電磁學(I)期中考(Chap. 1~Chap. 3.3)　　2022.11.8, 111(1)

(答案需有完整計算過程及標示必要的向量符號，數值請計算至小數點後第 2 位)

1. 給定三個向量 $\overline{A}=\hat{a}_x4-\hat{a}_z$、$\overline{B}=-\hat{a}_x2+\hat{a}_y+\hat{a}_z3$ 及 $\overline{C}=\hat{a}_x3-\hat{a}_y4+\hat{a}_z2$，請計算以下小題：
   (a) $\overline{A}\times\overline{B}$, (b) $\overline{A}\cdot(\overline{B}\times\overline{C})$, (c) θ_{BC}, (d) $|\overline{B}-\overline{A}|$。　　(每小題 4 分，共 16 分)

2. 給定一向量$\overline{B}=\hat{a}_R\dfrac{10}{R}+\hat{a}_\theta R\cos\theta+\hat{a}_\phi$，請計算以下小題：
   (a) 請將向量$\overline{B}$轉換為以直角座標系統表示。此外，請計算於點 P(−3, 4, 0)之$\overline{B}$。　(7 分)
   (b) 請將向量$\overline{B}$轉換為以圓柱座標系統表示。此外，請計算於點 Q(5, π/2, −2)之$\overline{B}$。　(7 分)

3. (a) 給定$V=x^2+yz+xyz$，請計算 V 朝著向量$\overline{A}=\hat{a}_x3+\hat{a}_y4+\hat{a}_z5$方向，在點 T(1, 1, −1)之空間增加率。　(7 分)
   (b) 給定$G=5r\sin\phi-6r^2z\cos\phi$，請計算在點 R(2, π, 3)之梯度$\nabla G$。　(7 分)

4. 給定一向量$\overline{E}=\hat{a}_r\dfrac{\cos\phi}{r}+\hat{a}_\phi r+\hat{a}_z e^{-Z}$及圓柱區域如圖 1 所示，圓柱半徑 3，高度為 4，圓柱中心位於座標原點。請驗證 Divergence theorem $\int_v(\nabla\cdot\overline{E})dv=\oint_s\overline{E}\cdot d\overline{s}$。　(15 分)

5. 給定一向量$\overline{F}=\hat{a}_x(x^2y+zx)+\hat{a}_y(4y^2+z)+\hat{a}_z z^2$及一封閉路徑如圖 2 所示，此路徑位於 yz 平面，請驗證 Stokes's theorem $\int_s(\nabla\times\overline{F})\cdot d\overline{s}=\oint_c\overline{F}\cdot d\overline{l}$。　(15 分)

6. 給定兩點電荷 Q_1 及 Q_2，Q_1 位於(1, 1, 1)位置，Q_2 位於(2, −1, 0)位置，請計算以下小題：
   (a) 若測試電荷位於(0, 0, z)位置，且 $Q_1=Q_2$，請計算 z 之值使測試電荷所受力無 y 分量。 (7 分)
   (b) 若測試電荷位於(0, 0, 2)位置，且 $Q_1\neq Q_2$，請計算 Q_1/Q_2 之值使測試電荷所受力無 x 分量。 (7 分)

7. 給定一均勻弧線段之線電荷ρ_l，位於 xy 平面，半徑為 1 m，角度介於 $0\le\phi\le\pi/4$，如圖 3 所示。若 $\rho_l=10\mu$C/m，請計算於位置(0, 0, z)處之電場$\overline{E}$。　(12 分)

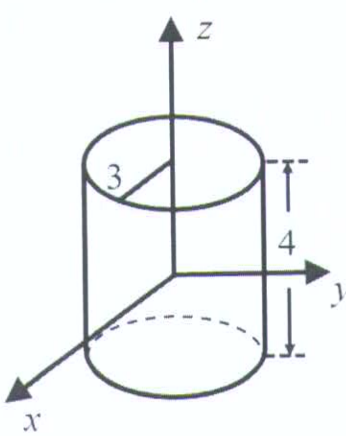


圖 1

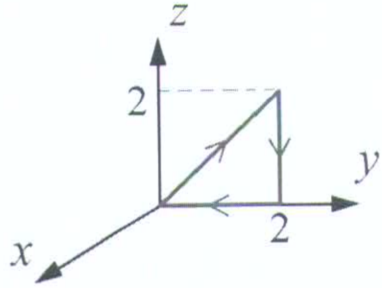


圖 2

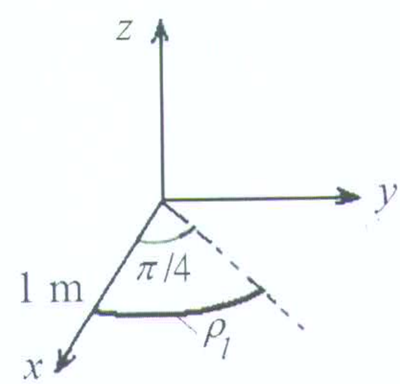


圖 3

1. 給定三個向量 $\bar{A} = \hat{a_x}4 - \hat{a_z}$、$\bar{B} = -\hat{a_x}2 + \hat{a_y} + \hat{a_z}3$、
$\bar{C} = \hat{a_x}3 - \hat{a_y}4 + \hat{a_z}2$

Sol):

(a) $\bar{A} \times \bar{B} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ 4 & 0 & -1 \\ -2 & 1 & 3 \end{vmatrix} = \hat{a_x} - \hat{a_y}10 + \hat{a_z}4$ #

(b) $\bar{A} \cdot (\bar{B} \times \bar{C})$

$\bar{B} \times \bar{C} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ -2 & 1 & 3 \\ 3 & -4 & 2 \end{vmatrix} = \hat{a_x}14 + \hat{a_y}13 + \hat{a_z}5$

$\bar{A} \cdot (\bar{B} \times \bar{C}) = 4 \times 14 - 5 = 51$ #

(c) θ_{BC} (向量$\bar{B}$與$\bar{C}$之夾角)

$\theta_{BC} = \cos^{-1} \frac{\bar{B} \cdot \bar{C}}{|\bar{B}||\bar{C}|} = \cos^{-1} \frac{-6-4+6}{\sqrt{14} \times \sqrt{29}}$

$= 101.45^\circ$ #

(d) $|\bar{B} - \bar{A}|$

$\bar{B} - \bar{A} = -\hat{a_x}6 + \hat{a_y} + \hat{a_z}4$

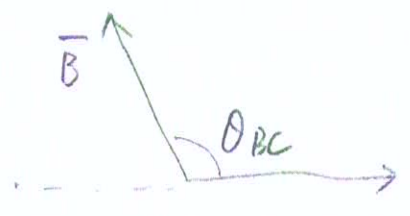
$|\bar{B} - \bar{A}| = |-\hat{a_x}6 + \hat{a_y} + \hat{a_z}4|$

$= \sqrt{(-6)^2 + 1 + 4^2}$

$= 7.28$ #

第一題(c)補充說明：

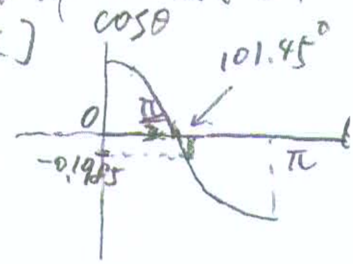
1. 兩個向量之夾角定義為 0～180°，且 < 180°



$\bar{B} = -\hat{a}_x 2 + \hat{a}_y + \hat{a}_z 3$, $\bar{C} = \hat{a}_x 3 - \hat{a}_y 4 + \hat{a}_z 2$

三角函數為多對一函數，反函數要有意義需有定義域

2. 反餘弦 $\cos^{-1}$ 函數之定義域為 $[0, \pi]$

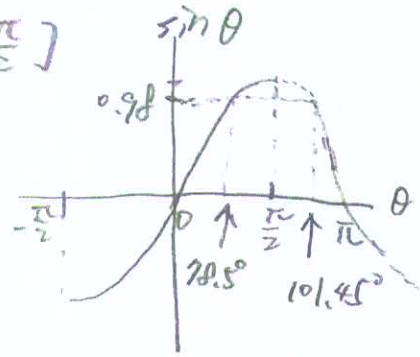


∴ 用反餘弦計算 $\theta_{BC} = \cos^{-1} \dfrac{\bar{B}\cdot\bar{C}}{|\bar{B}||\bar{C}|}$

$= \cos^{-1} \dfrac{-4}{\sqrt{14}\sqrt{29}} = \cos^{-1}(-0.1985)$

$= 101.45°$ 為正確

3. 反正弦 $\sin^{-1}$ 函數之定義域為 $[-\frac{\pi}{2}, \frac{\pi}{2}]$



∴ 用反正弦 $\theta_{BC} = \sin^{-1} \dfrac{|\bar{B}\times\bar{C}|}{|\bar{B}||\bar{C}|}$

$= \sin^{-1} \dfrac{\sqrt{390}}{\sqrt{14}\sqrt{29}} = \sin^{-1}(0.98)$

$= 78.5°$ 不正確

∵ 反正弦之定義域 $-\frac{\pi}{2}$～$\frac{\pi}{2}$，與向量夾角定義範圍 0～π 不同。

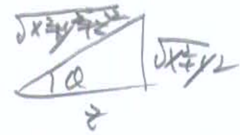
∴ 需 $\sin(78.5°) = \sin(180° - 78.5°)$

$= \sin(101.45°)$

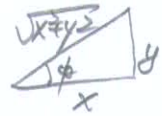
$\Rightarrow 101.45°$ 才正確

2. $\bar{B} = \hat{a_R}\,\frac{10}{R} + \hat{a_\theta}\, R\cos\theta + \hat{a_\phi}$

(a) $R = \sqrt{x^2+y^2+z^2}$, $\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, $\cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$



$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}$, $\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$



$$B_x = B_R \sin\theta\cos\phi + B_\theta\cos\theta\cos\phi - B_\phi\sin\phi$$

$$= \frac{10}{\sqrt{x^2+y^2+z^2}}\,\frac{\cancel{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2+z^2}}\,\frac{x}{\cancel{\sqrt{x^2+y^2}}} + \sqrt{x^2+y^2+z^2}\,\frac{z^2}{x^2+y^2+z^2}\,\frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{10x}{x^2+y^2+z^2} + \frac{z^2x}{\sqrt{(x^2+y^2+z^2)(x^2+y^2)}} - \frac{y}{\sqrt{x^2+y^2}}$$ ※

$$B_y = B_R\sin\theta\sin\phi + B_\theta\cos\theta\sin\phi + B_\phi\cos\phi$$

$$= \frac{10}{\sqrt{x^2+y^2+z^2}}\,\frac{\cancel{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2+z^2}}\,\frac{y}{\cancel{\sqrt{x^2+y^2}}} + \sqrt{x^2+y^2+z^2}\,\frac{z^2}{x^2+y^2+z^2}\,\frac{y}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{10y}{x^2+y^2+z^2} + \frac{z^2y}{\sqrt{(x^2+y^2+z^2)(x^2+y^2)}} + \frac{x}{\sqrt{x^2+y^2}}$$ ※

$$B_z = B_R\cos\theta - B_\theta\sin\theta = \frac{10}{\sqrt{x^2+y^2+z^2}}\,\frac{z}{\sqrt{x^2+y^2+z^2}} - \sqrt{x^2+y^2+z^2}\,\frac{z}{\sqrt{x^2+y^2+z^2}}\,\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

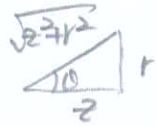
$$= \frac{10z}{x^2+y^2+z^2} - \frac{z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$ ※

$\therefore \bar{B} = \hat{a_x} B_x + \hat{a_y} B_y + \hat{a_z} B_z$

At $\bar{B}(-3,4,0)$, $x=-3$, $y=4$, $z=0$ 代入

$$\bar{B} = \hat{a_x}\left(\frac{-30}{25} - \frac{4}{5}\right) + \hat{a_y}\left(\frac{40}{25} - \frac{3}{5}\right) + \hat{a_z}(0)$$

$$= \hat{a_x}(-2) + \hat{a_y}$$ ※

(b) $R=\sqrt{r^2+z^2}$, $\theta=\tan^{-1}\frac{r}{z}$ 

$\therefore \sin\theta=\frac{r}{\sqrt{z^2+r^2}}$, $\cos\theta=\frac{z}{\sqrt{z^2+r^2}}$

$$B_r = B_R\sin\theta + B_\theta\cos\theta$$
$$= \frac{10}{\sqrt{r^2+z^2}}\frac{r}{\sqrt{z^2+r^2}} + \sqrt{r^2+z^2}\,\frac{z^2}{z^2+r^2}$$
$$= \underline{\frac{10r}{r^2+z^2} + \frac{z^2}{\sqrt{r^2+z^2}}} \quad ※$$

$$B_\phi = B_\phi = \underline{\underline{1}} \quad ※$$

$$B_z = B_R\cos\theta - B_\theta\sin\theta$$
$$= \frac{10}{\sqrt{r^2+z^2}}\frac{z}{\sqrt{r^2+z^2}} - \cancel{\sqrt{r^2+z^2}}\,\frac{z}{\cancel{\sqrt{r^2+z^2}}}\,\frac{r}{\sqrt{r^2+z^2}}$$
$$= \underline{\frac{10z}{r^2+z^2} - \frac{zr}{\sqrt{r^2+z^2}}} \quad ※$$

$$\therefore \bar{B} = \hat{a_r}\,B_r + \hat{a_\phi}\cdot 1 + \hat{a_z}\,B_z$$

At $\bar{B}(5,\frac{\pi}{2},-2)$ 代 $r=5$, $\phi=\frac{\pi}{2}$, $z=-2$

$$\bar{B} = \hat{a_r}\left(\frac{50}{29} + \frac{4}{\sqrt{29}}\right) + \hat{a_\phi} + \hat{a_z}\left(\frac{-20}{29} + \frac{10}{\sqrt{29}}\right)$$
$$= \hat{a_r}(2.467) + \hat{a_\phi} + \hat{a_z}\,1.167$$

3. (a) $V = x^2 + yz + xyz$

$$\nabla V = \hat{a_x}\frac{\partial}{\partial x}V + \hat{a_y}\frac{\partial V}{\partial y} + \hat{a_z}\frac{\partial V}{\partial z}$$

$$= \hat{a_x}(2x+yz) + \hat{a_y}(z+xz) + \hat{a_z}(y+xy), \text{ at } T(1,1,-1)$$

$$= \hat{a_x} + \hat{a_y}(-2) + \hat{a_z}2$$

$$\bar{A} = \hat{a_x}3 + \hat{a_y}4 + \hat{a_z}5 \quad \therefore \hat{a_A} = \frac{\bar{A}}{|\bar{A}|}$$

$$= \frac{\hat{a_x}3 + \hat{a_y}4 + \hat{a_z}5}{\sqrt{50}}$$

$$= \hat{a_x}0.42 + \hat{a_y}0.57 + \hat{a_z}0.71$$

$$\frac{dV}{dl} = \nabla V \cdot \hat{a_A}$$

$$= (\hat{a_x} - \hat{a_y}2 + \hat{a_z}2) \cdot (\hat{a_x}0.42 + \hat{a_y}0.57 + \hat{a_z}0.71)$$

$$= 0.42 - 1.14 + 1.42$$

$$= 0.7$$

※

(b) 給定 $G = 5r\sin\phi - 6r^2z\cos\phi$

$$\nabla G = \hat{a_r}\frac{\partial G}{\partial r} + \hat{a_\phi}\frac{1}{r}\frac{\partial G}{\partial \phi} + \hat{a_z}\frac{\partial G}{\partial z}$$

$$= \hat{a_r}(5\sin\phi - 12rz\cos\phi) + \hat{a_\phi}(5\cos\phi + 6rz\sin\phi)$$

$$+ \hat{a_z}(-6r^2\cos\phi)$$

At $R(2, \pi, 3)$

$$\nabla G = \hat{a_r}72 - \hat{a_\phi}5 + \hat{a_z}24$$

※

4.

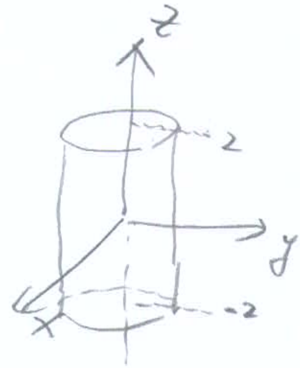
Sol): $\bar{E} = \hat{a_r}\frac{\cos\phi}{r} + \hat{a_\phi}\, r + \hat{a_z}\, e^{-z}$

驗证 Divergence theorem $\int_V (\nabla\cdot\bar{E})\,dv = \oint_S \bar{E}\cdot d\bar{s}$

- $\nabla\cdot\bar{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_\phi}{r\partial\phi} + \frac{\partial E_z}{\partial z}$

$= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\cos\phi}{r}) + \frac{\partial}{r\partial\phi} r + \frac{\partial}{\partial z}(e^{-z})$

$= -e^{-z}$



$\int_V (-e^{-z})\, r\,dr\,d\phi\,dz$

$= -\int_0^{2\pi} d\phi \int_0^3 r\,dr \int_{-2}^2 e^{-z} dz$

$= -2\pi \times \frac{r^2}{2}\Big|_0^3 \times (-e^{-z})\Big|_{-2}^2$

$= -2\pi \times \frac{9}{2} \times (-e^{-2} + e^2)$

$= 9\pi(e^{-2} - e^2)$ //

- $\oint_S \bar{E}\cdot d\bar{s} = \int_{頂(z=2)} \bar{E}\cdot \hat{a_z}\, r\,dr\,d\phi + \int_{底(z=-2)} \bar{E}\cdot(-\hat{a_z}\, r\,dr\,d\phi)$

$+ \int_{柱} \bar{E}\cdot\hat{a_r}\,(3\,d\phi\,dz)$

$= \int_0^{2\pi}\int_0^3 e^{-2}\, r\,dr\,d\phi - \int_0^{2\pi}\int_0^3 e^2\, r\,dr\,d\phi + \int_{-2}^2\int_0^{2\pi}\cos\phi\,d\phi\,dz$

$= e^{-2}\times\frac{9}{2}\times 2\pi - e^2\frac{9}{2}\times 2\pi + 4\sin\phi\Big|_0^{2\pi}$

$= 9\pi(e^{-2} - e^2)$ //

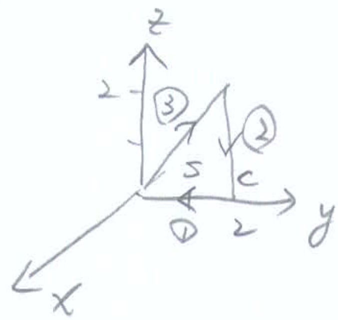
得证

5 Sol):

$\bar{F} = \hat{a_x}(x^2y + zx) + \hat{a_y}(4y^2 + z) + \hat{a_z} z^2$

驗證 Stokes's theorem

$\int_S (\nabla \times \bar{F}) \cdot d\bar{s} = \oint_C \bar{F} \cdot d\bar{l}$



• $\nabla \times \bar{F} = \hat{a_x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{a_y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{a_z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$

$= \hat{a_x}(-1) + \hat{a_y}\, x + \hat{a_z}(-x^2)$

$d\bar{s} = -\hat{a_x}\, dy\, dz$

$\int_S (\nabla\times\bar{F})\cdot d\bar{s} = \int_S (-\hat{a_x} + \hat{a_y} x - \hat{a_z} x^2)\cdot(-\hat{a_x}\, dy\, dz)$

$= \int_0^2 \int_0^z dy\, dz$, 斜線方程式為 $y=z$

$= \int_0^2 y\Big|_0^z dz$

$= \int_0^2 z\, dz = \frac{z^2}{2}\Big|_0^2 = 2$ ※

• $d\bar{l} = \hat{a_y}\, dy + \hat{a_z}\, dz$ (∵位於yz面,故無dx)

$\bar{F}\cdot d\bar{l} = (4y^2 + z)\, dy + z^2 dz$

$\oint_C \bar{F}\cdot d\bar{l} = \underbrace{\int_2^0 4y^2 dy}_{① \; z=0,\; dz=0} + \underbrace{\int_2^0 z^2 dz}_{② \; y=2,\; dy=0} + \underbrace{\int (4y^2+z)\, dy}_{③ \; y=z} + \underbrace{\int z^2 dz}_{③ \; y=z}$

$= \frac{4}{3}y^3\Big|_2^0 + \cancel{\frac{z^3}{3}\Big|_2^0} + \int_0^2 (4y^2+y)\, dy + \cancel{\frac{z^3}{3}\Big|_0^2}$

$= -\frac{32}{3} + \left(\frac{4y^3}{3} + \frac{y^2}{2}\right)\Big|_0^2 = \cancel{-\frac{32}{3}} + \cancel{\frac{32}{3}} + \frac{4}{2}$

$= 2$ ※

得證

6.

sol):

(a) $\bar{R} = \hat{a_z} z$

$\bar{R_1'} = \hat{a_x} + \hat{a_y} + \hat{a_z}$

$\bar{R_2'} = 2\hat{a_x} - \hat{a_y}$

$\therefore \bar{R} - \bar{R_1'} = -\hat{a_x} - \hat{a_y} + \hat{a_z}(z-1)$

$\bar{R} - \bar{R_2'} = -2\hat{a_x} + \hat{a_y} + \hat{a_z} z$

$$\therefore \bar{E} = \frac{Q_1(-\hat{a_x} - \hat{a_y} + \hat{a_z}(z-1))}{4\pi\varepsilon_0 [1+1+(z-1)^2]^{3/2}} + \frac{Q_2(-2\hat{a_x} + \hat{a_y} + \hat{a_z} z)}{4\pi\varepsilon_0 [4+1+z^2]^{3/2}}$$

$\because Q_1 = Q_2$ 且無 y 分量力

$$\Rightarrow \frac{-Q_1}{[2+(z-1)^2]^{3/2}} + \frac{Q_1}{[5+z^2]^{3/2}} = 0$$

$$\Rightarrow \frac{1}{[5+z^2]^{3/2}} = \frac{1}{[2+(z-1)^2]^{3/2}}$$

$$\Rightarrow 5+\cancel{z^2} = 2+(z-1)^2 = 2+\cancel{z^2}-2z+1$$

$2z = -2 \quad \therefore z = -1$ #

(b)
$$\frac{-Q_1}{[2+(z-1)^2]^{3/2}} - \frac{2Q_2}{[5+z^2]^{3/2}} = 0 \quad \text{代 } z=2$$

$$\Rightarrow \frac{Q_1}{(3)^{3/2}} = \frac{-2Q_2}{(9)^{3/2}}$$

$$\Rightarrow \frac{Q_1}{Q_2} = \frac{-2\sqrt{3}}{9} \text{ 或 } -0.385$$ #

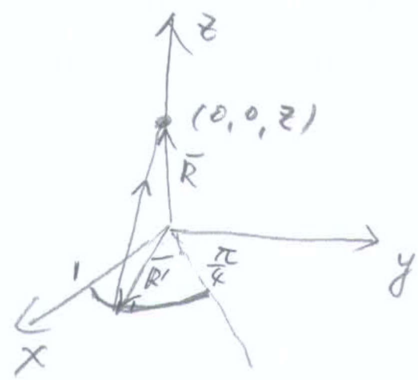
7.

Sol):

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int_{L'} \hat{a_R}\,\frac{\rho_\ell\, d\ell'}{|\bar{R}-\bar{R'}|^2} \quad , \quad \hat{a_R} = \frac{\bar{R}-\bar{R'}}{|\bar{R}-\bar{R'}|}$$

$\because$ ρ_ℓ uniform

$$\bar{E} = \frac{\rho_\ell}{4\pi\varepsilon_0}\int_{L'} \frac{\bar{R}-\bar{R'}}{|\bar{R}-\bar{R'}|^3}\, d\ell'$$



$$\bar{R} = \hat{a_z}\, z$$
$$\bar{R'} = \hat{a_r}\, r$$

$$\therefore \bar{R}-\bar{R'} = \hat{a_z}\, z - \hat{a_r}\, r \quad , \quad d\ell = r\, d\phi$$

$$\bar{E} = \frac{\rho_\ell}{4\pi\varepsilon_0}\int_0^{\frac{\pi}{4}} \frac{\hat{a_z}\, z - \hat{a_r}\, r}{|\hat{a_z}\, z - \hat{a_r}\, r|^3}\, r\, d\phi \quad , \quad \rho_\ell = 10\ \mu C$$

$$= \frac{10^{-5}}{4\pi \times \frac{1}{36\pi}\times 10^{-9}}\int_0^{\frac{\pi}{4}} \frac{\hat{a_z}\, z - \hat{a_r}}{(z^2+1)^{3/2}}\, d\phi$$

$$= \frac{9\times 10^4}{(z^2+1)^{3/2}}\left[\int_0^{\frac{\pi}{4}} \hat{a_z}\, z\, d\phi - \int_0^{\frac{\pi}{4}} \hat{a_r}\, d\phi\right]$$

$$= \frac{9\times 10^4}{(z^2+1)^{3/2}}\int_0^{\frac{\pi}{4}} \hat{a_z}\, z\, d\phi - \int_0^{\frac{\pi}{4}} (\hat{a_x}\cos\phi + \hat{a_y}\sin\phi)\, d\phi$$

$$= \frac{9\times 10^4}{(z^2+1)^{3/2}}\left[\hat{a_z}\, z\,\phi\Big|_0^{\frac{\pi}{4}} - \hat{a_x}\sin\phi\Big|_0^{\frac{\pi}{4}} + \hat{a_y}\cos\phi\Big|_0^{\frac{\pi}{4}}\right]$$

$$= \frac{9\times 10^4}{(z^2+1)^{3/2}}\left[\hat{a_z}\, z\,\frac{\pi}{4} - \hat{a_x}\frac{\sqrt{2}}{2} + \hat{a_y}\left(\frac{\sqrt{2}}{2}-1\right)\right]$$

$$= \frac{9\times 10^4}{(z^2+1)^{3/2}}\left[\hat{a_z}\, z\, 0.785 - \hat{a_x}\, 0.707 - \hat{a_y}\, 0.293\right] \quad V/m$$