ECE335 Summer 2019 - Assignment 4

Section 2.3 - More Operations on Sets

2.3.1 . Analyze the logical forms of the following statements

a. $\mathcal{F} \subseteq \mathcal{P}(A)$

Using the definition of subset we first expand

$\forall x(x \in \mathcal{F} \rightarrow x \in \mathcal{P}(A))$

Then applying the definition of powerset, i.e. that an element of the powerset is a *subset* gives

$\forall x(x \in \mathcal{F} \rightarrow \forall y(y \in x \rightarrow y \in A))$

b. $A \subseteq \{2n+1|n \in \mathbb{N}\}$

Using the definition of subset we first expand

$\forall x(x \in A \rightarrow x \in \{2n+1|n \in \mathbb{N}\}$

Then rewriting the elementhood test gives

$\forall x(x \in A \rightarrow \exists n \in \mathbb{N}(x = 2n+1))$

c. $\{n^2+n+1|n \in \mathbb{N}\} \subseteq \{2n+1|n \in \mathbb{N}\}$

Using the definition of subset we first expand

$\forall x(x \in \{n^2+n+1|n \in \mathbb{N}\} \rightarrow x \in \{2n+1|n \in \mathbb{N}\}$

Since the n's in both equations may be different for the same x we can more consisely write the statement by using m and n as

$\forall n \in \mathbb{N} \exists m \in \mathbb{N}(n^2+n+1 = 2m+1)$

d. $\mathcal{P}(\cup_{i \in I} A_i) \nsubseteq \cup_{i \in I} \mathcal{P}(A_i)$

Using the definition of not a subset (which implies *there exists* an element not in the superset) we first expand

$\exists x(x \in \mathcal{P}(\cup_{i \in I} A_i) \wedge x \notin \cup_{i \in I} \mathcal{P}(A_i))$

Now expanding the first term using the definition of powerset gives

$\exists x(\forall y(y \in x \rightarrow y \in (\cup_{i \in I} A_i)) \wedge \neg(x \notin \cup_{i \in I} \mathcal{P}(A_i)))$

Using the definition of indexed set union gives

$\exists x(\forall y(y \in x \rightarrow \exists i \in I(y \in A_i)) \wedge \neg \exists i \in I(x \in \mathcal{P}(A_i)))$

Expanding the second powerset term gives

$\exists x(\forall y(y \in x \to \exists i \in I(y \in A_i)) \wedge \neg\exists i \in I(\forall y(y \in x \to y \in A_i)))$

Negating the quantifier in the second term and using DeMorgan's law (after converting the conditional to $\vee$) gives

$\exists x(\forall y(y \in x \to \exists i \in I(y \in A_i)) \wedge \forall i \in I(\exists y(y \in x \wedge y \notin A_i)))$

2.3.8 . Let $I = \{2, 3\}$ and for each $i \in I$ let $A_i = \{i, 2i\}$ and $B_i = \{i, i + 1\}$

a. List the elements of the sets A_i and B_i for $i \in I$.

Substituting in the two values for i produces 4 sets

$A_2 = \{2, 4\}, A_3 = \{3, 6\}, B_2 = \{2, 3\}, B_3 = \{3, 4\}$

b. Find $\cap_{i\in I}(A_i \cup B_i)$ and $(\cap_{i\in I}A_i) \cup (\cap_{i\in I}B_i)$

First, to find $\cap_{i\in I}(A_i \cup B_i)$ we compute the unions for both indicies giving

$(A_2 \cup B_2) = \{2, 3, 4\}$ and $(A_3 \cup B_3) = \{3, 4, 6\}$

Taking the intersection of these two sets gives

$\cap_{i\in I}(A_i \cup B_i) = \{3, 4\}$

Second, to find $(\cap_{i\in I}A_i) \cup (\cap_{i\in I}B_i)$ we compute the two family intersections giving

$(\cap_{i\in I}A_i) = \emptyset$ and $(\cap_{i\in I}B_i) = \{3\}$

Taking the union of these two sets gives

$(\cap_{i\in I}A_i) \cup (\cap_{i\in I}B_i) = \{3\}$

which is clearly different than the first result.

c. What can you conclude about whether or not these statements are equivalent?

Since part (b) gives two sets that provide a *counterexample*

$\cap_{i\in I}(A_i \cup B_i) \neq (\cap_{i\in I}A_i) \cup (\cap_{i\in I}B_i)$

i.e. family intersection *does not* distribute over indexed set union.

2.3.10 . Show that for any sets A and B that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

We shall show this equivalency by showing that $x \in \mathcal{P}(A \cap B)$ is equivalent to $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

$$
\begin{aligned}
x \in \mathcal{P}(A \cap B) =& x \subseteq (A \cap B) && \text{definition of powerset} \\
=& \forall y(y \in x \to y \in (A \cap B)) && \text{definition of subset} \\
=& \forall y(y \in x \to (y \in A \wedge y \in B)) && \text{definition of intersection} \\
=& \forall y(\neg(y \in x) \vee (y \in A \wedge y \in B)) && \text{conditional law} \\
=& \forall y((\neg(y \in x) \vee (y \in A)) \wedge (\neg(y \in x) \vee (y \in B))) && \text{distributive law} \\
=& \forall y((\neg(y \in x) \vee (y \in A)) \wedge \forall y(\neg(y \in x) \vee (y \in B)) && \text{quantifier distribution} \\
=& \forall y(y \in x \to y \in A) \wedge \forall y(y \in x \to y \in B) && \text{conditional law} \\
=& (x \subseteq A) \wedge (x \subseteq B) && \text{definition of subset} \\
=& (x \in \mathcal{P}(A)) \wedge (x \in \mathcal{P}(B)) && \text{definition of powerset} \\
=& x \in \mathcal{P}(A) \cap \mathcal{P}(B) && \text{definition of intersection}
\end{aligned}
$$

Thus $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

Section 3.1 - Proof Strategies

3.1.8 . Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

Givens	*Goal*
$A \setminus B \subseteq C \cap D$	$x \notin D \to x \in B$
$x \in A$	

We can modify the goal by assuming $x \notin D$ as a given thus transforming the goal into

Givens	*Goal*
$A \setminus B \subseteq C \cap D$	$x \in B$
$x \in A$	
$x \notin D$	

At this point we begin by noting that since $x \notin D$ that $x \notin C \cap D$ and therefore since $A \setminus B \subseteq C \cap D \Rightarrow x \notin A \setminus B$. But since $x \in A$, for $x \notin A \setminus B$ we must have $x \in B$.

Formally,

Theorem. *Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. If $x \notin D$ then $x \in B$.*

Proof. Suppose $x \notin D$. Then this implies that $x \notin (C \cap D)$ and therefore $x \notin (A \setminus B)$. But since $x \in A$ for $x \notin (A \setminus B)$ means that $x \in B$. Thus if $x \notin D$ then $x \in B$.

3.1.11 . Suppose a, b, c, and d are real numbers, $0 < a < b$ and $d > 0$. Prove that if $ac \geq bd$ then $c > d$.

Givens	*Goal*
$a, b, c, d \in \mathbb{R}$	$ac \geq bd \to c > d$
$0 < a < b$	
$d > 0$	

Consider proving the contrapositive of the goal giving

Givens	*Goal*
$a, b, c, d \in \mathbb{R}$	$c \leq d \to ac < bd$
$0 < a < b$	
$d > 0$	

Then using the conditional proof technique, we move the first part of the goal to the givens

Givens	*Goal*
$a, b, c, d \in \mathbb{R}$	$ac < bd$
$0 < a < b$	
$d > 0$	
$c \leq d$	

Since $a > 0$ we can multiply both sides of the last given without reversing the inequality giving $ac \leq ad$. Likewise, since $d > 0$ we can multiply both sides of the second given without reversing the inequality giving $ad < bd$. Combining these two inequalities gives $ac \leq ad < bd \Rightarrow ac < bd$ which is the desired goal.

Formally

Theorem. *Suppose a, b, c, and d are real numbers, $0 < a < b$ and $d > 0$. If $ac \geq bd$ then $c > d$.*

Proof. We will prove by contrapositive, thus suppose $c \leq d$. Multiplying by the positve number a gives $ac \leq ad$. Similarly multiplying by the positive number d gives $ad < bd$. Combining $ac \leq ad$ and $ad < bd$ we can conclude that $ac < bd$. Thus if $ac \geq bd$ then $c > d$.

Section 3.2 - Proofs Involving Negations and Conditionals

3.2.5. Use the method of contradiction to prove that supposing $A \cap C \subseteq B$ and $a \in C$, that $a \notin (A \setminus B)$.

Since the initial goal is $a \notin (A \setminus B)$ we will instead use proof by contradiction by *assuming* $a \in (A \setminus B)$ and attempt to reach a contradiction. Thus our givens and goals are

Givens	*Goal*
$A \cap C \subseteq B$	*contradiction*
$a \in C$	
$a \in (A \setminus B)$	

Since $a \in (A \setminus B) \Rightarrow (a \in A \wedge a \notin B)$. Thus we can add those to our givens (since they both must be true)

Givens	*Goal*
$A \cap C \subseteq B$	*contradiction*
$a \in C$	
$a \in A$	
$a \notin B$	

Since by givens 2 and 3, $a \in A$ and $a \in C$, therefore $a \in (A \cap C)$. By given 1, this implies that $a \in B$ which contradicts the last given. Thus our initial assumption was incorrect, hence $a \notin (A \setminus B)$.

Formally,

Theorem. *Suppose $A \cap C \subseteq B$ and $a \in C$. Then $a \notin (A \setminus B)$.*

Proof. Suppose not, then $a \in (A \setminus B)$. This implies that $a \in A$ and $a \notin B$. Since $a \in C$ we have that $a \in (A \cap C)$. But since $A \cap C \subseteq B$, this means that $a \in B$ which contradicts the fact that $a \notin B$. Thus $a \notin (A \setminus B)$.