電磁學(I)期末考(Chaps. 3.4~4)

2023.1.3, 111(1)

(答案需有完整計算過程及標示必要的向量符號)

1. 給定一長直圓柱，長 5 m 及半徑 $a = 0.005$ m，表面均匀分布電荷$+5\mu$C。此圓柱外部有一同軸圓柱殼，長 5 m 及半徑 $b = 0.01$ m，表面均匀分布電荷-2.5μC，其剖面圖如圖 1 所示。可忽略邊緣效應，請計算所有區域之電場$\bar{E}$。 (16%)

2. 給定一長直同軸線，其內、外導體半徑分別為 $r_i = 0.5$ cm 及 $r_o = 1$ cm，導體間填充有同軸之橡膠(相對介電常數$\varepsilon_{rr} = 3.2$)及聚苯乙烯(相對介電常數 $\varepsilon_{rp} = 2.6$)，圖 2 所示為剖面圖，其中 $r_p = 0.75$ cm。為避免介電崩潰，介質能承受之最大電場強度需低於其介電強度之 25%，橡膠及聚苯乙烯之介電強度分別為 25×10^6 V/m 及 20×10^6 V/m。(a)請分別計算橡膠層及聚苯乙烯層各自所能承受之最大電場強度值 E (8%)，(b)請決定此同軸線內、外導體間最大允許工作電壓 V (8%)。

3. 給定一個厚介質球殼，內半徑為 a，外半徑為 b，具有極化向量$\bar{P} = \hat{a}_R \dfrac{k}{R}$，其中 k 為常數，假設無自由電荷。 (a)請計算束縛電荷密度ρ_{pv} 及ρ_{ps}。 (6%) (b)請計算總束縛電荷 Q_p。 ($Q_p = Q_{pv}+Q_{ps}$) (5%)。 (c)計算所有區域之電場$\bar{E}$。 (5%)

4. 給定同心導體球殼及導體球，導體球殼半徑 $b = 30$ cm，導體球半徑 $a = 8$ cm，兩導體間填充有兩層不同介質，兩介質之相對介電常數分別為$\varepsilon_{r1} = 5$ 及$\varepsilon_{r2} = 10$，兩層介質相接處半徑 $c = 20$ cm，如圖 3 所示。請計算其電容 C。 (16%)

5. 給定兩垂直連接導體板，兩板分別朝$+x$ 軸及$+y$ 軸方向無限延伸，也同時朝 z 軸方向無限延伸，擺置一 100 nC 之電荷 Q 於位置(3, 4, 0)處，如圖 4 所示，周圍為自由空間。(a)請計算於位置(3, 5, 0)之電位 V 及電場$\bar{E}$。 (8%) (b) 請計算此兩導體板位於 $z = 0$ 處之表面電荷密度ρ_s。 (8%)

6. 給定一中空圓柱電阻器，此材料電導率為σ，長度為 L，內、外徑分別為 r_a 及 r_b，如圖 5 所示。假設電流朝徑向方向流動，請計算中空圓柱內、外表面間之電阻值 R。 (20%)

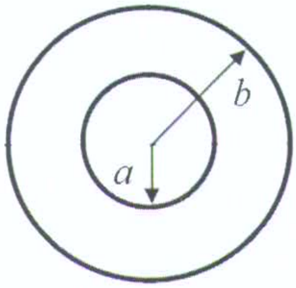


圖 1

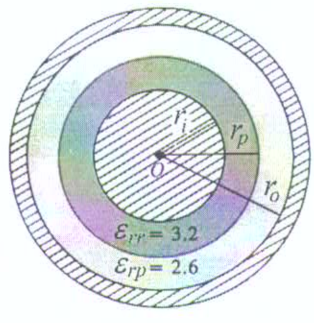


圖 2

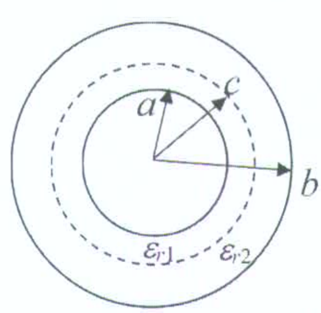


圖 3

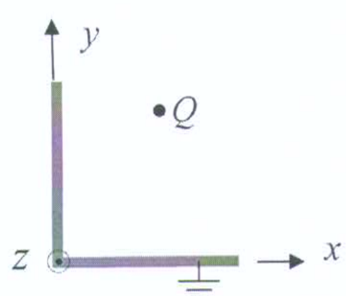


圖 4

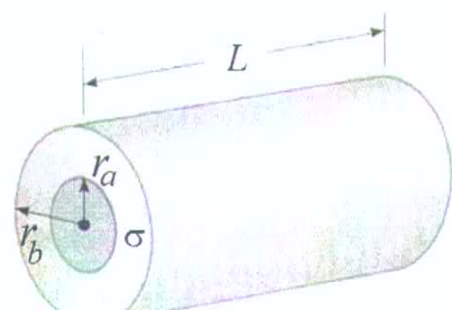


圖 5

1. 因具有對稱性，可選高斯面，套用高斯定律

- For $r<a$:

$$\oint_S \bar{E}\cdot d\bar{s} = \frac{Q}{\varepsilon_0} \Rightarrow 2\pi r L E_r = \frac{0}{\varepsilon_0}$$

$$\Rightarrow E_r = 0 \Rightarrow \underline{\bar{E}=0} \#$$

- For $a<r<b$:

$$\oint_S \bar{E}\cdot d\bar{s} = \frac{Q}{\varepsilon_0} \Rightarrow 2\pi r L E_r = \frac{q_a}{\varepsilon_0}$$

$$\Rightarrow E_r = \frac{q_a}{2\pi r L \varepsilon_0} = \frac{5\times 10^{-6}}{2\pi r\times 5\times \frac{1}{36\pi}\times 10^{-9}}$$

$$= \frac{1.8\times 10^{4}}{r}$$

$$\underline{\bar{E} = \hat{a_r}\,\frac{1.8\times 10^{4}}{r} \quad V/m} \#$$

- For $r>b$:

$$\oint_S \bar{E}\cdot d\bar{s} = \frac{Q}{\varepsilon_0}$$

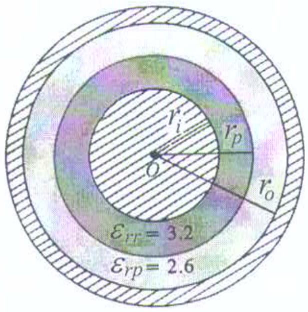
$$2\pi r L E_r = \frac{q_a+q_b}{\varepsilon_0}$$

$$\Rightarrow E_r = \frac{q_a+q_b}{2\pi r L\varepsilon_0} = \frac{5\times 10^{-6} - 2.5\times 10^{-6}}{2\pi r\times 5\times\frac{1}{36\pi}\times 10^{-9}}$$

$$= \frac{9\times 10^{3}}{r}$$

$$\therefore \underline{\bar{E} = \hat{a_r}\,\frac{9\times 10^{3}}{r} \quad V/m} \#$$

2.



圖二

Solution：

(a)參考課本 Example 3-12，$r_i = 0.5$ cm、$r_o = 1$ cm 及 $r_p = 0.75$ cm。

橡膠層 25%介電強度 ➔ $0.25 \times 25 \times 10^6 = 6.25 \times 10^6$ V/m

聚苯乙烯層 25%介電強度 ➔ $0.25 \times 20 \times 10^6 = 5 \times 10^6$ V/m

$$\text{Max.}\,E_r = \frac{\rho_l}{2\pi\varepsilon_o}\left(\frac{1}{3.2r_i}\right) \Rightarrow \frac{\rho_l}{2\pi\varepsilon_o} = 6.25\times10^6 \times 3.2r_i = 10^5 \quad \text{V/m}$$

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$$\text{Max.}\,E_p = \frac{\rho_l}{2\pi\varepsilon_o}\left(\frac{1}{2.6r_p}\right) \Rightarrow \frac{\rho_l}{2\pi\varepsilon_o} = 5\times10^6 \times 2.6r_p = 0.975\times10^5 \quad \text{V/m}$$

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(b)

由Max. E_p 決定可承受之最大$\rho_l / 2\pi\varepsilon_o \Rightarrow \dfrac{\rho_l}{2\pi\varepsilon_o} = 0.975\times10^5$ V/m

參考課本Example 3-12,

$$V_{\max} = \frac{\rho_l}{2\pi\varepsilon_o}\left(\frac{1}{\varepsilon_{rp}}\ln\frac{r_o}{r_p} + \frac{1}{\varepsilon_{rr}}\ln\frac{r_p}{r_i}\right)$$

$$= 0.975\times10^5\left(\frac{1}{2.6}\ln 1.333 + \frac{1}{3.2}\ln 1.5\right) = 2.31\times10^4 \text{ V}$$

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3.

sol):

(a) $\bar{P} = \hat{a_R} \frac{K}{R}$

$$\rho_{pv} = -\nabla \cdot \bar{P} = -\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{K}{R}\right) = \underline{-\frac{K}{R^2}} \quad \#$$

$$\rho_{ps\_outer} = \bar{P} \cdot \hat{a_n} = \hat{a_R}\frac{K}{b} \cdot \hat{a_R} = \underline{\frac{K}{b}} \quad \#$$

$$\rho_{ps\_inner} = \bar{P} \cdot \hat{a_n} = \hat{a_R}\frac{K}{a} \cdot (-\hat{a_R}) = \underline{-\frac{K}{a}} \quad \#$$

(b) $Q_p = Q_{pv} + Q_{ps} = \int_V \rho_{pv}\, dV + \oint_S \rho_{ps\_outer}\, ds + \oint_S \rho_{ps\_inner}\, ds$

$$= \int_0^{2\pi}\int_0^{\pi}\int_a^b \left(-\frac{K}{R^2}\right) R^2 \sin\theta\, dR\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi} \frac{K}{b} b^2 \sin\theta\, d\theta\, d\phi$$

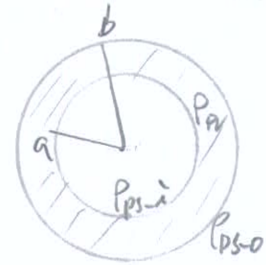
$$+ \int_0^{2\pi}\int_0^{\pi} \left(-\frac{K}{a}\right) a^2 \sin\theta\, d\theta\, d\phi$$

$$= -4\pi K R\Big|_a^b + 4\pi b K - 4\pi a K$$

$$= -4\pi K(b-a) + 4\pi b K - 4\pi a K$$

$$\underline{= 0} \quad \#$$

(c) 具對稱性，可選高斯面，套用高斯定律 $\oint_S \bar{E}\cdot d\bar{S} = \frac{Q+Q_p}{\varepsilon_0}$ 自由+束縛



- for $R<a$，無自由及束縛电荷 $\Rightarrow Q=0 \Rightarrow \bar{E}=0$ ※
- for $R>b$，無自由电荷，束縛电荷淨量 $Q_p=0$
  $\Rightarrow \bar{E}=0$ ※
- for $a<R<b$，

Method 1:

$$Q_p = \oint \rho_{ps-inner}\, ds + \int_0^{2\pi}\int_0^{\pi}\int_a^R \rho_{pv}\, dv$$

$$= 4\pi a^2\left(-\frac{k}{a}\right) + 4\pi(-k)R\Big|_a^R$$

$$= -4\pi aK - 4\pi K(R-a)$$

$$= -4\pi kR$$

$\because \oint_S \bar{E}\cdot d\bar{S} = \frac{Q+Q_p}{\varepsilon_0}$，假設 $\bar{E} = \hat{a_R} E_R$

$$4\pi R^2 E_R = -4\pi kR/\varepsilon_0$$

$$E_R = \frac{-k}{R\varepsilon_0}$$

$$\therefore \bar{E} = \hat{a_R}\left(\frac{-k}{R\varepsilon_0}\right)$$ ※

Method 2: 由 $\bar{D} = \varepsilon_0\bar{E} + \bar{P}$ $\quad\because a<R<b$ 無自由电荷 Q

$\therefore \oint_S \bar{D}\cdot d\bar{S} = Q \Rightarrow \bar{D}=0$

$$\therefore \bar{E} = -\frac{\bar{P}}{\varepsilon_0}$$

$$= -\hat{a_R}\frac{k}{R\varepsilon_0}$$ ※

4.

sol): 假設導体球上有电荷 Q

球殼上存 $-Q$。

由高斯定律 $\oint_s \bar{D}\cdot d\bar{s} = Q$

$$\Rightarrow \quad 4\pi R^2 D = Q$$

$$\Rightarrow \quad D = \frac{Q}{4\pi R^2}$$

$$\therefore \quad \bar{E} = \hat{a_R}\frac{Q}{4\pi\varepsilon R^2}, \quad \varepsilon_1 = \varepsilon_0\varepsilon_{r1}, \quad \varepsilon_2 = \varepsilon_0\varepsilon_{r2}$$

$$V_{ab} = -\left[\int_b^c \hat{a_R}\frac{Q}{4\pi\varepsilon_2 R^2}\cdot\hat{a_R}\,dR + \int_c^a \hat{a_R}\frac{Q}{4\pi\varepsilon_1 R^2}\cdot\hat{a_R}\,dR\right]$$

$$= \frac{Q}{4\pi\varepsilon_2}\left.\frac{1}{R}\right|_b^c + \frac{Q}{4\pi\varepsilon_1}\left.\frac{1}{R}\right|_c^a$$

$$= \frac{Q}{4\pi\varepsilon_2}\left(\frac{1}{c}-\frac{1}{b}\right) + \frac{Q}{4\pi\varepsilon_1}\left(\frac{1}{a}-\frac{1}{c}\right)$$

$$\therefore \quad C = \frac{Q}{V_{ab}} = \frac{1}{\frac{1}{4\pi\varepsilon_2}\left(\frac{1}{c}-\frac{1}{b}\right)+\frac{1}{4\pi\varepsilon_1}\left(\frac{1}{a}-\frac{1}{c}\right)}$$

$$= \frac{10^{-9}}{\frac{9}{10}\left(\frac{1}{0.2}-\frac{1}{0.3}\right)+\frac{9}{5}\left(\frac{1}{0.08}-\frac{1}{0.2}\right)}$$

$$= \frac{10^{-9}}{1.5+13.5} = \underline{6.67\times10^{-11}\ F}$$

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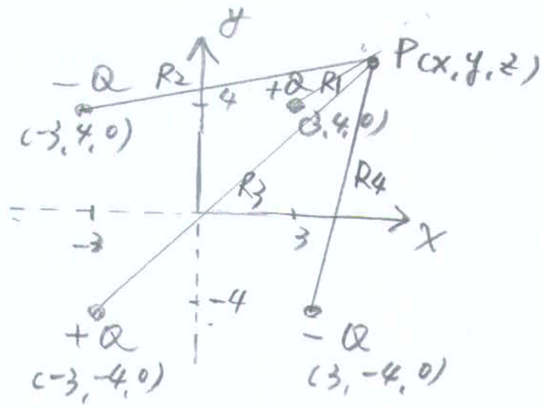
5. sol):

(a) 擺三顆鏡像电荷



$$R_1 = [(x-3)^2 + (y-4)^2 + z^2]^{1/2}$$
$$R_2 = [(x+3)^2 + (y-4)^2 + z^2]^{1/2}$$
$$R_3 = [(x+3)^2 + (y+4)^2 + z^2]^{1/2}$$
$$R_4 = [(x-3)^2 + (y+4)^2 + z^2]^{1/2}$$

At $P(3,5,0) \Rightarrow R_1 = 1,\ R_2 = 6.08,\ R_3 = 10.82,\ R_4 = 9$

$$V = \frac{Q}{4\pi\varepsilon_0 R} \Rightarrow V = \frac{100\times10^{-9}}{4\pi\times\frac{1}{36\pi}\times10^{-9}}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$$
$$= 900\left(1 - \frac{1}{6.08} + \frac{1}{10.82} - \frac{1}{9}\right)$$
$$= 735.2\ \text{V}$$

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$$\bar{E} = -\nabla V = -\hat{a_x}\frac{\partial V}{\partial x} - \hat{a_y}\frac{\partial V}{\partial y} - \hat{a_z}\frac{\partial V}{\partial z},\quad V = 900\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$$
$$= -900\left[-\frac{x-3}{R_1^3} + \frac{x+3}{R_2^3} - \frac{x+3}{R_3^3} + \frac{x-3}{R_4^3}\right]\hat{a_x}$$
$$-900\left[-\frac{y-4}{R_1^3} + \frac{y-4}{R_2^3} - \frac{y+4}{R_3^3} + \frac{y+4}{R_4^3}\right]\hat{a_y}$$
$$= -19.76\,\hat{a_x} + 891.28\,\hat{a_y}$$

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(b). 水平板 (y=0):

$$R_1 = [(x-3)^2 + 16]^{1/2} = R_4,\quad R_2 = [(x+3)^2 + 16]^{1/2} = R_3$$

$$\bar{E} = -900\left[\frac{8}{R_1^3} - \frac{8}{R_2^3}\right]\hat{a_y} = -900\left[\frac{8}{[(x-3)^2+16]^{3/2}} - \frac{8}{[(x+3)^2+16]^{3/2}}\right]\hat{a_y}$$

$$\rho_s = \varepsilon_0 E_n = -6.37\times10^{-8}\left[\frac{1}{[(x-3)^2+16]^{3/2}} - \frac{1}{[(x+3)^2+16]^{3/2}}\right]$$

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沿垂直板 $(x=0)$：

$$R_1 = \left[9+(y-4)^2\right]^{1/2} = R_2$$

$$R_3 = \left[9+(y+4)^2\right]^{1/2} = R_4$$

$$\bar{E} = -900\left[\frac{6}{\left[9+(y-4)^2\right]^{3/2}} - \frac{6}{\left[9+(y+4)^2\right]^{3/2}}\right]\hat{a}_x$$

$$\rho_s = \varepsilon_0 E_n = \underline{-4.77\times10^{-8}\left[\frac{1}{\left[9+(y-4)^2\right]^{3/2}} - \frac{1}{\left[9+(y+4)^2\right]^{3/2}}\right]}$$

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6.

Sol): 假設 $r=r_a$ 表面之电位 $V=V_0$，

而 $r=r_b$ 之电位 $V=0$，$\nabla^2 V=0$ $\because \frac{\partial^2 V}{\partial \phi^2}=0, \frac{\partial^2 V}{\partial z^2}=0$

$\therefore \frac{d}{dr}\left(r\frac{dV}{dr}\right)=0 \Rightarrow V=c_1 \ln r + c_2$

代入邊界條件：

$V_0 = c_1 \ln r_a + c_2$ —①

$0 = c_1 \ln r_b + c_2$ —②

①−② $V_0 = c_1(\ln r_a - \ln r_b) = c_1 \ln \frac{r_a}{r_b}$

$\therefore c_1 = \frac{V_0}{\ln \frac{r_a}{r_b}}$ 代回②

$c_2 = -c_1 \ln r_b = -\frac{V_0 \ln r_b}{\ln \frac{r_a}{r_b}}$

$\therefore V = c_1 \ln r + c_2 = \frac{V_0 \ln r}{\ln \frac{r_a}{r_b}} - \frac{V_0 \ln r_b}{\ln \frac{r_a}{r_b}}$

$= V_0 \frac{\ln \frac{r}{r_b}}{\ln \frac{r_a}{r_b}}$

$\bar{J} = \sigma \bar{E} = -\sigma \nabla V = -\hat{a_r}\sigma \frac{\partial V}{\partial r} = -\hat{a_r}\frac{\sigma V_0}{r \ln \frac{r_a}{r_b}}$

$I = \int_s \bar{J}\cdot d\bar{s} = \int_s \left(-\hat{a_r}\frac{\sigma V_0}{r \ln \frac{r_a}{r_b}}\right)\cdot \hat{a_r}\, r\,d\phi\, dz$

$= -\frac{\sigma V_0 2\pi L}{\ln \frac{r_a}{r_b}}$

$R = \frac{V_0}{I} = \frac{V_0}{-\frac{\sigma V_0 2\pi L}{\ln \frac{r_a}{r_b}}} = \frac{\ln \frac{r_a}{r_b}}{-\sigma 2\pi L}$

$= \frac{\ln \frac{r_b}{r_a}}{\sigma 2\pi L}$ ※