Simplex Method Example

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1 Simplex Method Example

Consider the following linear program:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \leq 10 \\ & 3x_1 + 2x_2 \leq 60 \\ & 2x_1 + 3x_2 \leq 6 \\ & x_1, x_2 \geq 0. \end{aligned}$$

1. First let us draw the feasible region of this LP in $\mathbb{R}^2$ in Figure 1.

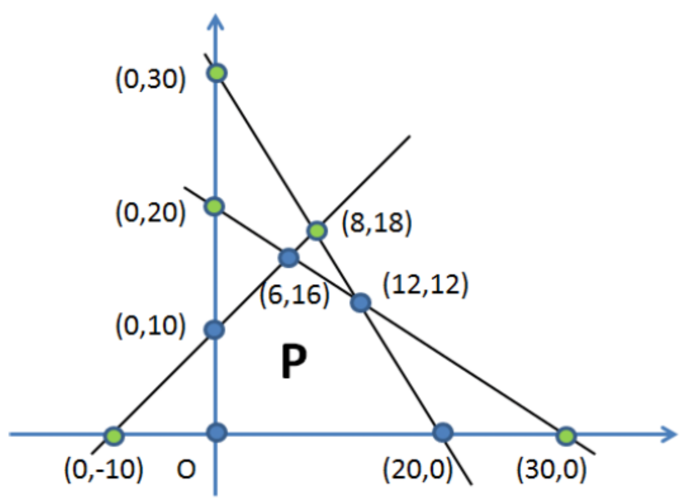


Figure 1: A simplex example.

   The blue dots are basic feasible solutions. The green dots are basic solutions but not feasible. So in total, there are 10 basic solutions.

2. Transform to a standard form LP: The simplex method works on standard form LPs, so let us first transform the above LP into the standard form.

$$
\begin{aligned}
\min \quad & -2x_1 - 3x_2 \\
\text{s.t.} \quad & -x_1 + x_2 + x_3 = 10 \\
& 3x_1 + 2x_2 + x_4 = 60 \\
& 2x_1 + 3x_2 + x_5 = 60 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}
$$

To facilitate the simplex method, it helps to write out explicitly the c, A, b:

$$
c = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 60 \\ 60 \end{bmatrix}.
$$

3. Start the simplex method:

   Iteration 1:

   (a) **Choose a starting BFS:** Let us select the basis matrix

   $$
   A_B = [A_3, A_4, A_5] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
   $$

   The corresponding basic solution is

   $$
   x_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = A_B^{-1} b = \begin{bmatrix} 10 \\ 60 \\ 60 \end{bmatrix}, \quad x_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
   $$

   The cost coefficients associated with basic and nonbasic variables:

   $$
   c_B = \begin{bmatrix} c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_N = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.
   $$

   Since $x_B \geq 0$ (and of course $x_N \geq 0$), the current basic solution is a basic feasible solution. So we are ready to start the simplex method.

   (b) **Compute reduced costs for nonbasic variables:**

   $$
   \bar{c}_1 = c_1 - c_B^T A_B^{-1} A_1 = -2, \qquad \bar{c}_2 = c_2 - c_B^T A_B^{-1} A_2 = -3.
   $$

   Both $\bar{c}_1$ and $\bar{c}_2$ are negative. Therefore, the current BFS is not optimal, and both x_1 and x_2 are candidates to enter the basis, i.e. to increase to a positive value. Let us take x_2 to enter the basis, and keep x_1 zero.

(c) **Compute feasible direction** $d = \begin{bmatrix} d_B \\ d_N \end{bmatrix}$**:** Since we decide to increase x_2 and keep x_1 at zero, the nonbasic variable part of the feasible direction is

$$d_N = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the basic variable part of the feasible direction is

$$d_B = \begin{bmatrix} d_3 \\ d_4 \\ d_5 \end{bmatrix} = -A_B^{-1} A_2 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}.$$

Since some components of d_B are negative, we do not have an unbounded optimal solution, and we need to decide how far to go along this direction while still remaining feasible.

(d) **Min-ratio test:** By going along the above calculated direction, we are going from the initial BFS x to a new point $x + \theta d$. Let us write it out componentwise:

$$x + \theta d = \begin{bmatrix} x_B + \theta d_B \\ x_N + \theta d_N \end{bmatrix} = \begin{bmatrix} x_3 + \theta d_3 \\ x_4 + \theta d_4 \\ x_5 + \theta d_5 \\ x_1 + 0 \\ x_2 + \theta \end{bmatrix} = \begin{bmatrix} 10 + \theta \cdot (-1) \\ 60 + \theta \cdot (-2) \\ 60 + \theta \cdot (-3) \\ 0 \\ \theta \end{bmatrix} = \begin{bmatrix} 10 - \theta \\ 60 - 2\theta \\ 60 - 3\theta \\ 0 \\ \theta \end{bmatrix}.$$

To decide the largest θ so that $x + \theta d \geq 0$, we need to do the min-ratio test:

$$\theta^* = \min_{i \in B : d_i < 0} \frac{-x_i}{d_i} = \min\left\{\tfrac{10}{1}, \tfrac{60}{2}, \tfrac{60}{3}\right\} = 10.$$

So x_3 exits the basis.

(e) **The new basis:** The new basis matrix $A_B = [A_2, A_4, A_5]$, which differs from the original basis only in one column: A_3 is replaced by A_2, i.e. x_3 exits the basis and x_2 enters the basis.

The new basic variables and nonbasic variables are

$$x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 40 \\ 30 \end{bmatrix}, \qquad x_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We are ready for a new iteration of the simplex method.

Iteration 2:

(a) Let us write the new basis and its inverse:

$$A_B = [A_2, A_4, A_5] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \qquad A_B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

The cost coefficients for basic and nonbasic variables:

$$c_B = \begin{bmatrix} c_2 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}, \qquad c_N = \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

(b) **Compute reduced costs:**

$$\bar{c}_1 = c_1 - c_B^T A_B^{-1} A_1 = -2 - \begin{bmatrix} -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = -2 - \begin{bmatrix} -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = -5,$$

$$\bar{c}_3 = c_3 - c_B^T A_B^{-1} A_3 = 0 - \begin{bmatrix} -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 - \begin{bmatrix} -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 3.$$

Since $\bar{c}_1 < 0$, the current BFS is not optimal, and x_1 enters the basis.

(c) **Feasible direction:**

$$d_N = \begin{bmatrix} d_1 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$d_B = \begin{bmatrix} d_2 \\ d_4 \\ d_5 \end{bmatrix} = -A_B^{-1} A_1 = - \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix}.$$

Since some components of d_B are negative, the optimal solution is not unbounded.

(d) **Min-ratio test:** Going along the direction calculated above, we move from the current BFS to a new point

$$x + \theta d = \begin{bmatrix} x_B + \theta d_B \\ x_N + \theta d_N \end{bmatrix} = \begin{bmatrix} x_2 + \theta d_2 \\ x_4 + \theta d_4 \\ x_5 + \theta d_5 \\ x_1 + \theta \\ x_3 + 0 \end{bmatrix} = \begin{bmatrix} 10 + \theta \cdot (1) \\ 40 + \theta \cdot (-5) \\ 30 + \theta \cdot (-5) \\ \theta \\ 0 \end{bmatrix} = \begin{bmatrix} 10 + \theta \\ 40 - 5\theta \\ 30 - 5\theta \\ \theta \\ 0 \end{bmatrix}.$$

Do the min-ratio test to decide how far to move to keep $x_B + \theta d_B \geq 0$:

$$\theta^* = \min_{i \in B: d_i < 0} \frac{-x_i}{d_i} = \min \left\{ \tfrac{40}{-(-5)}, \tfrac{30}{-(-5)} \right\} = 6.$$

x_5 becomes zero, so x_5 exits the basis.

(e) **The new basis:** $A_B = [A_2, A_4, A_1]$, since x_1 enters the basis and x_5 exits the basis. The new non-basis matrix $A_N = [A_5, A_3]$. The new BFS is

$$x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 6 \end{bmatrix}, \qquad x_N = \begin{bmatrix} x_5 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

To decide if this BFS is optimal, we need to start another iteration.

Iteration 3:

(a) Let us write the new basis and its inverse:

$$A_B = [A_2, A_4, A_1] = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 3 & 0 & 2 \end{bmatrix}, \qquad A_B^{-1} = \begin{bmatrix} 0.4 & 0 & 0.2 \\ 1 & 1 & -1 \\ -0.6 & 0 & 0.2 \end{bmatrix}.$$

The cost coefficients for basic and nonbasic variables:

$$c_B = \begin{bmatrix} c_2 \\ c_4 \\ c_1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -2 \end{bmatrix}, \qquad c_N = \begin{bmatrix} c_5 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(b) **Compute reduced costs:**

$$\bar{c}_5 = c_5 - c_B^T A_B^{-1} A_5 = 0 - \begin{bmatrix} -3 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0.4 & 0 & 0.2 \\ 1 & 1 & -1 \\ -0.6 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 - \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1,$$

$$\bar{c}_3 = c_3 - c_B^T A_B^{-1} A_3 = 0 - \begin{bmatrix} -3 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0.4 & 0 & 0.2 \\ 1 & 1 & -1 \\ -0.6 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 - \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0.$$

Since all the reduced costs are nonnegative, the current BFS is optimal. We are done! The final optimal solution is:

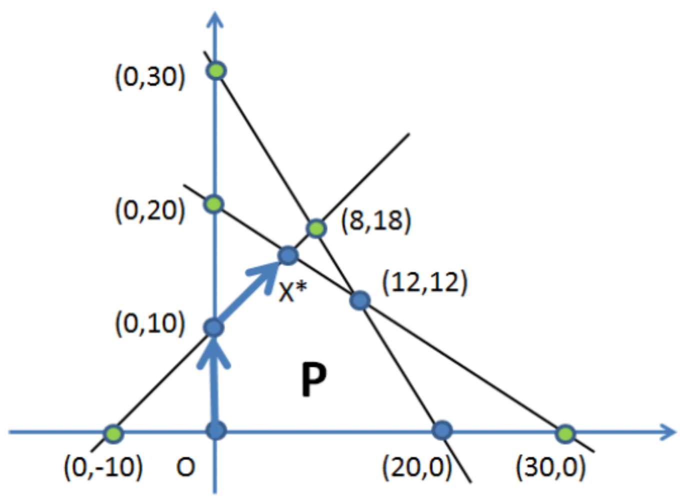
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ 0 \\ 10 \\ 0 \end{bmatrix}.$$

Now let us trace the trajectory of the above simplex iterations on the graph. We started at the initial BFS $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 10, 60, 60)$, which corresponds to the origin $(x_1, x_2) = (0, 0)$ on the 2-D graph.

After the first iteration, we moved to a new BFS $(x_1, x_2, x_3, x_4, x_5) = (0, 10, 0, 40, 30)$, which is the extreme point $(x_1, x_2) = (0, 10)$ on the x_2-axis. We decided this is not an optimal solution, so we did one more iteration of simplex.

This time, we reached the BFS $(x_1, x_2, x_3, x_4, x_5) = (6, 16, 0, 10, 0)$, which corresponds to $(x_1, x_2) = (6, 16)$ on the graph. It is optimal. This trajectory is shown on the following graph.

We can see, geometrically, the simplex method is traversing from one extreme point to another adjacent extreme point, while reducing the objective cost, until it reaches the optimal extreme point.



2 Two-Phase Simplex Method Example

Here is an example to form a Phase-I problem. The original LP is given as:

$$\begin{aligned} \min \quad & x_1 + 3x_2 + 2x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 = 3 \\ & -x_1 + 2x_2 - 6x_4 = 2 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0. \end{aligned}$$

The Phase-I problem is formulated as:

$$\begin{aligned} \min \quad & y_1 + y_2 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 + y_1 = 3 \\ & -x_1 + 2x_2 - 6x_4 + y_2 = 2 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ y_1 \geq 0,\ y_2 \geq 0. \end{aligned}$$

We can choose (y_1, y_2) to be the basic variables, which gives the basic feasible solution $(x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, y_1 = 3, y_2 = 2)$. Then, we can start the simplex method to solve the Phase-I problem.

Solving the Phase-I problem with the simplex method, we get an optimal solution $x^* = (0, \frac{3}{2}, 0, \frac{1}{6})$ and $y^* = (0, 0)$. The x variable part is a BFS for the original LP, with basic variables ($x_2 = \frac{3}{2}, x_4 = \frac{1}{6}$), and nonbasic variables ($x_1 = 0, x_3 = 0$). You can also go back to the original LP and verify this: The matrix $A_B = [A_2, A_4]$ is indeed invertible, therefore a basis matrix. The corresponding basic variable is

$$x_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = A_B^{-1} b = \begin{bmatrix} 3/2 \\ 1/6 \end{bmatrix},$$

which is positive, therefore it is a BFS of the original LP. Now, we can start solving the original LP with this BFS.