There are two elements for random walk processes, coin and shift operators. This gives us a hint that the whole system is composed of coin and position subspaces:

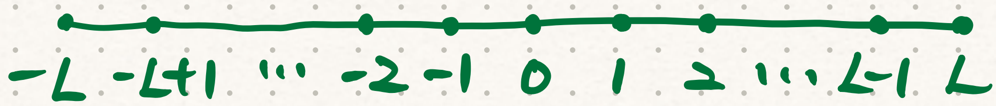
Hilbert space

$$\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p$$

dimension $2(2L+1)$ $\quad 2 \quad$ $2L+1$

two directions: forward & backward

a 1D line with half-lenght L:



totally $2L+1$ nodes

So, the system state contains two parts:

1) coin state $|\psi(t)\rangle_c$
2) position state $|\psi(t)\rangle_p$

$$|\Psi(t)\rangle = |\psi(t)\rangle_c \otimes |\psi(t)\rangle_p$$

Recall: Kronecker product

$$\underset{1\times 2}{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} \otimes \underset{3\times 3}{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}} = \underset{(1\times 3)\times(2\times 3)}{\begin{pmatrix} \alpha a & \alpha b & \alpha c \\ \alpha d & \alpha e & \alpha f \\ \alpha g & \alpha h & \alpha j \\ \beta a & \beta b & \beta c \\ \beta d & \beta e & \beta f \\ \beta g & \beta h & \beta j \end{pmatrix}}$$

For a 1D model, the common setup of two operators are

- coin : Hadamard coin

$$C = H = \frac{1}{\sqrt{2}}\left(|R\rangle\langle R| + |R\rangle\langle L| + |L\rangle\langle R| - |L\rangle\langle L|\right)$$

$$= \frac{1}{\sqrt{2}} \begin{array}{cc} & \begin{array}{cc} |R\rangle & |L\rangle \end{array} \\ & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{array}{c} \langle R| \\ \langle L| \end{array} \end{array}$$

$$A \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}|1\rangle\langle 1| + a_{12}|1\rangle\langle 2| + a_{21}|2\rangle\langle 1| + a_{22}|2\rangle\langle 2|$$

- shift

$$S := \sum_n \Big(\underbrace{|R\rangle\langle R| \otimes |n+1\rangle\langle n|}_{\text{forward}} + \underbrace{|L\rangle\langle L| \otimes |n-1\rangle\langle n|}_{\text{backward}} \Big)$$

example.

coin state : $|\psi\rangle_c = |L\rangle$

position state : $|\psi\rangle_p = |-2\rangle$

$$|\Psi\rangle = |\psi\rangle_c \otimes |\psi\rangle_p = |L\rangle \otimes |-2\rangle$$

after one step, we have

$$|\Psi'\rangle = (SC)|\Psi\rangle$$

$$= S \cdot \frac{1}{\sqrt{2}}\left(|R\rangle\langle R| + |L\rangle\langle R| + |R\rangle\langle L| - |L\rangle\langle L|\right)|L\rangle \otimes |-2\rangle$$

$(|R\rangle\langle R|)\cdot|L\rangle = |R\rangle\cdot(\langle R|L\rangle) = 0$

$$= S \cdot \frac{1}{\sqrt{2}}\left(|R\rangle - |L\rangle\right) \otimes |-2\rangle$$

$$= \frac{1}{\sqrt{2}} \sum_n \left(|L\rangle\langle L| \otimes |n-1\rangle\langle n| + |R\rangle\langle R| \otimes |n+1\rangle\langle n|\right)(|R\rangle - |L\rangle) \otimes |-2\rangle$$

$(|R\rangle\langle R|)\cdot|R\rangle = |R\rangle\cdot(\langle R|R\rangle)$

$= |R\rangle$

$$= \frac{1}{\sqrt{2}}\left(|R\rangle \otimes |-1\rangle - |L\rangle \otimes |-3\rangle\right)$$

$\because$ probability is the square of the coefficient (Born rule)

$\therefore$ 50% of being at $|-3\rangle$

50% of being at $|-1\rangle$

* For a orthonormal (orthogonal & normalized) basis, we have

$$\underbrace{\langle j|k\rangle}_{\text{inner product of } |j\rangle \text{ and } |k\rangle} = \delta_{jk} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

Now, we propagate the system for several time step, and compare the result of classical and quantum random walks.

· Quantum

initial coin state $|\psi(0)\rangle_c = |L\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

initial position state $|\psi(0)\rangle_p = |0\rangle$

$$\left.\begin{matrix} |\psi(0)\rangle_c = |L\rangle \\ |\psi(0)\rangle_p = |0\rangle \end{matrix}\right\} |\Psi(0)\rangle = |L\rangle \otimes |0\rangle$$

1^{st}-step:

$$|\Psi(1)\rangle = (SC)\,|\Psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|R\rangle \otimes |1\rangle - |L\rangle \otimes |-1\rangle\right)$$

$1/2$ $\quad$ $1/2$

2nd-step:

$$|\Psi(2)\rangle = (SC)|\Psi(1)\rangle = (SC)^2|\Psi(0)\rangle$$

$$= \frac{1}{2}\left[|R\rangle\otimes|2\rangle - (|R\rangle - |L\rangle)\otimes|0\rangle + |L\rangle\otimes|-2\rangle\right]$$

$1/4$ $\quad$ $1/2$ $\quad$ $1/4$

3rd-step: "propagator"

$$|\Psi(3)\rangle = (SC)|\Psi(2)\rangle = (SC)^3|\Psi(0)\rangle$$

$$= \frac{1}{2\sqrt{2}}\left(|R\rangle\otimes|3\rangle + |L\rangle\otimes|1\rangle + (|R\rangle - 2|L\rangle)\otimes|-1\rangle - |L\rangle\otimes|-3\rangle\right)$$

$1/8$ $\quad$ $1/8$ $\quad$ $5/8$ $\quad$ $1/8$

⋮

$$|\Psi(t)\rangle = (SC)^t|\Psi(0)\rangle$$

$$\equiv \mathcal{U}(t,0)|\Psi(0)\rangle$$

we can obtain the state of the system at any time t with initial state $|\Psi(0)\rangle$ and propagator $\mathcal{U}(t,0)$

- Classical

1st-step:

$50\%\,|1\rangle + 50\%\,|-1\rangle$

2nd-step:

$25\%\,|2\rangle + 50\%\,|0\rangle + 25\%\,|-2\rangle$

3rd-step:

$12.5\%\,|3\rangle + 37.5\%\,|1\rangle + 37.5\%\,|-1\rangle + 12.5\%\,|3\rangle$

⋮

binomial distribution!

step \ position	-3	-2	-1	0	1	2	3
0				1			
1			1/2		1/2		
2		1/4		1/2		1/4	
3	1/8		3/8 5/8		3/8 1/8		1/8

Quantum
Classical

When we start the walk with initial coin state $|\psi(0)\rangle_c = |L\rangle$, the probability distribution in position subspace is biased. A similar result will be given with the coin initially in $|\psi(0)\rangle_c = |R\rangle$ state, but in a opposite distribution. However, if we set the initial coin superposition

$$|\psi(0)\rangle_c = \frac{1}{\sqrt{2}}\left(|L\rangle \pm i|R\rangle\right),$$

we can obtain a symmetric distribution. Note that if the imaginary part is removed, it will go back to the cases with $|L\rangle$ and $|R\rangle$. This emphasize the importance of relative phase between $|L\rangle$ and $|R\rangle$.

We compare the two initial coin states with the same starting position $|\psi(0)\rangle_p = |0\rangle$.

- $|\psi(0)\rangle_c = \frac{1}{\sqrt{2}}(|L\rangle - i|R\rangle)$

$C = \frac{1}{\sqrt{2}}(|R\rangle\langle R| + |R\rangle\langle L| + |L\rangle\langle R| - |L\rangle\langle L|)$

$|\Psi(1)\rangle = (SC)|\Psi(0)\rangle$

$= S \cdot \frac{1}{2}\left[(1-i)|R\rangle - (1+i)|L\rangle\right] \otimes |0\rangle$

$= \frac{1}{2}\left[\underbrace{(1-i)|R\rangle \otimes |1\rangle}_{1/2} - \underbrace{(1+i)|L\rangle \otimes |-1\rangle}_{1/2}\right.$

$|\Psi(2)\rangle = (SC)|\Psi(1)\rangle$

$= S \cdot \frac{1}{2\sqrt{2}}\left[(1-i)|R\rangle|1\rangle + (1-i)|L\rangle|1\rangle - (1+i)|R\rangle|-1\rangle + (1+i)|L\rangle|-1\rangle\right)$

$= \frac{1}{2\sqrt{2}}\left[(1-i)|R\rangle|2\rangle + (1-i)|L\rangle|0\rangle - (1+i)|R\rangle|0\rangle + (1+i)|L\rangle|-2\rangle\right)$

$= \frac{1}{2\sqrt{2}}\left\{\underbrace{(1-i)|R\rangle \otimes |2\rangle}_{1/4} + \underbrace{\left[(1-i)|L\rangle - (1+i)|R\rangle\right] \otimes |0\rangle}_{1/2} + \underbrace{(1+i)|L\rangle \otimes |-2\rangle}_{1/4}\right\}$

- $|\psi(0)\rangle_c = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$

$|\Psi(1)\rangle = (SC)|\Psi(0)\rangle$

$= S \cdot \frac{1}{2}(-2|L\rangle \otimes |0\rangle)$

$= \underbrace{-|L\rangle \otimes |-1\rangle}_{1}$

The "i" matters!

$$|\Psi(2)\rangle = (SC)\,|\Psi(1)\rangle$$

$$= S\cdot\frac{1}{\sqrt{2}}(|R\rangle-|L\rangle)\otimes|-1\rangle$$

$$=\frac{1}{\sqrt{2}}(|R\rangle\otimes|0\rangle-|L\rangle\otimes|-2\rangle)$$

$1/2 \qquad 1/2$

From the above calculation, we can see that the imaginary coefficient separates the initial coin state $|L\rangle$ and $|R\rangle$ to the real axis and imaginary axis, and thus results in different wavefunction interference patterns.

step \ pos.	-2	-1	0	1	2
0			1		
1		1/2 1		1/2 0	
2	1/4 1/2		1/2 1/2		1/4 0

$|\psi(0)\rangle_c = \frac{1}{\sqrt{2}}(|L\rangle - i|R\rangle)$

$|\psi(0)\rangle_c = \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$