Introduction to Quantum Computing

IBM Quantum Computer Hub at NTU

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1 Quantum Gates

A quantum computer is built with quantum circuit, containing wires and quantum logic gates. The wires are used to carry information around the circuit, while the logic gates preform manipulations of the information, converting it from one form to another. Here we take a look at some basic quantum gates.

1.1 Single qubit gates

NOT gate It is also called as X gate. NOT gate converts the value of the qubit. For example, if the qubit is initially in $|0\rangle$ state, it flips to $|1\rangle$ after an X gate is applied. Therefore, another well-known name of X gate is NOT gate (since it is equivalent to NOT gate for classical computers). The operation equates to a rotation around the x-axis of the Bloch sphere by π radians. It can be represented by a Pauli-X matrix as

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

The default initial states of the qubits in IBM Quantum Experience are set to be the ground states $|0\rangle$s. In matrix representation, the $|0\rangle$ is interpreted by a column vector

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

***Exercise*:** Please find the matrix representation of the excited state $|1\rangle$ with the matrix multiplication: $X|0\rangle = |1\rangle$.

Y&Z gates Very similar as the X gate. The Y gate and the Z gate corresponds to rotations along y-axis and z-axis of the Bloch sphere by angle π, respectively. They have forms as

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2}$$

and

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

The matrices Eq.(1), Eq.(2) and Eq.(3) are the famous *Pauli matrices*.

Hadamard gate The Hadamard gate maps the basis state $|0\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|1\rangle$ to $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, which means that a measurement will have equal probabilities to become 1 or 0, i.e., creates a superposition. Its matrix representation is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4}$$

The Hadamard gate presents a rotation of π about the $x + y$ axis.
***Exercise*:** In addition to a rotation about the $x + y$ axis, The Hadamard gate can also be a combination of a rotation about y-axis of π and a rotation about z-axis with angle θ. Please find the angle θ. Note that we do the Y-rotation first.

Phase gate The next gate to mention is the phase gate, or sometimes known as the $\sqrt{Z}$ gate. It does a quarter-turn around the Bloch sphere. It is important to note that unlike every gate introduced so far, *the phase gate is not its own inverse*, but is still unitary. It has the matrix

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{5}$$

The name "$\sqrt{Z}$ gate" is due to the fact that two successively applied phase gates has the same effect as one Z gate: $SS\,|\psi\rangle = Z\,|\psi\rangle$. Anyone can verify the relation by linear algebra, or simply do a simulation via IBM Quantum Experience.

Phase shift gate This is a family of single-qubit gates that leave the basis state $|0\rangle$ unchanged and map $|1\rangle$ to $e^{i\phi}\,|1\rangle$. The probability of measuring a $|0\rangle$ or $|1\rangle$ is unchanged after applying this gate, however it modifies the phase of the quantum state. This is equivalent to tracing a horizontal circle (a line of latitude) on the Bloch sphere by ϕ radians. The phase shift gate has a matrix form of

$$R_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \tag{6}$$

where ϕ is the phase shift. You may notice that S gate is the one with $\phi = \pi/2$ and Z gate is the $\phi = \pi$ case.
***Exercise*:** For the case $\phi = \pi/4$, the gate is also called T gate. Show that the T gate is equivalent to $\sqrt[4]{Z}$ gate.

1.2 Two qubit gates

CNOT gate The two-qubit controlled NOT gate acts on two qubits, for which the first one is the *control qubit* that determine whether the action will be applied to the second one (*target qubit*) or not. If the control qubit has a value of $|1\rangle$ then the it performs an NOT operation to the target. Otherwise, it does nothing. It follows $|q_0, q_1\rangle \mapsto |q_0, q_0 \oplus q_1\rangle = |q_0, q_0 + q_1 \mod 2\rangle$. As an 4×4 unitary matrix, the CNOT gate reads

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7}$$

The only constraint for quantum logic gates is that quantum gates must be reversible. Mathematically, the statement indicates that the matrix representation of these *quantum gates should be unitary.*
***Exercise*:** Please prove this property by checking: $U^\dagger U = \mathbb{I}$, where $\mathbb{I}$ is the identity matrix, and U is the matrix representation of the quantum gate. Here we choose $U = \text{CNOT}$.

SWAP gate Sometimes we need to move information around in a quantum computer. For some qubit implementations, this could be done by physically moving them. Another option is simply to move the state between two qubits. The action is done by the SWAP gate. The SWAP gate has a matrix representation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

which is used to simulate an intrinsic swap operation. Though through this SWAP gate, we can simply exchange the information of two qubits. However, this is not an universal quantum gate, meaning that we can decompose an SWAP gate with standard basis gates (H gate, CNOT gate, S gate, T gate).
***Exercise*:** Please find a way to decompose the SWAP with the basis gates. You can first think of the case that one qubit is in state $|1\rangle$ and the other is in $|0\rangle$. How to exchange the values? Next step, you need to swap the states back to the original ones. Finally, combine the two processes by adding an ineffective gate from one onto the other.

2 Bell States

The *Bell states*, also known as *EPR states*or *EPR pairs*, are specific states of two qubits that represent the simplest and maximal examples of quantum entanglement. These four maximal entangled states form the "Bell basis", of a four-dimensional Hilbert space for two qubits. They are

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \tag{9}$$

$$\left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \tag{10}$$

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \tag{11}$$

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \tag{12}$$

where the first argument in the ket represents the state of the first qubit while the second indicates the other. For example, $|01\rangle$ means the first qubit (Q1) is in the ground state $|0\rangle$, and the second qubit (Q2) is in the (first) excited state: $|1\rangle$. They cannot be product states because there are no cross terms.

2.1 Entanglement

Consider one of the Bell states, $|\Psi^-\rangle$ as in Eq.(12). If we do a measurement to Q1 in the standard basis, it yields a 0 with probability $1/2$ and 1 with probability $1/2$. Likewise, measuring Q2 yields the same outcomes with the same probabilities. However, determining either qubit exactly determines the other. In this case, measurement of $|\Psi^-\rangle$ in any basis will yield opposite outcomes for the two individuals. Similarly, if we consider the state $|\Phi^+\rangle$ as Eq.(9), the outcomes of the two qubits will be definitely the same.

2.2 Quantum circuit

The simplest way to create entangled Bell states is through a quantum circuit shown in Fig.1.

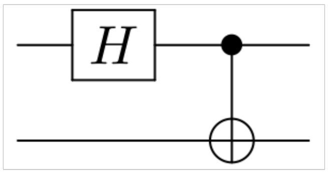

Figure 1: Quantum circuit to create Bell states.

Truth table As mentioned in class, the input information of the quantum circuit will affect the output. To obtain the four Bell states (9), (10), (11), (12), what are the initial state of the two qubits? We can cut the circuit into two parts: a Hadamard gate and a CNOT gate. Assume both qubits are initially in the ground state as $|0\rangle \otimes |0\rangle$, after a Hadamard operation, we will have

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \, ,$$

which can be simplified as $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$. Then we do the CNOT operation that only flips Q2 as Q1 has a value of 1. Follow the procedures, one can easily come out with the truth table for Bell states generator.
***Exercise*:** Please fill Table 1 to finish the truth table for Bell states generator.

Input		Output
Q1	Q2	Bell state
		$\lvert\Psi^+\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$
		$\lvert\Psi^-\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$
		$\lvert\Phi^+\rangle = \frac{1}{\sqrt{2}}(\lvert 01\rangle + \lvert 10\rangle)$
		$\lvert\Phi^-\rangle = \frac{1}{\sqrt{2}}(\lvert 01\rangle - \lvert 10\rangle)$

Table 1: Truth table for the Bell states generator. Please fill the blank with $|0\rangle$ and $|1\rangle$.

Matrix representation The operation of the circuit in Fig.1 can be described as a matrix. The system now is composed of two qubits, Q1 and Q2, which means the whole space contains two Hilbert spaces. One can imagine that we are now having a system consists of two Bloch spheres. We first apply a Hadamard gate on Q1 and then perform a CNOT on both. Since the Hadamard operation act only on Q1, we can write down

$$H^{(1)} = H \otimes \mathbb{I}, \tag{13}$$

where the superscript is the qubit's label. The identity $\mathbb{I}$ in Eq.(13) suggests that there is nothing performed on Q2. The symbol $\otimes$ is the tensor product operation. The order of the quantum gates of the circuit is important. On the diagram, we do the operations from left to right. However, when carrying calculations, the operations' order is inverse. For example, if the order of a sequence of quantum gates on a circuit is U_1, U_2 (from left to right), we can write down

$$|\psi_f\rangle = U_2 U_1 |\psi_i\rangle$$

with an input state $|\psi_i\rangle$ and the final state $|\psi_f\rangle$. Namely, do U_1 first and then U_2.
***Exercise*:** Please find the matrix representation corresponding to the quantum circuit in Fig.1. You may use Eq.(4) and Eq.(7). Whenever you have the matrix representation of the Bell states generator, you will be able to verify arbitrary input state in Table 1.

2.3 Measurement

In the above content, we do the measurements on the *standard basis*. But, what is that? The standard basis is the so-called *computational basis*, which is $\{|0\rangle, |1\rangle\}$ for a single qubit. Due to the fact that we are used to describing the state of a qubit in this basis, the computational basis are viewed as the standard basis. However, there are other common measurements that occur in quantum computing that, from a notational perspective, are convenient to express in terms of computational basis measurements. The most common kind of measurements that you'll run into will likely be *Pauli measurements*. In short, measuring the required qubit in the basis of the eigenstates of X, Y, and Z axes.

2.4 Single-qubit measurements

The standard basis is the one that project your state along the Z-axis. The default setting of IBM Quantum Experience is the Z-measurement, projecting your state onto the line connecting North pole and South pole. When aiming for measuring the state with other direction, we must do a frame transformation before doing so.

X-measurement To perform a X-measurement, we have to do a frame transformation before the classical Z-measurement. According to the operation of the Hadamard gate, it does mappings

$$\begin{cases} |0\rangle \mapsto \dfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \equiv |+\rangle \\ |1\rangle \mapsto \dfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \equiv |-\rangle \end{cases}. \tag{14}$$

that $\{|+\rangle, |-\rangle\}$ are actually the eigenvectors of Pauli-X matrix. This indicates that after appending an X gate before the (default) Z-measurement, we can then rotate the frame such that $+z \to +x$ and $-z \to -x$, so that the outcome will be the projection along X-axis, the goal of X-measurement. Remember that *the correct state would then be found by transforming back to the computational basis*, which amounts to applying $H^\dagger = H$ (Hadamard is a Hermitian matrix.) gate to the quantum state.

Y-measurement Similarly, we have to apple some gates in order to rotate the frame before doing Y-measurement.
***Exercise*:** What quantum gates do we need to append if one would like to project the state along Y-axis on the Bloch sphere? You may need information in Part 1.

2.5 Multiple-qubit measurements

Measurements of multi-qubit Pauli operators are defined similarly as the one-qubit case. Similar to the previous, all two-qubit Pauli-measurements can be written as $U^\dagger(Z \otimes \mathbb{I})U$ for 4×4 unitary U. We enumerate the transformations for two-qubit Pauli-measurements in Table 2.

It may be tempting to assume that measuring $Z \otimes Z$ is the same as sequentially measuring $Z \otimes \mathbb{I}$ and then $\mathbb{I} \otimes Z$, this assumption would be false. The reason is that measuring $Z \otimes Z$ projects the quantum state into either the $+1$ or -1 eigenstate of these operators, but measuring $Z \otimes \mathbb{I}$ and then $\mathbb{I} \otimes Z$ projects the quantum state vector first onto a half space of $Z \otimes \mathbb{I}$ and then onto a half space of $\mathbb{I} \otimes Z$. As there are four computational basis vectors in a two-qubit system $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, performing both measurements reduces the state to a quarter-space and hence reduces it to a single computational basis vector.

Pauli measurement	Unitary transformation
$Z \otimes \mathbb{I}$	$\mathbb{I} \otimes \mathbb{I}$
$X \otimes \mathbb{I}$	$H \otimes \mathbb{I}$
$Y \otimes \mathbb{I}$	$HS^\dagger \otimes \mathbb{I}$
$\mathbb{I} \otimes Z$	$\textsc{swap}$
$\mathbb{I} \otimes X$	$(H \otimes \mathbb{I})\textsc{swap}$
$\mathbb{I} \otimes Y$	$(HS^\dagger \otimes \mathbb{I})\textsc{swap}$
$Z \otimes Z$	$\textsc{cnot}_{10}$
$X \otimes Z$	$\textsc{cnot}_{10}(H \otimes \mathbb{I})$
$Y \otimes Z$	$\textsc{cnot}_{10}(HS^\dagger \otimes \mathbb{I})$
$Z \otimes X$	$\textsc{cnot}_{10}(\mathbb{I} \otimes H)$
$X \otimes X$	$\textsc{cnot}_{10}(H \otimes H)$
$Y \otimes X$	$\textsc{cnot}_{10}(HS^\dagger \otimes H)$
$Z \otimes Y$	$\textsc{cnot}_{10}(\mathbb{I} \otimes HS^\dagger)$
$X \otimes Y$	$\textsc{cnot}_{10}(H \otimes HS^\dagger)$
$Y \otimes Y$	$\textsc{cnot}_{10}(HS^\dagger \otimes HS^\dagger)$

Table 2: In this table, we list the transformation for doing different two-qubit Pauli-measurements. Symbol $\textsc{cnot}_{10}$ means the control qubit is Q1 and the target is Q0.

***Exercise**:** Why does the $\textsc{cnot}_{10}$ appear in Table 2?
***Exercise*:** What if we want to do a Bell-state measurement? i.e., How to do a measurement along the basis of the Bell states?

3 Experiments

In this part, we realize the content in lecture on IBM Quantum Experience. The cloud platform allows users doing quantum programs with simulators or real quantum processors of IBM Company. Qiskit (Quantum Information Software Kit) is an open source released by IBM Quantum Team based on Python, and can be used in programming quantum computing experiments and computations. The official Qiskit textbook gives detailed tutorials on how to execute your experiments locally or remotely. For very beginners, you can try GUI to construct quantum algorithms and view the results. The advanced users are encouraged to do the experiments based on Python which allows you modifying the more complicated programs. To access it, you will need to

1. apply an IBM Quantum Experience account
2. install the Qiskit package
3. make sure you have a Python (3.5 or later) platform in your computer.

You may find some useful instructions on Qiksit Document.

***Exercise*:** Compose a quantum circuit to generate Bell states, with either graphic interface or Notebook. How to set the initial state of the qubits, if we would like the qubits initially in the state $|10\rangle$? Run the program on both simulator and a real machine for 1024 shots, and compare the results. What cause the difference between the two backends? Consider the case that the first qubit is initially in $1/\sqrt{2}\,|0\rangle - |1\rangle$ and the second is in $|1\rangle$. What is the expected output state of the circuit? Now do the experiment via Qiskit and view the transpiled circuit, what do you see? Is the printed circuit diagram equivalent to yours? Append *barriers* to your circuit, what do you get now? Start your Bell states generator with input state $|1\rangle \otimes |0\rangle$. Please measure with Bell states basis, the Pauli measurement $X \otimes Y$ and the default setting. Compare the outcomes.

3.1 Relaxation

The identity gate has a matrix representation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{15}$$

which actually do nothing with the qubit. With the help of identity gate, we can test the *lifetime* of qubits. Consider two cases:

- X gate & measurement
- X gate & 100 identity gates & measurement

The expected outcomes are the same: $|1\rangle$. However, when running the program on a real device, the measurement outcome (with same number of shots, let's say, 1024 shots) are different. The second case has the higher probability on the state $|0\rangle$ than the first one. It is because we do the measurement after 100 times of gate length (gate time) of an identity gate. During the period, the information stored in the qubit leaks out.

***Exercise*:** What if we want to exam the lifetime of a two-qubit system (including the interaction between them)? Please construct a circuit for this.