DDA5002 Optimization
Lecture 7 Sensitivity Analysis
IP Modeling

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Announcements

- **Homework 3:** Duality Theory + IP Modeling — due **Oct 30**
- **Nov 2 (Sunday, 10am–12pm): Midterm Exam**
  - Covers everything up to IP Modeling
  - Bad News: Closed-book, Closed-notes, No Cheat Sheet
  - Sample Midterm is posted on Blackboard.
  - Review Session during next Wednesday's class.

Outline

Outline

Table to Form Dual Problem

Min $\longrightarrow$

$\longleftarrow$ Max

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

Primal and Dual Pair in Standard Form

$$
\begin{aligned}
\text{(P)} \quad \min \quad & c^\mathsf{T} x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{(D)} \quad \max \quad & b^\mathsf{T} y \\
\text{s.t.} \quad & A^\mathsf{T} y \leq c \\
& y \text{ free}
\end{aligned}
$$

Weak and Strong Duality Theorems

Primal		**Dual**	
min	$\boldsymbol{c}^T\boldsymbol{x}$	max	$\boldsymbol{b}^T\boldsymbol{y}$
s.t.	$A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$	s.t.	$A^T\boldsymbol{y} \leq \boldsymbol{c}$

Theorem (Weak Duality Theorem)

If $\boldsymbol{x}$ is feasible to the primal and $\boldsymbol{y}$ is feasible to the dual, then

$$\boldsymbol{b}^T\boldsymbol{y} \leq \boldsymbol{c}^T\boldsymbol{x}$$

Theorem (Strong Duality Theorem)

If a linear program has an optimal solution, so does its dual, and the optimal values of the primal and dual are equal $c^T x^* = b^T y^*$

LP Optimality via Duality

Based on the strong duality theorem, we know that (x, y) is optimal to the primal and dual respectively if and only if

- x is primal feasible,
- y is dual feasible,
- They achieve the same objective value.

Therefore solving LP (in standard form) is in fact equivalent as solving the following linear system:

- 1 $Ax = b$, $x \geq 0$
- 2 $A^T y \leq c$
- 3 $b^T y = c^T x$

X and y are optimal

$\Updownarrow$

1 2 3

Table of Possibles and Impossibles

The primal and dual LPs can be finite optimal, or unbounded, or infeasible. So, there are in total 9 combinations. Are all these 9 combinations possible?

	Finite Optimal	Unbounded	Infeasible
Finite Optimal	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Notice this table is exactly symmetric, because the dual of the dual is the primal.

Complementary Slackness Conditions

Consider the primal-dual pair:

Primal			**Dual**		
minimize	$c^T x$		maximize	$b^T y$	
subject to	$a_i^T x \geq b_i,$	$i \in M_1,$	subject to	$y_i \geq 0,$	$i \in M_1$
	$a_i^T x \leq b_i,$	$i \in M_2,$		$y_i \leq 0,$	$i \in M_2$
	$a_i^T x = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \geq 0,$	$j \in N_1,$		$A_j^T y \leq c_j,$	$j \in N_1$
	$x_j \leq 0,$	$j \in N_2,$		$A_j^T y \geq c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$		$A_j^T y = c_j,$	$j \in N_3$

Theorem

Let x and y are feasible solutions to the primal and dual problems respectively. Then x and y are optimal if and only if

dual constraint

dual variable   Primal constraint   Primal variable

$$y_i \cdot (a_i^T x - b_i) = 0, \quad \forall i; \quad x_j \cdot (A_j^T y - c_j) = 0, \quad \forall j.$$

LP Optimality via Complementary Slackness

Recall that the optimality conditions for LPs were given by:

1 1. x is primal feasible,
2 2. y is dual feasible,
3 3. The objective values coincide, i.e., $c^\top x = b^\top y$.

Now with the complementary slackness, we can write an equivalent iff optimality conditions:

X and Y are optimal

$\Updownarrow$

KKT conditions

1 2 3

$\Updownarrow$

3'

1 1. x is primal feasible,
2 2. y is dual feasible,
3' 3. The complementary slackness conditions are satisfied.

Primal feasibility, dual feasibility, and complementary slackness form the Karush-Kuhn-Tucker (KKT) optimality conditions for LP.

Shadow Price Interpretation

$\cdots \geq b_i$ demand

$\cdots \leq b_i$ supply/resource

- Dual variables represent the unit economic evaluation of the demand/resource.
- In a diet problem, dual variables evaluate how much an additional unit of demand (nutrient) costs.
- In a production planning problem, dual variables evaluate how much an additional unit of resource (workers, fertilizers, raw materials etc.) is worth.
- The shadow price can be viewed as a **unit/fair price** of the demand/resource that you would be willing to pay/purchase.

y_i^*: optimal cost change when $b_i \to b_i + 1$

Alternative Systems

Given a set of linear inequalities:

$$A^T y \leq c$$

An important question is: whether the system has a solution?

- It is easy to verify that it has a solution, one only needs to find a solution (we call it a *certificate*).
- To disprove the existence, can we also have such a certificate?

The answer: Yes.

- If we can find a vector x satisfying

$$Ax = 0, \quad x \geq 0, \quad c^T x < 0$$

  Then there must be no solution to the system $A^T y \leq c$

Thm: If $\exists\, x$ s.t. $Ax=0,\ x\geq 0,\ c^T x<0$,
then $\{y: A^T y \leq c\} = \emptyset$.

Proof: Consider the following LP:

$$(P)\quad \begin{aligned} \min.\ & c^T x \\ \text{s.t. } & Ax=0 \quad (y) \\ & x\geq 0 \end{aligned} \qquad\qquad (D)\quad \begin{aligned} \max.\ & 0^T y = 0 \\ \text{s.t. } & A^T y \leq c \\ & y \text{ free} \end{aligned}$$

If (P) is feasible and $c^T x<0$

$\exists\, x: Ax=0,\ x\geq 0,\ c^T x<0$

$x \to 2x$ $\quad A\cdot(2x)=0,\ 2x\geq 0,\ c^T(2x)<0$

$\Rightarrow A\cdot(kx)=0,\ kx\geq 0,\ c^T(kx)<0,\ \forall k>0$

kx is feasible for (P) and $c^T(kx)<0,\ \forall k>0$

I can scale x to ∞x, $c^T(\infty x)=\infty\cdot \underbrace{c^T x}_{<0} \to -\infty$

This means (P) is unbounded.

By the table of possibles and impossibles,
(D) must be infeasible.

Thus $\{y: A^T y\leq c\} = \emptyset$

Farkas' Lemma (Theorem of Alternative)

Theorem (Farkas' Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following two alternatives hold:

(I) $P := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \neq \emptyset$

(II) $Q := \{y \in \mathbb{R}^m : A^\top y \geq 0, b^\top y < 0\} \neq \emptyset$

(a) If $P \neq \emptyset$, then $Q = \emptyset$

(b) If $P = \emptyset$, then $Q \neq \emptyset$

$$\text{(P)}\quad \begin{aligned} \max.\ & 0^T x = 0 \\ \text{s.t.}\ & Ax = b \quad (y) \\ & x \geq 0 \end{aligned} \qquad \text{(D)}\quad \begin{aligned} \min.\ & b^T y \\ \text{s.t.}\ & A^T y \geq 0 \\ & y \text{ free} \end{aligned}$$

(a) If $P \neq \emptyset$, $\exists\, x$ s.t.

$Ax = b$, $x \geq 0$,

then (P) is feasible and (P) must have optimal sol^n (because obj $= 0$)

By weak duality, we have $b^T y \geq 0$

for any y satisfying $A^T y \geq 0$.

$\Rightarrow$ If $A^T y \geq 0$, then $b^T y \geq 0$

So $Q = \emptyset$

(b) If $P = \emptyset$, then (P) is infeasible.

By the table of possibles and impossibles,

(D) is unbounded or ~~infeasible~~.

Since $y = 0$ is feasible to (D), so

(D) must be unbounded.

$\exists\, y$ satisfying $A^T y \geq 0$ and $b^T y \to -\infty$

This means $\exists\, y$: $A^T y \geq 0$ and $b^T y < 0$

So $Q \neq \emptyset$.

Alternative Systems

One can construct many more pairs of such alternative systems.

- It is hard to directly prove something is not feasible.
- LP duality provides an alternative approach, transforming the problem to proving something is feasible.

Outline

Sensitivity Analysis

min. $c^T x$
s.t. $Ax = b$
$x \geq 0$

One important question when studying LP is as follows:

- How do the optimal solution and the optimal value change when the input changes?

This type of problems is called the *Sensitivity Analysis* of LP.

- We first study this question from a local perspective, and then globally.
- Local sensitivity analysis means the change of input is small such that the optimal solution does not change.

Local Sensitivity

Consider the standard form LP: optimal value

$$\begin{aligned} V = \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Define the optimal value by V.

- Given A and $\boldsymbol{c}$ fixed, V can be viewed as a function of $\boldsymbol{b}$: $V(\boldsymbol{b})$

Theorem

If the dual has a unique optimal solution $\boldsymbol{y}^*$, then $\nabla_b V(\boldsymbol{b}) = \boldsymbol{y}^*$.

- If the dual optimal solution is not unique (or the dual problem is unbounded or infeasible), then the gradient does not exist.
- If one changes b_i by a small amount Δb_i, then the change of the objective value will be $\Delta b_i y_i^*$

$$f(x) = f(x_1, \cdots, x_n)$$

Gradient of $f(x)$

$$\nabla f(x) = \begin{bmatrix} \dfrac{\partial f(x)}{\partial x_1} \\ \vdots \\ \dfrac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

Explanation

Strong duality $V = c^T x^* = b^T y^* = V(b)$

We know that the optimal value V is also the optimal value of the dual problem:

$$\max_{\boldsymbol{y}} \quad \boldsymbol{b}^T \boldsymbol{y}$$
$$\text{s.t.} \quad A^T \boldsymbol{y} \le \boldsymbol{c}$$

i.e., $V(\boldsymbol{b}) = \boldsymbol{b}^T \boldsymbol{y}^*$. $\Rightarrow \nabla V(b) = y^*$

If we change $\boldsymbol{b}$ by a small amount $\Delta \boldsymbol{b}$, such that the optimal solution does not change, then the change to V must be $\Delta \boldsymbol{b}^T \boldsymbol{y}^*$.

$* \; b \to b + \Delta b \Rightarrow V \to V + \Delta b^T y^*$

Local Sensitivity

$$
\begin{aligned}
V = \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

$V(c) = c^T x^*$

Similarly, given A and $\boldsymbol{b}$ fixed, V can be viewed as a function of $\boldsymbol{c}$.

Theorem

If the primal problem has a unique optimal solution $\boldsymbol{x}^*$, then $\nabla V(\boldsymbol{c}) = \boldsymbol{x}^*$.

If one changes c_i by a small amount Δc_i, then the change of the objective value will be $\Delta c_i x_i^*$

$* \; c \to c + \Delta c \Rightarrow V \to V + \Delta c^T x^*$

- Reason: If we change $\boldsymbol{c}$ by a small amount $\Delta \boldsymbol{c}$, such that the optimal solution does not change, then the change to V must be $\Delta \boldsymbol{c}^T \boldsymbol{x}^*$.

Local Sensitivity

The above results also hold for inequality constraints (or maximization problem) such as follows:

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

We have:

1. If the dual has a unique optimal solution $\boldsymbol{y}^*$, then $\nabla V(\boldsymbol{b}) = \boldsymbol{y}^*$
2. If the primal has a unique optimal solution $\boldsymbol{x}^*$, then $\nabla V(\boldsymbol{c}) = \boldsymbol{x}^*$

- To see why this must be true, one can add a slack variable and transform it back to the standard form and then one can use the earlier result.

Example

(P)

$$
\begin{array}{llllll}
\text{maximize} & x_1 & +2x_2 & & & \\
\text{subject to} & x_1 & & \leq 100 & & (y_1) \\
 & & 2x_2 & \leq 200 & & (y_2) \\
 & x_1 & +x_2 & \leq 150 & & (y_3) \\
 & x_1, & x_2 & \geq 0 & &
\end{array}
$$

The optimal solution is $\boldsymbol{x}^* = (\underset{x_1^*}{50}, \underset{x_2^*}{100})$ with optimal value 250. ✓

The dual problem is

(D)

$$
\begin{array}{lllll}
\text{minimize} & 100y_1 & +200y_2 & +150y_3 & \\
\text{subject to} & y_1 & & +y_3 & \geq 1 \\
 & & 2y_2 & +y_3 & \geq 2 \\
 & y_1, & y_2, & y_3 & \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{y}^* = (\overset{y_1^*}{0}, \overset{y_2^*}{0.5}, \overset{y_3^*}{1})$ with optimal value 250. ✓

Example Continued

$$V = \begin{array}{llll} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \le 100 \\ & & 2x_2 & \le 200 \rightarrow 202 \\ & x_1 & +x_2 & \le 150 \\ & x_1, & x_2 & \ge 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

1. What would be the optimal value if we change the RHS of second constraint to 202?

- It will change by $\underbrace{\Delta b_2}_{=2} \overset{0.5}{y_2^*} = 1$. Therefore, the optimal value would be 251

Example Continued

$$V = \begin{array}{llll} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \rightarrow 99 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

2. What would be the optimal value if we change the RHS of first constraint to 99?

- It will change by $\underbrace{\Delta b_1}_{\overset{-1}{\shortparallel}} \underbrace{y_1^*}_{\overset{0}{\shortparallel}} = 0$. Therefore, the optimal value would be unchanged.

Example Continued

$$V = \begin{array}{lrrl} \text{maximize} & 1.02\,x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

3. What would be the optimal value if the cost coefficient of x_1 becomes 1.02?

- It will increase by $\underbrace{\Delta c_1}_{0.02} \underbrace{x_1^*}_{50} = 1$. Therefore, the optimal value would be 251

Example Continued

1.97

$$
\begin{array}{rlll}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

4. What would be the optimal value if the cost coefficient of x_2 becomes 1.97?

-0.03 100

- It will decrease by $\Delta c_2 x_2^* = -3$. Therefore, the optimal value would be 247

Property: Inactive Constraints

$$
\begin{aligned}
\max_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} \le \boldsymbol{b} \\
& \boldsymbol{x} \ge 0
\end{aligned}
$$

$a_i^T x \le b_i$ $(y_i^* = 0)$

$\underbrace{y_i^*}_{=0} \cdot \underbrace{(a_i^T x^* - b_i)}_{<0} = 0$

resource

If at optimal $\boldsymbol{x}^*$, $\boldsymbol{a}_i^T \boldsymbol{x}^* < b_i$, then what happens if we change b_i?

- By the complementary slackness conditions, the corresponding dual variable y_i^* must be 0.
- Therefore, changing the right-hand-side of an inactive constraint by a small amount won't affect the optimal value (also the optimal solution).
- Intuition: If a resource is already redundant, then adding or reducing a small amount wouldn't matter.

Shadow Prices

Recall that

- $\nabla V(\boldsymbol{b}) = \boldsymbol{y}^*$, where $\boldsymbol{y}^*$ is the optimal dual solution

We call $\boldsymbol{y}^*$ the shadow prices of $\boldsymbol{b}$.

- The shadow price of a resource corresponds to the increment of profit if there is one unit more of that resource (locally).
- Therefore, it can be viewed as the *unit value* or *unit fair price* for that resource.

Caveat

optimal soln does not change

The above analysis is *local*, meaning that it can only deal with small changes.

- Basically, it is valid as long as the optimal basis does not change.
- Otherwise, it may not be true.

In the above example, if the RHS of first constraint reduces to 0, then the optimal solution will be $(0, 100)$, with optimal value 200 (reduced by 50). This difference would be different from $\Delta b_1 y_1^* = 0$.

- We want to study what ranges of changes belong to *small* changes.
- This will be the *global sensitivity analysis*.

Outline

Global Sensitivity

min. $c^T x$
s.t. $Ax = b$
$x \geq 0$

Now we study what will happen if

1. $\boldsymbol{b}$ changes to $\boldsymbol{b} + \Delta\boldsymbol{b}$
2. $\boldsymbol{c}$ changes to $\boldsymbol{c} + \Delta\boldsymbol{c}$

$\bar{c} =$

- At optimal, the reduced costs $\boldsymbol{c}^T - \boldsymbol{c}_B^T A_B^{-1} A \geq 0$.
- $x_B^* = A_B^{-1}\boldsymbol{b}$ is the basic part of the optimal primal solution.
- $y^* = (A_B^{-1})^T \boldsymbol{c}_B$ is the optimal dual solution.

Changing on b

Suppose b becomes $\tilde{b} = b + \Delta b$. Consider the original optimal basis B, we still have

$$\bar{c} = c^\top - c_B^\top A_B^{-1} A \geq 0$$

since the reduced costs do not depend on b.

If we want to ensure the original basis B remains optimal for the new problem, we just need to require the new basic solution is feasible (a BFS):

$$\tilde{x}_B = A_B^{-1}\tilde{b} = A_B^{-1}(b + \Delta b) = x_B^* + A_B^{-1}\Delta b \geq 0.$$

Then B is still the optimal basis and the new primal optimal solution is

$$\tilde{x} = [\tilde{x}_B; 0],$$

$\| \; x_N$

with the new optimal value:

$$V(\tilde{b}) = \tilde{b}^\top y^* = (b + \Delta b)^\top y^* = V^* + (y^*)^\top \Delta b,$$

since the original dual optimal solution $y^* = (A_B^{-1})^\top c_B$ is still dual optimal.

Conclusion: If the original optimal basis is still optimal, then the local sensitivity analysis holds (this justifies the local theorem).

Change on b

Now we study when the change only occurs to one component of $\boldsymbol{b}$, what ranges of changes qualify for a *small* change (i.e., the local sensitivity analysis holds).

$b_i \to b_i + \lambda$

Assume $\Delta \boldsymbol{b} = \lambda \boldsymbol{e}_i$ ($\boldsymbol{e}_i$ is a vector with 1 at position i). Then we need to have

$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$$\boldsymbol{x}_B^* + \lambda A_B^{-1} \boldsymbol{e}_i \geq 0$$

so that the optimal basis remains the same. We can then find the range of λ by solving these inequalities.

Example

Consider the example:

$$
\begin{array}{lrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

$\to 150+\lambda$

min. $-x_1 - 2x_2$
s.t. $x_1 + s_1 = 100$
$2x_2 + s_2 = 200$
$x_1 + x_2 + s_3 = 150$
$x_1, x_2, s_1, s_2, s_3 \geq 0$

$x_B = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \end{bmatrix}$

At optimal, the basis is $\{1, 2, 3\}$, and the optimal solution is $(50, 100, 50, 0, 0)$

- How much can we change the 3rd right hand side coefficient (150) such that the optimal basis remains the same?

$A_B = [A_1 \quad A_2 \quad A_3]$

Example Continued

The final simplex tableau is

B						
B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

Thus $A_B^{-1} = \begin{pmatrix} 0 & -0.5 & 1 \\ 1 & 0.5 & -1 \\ 0 & 0.5 & 0 \end{pmatrix}$. If $\boldsymbol{b}$ changes to $\boldsymbol{b} + \lambda \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{e_3}$, then

$$\tilde{\boldsymbol{x}}_B = \boldsymbol{x}_B^* + \lambda A_B^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \\ 100 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \geq 0$$

In order for this to be positive, we need $-50 \leq \lambda \leq 50$.

Changes in $\boldsymbol{c}$

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j \quad \begin{cases} j \in B \\ j \in N \end{cases}$$

Now suppose $\boldsymbol{c}$ changes to $\tilde{\boldsymbol{c}} = \boldsymbol{c} + \Delta\boldsymbol{c}$. In order for the basic solution to be still optimal, we need to guarantee that the reduced costs are nonnegative:

basic variable — $\tilde{\boldsymbol{c}}_B^T - \tilde{\boldsymbol{c}}_B^T \cancel{A_B^{-1} A_B}^{I} = 0$ for sure, no need to consider

non-basic variable — $\tilde{\boldsymbol{c}}_N^T - \tilde{\boldsymbol{c}}_B^T A_B^{-1} A_N \geq 0$

Note that this basis still provides a basic feasible solution since the feasibility doesn't depend on $\boldsymbol{c}$.

$c_j \to c_j + \lambda$

Next we assume $\Delta\boldsymbol{c} = \lambda \boldsymbol{e}_j$. We discuss two cases: $j \in B$ and $j \in N$. We study how to find the ranges of λ such that the original basis is still optimal (and thus one can apply the local sensitivity analysis).

Case 1: $j \in B$

$c_j \to c_j + \lambda,\ j \in B$ $\quad \tilde{c}_N^T - \tilde{c}_B^T A_B^{-1} A_N$

In this case, the reduced costs are

$$
\begin{aligned}
& c_N^T - (\underbrace{c_B^T + \lambda e_j^T}) A_B^{-1} A_N \\
= \; & c_N^T - c_B^T A_B^{-1} A_N - \lambda e_j^T A_B^{-1} A_N
\end{aligned}
$$

non-basic variables

Note that $c_N^T - c_B^T A_B^{-1} A_N$ is the reduced costs for the original problem. We denote it by r_N^T. Therefore, in order to maintain the optimality of the current basis, we need to have

$$
r_N^T - \lambda e_j^T A_B^{-1} A_N \geq 0
$$

- We can solve the range of λ from the above set of inequalities.

Case 2: $j \in N$

$c_j \to c_j + \lambda, \ j \in N$

In this case, the reduced costs are:

$$\tilde{\mathbf{c}}_N^T - \tilde{\mathbf{c}}_B^T A_B^{-1} A_N = \mathbf{c}_N^T + \lambda \mathbf{e}_j^T - \mathbf{c}_B^T A_B^{-1} A_N = r_N^T + \lambda \mathbf{e}_j^T$$

Therefore, in order to maintain the optimality of the current basis, we need to have

$$\underline{r_N + \lambda \mathbf{e}_j \geq 0}$$

- We can solve the range of λ from the above set of inequalities.

Example

Consider the same example:

$1+\lambda$

$$\begin{array}{rrrl} \text{maximize} & 1x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

$\Rightarrow$ min. $-x_1 - 2x_2$

$-1-\lambda$

x_1 is a basic variable

The final simplex tableau is

B						
B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

How much can we change the first objective coefficient so that we can use the local sensitivity analysis?

Example Continued

We have

$$A_B^{-1} = \begin{pmatrix} 0 & -0.5 & 1 \\ 1 & 0.5 & -1 \\ 0 & 0.5 & 0 \end{pmatrix}; \quad A_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad r_N = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

Assume we change the c_1 from 1 to $1+\lambda$ (i.e., $-1-\lambda$ in the standard form). Then we need

$$\underline{r_N} - (-\lambda)\underline{A_N^T}(\underline{A_B^{-1}})^T \overbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}^{e_1} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} 0.5 \\ -1 \end{pmatrix} \geq 0$$

Therefore, $-1 \leq \lambda \leq 1$

- It means that when c_1 is between 0 and 2, we can use the local sensitivity theorem to compute the optimal value.

What if the Change is Outside the Range

If the change of $\boldsymbol{c}$ is so much that the reduced cost of the current solution contains negative number, then

- We can continue with the simplex method (tableau) until it reaches optimal solution.

If the change of $\boldsymbol{b}$ is so much that the solution corresponding to the original optimal basis B is no longer feasible, then: $x_B = A_B^{-1} b \not\geq 0$

- We may need to solve the problem from the start.
- However, we can also have a dual perspective: it can be viewed as that the objective coefficients of the dual problem changed. Then one can use the method that deals with changes in objective coefficients for the dual problem.

Changes to A

Case 1: change A_j, $j \in N$

If the change is for a number in a non-basic column, say A_j, then the original optimal solution is still feasible (since the non-basic $x_N = 0$). The only change is to the reduced cost of jth variable.

- Recompute $\bar{c}_j$. If it is still nonnegative, then the original optimal solution stays optimal. Otherwise, update the simplex method (tableau) for the jth column as well as the reduced cost and continue from there.

Case 2: change A_j, $j \in B$

If the change is for a number in a basic column, then nearly all the numbers in the simplex method (tableau) will change. In general, there is not a simple way to deal with it.

Other Changes

Adding a variable (the rest are kept the same):

- Assign the added variable to 0 and to be a non-basic variable. The original BFS is still a BFS, the reduced cost is unchanged.
- We only need to check the reduced cost corresponding to the new variable. If it is non-negative, then the original optimal solution plus the added non-basic variable is still optimal; otherwise continue the simplex method from there.

Adding a constraint:

- If the original optimal solution satisfies the constraint, then it is still optimal.
- If not, then the best way to deal with it is to think it as adding a dual variable, then use the simplex method (tableau) for the dual problem to continue calculations.

End of Linear Program! Next Topic: Integer Program.

Outline

Integer Linear Program

An integer linear program is a linear program with the additional constraint that all variables must be integers:

$$\begin{aligned} \text{minimize}_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \\ & \underline{\boldsymbol{x} \in \mathbb{Z}^n} \end{aligned}$$

Here we use $\mathbb{Z}$ to denote the set of integers.

Mixed Integer Linear Program (MILP)

LP

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & c^\mathsf{T} x \\
\text{s.t.} \quad & a_j^\mathsf{T} x \le b_j, \quad j = 1, 2, .., m \\
& x \in \underbrace{\mathbb{R}^{n-p}}_{\text{continuous}} \times \underbrace{\mathbb{Z}^p}_{\text{integers}}
\end{aligned}
$$

- In mixed integer linear programs (MILPs), one set of variables must be integer and the rest are allowed to be continuous.

A Special Case: Binary IP

A special and important class of integer program is those where the integer variables are required to be binary, that is, they are required to take values of 0 or 1.

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_j(x) \leq 0, \quad j = 1, 2, ..., m \\
& x \in \mathbb{R}^{n-p} \times \{0, 1\}^p
\end{aligned}
$$

The variables in a binary IP are also called indicator variables. They are powerful in modeling.

binary variable

- Indivisible decisions
- Yes/No choices
- Logical conditions
- Nonconvex functions and sets

define $x = \begin{cases} 1 & \text{if} \cdots \\ 0 & \text{otherwise} \end{cases}$

$x \in \{0, 1\}$

Example: Knapsack Problem

John is planning for a trip. There are n items he would like to bring with him.

- The ith item has value v_i
- The weight of ith item is a_i
- His bag has a maximum allowable weight C
- He wants to bring as much value as possible

$$X_i = \begin{cases} 1 & \text{if bring item } i \\ 0 & \text{o.w.} \end{cases}$$

Decision variables:

- x_i: whether to bring ith item or not. $x_i \in \{0, 1\}$

Optimization problem:

$$\begin{array}{ll} \text{maximize}_{\boldsymbol{x}} & \sum_{i=1}^{n} v_i x_i \\ \text{subject to} & \sum_{i=1}^{n} a_i x_i \leq C \\ & x_i \in \{0,1\} \quad \forall i = 1, \ldots n \end{array}$$

It is a binary optimization problem

Greedy Method: Enumeration

- Small discrete optimization problems can be solved by enumerating all possibilities
- Example:

$$
\begin{aligned}
\text{maximize}_{\boldsymbol{x}} \quad & \textstyle\sum_{i=1}^{n} v_i x_i \\
\text{subject to} \quad & \textstyle\sum_{i=1}^{n} a_i x_i \leq C \\
& x_i \in \{0,1\} \quad \forall i = 1, \ldots, n
\end{aligned}
$$

- There are 2^n possible values of the binary variables
- If n is small, we can check each to see if it is feasible, and then choose a feasible solution with maximum objective value

Enumeration Result

```python
# Solving an example discrete optimization problem by enumeration
import itertools
from random import randint
import time

# generate random problem instance
n = 25
r = [randint(0,9) for j in range(n)]
c = [randint(0,9) for j in range(n)]
B = sum(c)/2.0

# all possible solutions
solutions = list(itertools.product([0, 1], repeat=n))

xbest = []
vbest = -1

# check each solution
start_time = time.time()
for x in solutions:
    # check feasibility
    val = sum( [c[j]*x[j] for j in range(n)])
    if val <= B:
        obj = sum( [r[j]*x[j] for j in range(n)])
        if obj > vbest:
            vbest = obj
            xbest = x

print 'n:', n
print 'Time:', (time.time() - start_time)

```

n	Time (secs)
5	0.0012
10	0.0049
15	0.1943
20	7.4745
25	311.9747

With n=50 estimated time > 330 years!!

Computation Complexity for Optimization Problems

- Linear programming and many important classes of convex optimization problems (e.g. conic programming and semidefinite programming) are known to be in **P**.
- Most integer optimization problems are **NP-hard**.
- It is widely believed that there is little hope of finding a polynomial time algorithm for discrete optimization problems.

Outline

If-Then Condition I

1) - If A is chosen, then B is chosen.

$$x_A \le x_B$$

2) - If A is chosen, then B is not chosen.

$$x_A + x_B \le 1$$

3) - If A is chosen, then ~~both~~ B or C are chosen.

$$x_A \le x_B + x_C$$

4) - If B or C is chosen, then A is chosen.

$$x_A \ge x_B, \quad x_A \ge x_C$$

$$X_A = \begin{cases} 1 & \text{if } A \text{ is chosen} \\ 0 & \text{o.w.} \end{cases}$$

$$X_B = \begin{cases} 1 & \text{if } B \text{ is chosen} \\ 0 & \text{o.w.} \end{cases}$$

1. $X_A \le X_B$

$$X_A = 1 \Rightarrow 1 \le X_B \xRightarrow{X_B \in \{0,1\}} X_B = 1$$

2. $X_A + X_B \le 1$

$$X_A = 1 \Rightarrow 1 + X_B \le 1 \Rightarrow X_B \le 0 \xRightarrow{X_B \in \{0,1\}} X_B = 0$$

3. $X_A \le X_B + X_C$

$$X_A = 1 \Rightarrow 1 \le X_B + X_C \Rightarrow X_B = 1 \text{ or } X_C = 1 \text{ or both}$$

4. $X_A \ge X_B, \quad X_A \ge C$

$$X_B = 1 \Rightarrow X_A \ge 1 \Rightarrow X_A = 1$$

$$X_C = 1 \Rightarrow X_A \ge 1 \Rightarrow X_A = 1$$

If A is chosen, then B is not chosen.

Right $\rightarrow X_A + X_B \le 1$

A. $\frac{X_A}{3} + X_B \le \frac{2}{3}$ WRONG!

B. $X_A + X_B \le \frac{4}{3}$ RIGHT!

$$X_A = 1 \Rightarrow \frac{1}{3} + X_B \le \frac{2}{3} \Rightarrow X_B \le \frac{1}{3} \Rightarrow X_B = 0$$

$$X_A = 1 \Rightarrow 1 + X_B \le \frac{4}{3} \Rightarrow X_B \le \frac{1}{3} \Rightarrow X_B = 0$$

A. $\frac{X_A}{3} + X_B \leq \frac{2}{3}$

. $X_A = 0 \Rightarrow X_B \leq \frac{2}{3} \Rightarrow X_B = 0$

This is additional information (constraint)

If-Then Condition II

- If A is chosen, then both B and C are chosen.

$$x_A \le x_B, \quad x_A \le x_C$$

- If B and C are chosen, then A is chosen.

$$x_A \ge x_B + x_C - 1$$

- A is chosen if and only if B and C are chosen.

$$x_A \le x_B, \quad x_A \le x_C$$
$$x_A \ge x_B + x_C - 1$$

$$\Downarrow$$

$$x_A = x_B \cdot x_C$$

nonlinear

If-Then Condition III

1. If A is chosen, then $x \leq 8$.

$$x \leq 8 + M(1 - y)$$

2. If A is chosen, then $x \geq 8$.

$$x \geq 8 - M(1 - y)$$

3. If $x < 8$, then A is chosen.

$$x \geq 8 - My$$

4. If $x > 8$, then A is chosen.

$$x \leq 8 + My$$

$y = \begin{cases} 1 & \text{if } A \text{ is chosen} \\ 0 & \text{o.w.} \end{cases}$

M: 'Big' Number

$M = +\infty$

Big-"M" Notation

① $x \le 8 + M(1-y)$

$y=1 \Rightarrow x \le 8$

Try: $x \le 8y$ → WRONG!

$y=1 \Rightarrow x \le 8$

$y=0 \Rightarrow x \le 0$ → Additional!

$y=0 \Rightarrow x \le 8+M \Rightarrow x \le +\infty$ 'redundant'

② $x \ge 8 - M(1-y)$

- $y=1 \Rightarrow x \ge 8$
- $y=0 \Rightarrow x \ge 8-M \Rightarrow x \ge -\infty$

③ If $x<8$, then A is chosen.

Contra-Positive Statement:

If A, then B

⇕

If not B, then not A

If A is not chosen, then $x \ge 8$

$x \ge 8 - My$

- $y=0 \Rightarrow x \ge 8$
- $y=1 \Rightarrow x \ge 8-M \Rightarrow x \ge -\infty$

④ If $x > 8$, then A is chosen.

$\Updownarrow$

If A is not chosen, then $x \le 8$

$$x \le 8 + My$$

- $y = 0 \Rightarrow x \le 8$
- $y = 1 \Rightarrow x \le 8 + M \Rightarrow x \le +\infty$

- Check if $f(x) \ge b$ is satisfied.
- If $y = 1$, then $f(x) \ge b$

  A. $f(x) \ge by$ → WRONG

  B. $f(x) \ge b - M(1-y)$ → Correct!

Take-home exercise

If $y = 0$, then $g(x) \le b$

If-Then Condition IV

- If $x_1 > 4$, then $x_2 \le 8$.

If $x_1 > 4$, then A is chosen

$$\begin{cases} x_1 \le 4 + My \\ x_2 \le 8 + M(1-y) \\ y \in \{0,1\} \end{cases}$$

If A is chosen, then $x_2 \le 8$

- If $x_1 < 4$, then $x_2 \ge 8$.

$$\begin{cases} x_1 \ge 4 - My \\ x_2 \ge 8 - M(1-y) \\ y \in \{0,1\} \end{cases}$$

If $x_1 > 4$, then $x_2 \le 8$

$$\Updownarrow$$

$$\begin{cases} \text{If } x_1 > 4, \text{ then } A \text{ is chosen.} \\ \text{If } A \text{ is chosen, then } x_2 \le 8. \end{cases}$$

If $x_1 < 4$, then $x_2 \ge 8$

$$\Updownarrow$$

$$\begin{cases} \text{If } x_1 < 4, \text{ then } A \text{ is chosen.} \\ \text{If } A \text{ is chosen, then } x_2 \ge 8. \end{cases}$$

General "If ... then ... " Conditions

If $f(x) > a$, then A is chosen.
If A is chosen, then $g(x) \geq b$.

If $f(x_1, x_2, ..., x_n) > a$, then $g(x_1, x_2, ..., x_n) \geq b$.

$$\begin{cases} f(x_1, x_2, ..., x_n) \leq a + My \\ g(x_1, x_2, ..., x_n) \geq b - M(1 - y) \\ y \in \{0, 1\} \end{cases}$$

Disjunctive conditions: "Either ... Or ..." Conditions

Disjunctive constraints: **at least one** of a set of constraints is satisfied. It's possible the set of constraints are all satisfied.

$$\begin{aligned}&\text{Either A or B} \Longleftrightarrow \text{If not A, then B}\\&\text{Either A or B} \Longleftrightarrow \text{If not B, then A}\end{aligned}$$

- Either $x_1 \leq 4$ or $x_2 \leq 8$

$$\text{Either } x_1 \leq 4 \text{ or } x_2 \leq 8$$
$$\Updownarrow$$
$$\text{If } x_1 > 4, \text{ then } x_2 \leq 8$$
$$\Updownarrow$$
$$\begin{cases} x_1 \leq 4 + My \\ x_2 \leq 8 + M(1-y) \\ y \in \{0,1\} \end{cases}$$

General Disjunctive Constraints

$$\text{Either } f(x_1, x_2, ..., x_n) \le a \text{ or } g(x_1, x_2, ..., x_n) \le b$$

$$\Updownarrow$$

$$\text{If } f(x_1, x_2, ..., x_n) > a, \text{ then } g(x_1, x_2, ..., x_n) \le b$$

$$\Updownarrow$$

$$\begin{cases} f(x_1, x_2, ..., x_n) \le a + My \\ g(x_1, x_2, ..., x_n) \le b + M(1 - y) \\ y \in \{0, 1\} \end{cases}$$

K-out-of-N Must Hold

At least K of them be satisfied.

Out of the N following constraints, K must be satisfied:

N requirements

$$\begin{cases} f_1(x_1, \ldots, x_n) \le b_1 \\ f_2(x_1, \ldots, x_n) \le b_2 \\ \vdots \\ f_N(x_1, \ldots, x_n) \le b_N \end{cases}$$

- Check if $f(x) \le b$ is satisfied or not

$$f(x) \le b + M(1-y)$$

- If $y=1$, $f(x) \le b$ is satisfied
- If $y=0$, $f(x) \le b$ is not satisfied

WRONG

- If $y=0$, $f(x) \le b + M \Rightarrow f(x) \le +\infty$

K-out-of-N Must Hold

$$
\begin{aligned}
&f_1(x_1, \ldots, x_n) \leq b_1 + M(1 - y_1) \\
&f_2(x_1, \ldots, x_n) \leq b_2 + M(1 - y_2) \\
&\vdots \\
&f_N(x_1, \ldots, x_n) \leq b_N + M(1 - y_N) \\
&\sum_{j=1}^{N} y_j = K
\end{aligned}
$$

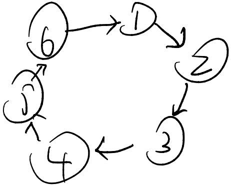
$\geq$

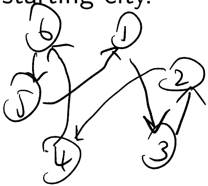
$$
y_j \in \{0, 1\}, \quad j = 1, 2, \ldots, N.
$$

Outline

Travelling Salesman Problem (TSP)

- The Traveling Salesman Problem (TSP) is a very well-known integer optimization problem, one of the most intensively studied, most widely applied, and the most difficult integer programs.
- Given a collection of n cities, TSP asks to find a tour of minimum distance that starts from a city and visits each of the other $n - 1$ cities exactly once and comes back to the starting city.





TSP Model

- Let $N = \{1, 2, ..., n\}$ be the index set of cities.
- Define $x_{ij} = 1$ if the travelling salesman goes from city i directly to city j and 0 otherwise.

$$
\begin{aligned}
\min \quad & \underline{\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij}} = \text{total distance} \\
\text{s.t} \quad & \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i = 1, ..., n \\
& \sum_{i=1}^{n} x_{ij} = 1 \quad \forall j = 1, ..., n \\
& \sum_{i \in S} \sum_{j \in S} x_{ij} \le |S| - 1 \quad \forall S \subset N, |S| \le n - 1 \\
& x_{ij} \in \{0, 1\} \quad \forall i, j = 1, ..., n
\end{aligned}
$$

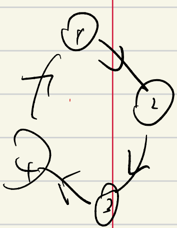
$$x_{ij} = \begin{cases} 1 & \text{if travel from city } i \text{ to city } j \\ 0 & \text{o.w.} \end{cases}$$

d_{ij} : distance between city i and city j

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1, j \neq i}^{n} x_{ij} = 1, \quad \forall i = 1, \cdots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad \forall j = 1, \cdots, n$$

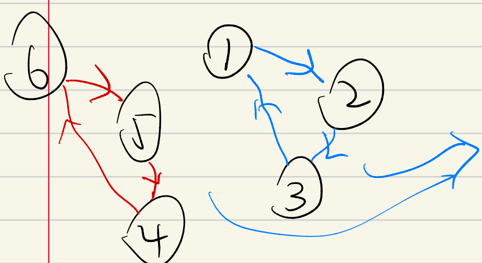


$$x_{ij} \in \{0, 1\}, \quad \forall i, j = 1, \cdots, n$$

subtour elimination

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1, \quad \forall S \subset N, \; |S| \leq n-1$$

Cardinality



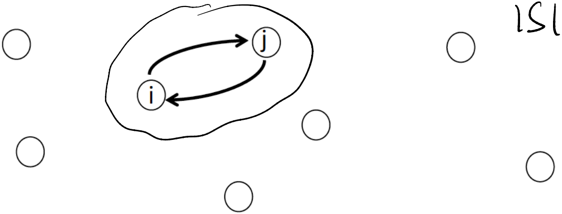
Subtours

Subtour Elimination

Subtour elimination constraints: limit the number of selected arcs in each subset of cities S to be no greater than $|S| - 1$

$S = \{i, j\}$

$|S| = 2$

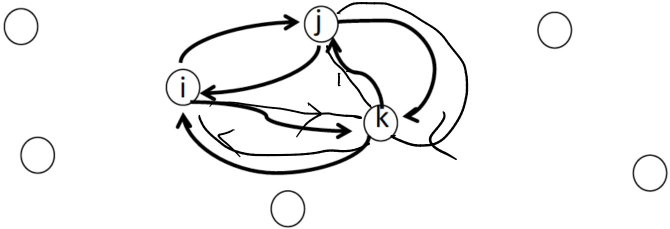


$$S = \{i, j\}$$
$$x_{ij} + x_{ji} \leq 1$$

$x_{ij}, x_{ji} \in \{0, 1\}$

Subtour Elimination

$S=\{i,j,k\}$
$|S|=3$



$$S = \{i, j, k\}$$
$$x_{ij} + x_{ik} + x_{ji} + x_{jk} + x_{ki} + x_{kj} \leq 2$$