ECE335 Summer 2019 - Lecture 17 Examples

Example 1: Find a closed form formula for

$$a_1 = 1$$
$$a_{n+1} = 3a_n + 1$$

and prove it is correct using induction.

Expand the first few terms of the recursion to identify a pattern, then derive a formula for the pattern.

$a_2 = 3a_1 + 1 = 3 \cdot 1 + 1 = 3 + 1 = 4$

$a_3 = 3a_2 + 1 = 3(3+1) + 1 = 9 + 3 + 1 = 13$

$a_4 = 3a_3 + 1 = 3(9+3+1) + 1 = 27 + 9 + 3 + 1 = 40$

$a_5 = 3a_4 + 1 = 3(27+9+3+1) + 1 = 81 + 27 + 9 + 1 = 118$

$a_n = \sum_{i=0}^{n-1} 3^i = \frac{3^n - 1}{3 - 1} = \frac{3^n - 1}{2}$ (from the formula $\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$)

Proof By Induction

Step 1: Write the induction given closed form for n

$$a_n = \frac{3^n - 1}{2}$$

Step 2: Show the induction given is true for a base case

For $n = 1$ $\quad a_1 = 1 \quad \frac{3^1 - 1}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1$ ✓

Step 3: Write the induction goal closed form for $n + 1$

$$a_{n+1} = 3a_n + 1 = \frac{3^{n+1} - 1}{2}$$

Step 4: Assume the induction formula from step 1 is true and substitute the closed form solution into step 3

$$3a_n + 1 = 3\left(\frac{3^n - 1}{2}\right) + 1 = \frac{3^{n+1} - 1}{2}$$

Step 5: Perform any necessary algebra to show the closed form solution from step 3 for $n + 1$

$$3\left(\frac{3^n - 1}{2}\right) + 1 = \frac{3^{n+1} - 3}{2} + 1$$

$$= \frac{3^{n+1} - 3 + 2}{2}$$

$$= \frac{3^{n+1} - 1}{2}$$

Example 2: Find a closed form formula for

$$a_0 = 1$$
$$a_{n+1} = a_n + 2$$

and prove it is correct using induction.

Expand the first few terms of the recursion to identify a pattern, then derive a formula for the pattern.

$a_1 = a_0 + 2 = 1 + 2 = 3$

$a_2 = a_1 + 2 = 1 + 2 + 2 = 5$

$a_3 = a_2 + 2 = 1 + 2 + 2 + 2 = 7$

$a_4 = a_3 + 2 = 1 + 2 + 2 + 2 + 2 = 9$

$a_n = 2n + 1$

Proof By Induction

Step 1: Write the induction given closed form for n

$a_n = 2n + 1$

Step 2: Show the induction given is true for a base case

For $n = 0$

$a_0 = 1 \qquad 2(0) + 1 = 1$ ✓

Step 3: Write the induction goal closed form for $n + 1$

$a_{n+1} = a_n + 2 = 2(n+1) + 1 = 2n + 3$

Step 4: Assume the induction formula from step 1 is true and substitute the closed form solution into step 3

$$a_n + 2 = (2n+1) + 2 = 2n+3$$

Step 5: Perform any necessary algebra to show the closed form solution from step 3 for $n+1$

$$(2n+1) + 2 = 2n + 1 + 2$$

$$= \underline{2n+3}$$