State vector

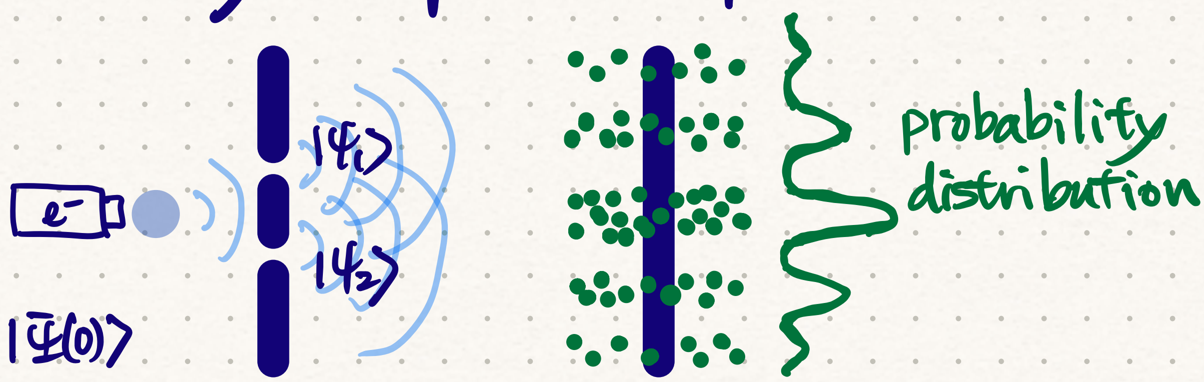
At each instant, the state of a physical system is represented by a ket $|\psi\rangle$ in the space of states.

- If $|\psi_1\rangle$ and $|\psi_2\rangle$ are possible states of a system, so is

$$|\Psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle,$$

where c_1 and c_2 are complex numbers.

- Double-slit diffraction experiment
  - 2 possible positions of an electron
  - each may correspond to a possible state



$$|\Psi(t)\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$$

- This postulate is radical because it implies that the superposition of states is again a state of the system.

- Inner product reduces to the integral overlap of the two states:

$$(|\psi\rangle, |\phi\rangle) := \langle\psi|\phi\rangle = \int dx\, \psi^*(x)\, \phi(x)$$

Observable

Every observable attribute of a physical system is described by an operator, which acts on the kets that describe the system.

- An operator $\hat{A}$ acting on a ket $|\Psi\rangle$ is denoted by

$$\hat{A} : |\Psi\rangle \mapsto |\Psi'\rangle = \hat{A}|\Psi\rangle$$

Hence, in general, acting with an operator on a state changes the state.

- For every operator, there exist some special states that are not changed by the actor of itself,

$$\hat{A}|\psi_a\rangle = a|\psi_a\rangle,$$

except for being multiplied by a constant. These special states are the eigenstates and the numbers are the eigenvalues of the operator.

$$\underset{\text{operator}}{\hat{A}} \longrightarrow \begin{cases} \{a\} = \text{eigenvalues} \\ \{|\psi_a\rangle\} = \text{eigenstates} \end{cases}$$

Measurement

The only possible measurement outcome of an observable $\hat{A}$ is one of the eigenvalues a of the corresponding operator $\hat{A}$.

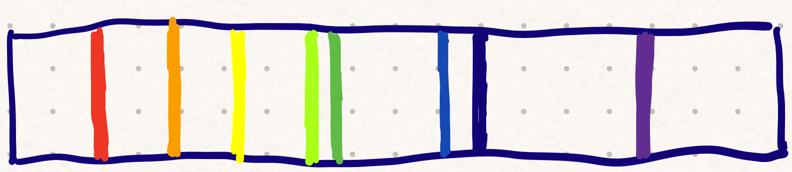
- Since we measure only real numbers, the eigenvalues of operators corresponding to observables had better be real.

- Operators with real eigenvalues are Hermitian.
  - The eigenstates of a Hermitian operator are orthogonal:

$$\langle \psi_j | \psi_k \rangle = \int dx\, \psi_j^*(x)\, \psi_k(x) = \delta_{jk} = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases}.$$

  - The eigenstates of a Hermitian operator form a basis, i.e., they span the space of states.

- Discrete spectral lines
  - only one of a discrete set of values (lines) are observed for a kind of atom.



unique for every atom!

Born rule

When a measurement of an observable $\hat{A}$ is made on a generic state $|\Psi\rangle$, the probability of obtaining an eigenvalues a is given by

$$P_a := |\langle\psi_a|\Psi\rangle|^2 .$$

- The states are assumed to be normalized.

- The complex number $\langle\psi_a|\Psi\rangle$ is known as the probability amplitude, to measure a as the value for $\hat{A}$ in the state $|\Psi\rangle$.

example.

$$|\Psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \qquad \langle\psi_1|\Psi\rangle = \langle\psi_1|\left(c_1|\psi_1\rangle + c_2|\psi_2\rangle\right)$$

$$P_1 = |\langle\psi_1|\Psi\rangle|^2 = |c_1|^2 \qquad = c_1\underbrace{\langle\psi_1|\psi_1\rangle}_{1} + c_2\underbrace{\langle\psi_1|\psi_2\rangle}_{0}$$

$$P_2 = |\langle\psi_2|\Psi\rangle|^2 = |c_2|^2 \qquad = c_1$$

- Any state can be expanded as a superposition of $\hat{A}$-eigenstates:

$$|\Psi\rangle = \sum_a c_a|\psi_a\rangle$$
$$= \sum_a \langle\psi_a|\Psi\rangle \cdot |\psi_a\rangle .$$

The weighting (or the component) of $|\Psi\rangle$ along the direction of the eigenstate $|\psi_a\rangle$ is given by

$$c_a \equiv \langle\psi_a|\Psi\rangle .$$

• Probability conservation implies

$$\sum_a |c_a|^2 = 1 .$$

• Expectation value of the observable $\hat{A}$ reads

$$\langle \hat{A} \rangle = \sum_a |c_a|^2 \cdot a$$

$$= \langle \Psi | \hat{A} | \Psi \rangle ,$$

where $|c_a|^2$ is the probability of observing a, i.e., c_a is the probability amplitude with respect to a. The symbol $|\Psi\rangle$ denotes the state vector of the system.

example.

	possible outcome	probability
eigen-values	5 eV	50%
	7 eV	25%
	13 eV	25%

$$\langle \hat{A} \rangle = 5 \times 0.5 + 7 \times 0.25 + 13 \times 0.25$$
$$= 7.5 \text{ eV}$$

$$c_5 = \frac{1}{\sqrt{2}}, \quad c_7 = \frac{1}{2}, \quad c_{13} = \frac{1}{2}$$

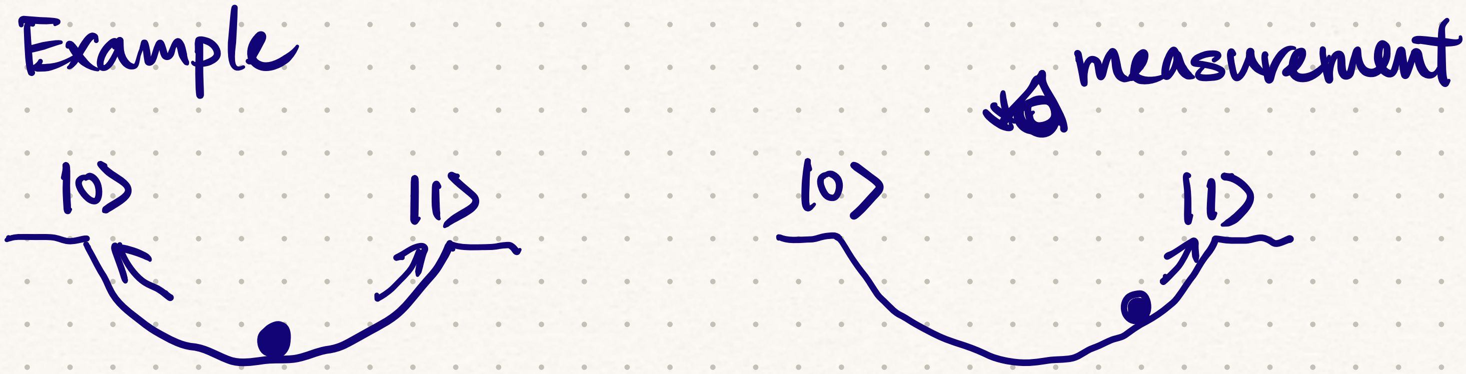
eigenstates: $\{ |\psi_5\rangle, |\psi_7\rangle, |\psi_{13}\rangle \}$

Wavefunction collapse

Immediately after the measurement of an observable $\hat{A}$ has yielded a value a, the state of the measured system is the normalized eigenstate $|\psi_a\rangle$.

- The collapse of the wavepacket preserves the normalization of the state. If $|\Psi\rangle$ and $|\psi_a\rangle$ are both normalized to unity, then the measurement process replaces $|\Psi\rangle$ by $|\psi_a\rangle$, instead of $|\langle\psi_a|\Psi\rangle|^2 \cdot |\psi_a\rangle$.

- Example



$|\Psi(0)\rangle = c_0|0\rangle + c_1|1\rangle$ $\qquad$ $|\Psi(t)\rangle = |1\rangle$

Time evolution

The time evolution of a quantum system preserves the normalization of the associated ket. The time evolution of the state of a quantum system is described by

$$|\Psi(t)\rangle = \underbrace{\hat{U}(t,t_0)}_{\text{propagator}} |\Psi(t_0)\rangle$$

for some unitary operator $\hat{U}(t,t_0)$.

- Schrödinger's equation:

$$\boxed{i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle,}$$

where $\hat{H}$ is the Hamiltonian of the system.

- What does it mean for unitary $\hat{U}(t,t_0)$?

$$\hat{U}(t,t_0) : \ |\Psi(t_0)\rangle \mapsto |\Psi(t)\rangle$$

$$\hat{U}^{-1}(t,t_0) = \hat{U}(t_0,t) : \ |\Psi(t)\rangle \mapsto |\Psi(t_0)\rangle$$

$$\Rightarrow \hat{U}(t,t_0) \cdot \hat{U}^{-1}(t,t_0) = \mathbb{1} \ !$$