ECE335 Summer 2019 - Assignment 7

Section 6.3 - Recursion

6.3.12. Prove the following for all $n \in \mathbb{N}$

(a) $2^n > n$.

Instead of proving $2^n > n$, we will instead prove that $2^n > n + 1$ (which clearly is then $> n$) using induction.

Base Case: Using $2^n > n + 1$ unfortunately is *not* true for $n = 1, 2$. However for $n = 3$, $2^3 = 8 > 4$.

Induction Step: Let n be arbitrary $n > 3$ and assume $2^n > n + 1$. Then

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &> 2 \cdot (n + 1) \\ &= (n + 1) + (n + 1) \\ &> (n + 2) + n \\ &> (n + 2) \end{aligned}$$

Then from the original inequality, for $n = 0$ $2^0 = 1 > 1$, for $n = 1$ $2^1 = 2 > 1$, and for $n = 2$ $2^2 = 4 > 2$. Thus $2^n > n$ is true for all $n \in \mathbb{N}$.

(b) For $n \geq 9$, $n! \geq (2^n)^2$.

Using induction

Base Case: For $n = 9$, $9! = 362880 > (2^9)^2 = 262144$.

Induction Step: Let n be arbitrary such that $n > 9$ and assume $n! \geq (2^n)^2$. Then since $(n + 1) > 4$ for $n > 9$

$$\begin{aligned} (n + 1)! &= (n + 1) \cdot n! \\ &> (n + 1)(2^n)^2 \\ &= (n + 1) \cdot 2^{2n} \\ &> 4 \cdot 2^{2n} \\ &= 2^2 \cdot 2^{2n} \\ &= 2^{2n+2} \\ &= (2^{n+1})^2 \end{aligned}$$

(c) $n! \leq 2^{(n^2)}$.

Using induction

Base Case: For $n = 0$, $0! = 1 \leq 2^{(0^2)} = 1$.

Induction Step: Let n be arbitrary and assume $n! \leq 2^{(n^2)}$. Then using the fact that $n \leq 2^n \Rightarrow (n+1) \leq 2^{n+1}$

$$\begin{aligned}
(n+1)! &= (n+1) \cdot n! \\
&\leq (n+1) \cdot 2^{(n^2)} \\
&\leq 2^{n+1} \cdot 2^{(n^2)} \\
&= 2^{(n^2+n+1)} \\
&< 2^{(n^2+n)} \cdot 2^n \\
&= 2^{(n^2+2n+1)} \\
&= 2^{(n+1)^2}
\end{aligned}$$

Section 4.1 - Ordered Pairs and Cartesian Products

4.1.8. Is it true that for any sets A, B, and C, $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$?

This equivalency is true as can be proven by showing each side is a subset of the other which gives a biconditional proof

$(\rightarrow)$ Let $(x, y) \in A \times (B \setminus C)$ be an arbitrary ordered pair. Then $x \in A$ and $y \in B \setminus C$. Since $y \in B \setminus C$ gives $y \in B$ and $y \notin C$. Because $x \in A$ and $y \in B$ we know $(x, y) \in A \times B$. Since $y \notin C$ we know that $(x, y) \notin A \times C$ and thus $(x, y) \in (A \times B) \setminus (A \times C)$. Since (x, y) was arbitrary, $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$.

$(\leftarrow)$ Let $(x, y) \in (A \times B) \setminus (A \times C)$ be an arbitrary ordered pair. Then $(x, y) \in A \times B$ and $(x, y) \notin A \times C$. Since $(x, y) \in A \times B$ we have $x \in A$ and $y \in B$. Furthermore, since $(x, y) \notin A \times C$ gives that *either* $x \notin A$ or $y \notin C$. But since we already have $x \in A$, it must be true that $y \notin C$. Thus since $y \in B$ and $y \notin C$ we know $y \in B \setminus C$ and since $x \in A$ gives $(x, y) \in A \times (B \setminus C)$. Since (x, y) was arbitrary, $(A \times B) \setminus (A \times C) \subseteq A \times (B \setminus C)$.

Since $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$ and $(A \times B) \setminus (A \times C) \subseteq A \times (B \setminus C)$ $\Rightarrow A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

Section 4.3 - More About Relations

4.3.17. Suppose R and S are symmetric relations on A. Prove that $R \circ S$ is symmetric iff $R \circ S = S \circ R$.

($\rightarrow$) Suppose R and S are symmetric relations on A and that $R \circ S$ is symmetric. Let $(x, y) \in R \circ S$. Then there exists z such that $(x, z) \in S$ and $(z, y) \in R$. Because R and S are symmetric gives $(z, x) \in S$ and $(y, z) \in R$ which implies $(y, x) \in S \circ R$. But since $R \circ S$ is symmetric implies $(y, x) \in R \circ S$. Hence $R \circ S \subseteq S \circ R$. By similar reasoning, if $(u, v) \in S \circ R$ then there exists w such that $(u, w) \in R$ and $(w, v) \in S$. Since R and S are symmetric gives $(w, u) \in R$ and $(v, w) \in S$. Thus $(v, u) \in R \circ S$ and since $R \circ S$ is symmetric $(u, v) \in R \circ S$. Hence $S \circ R \subseteq R \circ S$. Because $R \circ S \subseteq S \circ R$ and $S \circ R \subseteq R \circ S$ gives $R \circ S = S \circ R$.

($\leftarrow$) Suppose R and S are symmetric relations on A and that $R \circ S = S \circ R$. Since R is symmetric $R = R^{-1}$ and since S is symmetric $S = S^{-1}$. Thus $S \circ R = S^{-1} \circ R^{-1} = (R \circ S)^{-1}$. But since $S \circ R = R \circ S \Rightarrow R \circ S = (R \circ S)^{-1}$ and thus $R \circ S$ is symmetric.

Section 5.1 - Functions

5.1.11. Suppose $f : A \to B$ and S is a relation on B. Define a relation R on A as follows:

$$R = \{(x, y) \in A \times A | (f(x), f(y)) \in S\}$$

a. Prove that if S is reflexive, then so is R.

Assume S is reflexive, thus for all $b \in B \Rightarrow (b, b) \in S$. Let $x \in A$ be arbitrary with $b = f(x)$. Since $(b, b) \in S \Rightarrow (f(x), f(x)) \in S$ and thus $(x, x) \in R$. Since x was arbitrary, R is reflexive.

b. Prove that if S is symmetric, then so is R.

Assume S is symmetric, thus for all $a, b \in B \Rightarrow (a, b) \in S \rightarrow (b, a) \in S$. Let $x, y \in A$ be arbitrary with $a = f(x)$ and $b = f(y)$. Since $(a, b) \in S \Rightarrow (f(x), f(y)) \in S$ and thus $(x, y) \in R$. But since S is symmetric $(b, a) \in S \Rightarrow (f(y), f(x)) \in S$ and thus $(y, x) \in R$. Hence since x, y are arbitrary and $(x, y) \in R$ and $(y, x) \in R$ we can conclude R is symmetric.

c. Prove that if S is transitive, then so is R.

Assume R is transitive, thus for all $a, b, c \in B \Rightarrow ((a, b) \in S \wedge (b, c) \in S) \rightarrow (a, c) \in S$. Let $x, y, z \in A$ be arbitrary with $a = f(x)$, $b = f(y)$, $c = f(z)$. Since $(a, b) \in S \Rightarrow (f(x), f(y)) \in S$ and thus $(x, y) \in R$. Likewise, since $(b, c) \in S \Rightarrow (f(y), f(z)) \in S$ and thus $(y, z) \in R$. But since S is transitive $(a, c) \in S \Rightarrow (f(x), f(z)) \in S$ and thus $(x, z) \in R$. Hence since x, y, z are arbitrary and $((x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$, therefore R is transitive.

Section 5.3 - Inverses of Functions

5.3.2. Let F be the function defined as $A = \{1,2,3\}$, $B = \mathcal{P}(A)$, and $F : B \to B | F(X) = A \setminus X$. If $X \in B$, what is $F^{-1}(X)$?

First we must show that F is *one-to-one* and *onto* such that F^{-1} exists. Enumerating B (of which there are $2^3 = 8$ elements)

$$B = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Thus F is found by taking each set $X \in B$ and computing $A \setminus X$

$$\begin{aligned} F = &\{(\emptyset, \{1,2,3\}), (\{1\}, \{2,3\}), (\{2\}, \{1,3\}), (\{3\}, \{1,2\}), (\{1,2\}, \{3\}) \\ &(\{1,3\}, \{2\}), (\{2,3\}, \{1\}), (\{1,2,3\}, \emptyset)\} \end{aligned}$$

One-to-one

Clearly we can see above that every element in the domain B has a *unique* element in the range. We could formally prove this by assuming there exist two sets $X, Y \in B$. By the definition of F we then have $F(X) = A \setminus X = F(Y) = A \setminus Y$. But then $A \setminus X = A \setminus Y \Rightarrow X = Y$. Thus F is one-to-one.

Onto

Again by looking at the elements in the range of F (i.e. second components in the ordered pairs) we see that all the elements of B are present. Formally we let $Y \in B$ be arbitrary and choose $X = A \setminus Y$. Then $F(X) = F(A \setminus Y) = A \setminus (A \setminus Y) = A \cap Y$ (by the equivalence in exercise 3.5.3). However since $Y \in B \Rightarrow Y \subseteq A$ and thus $A \cap Y = Y$. Thus since Y was an arbitrary element in the range of F, F is onto.

Inverse

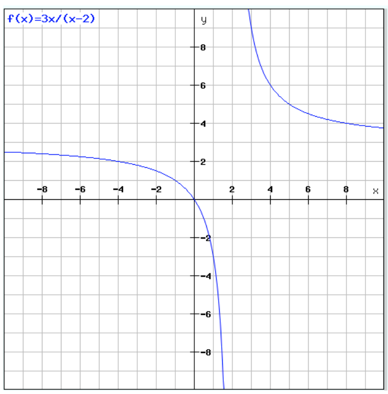
Since F is one-to-one and onto, F^{-1} exists. We could write out the ordered pairs for F^{-1} by simply switching the components of each ordered pair in F. However it should be clear that this would give the *same* ordered pairs and thus $F^{-1} = F$. Hence $F^{-1}(X) = F(X) = A \setminus X$. We can prove this by letting $X \in B$ be arbitrary such that $Y = F(X) = A \setminus X$. Then if we compute $F(Y) = F(A \setminus X) = A \setminus (A \setminus X) = A \cap X = X = F^{-1}(Y)$.

5.3.6. Let $A = \mathbb{R} \setminus \{2\}$, and let f be the function with domain A defined by the formula

$$f(x) = \frac{3x}{x-2}$$

a. Show that f is a one-to-one, onto function from A to B for some set $B \subseteq \mathbb{R}$. What is the set B?

Graphing the function gives



One-to-one

Hence we see that f is one-to-one for $B = \mathbb{R}$ since no two values for x give the same value for $f(x)$. Formally assume there exists two values x_1, x_2 such that $f(x_1) = f(x_2)$. Then

$$\begin{aligned}
f(x_1) = \frac{3x_1}{x_1 - 2} &= \frac{3x_2}{x_2 - 2} = f(x_2) \\
x_1(x_2 - 2) &= x_2(x_1 - 2) \\
x_1x_2 - 2x_1 &= x_2x_1 - 2x_2 \\
-2x_1 &= -2x_2 \\
x_1 &= x_2
\end{aligned}$$

Onto

However we see from the graph that there is a value for $f(x)$ that is never achieved, as shown by the horizontal asymptote in the graph. This value occurs as $x \to \infty$ giving the value 3. Hence for $f(x)$ to be onto, $B = \mathbb{R} - \{3\}$. Formally, assume $y \in \mathbb{R} - \{3\}$ and let $x = \frac{2y}{y-3}$ which exists for $y \neq 3$. Then

$$
\begin{aligned}
f(x) &= \frac{3x}{x-2} \\
&= \frac{3\frac{2y}{y-3}}{\frac{2y}{y-3}-2} \\
&= \frac{\frac{6y}{y-3}}{\frac{2y-2(y-3)}{y-3}} \\
&= \frac{6y}{2y-2y+6} \\
&= \frac{6y}{6} \\
&= y
\end{aligned}
$$

b. Find a formula for $f^{-1}(x)$?

Inverse

For the range $B = \mathbb{R} - \{3\}$, f is one-to-one and onto, thus f^{-1} exists. Using the formula for x in the proof of onto we have $f^{-1} : B \to A$ where $B = \mathbb{R} - \{3\}$ defined as

$$f^{-1}(y) = \frac{2y}{y-3}$$