DDA5002 Optimization
Lecture 2 Linear Program Modeling

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Announcements

- Homework 1 is posted and due next Thursday (September 18) at 11:59pm.
- We have a make-up class on September 28 (Sunday), same as the class time 6 – 9pm.
- Midterm is on November 2 (Sunday) from 10am to 12pm.

Wechat Group

群聊: DDA5002 Fall25 L02

该二维码7天内(9月17日前)有效，重新进入将更新

Outline

Outline

Optimization Generic Formulation

Mathematically, an optimization problem is usually represented as:

Generic Formulation

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in X \end{aligned}$$

- x: decision variable or optimization variable
- $f(\cdot)$: objective function
- X: feasible region/set (constraints)
- $x \in X$: a feasible solution (satisfies all constraints)
- Sometimes, we express the problem using the abstract format: $\min\{f(x) : \; x \in X\}$.

Mathematical Formulation

Mathematical Formulation

$$
\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) \\
\text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0, \quad \forall i = 1, 2, \ldots, m \\
& h_j(\boldsymbol{x}) = 0, \quad \forall j = 1, 2, \ldots, p
\end{aligned}
$$

- Optimization variables: $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$
- Objective function: $f\colon \mathbb{R}^n \to \mathbb{R}$
- Inequality constraints functions: $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \ldots, m$
- Equality constraints functions: $h_j\colon \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \ldots, p$
- Sometimes, we write feasible region as $X = \{\boldsymbol{x} : \boldsymbol{g}(\boldsymbol{x}) \leq 0, \ \boldsymbol{h}(\boldsymbol{x}) = 0\}$, where

$$
\boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} g_1(\boldsymbol{x}) \\ \vdots \\ g_m(\boldsymbol{x}) \end{bmatrix} \quad \text{and} \quad \boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} h_1(\boldsymbol{x}) \\ \vdots \\ h_p(\boldsymbol{x}) \end{bmatrix}
$$

Four Outcomes of Optimization Problem

Mathematical Formulation

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in X \end{aligned}$$

1. Infeasible: $X = \phi$ $\{x : x \geq 1, x \leq 0\}$
2. Unbounded: $\exists \{x^i\} \in X$, s.t. $f(x^i) \to -\infty$ min. e^x
3. Feasible and bounded but the ~~minimizer~~ optimal solution is not achieved (attained)
4. An optimal solution x^* exists

Linear Program

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^\top \boldsymbol{x} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
\text{s.t.} \quad & \boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i, \quad i = 1, 2, .., m
\end{aligned}
$$

- Linear objective function
- n continuous decision variables
- m linear constraints
- Optimize a linear function over a polyhedron
- Matrix form

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}
\end{aligned}
$$

Outline

Scheduling

- Hospital wants to make weekly nightshift for its nurses
- D_j: demand for nurses on day j, $j = 1, .., 7$
- Every nurse works 5 days in a row

Goal: hire minimum number of nurses to satisfy all demands

X_j: # nurses working on day j

$$X_j \geq D_j, \quad \forall j = 1, \cdots, 7$$

Total # of nurses = ?

e.g. $\left.\begin{array}{l} X_1 = 20 \\ X_2 = 30 \end{array}\right\}$

NOT a good decision variable

X_j : # nurses Start working on day j

$$\min. \sum_{j=1}^{7} X_j$$

$$\text{s.t.} \quad \begin{array}{rl}
X_1 + X_4 + X_5 + X_6 + X_7 & \geq d_1 \\
X_1 + X_2 + X_5 + X_6 + X_7 & \geq d_2 \\
X_1 + X_2 + X_3 + X_6 + X_7 & \geq d_3 \\
X_1 + X_2 + X_3 + X_4 + X_7 & \geq d_4 \\
X_1 + X_2 + X_3 + X_4 + X_5 & \geq d_5 \\
X_2 + X_3 + X_4 + X_5 + X_6 & \geq d_6 \\
X_3 + X_4 + X_5 + X_6 + X_7 & \geq d_7
\end{array}$$

$$X_j \geq 0, \quad \forall j = 1, \cdots, 7$$

Airline Revenue Management

- Before deregulation, carriers were only allowed to fly certain routes (e.g., Northwest, Eastern, Southwest). Fares were determined by the Civil Aeronautics Board (CAB) based on mileage and other costs (CAB no longer exists).
- After deregulation (1978), any carrier can fly anywhere, fares are determined by the carrier and market dynamics.
- Economics of the Airline Industry:
  - Huge sunk and fixed costs. Very low variable costs per passenger (e.g., $10 or less).
  - Highly competitive market environment.
  - Near-perfect information and negligible cost of information.
  - Highly perishable inventory (e.g., unsold seats lose value after departure).
  - Result: Airlines implement multiple fare structures. Dynamic pricing strategies to maximize revenue.

Ticketing Problem

- n routes
- 2 classes: Q class, Y class
- Revenue: r_i^Q, r_i^Y on route $i = 1, ..n$
- Capacities: C_i on route $i = 1, ..n$
- Expected demand: D_i^Q, D_i^Y on route $i = 1, ..n$

Goal: find open tickets in each class on each route with maximized revenue

Q_i: # of Q-class tickets open on route i

Y_i: # of Y-class tickets —— ——

$$\max. \sum_{i=1}^{n} r_i^Q Q_i + r_i^Y Y_i$$

$>$

$$\text{s.t.} \quad Q_i + Y_i \leq C_i, \quad \forall i=1,\cdots,n$$

$$0 \leq Q_i \leq D_i^Q, \quad \forall i=1,\cdots,n$$

$$0 \leq Y_i \leq D_i^Y, \quad \forall i=1,\cdots,n$$

Capacity Expansion

- D_t: forecast demand for electricity at year t
- E_t: existing capacity (in oil) available at t
- c_t: cost to construct 1 MW power using coal capacity
- n_t: cost to construct 1MW using nuclear capacity
- No more than 20% nuclear
- Coal plants last 20 years
- Nuclear plants last 15 years
- Consider a T-year time horizon

Goal: find the optimal coal and nuclear capacity for each year with lowest total costs

X_t: # MW Coal capacity to build in Year t
Y_t: # MW Nuclear capacity to build in Year t
W_t: # MW Coal capacity available in Year t
Z_t: # MW Nuclear capacity available in Year t

$$\min. \sum_{t=1}^{T} C_t X_t + n_t Y_t$$

$$\text{s.t. } \underline{E_t} + \underline{W_t} + \underline{Z_t} \ge D_t, \quad \forall t = 1, \dots, T$$

$$Z_t \le 20\% \, (E_t + W_t + Z_t), \quad \forall t = 1, \dots, T$$

$$- \; W_t = \sum_{s=\max\{1, t-19\}}^{t} X_s, \quad \forall t = 1, \dots, T$$

$$- \; Z_t = \sum_{s=\max\{1, t-14\}}^{t} Y_s, \quad \forall t = 1, \dots, T$$

Year	total coal available
1	X_1
2	$X_1 + X_2$
3	$X_1 + X_2 + X_3$
⋮	
20	$X_1 + X_2 + \dots + X_{20}$
21	$X_2 + X_3 + \dots + X_{21}$
22	$X_3 + X_4 + \dots + X_{22}$

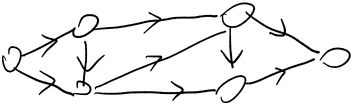
$$t: \sum_{s=\max\{1, t-19\}}^{t} X_s$$

Year	total nuclear available
1	Y_1
2	$Y_1 + Y_2$
⋮	
15	$Y_1 + \dots + Y_{15}$
16	$Y_2 + \dots + Y_{16}$
17	$Y_3 + \dots + Y_{17}$

$$t: \sum_{s=\max\{1, t-14\}}^{t} Y_s$$

Graph

Network

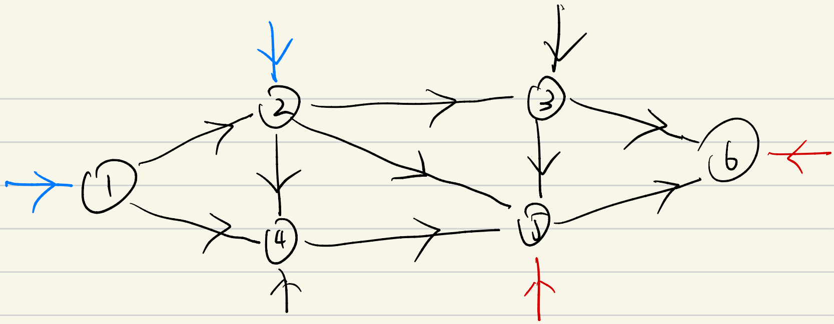


- Undirected Graph $G = \{N, E\}$: a set of N nodes and a set of E undirected edges (unordered pair (i, j))
- Directed Graph $G = \{N, A\}$: a set of N nodes and a set of A directed arc (ordered pair (i, j))

$\begin{cases} i \to j \\ j \to i \end{cases}$

Min-cost Network Flow Problem

- A directed network with at least one supply node ($b_i > 0$), at least one demand node ($b_i < 0$), and transshipment nodes ($b_i = 0$).
- Flow through an arc (ij) is allowed only in the direction indicated by the arrowhead, where the maximum amount of flow is given by the capacity of that arc (u_{ij}).
- The cost of the flow through each arc is proportional to the amount of that flow, where the cost per unit flow is known (c_{ij}).
- The objective is to minimize the total cost of sending the available supply through the network to satisfy the given demand.



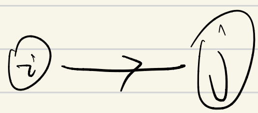
$\downarrow b_i$ - supply

$$b_i \begin{cases} >0 & \text{supply} \\ <0 & \text{demand} \\ =0 & \text{transshipment} \end{cases}$$

\$ c_{ij} / item

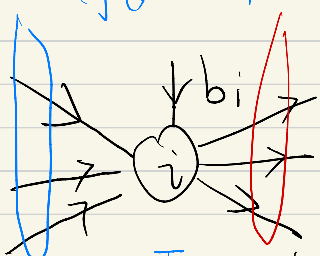
$i \longrightarrow j$

$\le u_{ij}$

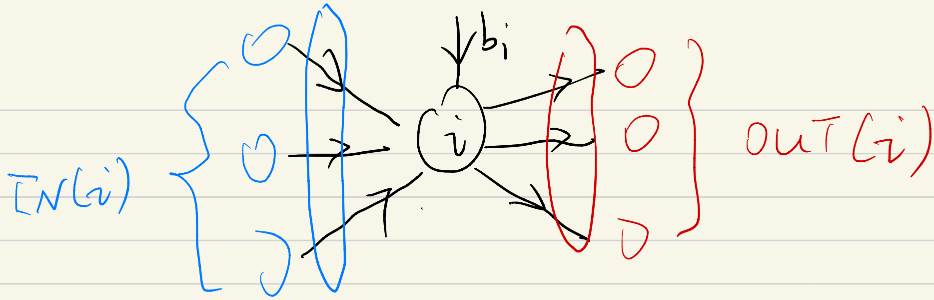


'Balance' Network : $\sum_{i \in N} b_i = 0$

X_{ij} : $\underbrace{\text{\# items delivered on arc } (i,j)}_{\text{flow on arc } (i,j)}$



Balance constraint IN + b_i = OUT



$$IN + b_i = OUT \quad \forall i$$

$$\min. \sum_{ij \in A} c_{ij} x_{ij}$$

$$\text{s.t.} \sum_{j \in IN(i)} x_{ji} + b_i = \sum_{j \in OUT(i)} x_{ij}, \quad \forall i \in N$$

$$0 \le x_{ij} \le u_{ij}, \quad \forall ij \in A$$

$$\cancel{x_{ij} \in \mathbb{Z}, \quad \forall ij \in A}$$

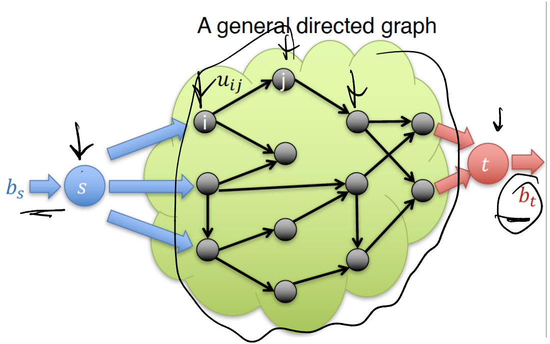
NO NEED

Take-home Exercise

What if we have an unbalanced network?

$\sum_{i \in N} b_i \neq 0$, mere supply than demand.

Maximum Flow Problem



- The capacity of an edge (u_{ij}) is the maximum amount of flow that can pass through an edge.
- The sum of the flows entering a node must equal the sum of the flows exiting that node, except for the source (s) and the sink (t).
- Question: What's the largest supply b_s can be transported from source to sink through the network with limited arc capacity?

Decision variables:

x_{ij} : flow on arc (i,j)

b_i : supply for node i, $\forall i \in N$

$$\max \quad b_s$$

$$\text{s.t.} \sum_{j \in IN(i)} x_{ji} + b_i = \sum_{j \in OUT(i)} x_{ij}, \quad \forall i \in N$$

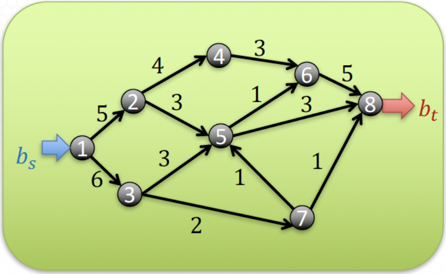
$$0 \le x_{ij} \le u_{ij}, \quad \forall ij \in A$$

$$b_i = \begin{cases} b_s, & i = s \\ -b_s, & i = t \\ 0 & i \neq s,t \end{cases}$$

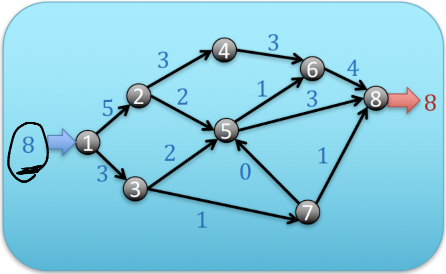
constraints

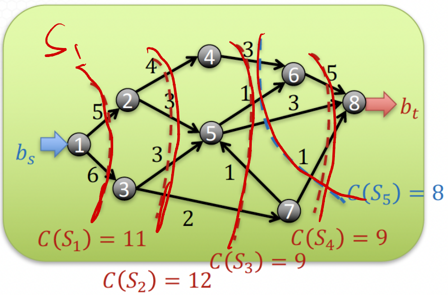
Concrete Example

Capacity constrained network



Max flow solution





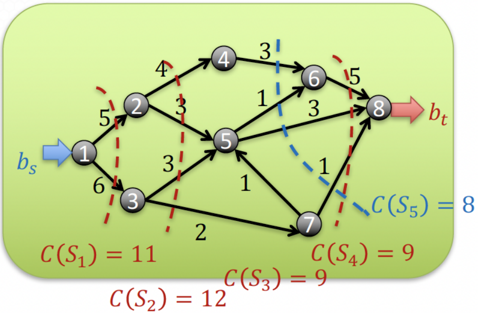
A s-t **cut** S is a subset of nodes
Such that $s \in S$ and $t \notin S$

So a **cut** S is a separation of
Source node from target node

Capacity of a cut S is the total capacity
of arcs that cross from S to its complement
Denoted as $C(S) := \sum_{(i,j)\in A, i\in S, j\notin S} c_{ij}$

A million-dollar question:
Can you find a cut with minimum capacity?

Minimum cut $= C(S_5) = 8 =$ Max Flow



Is this a Coincidence?

Not at all! There is a deep theory behind it – LP duality.

Max-flow and min-cut are two LPs dual to each other.

Intuitively, it makes sense too.

Outline

Linearize Nonlinear Objective

General principle:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

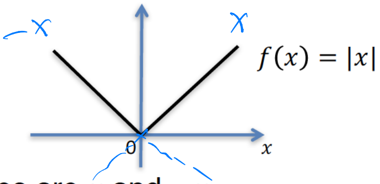
$$\Updownarrow$$

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & f(x) \leq z \\ & x \in X \end{aligned}$$

Absolute Value Function

|x| is not linear

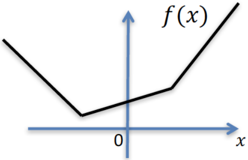
- An absolute value function $f(x) = |x|$ has two linear pieces



- The two linear pieces are x and $-x$.
- $|x|$ can be written as $|x| = \mathbf{max}\{\boldsymbol{x}, -\boldsymbol{x}\}$
- This is called a **piecewise linear function.**
- Clearly, $|x|$ is also a convex function.
- Therefore, $|x|$ is a **convex piecewise linear function**.

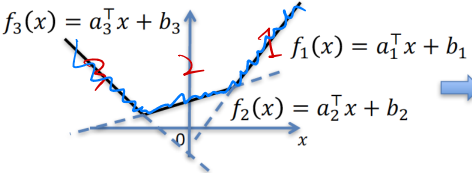
Convex Piecewise Linear Function

- A convex piecewise linear function looks like:



Any convex PWL $f(x)$ can be written as max of a finite number of linear functions

$$f(x) = \max\{a_1^\top x + b_1, \ldots, a_m^\top x + b_m\}$$



$$f(x) = \max\big(f_1(x), f_2(x), f_3(x)\big)$$
$$= \max(a_1^\top x + b_1, a_2^\top x + b_2, a_3^\top x + b_3)$$

$$\min_{x} \ \max\{a_1^T x + b_1, \cdots, a_m^T x + b_m\}$$

$\Updownarrow$ General Principle

$$\min_{x,z} \ z$$

$$\text{s.t.} \ \max\{a_1^T x + b_1, \cdots, a_m^T x + b_m\} \le z$$

$\Updownarrow$

LP

$$\min \ z$$

$$\text{s.t.} \ z \ge a_i^T x + b_i, \quad \forall i = 1, \cdots, m$$

Minimizing Convex PWL as LP

- Minimization of a convex PWL function:
  - $\min_{x \in X} f(x) = \max\{a_1^\top x + b_1, \ldots, a_m^\top x + b_m\}$

  can be reformulated by introducing a new variable and putting objective into constraint as follows:

- An equivalent reformulation:

  - $$\min_{x,z} z \quad \text{s.t. } f(x) \le z, \quad x \in X$$

  - $$\min_{x \in X, z} z \quad \text{s.t. } \max\{a_1^\top x + b_1, \ldots, a_m^\top x + b_m\} \le z$$

  - $$\min_{x \in X, z} z \quad \text{s.t. } a_i^\top x + b_i \le z \ \ \forall i = 1, \ldots, m.$$

This is an LP!

Linearize Absolute Value Function

- Absolute value function $|x| \leq z$ can be reformulated as
  - $\max\{x, -x\} \leq z$
  - $x \leq z,\ \ -x \leq z$
  - Or equivalently, $-z \leq x \leq z$
- If we have $\left|a^\top x - b\right| \leq z$, then we can reformulate as
  - $-z \leq a^\top x - b \leq z$

Linearize Absolute Value Function Example

$$\begin{aligned} \min \quad & |x_1 - x_2| \\ \text{s.t} \quad & |2x_1 - 3x_2| \leq 5 \end{aligned}$$

$$\begin{aligned} \min.\quad & Z \\ \text{s.t.}\quad & |X_1 - X_2| \leq Z \\ & |2X_1 - 3X_2| \leq 5 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \min.\quad & Z \\ \text{s.t.}\quad & -Z \leq X_1 - X_2 \leq Z \\ & -5 \leq 2X_1 - 3X_2 \leq 5 \end{aligned}$$

A Modeling Tool

Consider two optimization problems:

$$\min_{x} \quad \sum_{i=1}^{m} f_i(x) \quad \text{s.t. } x \in X \tag{1}$$

$$\Updownarrow$$

$$\min_{x,t} \quad \sum_{i=1}^{m} t_i \quad \text{s.t. } x \in X, \ \underline{f_i(x) \leq t_i}, \ \ \forall i \tag{2}$$

Lemma

The problems (1) and (2) are equivalent in the following way:

- If x^* is an optimal solution of (1), then $(x^*, \underbrace{f_1(x^*), \ldots, f_m(x^*)}_{t^*})$ is an optimal solution of (2).
- If (x^*, t^*) is an optimal solution of (2), then x^* is an optimal solution of (1).
- Both problems have the same optimal value.

Dealing with Absolute Values

Problems with absolute values might be handled as well by LP.

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{i=1}^n |x_i| \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

$\Downarrow$

$$\begin{aligned} \min. \quad & \sum_{i=1}^n y_i \\ \text{s.t.} \quad & Ax=b \\ & |x_i| \le y_i, \forall i \end{aligned}$$

This can be equivalently written as (why?)

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{i=1}^n y_i \\ \text{s.t.} \quad & y_i \geq x_i \\ & y_i \geq -x_i \quad \forall i \\ & A\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

How About Maximizing Absolute Values?

Consider a similar problem:

$$\begin{aligned} \text{maximize}_x \quad & \sum_{i=1}^{n} |x_i| \\ \text{s.t.} \quad & Ax = b. \end{aligned}$$

Can we use the similar idea and transform it into?

$$\begin{aligned} \text{maximize}_{x,t} \quad & \sum_{i=1}^{n} t_i \\ \text{s.t.} \quad & t_i \geq x_i, \\ & t_i \geq -x_i, \\ & Ax = b. \end{aligned}$$

X WRONG

- **Answer: No.** There is some intrinsic property that prevents us from formulating it as an LP (**non-convexity**).

- max. $|a^Tx+b|$
- max. $\sum_i |a_i^Tx+b_i|$
- $|a^Tx+b| \geq t$

NOT ABLE TO TRANSFORM INTO LP

$$\begin{cases} x \geq b \\ x \leq a \end{cases}$$

PUT INTO OPT?

$$|x| \geq 1 \Rightarrow \begin{cases} x \geq 1 \\ x \leq -1 \end{cases} \Rightarrow \text{infeasible}$$

How About Maximin? Air Traffic Control

An air traffic controller needs to control the landing time of n aircrafts

- Flights must land in the order $1, \ldots, n$
- Flight j must land in time interval $[a_j, b_j]$
- The objective is to maximize the minimum *separation time*, which is the interval between two landings

An Optimization Formulation

Decision variable

- Let t_j be the landing time of flight j

Optimization problem:

min separation time

$$
\begin{array}{lll}
\max & \min_{j=1,\ldots,n-1}\{t_{j+1}-t_j\} & \\
\text{s.t.} & a_j \le t_j \le b_j, & j=1,\ldots,n \\
& t_j \le t_{j+1}, & j=1,\ldots,n-1
\end{array}
$$

nonlinear

The objective function is not a linear function. We call it a maximin objective.

LP Formulation

Define

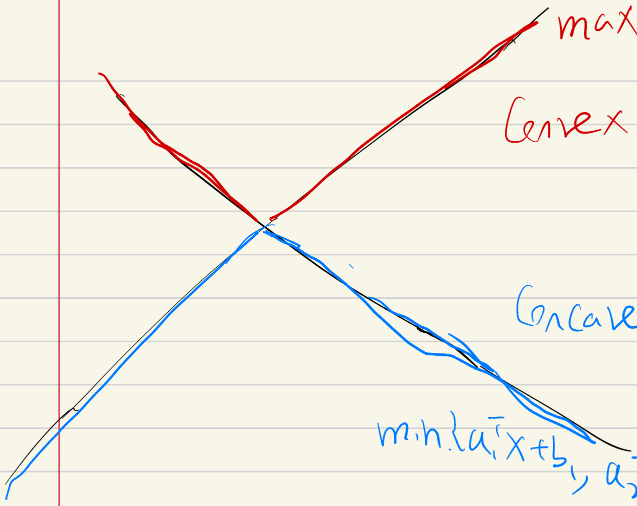
$$\Delta = \min_{j=1,\dots,n-1} \{t_{j+1} - t_j\}$$

Therefore, $t_{j+1} - t_j \geq \Delta$, $\forall j$.

Write an LP:

$$\begin{array}{lll} \text{maximize} & \Delta & \\ \text{subject to} & t_{j+1} - t_j - \Delta \geq 0, & j = 1, \dots, n-1 \\ & a_j \leq t_j \leq b_j, & j = 1, \dots, n \\ & t_j \leq t_{j+1}, & j = 1, \dots, n-1 \end{array}$$

At optimal, Δ must equal the minimal separation (since we try to maximize Δ).



$\max\{a_1^T x + b_1, a_2^T x + b_2\}$

Convex Piecewise-linear

Concave Piecewise-linear

$\min\{a_1^T x + b_1, a_2^T x + b_2\}$

Maximin Reformulation

$$\begin{aligned} \max \quad & f(x) = \min\{a_1^T x + b_1, \ldots, a_m^T x + b_m\} \\ & \Updownarrow \qquad\qquad z = \min\{a_1^T x + b_1, \cdots, a_m^T x + b_m\} \\ \max \quad & z \\ \text{s.t.} \quad & z \le a_i^T x + b_i \ \forall i = 1, \ldots, m \end{aligned}$$

Outline

Fractional Programming

$$
\begin{aligned}
\min_{x} \quad & \frac{c^\top x + d}{e^\top x + a} \\
\text{s.t.} \quad & Ax \leq b \\
& e^\top x + a \geq 0
\end{aligned}
$$

- We assume that $e^\top x + a > 0$ for any x satisfying $Ax \leq b$.

Frame Title

Define:

$$y = \frac{x}{e^\top x + a}, \qquad z = \frac{1}{e^\top x + a}.$$

We can write the fractional Programming as

$$\begin{aligned} \text{minimize}_{\ y,z} \quad & c^\top y + dz \\ \text{s.t.} \quad & Ay - bz \le 0, \\ & e^\top y + az = 1, \\ & z \ge 0 \end{aligned}$$

- This is a LP.
- Why are they equivalent?
  See the textbook 'Convex Optimization' for details (page 151).

Outline

Linear Program Standard Form

$$
\begin{aligned}
\min \quad & c^\mathsf{T} x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

- $x \in \mathbb{R}^n$, i.e. there are n variables
- $A \in \mathbb{R}^{m \times n}$, i.e. there are m equality constraints
- We always assume all the m equality constraints are linearly independent, otherwise we can remove all redundant linearly dependent constraints.
- Always assume $n > m$, i.e. more variables than constraints

Standard Form LP

$$
\begin{aligned}
\min \quad & c^\intercal x \quad \text{[Minimization]} \\
\text{s.t.} \quad & Ax = b \quad \text{[Only equality constraints]} \\
& x \geq 0 \quad \text{[All variables nonnegative]}
\end{aligned}
$$

$\max \boldsymbol{c}^\top \boldsymbol{x}$	$\Leftrightarrow$	$-\min(-\boldsymbol{c}^\top \boldsymbol{x})$
$\boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i$	$\Leftrightarrow$	$\boldsymbol{a}_i^\top \boldsymbol{x} - s_i = b_i, s_i \geq 0$
$\boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i$	$\Leftrightarrow$	$\boldsymbol{a}_i^\top \boldsymbol{x} + s_i = b_i, s_i \geq 0$
$x_j \leq 0$	$\Leftrightarrow$	$-x_j \geq 0$
x_j free	$\Leftrightarrow$	$x_j = x_j^+ - x_j^-, x_j^+ \geq 0, x_j^- \geq 0$

Example I

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
& & 2x_2 & \leq 200 \\
& x_1 & +x_2 & \leq 150 \\
& x_1, & x_2 & \geq 0
\end{array}
$$

Standard form

$$
\begin{array}{rrrrrrl}
\text{minimize} & -x_1 & -2x_2 & & & & \\
\text{subject to} & x_1 & & +s_1 & & & = 100 \\
& & 2x_2 & & +s_2 & & = 200 \\
& x_1 & +x_2 & & & +s_3 & = 150 \\
& x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0
\end{array}
$$

Example II

$$\begin{aligned} \text{maximize} \quad & x_1 + 4x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + x_3 \leq 4, \\ & x_1 - x_3 \geq 1, \\ & x_1,\, x_2 \geq 0 \end{aligned}$$

Standard form

$$\begin{aligned} \text{minimize} \quad & -x_1 - 4x_2 - x_4 + x_5 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + x_4 - x_5 + x_6 = 4, \\ & x_1 - x_4 + x_5 - x_7 = 1, \\ & x_1,\, x_2,\, x_4,\, x_5,\, x_6,\, x_7 \geq 0 \end{aligned}$$

Standard Form LP

- Standard form is mainly used for analysis purposes. We don't need to write a problem in standard form unless necessary. Usually just write in a way that is easy to understand.
- However, being able to transform an LP into the standard form is an important skill. It is helpful for analyzing LP problems as well as using some software to solve it.

Outline

Starting Point: Graphical Solutions to LP

It is very helpful to study a small LP from a graphical point of view.

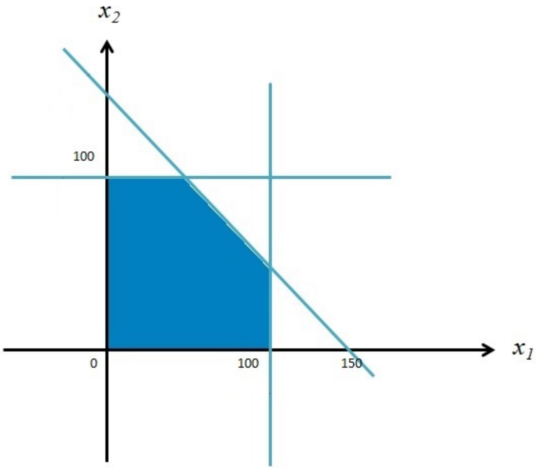
Recall the production problem:

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

How can we solve this from a graph?

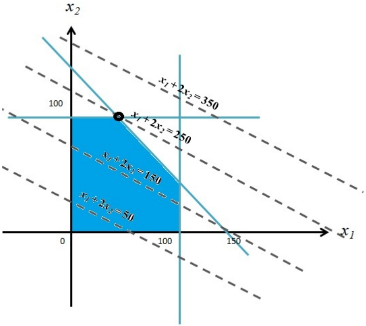
Solve LP from Graph

We first draw the feasible region.



To Maximize $x_1 + 2x_2$...

Then we draw the function $x_1 + 2x_2 = c$ for different values of c.



- The optimal solution is the highest one among these lines that touch the feasible region
- The coordinates: $(50, 100)$. Objective value: 250
- What if the objective changes to $\max x_1 + x_2$?

Some Observations

- The feasible region of LP is a polyhedron.
- **The optimal solution must be at a corner of the feasible region.**
- Some constraints are *active* at the optimal solution ($x_2 \leq 100$, $x_1 + x_2 \leq 150$), some are not ($x_1 < 100$).

Next we will formalize these observations and study algorithms for solving LPs that can

- Guarantee to find the optimal solution
- Run within a certain (reasonable) amount of time