ECE 335 Summer 2019 - Assignment 3

Section 1.3 - Variables and Sets

1.3.2 . Analyze the logical forms of the following statements

a. x and y are men, and either x is taller than y or y is taller than x

Let $M(x)$ represent "x is a man" and $T(x, y)$ represent "x is taller than y"

Then the logical form becomes

$(M(x) \wedge M(y)) \wedge (T(x, y) \vee T(y, x))$

b. Either x or y has brown eyes, and either x or y has red hair.

Let $B(x)$ represent "x has brown eyes" and $R(x)$ represent "x has red hair" Then the logical form becomes

$(B(x) \vee B(y)) \wedge (R(x) \vee R(y))$

c. Either x or y has both brown eyes and red hair.

Let $B(x)$ represent "x has brown eyes" and $R(x)$ represent "x has red hair" Then the logical form becomes

$(B(x) \wedge R(x)) \vee (B(y) \wedge R(y))$

1.3.5 . Simplify the following statements noting free and bound variables and the truth of statements without free variables.

a. $-3 \in \{x \in \mathbb{R} | 13 - 2x > 1\}$

Here x is a bound variable since it is used to define the elementhood test and there are no free variables. Thus the statement can be written in logical form as

$(-3 \in \mathbb{R}) \wedge (13 - 2(-1) > 1)$

Since -3 is a real number and $13 - 2(-3) = 19 > 1$, the statement is **true**.

b. $4 \in \{x \in \mathbb{R}^- | 13 - 2x > 1\}$

Here x is a bound variable since it is used to define the elementhood test and there are no free variables. Thus the statement can be written in logical form (noting that $\mathbb{R}^-$ means *negative* real numbers) as

$(4 \in \mathbb{R}) \wedge (4 < 0) \wedge (13 - 2(4) > 1)$

Since 4 is a real number but is **not** negative, even though $13 - 2(4) = 5 > 1$, the statement is **false**.

c. $5 \notin \{x \in \mathbb{R} | 13 - 2x > c\}$

Here x is a bound variable since it is used to define the elementhood test but c is a free variable since it determines the truth of the elementhood test. Thus the statement can be written in logical form

$\neg[(5 \in \mathbb{R}) \wedge (13 - 2(5) > c)]$

Since c is a free variable, the truth of the statement will depend on the particular value of c.

1.3.8 . What are the truth sets of the following statements?

a. x is a real number and $x^2 - 4x + 3 = 0$

Thus this statement means $P(x) = \{x \in \mathbb{R} | x^2 - 4x + 3 = 0\}$.

$x^2 - 4x + 3 = (x - 3)(x - 1)$ so the values 1 and 3 satisfy this equation. Since both values are also real numbers, they give the truth set $\{1, 3\}$

b. x is a real number and $x^2 - 2x + 3 = 0$

Thus this statement means $P(x) = \{x \in \mathbb{R} | x^2 - 2x + 3 = 0\}$.

$x^2 - 2x + 3$ has roots $1 \pm \sqrt{2}i$. Since neither of these values are real numbers, the truth set is $\emptyset$, i.e. the empty set.

c. x is a real number and $5 \in \{y \in \mathbb{R} | x^2 + y^2 < 50\}$

Thus this statement means $P(x) = \{x \in \mathbb{R} | 5 \in \{y \in \mathbb{R} | x^2 + y^2 < 50\}\}$.

Hence we want to know for $x = 5$ (since x is a bound variable with $5 \in \mathbb{R}$) when $(5)^2 + y^2 < 50 \Rightarrow y^2 < 25 \Rightarrow -5 < y < 5$. Since every value in this range is a real number, the truth set (in terms of the free variable y) is $\{y \in \mathbb{R} | -5 < y < 5\}$.

Section 1.4 - Operations on Sets

1.4.1. $A = \{1, 3, 12, 35\}$, $B = \{3, 7, 12, 20\}$, and $C = \{x | x \text{is a prime number}\}$. List the elements of the following sets and note if any of them are subsets of any others.

a. $A \cap B$

From the list of elements, A and B have the values 3 and 12 in common. Thus $A \cap B = \{3, 12\}$. Since all the elements of the intersection are in both sets, the intersection must be a subset of both A and B, i.e. $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Note since 12 is not prime $A \cap B \nsubseteq C$.

b. $(A \cup B) \setminus C$

From the list of elements, combining A and B (removing any duplicates) gives $A \cup B = \{1, 3, 7, 12, 20, 35\}$. Removing any elements that are in C, i.e. any prime numbers, gives $(A \cup B) \setminus C = \{1, 12, 20, 35\}$ Since this set contains at least one element not in each of the original sets, it is *not* a subset of A, B, or C.

c. $A \cup (B \setminus C)$

From the list of elements, removing any elements of C (i.e. prime numbers) from B gives $B \setminus C = \{12, 20\}$. Combining this with A gives $A \cup (B \setminus C) = \{1, 3, 12, 20, 35\}$ Since this set contains at least one element not in each of the original sets, it is *not* a subset of A, B, or C. But we can see that both of the sets from (a) and (b) are a subset of (c), i.e. $A \cap B \subseteq A \cup (B \setminus C)$ and $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

1.4.5. Verify the following identities

a. $A \setminus (A \cap B) = A \setminus B$

Using the element notation and logical equivalences gives

$$\begin{aligned}
A \setminus (A \cap B) =& x \in A \wedge x \notin (A \cap B) && \text{definition of set difference} \\
=& x \in A \wedge \neg(x \in A \wedge x \in B) && \text{definition of set intersection} \\
=& x \in A \wedge (x \notin A \vee x \notin B) && \text{DeMorgan's law} \\
=& (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B) && \text{distributive law} \\
=& (\mathit{contradiction} \vee (x \in A \wedge x \notin B) && \text{contradiction} \\
=& x \in A \wedge x \notin B && \text{contradiction law} \\
=& A \setminus B && \text{definition of set difference}
\end{aligned}$$

b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (note this is distribution of union over intersection)

Using the element notation and logical equivalences gives

$$\begin{aligned}
A \cup (B \cap C) =& x \in A \vee x \in (B \cap C && \text{definition of set union} \\
=& x \in A \vee (x \in B \wedge x \in C) && \text{definition of set intersection} \\
=& (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) && \text{distributive law} \\
=& (x \in (A \cup B)) \wedge (x \in (A \cup C)) && \text{definition of set union} \\
=& (A \cup B) \cap (A \cup C) && \text{definition of set intersection}
\end{aligned}$$

1.4.9. Give an example of two sets A and B for which $(A \cup B) \setminus B \neq A$

This will be true if there is an element in $A \cap B$ since those elements will by definition be in B and thus *removed* from $A \cup B$. But because those elements were in $A \cap B$ means they will *not* be in $(A \cup B) \setminus B$ and thus that set will *not* be the same as A (even though it will be a subset of A). For example, consider the sets

$A = \{1, 2, 3\}$ and $B = \{1, 4\}$

$A \cup B = \{1, 2, 3, 4\}$ and $(A \cup B) \setminus B = \{2, 3\}$ which is clearly $\neq A$ (but is $\subseteq A$ as stated in Theorem 1.4.7).