

ECE335 Summer 2019 - Lecture 17 Examples

Example 1: Find a closed form formula for

$$a_1 = 1$$

$$a_{n+1} = 3a_n + 1$$

and prove it is correct using induction.

Expand the first few terms of the recursion to identify a pattern, then derive a formula for the pattern.

$$a_2 = 3a_1 + 1 = 3 \cdot 1 + 1 = 3 + 1 = 4$$

$$a_3 = 3a_2 + 1 = 3(3+1) + 1 = 9 + 3 + 1 = 13$$

$$a_4 = 3a_3 + 1 = 3(9+3+1) + 1 = 27 + 9 + 3 + 1 = 40$$

$$a_5 = 3a_4 + 1 = 3(27+9+3+1) + 1 = 81 + 27 + 9 + 1 = 118$$

$$a_n = \sum_{i=0}^{n-1} 3^i = \frac{3^n - 1}{3 - 1} = \frac{3^n - 1}{2} \quad \left(\text{from the formula } \sum_{i=0}^n x^i = \frac{x^{n+1} - 1}{x - 1} \right)$$

Proof By Induction

Step 1: Write the induction given closed form for n

$$a_n = \frac{3^n - 1}{2}$$

Step 2: Show the induction given is true for a base case

$$\text{For } n=1 \quad a_1 = 1 \quad \frac{3^1 - 1}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1 \quad \checkmark$$

Step 3: Write the induction goal closed form for $n + 1$

$$a_{n+1} = 3a_n + 1 = \frac{3^{n+1} - 1}{2}$$

Step 4: Assume the induction formula from step 1 is true and substitute the closed form solution into step 3

$$3a_n + 1 = 3\left(\frac{3^n - 1}{2}\right) + 1 = \frac{3^{n+1} - 1}{2}$$

Step 5: Perform any necessary algebra to show the closed form solution from step 3 for $n + 1$

$$3\left(\frac{3^n - 1}{2}\right) + 1 = \frac{3^{n+1} - 3}{2} + 1$$

$$= \frac{3^{n+1} - 3 + 2}{2}$$

$$= \frac{3^{n+1} - 1}{2}$$