DDA5002 Optimization
Lecture 6 Duality Theory

Yuang Chen

School of Data Science
The Chinese University of Hong Kong, Shenzhen

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Agenda for October

- **Oct 15:** LP Duality Theory and Sensitivity Analysis
- **Oct 22:** Sensitivity Analysis and Integer Programming (IP) Modeling
- **Oct 29:** LP Relaxation, Total Unimodularity, and Midterm Review
- **Nov 2 (Sunday, 10am–12pm): Midterm Exam**
  Covers everything up to IP Modeling
- **Homework 3:** Duality Theory + IP Modeling — due **Oct 30**

Outline

1. Optimality Certificate and Optimality Gap
2. LP Duality
3. Weak and Strong Duality Theorems
4. Table of Possibles and Impossibles
5. Complementary Slackness
6. Dual Applications
7. Local Sensitivity Analysis
8. Global Sensitivity

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1. Optimality Certificate and Optimality Gap
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Optimality Certificate

- Optimization algorithms typically search for an optimal solution by moving from one solution to another.
- An important question is how to know when an optimal solution or a "near-optimal" solution has been found and the search can stop.
- An **optimality certificate** or a **stopping condition** is an easily checkable condition such that, if the current solution satisfies this condition, it is **guaranteed** to be optimal or near-optimal.
- Then the algorithm can check the condition every time it finds a new solution and stop when it is satisfied.

Example

lower bound

- Consider the following optimization problem:

$$\min_{x,y} \left\{ e^{\left(x^2-4x+y^3+4\right)^2} \right\} \geq 1$$

- Suppose we have found a solution $x = 1.0$, $y = 0.2$. Is it optimal?
- The objective value of this solution is around 1.002.
- The least possible value the objective function can take is 1.
- We do not know if this solution is optimal or not.
- However, we know that it is off by at most 0.2% from being optimal.

$x^* = 3$ $\quad$ $x^* = 3.04$

Example (contd.)

- Consider the same problem:

$$\min_{x,y} \left\{ e^{\left(x^2-4x+y^3+4\right)^2} \right\} \geq 1$$

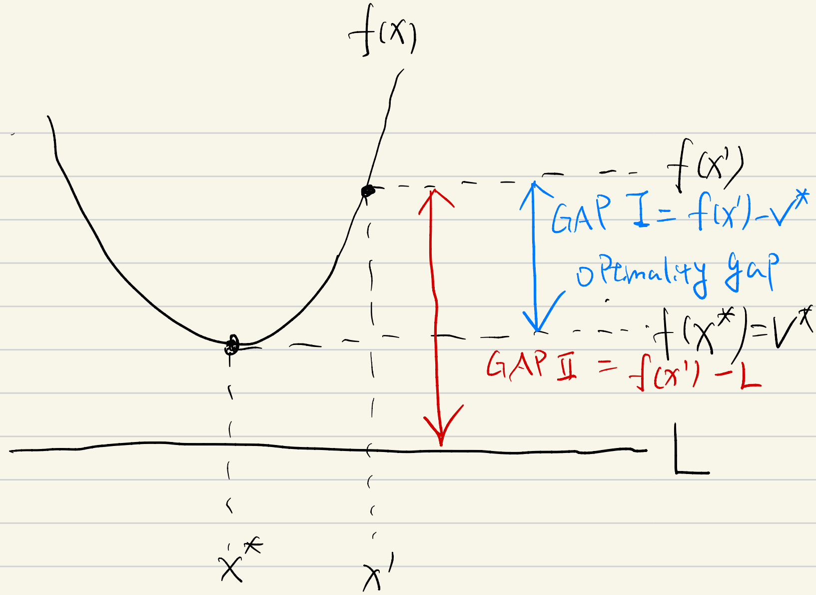
- Is the solution $\underbrace{x = 2, y = 0}$ optimal?
- Note that the objective value of this solution is 1
- Therefore, this solution $(x = 2, y = 0)$ must be optimal!

Lower Bound

$$\min_{x \in X} f(x) \qquad f(x) \geq \underline{L}, \ \forall x \in X$$

lower bound

- In the previous example, we knew (a priori) that the objective value of any solution to the problem cannot be lower than 1.
- Thus, we could compare the objective of any given solution to this lower bound.
- If the solution has an objective close to this lower bound, then we know we found a (near)-optimal solution.
- Therefore, the lower bound of 1 is an easily checkable certificate.



$$0 \le \text{GAP I} \le \text{GAP II}$$

$$\underset{\text{small}}{\downarrow} \quad \longleftarrow \quad \underset{\text{small}}{\downarrow}$$

If GAP II is small enough $(\le \varepsilon)$
then GAP I is small enough $(\le \varepsilon)$.

$x^2 \ge -1$

prefer larger lower bound

v^*
$=$
best lower bound
$\downarrow$
largest

Optimality Gap

- Suppose we have a feasible solution x' to an minimization optimization problem with an objective value of $f(x')$.
- Suppose the optimal objective value of the problem is v^*.
- Then the (absolute) optimality gap of the solution x' is:

$$\text{gap}(x') = f(x') - v^* \geq 0.$$

- But we do not know v^*. Suppose we know a lower bound $L \leq v^*$, then the following holds:

$$L \leq v^* \leq f(x')$$

$$0 \leq \text{gap}(x') = f(x') - v^* \leq f(x') - L.$$

- Thus, a lower bound allows us to get an upper bound on the optimality gap and helps us know how our solution is.
- Question: how to find a (good/best) lower bound for a general optimization?

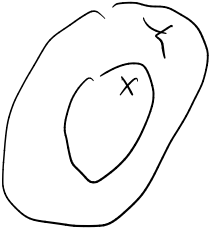
Relaxation

origin
Primal V_P^* $\qquad\qquad\qquad\qquad$ V_Q^*

$$\underbrace{(P)}_{}:\ \min_x\{f(x) : x \in X\} = V_P^* \qquad\qquad (Q) : \min_x\{g(x) : x \in Y\} = V_Q^*$$

Problem (Q) is a relaxation of (P) if:

- $X \subseteq Y$
- $f(x) \geq g(x) \quad \forall x \in X$



$$V_P^* \geq V_Q^* \Leftarrow \text{lower bound}$$

Relaxation and Lower bound

- The relaxation of an optimization problem should be easier to solve.
- The optimal value of the relaxation provides a lower bound on the original problem.
- If the relaxation is infeasible, then clearly the original problem is also infeasible. $Y = \emptyset \Rightarrow X = \emptyset$
- Suppose only the constraints are relaxed. If a solution to the relaxation is feasible for the original problem, then it must be an optimal solution to the original problem.
- The most famous/commonly used relaxation method is called **Lagrangian relaxation**, we will learn it after midterm.

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Motivation of Dual

$$
\begin{aligned}
\text{(P)} \quad \min \quad & c^T x \\
\text{s.t.} \quad & Ax \leq b
\end{aligned}
$$

- Any feasible solution of a minimization LP (P) provides an upper bound. The dual problem (D) of (P) is to find a lower bound (the best lower bond) to the optimal cost of (P).
- This lower bound is useful, because if the lower bound is very close to an upper bound, we have a good estimate of the true optimal.
- However, to get a lower bound, we need to modify the original LP. In particular, we need to **relax** the problem.

Lagrangian Relaxation and Lagrangian Dual

The process of formulating the dual that will give the best lower bound to the optimal cost of (P) is called **relaxation**. Relax a minimization problem (P) involves three principal steps:

- Relax the objective function: by constructing a new objective function that is always smaller or equal to the original objective function of (P) over the feasible region of (P).
- Relax the feasible region of (P): by enlarging the feasible region of the original problem (P).
- Maximize the lower bound over Lagrangian multipliers so that we get the best lower bound.

This final maximization problem is called the Lagrangian dual problem, or the dual problem for short.

How to enlarge the feasible region?

A. add constraint(s)

B. delete constraint(s)

Primal LP

$$
\begin{aligned}
Z_p = \min \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \forall i = 1, \ldots, m_1 \\
& \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \forall i = m_1 + 1, \ldots, m_2 \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall i = m_2 + 1, \ldots, m \\
& x_j \geq 0 \quad \forall j = 1, \ldots, n_1 \\
& x_j \text{ free} \quad \forall j = n_1 + 1, \ldots, n_2 \\
& x_j \leq 0 \quad \forall j = n_2 + 1, \ldots, n
\end{aligned}
$$

delete

Step 1: Relax the objective function

By relaxing the objective function, we mean to formulate a new objective function that is smaller than or equal to the original primal objective function.

$$\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \overbrace{y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right)}^{\leq 0} \leq \sum_j c_j x_j$$

where y_i's are **Lagrangian multipliers**. Since we require the new objective is smaller, we need $y_i \cdot (b_i - \sum_{j=1}^{n} a_{ij} x_j) \leq 0$.

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \rightarrow y_i \geq 0$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \rightarrow y_i \text{ free}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \rightarrow y_i \leq 0$$

$\rightarrow$ designated y_i's

Step 2: Relax the primal constraints

We want to enlarge the feasible region of an optimization problem by removing constraints. The relaxed problem becomes

$\le \sum_j c_j x_j$

$$
\begin{aligned}
(LR)\ Z(y) = \min_{x} \quad & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} y_i \cdot (b_i - \sum_{j=1}^{n} a_{ij} x_j) \\
\text{s.t.} \quad & x_j \geq 0 \quad \forall j = 1, ..., n_1 \\
& x_j \text{ free} \quad \forall j = n_1 + 1, ..., n_2 \\
& x_j \leq 0 \quad \forall j = n_2 + 1, ..., n
\end{aligned}
$$

This problem (LR) is called the **Lagrangian relaxation problem** of the original primal problem (P). The objective function of (LR) is called the **Lagrangian function**. We have $Z(y) \leq Z_P$ for designated y's.

$$\min_{x} \sum_j c_j x_j + \sum_i y_i \left(b_i - \sum_j a_{ij} x_j\right)$$

$$\text{s.t.} \quad x_j \geq 0, \quad \forall j = 1, \cdots, n_1$$

$$x_j \text{ free}, \quad \forall j = n_1+1, \cdots, n_2$$

$$x_j \leq 0, \quad \forall j = n_2+1, \cdots, n$$

$$\Downarrow$$

$$\sum_i y_i b_i + \min_x \sum_j c_j x_j - \sum_i y_i \sum_j a_{ij} x_j$$

$$\text{s.t.} \quad \vdots$$

$$\Downarrow$$

$$\sum_i y_i b_i + \min \sum_j \Big(\underbrace{c_j - \sum_i y_i a_{ij}}_{= \delta_j}\Big) x_j$$

$$\text{s.t.} \quad x_j \geq 0, \quad \forall j = 1, \cdots, n_1$$

seperable $\quad x_j \text{ free}, \quad \forall j = n_1+1, \cdots, n_2$

$$x_j \leq 0, \quad \forall j = n_2+1, \cdots, n$$

$$\Downarrow$$

$$\sum_i y_i b_i + \sum_j \begin{pmatrix} \min_{x_j} \delta_j x_j \\ \text{s.t. } x_j \text{ type} \end{pmatrix}$$

min. $2x$
s.t. $x \geq 0$

min. $-2x$
s.t. $x \geq 0$

$$\min_{x_j \geq 0} \delta_j x_j = \begin{cases} 0 & \text{if } \delta_j \geq 0 \\ -\infty & \text{if } \delta_j < 0 \end{cases}$$

(a)

(b)

$$\min_{x_j \text{ free}} \delta_j x_j = \begin{cases} 0 & \text{if } \delta_j = 0 \\ -\infty & \text{if } \delta_j \neq 0 \end{cases}$$

(c)

min. $2x$
s.t. $x \leq 0$

min. $-2x$
s.t. $x \leq 0$

$$\min_{x_j \leq 0} \delta_j x_j = \begin{cases} 0 & \text{if } \delta_j \leq 0 \\ -\infty & \text{if } \delta_j > 0 \end{cases}$$

Lagrangian relaxation problem

$$
\begin{aligned}
(LR)\ Z(y) = \min_{x} \quad & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} y_i \cdot (b_i - \sum_{j=1}^{n} a_{ij} x_j) \\
\text{s.t.} \quad & x_j \geq 0 \quad \forall j = 1, ..., n_1 \\
& x_j \text{ free} \quad \forall j = n_1 + 1, ..., n_2 \\
& x_j \leq 0 \quad \forall j = n_2 + 1, ..., n
\end{aligned}
$$

$$
\begin{aligned}
(LR)\ Z(y) = \sum_{i=1}^{m} y_i b_i + \min_{x_j} \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} y_i \sum_{j=1}^{n} a_{ij} x_j & \\
\text{s.t.} \quad x_j \geq 0 \quad \forall j = 1, ..., n_1 & \\
x_j \text{ free} \quad \forall j = n_1 + 1, ..., n_2 & \\
x_j \leq 0 \quad \forall j = n_2 + 1, ..., n &
\end{aligned}
$$

Separable Lagrangian relaxation problem

The Lagrangian relaxation problem is separable and it can be divided into n smaller problems.

$$Z(y) = \sum_{i=1}^{m} y_i b_i + \sum_{j=1}^{n} \min_{x_j} c_j x_j - \sum_{i=1}^{m} y_i a_{ij} x_j$$

$$\min_{x_j \geq 0} \; (c_j - \sum_{i=1}^{m} y_i a_{ij}) x_j = \begin{cases} 0 & \text{if } c_j - \sum_{i=1}^{m} y_i a_{ij} \geq 0 \; (a) \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x_j \text{ free}} \; (c_j - \sum_{i=1}^{m} y_i a_{ij}) x_j = \begin{cases} 0 & \text{if } c_j - \sum_{i=1}^{m} y_i a_{ij} = 0 \; (b) \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x_j \leq 0} \; (c_j - \sum_{i=1}^{m} y_i a_{ij}) x_j = \begin{cases} 0 & \text{if } c_j - \sum_{i=1}^{m} y_i a_{ij} \leq 0 \; (c) \\ -\infty & \text{otherwise} \end{cases}$$

Lagrangian relaxation problem

$$Z(y) = \sum_{i=1}^{m} y_i b_i + \sum_{j=1}^{n} \min_{x_j} c_j x_j - \sum_{i=1}^{m} y_i a_{ij} x_j$$

$$= \begin{cases} \sum_{i=1}^{m} y_i b_i & \text{if (a), (b), (c) hold for all } x_j \\ -\infty & \text{otherwise} \end{cases}$$

With designated y's, $Z(y)$ is a lower bound on Z_P , i.e. $\sum_{i=1}^{m} y_i b_i \leq Z_P$.

Step 3: Finding the best lower bound

The best lower bound means the lower bound that is the closest to Z_P .
The following LP finds the best lower bound by maximizing $Z(y)$ over the constraints (a), (b), (c), and the sign constraints on y's.

$$Z_D = \max_{y_i} \quad Z(y) = \sum_i y_i b_i$$

$$\text{s.t.} \quad (a), (b), (c)$$

designated y's

$$\left.\begin{aligned} &y_i \geq 0 \quad \forall i = 1, ..., m_1 \\ &y_i \text{ is free} \quad \forall i = m_1 + 1, ..., m_2 \\ &y_i \leq 0 \quad \forall i = m_2 + 1, ..., m \end{aligned}\right.$$

Dual problem

$$
\begin{aligned}
(D)\ Z_D = \max_{y_i} \quad & \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \leq c_j \quad j = 1, \ldots, n_1 \qquad \text{(a)} \\
& \sum_{i=1}^{m} a_{ij} y_i = c_j \quad j = n_1 + 1, \ldots, n_2 \qquad \text{(b)} \\
& \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad j = n_2 + 1, \ldots, n \qquad \text{(c)} \\
& \left.\begin{aligned}
& y_i \geq 0 \quad \forall i = 1, \ldots, m_1 \\
& y_i \text{ is free} \quad \forall i = m_1 + 1, \ldots, m_2 \\
& y_i \leq 0 \quad \forall i = m_2 + 1, \ldots, m
\end{aligned}\right\} \text{designated } y\text{'s}
\end{aligned}
$$

General LP Dual

Primal constraints

dual variable type

Primal			Dual		
minimize	$c^T x$		maximize	$b^T y$	
subject to	$a_i^T x \geq b_i$,	$i \in M_1$,	subject to	$y_i \geq 0$,	$i \in M_1$
	$a_i^T x \leq b_i$,	$i \in M_2$,		$y_i \leq 0$,	$i \in M_2$
	$a_i^T x = b_i$,	$i \in M_3$,		y_i free,	$i \in M_3$
	$x_j \geq 0$,	$j \in N_1$,		$A_j^T y \leq c_j$,	$j \in N_1$
	$x_j \leq 0$,	$j \in N_2$,		$A_j^T y \geq c_j$,	$j \in N_2$
	x_j free,	$j \in N_3$,		$A_j^T y = c_j$,	$j \in N_3$

Primal variable type

dual constraints

- a_i^T is the ith row of A, A_j is the jth column of A
- Each primal constraint corresponds to a dual variable
- Each primal variable corresponds to a dual constraint
- Equality constraints always correspond to free variables

dual variables = # Primal Constraints

dual Constraints = # Primal variables

Table to Form Dual Problem

min. $\longrightarrow$

$\longleftarrow$ max

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$	≥ 0	Variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
Variables	≥ 0	$\leq c_j$	Constraints
	≤ 0	$\geq c_j$	
	free	$= c_j$	

Example

$$
\begin{aligned}
(P) \quad Z_P = \min \quad & x_1 + 2x_2 + 3x_3 \\
\text{s.t.} \quad & x_1 + 5x_2 + 4x_3 \geq 6 \\
& 2x_1 + 3x_2 - x_3 = 3 \\
& x_1 + x_2 - 2x_3 \leq 4 \\
& x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}
\end{aligned}
$$

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$	≥ 0	Variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
Variables	≥ 0	$\leq c_j$	Constraints
	≤ 0	$\geq c_j$	
	free	$= c_j$	

$$
\begin{aligned}
\min\ & x_1 + 2x_2 + 3x_3 \\
\text{s.t. } & x_1 + 5x_2 + 4x_3 \geq 6 \quad (y_1) \\
& 2x_1 + 3x_2 - x_3 = 3 \quad (y_2) \\
& x_1 + x_2 - 2x_3 \leq 4 \quad (y_3) \\
& x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ free}
\end{aligned}
$$

$$
\begin{aligned}
\max.\ & 6y_1 + 3y_2 + 4y_3 \\
\text{s.t. } & y_1 + 2y_2 + y_3 \leq 1 \\
& 5y_1 + 3y_2 + y_3 \geq 2 \\
& 4y_1 - y_2 - 2y_3 = 3 \\
& y_1 \geq 0,\ y_2 \text{ free},\ y_3 \leq 0
\end{aligned}
$$

Example

$$
\begin{aligned}
\max \quad & x_1 + 2x_2 + x_3 + x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 2 \\
& x_2 + x_4 \leq 1 \\
& x_1 + 2x_3 \geq 1 \\
& x_1, x_3 \geq 0, x_2, x_4 \text{ free}
\end{aligned}
$$

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

$$
\begin{array}{lll}
\max & x_1 + 2x_2 + x_3 + x_4 & \\
\text{s.t.} & x_1 + 2x_2 + 3x_3 \leq 2 & (y_1) \\
& x_2 + x_4 \leq 1 & (y_2) \\
& x_1 + 2x_3 \geq 1 & (y_3) \\
& x_1, x_3 \geq 0,\ x_2, x_4 \text{ free} &
\end{array}
$$

$$
\begin{array}{llll}
\min. & 2y_1 + y_2 + y_3 & & \\
\text{s.t.} & y_1 + y_3 \geq 1 & & (x_1) \\
& 2y_1 + y_2 = 2 & & (x_2) \\
& 3y_1 + 2y_3 \geq 1 & & (x_3) \\
& y_2 = 1 & & (x_4) \\
& y_1 \geq 0,\ y_2 \geq 0,\ y_3 \leq 0 & &
\end{array}
$$

Primal and Dual Pair in Standard Form

$$
\text{(P)} \quad \begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}
\qquad\qquad
\text{(D)} \quad \begin{aligned} \max \quad & b^\mathsf{T} y \\ \text{s.t.} \quad & A^\mathsf{T} y \leq c \\ & y \text{ free} \end{aligned}
$$

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

$$\min.\ c^T x \qquad c \in \mathbb{R}^n$$

$$\text{s.t.}\ Ax = b \quad (y) \qquad x \in \mathbb{R}^n$$

$$A \in \mathbb{R}^{m \times n}$$

$$x \geq 0 \qquad b \in \mathbb{R}^m$$

$$\max.\ b^T y \qquad y \in \mathbb{R}^m$$

$$\text{s.t.}\ A^T y \leq c$$

$$y \text{ free} \qquad \in \mathbb{R}^n$$

$$\underset{m \times n}{A}\ \underset{m \times 1}{y} \rightarrow \underset{n \times m}{A^T}\ \underset{m \times 1 \rightarrow n \times 1}{y}$$

dual

$$\min.\ c^T x$$

$$\text{s.t.}\ Ax = b$$

$$x \geq 0$$

Invariance of Transformations

Theorem

If we transform a linear program to an equivalent one (such as by replacing free variables, adding slack variables, etc), then the dual of the two problems will be equivalent.

Theorem

If we transform the primal to its dual, then transform the dual to its dual, then we will obtain a problem equivalent to the primal problem, that is, the dual of dual is the primal.

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Weak Duality Theorem

Primal		Dual	
min	$\boldsymbol{c}^T\boldsymbol{x}$	max	$\boldsymbol{b}^T\boldsymbol{y}$
s.t.	$A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$	s.t.	$A^T\boldsymbol{y} \leq \boldsymbol{c}$

Theorem (Weak Duality Theorem)

If $\boldsymbol{x}$ is feasible to the primal and $\boldsymbol{y}$ is feasible to the dual, then

$$\boldsymbol{b}^T\boldsymbol{y} \leq \boldsymbol{c}^T\boldsymbol{x}$$

If the primal is a minimization and dual is a maximization, then

- Any dual feasible solution will give a lower bound on the primal optimal value.
- Any primal feasible solution will give an upper bound on the dual optimal value.
- The optimal value of primal is no less than that of dual.

Assume $\boldsymbol{x}$ is feasible to the primal problem and $\boldsymbol{y}$ is feasible to the dual problem. Then we have

from dual constraint

$\le C$

$$\boldsymbol{b}^T \boldsymbol{y} = (A\boldsymbol{x})^T \boldsymbol{y} = \boldsymbol{x}^T (A^T \boldsymbol{y}) \le \boldsymbol{c}^T \boldsymbol{x}$$

The last inequality is because that $\boldsymbol{x} \ge 0$ and $A^T \boldsymbol{y} \le \boldsymbol{c}$. □

Corollary

- *If the primal problem is unbounded (i.e., the optimal value is $-\infty$), then the dual problem must be infeasible*
- *If the dual problem is unbounded (i.e., the optimal value is $+\infty$), then the primal problem must be infeasible*

Weak duality

$$\underset{\text{dual}}{b^T y} \le \underset{\text{Primal}}{c^T x}$$

- If Primal is unbounded, $c^T x \to -\infty$, then

$$b^T y \le -\infty$$

no such y exists

$\Rightarrow$ Dual is infeasible.

- If Dual is unbounded, $b^T y \to +\infty$, then

$$c^T x \ge +\infty$$

no such x exists.

$\Rightarrow$ Primal is infeasible

LP Optimality via Duality

Corollary

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be feasible solutions to the primal and dual problems respectively. If $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$, then $\boldsymbol{x}$ and $\boldsymbol{y}$ must be optimal solutions to the primal and dual, respectively.

Optimality conditions for LP: If $\boldsymbol{x}$, $\boldsymbol{y}$ satisfy:

1. $\boldsymbol{x}$ is primal feasible,
2. $\boldsymbol{y}$ is dual feasible,
3. The objective values are the same, i.e., $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$.

Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal solutions to the primal and dual problems respectively.

The reverse is also true (see the next theorem).

Strong Duality Theorem

Theorem (Strong Duality Theorem)

If a linear program has an optimal solution, so does its dual, and the optimal values of the primal and dual are equal

- We present a constructive proof. That is, for a given primal optimal solution, we construct a dual optimal solution and show that their objective values are equal
- We use simplex method in our proof
- In the proof, we will see that the simplex method actually also finds the dual optimal solution when it finishes

Proof

We prove by using the simplex method. Without loss of generality, we assume the primal problem is in the standard form.

If the primal problem has an optimal solution $\boldsymbol{x}^*$, then it must be associated with some optimal basis B such that $\boldsymbol{x}_B = A_B^{-1}\boldsymbol{b}$ ($\boldsymbol{x}_B^*$ is the basic part of $\boldsymbol{x}^*$). Also, when the simplex method terminates, the reduced costs

reduced costs

$$\boldsymbol{c}^T - \overbrace{\boldsymbol{c}_B^T A_B^{-1}}^{y^T} A \geq 0 \tag{1}$$

Now we define $\boldsymbol{y}^T = \boldsymbol{c}_B^T A_B^{-1}$. By (1), $A^T\boldsymbol{y} \leq \boldsymbol{c}$, i.e., $\boldsymbol{y}$ is feasible to the dual problem. In addition

dual constraint

$$\boldsymbol{b}^T\boldsymbol{y} = \boldsymbol{c}_B^T A_B^{-1}\boldsymbol{b} = \boldsymbol{c}_B^T\boldsymbol{x}_B = \boldsymbol{c}^T\boldsymbol{x}^*$$

$$= c_B^T x_B + c_N^T x_N$$

Therefore by the weak duality theorem, $\boldsymbol{y}$ must be optimal to the dual problem and the theorem holds. □

Discussion

Remark:
From the proof, we see the term $\boldsymbol{c}_B^T A_B^{-1}$ will be the dual optimal solution (if primal has an optimal solution), where B is the basis corresponding to the primal optimal solution x^*. Therefore, when we solve the primal problem, the dual is also solved.

This is not a coincidence. Nearly all LP algorithms (simplex method, interior point method or ellipsoid method) solve both primal and dual problems simultaneously.

LP Optimality via Duality

Based on the strong duality theorem, we know that (x, y) is optimal to the primal and dual respectively if and only if

- x is primal feasible,
- y is dual feasible,
- They achieve the same objective value. $c^T x = b^T y$

Therefore solving LP (in standard form) is in fact equivalent as solving the following linear system:

- $Ax = b$, $x \geq 0$ Primal feasible
- $A^T y \leq c$ Dual feasible
- $b^T y = c^T x$ Same obj val

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Table of Possibles and Impossibles

The primal and dual LPs can be finite optimal, or unbounded, or infeasible. So, there are in total 9 combinations. Are all these 9 combinations possible?

	Finite Optimal	Unbounded	Infeasible
Finite Optimal	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Notice this table is exactly symmetric, because the dual of the dual is the primal.

strong duality

(P) \ (D)	optimal	unbounded	infeasible
optimal	✓	✗	✗
unbounded	✗	✗	✓
infeasible	✗	✓	✓

weak duality

(a) Derive the dual

(b) What is the outcome for the dual

(P) is infeasible $\Rightarrow$ (D) unbounded or infeasible

$\Downarrow$ one feasible solution for (D)

(D) unbounded

Figuring out the possibilities

- If one of the primal or dual is feasible, then the other problem must also be feasible and have optimal solutions (by strong duality).
- If one of the primal or dual is unbounded, then the other problem must be infeasible (by weak duality).
- Both primal and dual problems may be infeasible. Check the following example.

$$
(P)\quad \begin{array}{rll} \min & x_1 + 2x_2 & \\ \text{s.t.} & x_1 + x_2 = 1 & (y_1) \\ & 2x_1 + 2x_2 = 3 & (y_2) \\ & x_1, x_2 \text{ free.} & \end{array}
\qquad
(D)\quad \begin{array}{rl} \max & y_1 + 3y_2 \\ \text{s.t.} & y_1 + 2y_2 = 1 \\ & y_1 + 2y_2 = 2 \\ & y_1, y_2 \text{ free.} \end{array}
$$

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Complementarity Conditions

Consider the primal-dual pair:

Primal			**Dual**		
minimize	$c^T x$		maximize	$b^T y$	
subject to	$a_i^T x \geq b_i,$	$i \in M_1,$	subject to	$y_i \geq 0,$	$i \in M_1$
	$a_i^T x \leq b_i,$	$i \in M_2,$		$y_i \leq 0,$	$i \in M_2$
	$a_i^T x = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \geq 0,$	$j \in N_1,$		$A_j^T y \leq c_j,$	$j \in N_1$
	$x_j \leq 0,$	$j \in N_2,$		$A_j^T y \geq c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$		$A_j^T y = c_j,$	$j \in N_3$

Theorem

Let x and y are feasible solutions to the primal and dual problems respectively. Then x and y are optimal if and only if

$$y_i \cdot (a_i^T x - b_i) = 0, \quad \forall i; \quad x_j \cdot (A_j^T y - c_j) = 0, \quad \forall j.$$

x^* and y^* are optimal

- $y_i^*(a_i^T x^* - b_i) = 0$ $<$ $y_i^* = 0$, $a_i^T x^* = b_i$

- $x_j^*(A_j^T y^* - c_j) = 0$ $<$ $x_j^* = 0$, $A_j^T y^* = c_j$

Complementary Slackness

Let x and y be feasible solutions to the primal and dual problem, respectively. Then x and y are optimal solutions for the two respective problems if and only if they satisfy the following conditions:

- Primal Complementary Slackness: $y_i(a_i^\intercal x - b_i) = 0$ for all i. In words, either the i-th primal constraint is active (binding, tight) so $a_i^\intercal x = b_i$, or the corresponding dual variable $y_i = 0$.
- Dual Complementary Slackness: $x_j(A_j^T y - c_j) = 0$ for all j. In words, either the j-th dual constraint is active (binding, tight) so $A_j^T y = c_j$, or the corresponding primal variable $x_j = 0$.

Example

$$\begin{aligned} \min\ & 13x_1 + 10x_2 + 6x_3 \\ \text{s.t. } & 5x_1 + x_2 + 3x_3 = 8 \\ & 3x_1 + x_2 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \max\ & 8y_1 + 3y_2 \\ \text{s.t. } & 5y_1 + 3y_2 \leq 13 \\ & y_1 + y_2 \leq 10 \\ & 3y_1 \leq 6 \end{aligned}$$

- Someone solved the dual problem graphically, and told us the dual optimal solution $y_1^* = 2$, $y_2^* = 1$. We want to use this information and Complementary Slackness to get the optimal solution of the primal.
- If we are given a primal feasible solution $x_1^* = 1$, $x_2^* = 0$; $x_3^* = 1$, can we verify this is an optimal solution?

$$(P)\quad \begin{aligned} \min\ & 13x_1 + 10x_2 + 6x_3 \\ \text{s.t.}\ & 5x_1 + x_2 + 3x_3 = 8 \quad (y_1) \\ & 3x_1 + x_2 = 3 \quad (y_2) \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \qquad (D)\quad \begin{aligned} \max\ & 8y_1 + 3y_2 \\ \text{s.t.}\ & 5y_1 + 3y_2 \leq 13 \quad (x_1) \\ & y_1 + y_2 \leq 10 \quad (x_2) \\ & 3y_1 \leq 6 \quad (x_3) \end{aligned}$$

(D): $y_1^* = 2 \qquad y_2^* = 1$

Primal Complementary Slackness

$$\begin{cases} y_1^* (5x_1^* + x_2^* + 3x_3^* - 8) = 0 \\ y_2^* (3x_1^* + x_2^* - 3) = 0 \end{cases}$$

Dual Complementary Slackness

$$\begin{cases} x_1^* \cdot (5y_1^* + 3y_2^* - 13) = 0 \Rightarrow x_1^* \cdot 0 = 0 \\ x_2^* \cdot (y_1^* + y_2^* - 10) = 0 \Rightarrow x_2^* \cdot (2+1-10) = 0 \\ x_3^* \cdot (3y_1^* - 6) = 0 \end{cases}$$

$$\Downarrow$$

$$x_2^* = 0$$

$$\Downarrow$$

$$x_3^* \cdot 0 = 0$$

$$\Rightarrow 1 \cdot (3x_1^* + 0 - 3) = 0$$

$$\Rightarrow x_1^* = 1$$

$$\Rightarrow 2(5 \cdot 1 + 0 + 3x_3^* - 8) = 0$$

$$\Rightarrow x_3^* = 1$$

Complementary Slackness in Another Form

$\max\ b^\top y$

s.t. $A^\top y \le c$

y free

Sometimes, we write the dual problem (equivalently) as:

Primal	**Dual**
$\min\ c^\top x$	$\max\ b^\top y$
s.t. $Ax = b,\ x \ge 0$	s.t. $A^\top y + s = c,\ s \ge 0$

$x_i(A_i^\top y - c_i) = 0$

We call s the dual slack variables. Then, the complementarity conditions can be written as:

$$x_i \cdot s_i = 0 \quad \forall i = 1, \ldots, n.$$

That's the reason we call the complementarity conditions the complementary slackness conditions.

LP Optimality via Complementary Slackness

Recall that the optimality conditions for LPs were given by:

1. x is primal feasible,
2. y is dual feasible,
3. The objective values coincide, i.e., $c^\top x = b^\top y$.

Now with the complementary slackness, we can write an equivalent iff optimality conditions:

1. x is primal feasible,
2. y is dual feasible,
3. The complementary slackness conditions are satisfied.

KKT Conditions

Primal feasibility, dual feasibility, and complementary slackness form the Karush-Kuhn-Tucker (KKT) optimality conditions for LP.

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Diet Problem

Suppose there are n types of foods. Each food j costs c_j dollars per unit to purchase, and has a_{ij} units of nutrient i for $i = 1, .., m$. The goal is to combine n types of food with the minimum cost to produce an ideal food combination that contains b_i units of nutrient i for $i = 1, .., m$.

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \forall i = 1, .., m \\ & x_j \geq 0 \quad \forall j = 1, ..., n \end{aligned}$$

We want to study how the change of right-hand side of the constraint, namely b_i's, would affect the optimal cost of the diet problem. In particular, if b_i of a specific nutrient i increases by a unit, while holding all other components of the RHS constant, how would the minimum cost change? Should it increase or decrease?

$$b_i \to b_i + 1$$

Q: How the optimal cost will change?

(P)
$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \forall i = 1, .., m \\ & x_j \geq 0 \quad \forall j = 1, \dots, n \end{aligned}$$

(D)
$$\begin{aligned} \max \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \leq c_j \quad \forall j = 1, .., n \\ & y_i \geq 0 \quad \forall i = 1, \dots, m \end{aligned}$$

From strong duality,

$$\text{old cost} = \sum_j c_j x_j^* = \underline{\sum_i b_i y_i^*}$$

If b_i becomes $b_i + 1$ for one i,

$$\text{New cost} = \sum_{k \neq i} b_k y_k^* + (b_i + 1) y_i^*$$

$$= \sum_i b_i y_i^* + y_i^*$$

$\Rightarrow$ Cost change = New cost − old cost

$$= y_i^*$$

shadow Price

Shadow Price

Shadow Price

The shadow price of a constraint is the change of the objective cost from the current optimal level if the right-hand side of this constraint is increased by a unit from the current level. $b_i \to b_i + 1$

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \le c_j \quad \forall j = 1, .., n \\ & y_i \ge 0 \quad \forall i = 1, ..., m \end{aligned}$$

Shadow Price Interpretation

$Ax \geq b \leftarrow$ demand
$Ax \leq b \leftarrow$ supply/resource

- Dual variables represent the unit economic evaluation of the demand/resource.
- In a diet problem, dual variables evaluate how much an additional unit of demand (nutrient) costs.
- In a production planning problem, dual variables evaluate how much an additional unit of resource (workers, fertilizers, raw materials etc.) is worth.
- The shadow price can be viewed as a **unit/fair price** of the demand/resource that you would be willing to pay/purchase.

Dynamic Pricing

Alternative Systems

Given a set of linear inequalities:

Constrant

$\{y : A^T y \leq c\} = \emptyset$ $\quad A^T y \leq c \quad \rightarrow$ Programming

An important question is: whether the system has a solution?

- It is easy to verify that it has a solution, one only needs to find a solution (we call it a *certificate*).
- To disprove the existence, can we also have such a certificate?

The answer: Yes.

- If we can find a vector x satisfying

$$Ax = 0, \quad x \geq 0, \quad c^T x < 0$$

Then there must be no solution to the system $A^T y \leq c$

Thm: If $\exists x$ s.t. $Ax=0$, $x\geq 0$, $c^Tx<0$

Then $\{y : A^Ty \leq c\} = \emptyset$

Farkas' Lemma (Theorem of Alternative)

Theorem (Farkas' Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following two alternatives hold:

(I) $P := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \neq \emptyset$

(II) $Q := \{y \in \mathbb{R}^m : A^\top y \geq 0, b^\top y < 0\} \neq \emptyset$

Alternative Systems

One can construct many more pairs of such alternative systems.

- It is hard to directly prove something is not feasible.
- LP duality provides an alternative approach, transforming the problem to proving something is feasible.

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Sensitivity Analysis

One important question when studying LP is as follows:

- How do the optimal solution and the optimal value change when the input changes?

This type of problems is called the *Sensitivity Analysis* of LP.

- We first study this question from a local perspective, and then globally.

Local Sensitivity

Consider the standard form LP:

$$\begin{aligned} V = \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Define the optimal value by V.

- Given A and $\boldsymbol{c}$ fixed, V can be viewed as a function of $\boldsymbol{b}$: $V(\boldsymbol{b})$

Theorem

If the dual has a unique optimal solution $\boldsymbol{y}^*$, then $\nabla_b V(\boldsymbol{b}) = \boldsymbol{y}^*$.

- If the dual optimal solution is not unique (or the dual problem is unbounded or infeasible), then the gradient does not exist.
- If one changes b_i by a small amount Δb_i, then the change of the objective value will be $\Delta b_i y_i^*$

Explanation

We know that the optimal value V is also the optimal value of the dual problem:

$$\begin{aligned} \max_{\boldsymbol{y}} \quad & \boldsymbol{b}^T \boldsymbol{y} \\ \text{s.t.} \quad & A^T \boldsymbol{y} \leq \boldsymbol{c} \end{aligned}$$

i.e., $V(\boldsymbol{b}) = \boldsymbol{b}^T \boldsymbol{y}^*$.

If we change $\boldsymbol{b}$ by a small amount $\Delta \boldsymbol{b}$, such that the optimal solution does not change, then the change to V must be $\Delta \boldsymbol{b}^T \boldsymbol{y}^*$.

Local Sensitivity

$$
\begin{aligned}
V = \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

Similarly, given A and $\boldsymbol{b}$ fixed, V can be viewed as a function of $\boldsymbol{c}$.

Theorem

If the primal problem has a unique optimal solution $\boldsymbol{x}^*$, then $\nabla V(\boldsymbol{c}) = \boldsymbol{x}^*$.

If one changes c_i by a small amount Δc_i, then the change of the objective value will be $\Delta c_i x_i^*$

- Reason: If we change $\boldsymbol{c}$ by a small amount $\Delta \boldsymbol{c}$, such that the optimal solution does not change, then the change to V must be $\Delta \boldsymbol{c}^T \boldsymbol{x}^*$.

Local Sensitivity

The above results also hold for inequality constraints (or maximization problem) such as follows:

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

We have:

1. If the dual has a unique optimal solution $\boldsymbol{y}^*$, then $\nabla V(\boldsymbol{b}) = \boldsymbol{y}^*$
2. If the primal has a unique optimal solution $\boldsymbol{x}^*$, then $\nabla V(\boldsymbol{c}) = \boldsymbol{x}^*$

- To see why this must be true, one can add a slack variable and transform it back to the standard form and then one can use the earlier result.

Example

$$\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
& & 2x_2 & \leq 200 \\
& x_1 & +x_2 & \leq 150 \\
& x_1, & x_2 & \geq 0
\end{array}$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250.

The dual problem is

$$\begin{array}{rrrrl}
\text{minimize} & 100y_1 & +200y_2 & +150y_3 & \\
\text{subject to} & y_1 & & +y_3 & \geq 1 \\
& & 2y_2 & +y_3 & \geq 2 \\
& y_1, & y_2, & y_3 & \geq 0
\end{array}$$

The optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$ with optimal value 250.

Example Continued

$$
\begin{array}{lrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

1. What would be the optimal value if we change the RHS of second constraint to 202?

- It will change by $\Delta b_2 y_2^* = 1$. Therefore, the optimal value would be 251

Example Continued

$$\begin{array}{lrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

2. What would be the optimal value if we change the RHS of first constraint to 99?

- It will change by $\Delta b_1 y_1^* = 0$. Therefore, the optimal value would be unchanged.

Example Continued

$$
\begin{array}{rccl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

3. What would be the optimal value if the cost coefficient of x_1 becomes 1.02?

- It will increase by $\Delta c_1 x_1^* = 1$. Therefore, the optimal value would be 251

Example Continued

$$
\begin{array}{rlll}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

4. What would be the optimal value if the cost coefficient of x_2 becomes 1.97?

- It will decrease by $\Delta c_2 x_2^* = -3$. Therefore, the optimal value would be 247

Property: Inactive Constraints

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge 0 \end{aligned}$$

If at optimal $\boldsymbol{x}^*$, $\boldsymbol{a}_i^T \boldsymbol{x}^* < b_i$, then what happens if we change b_i?

- By the complementary slackness conditions, the corresponding dual variable y_i^* must be 0.
- Therefore, changing the right-hand-side of an inactive constraint by a small amount won't affect the optimal value (also the optimal solution).
- Intuition: If a resource is already redundant, then adding or reducing a small amount wouldn't matter.

Shadow Prices

Recall that

- $\nabla V(\boldsymbol{b}) = \boldsymbol{y}^*$, where $\boldsymbol{y}^*$ is the optimal dual solution

We call $\boldsymbol{y}^*$ the shadow prices of $\boldsymbol{b}$.

- The shadow price of a resource corresponds to the increment of profit if there is one unit more of that resource (locally).
- Therefore, it can be viewed as the *unit value* or *unit fair price* for that resource.

Caveat

The above analysis is *local*, meaning that it can only deal with small changes.

- Basically, it is valid as long as the optimal basis does not change.
- Otherwise, it may not be true.

In the above example, if the RHS of first constraint reduces to 0, then the optimal solution will be $(0, 100)$, with optimal value 200 (reduced by 50). This difference would be different from $\Delta b_1 y_1^* = 0$.

- We want to study what ranges of changes belong to *small* changes.
- This will be the *global sensitivity analysis*.

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Global Sensitivity

Now we study what will happen if

1. $\boldsymbol{b}$ changes to $\boldsymbol{b} + \Delta \boldsymbol{b}$
2. $\boldsymbol{c}$ changes to $\boldsymbol{c} + \Delta \boldsymbol{c}$

- At optimal, the reduced costs $\boldsymbol{c}^T - \boldsymbol{c}_B^T A_B^{-1} A \geq 0$.
- $x_B^* = A_B^{-1}\boldsymbol{b}$ is the basic part of the optimal primal solution.
- $y^* = (A_B^{-1})^T \boldsymbol{c}_B$ is the optimal dual solution.

Changing on b

Suppose b becomes $\tilde{b} = b + \Delta b$. Consider the original optimal basis B, we still have

$$\bar{c} = c^\top - c_B^\top A_B^{-1} A \geq 0$$

since the reduced costs do not depend on b.
If we want to ensure the original basis B remains optimal for the new problem, we just need to require the new basic solution is feasible (a BFS):

$$\tilde{x}_B = A_B^{-1} \tilde{b} = A_B^{-1}(b + \Delta b) = x_B^* + A_B^{-1} \Delta b \geq 0.$$

Then B is still the optimal basis and the new primal optimal solution is

$$\tilde{x} = [\tilde{x}_B; 0],$$

with the new optimal value:

$$V(\tilde{b}) = \tilde{b}^\top y^* = (b + \Delta b)^\top y^* = V^* + (y^*)^\top \Delta b,$$

since the original dual optimal solution $y^* = (A_B^{-1})^\top c_B$ is still dual optimal.
Conclusion: If the original optimal basis is still optimal, then the local sensitivity analysis holds (this justifies the local theorem).

Change on b

Now we study when the change only occurs to one component of $\boldsymbol{b}$, what ranges of changes qualify for a *small* change (i.e., the local sensitivity analysis holds).

Assume $\Delta\boldsymbol{b} = \lambda\boldsymbol{e}_i$ ($\boldsymbol{e}_i$ is a vector with 1 at position i). Then we need to have

$$\boldsymbol{x}_B^* + \lambda A_B^{-1}\boldsymbol{e}_i \geq 0$$

so that the optimal basis remains the same. We can then find the range of λ by solving these inequalities.

Example

Consider the example:

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

At optimal, the basis is $\{1, 2, 3\}$, and the optimal solution is $(50, 100, 50, 0, 0)$

- How much can we change the 3rd right hand side coefficient (150) such that the optimal basis remains the same?

Example Continued

The final simplex tableau is

B						
B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

Thus $A_B^{-1} = \begin{pmatrix} 0 & -0.5 & 1 \\ 1 & 0.5 & -1 \\ 0 & 0.5 & 0 \end{pmatrix}$. If $\boldsymbol{b}$ changes to $\boldsymbol{b} + \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then

$$\tilde{\boldsymbol{x}}_B = \boldsymbol{x}_B^* + \lambda A_B^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \\ 100 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

In order for this to be positive, we need $-50 \leq \lambda \leq 50$.

Changes in c

Now suppose $\boldsymbol{c}$ changes to $\tilde{\boldsymbol{c}} = \boldsymbol{c} + \Delta\boldsymbol{c}$. In order for the basic solution to be still optimal, we need to guarantee that the reduced costs are nonnegative:

$$\tilde{\boldsymbol{c}}_B^T - \tilde{\boldsymbol{c}}_B^T A_B^{-1} A_B = 0 \quad \text{for sure, no need to consider}$$
$$\tilde{\boldsymbol{c}}_N^T - \tilde{\boldsymbol{c}}_B^T A_B^{-1} A_N \geq 0$$

Note that this basis still provides a basic feasible solution since the feasibility doesn't depend on $\boldsymbol{c}$.

Next we assume $\Delta\boldsymbol{c} = \lambda \boldsymbol{e}_j$. We discuss two cases: $j \in B$ and $j \in N$. We study how to find the ranges of λ such that the original basis is still optimal (and thus one can apply the local sensitivity analysis).

Case 1: $j \in B$

In this case, the reduced costs are

$$\begin{aligned} & \boldsymbol{c}_N^T - (\boldsymbol{c}_B^T + \lambda \boldsymbol{e}_j^T) A_B^{-1} A_N \\ = \quad & \boldsymbol{c}_N^T - \boldsymbol{c}_B^T A_B^{-1} A_N - \lambda \boldsymbol{e}_j^T A_B^{-1} A_N \end{aligned}$$

Note that $\boldsymbol{c}_N^T - \boldsymbol{c}_B^T A_B^{-1} A_N$ is the reduced costs for the original problem. We denote it by r_N^T. Therefore, in order to maintain the optimality of the current basis, we need to have

$$r_N^T - \lambda \boldsymbol{e}_j^T A_B^{-1} A_N \geq 0$$

- We can solve the range of λ from the above set of inequalities.

Case 2: $j \in N$

In this case, the reduced costs are:

$$\tilde{\boldsymbol{c}}_N^T - \boldsymbol{c}_B^T A_B^{-1} A_N = \boldsymbol{c}_N^T + \lambda \boldsymbol{e}_j^T - \boldsymbol{c}_B^T A_B^{-1} A_N = r_N^T + \lambda \boldsymbol{e}_j^T$$

Therefore, in order to maintain the optimality of the current basis, we need to have

$$r_N + \lambda \boldsymbol{e}_j \geq 0$$

- We can solve the range of λ from the above set of inequalities.

Example

Consider the same example:

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
& & 2x_2 & \leq 200 \\
& x_1 & +x_2 & \leq 150 \\
& x_1, & x_2 & \geq 0
\end{array}
$$

The final simplex tableau is

B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

How much can we change the first objective coefficient so that we can use the local sensitivity analysis?

Example Continued

We have

$$A_B^{-1} = \begin{pmatrix} 0 & -0.5 & 1 \\ 1 & 0.5 & -1 \\ 0 & 0.5 & 0 \end{pmatrix}; \quad A_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad r_N = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

Assume we change the c_1 from 1 to $1 + \lambda$ (i.e., $-1 - \lambda$ in the standard form). Then we need

$$r_N - (-\lambda) A_N^T (A_B^{-1})^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} 0.5 \\ -1 \end{pmatrix} \geq 0$$

Therefore, $-1 \leq \lambda \leq 1$

- It means that when c_1 is between 0 and 2, we can use the local sensitivity theorem to compute the optimal value.

What if the Change is Outside the Range

If the change of $\boldsymbol{c}$ is so much that the reduced cost of the current solution contains negative number, then

- We can continue with the simplex method (tableau) until it reaches optimal solution.

If the change of $\boldsymbol{b}$ is so much that the solution corresponding to the original optimal basis B is no longer feasible, then:

- We may need to solve the problem from the start.
- However, we can also have a dual perspective: it can be viewed as that the objective coefficients of the dual problem changed. Then one can use the method that deals with changes in objective coefficients for the dual problem.

Changes to A

If the change is for a number in a non-basic column, say A_j, then the original optimal solution is still feasible (since the non-basic $x_N = 0$). The only change is to the reduced cost of jth variable.

- Recompute $\bar{c}_j$. If it is still nonnegative, then the original optimal solution stays optimal. Otherwise, update the simplex method (tableau) for the jth column as well as the reduced cost and continue from there.

If the change is for a number in a basic column, then nearly all the numbers in the simplex method (tableau) will change. In general, there is not a simple way to deal with it.

Other Changes

Adding a variable (the rest are kept the same):

- Assign the added variable to 0 and to be a non-basic variable. The original BFS is still a BFS, the reduced cost is unchanged.
- We only need to check the reduced cost corresponding to the new variable. If it is non-negative, then the original optimal solution plus the added non-basic variable is still optimal; otherwise continue the simplex method from there.

Adding a constraint:

- If the original optimal solution satisfies the constraint, then it is still optimal.
- If not, then the best way to deal with it is to think it as adding a dual variable, then use the simplex method (tableau) for the dual problem to continue calculations.