DDA5002 Optimization
Lecture 5 Simplex Method
Simplex Tableau

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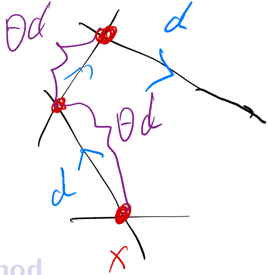
Announcement

- Homework 2 is posted and due October 12 (Sunday).

Outline

1. Simplex Method
2. One Iteration of Simplex Method Example
3. Degeneracy
4. Two-Phase Simplex Method
5. Correctness and Complexity of Simplex Method
6. Simplex Tableau
7. Two-Phase Method in Simplex Tableau

Outline



The Simplex Method

most difficult

We start from a BFS $\boldsymbol{x}$ (with corresponding basis B)

$$\begin{cases} x_B = A_B^{-1} b \geq 0 \\ x_N = 0 \end{cases}$$

1. We first compute the reduced costs $\bar{\boldsymbol{c}}$ for all nonbasic variables

$$\bar{c}_j = c_j - \boldsymbol{c}_B^T A_B^{-1} A_j$$

   - If $\bar{\boldsymbol{c}} \geq 0$, then $\boldsymbol{x}$ is already optimal
   - Otherwise choose some j such that $\bar{c}_j < 0$

   select x_j to enter the basis

2. Compute the jth basic direction $\boldsymbol{d} = \begin{bmatrix} -A_B^{-1} A_j \\ e_j \end{bmatrix} \begin{matrix} = d_B \\ = d_N \end{matrix}$

   - If $\boldsymbol{d} \geq 0$, then the problem is unbounded.
   - Otherwise, compute $\theta^* = \min_{i \in B, d_i < 0} \{-\frac{x_i}{d_i}\}$

   select x_i to exit basis

3. Let $\boldsymbol{y} = \boldsymbol{x} + \theta^* \boldsymbol{d}$. Then $\boldsymbol{y}$ is the new BFS with index j replacing i in the basis, where i is the index attaining the minimum in θ^*.

4. Repeat these procedures until one stopping criteria is met.

New solution: j enters and i exits basis

Simplex iteration:

$$y = x + \theta^* d$$

$$k \in N \qquad y_k = \begin{cases} 0 & k \in N \backslash j \\ \theta^* & k = j \end{cases}$$

$$k \in B \qquad y_k = \begin{cases} x_k + \theta^* d_k & k \in B \backslash i \\ 0 & k = i \end{cases}$$

- Basis for current iteration BFS x:

$$B = \{B(1), \ldots, B(\ell - 1), i, B(\ell + 1), \ldots, B(m)\}$$

- Basis for next iteration BFS y:

$$B' = \{B(1), \ldots, B(\ell - 1), j, B(\ell + 1), \ldots, B(m)\}$$

Outline

Example - Find BFS

$m = 3$

$n = 5$

$$
\begin{array}{llllllll}
\text{minimize} & -x_1 & -2x_2 & & & & & \\
\text{subject to} & x_1 & & +x_3 & & & = & 100 \\
 & & 2x_2 & & +x_4 & & = & 200 \\
 & x_1 & +x_2 & & & +x_5 & = & 150 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0
\end{array}
$$

Current basis: $\{x_2, x_3, x_5\}$, corresponding to columns 2, 3, and 5 of A.

$$
A_B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{T}
$$

The current basic feasible solution is

$$
\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \end{bmatrix} = A_B^{-1} b = \begin{bmatrix} 100 \\ 100 \\ 50 \end{bmatrix} \geq 0, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Example: Reduced Costs

$$
\begin{array}{lrrrrrl}
\text{minimize} & -x_1 & -2x_2 & +0x_3 & & +0\cdot x_5 & \\
\text{subject to} & x_1 & & +x_3 & & & = 100 \\
& & 2x_2 & & +x_4 & & = 200 \\
& x_1 & +x_2 & & & +x_5 & = 150 \\
& x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0
\end{array}
$$

$$x_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \qquad c_B = \begin{bmatrix} c_2 \\ c_3 \\ c_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

If we are at basis $\{2, 3, 5\}$, then the reduced costs are:

$$c_1 - c_B^T A_B^{-1} A_1$$

$$\bar{c}_1 = -1 - \begin{bmatrix} -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -1 < 0$$

$$c_4 - c_B^T A_B^{-1} A_4$$

$$\bar{c}_4 = 0 - \begin{bmatrix} -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 > 0$$

Therefore only including x_1 in the basis in the next iteration will reduce the objective value.

→ select x_1 to enter basis

Example - Directions

$j = 1$

$$\begin{array}{llllllll} \text{minimize} & -x_1 & -2x_2 & & & & & \\ \text{subject to} & x_1 & & +x_3 & & & = & 100 \\ & & 2x_2 & & +x_4 & & = & 200 \\ & x_1 & +x_2 & & & +x_5 & = & 150 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0 \end{array}$$

$x_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$

If we are at basis $\{2, 3, 5\}$ and select x_1 to enter the basis in the next iteration.

$$d_N = \begin{bmatrix} d_1 \\ d_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1$$

x_1

$$d_B = \begin{bmatrix} d_2 \\ d_3 \\ d_5 \end{bmatrix} = -A_B^{-1} A_1 = -\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

Example - Min-ratio Test

$$\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 50 \end{bmatrix}, \qquad d_B = \begin{bmatrix} d_2 \\ d_3 \\ d_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \begin{array}{l} \\ < 0 \\ < 0 \end{array}$$

Since some components of d_B is negative, we do not have an unbounded optimal solution, and we need to decide how far to go along this direction while still remaining feasible.

$$\theta^* = \min_{\{i \in B : d_i < 0\}} \frac{-x_i}{d_i} = \min\{\frac{-x_3}{d_3}, \frac{-x_5}{d_5}\} = \min\{\frac{-100}{-1}, \frac{-50}{-1}\} = 50.$$

Therefore x_5 exits the basis (becomes a non-basic variable) in the next iteration.

x_5

Example - New BFS

$$y = x + \theta^* d = \begin{bmatrix} x_B \\ x_N \end{bmatrix} + \theta^* \begin{bmatrix} d_B \\ d_N \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \\ x_1 \\ x_4 \end{bmatrix} + \theta^* \begin{bmatrix} d_2 \\ d_3 \\ d_5 \\ d_1 \\ d_4 \end{bmatrix} \begin{array}{l} d_B \\ \\ d_N \end{array}$$

$$= \begin{bmatrix} 100 \\ 100 \\ {\color{blue}50} \\ {\color{red}0} \\ 0 \end{bmatrix} + 50 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 100 \\ 50 \\ {\color{blue}0} \\ {\color{red}50} \\ 0 \end{bmatrix} \begin{array}{l} \to x_2 \\ \to x_3 \\ \to x_5 \\ \to x_1 \\ \to x_4 \end{array}$$

OUT IN x_B x_N

- Old Basis: $\{2, 3, 5\} \longrightarrow$ New Basis: $\{1, 2, 3\}$
- Old BFS: $(0, 100, 100, 0, 50) \longrightarrow$ New BFS: $(\overset{x_1}{50}, \overset{x_2}{100}, \overset{x_3}{50}, \overset{x_4}{0}, \overset{x_5}{0})$

Detail Example

Please see "Simplex Method Example" file on Blackboard.

Outline

Degeneracy

$x_B > 0$

- In standard-form LP, we call a basic feasible solution x **degenerate** if some of the basic variables are 0. Otherwise, it is **non-degenerate**.
- Simplex method may stuck in cycle when it has degeneracy BFS.
- **Bland's Rule:** If we use both the **smallest index rule** for choosing the entering basis and the exiting basis, then no cycle will occur in the simplex algorithm.
- Using the Bland's rule when applying the simplex method, we can guarantee to stop within a finite number of iterations at an optimal solution, or find the LP is unbounded.

Outline

Finding an Initial BFS

In our previous discussion, we assumed that we start with a certain BFS

- This can be done easily if the standard form is derived by adding slacks to each constraint and the right hand side is all nonnegative. (Why?)

However, in general, it is not necessarily easy to get an initial BFS from the standard form. For example,

$$
\begin{array}{llllll}
\text{minimize} & x_1 & +x_2 & +x_3 & & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & = & 3 \\
& & -4x_2 & -9x_3 & & = & -5 \\
& & & 3x_3 & +x_4 & = & 1 \\
& x_1 & ,x_2 & ,x_3 & ,x_4 & \geq & 0
\end{array}
$$

Finding an Initial BFS

- One could test different basis B, to see if $A_B^{-1}b \geq 0$.
- However, this may take a long time.
- In fact, in terms of computational complexity (which we will define later), finding one BFS is as hard as finding the optimal solution!

We will discuss an initialization method next — two-phase method.

Two-Phase Simplex Method

In the two-phase simplex method, we first solve an auxiliary problem (e means an all-one vector).

original problem:

$$\min_{x} \; c^T x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0$$

Phase-I LP:

$$\begin{aligned} \min_{x,y} \quad & e^T y \\ \text{s.t.} \quad & Ax + y = b \\ & x, y \geq 0 \end{aligned}$$

Without loss of generality, we assume $b \geq 0$ (otherwise, we pre-multiply that row by -1).

BFS

There is a trivial BFS to the auxiliary problem: $(x = 0, y = b \geq 0)$ so one can apply the Simplex method to solve it.

Theorem

The original problem is feasible if and only if the optimal value of the auxiliary problem is 0.

Standard form LP ✓

$$\min. \quad e^T y = [1 \; 1 \cdots 1] \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = y_1 + y_2 + \cdots + y_m$$

$$\text{s.t.} \quad Ax + y = b$$

$$x, y \geq 0$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$

$y \in \mathbb{R}^m$, $e \in \mathbb{R}^m$

BFS : $x = 0 \in \mathbb{R}^n$, $y = b \in \mathbb{R}^m$ ≥ 0

m equality constraints $\rightarrow$ $\underline{m}$ basic variables

$m+n$ variables $\rightarrow$ $\underline{n}$ nonbasic variables

Proof

For the " $\Longrightarrow$ " side, for any x feasible to the original problem, i.e.,

$$Ax = b, \; x \geq 0.$$

$y_1 + y_2 + \cdots + y_m \geq 0$

Then, it is easy to observe that $(x, y = 0)$ is feasible to the auxiliary problem. This immediately gives that $(x, y = 0)$ is also optimal to the auxiliary problem, as the function value of the auxiliary problem is always non-negative. For the " $\Longleftarrow$ " side, suppose that $(\hat{x}, \hat{y} = 0)$ is optimal to the auxiliary problem. Then, we have

$$A\hat{x} + \overset{0}{\cancel{\hat{y}}} = A\hat{x} = b,$$
$$\hat{x} \geq 0.$$

constraint for original problem

This means that $\hat{x}$ is feasible to the original problem.

Two-Phase Simplex Method

By above Theorem, we can solve the auxiliary problem by the Simplex method, and

1. If the optimal value is not 0, then we can claim that the original problem is infeasible;
2. If the optimal value is 0 with solution $(x^* = \hat{x}, y^* = 0)$. Then we know that $\hat{x}$ must be a BFS for the auxiliary problem. Then it must be a BFS for the original problem as well. And we can start from there to initialize the simplex method.

Original Problem:

$$\min \ c^T x$$
$$\text{s.t.} \ Ax = b$$
$$x \geq 0$$

Auxiliary Problem:

$$\min \ e^T y$$
$$\text{s.t.} \ Ax + y = b$$
$$x, y \geq 0$$

$\downarrow$ solve

BFS

$\begin{cases} x^* = \hat{x} \in R^n \\ y^* = 0 \in R^m \end{cases}$ optimal $\Leftrightarrow$ BFS

BFS $\begin{cases} \underline{m} \text{ basic variables} \\ \underline{n-m} \text{ nonbasic variables} \end{cases}$

BFS $\begin{cases} \underline{m} \text{ basic variables} \\ \underline{n} \text{ nonbasic variables} \end{cases}$

$\underline{n}$ variables are zero.

I have $\underline{m}$ y variables are zero.

I still have $\underline{n-m}$ variables are zero coming from $\hat{x}$.

$\Downarrow$

There are at least $\underline{n-m}$ variables in $\hat{x}$ are zeros. call $\hat{x}_N$

The remaining m variables in $\hat{x}$ call $\hat{x}_B$.

$$\hat{x} = \begin{bmatrix} \hat{x}_B \\ \hat{x}_N \end{bmatrix} \begin{matrix} \in R^m, \geq 0 \\ \in R^{n-m}, = 0 \end{matrix}$$

optimal: $\begin{cases} x^* = \hat{x} \\ y^* = 0 \end{cases} \Rightarrow \begin{cases} \hat{x} = \begin{bmatrix} \hat{x}_B \\ \hat{x}_N \end{bmatrix} \begin{matrix} \in \mathbb{R}^m & \geq 0 \\ \in \mathbb{R}^{n-m} & = 0 \end{matrix} \\ y^* = 0 \in \mathbb{R}^m \end{cases}$

If there is some zero is $\hat{x}_B$, then this optimal BFS is degenerate.

e.g. Basis for Phase-I LP basic variables: $m-1$ $\hat{x}_B$'s $+$ 1 y^*

select 1 $\hat{x}_B$ variable which is equal to zero as the last basic variable for the original problem.

Basis for Phase-I LP: some $\hat{x}_B$'s + some y^*

Select $\hat{x}_B$ variables which are equal to zero as the remaining basic variables for the original problem.

Degenerate Case

If there is at least one basic variable in y^* (degenerate case), $x^* = \hat{x}$ will not contain the full basis B. What can we do in this case?

Let us assume one basic index is in y^* (say $B(m)$) w.l.o.g.:

- We still have $A\hat{x} = b$, $\hat{x} \geq 0$ (feasibility of $\hat{x}$).
- $\hat{x}$ has $m-1$ basic variables corresponding to $x_{B(1)}, \ldots, x_{B(m-1)}$. In addition, other parts in $\hat{x}$ must be zeros since they are non-basic.
- Then one can pick any other column from A (such that it forms a full-rank matrix with $A_{B(1)}, \ldots, A_{B(m-1)}$) in the non-basic part in $\hat{x}$ to make it a BFS for the original problem.

Procedure of the Two-Phase Method

Phase I:

1. Construct the auxiliary problem such that $b \geq 0$.
2. Solve the auxiliary problem using the Simplex method.
   - If we reach an optimal solution with optimal value greater than 0, then the original problem is infeasible.
3. If the optimal value is 0 with optimal solution $(x^* = \hat{x}, y^* = 0)$, then we enter phase II.

initial BFS
$x=0$ $y=b$

Phase II: Solve the original problem starting from the BFS $\hat{x}$.

- If $(x^* = \hat{x}, y^* = 0)$ for the auxiliary problem has one or more basic indices appearing in ~~y~~ y^*, then we need to supplement some indices from the non-basic part in $\hat{x}$ to make $\hat{x}$ a BFS for the original problem.

The Big-M method

There is another method that can be used to solve LP without a starting BFS. Consider the following auxiliary problem:

A large number

$$\begin{array}{ll} \min\limits_{x,y} & c^T x + M \sum_{i=1}^m y_i \\ \text{s.t.} & Ax + y = b \\ & x, y \geq 0 \end{array}$$

This problem has an initial BFS $x = 0, y = b \geq 0$ (assuming $b \geq 0$). Now we can use simplex to solve it. In the simplex procedure, pretend that M is a very large value (larger than any specified number).

- If the original problem is feasible, then optimal y must be 0.

Outline

Correctness and Complexity of Simplex Method

- If every basic feasible solution of a standard form linear program is nondegenerate, then the simplex method always terminates in a finite number of steps.
- For all the known versions of simplex methods, the problem instances such that the algorithm will run exponential iterations in terms of n and m to solve that problem. Simplex is not a **polynomial-time algorithm**.
- This is only the worst-case performance. The practical performance of simplex method is quite good. Typically it needs a small multiple of m iterations to stop.
- One can prove that on average, simplex method stops in a polynomial number of iterations.
- It is still one of the most widely-used algorithms to solve LP (in history).

Complexity of LP

- Simplex method is not be a polynomial-time algorithm, but this doesn't mean that LP is not polynomial-time solvable.
- Whether LP is in P was a major maths problem in the 20th century. It was finally solved by Soviet Union mathematician Khachiyan in 1979, who showed a first polynomial-time algorithm for LP. His method is called the *ellipsoid method*.


ARCHIVES | A Soviet Discovery Rocks World of Mathematics

ARCHIVES | 1979

A Soviet Discovery Rocks World of Mathematics

By MALCOLM W. BROWNE NOV. 7, 1979

The New York Times

A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics and computer analysis, and experts have begun exploring its practical applications.

Mathematicians describe the discov ery by L.G. Khachian as a method by which computers can find guaranteed solutions to a class of very difficult problems that have hitherto been tackled on a kind of hit–or–miss basis.

Apart from its profound theoretical interest, the discovery may be applicable in weather prediction, complicated

This concludes the Simplex Method. You are not required to solve an LP by hand using the simplex tableau in this class. However, learning simplex tableau may be helpful for solving homework and exam problems.

Outline

Simplex Tableau

A_B^{-1}

Now we have obtained the algebraic procedures for the simplex method.

- We want to have a simpler implementation of it — in particular, we want to avoid explicit matrix inversion in the calculation

We are going to introduce the simplex tableau, which is a practical way to implement the simplex method.

- The simplex tableau maintains a table of numbers
- It visualizes the procedures of the simplex algorithm and facilitates the computation
- After learning the simplex tableau, one should be able to solve small-sized linear optimization problems by hand

Simplex Tableau

The simplex tableau is a table with the following structure (the corresponding basis matrix and objective coefficients are A_B and c_B):

$c^T - c_B^T A_B^{-1} A$	$-c_B^T A_B^{-1} b$
$A_B^{-1} A$	$A_B^{-1} b$

In the following, we take a closer look at what each part of the tableau means (and looks like) and how we can update the tableau efficiently in each iteration.

Simplex Tableau

$c^T - c_B^T A_B^{-1} A$	$-c_B^T A_B^{-1} b$
$A_B^{-1} A$	$A_B^{-1} b$

directions $\rightarrow$ $A_B^{-1}A$; $A_B^{-1}b \rightarrow x_B$

The lower part of the tableau can be viewed as a transformation of the constraint $Ax = b$ to $A_B^{-1}Ax = A_B^{-1}b$. It is equivalent to the original constraint $Ax = b$.

Furthermore, if we write $A = [A_B, A_N]$, then

$$A_B^{-1}A = A_B^{-1}[A_B, A_N] = [I, A_B^{-1}A_N]$$

Therefore, this part must contain an identity matrix.

Also when the basis is B, the current basic feasible solution is

$$x = [x_B; x_N] = [A_B^{-1}b; 0]$$

Therefore the lower right corner gives x_B of the current BFS.

$$Ax = b$$

$$\Rightarrow A_B^{-1} A x = A_B^{-1} b$$

$$\Rightarrow A_B^{-1} [A_B \quad A_N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = A_B^{-1} b$$

$$A_B^{-1} A_B x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

$$I x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

value

$\rightarrow x_B$

Simplex Tableau

reduced costs

basic variables

non-basic variables

$c^T - c_B^T A_B^{-1} A$	$-c_B^T A_B^{-1} b$
$A_B^{-1} A$	$A_B^{-1} b$

negative objective function value

0

$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$

The term $c^T - c_B^T A_B^{-1} A$ is the reduced cost at this basis.

- Reduced costs for basic variables are 0's. Therefore, this part are 0's for the basic indices.

Lastly, the term

$$-c_B^T A_B^{-1} b = -c_B^T x_B = -(\underline{c_B^T x_B + c_N^T \cancel{x_N}}) = -c^T x$$

?

is the negative of the objective value at this BFS.

Simplex Tableau

Therefore, the simplex tableau should look like (after reordering the columns)

reduced costs

x_B	x_N >0	
0_m	$c_N^T - c_B^T A_B^{-1} A_N$	$-c_B^T x_B$
I_m	$A_B^{-1} A_N$	x_B

negative directions

negative obj val

basic variables

Here 0_m is a vector of m zeros and I_m is the m-dimensional identity matrix.

This form of LP is called the *canonical* form.

- A canonical form in the simplex tableau corresponds to an BFS and its status.
- The constraint matrix for the basic variables (not necessarily the first m columns) is an identity matrix.
- The reduced costs part for the basic variables is zero.
- In a simplex tableau of canonical form, the numbers in the top-left block are all nonnegative, then we have reached optimality.

Simplex Tableau

The simplex tableau will maintain a canonical form for each iteration until we reach optimality.

- If in a simplex tableau, the numbers in the left top row are all nonnegative, then we have reached optimality!
- For now we assume that we already have a canonical form to start with.

We want to map each algebraic step of the simplex method to an operation on the simplex tableau. Here are the key steps:

1. Compute the reduced costs.
2. Choose the incoming non-basic index j.
3. Compute the step size θ^* and choose the leaving basic index i.
4. Update the tableau with the new basis.

We call the procedure of transforming from one canonical form (one set of basis) to another one pivoting.

Simplex Tableau Start Setting

Let's assume we have an LP in the following format with $b \geq 0$:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

It can be transferred into a standard form LP in canonical form:

$s \in R^m$

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{s.t.} \quad & Ax + Is = b \\ & x, s \geq 0 \end{aligned}$$

m equality constraints

m+n variables

m basic variables

n nonbasic variables

- The above standard form LP has a BFS: $x = 0$ $\in R^n$ (nonbasic variables) and $s = b$ $\in R^m$ (basic variables).
- The simplex tableau starts from this BFS.
- The objective coefficients for basic variables s are 0's.
- The reduced costs for nonbasic variables x are the vector c. (why?)

$A_B = I_{m \times m}$

$$\min \quad c^T x + 0 \cdot s$$
$$\text{s.t.} \quad Ax + s = b$$
$$x, s \geq 0$$

$$x_B = s = b$$
$$x_N = x = 0$$

Q1: What is the reduced cost for x?

Q2: What is the reduced cost for s?

$$\bar{c}_x = c_x - c_B^T A_B^{-1} A_x$$
$$= c - 0 = c$$

Reduced costs for basic variables are always 0!

Example

The production problem

$$
\begin{array}{llllllll}
\text{minimize} & -x_1 & -2x_2 & & & & \\
\text{subject to} & x_1 & & +s_1 & & & = 100 \\
& & 2x_2 & & +s_2 & & = 200 \\
& x_1 & +x_2 & & & +s_3 & = 150 \\
& x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0
\end{array}
$$

It is already in the canonical form as follows:

reduced costs

	x_1	x_2	s_1	s_2	s_3	
	-1	-2	0	0	0	0
s_1	1	0	1	0	0	100
s_2	0	2	0	1	0	200
s_3	1	1	0	0	1	150

direction

$$X_B = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = b = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$$

$$X_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Is it an optimal solution?

- No. Because the top row contains negative entries (there exist negative reduced costs)

Pivoting Step I: Choose the Entering Index j

In the canonical form, the reduced costs are simply the coefficients in the top-left block.

- We can choose any column with negative reduced cost to be the entering basic index.

non basic basic

Consider the example:

		x_1	x_2	s_1	s_2	s_3	
	B	-1	-2	0	0	0	0
s_1	3	1	0	1	0	0	100
s_2	4	0	2	0	1	0	200
s_3	5	1	1	0	0	1	150

I_3

We can choose either the first or second column as the entering index j (if we use Bland's rule, then we choose the first one).

$\bar{A}$

Pivoting Step II: θ^* and leaving Index i

Assume we have chosen column j as the entering index.

We need to make sure that the next BFS is still feasible (≥ 0). The step size θ^* was determined via:

$$\theta^* = \min_{d_i<0, i\in B} -\frac{x_i}{d_i}$$

where x_i is the ith entry of the basic solution and $d_B = -A_B^{-1}A_j$.

In the simplex tableau, this is equivalent to

$$\theta^* = \min_i \left\{ \frac{x_i}{\bar{A}_{ij}} : \bar{A}_{ij} > 0 \right\}$$

where $x_i \in x_B$ is the lower right column and $\bar{A}$ is the lower left part of the tableau.

- This is called the *Minimal Ratio Test (MRT).*

Pivoting Step II: Compute θ^* and Leaving Index i

If $\bar{A}_{ij} \leq 0$ for all i, then the problem is unbounded.

Otherwise, assume index i achieves the minimum in:

$$\theta^* = \min \left\{ \frac{\bar{b}_i}{\bar{A}_{ij}} : \bar{A}_{ij} > 0 \right\}$$

Example: If we choose column 1 to be the entering index:

		x_1	x_2	s_1	s_2	s_3	
	B	-1	-2	0	0	0	0
s_1	3	1	0	1	0	0	100
s_2	4	0	2	0	1	0	200
s_3	5	1	1	0	0	1	150

b

$\frac{100}{1}$ small

$\frac{150}{1}$

- By testing $x_i/\bar{A}_{ij}$ for $j = 1$ and $i = 3$ and $i = 5$, we have $i = 3$ achieves MRT and hence basic index $i = 3$ is the leaving index.
- Then, MRT will choose the third column in A (also $\bar{A}$) to leave the basis.

Example Continued

	x_1	x_2	s_1	s_2	s_3	
B	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

$\frac{200}{2} = 100$

$\frac{150}{1} = 150$

What if we choose column 2 to be the entering basis?

- Then the MRT will choose the fourth column as the leaving index.
- We choose the basis whose second row element is 1 to be the leaving column, which in this case is column 4.

Pivot Column, Row, Element

- We call the entering column the **pivot column**
- We call the row that achieves the MRT the **pivot row** (determines the leaving index/column)
- The intersection element of the pivot column and the pivot row is called the **pivot element**

	B	x_1					
	B	-1	-2	0	0	0	0
S_1	3	1	0	1	0	0	100
S_2	4	0	2	0	1	0	200
S_3	5	1	1	0	0	1	150

Update the Tableau

Assume we have determined the entering and leaving index (pivoting element $\bar{A}_{ij}$).

Then we perform the following two steps:

1. Divide each element in the pivot row by the pivot element.
2. Add proper multiples (could be negative) of the pivot row (after the first step) to each other rows, including the top row of reduced costs and objective coefficients, such that all other elements in the pivot column become zeros (including the top row).

- The above two operations include the right-hand-side columns.
- After this procedure, the new pivot column should be $(0; \dots; 0; 1; 0; \dots; 0)$ with 1 at the pivot row.
- The new resulting tableau will still be in a canonical form, however, with the new basis.

Simplex Method in the Tableau

We have shown how to get from one canonical form to another, we then repeat this procedure until we reach optimality or find LP unbounded.

- When choosing the entering and the leaving index, we use the smallest index rule .
- This will guarantee that the simplex iterations will terminate in a finite number of steps.

We also attach the index of the basis to the left of the tableau to indicate the current basis (just for clarity).

Example - Iteration 1

Consider the example. The initial simplex tableau is:

B						
	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

We use the smallest index rule. The pivot column (entering index $j = 1$) is the first column, the pivot row is the first row (leaving index is $i = 3$ or $i = x_3$), the pivot element is 1 (in blue).

- Divide the pivot row by the pivot element
- Add proper multiples of row 1 to other rows (including the top row) such that all other elements in the pivot column become zero (including the top element).

Example - Iteration 2

The tableau becomes:

	x_1	x_2	S_1	S_2	S_3		
B	0	-2	1	0	0	100	
1	1	0	1	0	0	100	
4	0	2	0	1	0	200	$\frac{200}{2}$
5	0	1	-1	0	1	50	$\frac{50}{1}$

- It is not optimal since there is one negative reduced cost.
- The only choice for the pivot column is column 2.
- Use MRT, the pivot row should be row 3 (leaving index is $i = 5$ or $i = x_5$).

Then we apply the same procedure

- Add $2\times$ row 3 to the very top row, and $-2\times$ row 3 to the second row in the constraint.

Example - Iteration 3

The tableau becomes

	x_1	x_2	s_1	s_2	s_3	
B	0	0	-1	0	2	200
1	1	0	1	0	0	100
4	0	0	2	1	-2	100
2	0	1	-1	0	1	50

$\frac{100}{1}$

$\frac{100}{2} = 50$

- It is still not optimal since there is one negative reduced cost.
- The only choice for the pivot column is column 3.
- Use MRT, the pivot row should be row 2 (leaving index is $i = 4$ or $i = x_4$).

We apply the same procedure again

- Divide row 2 by 2, then add $1\times$ row 2 to the very top row, add $-1\times$ row 2 to the first row in the constraint, add $1\times$ row 2 to the last row.

Example - Iteration 4

≥ 0

The tableau becomes: → neg obj

	B						
	B	0	0	0	1/2	1	250
x_1	1	1	0	0	-1/2	1	50
s_1	3	0	0	1	1/2	-1	50
x_2	2	0	1	0	1/2	0	100

All the reduced costs are positive now

- It is optimal.
  x_1 x_2 s_1 s_2 s_3
- The optimal solution is $(\underbrace{50, 100, 50}_{x_B}, 0, 0)$ with optimal value -250.

Another Example

Consider the linear optimization problem:

$$\begin{aligned} \text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 \leq 20 \\ & 2x_1 + x_2 + 2x_3 \leq 20 \\ & 2x_1 + 2x_2 + x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$Ax \leq b,\ b \geq 0$

First, we write down the standard form:

$$\begin{array}{lccccccc} \text{minimize} & -10x_1 & -12x_2 & -12x_3 & & & & \\ \text{s.t.} & x_1 & +2x_2 & +2x_3 & +s_1 & & & = 20 \\ & 2x_1 & +x_2 & +2x_3 & & +s_2 & & = 20 \\ & 2x_1 & +2x_2 & +x_3 & & & +s_3 & = 20 \\ & x_1, & x_2, & x_3, & s_1, & s_2, & s_3 & \geq 0 \end{array}$$

$$
\begin{array}{llllllll}
\text{minimize} & -10x_1 & -12x_2 & -12x_3 & & & & \\
\text{s.t.} & x_1 & +2x_2 & +2x_3 & +s_1 & & & = 20 \\
 & 2x_1 & +x_2 & +2x_3 & & +s_2 & & = 20 \\
 & 2x_1 & +2x_2 & +x_3 & & & +s_3 & = 20 \\
 & x_1, & x_2, & x_3, & s_1, & s_2, & s_3 & \geq 0
\end{array}
$$

reduced cost

B	x_1	x_2	x_3	s_1	s_2	s_3		
	-10	-12	-12	0	0	0	0	
s_1	1	2	2	1	0	0	20	$\frac{20}{1}$
s_2	2	1	2	0	1	0	20	$\frac{20}{2}$
s_3	2	2	1	0	0	1	20	$\frac{20}{2}$

B	x_1	x_2	x_3	s_1	s_2	s_3		
	0	-7	-2	0	5	0	100	
s_1	0	3/2	1	1	-1/2	0	10	$10/\frac{3}{2}$
x_1	1	1/2	1	0	1/2	0	10	$10/\frac{1}{2}$
s_3	0	1	-1	0	-1	1	0	0/1

	x_1	x_2	x_3	S_1	S_2	S_3		
B	0	0	-9	0	-2	7	100	
S_1	0	0	$5/2$	1	1	$-\frac{3}{2}$	10	$10/\frac{5}{2}$
X_1	1	0	$3/2$	0	1	$-\frac{1}{2}$	10	$10/\frac{3}{2}$
X_2	0	1	-1	0	-1	1	0	

B	0	0	0	$18/5$	$8/5$	$\frac{8}{5}$	136
X_3	0	0	1	$2/5$	$2/5$	$-\frac{3}{5}$	4
X_1	1	0	0	$-\frac{3}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	4
X_2	0	0	0	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{2}{5}$	4

≥ 0

$obj = -136$

$x_3 = 4, \; x_1 = 4, \; x_2 = 4$

$S_1 = 0, \; S_2 = 0 \; S_3 = 0$

optimal

Another Example - Iteration I

We write down the initial tableau:

B							
B	-10	-12	-12	0	0	0	0
4	1	2	2	1	0	0	20
5	2	1	2	0	1	0	20
6	2	2	1	0	0	1	20

- By the smallest index rule, we choose column 1 to enter the basis.
- Use MRT, we have two candidates to leave the basis: 5th column (row 2) or 6th column (row 3).
- By the smallest index rule again, we choose 5th column to exit (pivot row is row 2).
- Divide 2 to each element in row 2
- Add $10\times$ new row 2 to the top row, $-1\times$ new row 2 to the first constraint row, and $-2\times$ new row 2 to the last row.

Another Example - Iteration II

Then the tableau becomes:

B	0	-7	-2	0	5	0	100
4	0	3/2	1	1	-1/2	0	10
1	1	1/2	1	0	1/2	0	10
6	0	1	-1	0	-1	1	0

- Column 2 is the pivot column. By MRT, the pivot row is row 3.
- Here we encounter a degeneracy case where the minimal ratio is 0. It means that in this pivoting, we can't strictly improve the objective value.
- But we can still proceed as normal (no cycle will occur if we use the Bland's rule).
  - We add $7\times$ row 3 to the top row, $-3/2\times$ row 3 to second row.

Another Example - Iteration III

Then the tableau becomes:

B							
B	0	0	-9	0	-2	7	100
4	0	0	5/2	1	1	-3/2	10
1	1	0	3/2	0	1	-1/2	10
2	0	1	-1	0	-1	1	0

- We choose column 3 to enter the basis. By MRT, the pivot row is row 1 (column 4 or s_1 is leaving basis).
- We multiply 2/5 to each number in row 1, then add $9\times$ row 1 to the top row, $-3/2\times$ row 1 to the second row and $1\times$ row 1 to the last row.

Another Example - Iteration IV

Then the tableau becomes:

B							
	0	0	0	18/5	8/5	8/5	136
3	0	0	1	2/5	2/5	-3/5	4
1	1	0	0	-3/5	2/5	2/5	4
2	0	1	0	2/5	-3/5	2/5	4

This is optimal since all reduced costs are non-negative. The optimal solution is $(4, 4, 4, 0, 0, 0)$ with optimal value -136.

Outline

Two-Phase Method in Simplex Tableau

For the simplex tableau, when there is no obvious initial basic feasible solution, we still need to use the two-phase method.

To carry out the two-phase methods in the simplex tableau, we need to solve some additional issues.

$$\begin{aligned} \min_{x,y} \quad & e^T y \\ \text{s.t.} \quad & Ax + y = b \\ & x, y \geq 0 \end{aligned}$$

Although there is an identity matrix in the constraints (corresponding to y), the auxiliary problem is not in the canonical form - the corresponding objective coefficients are not 0.

- Therefore, we need to calculate the top row of the initial tableau for the Phase I problem.

Two-Phase Method in Simplex Tableau

To compute the simplex tableau for the Phase I problem

- The bottom part can use the constraint matrix, and the basis is corresponding to y part
- For basic part, the reduced costs are 0
- For nonbasic part, $\bar{c}_j = c_j - c_B^T A_B^{-1} A_j = -e^T A_j$, so the jth reduced cost is the negative of the sum of that column
- This also applies to the initial objective value, which equals the negative of the sum of the right hand side vector.

Example

$$
\begin{array}{lrrrrcr}
\text{minimize} & x_1 & +x_2 & +x_3 & & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & = & 3 \\
& & -4x_2 & -9x_3 & & = & -5 \\
& & & +3x_3 & +x_4 & = & 1 \\
& x_1 & , x_2 & , x_3 & , x_4 & \geq & 0
\end{array}
$$

First, make $\boldsymbol{b}$ positive and construct the auxiliary problem:

$$
\begin{array}{lrrrrrrrcr}
\text{minimize} & & & & & y_1 & +y_2 & +y_3 & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & +y_1 & & & = & 3 \\
& & 4x_2 & +9x_3 & & & +y_2 & & = & 5 \\
& & & +3x_3 & +x_4 & & & +y_3 & = & 1 \\
& x_1, & x_2, & x_3, & x_4, & y_1, & y_2, & y_3 & \geq & 0
\end{array}
$$

Example Continued

Construct the initial tableau for the auxiliary problem

B	-1	-6	-15	-1	0	0	0	-9
y_1	1	2	3	0	1	0	0	3
y_2	0	4	9	0	0	1	0	5
y_3	0	0	3	1	0	0	1	1

Carry out the simplex method (Step 1):

B	0	-4	-12	-1	1	0	0	-6
x_1	1	2	3	0	1	0	0	3
y_2	0	4	9	0	0	1	0	5
y_3	0	0	3	1	0	0	1	1

Example Continued

Step 2:

B	0	0	-3	-1	1	1	0	-1
x_1	1	0	-3/2	0	1	-1/2	0	1/2
x_2	0	1	9/4	0	0	1/4	0	5/4
y_3	0	0	3	1	0	0	1	1

Step 3:

B	0	0	0	0	1	1	1	0
x_1	1	0	0	1/2	1	-1/2	1/2	1
x_2	0	1	0	-3/4	0	1/4	-3/4	1/2
x_3	0	0	1	1/3	0	0	1/3	1/3

This is optimal for the auxiliary problem.
$x = (x_1, x_2, x_3, x_4) = (1, 1/2, 1/3, 0)$ is a BFS for the original problem ($B = \{1, 2, 3\}$).

Example Continued

B	0	0	0	0	1	1	1	0
x_1	1	0	0	1/2	1	-1/2	1/2	1
x_2	0	1	0	-3/4	0	1/4	-3/4	1/2
x_3	0	0	1	1/3	0	0	1/3	1/3

We drop all the columns for auxiliary variables. Then we recompute the reduced cost for the original problem for $B = \{1, 2, 3\}$:

$$\begin{aligned}\bar{c} &= c^T - c_B^T A_B^{-1} A \\ &= c^T - c_B^T \begin{bmatrix} 1/2 & 1 & -1/2 & 1/2 & 1 \\ -3/4 & 0 & 1/4 & -3/4 & 1/2 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \end{bmatrix} \\ &= (0, 0, 0, -1/12)\end{aligned}$$

We also need to compute the current objective value: $11/6$

Example Continued

Now the Simplex tableau becomes:

B					
B	0	0	0	-1/12	-11/6
x_1	1	0	0	1/2	1
x_2	0	1	0	-3/4	1/2
x_3	0	0	1	1/3	1/3

Example Continued

Then we continue from the new simplex tableau:

B					
B	0	0	0	-1/12	-11/6
x_1	1	0	0	1/2	1
x_2	0	1	0	-3/4	1/2
x_3	0	0	1	1/3	1/3

The next pivot:

B	0	0	1/4	0	-7/4
1	1	0	-3/2	0	1/2
2	0	1	9/4	0	5/4
4	0	0	3	1	1

This is optimal. The optimal solution is $x = (1/2, 5/4, 0, 1)$. The optimal value is $7/4$.