ECE335 Summer 2019 - Assignment 6

Section 3.5 - Proofs Involving Disjunctions

3.5.2 . Suppose A, B, and C are sets. Prove that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

Givens	*Goal*
A, B, and C are sets	$(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$

Applying the definition of subset to the goal gives

Givens	*Goal*
A, B, and C are sets	$\forall x(x \in (A \cup B) \setminus C \to x \in A \cup (B \setminus C))$

Letting x be arbitrary (universal instantiation) and bringing the left hand side of the conditional to the givens gives

Givens	*Goal*
A, B, and C are sets	$x \in A \cup (B \setminus C))$
$x \in (A \cup B) \setminus C$	

Expanding the given using the definition of set difference (separating the **and**) and the goal using the definition of union then gives

Givens	*Goal*
A, B, and C are sets	$x \in A \vee x \in (B \setminus C))$
$x \in A \cup B$	
$x \notin C$	

Lastly we can expand the second given using the definition of set union giving the disjunction

Givens	*Goal*
A, B, and C are sets	$x \in A \vee x \in (B \setminus C))$
$x \in A \vee x \in B$	
$x \notin C$	

Now we will consider the two cases from the disjunction in the given:

Case 1: Suppose $x \in A$. Then clearly the first part of the goal is satisfied (and $x \in A \cup (B \setminus C))$.

Case 2: Suppose $x \in B$. Then since $x \notin C$ gives $x \in B \setminus C$ which is the second part of the goal (and again $x \in A \cup (B \setminus C)$.

Formally,

Theorem. *Suppose A, B, and C are sets. Then $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.*

Proof. Suppose that x is arbitrary and $x \in (A \cup B) \setminus C$. Then $x \in A \cup B$ and $x \notin C$. Since $x \in A \cup B$ we know that either $x \in A$ or $x \in B$ so consider two cases:

Case 1: Suppose $x \in A$. Then clearly $x \in A \cup (B \setminus C)$.

Case 2: Suppose $x \in B$. Then since $x \notin C$ we have that $x \in B \setminus C$ and thus $x \in A \cup (B \setminus C)$.

Thus in either case $x \in A \cup (B \setminus C)$ and since x was arbitrary that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

3.5.27. Consider the following putative theorem.

Theorem? *For any sets A, B, and C, if $A \setminus B \subseteq C$ and $A \not\subseteq C$ then $A \cap B \neq \emptyset$.*

Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Proof. Since $A \not\subseteq C$, we can choose some x such that $x \in A$ and $x \notin C$. Since $x \notin C$ and $A \setminus B \subseteq C$, $x \notin A \setminus B$. Therefore either $x \notin A$ or $x \in B$. But we already know that $x \in A$, so it follows that $x \in B$. Since $x \in A$ and $x \in B$, $x \in A \cap B$. Therefore $A \cap B \neq \emptyset$.

The proof (and thus the theorem) is correct. First since $A \not\subseteq C$ means $\exists x(x \in A \wedge x \notin C$ and thus it employs existential instantiation to assert that an x can be chosen (although using x_0 may have been a better choice of symbol). Then it uses a modus tollens argument since $x \notin C$ but $A \setminus B \subseteq C$ (i.e. $\forall x(x \in A \setminus B \rightarrow x \in C)$ to assert that $x \notin A \setminus B$. Using the definition of set difference and applying DeMorgan's law then asserts that $x \notin A \vee x \in B$. We can then use a variant of modus ponens (since $x \notin A \vee x \in B \Rightarrow x \in A \rightarrow x \in B$ and $x \in A$) to assert that $x \in B$. Finally using the definition of intersection gives that $x \in A$ and $x \in B$ means $x \in A \cap B$ and therefore $A \cap B \neq \emptyset$.

Section 6.1 - Proof By Induction

6.1.2. Prove that for all $n \in \mathbb{N}$, $0^2 + 1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.

First we show the base case for $n = 0$ which gives $0(1)(1)/6 = 0$.

Next we need to show the induction step that if $0^2 + 1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ then $0^2 + 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = (n+1)(n+2)(2(n+1)+1)/6 = (n+1)(n+2)(2n+3)$. Therefore

$$\begin{aligned}
0^2 + 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= (0^2 + 1^2 + 2^2 + \cdots + n^2) + (n+1)^2 \\
&= n(n+1)(2n+1)/6 + (n+1)^2 \\
&= n(n+1)(2n+1)/6 + 6(n+1)^2/6 \\
&= (n(n+1)(2n+1) + 6(n+1)^2)/6 \\
&= (n+1)(n(2n+1) + 6(n+1))/6 \\
&= (n+1)(2n^2 + n + 6n + 6)/6 \\
&= (n+1)(2n^2 + 7n + 6)/6 \\
&= (n+1)(n+2)(2n+3)/6
\end{aligned}$$

Formally,

Theorem. *For every natural number n, $0^2 + 1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.*

Proof. Using mathematical induction:

Base Case: Letting $n = 0$ gives $0(1)(1)/6 = 0$.

Induction Step: Let n be arbitrary and assume $0^2 + 1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$. Then

$$\begin{aligned}
0^2 + 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= (0^2 + 1^2 + 2^2 + \cdots + n^2) + (n+1)^2 \\
&= n(n+1)(2n+1)/6 + (n+1)^2 \\
&= n(n+1)(2n+1)/6 + 6(n+1)^2/6 \\
&= (n(n+1)(2n+1) + 6(n+1)^2)/6 \\
&= (n+1)(n(2n+1) + 6(n+1))/6 \\
&= (n+1)(2n^2 + n + 6n + 6)/6 \\
&= (n+1)(2n^2 + 7n + 6)/6 \\
&= (n+1)(n+2)(2n+3)/6
\end{aligned}$$

6.1.14. Prove that for all $n \geq 10$, $2^n > n^3$.

First verify the base case for $n = 10$. $2^{10} = 1024$ and $10^3 = 1000$, thus $2^n > n^3$ for $n = 10$.

Next for the induction step we will observe that for $n > 10$ that $n^3 = n \cdot n^2 > 10n^2$. Noting that $(n+1)^3 = n^3 + 3n^2 + 3n + 1$ we have

$$\begin{aligned}
2^{n+1} &= 2 \cdot 2^n \\
&> 2 \cdot n^3 \\
&= n^3 + n^3 \\
&> n^3 + 10n^2 \\
&= n^3 + 3n^2 + 7n^2 \\
&> n^3 + 3n^2 + 7(10)n \\
&= n^3 + 3n^2 + 70n \\
&= n^3 + 3n^2 + 3n + 67n \\
&> n^3 + 3n^2 + 3n + 1 \\
&= (n+1)^3
\end{aligned}$$

Formally,

Theorem. *For every natural number $n > 10$, $2^n > n^3$.*

Proof. Using mathematical induction:

Base Case: Letting $n = 10$ gives $2^{10} = 1024$ and $10^3 = 1000$. Thus $2^n > n^3$ for $n = 10$.

Induction Step: Let n be arbitrary such that $n > 10$ and assume $2^n > n^3$. Then clearly $n^3 = n \cdot n^2 > 10n^2$. Thus

$$\begin{aligned}
2^{n+1} &= 2 \cdot 2^n \\
&> 2 \cdot n^3 \\
&= n^3 + n^3 \\
&> n^3 + 10n^2 \\
&= n^3 + 3n^2 + 7n^2 \\
&> n^3 + 3n^2 + 7(10)n \\
&= n^3 + 3n^2 + 70n \\
&= n^3 + 3n^2 + 3n + 67n \\
&> n^3 + 3n^2 + 3n + 1 \\
&= (n+1)^3
\end{aligned}$$

Section 6.3 - Recursion

6.3.1. Find a formula for $\sum_{i=1}^{n} 1/i(i+1)$ and prove it is correct.

Computing the first several terms gives

$$\begin{aligned} n = 1 &\Rightarrow 1/(1)(2) = 1/2 \\ n = 2 &\Rightarrow 1/2 + 1/(2)(3) = 4/6 = 2/3 \\ n = 3 &\Rightarrow 2/3 + 1/(3)(4) = 9/12 = 3/4 \\ n = 4 &\Rightarrow 3/4 + 1/(4)(5) = 16/20 = 4/5 \end{aligned}$$

Thus we guess that the formula is $\sum_{i=1}^{n} 1/i(i+1) = n/(n+1)$. To *prove* this guess we will use induction.

Base Case: For $n = 1$, $\sum_{i=1}^{1} 1/i(i+1) = 1/(1)(2) = 1/2$.

Induction Step: Let n be arbitrary and assume $\sum_{i=1}^{n} 1/i(i+1) = n/(n+1)$. Then removing the last $n+1$ term from the summation gives

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2)+1}{(n+1)(n+2)} \\ &= \frac{n^2+2n+1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} \\ &= \frac{n+1}{n+2} \end{aligned}$$

6.3.6. Prove that for all $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

Applying induction

Base Case: For $n = 1$, $\sum_{i=1}^{1} \frac{1}{i^2} = \frac{1}{1} = 1$ and $2 - \frac{1}{1} = 1$ so $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ holds for $n = 1$.

Induction Step: Let n be arbitrary and assume $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$. Then removing the last $n+1$ term from the summation gives

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{i^2} &= \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \\
&\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\
&= 2 - \frac{(n+1)^2 - n}{n(n+1)^2} \\
&= 2 - \frac{n^2 + n + 1}{n(n+1)^2} \\
&= 2 - \frac{n^2 + n}{n(n+1)^2} - \frac{1}{n(n+1)^2} \\
&\leq 2 - \frac{n^2 + n}{n(n+1)^2} (\text{since } \tfrac{1}{n(n+1)^2} > 0) \\
&= 2 - \frac{n(n+1)}{n(n+1)^2} \\
&= 2 - \frac{1}{n+1}
\end{aligned}$$