ECE335 Summer 2019 - Assignment 5

Section 3.2 - Proofs Involving Negations and Conditionals

3.2.3 . Suppose $A \subseteq C$, and B and C are disjoint. Prove that if $x \in A$ then $x \notin B$.

Givens	*Goal*
$A \subseteq C$	$x \in A \to x \notin B$
B and C are disjoint	

First we bring the left-hand side of the goal into the givens

Givens	*Goal*
$A \subseteq C$	$x \notin B$
B and C are disjoint	
$x \in A$	

Since our goal is a negated statement, we will try proof by contradiction by *assuming* $x \in B$ and trying to prove a contradiction. Thus

Givens	*Goal*
$A \subseteq C$	*(contradiction)*
B and C are disjoint	
$x \in A$	
$x \in B$	

Now we can reason that since $x \in A$ and $A \subseteq C$ means that $x \in C$. Then since $x \in C$ and $x \in B$ means that $x \in B \cap C$. But since B and C are disjoint, we know that $B \cap C = \emptyset$ which is a contradiction. Thus $x \notin B$.

Formally,

Theorem. *Suppose $A \subseteq C$, and B and C are disjoint. If $x \in A$ then $x \notin B$*

Proof. Suppose that $x \in A$ and assume $x \in B$. Then since $x \in A$ and $A \subseteq C$ gives that $x \in C$. Since $x \in C$ and $x \in B$ we know that $x \in B \cap C$. However since B and C are disjoint means that $B \cap C = \emptyset$ which is a contradiction. Therefore $x \notin B$ and we can conclude that if $x \in A$ then $x \notin B$.

3.2.12 . Consider the following incorrect theorem

Incorrect Theorem. *Suppose that $A \subseteq C$, $B \subseteq C$, and $x \in A$. Then $x \in B$.*

a. What's wrong with the following proof of the theorem?

Proof. Suppose that $x \notin B$. Since $x \in A$ and $A \subseteq C$, $x \in C$. Since $x \notin B$ and $B \subseteq C$, $x \notin C$. But now we have proven both $x \in C$ and $x \notin C$, so we have reached a contradiction. Therefore $x \in B$.

The problem with this proof lies in the assertion that "Since $x \notin B$ and $B \subseteq C$, $x \notin C$". Subsets only imply that *if* and element is in the subset, *then* it is in the superset. An element could be in the superset but not in the subset and thus the conclusion that $x \notin C$ is invalid (i.e. $P \to Q$ and $\neg P$ does **not** imply $\neg Q$).

b. Show that the theorem is incorrect by finding a counterexample.

Let $A = \{1\}$, $B = \{2\}$, and $C = \{1, 2, 3\}$. Clearly $A \subseteq C$ and $B \subseteq C$. Select $x = 1$ which gives $x \in A$ but $x \notin B$.

Section 3.3 - Proofs Involving Quantifiers

3.3.2 . Prove that if A and $B \setminus C$ are disjoint, then $A \cap B \subseteq C$.

Givens	*Goal*
A and $B \setminus C$ are disjoint	$A \cap B \subseteq C$

Applying the definition of subset to the goal gives $\forall x(x \in A \cap B \to x \in C)$

Givens	*Goal*
A and $B \setminus C$ are disjoint	$\forall x(x \in A \cap B \to x \in C)$

Since the goal contains a universal quantifier, we will let x be arbitrary such that $x \in A \cap B$ and then applying the conditional transformation gives

Givens	*Goal*
A and $B \setminus C$ are disjoint $x \in A \cap B$	$x \in C$

A and $B \setminus C$ disjoint means that $\neg \exists y(y \in A \cap (B \setminus C))$

Since $x \in A \cap B$ we know that $x \in A$ and $x \in B$. But A and $B \setminus C$ are disjoint so $x \in A$ implies that $x \notin B \setminus C$. However since $x \in B$, for $x \notin B \setminus C$ means that $x \in C$.

Theorem. *If A and $B \setminus C$ are disjoint, then $A \cap B \subset C$.*

Proof. Suppose that x is arbitrary and $x \in A \cap B$. Since $x \in A \cap B$ means that $x \in A$ and $x \in B$. Because A and $B \setminus C$ are disjoint, since $x \in A$ implies that $x \notin B \setminus C$. Thus since $x \in B$ we can conclude that for $x \notin B \setminus C$ we must have $x \in C$. Therefore since x was arbtrary $A \cap B \subseteq C$.

3.3.12. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.

Givens	*Goal*
$\mathcal{F}$ and $\mathcal{G}$ are families	$\cup\mathcal{F} \subseteq \cup\mathcal{G}$
$\mathcal{F} \subseteq \mathcal{G}$	

Applying the definition of subset to the goal gives the universal quantified statement

Givens	*Goal*
$\mathcal{F}$ and $\mathcal{G}$ are families	$\forall x(x \in \cup\mathcal{F} \to x \in \cup\mathcal{G})$
$\mathcal{F} \subseteq \mathcal{G}$	

Therefore we let x be an arbitrary element and separate the conditional giving

Givens	*Goal*
$\mathcal{F}$ and $\mathcal{G}$ are families	$x \in \cup\mathcal{G}$
$\mathcal{F} \subseteq \mathcal{G}$	
$x \in \cup\mathcal{F}$	

Now rewriting the goal by applying the definition of family union gives

Givens	*Goal*
$\mathcal{F}$ and $\mathcal{G}$ are families	$\exists A \in \mathcal{G}(x \in A)$
$\mathcal{F} \subseteq \mathcal{G}$	
$x \in \cup\mathcal{F}$	

Applying the same definition of family union to the given and letting A_0 be the particular set in $\mathcal{F}$ that contains x gives

Givens	*Goal*
$\mathcal{F}$ and $\mathcal{G}$ are families	$\exists A \in \mathcal{G}(x \in A)$
$\mathcal{F} \subseteq \mathcal{G}$	
$x \in A_0$	
$A_0 \in \mathcal{F}$	

Therefore since $\mathcal{F} \subset \mathcal{G}$ and $A_0 \in \mathcal{F}$ implies that $A_0 \in \mathcal{G}$. Then since $x \in A_0$ means that *there exists* a set in $\mathcal{G}$ (namely A_0 that contains x and therefore since x was arbitrary $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.

Theorem. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\mathcal{F} \subseteq \mathcal{G}$ then $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.*

Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$ and let x be arbitrary with $x \in \cap\mathcal{F}$. Since $x \in \cap\mathcal{F}$, let $A_0 \in \mathcal{F}$ such that $x \in A_0$. Hence, since $A_0 \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{G}$ gives that $A_0 \in \mathcal{G}$ and thus $x \in \cup\mathcal{G}$. Therefore since x was arbitrary, $\cup\mathcal{F} \subseteq \cup\mathcal{G}$.

Section 3.4 - Proofs Involving Conjunctions and Biconditionals

3.4.3 . Suppose $A \subseteq B$. Prove that for every set C, $C \setminus B \subseteq C \setminus A$.

Givens	*Goal*
$A \subseteq B$	$C \setminus B \subseteq C \setminus A$

Expanding using the definition of subset gives

Givens	*Goal*
$\forall x(x \in A \to x \in B)$	$\forall x(x \in (C \setminus B) \to x \in (C \setminus A))$

We can now let x be arbitrary in both the given and goal and bring the left hand side of the goal to the givens

Givens	*Goal*
$x \in A \to x \in B$	$x \in C \setminus A$
$x \in C \setminus B$	

Finally by set difference we have

Givens	*Goal*
$x \in A \to x \in B$	$x \in C \wedge x \notin A$
$x \in C \wedge x \notin B$	

Separating the **and** statements gives

Givens	*Goal*
$x \in A \to x \in B$	$x \in C$
$x \in C$	$x \notin A$
$x \notin B$	

Thus we will prove each goal separately. First, since $x \in C$ the first goal is clearly satisfied. Second, since $x \notin B$ by modus tollens since $x \in A \to x \in B$ gives $x \notin A$ which proves the second goal.

Formally,

Theorem. *Suppose $A \subseteq B$. Then for every set C, $C \setminus B \subseteq C \setminus A$.*

Proof. Let x be an arbitrary element with $x \in C \setminus B$. Then $x \in C$ and $x \notin B$. Since $x \notin B$ and $A \subseteq B$ we know that $x \notin A$. But then since $x \in C$ and $x \notin A$ we can conclude that $x \in C \setminus A$ and because x was arbitrary that $C \setminus B \subseteq C \setminus A$.

3.4.7. Prove that for any sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

The equality can be interpreted as $\leftrightarrow$ giving

Givens	*Goal*
A and B are sets	$x \in \mathcal{P}(A \cap B) \leftrightarrow x \in (\mathcal{P}(A) \cap \mathcal{P}(B))$

Therefore since $x \in \mathcal{P}(A \cap B) \leftrightarrow x \in (\mathcal{P}(A) \cap \mathcal{P}(B))$ means

$(x \in \mathcal{P}(A \cap B) \to x \in (\mathcal{P}(A) \cap \mathcal{P}(B))) \wedge (x \in (\mathcal{P}(A) \cap \mathcal{P}(B)) \to x \in \mathcal{P}(A \cap B))$

We can separate the goal into two separate conditional goals

Givens	*Goal*
A and B are sets	$x \in \mathcal{P}(A \cap B) \to x \in (\mathcal{P}(A) \cap \mathcal{P}(B))$
	$x \in (\mathcal{P}(A) \cap \mathcal{P}(B)) \to x \in \mathcal{P}(A \cap B)$

We can now prove each goal separately.

$(\to)$ *Givens*	*Goal*
A and B are sets	$x \in (\mathcal{P}(A) \cap \mathcal{P}(B))$
$x \in \mathcal{P}(A \cap B)$	

Applying the definition of intersection and powerset gives

$(\to)$ *Givens*	*Goal*
A and B are sets	$(x \subseteq A) \wedge (x \subseteq B)$
$x \subseteq (A \cap B)$	

Using the definition of subset then gives

($\rightarrow$) *Givens*	*Goal*
A and B are sets	$\forall y(y \in x \rightarrow y \in A) \wedge \forall y(y \in x \rightarrow y \in B)$
$\forall y(y \in x \rightarrow y \in (A \cap B)$	

Letting y be arbitrary then gives

($\rightarrow$) *Givens*	*Goal*
A and B are sets	$(y \in x \rightarrow y \in A) \wedge (y \in x \rightarrow y \in B)$
$y \in x \rightarrow y \in (A \cap B)$	

Separating both conditionals of the goal gives $y \in x$ and thus by modus ponens gives $y \in (A \cap B)$ which means $y \in A$ and $y \in B$. Since y was arbitrary means $x \subseteq (A \cap B)$ implies $x \subseteq A$ and $x \subseteq B$ and thus $x \in \mathcal{P}(A \cap B)$ implies $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$. Therefore $x \in (\mathcal{P}(A) \cap \mathcal{P}(B))$.

Using the same logical forms but swapping the given and goal from the first proof gives

($\leftarrow$) *Givens*	*Goal*
A and B are sets	$y \in x \rightarrow y \in (A \cap B)$
$(y \in x \rightarrow y \in A) \wedge (y \in x \rightarrow y \in B)$	

Again letting y be arbitrary and letting $y \in x$ gives that $y \in A$ and $y \in B$. Hence $y \in (A \cap B)$. Since y was arbitrary means $x \subseteq A$ and $x \subseteq B$ implies $x \subseteq (A \cap B)$ and thus $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$ implies $x \in \mathcal{P}(A \cap B)$.

Theorem. *For any sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.*

Proof. ($\rightarrow$) Suppose x is arbitrary such that $x \in \mathcal{P}(A \cap B)$. Thus $x \subseteq A \cap B$. Let y be an arbitrary element of x, then since $x \subseteq A \cap B$, $y \in A \cap B$ and thus $y \in A$ and $y \in B$. Since y was arbitrary, $y \in A$ means $x \subseteq A$ and $y \in B$ means $x \subseteq B$. Therefore $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$ and therefore $x \in (\mathcal{P}(A) \cap \mathcal{P}(B))$.

($\leftarrow$) Now suppose x is arbitrary such that $x \in (\mathcal{P}(A) \cap \mathcal{P}(B))$. Thus $x \subseteq A$ and $x \subseteq B$. Let y be an arbitrary element of x, then since $x \subseteq A$ means $y \in A$. Similarly, since $x \subseteq B$ means $y \in B$. Since $y \in A$ and $y \in B$ means $y \in A \cap B$. Thus because y is arbitrary, we can conclude that $x \subseteq A \cap B$ and hence $x \in \mathcal{P}(A \cap B)$.