CS350: Data Structures

AA Trees

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Introduction to AA Trees

- **A type of balanced binary search tree**

- **Developed as a simpler alternative to red-black trees and other balanced trees**
  - Introduced by Arne Andersson (hence the AA) in 1993
  - Eliminates many of the conditions that need to be considered to maintain a red-black tree
  - Fewer conditions means AA trees are easier to implement
  - Comparable in performance to red-black trees

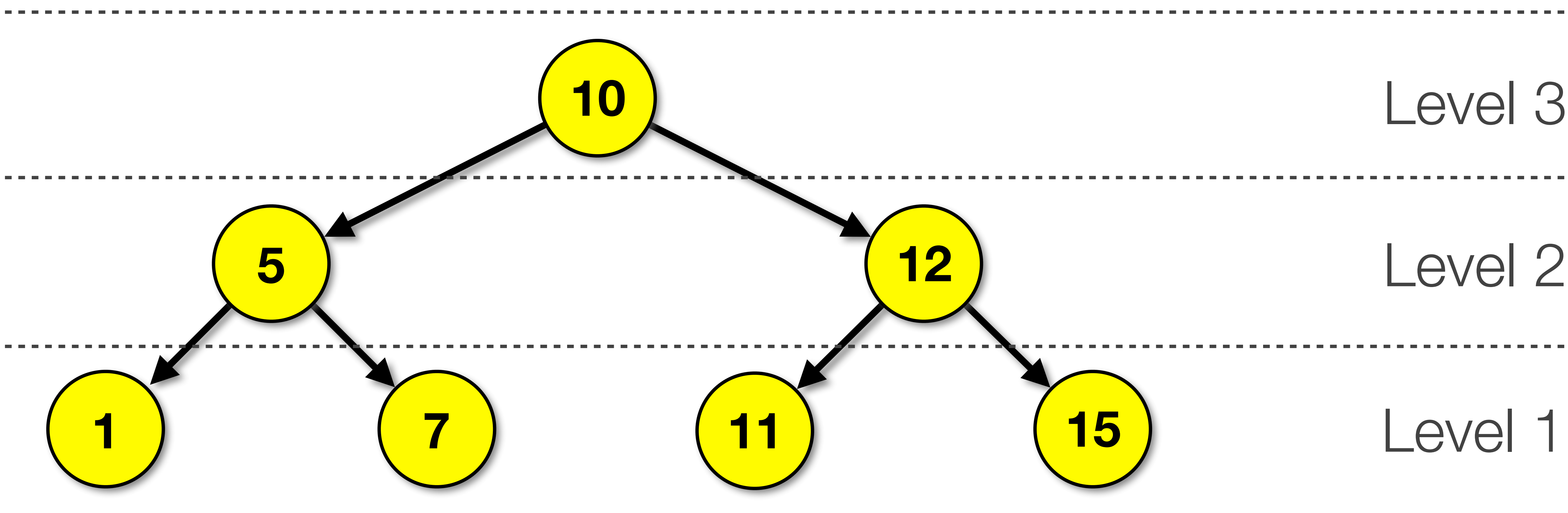
Introduction to AA Trees

- **Similar to red-black trees, but with the addition of a single new rule**
    - Every node is colored either red or black
    - The root node is black
    - If a node is red, its children must be black
    - Every path from a node to a `null` link must contain the same number of black nodes
    - Left children may not be red

New Rule

Levels in AA Trees

- **AA trees utilize the concept of *levels* to aid in balancing binary trees**
  - The level of a node represents the number of left links on the path to the `nullNode` (sentinel node)
- **All leaf nodes are at *level 1***



AA Tree Invariants

- **AA trees must always satisfy the following <u>five</u> invariants:**

1) The *level* of a leaf node is 1

2) The *level* of a left child is strictly <u>less than</u> that of its parent

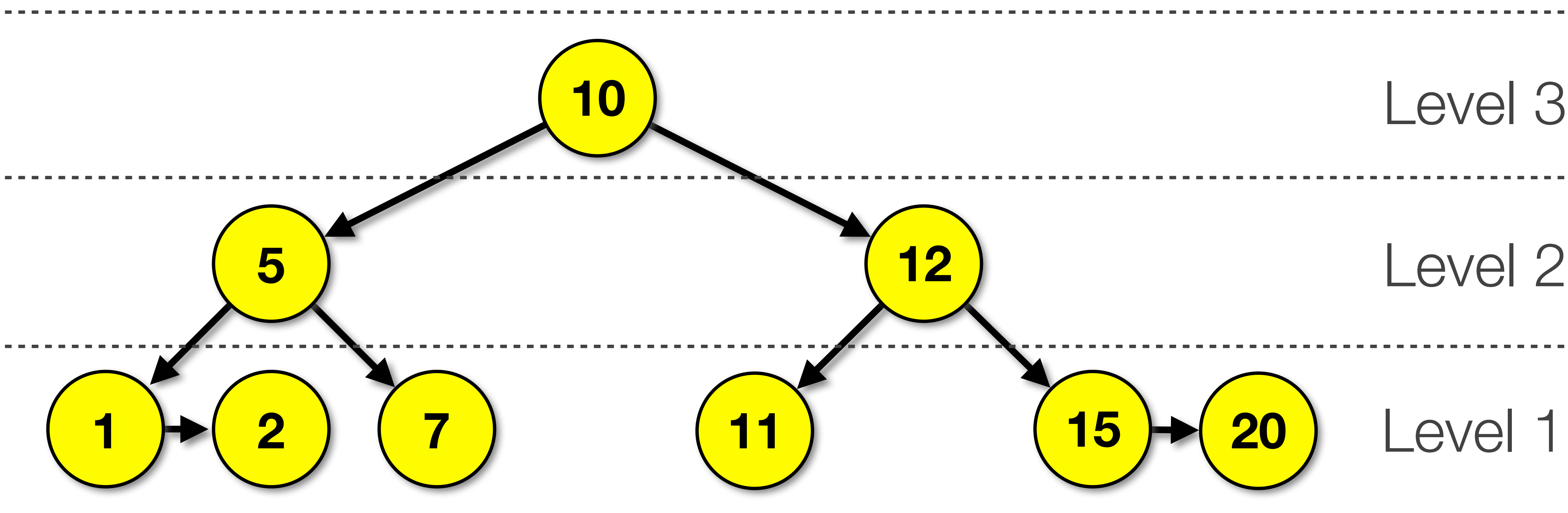
3) The *level* of a right child is <u>less than or equal</u> to that of its parent

4) The *level* of a right grandchild is strictly <u>less than</u> that of its grandparent

5) Every node of *level* greater than one must have two children

Inserting Into AA Trees

- **All nodes are initially inserted as leaf nodes using the standard BST insertion algorithm (tree may require rebalancing after insert)**
- **Since a parent and its <u>right child</u> can be on the same level (*rule #3*), *horizontal links* are possible**



Horizontal Links in AA Trees

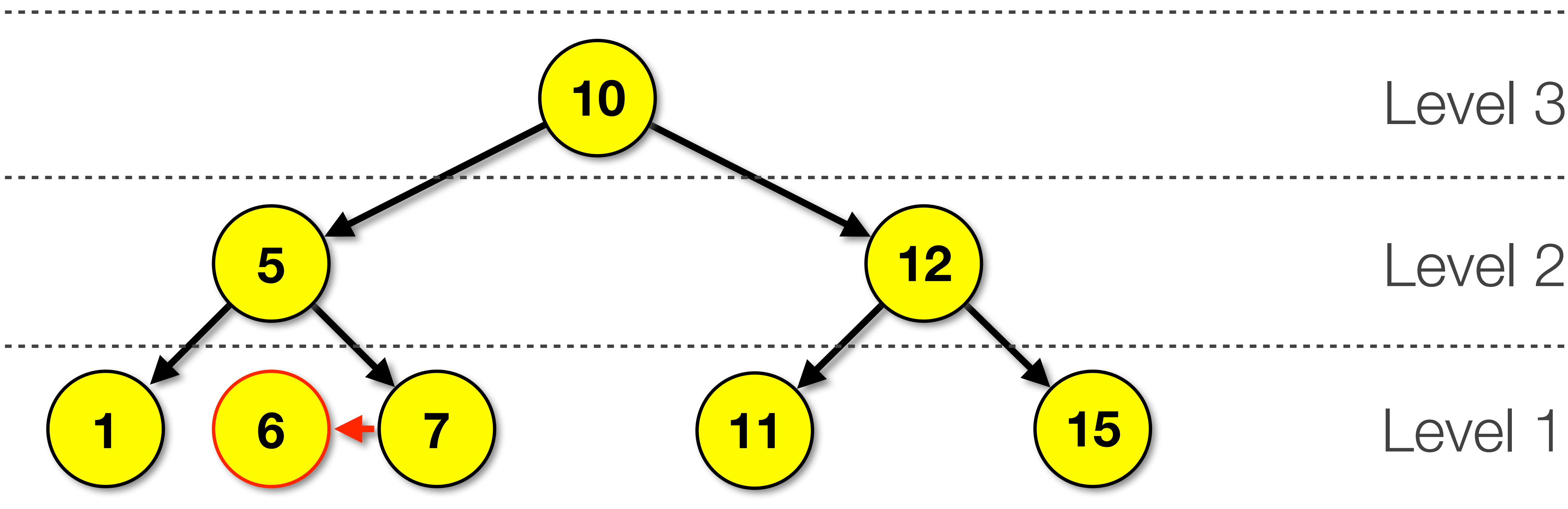
- **The five invariants of AA trees impose restrictions on horizontal links**

- **If any of the invariants are violated the tree must be modified until it once again satisfies all five invariants**

- **Only two cases need to be considered and corrected to maintain the balance of an AA tree**

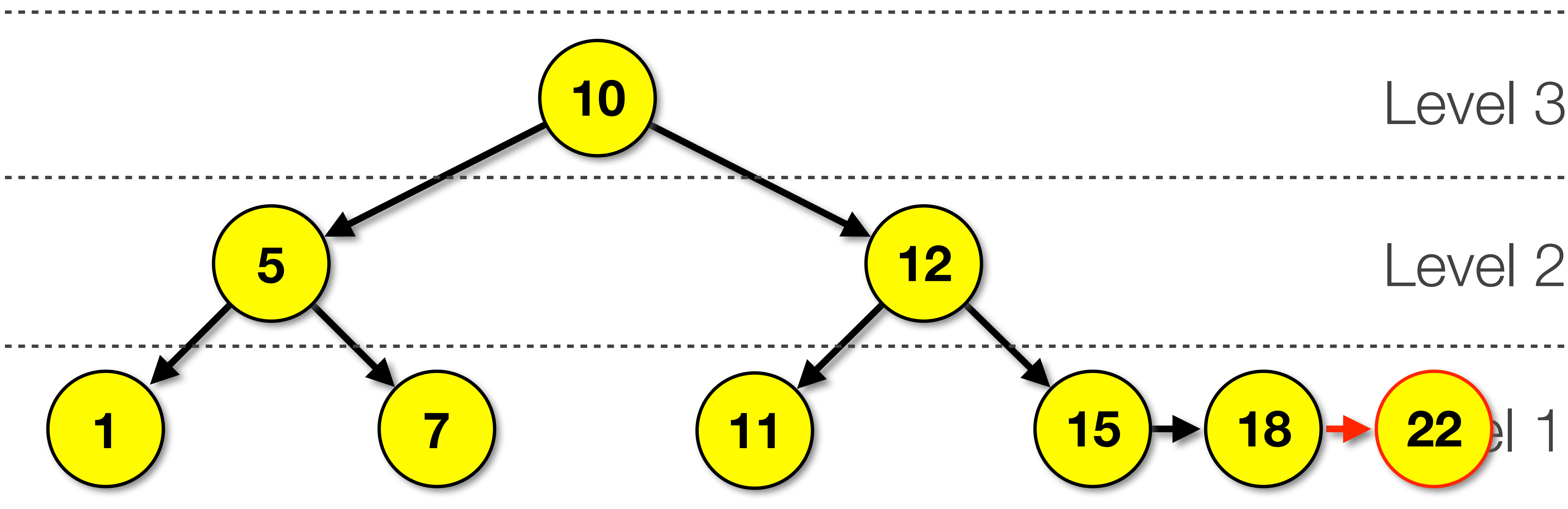
Horizontal Links in AA Trees

- **<u>Case #1</u> - Left horizontal links are <u>NOT</u> allowed**
  - Violates rule #2 - the *level* a left child is strictly less than that of its parent
  - A ***<u>skew</u>*** operation will be introduced to handle this case



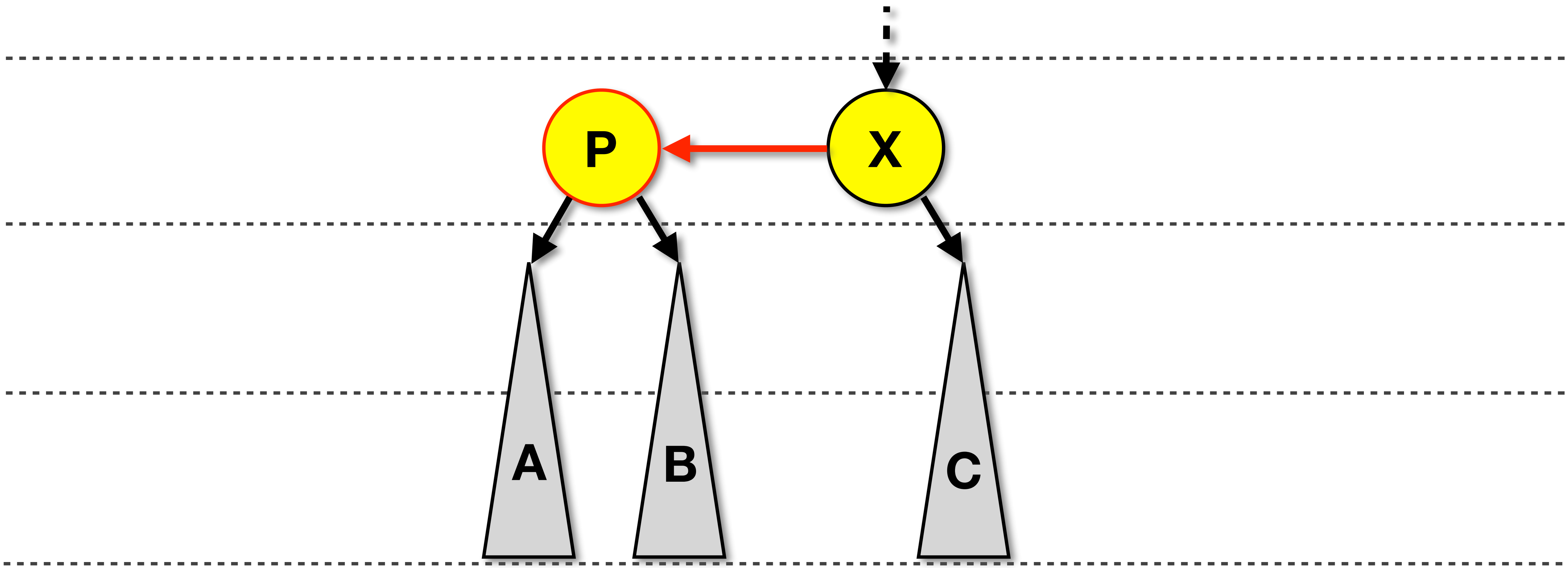
Horizontal Links in AA Trees

- **Case #2 - Two consecutive right horizontal links are NOT allowed**
  - Violates rule #4 - the *level* of a right grandchild is strictly less than that of its grandparent
  - A ***split*** operation will be introduced to handle this case



The ***skew*** Operation

- **The *skew* operation is a single right rotation when an insertion (or deletion) creates a left horizontal link**
  - Removes the left horizontal link
  - May create consecutive right horizontal links in process



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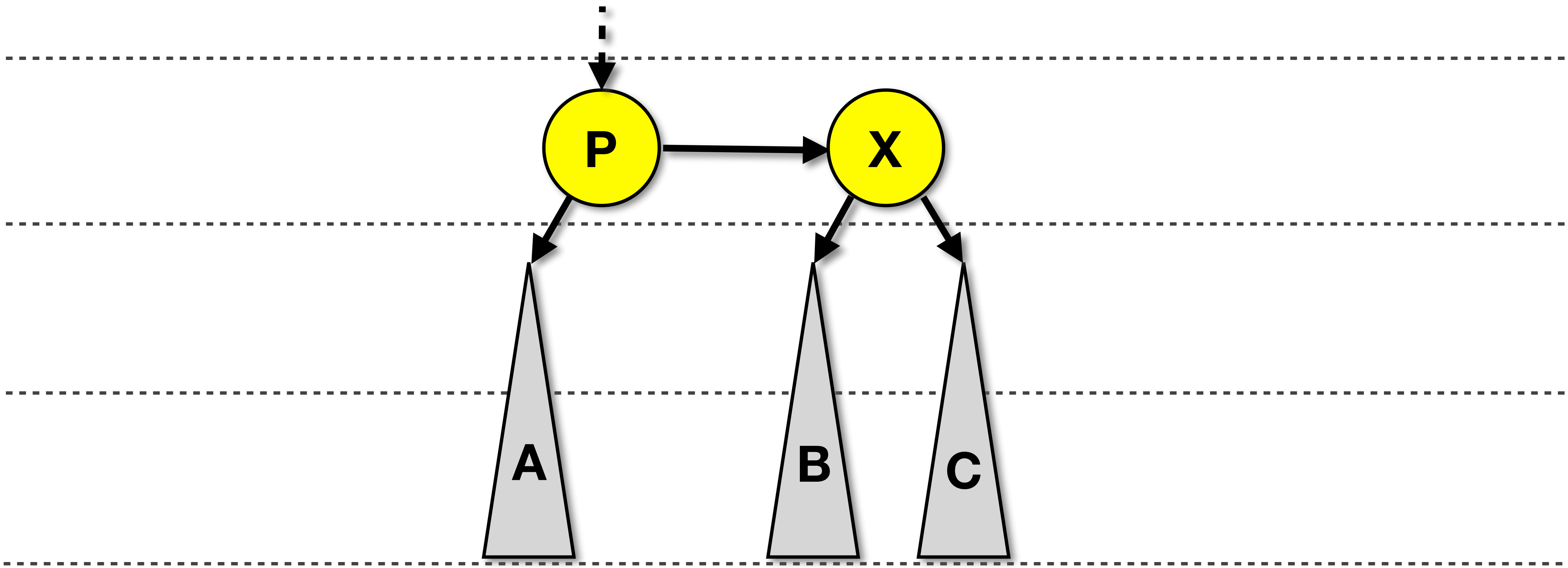

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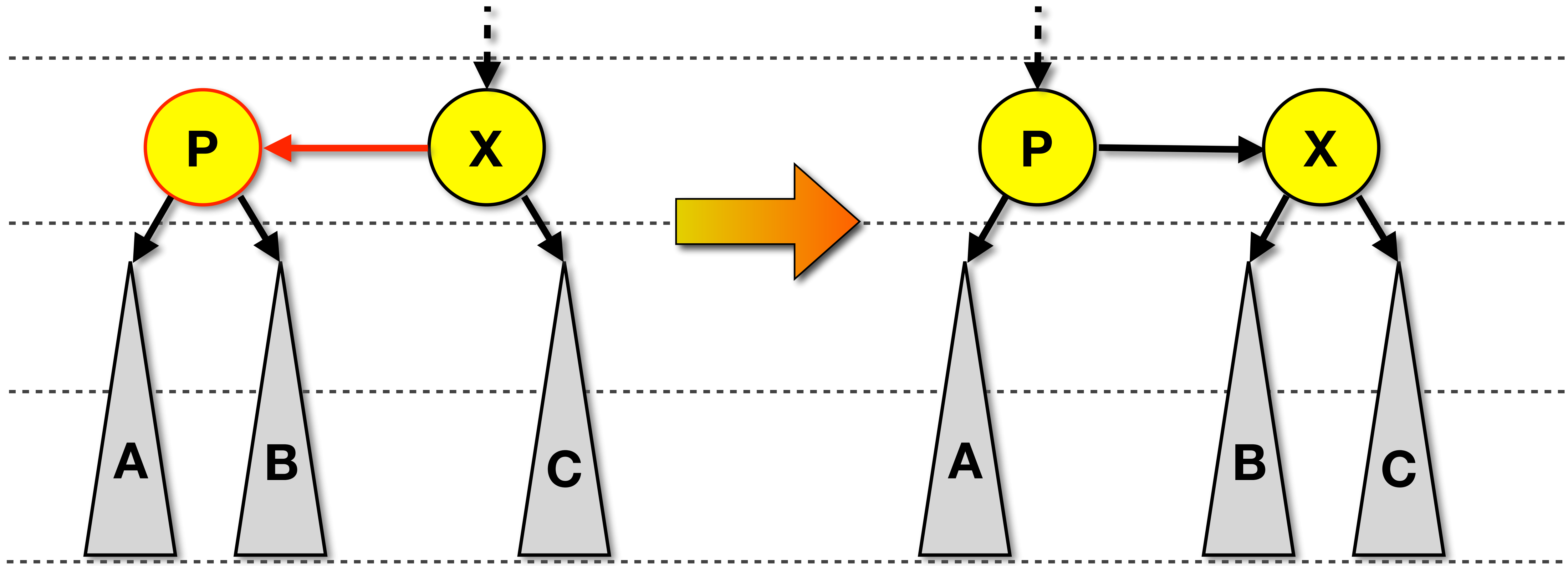
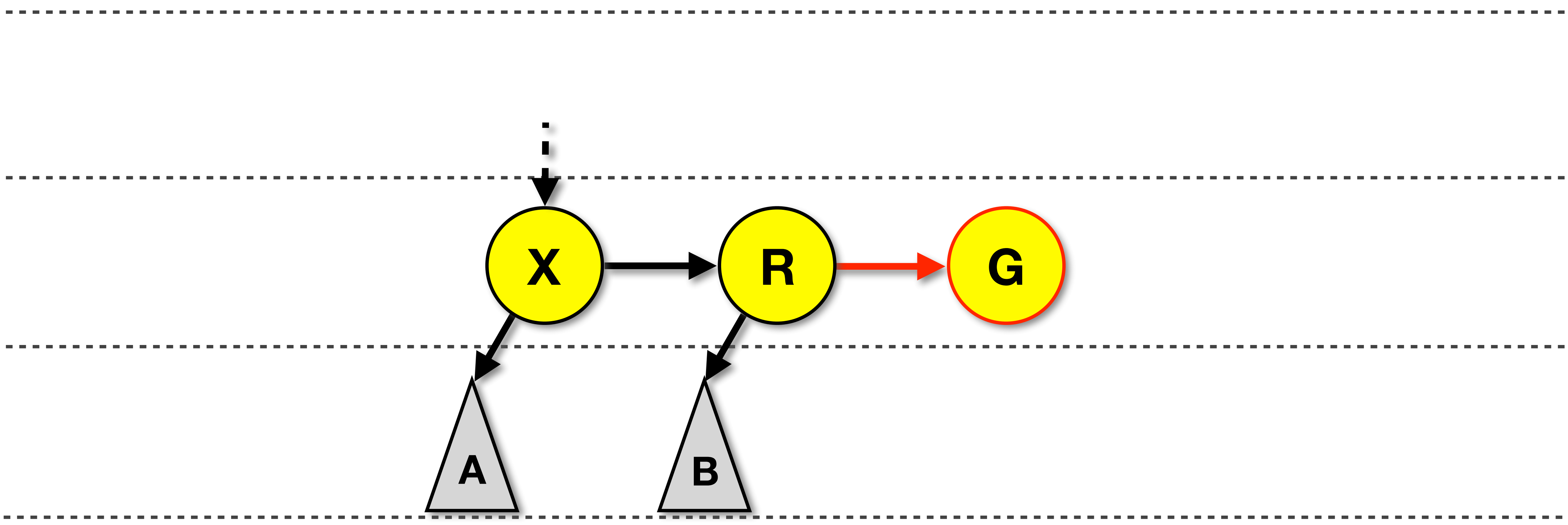


The *split* Operation

- **The *split* operation is a single left rotation when an insertion (or deletion) creates two consecutive right horizontal links**
  - Removes two consecutive right horizontal links
  - Increases level of middle node which may cause problems invalid horizontal links higher in tree



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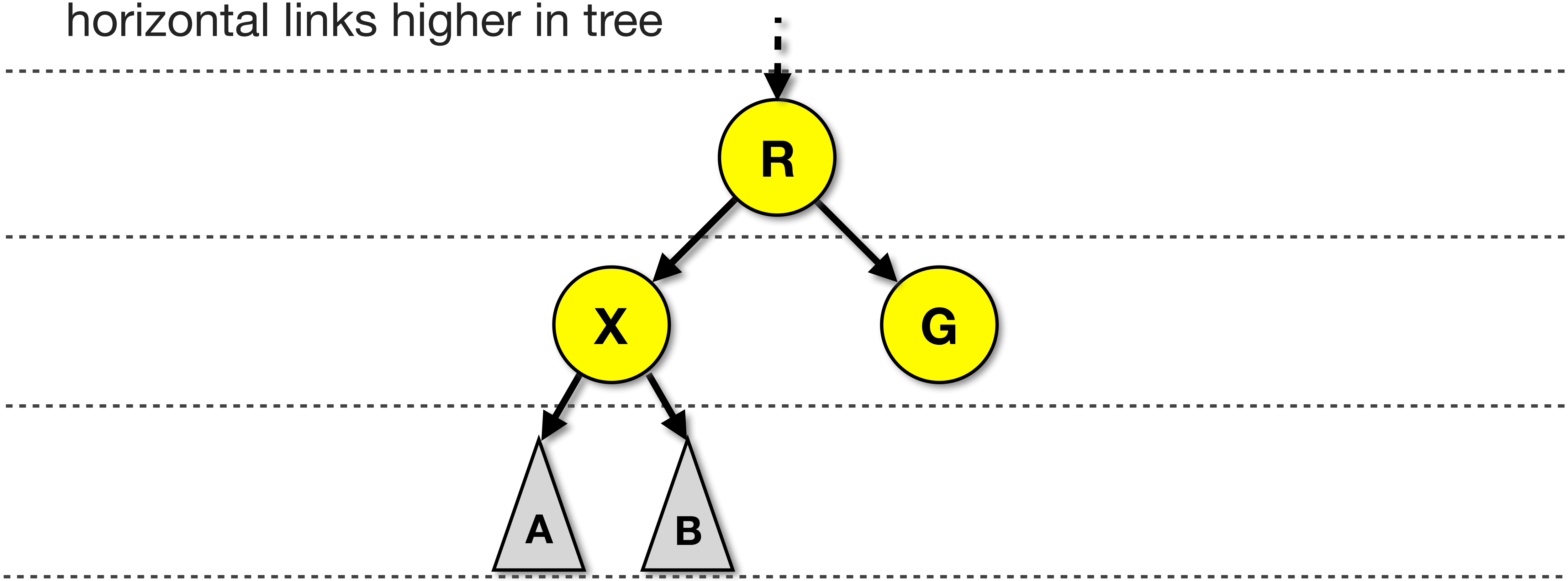

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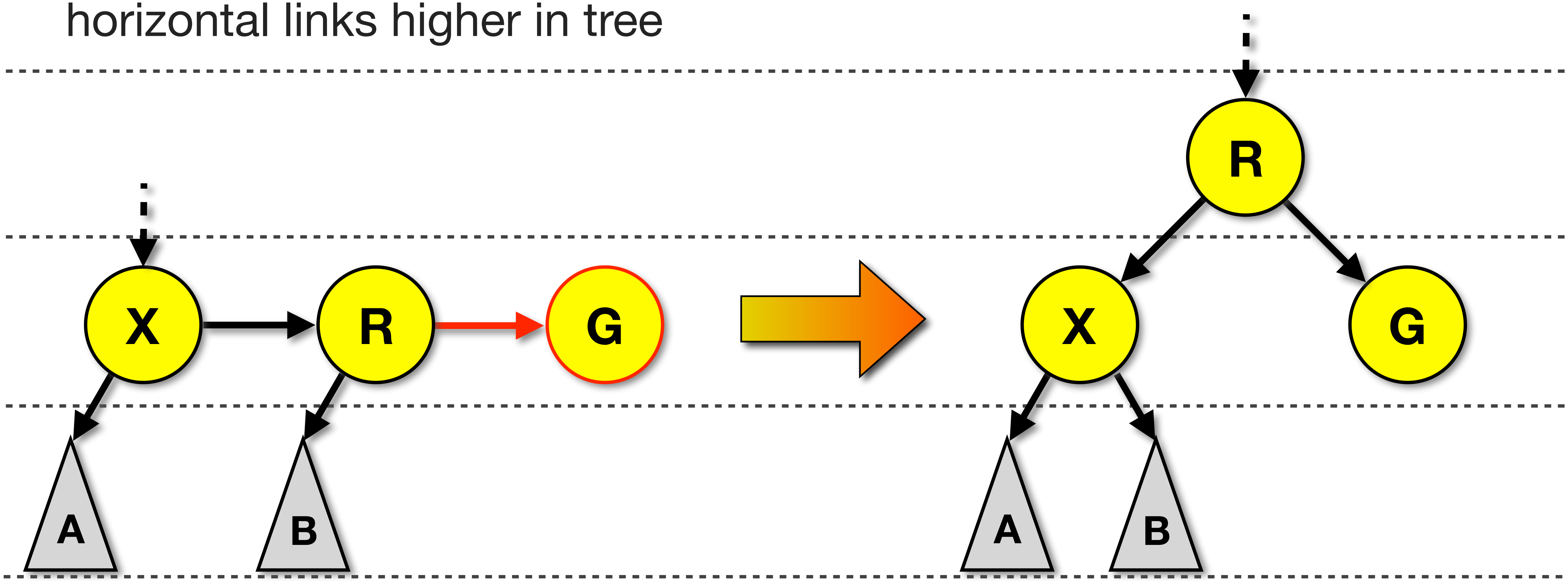


Implementation of ***insert***

```java
/**
   * Internal method to insert into a subtree.
   * @param x the item to insert.
   * @param t the node that roots the tree.
   * @return the new root.
   * @throws DuplicateItemException if x is already present.
   */
  private AANode<AnyType> insert( AnyType x, AANode<AnyType> t )
  {
      if( t == nullNode )
          t = new AANode<AnyType>( x, nullNode, nullNode );
      else if( x.compareTo( t.element ) < 0 )
          t.left = insert( x, t.left );
      else if( x.compareTo( t.element ) > 0 )
          t.right = insert( x, t.right );
      else
          throw new DuplicateItemException( x.toString( ) );

      t = skew( t );
      t = split( t );
      return t;
  }
```

Implementation of ***skew***

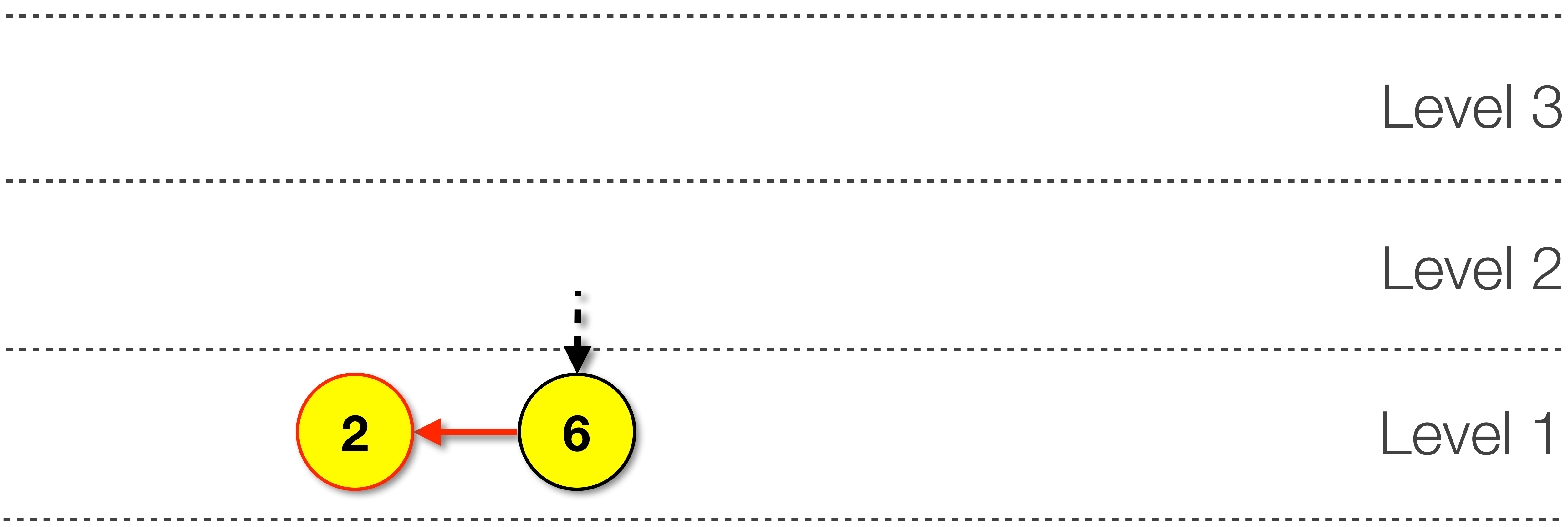
```java
/**
 * Skew primitive for AA-trees.
 * @param t the node that roots the tree.
 * @return the new root after the rotation.
 */
private static AANode<AnyType> skew( AANode<AnyType> t )
{
    if( t.left.level == t.level )
        t = rotateWithLeftChild( t );
    return t;
}
```

Implementation of ***split***

```java
/**
  * Split primitive for AA-trees.
  * @param t the node that roots the tree.
  * @return the new root after the rotation.
  */
private static AANode<AnyType> split( AANode<AnyType> t )
{
    if( t.right.right.level == t.level )
    {
        t = rotateWithRightChild( t );
        t.level++;
    }
    return t;
}
```

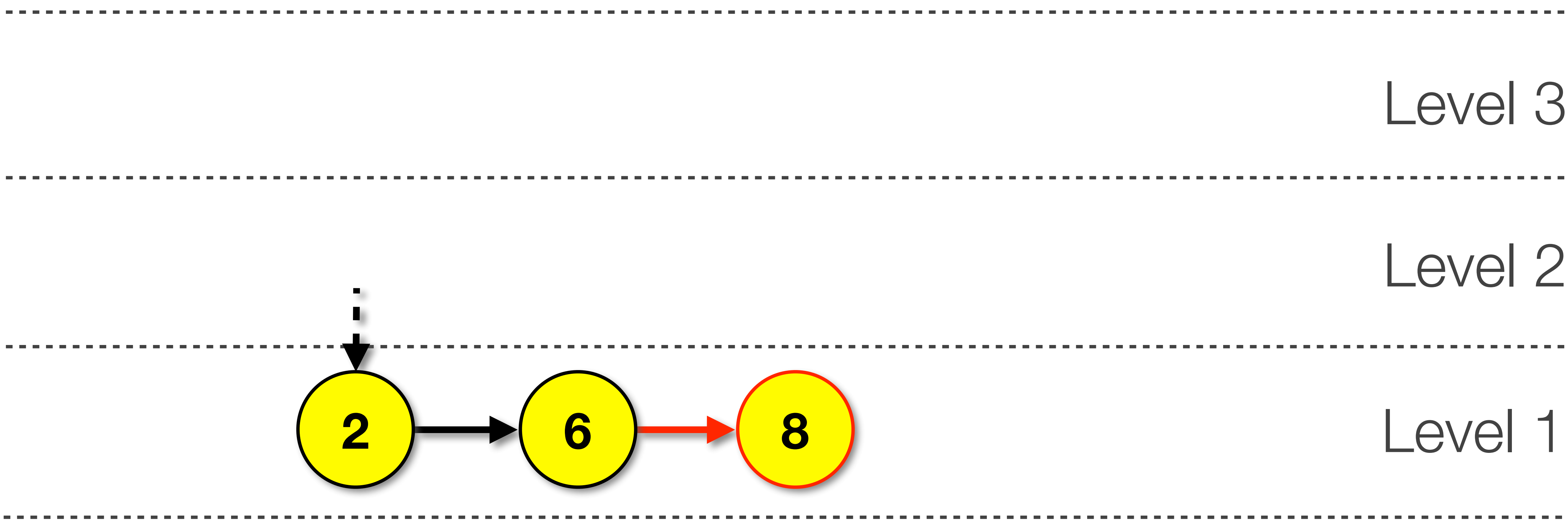
Example of Insertion

Inserts: 6 2



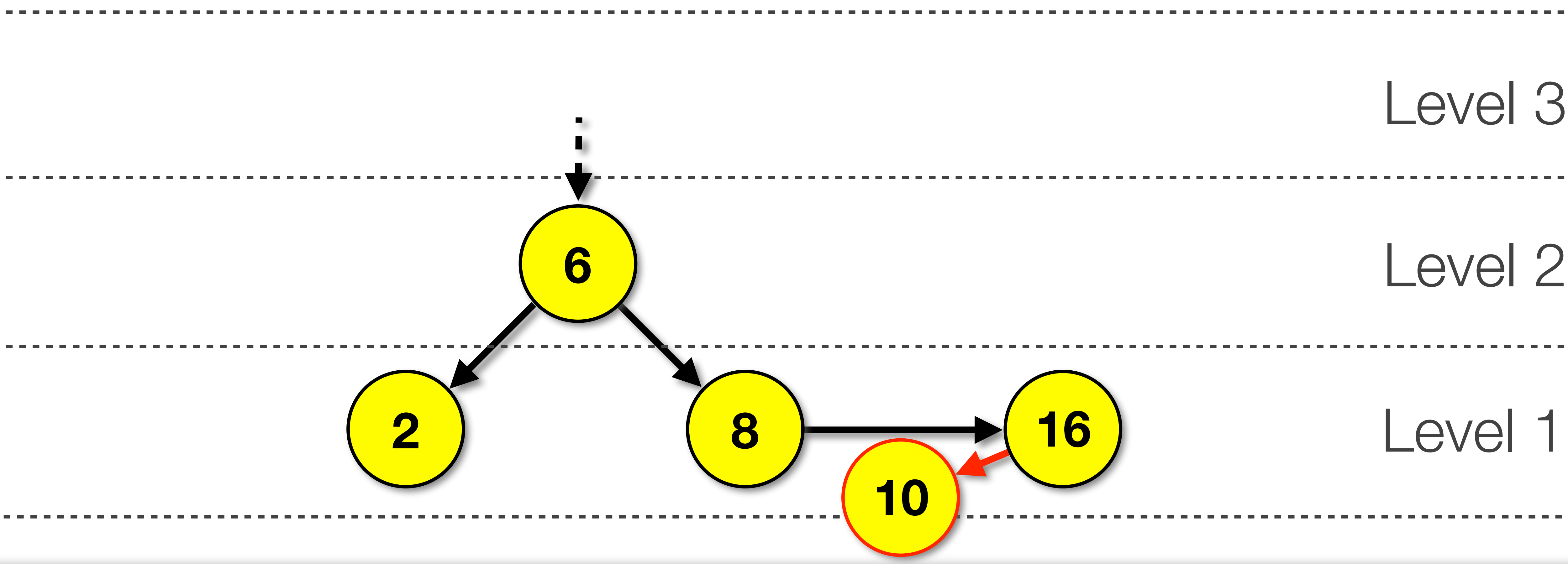
Example of Insertion

Inserts: 6 2 8



Example of Insertion

Inserts: 6 2 8 16 10



Example of Insertion

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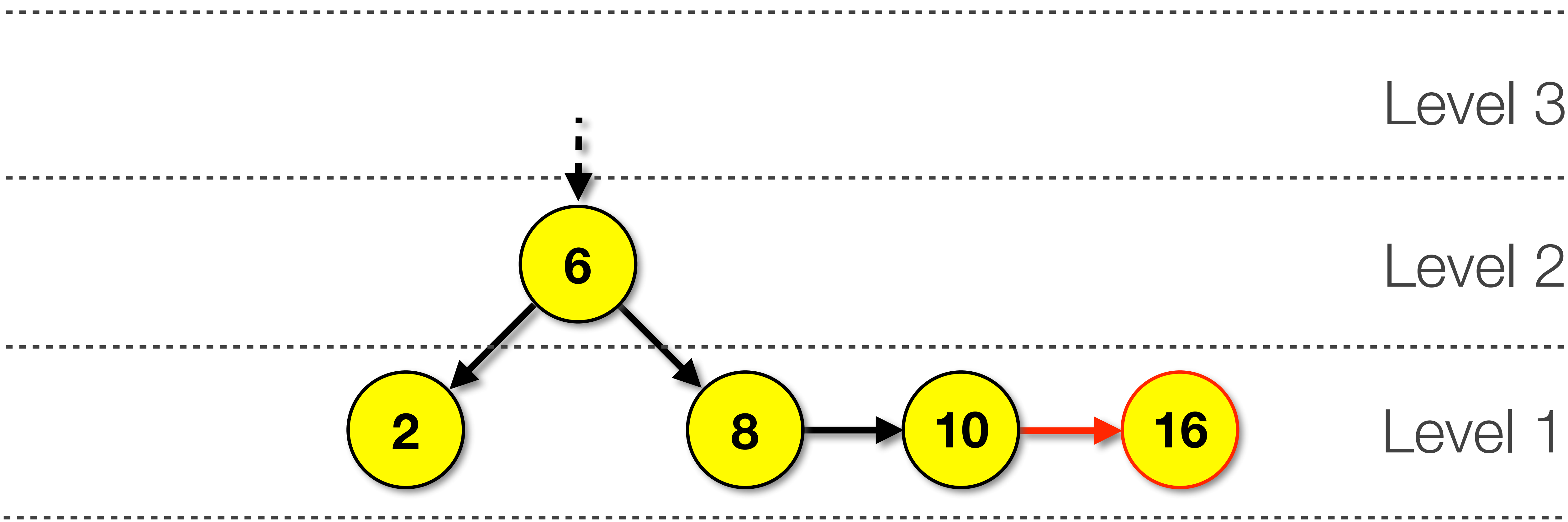
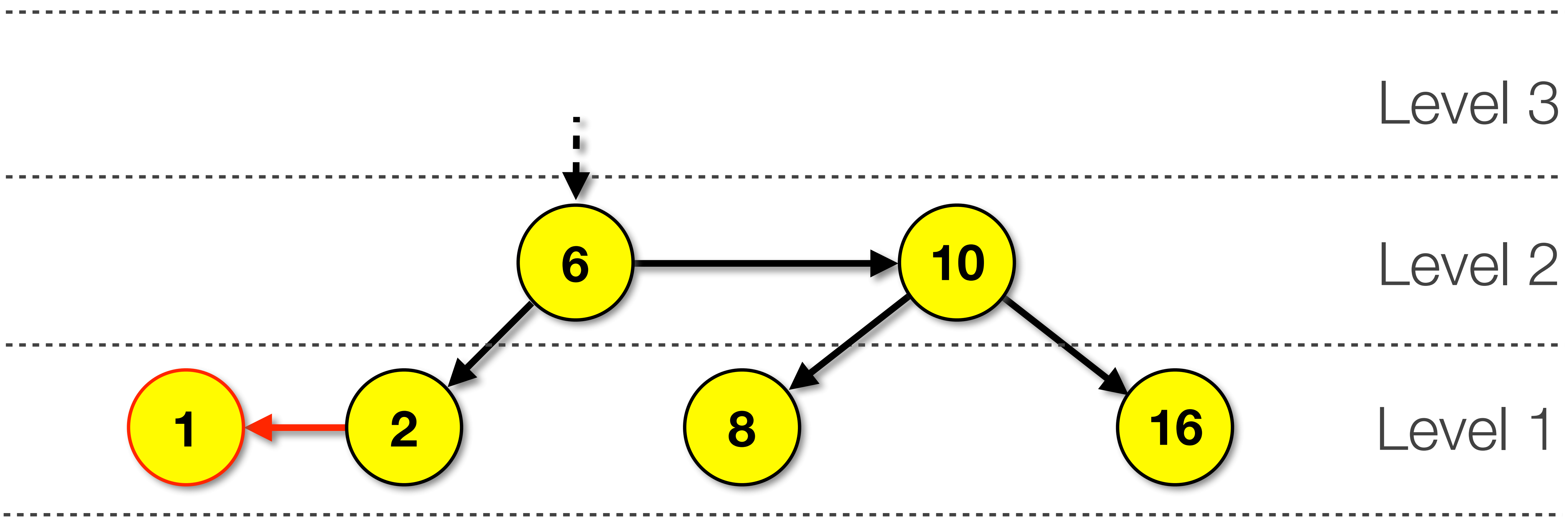


Example of Insertion

Inserts: **6 2 8 16 10 1**



Example of Insertion

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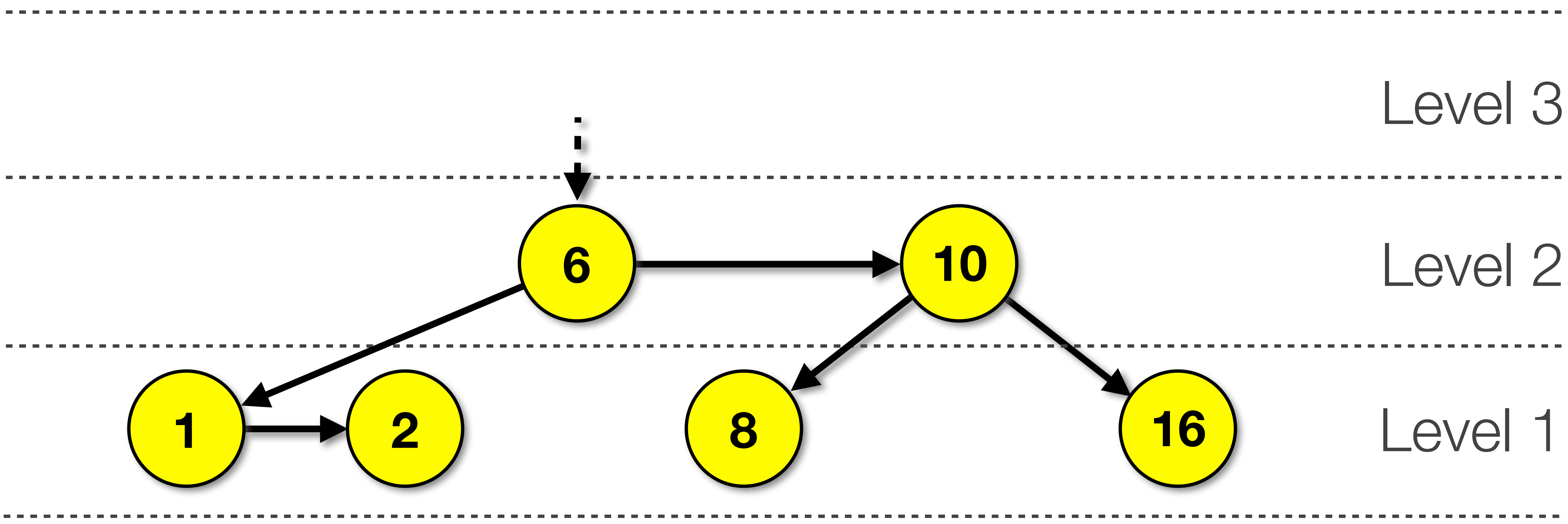


Deleting From AA Trees

- **Perform a *recursive* deletion just like on other BSTs:**
  - To delete a leaf node with no children, simply remove the node
  - To delete a leaf node with one child, replace node with child (in AA trees the child node will be a right child / both nodes at level 1)
  - To delete an internal node, replace that node with either its successor or predecessor

- **May need to rebalance AA tree after a deletion occurs**

Fixing an Unbalanced AA Tree

1) Decrease the level of a node when:

- Either of the nodes children are more than one level down
  (Note that a null sentinel node is at level 0)
- A node is the right horizontal child of another node whose level was decreased

2) Skew the level of the node whose level was decremented (3 skews)

- Skew the subtree from the root, where the decremented node is the root
  (may alter the root node of the subtree)
- Skew the root node's right child
- Skew the root node's right-right child

3) Split the level of the node whose level was decremented (2 splits)

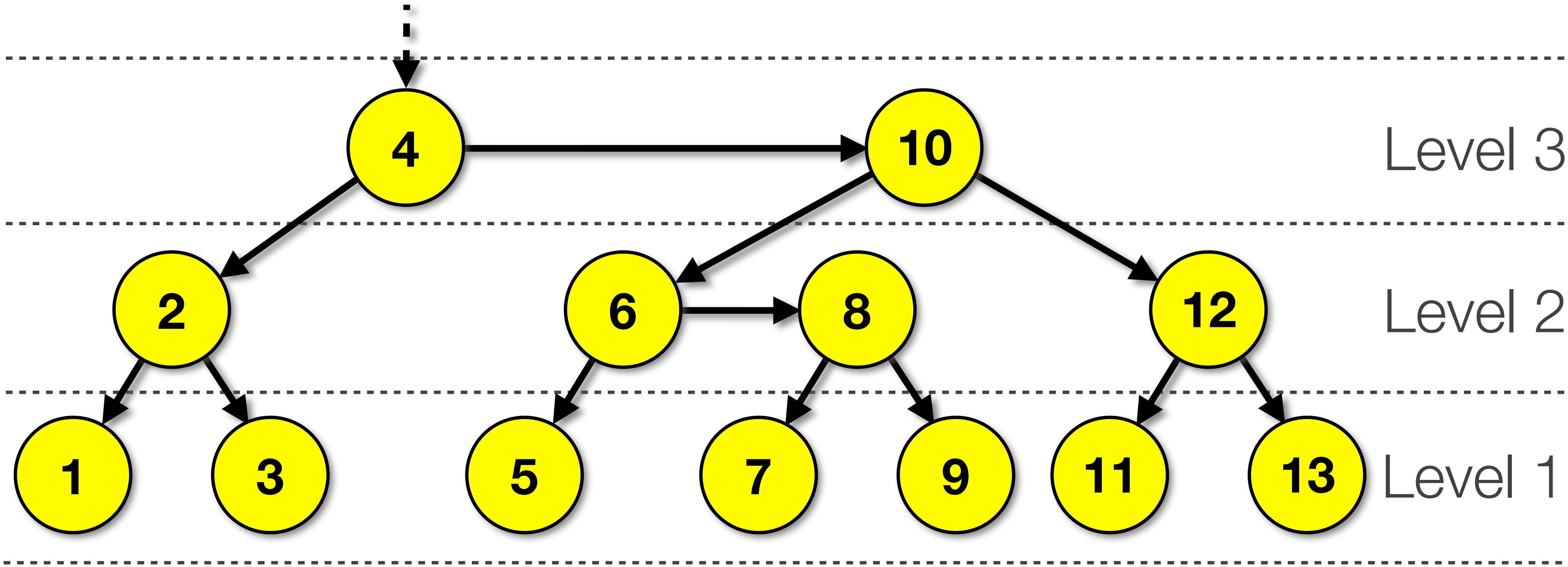
- Split the root node of the subtree
  (may alter the root node of the subtree)
- Split the root node's right child

Excerpt From ***remove***

```java
// Rebalance tree
if( t.left.level < t.level - 1 || t.right.level < t.level - 1 )  // check level of children
{
    if( t.right.level > --t.level )         // check level of right horizontal children
        t.right.level = t.level;            //       and decrement if necessary
    t = skew( t );                          // First skew (may alter current root)
    t.right = skew( t.right );              // Second skew
    t.right.right = skew( t.right.right );  // Third skew
    t = split( t );                         // First split (may alter current root)
    t.right = split( t.right );             // Second split
}
```

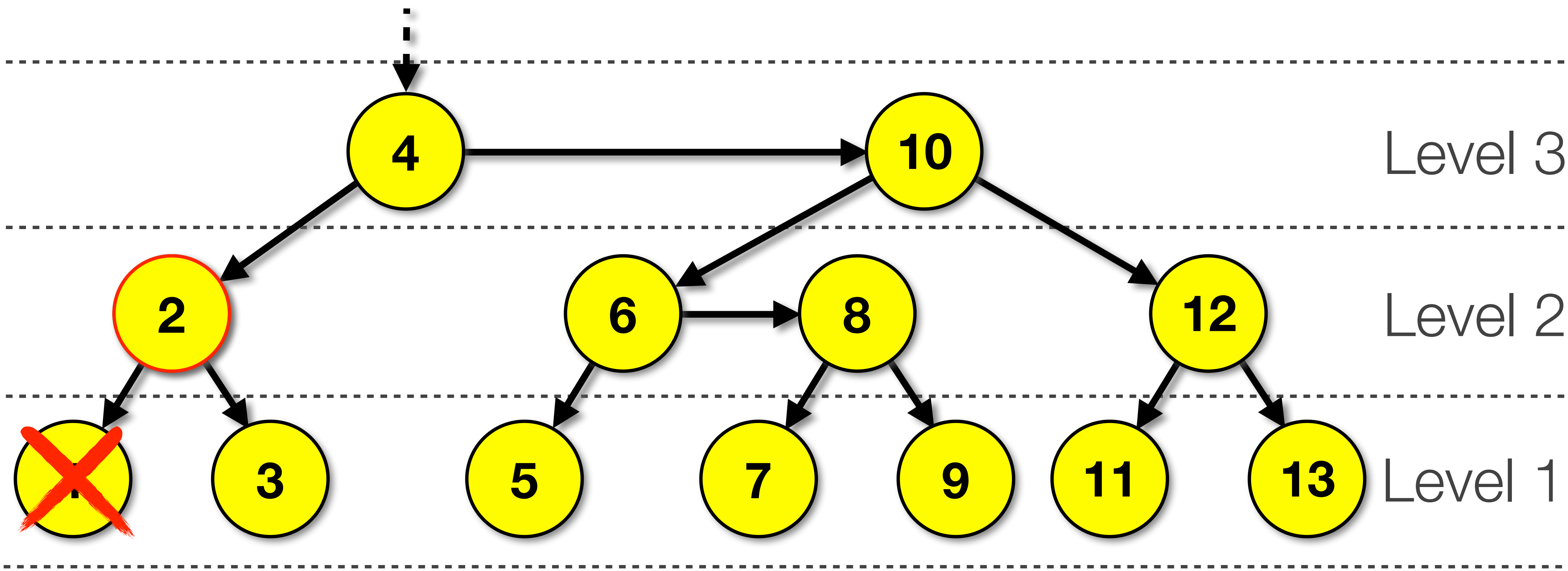
Example of Deletion

This tree can be recreated with the following sequence of inserts:
4, 10, 2, 6, 12, 3, 1, 8, 13, 11, 5, 9, 7



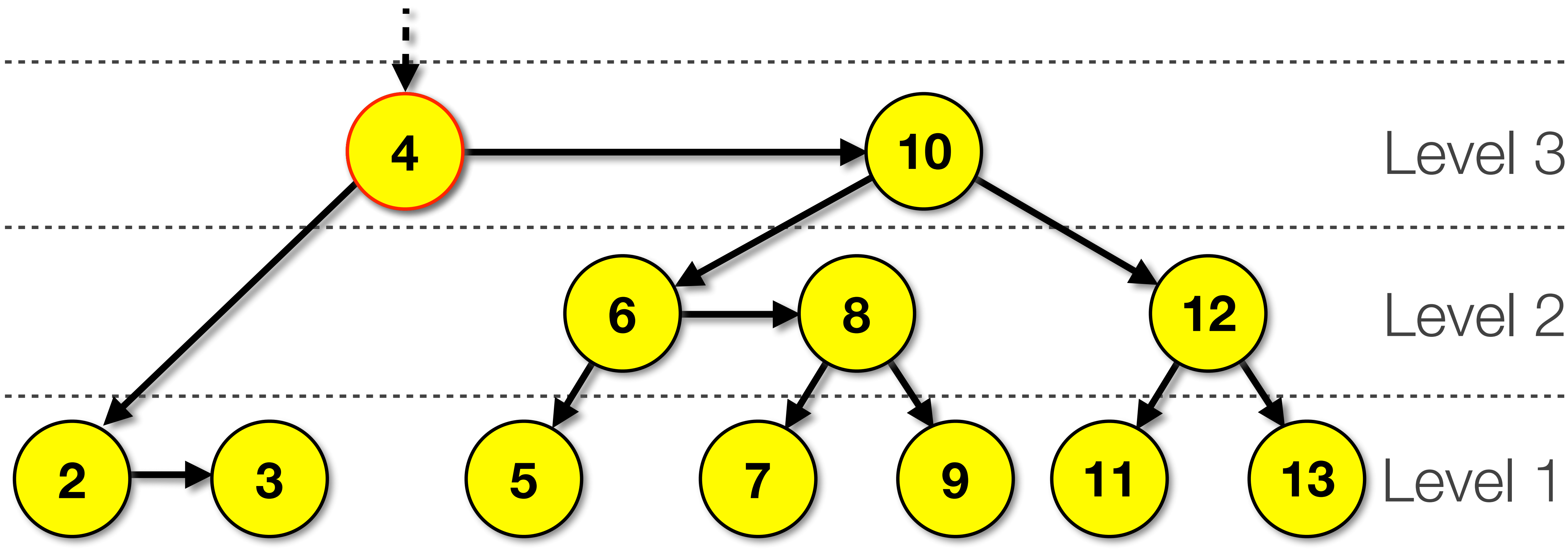
Example of Deletion

Delete node 1
Node 2 now violates rule #5



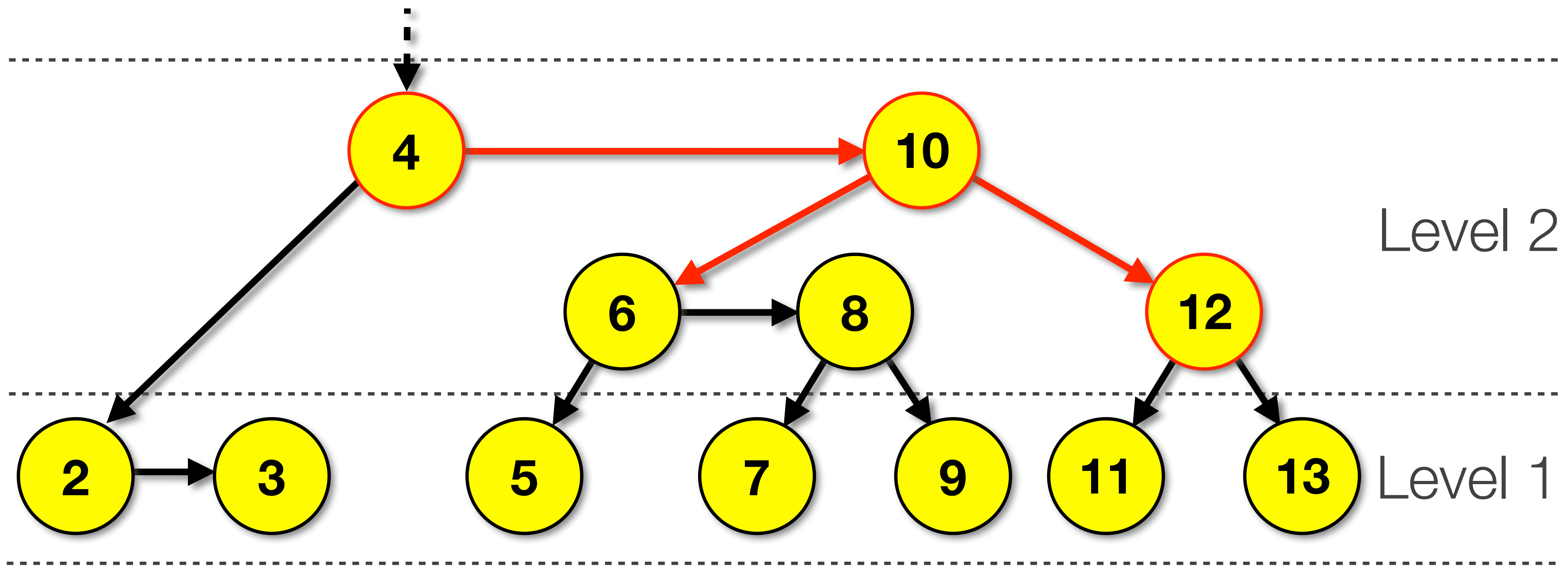
Example of Deletion

Decrement the level of node 2
Node 4 is more than one level above child



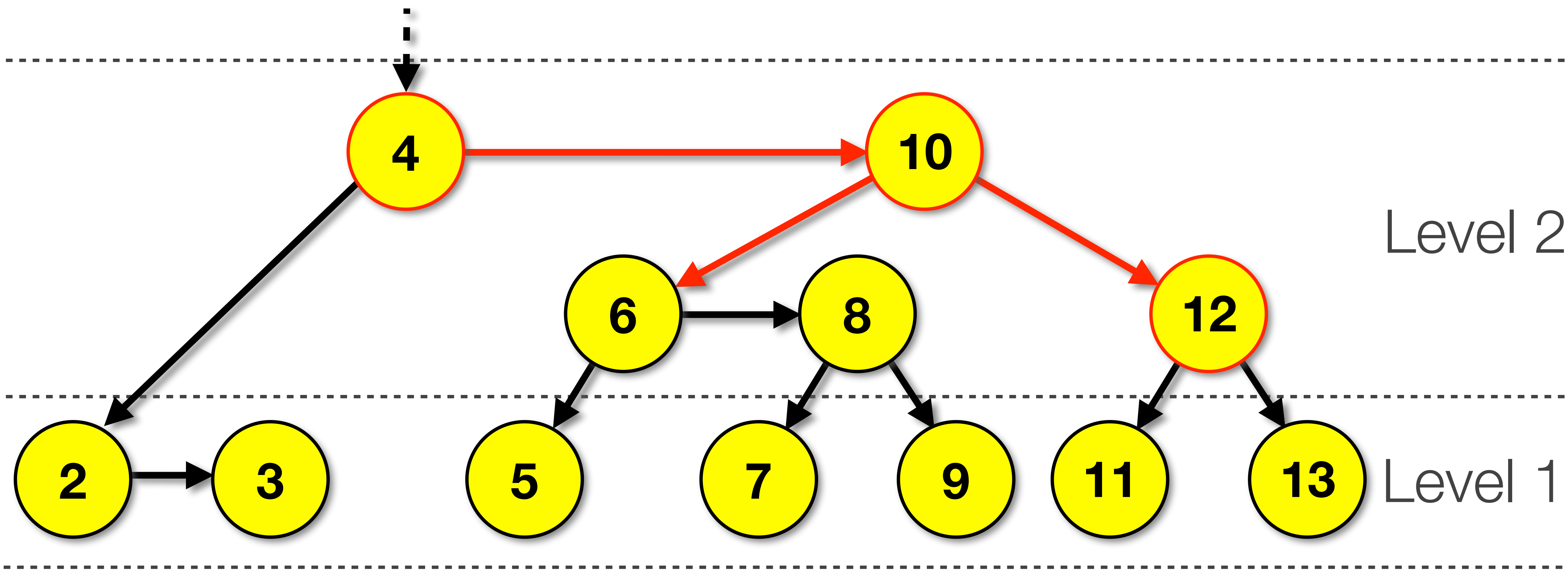
Example of Deletion

Decrement the level of nodes 4 and 10
Node 4 now has two consecutive right links
Node 10 now has left horizontal link



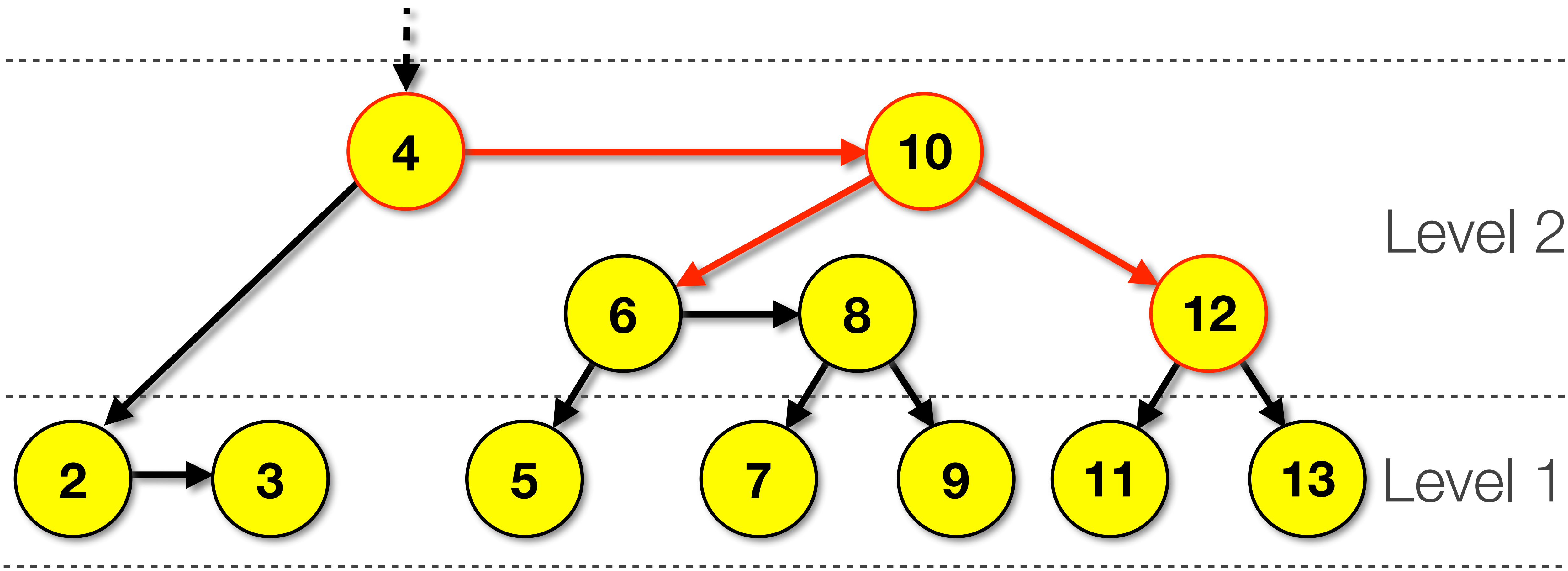
Example of Deletion

No more level decrementing necessary
Start triple-skew, double-split process



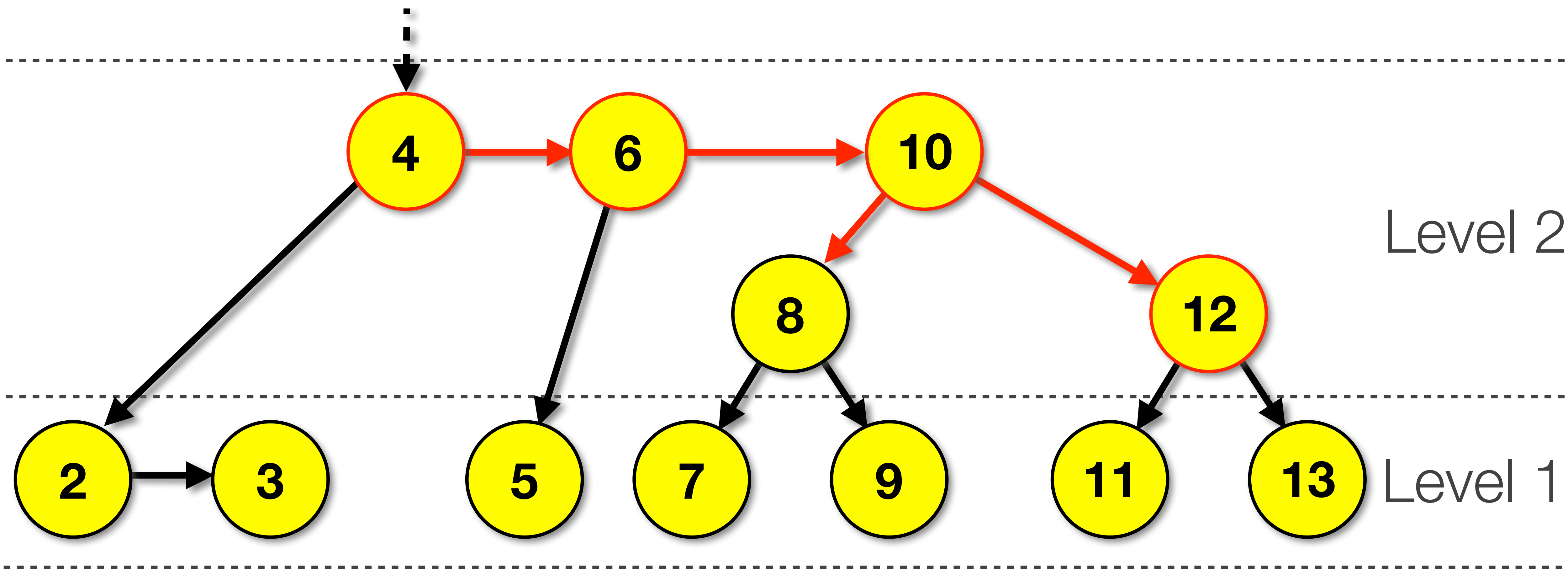
Example of Deletion

Skew node 4 (does nothing)
Next skew 4.right (node 10)



Example of Deletion

After skew 4.right (node 10)
Next skew 4.right.right (node 10 again)



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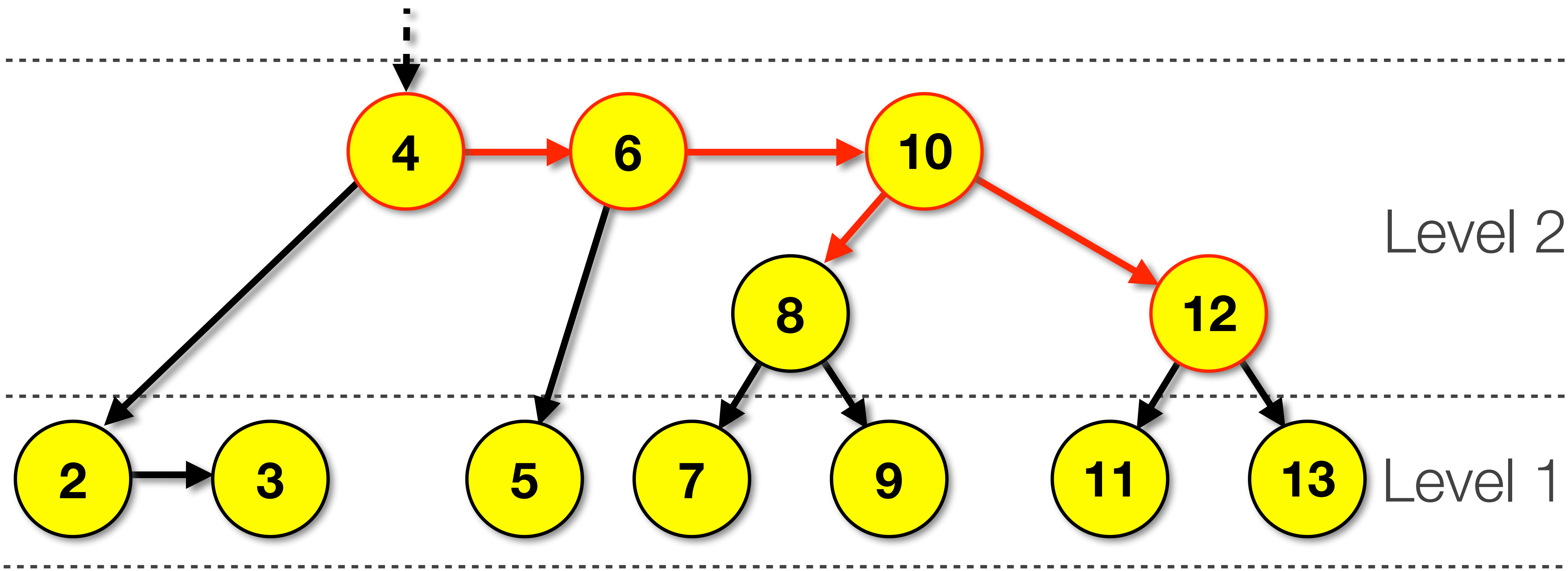
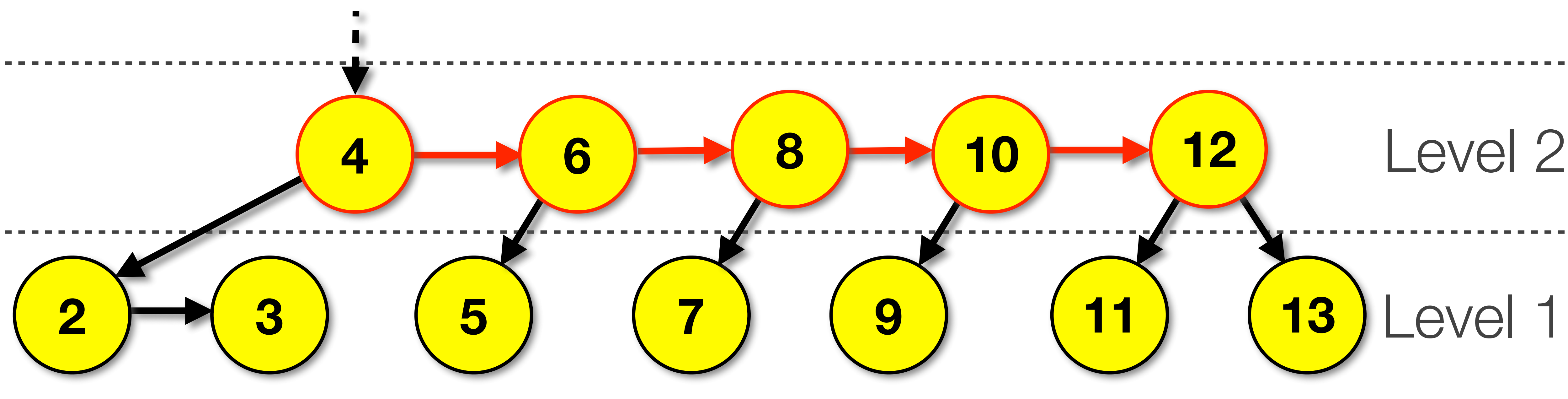
After skew 4.right (node 10)
Next skew 4.right.right (node 10 again)



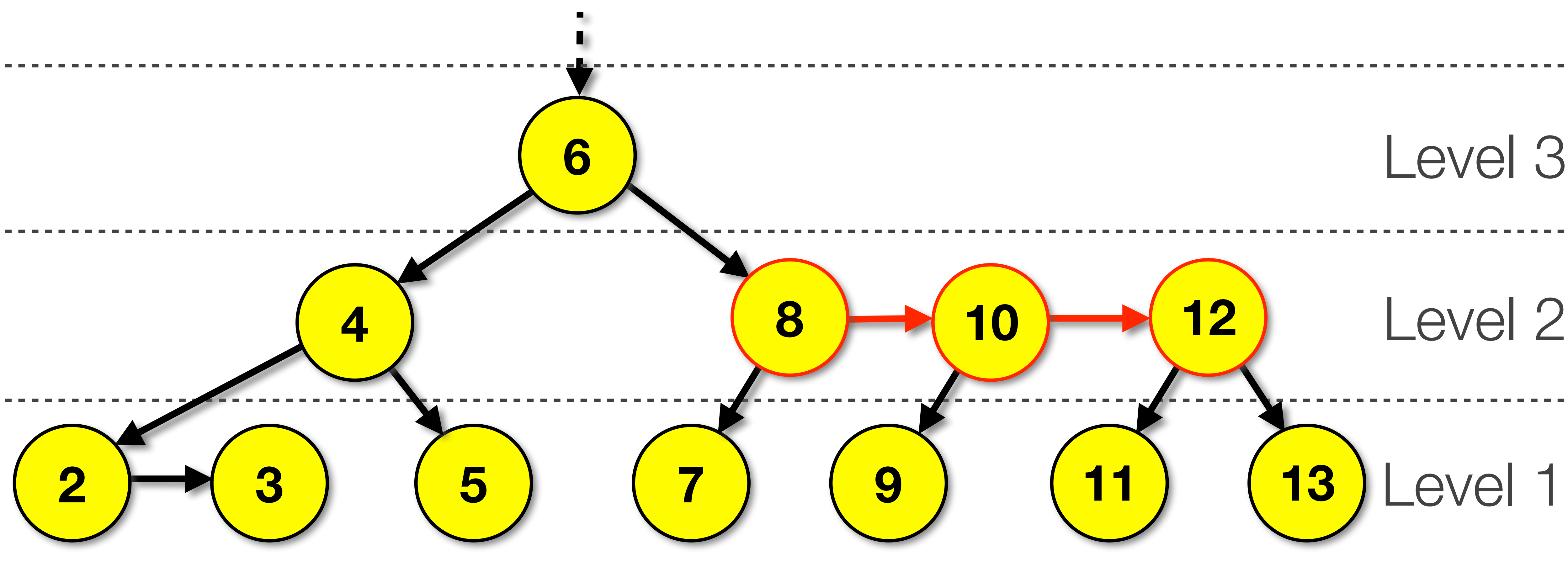
Example of Deletion

After skew 4.right.right (node 10)
Next split node 4



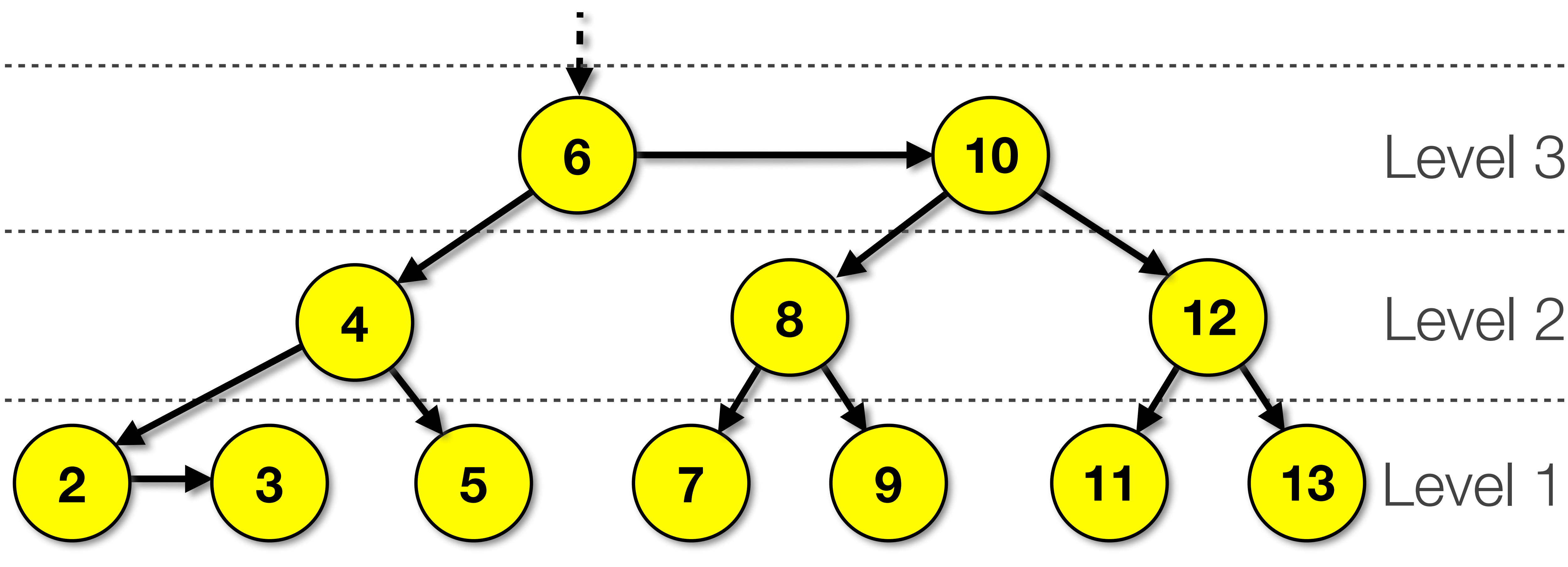
Example of Deletion

After split node 4 (new subtree root)
Next split node 6.right (node 8)



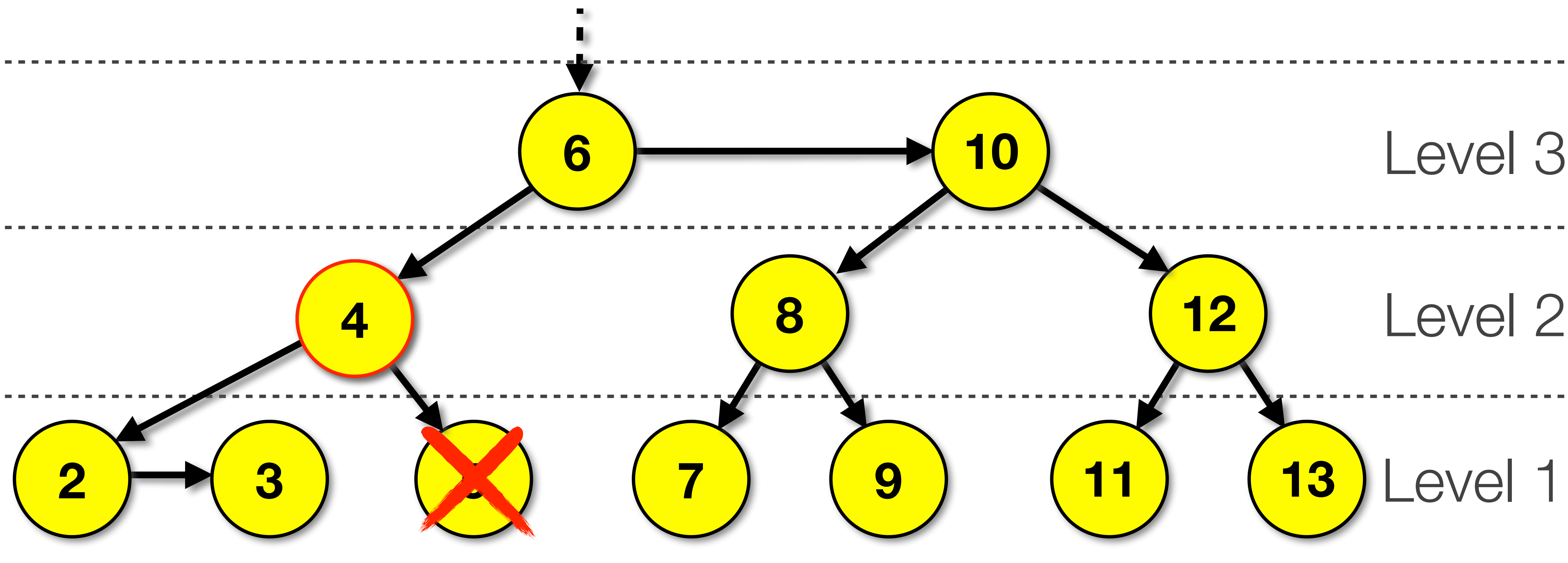
Example of Deletion

After split node 6
Tree is balanced



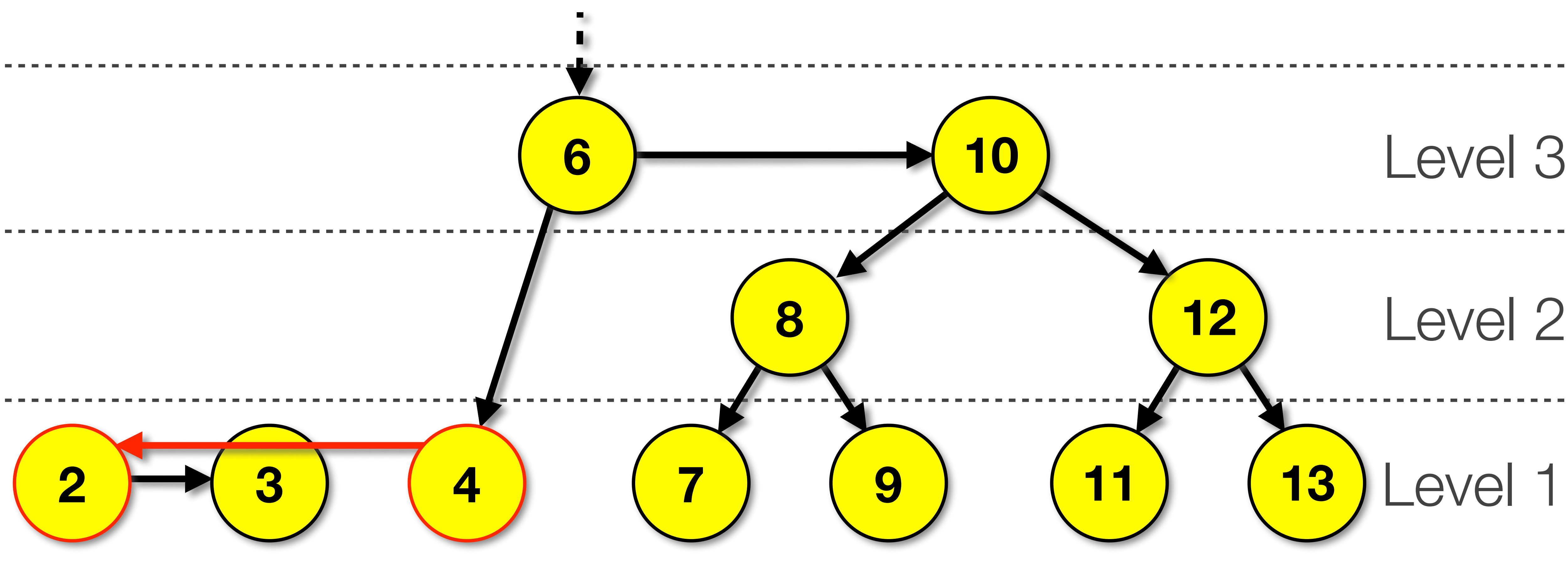
Example of Deletion

Delete node 5
Node 4 is now violates rule #5



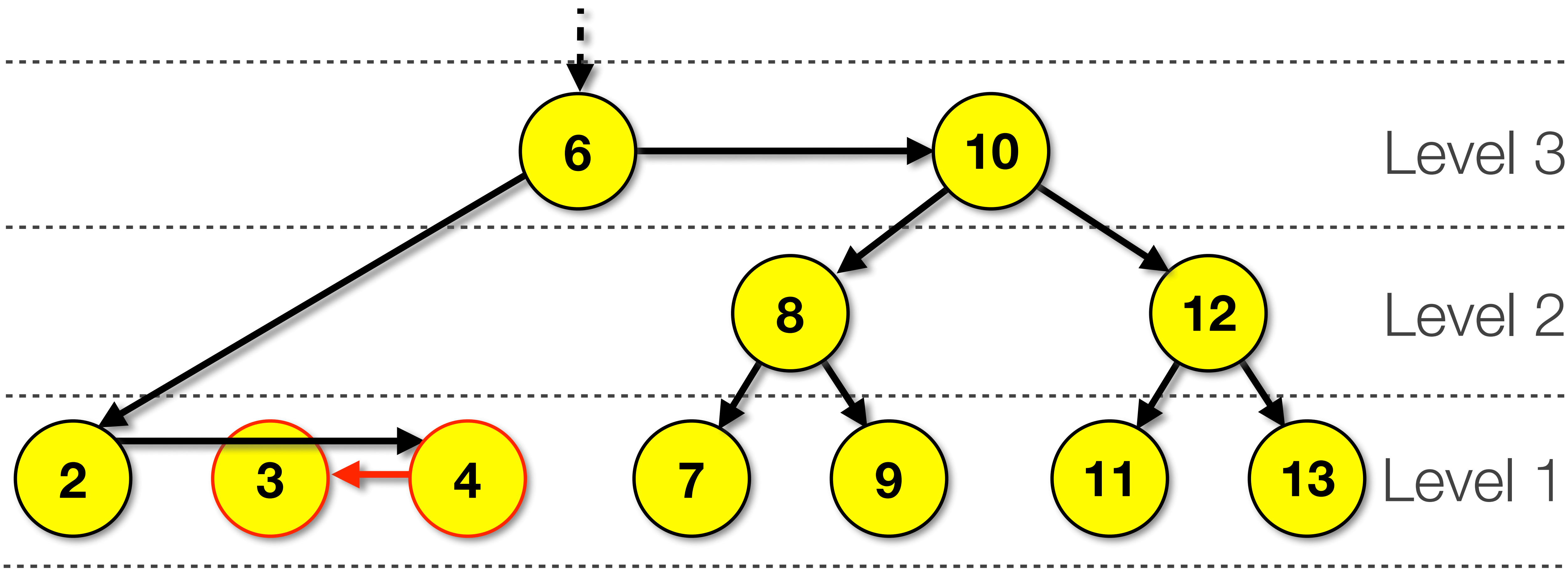
Example of Deletion

Decrement the level of node 4
Node 4 now has left horizontal link
Next skew node 4



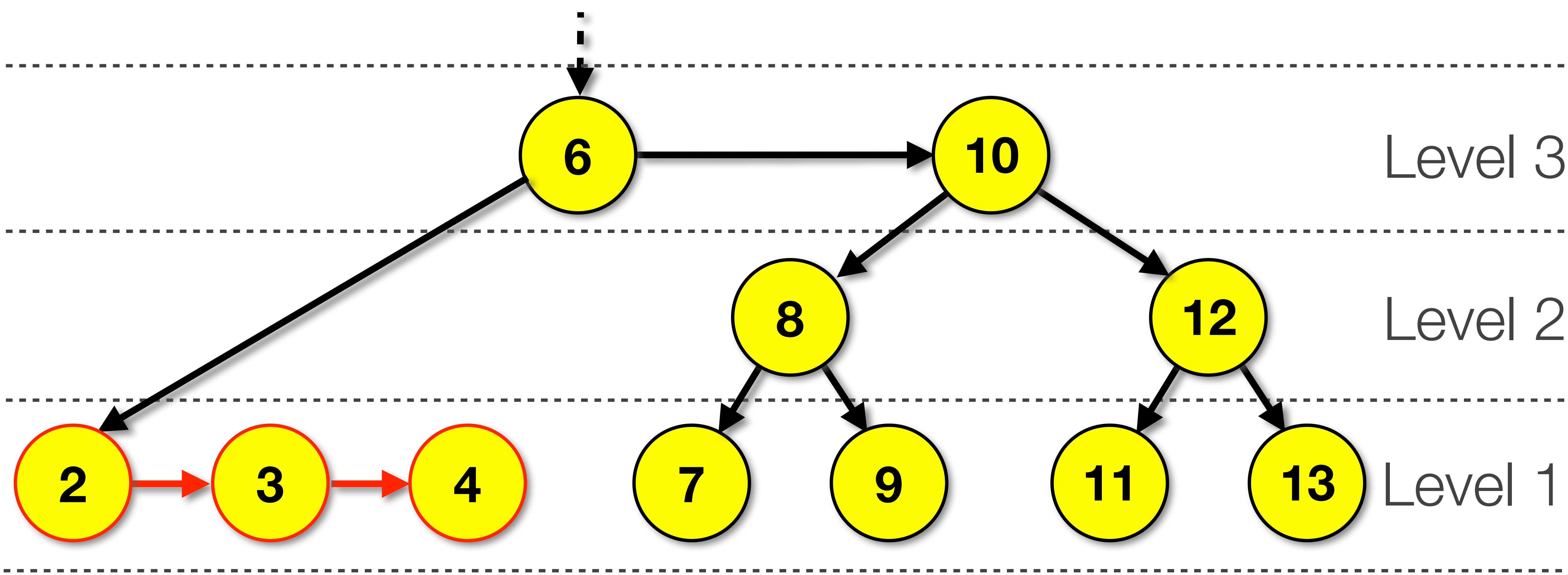
Example of Deletion

After skew node 4 (new subtree root)
Next skew node 2.right (node 4 again)



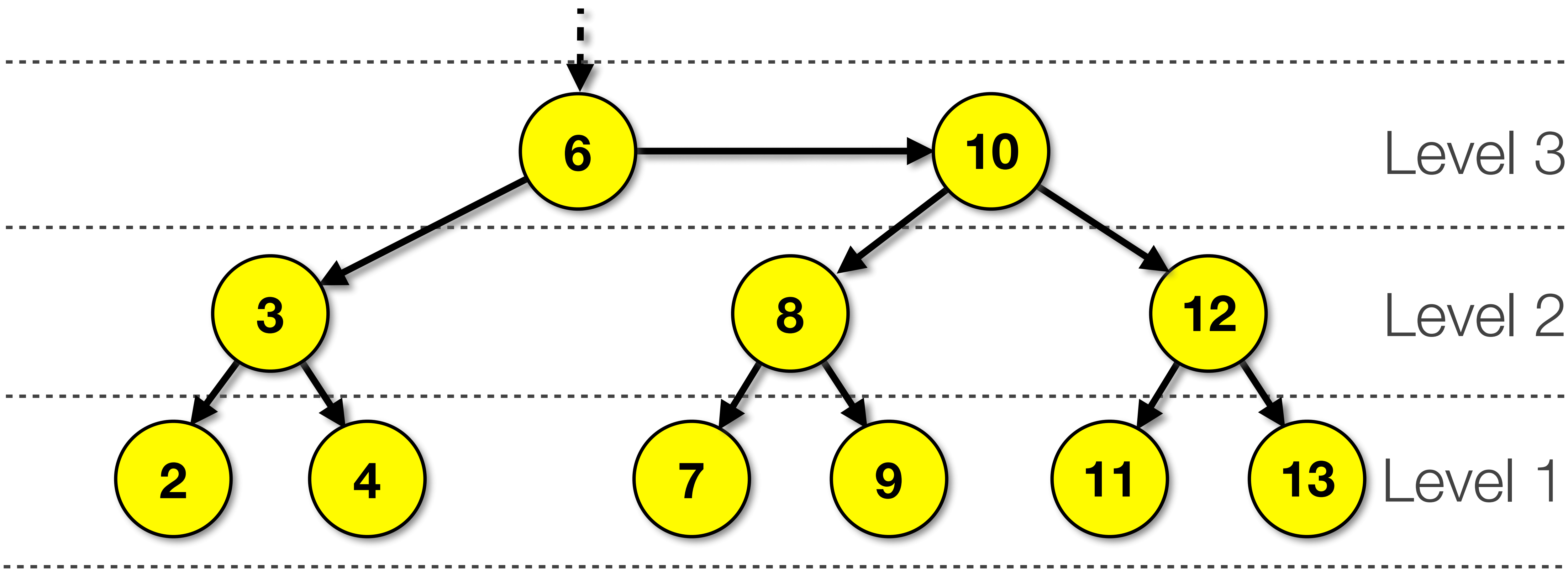
Example of Deletion

After skew node 2.right
Skew node 2.right.right (does nothing)
Split node 2



Example of Deletion

After split node 2 (new subtree root)
Split node 3.right (does nothing)
Tree is balanced



Additional Slides

Implementation of Child Rotations

```java
/**
 * Rotate binary tree node with left child.
 * For AVL trees, this is a single rotation for case 1.
 */
static BinaryNode<AnyType> rotateWithLeftChild( BinaryNode<AnyType> k2 )
{
    BinaryNode<AnyType> k1 = k2.left;
    k2.left = k1.right;
    k1.right = k2;
    return k1;
}
```

```java
/**
 * Rotate binary tree node with right child.
 * For AVL trees, this is a single rotation for case 4.
 */
static BinaryNode<AnyType> rotateWithRightChild( BinaryNode<AnyType> k1 )
{
    BinaryNode<AnyType> k2 = k1.right;
    k1.right = k2.left;
    k2.left = k1;
    return k2;
}
```