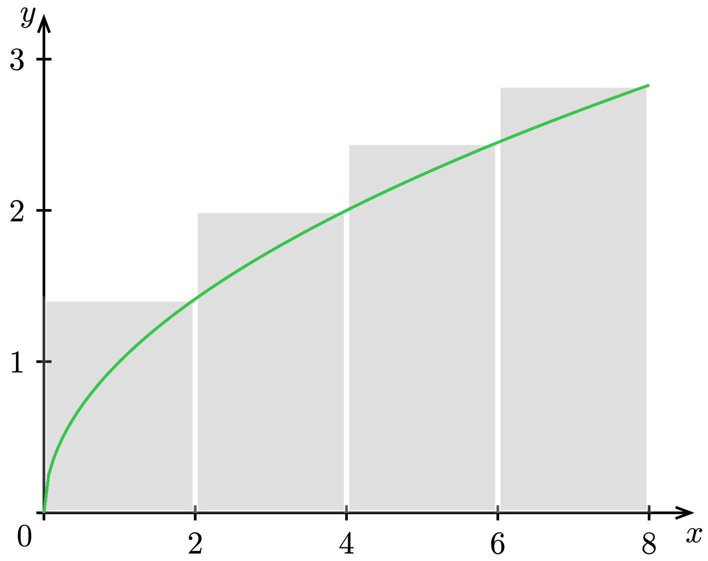
$$\int_0^8 \sqrt{x}\,\mathrm{d}x\,. \tag{4}$$

Using RRS with 4 rectangles, we have:

1. each rectangle has width $\frac{8-0}{4} = 2$,
2. the right endpoints are $2, 4, 6, 8$,
3. the heights are $\sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}$.

That gives us the estimation:

$$\begin{aligned} \sum_{i=1}^{4} 2\sqrt{2i} &= 2\sqrt{2} + 2\sqrt{4} + 2\sqrt{6} + 2\sqrt{8} && (5.1) \\ &\approx 2.83 + 4 + 5.29 + 5.66 && (5.2) \\ &= 17.78. && (5.3) \end{aligned}$$



Similarly, Left Riemann Sum (LRS), Midpoint Riemann Sum (MRS), and Trapezoidal Riemann Sum (TRS) exist.

The generalized formula with n rectangles/trapeziums from a to b are:

$$\mathrm{RRS}(a,b,n) = \sum_{i=1}^{n} f(x_i)\Delta x \tag{6.1}$$

$$\mathrm{LRS}(a,b,n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x \tag{6.2}$$

$$\mathrm{MRS}(a,b,n) = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x \tag{6.3}$$

$$\mathrm{TRS}(a,b,n) = \sum_{i=1}^{n} (f(x_{i-1})+f(x_i))\frac{\Delta x}{2}, \tag{6.4}$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

For an increasing function ($f'(x) > 0$), RRS is an overestimation, and LRS is an underestimation. For a function concave up ($f''(x) > 0$), TRS is an overestimation.

2.2. Signed Area

If the ‘area under the curve’ is below the x-axis, it can be called ‘negative’. Hence, the integral of a function can be interpreted as the signed area of a curve, which can be positive or negative.

Say, if we have an odd function (π rotation symmetry about the origin), then its signed area/integral over a symmetric interval is 0.

2.3. Precise Calculation

Using Riemann Sums, we can see that the more rectangles we use, the closer the estimation is to the actual value.

So, let’s bust up n to infinity:

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{RRS}(a,b,n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x && (7.1)\\
&= \text{the actual signed area} && (7.2)\\
&= \text{wait isn't this the definition of the integral?} && (7.3)\\
&= \int_a^b f(x)\,\mathrm{d}x\,. && (7.4)
\end{aligned}$$

3. Fundamental Theorem of Calculus

3.1. Part 1

The Fundamental Theorem of Calculus (FTC) states that the derivative of the integral of a function is the function itself.

$$\left(\int_a^x f(t)\,\mathrm{d}t\right)' = f(x). \tag{8}$$

3.2. Integral Properties

Integral with range a to b is a linear operator:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x \tag{9.1}$$

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x\,. \tag{9.2}$$

Scaling and summing are also allowed:

$$\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x \tag{10.1}$$

$$\int_a^b (f(x)+g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x\,. \tag{10.2}$$

However, the integral of a product is not the product of integrals:

$$\int_a^b f(x)g(x)\,\mathrm{d}x \neq \int_a^b f(x)\,\mathrm{d}x \int_a^b g(x)\,\mathrm{d}x, \tag{11}$$

nor can an integral ‘scale’ like a scalar:

$$\int_a^b f(x)g(x)\,\mathrm{d}x \neq g(x)\int_a^b f(x)\,\mathrm{d}x\,. \tag{12}$$

3.3. Anti-derivative

The anti-derivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

For example, the anti-derivative of x^n is $\left(\frac{1}{n+1}\right)x^{n+1} + c$, where c is a constant. c can be any number, since the derivative of a constant is 0.

Unfortunately, there is no systematic way to find the anti-derivative of a function, but there are some common rules to follow.

Function	Anti-derivative
$\frac{1}{x}$	$\ln(\lvert x\rvert) + c$
e^x	$e^x + c$
$\ln(x)$	$x\ln(x) - x + c$
$\tan(x)$	$-\ln(\lvert\cos(x)\rvert) + c$
$\sin(x)$	$-\cos(x) + c$
$\cos(x)$	$\sin(x) + c$
$\sec^2(x)$	$\tan(x) + c$
$\csc^2(x)$	$-\cot(x) + c$
$\sec(x)\tan(x)$	$\sec(x) + c$
$\csc(x)\cot(x)$	$-\csc(x) + c$

Also, there are functions we can't find the anti-derivative for, like e^{-x^2}.

3.4. Part 2

The second part of the Fundamental Theorem of Calculus states that the integral of a function can be calculated by finding an anti-derivative of the function. For a definite integral, such operation will cancel out the constant c:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - c - F(a) - (-c) \tag{13.1}$$

$$= F(b) - F(a). \tag{13.2}$$

To put it in other words, once c is fixed, the integral has bounds, and is definite.

3.5. Even and Odd Functions

Even function $f(x) = f(-x)$, symmetrical about the y-axis.
Odd function $f(x) = -f(-x)$, symmetrical about the origin.

For an even function, the integral over a symmetric interval is twice the integral over half the interval:

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x\,. \tag{14}$$

For an odd function, the integral over a symmetric interval is 0:

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0. \tag{15}$$

3.6. Integration by Substitution

In essence, integration by substitution is the chain rule in reverse.

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x) \tag{16.1}$$

$$\int f'(g(x)) \cdot g'(x) \,\mathrm{d}x = f(g(x)) + c. \tag{16.2}$$

When we have $f'(g(x))$ and $g'(x)$, we can substitute $u(x) = g(x)$ and $u'(x) = g'(x)\,\mathrm{d}x$:

$$\begin{aligned} \int f'(g(x)) \cdot g'(x) \,\mathrm{d}x &= \int f'(u(x)) u'(x) \,\mathrm{d}x && (17.1) \\ &= \int g(u) \,\mathrm{d}u \bigg|_{u=u(x)} && (17.2) \\ &= f(u) + c. && (17.3) \end{aligned}$$

u would typically be something that is easy to differentiate.

For example, consider $\int 9\sin^8(x)\cos(x)\,\mathrm{d}x$. Let $u = \sin(x)$,

$$\begin{aligned} \mathrm{d}u &= u'(x)\,\mathrm{d}x && (18.1) \\ &= \cos(x)\,\mathrm{d}x\,. && (18.2) \end{aligned}$$

$$\begin{aligned} \int 9\sin^8(x)\cos(x)\,\mathrm{d}x &= 9\int \sin^8(x)\underline{\cos(x)\,\mathrm{d}x} && (19.1) \\ &= 9\int \sin^8(x)\,\underline{\mathrm{d}u} && (19.2) \\ &= 9\int u^8\,\mathrm{d}u && (19.3) \\ &= 9 \times \left(\frac{1}{9}\right) u^9 + c && (19.4) \\ &= \sin^9(x) + c. && (19.5) \end{aligned}$$

The expression to become $\mathrm{d}u$ would not always be so obvious, so it is a good idea to first factor the integrand. Also, you will want u to be some function with arguments more than just x, like $\sqrt{x}$ or $(x^2 + 1)$.

3.6.1. Substitution with Definite Integrals

When substituting in a definite integral, the bounds must be changed accordingly:

$$\int_a^b f'(g(x)) \cdot g'(x)\,\mathrm{d}x = \int_{u(a)}^{u(b)} f'(u)\,\mathrm{d}u\,. \tag{20}$$

Alternatively, we can first find the indefinite integral, then take in the bounds.

The general steps are:
1. Determine u.
2. *Do not forget to* substitute bounds.
3. Substitute to $\mathrm{d}u$.
4. Substitute the integrand with u.

3.6.2. Special Integrals

Function	Anti-derivative	Note
$\frac{1}{1+x^2}$	$\arctan(x)+c$	For denominator a^2+x^2, use $\frac{1}{a}\arctan\left(\frac{x}{a}\right)$. (Time a^2 to inside and substitute $u=\frac{x}{a}$.)
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)+c$	For denominator $\sqrt{a-(x-b)^2}$, use $\arcsin\left(\frac{x-b}{\sqrt{a}}\right)$. (Time $\sqrt{a}$ to inside and substitute $u=\frac{x-b}{\sqrt{a}}$.)

Table 2: Integrals to Memorize

3.7. Area Between Curves

Turns out, we were integrating some others functions against function $y=0$ all this time! So to find area between curves, we can just subtract the two functions and integrate the result.

Note that when asked to find the area, *not signed area*, we should make sure to subtract the function with smaller value in bounds from the one with larger value. Usually it is done by finding intersections of the two functions and comparing the values around those points, or the derivatives at those points.

Sometimes, we will even break up the integral into multiple parts, where the functions are integrable. When it is hard to integrate in terms of x, try taking an inverse approach and integrate in terms of y.

3.8. Trigonometric Substitution

When dealing with integrals with square roots, we can use trigonometric substitution to simplify the integrand.

For example, consider $\int\sqrt{1-x^2}\,\mathrm{d}x$. If the bounds like 0 to 1, we know it is a nice part area of circle, but what if not?

Forms of

$$\sqrt{a^2-x^2}, \sqrt{a^2+x^2}, \sqrt{x^2-a^2} \tag{25}$$

can be simplified by using trigonometric substitution: Known that

$$\sin^2(\theta)+\cos^2(\theta)=1 \tag{26.1}$$

$$\tan^2(\theta)+1=\sec^2(\theta), \tag{26.2}$$

then let $x=a\sin(\theta)$ where $a>0$,

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2(\theta)} \tag{27.1}$$

$$=\sqrt{a^2\cos^2(\theta)} \tag{27.2}$$

$$=a|\cos(\theta)| \tag{27.3}$$

$$= a\cos(\theta) \text{ when } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}. \tag{27.4}$$

We can thus apply it, for example,

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \int \sqrt{1-a^2\sin^2(\theta)}\ a\cos(\theta)\,\mathrm{d}\theta \tag{28.1}$$

$$= \int a^2\cos^2(\theta)\,\mathrm{d}\theta \tag{28.2}$$

$$= a^2 \int \cos^2(\theta)\,\mathrm{d}\theta \tag{28.3}$$

$$= a^2 \int \frac{1+\cos(2\theta)}{2}\,\mathrm{d}\theta \tag{28.4}$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + c \tag{28.5}$$

$$= \frac{a^2}{2}(\theta + \sin(\theta)\cos(\theta)) + c, \tag{28.6}$$

$$\frac{x}{a} = \sin(\theta) \tag{28.7}$$

$$\theta = \arcsin\left(\frac{x}{a}\right), \tag{28.8}$$

$$\cos(\theta) = \sqrt{1-\sin^2(\theta)} \tag{28.9}$$

$$= \sqrt{1-\frac{x^2}{a^2}}, \tag{28.10}$$

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right) + x\sqrt{1-\frac{x^2}{a^2}}\right) + c. \tag{28.11}$$

Memorize trigonometric identities!

3.9. Integration by Parts

Integration by parts is the reverse of the product rule.

$$(uv)' = u'v + uv'. \tag{29}$$

When we have u and v', we can integrate by parts:

$$\int uv\,\mathrm{d}x = u\int v\,\mathrm{d}x - \int u'\left(\int v\,\mathrm{d}x\right)\mathrm{d}x \tag{30.1}$$

$$\int u'v\,\mathrm{d}x = uv - \int uv'\,\mathrm{d}x\,. \tag{30.2}$$

3.10. Simpson's rule

We already know that integrals can be estimated by Riemann sums, but all those sums can be rather inaccurate. What if, instead of using 2 points to form trapezoids or rectangles, we use 3 points to form parabolas?

Simpson's rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots \\ +2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \tag{31}$$

where n is an *even* number.

4. Improper Integrals

Improper integral An integral either with an unbounded integrand, or taken over an infinite interval.

For example, $\int_{-1}^{1} \frac{1}{x^2}$ and $\int_{0}^{\infty} e^{-x}$ are improper.

4.1. Redefining Unbounded Improper Integral

If the integrand is discontinuous at some point or in some region, we can redefine it as one or more integrals with bounds approaching the discontinuities.

For example, given that $f(x)$ has discontinuity at $c \in (a, b)$,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x = \int_c^b f(x)\,\mathrm{d}x \tag{32.1}$$

$$= \lim_{s \to a^-} \int_a^s f(x)\,\mathrm{d}x + \lim_{t \to c^+} \int_t^b f(x)\,\mathrm{d}x\,. \tag{32.2}$$

Notably, the LHS converges when both expressions in the RHS converge.

For example, given

$$I = \int_0^5 \frac{-\cos\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x, \tag{33}$$

notice that I discontinues at $x = 0$.

$$I = \lim_{t \to 0^+} \int_t^5 \frac{-\cos\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x \tag{34.1}$$

$$= \lim_{t \to 0^+} \int_{\frac{1}{t}}^{\frac{1}{5}} -\cos(u)\,\mathrm{d}u \tag{34.2}$$

$$= \lim_{t \to 0^+} \left(\sin\left(\frac{1}{5}\right) - \sin\left(\frac{1}{t}\right)\right). \tag{34.3}$$

The integral does not exist.

4.2. Redefining Improper Integral with Infinite Interval

$$\int_a^{\infty} f(x)\,\mathrm{d}x = \lim_{R \to \infty} \int_a^R f(x)\,\mathrm{d}x \tag{35.1}$$

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{T \to -\infty} \int_T^b f(x)\,\mathrm{d}x\,. \tag{35.2}$$

The LHS converges when the RHS converge.

For example,

$$\int_0^\infty e^{-x}\,\mathrm{d}x = \lim_{R\to\infty}\int_0^R e^{-x}\,\mathrm{d}x \tag{36.1}$$
$$= \lim_{R\to\infty}\left[-e^{-x}\right]_0^R \tag{36.2}$$
$$= \lim_{R\to\infty}\left(-e^{-R}+e^0\right) \tag{36.3}$$
$$= 1. \tag{36.4}$$

4.2.1. When Both Bounds Are Infinity

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x \neq \lim_{R\to\infty}\int_{-R}^R f(x)\,\mathrm{d}x\,. \tag{37}$$

To realize this, think of $\int_{-\infty}^\infty (\sin(x)-\cos(x))\,\mathrm{d}x$ geometrically.

Instead,

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{T\to-\infty}\int_T^c f(x)\,\mathrm{d}x + \lim_{R\to\infty}\int_c^R f(x)\,\mathrm{d}x \tag{38}$$

where $c \in \mathbb{R}$.

For example,

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{T\to-\infty}\int_T^0 \frac{1}{1+x^2}\,\mathrm{d}x + \lim_{R\to\infty}\int_0^R \frac{1}{1+x^2}\,\mathrm{d}x \tag{39.1}$$
$$= \lim_{T\to-\infty}\left[\arctan(x)\right]_T^0 + \lim_{R\to\infty}\left[\arctan(x)\right]_0^R \tag{39.2}$$
$$= \left(0-\left(-\frac{\pi}{2}\right)\right)+\left(\frac{\pi}{2}-0\right) \tag{39.3}$$
$$= \pi. \tag{39.4}$$

4.3. Determining Convergence

4.3.1. Value Comparison Test

Let function f, g be integrable on any interval contained in (a, b), then:
- If $|f(x)| \leq g(x)$ for $x \in (a, b)$ and $\int_a^b g(x)\,\mathrm{d}x$ converges, then $\int_a^b f(x)\,\mathrm{d}x$ converges.
- If $f(x) \geq g(x) \geq 0$ for $x \in (a, b)$ and $\int_a^b g(x)\,\mathrm{d}x$ diverges, then $\int_a^b f(x)\,\mathrm{d}x$ diverges.

4.3.2. Limit Comparison Test

Let $-\infty < a < \infty$. Let f, g be functions that are defined and continuous for all $x \geq a$ and assume that $g(x) \geq 0$ for $x \geq a$.

- If $\int_a^\infty g(x)\,\mathrm{d}x$ converges and $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ exists, then $\int_a^\infty f(x)\,\mathrm{d}x$ converges.
- If $\int_a^\infty g(x)\,\mathrm{d}x$ diverges and $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ exists and *is nonzero*, then $\int_a^\infty f(x)\,\mathrm{d}x$ diverges.

Definitions

Even function° $f(x) = f(-x)$, symmetrical about the y-axis.

Odd function° $f(x) = -f(-x)$, symmetrical about the origin.

Simpson's rule°

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots \\ +2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \tag{40}$$

where n is an *even* number.

Improper integral° An integral either with an unbounded integrand, or taken over an infinite interval.