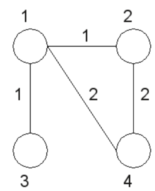
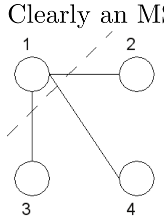
CS360 Assignment 6

23.1-2 Let $G(V, E)$ be a connected, undirected graph with real valued weights $w(u, v)$. Let A be a subset of E that is included in some MST for G, $(S, V - S)$ be any cut of G that respects A, and (u, v) be a safe edge for A that crosses $(S, V - S)$. Show that (u, v) is *not necessarily* a light edge. (Note: this is showing that the *converse* of Theorem 23.1 is *not* true.)

Consider the following graph:



Clearly an MST for this graph is:



If we let $S = \{1\}$ and $A = \emptyset$, then the cut $\{1\}, \{2, 3, 4\}$ trivially respects A. Furthermore, all three edges $(1, 2), (1, 3), (1, 4)$ are *safe* edges (since they are in the MST). However edge $(1, 4)$ is **not** a light edge and hence the converse of Theorem 23.1 is not true, i.e. light edge $\Rightarrow$ safe edge, but safe edge $\not\Rightarrow$ light edge).

23.2-4 Suppose all edge weights are integers in the range $1 \rightarrow |V|$. How fast can Kruskal's algorithm be made to run? What if the weights are integers in the range $1 \rightarrow W$ for a *constant* W?

The running time for a general implementation of Kruskal's algorithm consists of:

$$\begin{array}{rll} & O(V) & \text{- Initialization} \\ & O(E \lg E) & \text{- Edge sorting} \\ + & O(E\alpha(V)) & \text{- Edge processing} \\ \hline & O(E \lg E) & \end{array}$$

Knowing the range of the weights allows for the use of *counting sort* to improve the time for the sorting step - which is the dominant term in the general implementation. If the weights are in the range $1 \rightarrow |V|$, sorting of the edges using counting sort becomes $O(n + k) = O(E + V)$. For connected graphs

$V = O(E)$ giving a sort time of $O(E)$. Hence the total running time becomes $O(V) + O(E) + O(E\alpha(V)) = O(E) + O(E) + O(E\alpha(V)) = O(E\alpha(V))$
If the weights are in the range $1 \to W$ (where W is a *constant*), then sorting becomes $O(n + k) = O(E + W) = O(E)$ (same as above) which again gives a running time of $O(E\alpha(V))$.

23.2-5 Suppose all edge weights are integers in the range $1 \to |V|$. How fast can Prim's algorithm be made to run? What if the weights are integers in the range $1 \to W$ for a *constant* W?

The running time for a general implementation of Prim's algorithm (assuming a Fibonacci heap implementation) consists of:

$$\begin{array}{lll} & VO(\lg V) & \text{- Extract-min for each vertex} \\ + & EO(1) & \text{- Decrease key for each edge} \\ \hline & O(V \lg V + E) & \end{array}$$

Implementing the queue as an array of linked-lists with one element per weight value changes the running time of Extract-min() which now consists of *scanning* the array for the first non-empty list element. (Note: Decrease-key() remains $O(1)$ since it simply involves moving the element from one list to another.) If the weights range from $1 \to |V|$ then Extract-min() becomes $O(V)$ per scan giving a total time $O(V(V) + E) = O(V^2)$ which is *worse* than using a heap!

However if the weights range from $1 \to W$ (where W is a *constant*), then Extract-min() becomes $O(1)$ per scan giving a total time of $O(V(1) + E) = O(V + E) = O(E)$ (for a connected graph) which turns out to be the best we can do.

24.1-1 Run the Bellman-Ford algorithm on the directed graph of Fig 24.4 using the vertex z as the source, relaxing edges in the same order as in the figure, giving the d and π values after each pass. Change the weight of edge $(z, x) = 4$ and rerun the algorithm with s as the source.

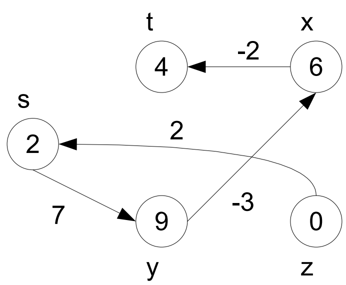
Considering edges in the order $(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$ gives the following table:

	d						π				
pass	s	t	x	y	z		s	t	x	y	z
0	∞	∞	∞	∞	0		/	/	/	/	/
1	2	8	7	9	0		z	s	z	s	/
2	2	5	6	9	0		z	x	y	s	/
3	2	4	6	9	0		z	x	y	s	/
4	2	4	6	9	0		z	x	y	s	/

Checking edges

$$
\begin{aligned}
d[x] = 6 &\quad < \quad d[t] = 4 + 5 \\
d[y] = 9 &\quad < \quad d[t] = 4 + 8 \\
d[z] = 0 &\quad = \quad d[t] = 4 - 4 * \\
d[t] = 4 &\quad = \quad d[x] = 6 - 2 * \\
d[x] = 6 &\quad = \quad d[y] = 9 - 3 * \\
d[z] = 0 &\quad < \quad d[y] = 9 + 9 \\
d[x] = 6 &\quad < \quad d[z] = 0 + 7 \\
d[s] = 2 &\quad = \quad d[z] = 0 + 2 * \\
d[t] = 4 &\quad < \quad d[s] = 2 + 6 \\
d[y] = 9 &\quad = \quad d[s] = 2 + 7*
\end{aligned}
$$

All edges are OK so routine returns *TRUE*. Note: All $*$ edges (equalities) are ones that appear in the final graph shown below.



Setting edge $(z, x) = 4$ and rerunning the algorithm gives:

	d						π				
pass	s	t	x	y	z		s	t	x	y	z
0	0	∞	∞	∞	∞		/	/	/	/	/
1	0	6	∞	7	∞		/	s	/	s	/
2	0	6	(11)4	7	2		/	s	$(t)y$	s	/
3	0	2	4	7	2		/	x	y	s	t
4	0	2	2	7	-2		/	x	z	s	t

Checking edges

$$
\begin{aligned}
d[x] = 2 &\quad < \quad d[t] = 2 + 5 \\
d[y] = 7 &\quad < \quad d[t] = 2 + 8 \\
d[z] = -2 &\quad = \quad d[t] = 2 - 4 * \\
d[t] = 2 &\quad > \quad d[x] = 2 - 2X
\end{aligned}
$$

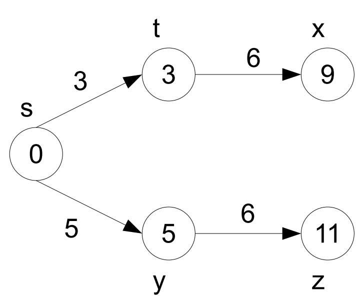
Check *fails* so routine returns *FALSE*. Note: $< x, t, z >$ is a negative weight cycle.

24.3-1 Run Dijkstra's algorithm on the directed graph of Figure 24.2 first using vertex s then vertex z as source. Show the d and π values and the vertices in S after each iteration.

Using vertex s

Q	S	d						π				
		s	t	x	y	z		s	t	x	y	z
$< s, t, x, y, z >$	$\emptyset$	**0**	∞	∞	∞	∞		/	/	/	/	/
$< t, y, x, z >$	$< s >$	0	**3**	∞	5	∞		/	s	/	s	/
$< y, x, z >$	$< s, t >$	0	3	9	**5**	∞		/	s	t	s	/
$< x, z >$	$< s, t, y >$	0	3	**9**	5	11		/	s	t	s	y
$< z >$	$< s, t, y, x >$	0	3	9	5	**11**		/	s	t	s	y
$\emptyset$	$< s, t, y, x, z >$	0	3	9	5	11		/	s	t	s	y

Giving the following graph:



Using vertex z

Q	S	d						π				
		s	t	x	y	z		s	t	x	y	z
$< z, s, t, x, y >$	$\emptyset$	∞	∞	∞	∞	**0**		/	/	/	/	/
$< s, x, t, y >$	$< z >$	**3**	∞	7	∞	0		z	/	z	/	/
$< t, x, y >$	$< z, s >$	3	**6**	7	8	0		z	s	z	s	/
$< x, y >$	$< z, s, t >$	3	6	**7**	8	0		z	s	z	s	/
$< y >$	$< z, s, t, x >$	3	6	7	**8**	0		z	s	z	s	/
$\emptyset$	$< z, s, t, x, y >$	3	6	7	8	0		z	s	z	s	/

Giving the following graph: