Solution of Electromagnetic (I) Midterm Ex. #1

1. 在空間中有兩點以圓柱座標表示為 A(3,0°,3)與 B(2,45°,$\sqrt{2}$)，請找出一個單位向量$\overrightarrow{U}$與$\overrightarrow{OA}$與$\overrightarrow{OB}$都垂直. 20%

Solution: A(3,0°,3)與 B(2,45°,$\sqrt{2}$)在直角座標為 A(3,0,3)與 B($\sqrt{2}$,$\sqrt{2}$,$\sqrt{2}$)

$$\overrightarrow{U}=\frac{\overrightarrow{OA}\times\overrightarrow{OB}}{\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|}$$

$$\overrightarrow{OA}\times\overrightarrow{OB}=\begin{vmatrix}\vec{a}_x & \vec{a}_y & \vec{a}_z\\ 3 & 0 & 3\\ \sqrt{2} & \sqrt{2} & \sqrt{2}\end{vmatrix}$$

$$=\vec{a}_x(-\sqrt{2}\times 3)+\vec{a}_y(\sqrt{2}\times 3-\sqrt{2}\times 3)+\vec{a}_z(\sqrt{2}\times 3)=\vec{a}_x(-3\sqrt{2})+\vec{a}_z(3\sqrt{2})$$

$$\overrightarrow{U}=\frac{\vec{a}_x(-3\sqrt{2})+\vec{a}_z(3\sqrt{2})}{\sqrt{(-3\sqrt{2})^2+(3\sqrt{2})^2}}=\vec{a}_x(-\frac{\sqrt{2}}{2})+\vec{a}_z(\frac{\sqrt{2}}{2})$$

Or $\overrightarrow{U}'=\dfrac{\overrightarrow{OB}\times\overrightarrow{OA}}{\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|}$=-$\overrightarrow{U}$

2. 空間中有兩點 A(3,0,-4)與 B(2,1,-1)，(a)請將$\overrightarrow{OB}$表示成平行與垂直$\overrightarrow{OA}$的兩個向量。

Solution:

$\overrightarrow{OB}=\overrightarrow{OA}_{\parallel}+\overrightarrow{OA}_{\perp}$ ， $\overrightarrow{OA}_{\parallel}=\overrightarrow{OB}\cdot\overrightarrow{OA}\times\overrightarrow{U}_{OA}=\dfrac{(6+4)}{5}(\vec{a}_x\dfrac{3}{5}-\vec{a}_z\dfrac{4}{5})=\vec{a}_x\dfrac{6}{5}-\vec{a}_z\dfrac{8}{5}$

$\overrightarrow{OA}_{\perp}=\overrightarrow{OB}$ - $\overrightarrow{OA}_{\parallel}$=($\vec{a}_x 2+\vec{a}_y-\vec{a}_z$)-($\vec{a}_x\dfrac{6}{5}-\vec{a}_z\dfrac{8}{5}$)=$\vec{a}_x\dfrac{4}{5}+\vec{a}_y+\vec{a}_z\dfrac{3}{5}$

3. 球座標中有一向量函數$\overrightarrow{A}=\vec{a}_R R^2\cos\theta\sin\phi-\vec{a}_\theta R^2\cos\theta$在圓柱座標中其向量函數為何?請問在點(4,30°,1)上其向量值為何? 20%

Solution:

$$\begin{bmatrix}A_r\\A_z\end{bmatrix}=\begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & -\sin\theta\end{bmatrix}\begin{bmatrix}A_R\\A_\theta\end{bmatrix}\Rightarrow\begin{bmatrix}A_r\\A_z\end{bmatrix}=\begin{bmatrix}R^2\cos\theta\sin\theta\sin\phi-R^2\cos^2\theta\\ R^2\cos^2\theta\sin\phi+R^2\cos\theta\sin\theta\end{bmatrix}$$

$$=\begin{bmatrix}A_r\\A_z\end{bmatrix}=\begin{bmatrix}rz\sin\phi-z^2\\ z^2\sin\phi+rz\end{bmatrix}(R\cos\theta=z,\ R\sin\theta=r)$$

$$\overrightarrow{A}=\vec{a}_r(4\times\frac{1}{2}-1)+\vec{a}_z(1^2\times\frac{1}{2}+4)=\vec{a}_r+\vec{a}_z\frac{9}{2}$$

4. 若有函數在圓柱座標為$F(r,\phi,z)=r^2z\cos\phi$請問它的 gradient 函數為何?在直角座標中點(0,2,1)其梯度向量值為何? 20%

Solution:

$$\vec{G}=-\nabla F\Rightarrow\vec{G}=\vec{a}_r\frac{r^2z\cos\phi}{\partial r}+\vec{a}_\phi\frac{r^2z\cos\phi}{r\partial\phi}+\vec{a}_z\frac{r^2z\cos\phi}{\partial z}$$

$$\vec{G}=\vec{a}_r 2rz\cos\phi-\vec{a}_\phi rz\sin\phi+\vec{a}_z r^2\cos\phi$$

(0,2,1)在圓柱座標為(2,90°,1)帶入上式可得$\vec{G}(2,90°,1)=-\vec{a}_\phi 2$

5. 若有立方體的各角的座標為(0,0,0),(2,0,0),(0,2,0),(2,2,0),(0,0,2),(2,0,2),(0,2,2)及(2,2,2),且其中有一向量$\vec{P}=\vec{a}_x x+\vec{a}_y y+\vec{a}_z z$；請証明此向量在此空間中可滿足 Divergence theorem。

20%

Solution:

$\vec{P}=\vec{a}_x x+\vec{a}_y y+\vec{a}_z z$

$\rho_{ps}=\vec{a}_n\cdot\vec{P}$

$\rho_{ps1}=\vec{a}_x\cdot\vec{P}\Big|_{x=2}=2(C/m^2),\rho_{ps2}=\vec{a}_y\cdot\vec{P}\Big|_{y=2}=2(C/m^2),\rho_{ps3}=\vec{a}_z\cdot\vec{P}\Big|_{z=2}=2(C/m^2),$

$\rho_{ps4}=-\vec{a}_x\cdot\vec{P}\Big|_{x=0}=0,\rho_{ps5}=-\vec{a}_y\cdot\vec{P}\Big|_{y=0}=0,\rho_{ps6}=-\vec{a}_z\cdot\vec{P}\Big|_{z=0}=0$

$\oint\rho_{ps}ds=\int_0^2\int_0^2 2dydz+\int_0^2\int_0^2 2dxdz+\int_0^2\int_0^2 2dxdy=24$

$\nabla\cdot\vec{P}=(\frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial x}y)=3$

$\int_v\rho_{pv}dv=\int_0^2\int_0^2\int_0^2 3dxdydz=3xyz\Big|_{x=0}^{2}\Big|_{y=0}^{2}\Big|_{z=0}^{2}=24$