Final Project: Modal Logic

Robert Liu, CMSC 22400

In this project, I formalized modal logic in Coq using the S5 system, proved facts in modal logic using S5, and formalized various philosophical arguments from Leibniz, Moore, Kant, Spinoza, Gödel, and Fitch.

I explored different modal logic systems such as K, S4, and S5. I ultimately decided on implementing S5 as it was the strongest given that its $\Box$ operator implies that P holds in all possible worlds. These benefits are shown directly in results I proved like $\diamond\ (x =_M y) \to_M \Box(x =_M y)$ which can be done quite easily in S5 but could not have been possible in some other systems. I defined the box operator $\Box$ (necessarily), the diamond operator $\diamond$ (possibly), an equivalence relation between worlds which is implicit in $\Box$ and $\diamond$, as well as other connectives and operators such as $\wedge_M, \vee_M, =_M, \neq_M, \to_M, \leftrightarrow_M, \sim_M, \forall_M, \exists_M$.

Included Lemmas and Proofs

Basic Modal Logic (ModalLogic.v)

I started by proving various lemmas within the modal logic system. All of these followed very easily from S5 and verified that there were no obvious issues with my implementation.

1. In all worlds w, $p \wedge_M q \to_M p$ (testing if the modal conjunction works)
2. In all worlds w, $p \vee_M q \to_M (\sim_M (p \vee_M q) \to_M \text{False})$ (testing if the modal disjunction and negation works)
3. In all worlds w, $p \to_M (q \to_M p)$ (testing if implication works)
4. In all worlds w, $p \to_M q \wedge_M p \to_M q$ (modus ponens with modal operators)
5. In all worlds w, $\Box p \to_M p$ (if p necessarily holds then p holds)
6. In all worlds w, $\Box p \to_M \Box(\Box p)$ (if p necessarily holds then it necessarily necessarily holds)
7. In all worlds w, $p \to_M \Box(\diamond\ p)$ (if p holds then it is necessarily the case that p possibly holds)
8. In all worlds w, $\diamond\ p \to_M \Box(\diamond\ p)$ (if p possibly holds then it is necessarily the case that p possibly holds)
9. In all worlds w, $\diamond\ p \leftrightarrow_M \sim_M\big(\Box(\sim_M p)\big)$ (p possibly holds if any only if it is not the case that p necessarily does not hold)
10. In all worlds w, $\sim_M(\diamond\ p) \to_M \Box(\sim_M p)$ (if it is not the case that p possibly holds, then it is necessarily the case that p does not hold)
11. In all worlds w, $\diamond\ p \to_M (p \to_M q) \to_M \diamond\ p$ (modus ponens with diamond)
12. In all worlds w, $\Box(p \vee_M \sim_M p)$ (law of the excluded middle)
13. In all worlds w, $\Box(\sim_M p) \to_M \sim_M(\diamond\ p)$ (if p necessarily does not hold then necessarily p cannot possibly hold)
14. In all worlds w, $\Box(\sim_M p) \to_M \sim_M p$
15. In all worlds w, $\diamond\ (\Box p) \to_M p$ (if possibly p necessarily holds then p holds)
16. In all worlds w, $\diamond\ (\Box p) \to_M \Box p$ (if possibly p necessarily holds then necessarily p holds)
17. In all worlds w, $\diamond\ (\Box p) \to_M \Box\big(\diamond\ (\Box p)\big)$
18. In all worlds w, $\Box(p \to_M q) \to_M \big((\Box p) \to_M (\Box q)\big)$ (box distributes)
19. For all propositions p, in all worlds w, $\Box\big(\forall_M x, (p\ x)\big) \leftrightarrow_M \big(\forall_M x, \Box(p\ x)\big)$ (bidirectional Barcan formula)
20. In all worlds w, $\diamond\ (x =_M y) \to_M \Box(x =_M y)$ (if possibly $x =_M y$ then $x =_M y$ in every world)
21. In all worlds w, $(x =_M y) \to_M \Box(x =_M y)$
22. For all propositions P, in all worlds w, $x =_M y \wedge_M \Box(P\ x) \to_M \Box(P\ y)$ (substitution inside box)
23. In all worlds w, $\Box p \to_M \diamond\ p$ (if p necessarily holds then p possibly holds)
24. In all worlds w, $\diamond\diamond\ p \to_M \diamond\ p$ (if p necessarily holds then p possibly holds)
25. For all propositions P, in all worlds w, $(\exists_M x, \Box P\ x) \to_M \Box(\exists_M x, P\ x)$
26. For all propositions P, in all worlds w, $\diamond\ (\exists_M x, \Box P\ x) \to_M \diamond\ \Box(\exists_M x, P\ x)$
27. In all worlds w, $\sim_M(\Box p) \to_M \diamond\ (\sim_M p)$

Leibniz's Law (Leibniz.v)

I then proved Leibniz's Law using modal logic. This followed very easily from S5.

28. In all worlds w, $\forall_M a, b\big((\forall_M p, p\ a \leftrightarrow_M p\ b) \rightarrow_M a =_M b\big)$ (identity of indiscernibles)
29. In all worlds w, $\forall_M a, b\big(a =_M b \rightarrow_M (\forall_M p, p\ a \leftrightarrow_M p\ b)\big)$ (indiscernibility of identicals)
30. Together, #28 and #29 give the bidirectional Leibniz's Law

Moore's Paradox (Moore.v)

I also explored Moore's Paradox and showed that it is sound in the S5 system, where all worlds are accessible to one another. I formalized belief as a relation on worlds, Bel.

31. Given p which holds in one world, we can have $p \wedge_M \sim_M (Bel\ p)$.

DDL + Kantian Ethics (Kant.v)

I extended S5 to get Dyadic Deontic Logic (DDL) which has new operators for obligation and permission. This allows me to formalize ideas in Kantian ethics. Furthermore, I can create statements that combine metaphysical necessity (□) and deontic necessity (O). I defined moral permissibility as the fact that an action (which is an object, from my S5 implementation) can be universalized and there is no contradiction. I took it to be an axiom that if an agent performs an action in world w, then there is an accessible world w' where everyone can act and that universalization is consistent.

32. In all worlds w, $O(p \rightarrow_M q) \rightarrow_M (O\ p \rightarrow_M O\ q)$ (K-schema)
33. For all propositions p, $\forall w, p\ w \rightarrow \forall w, O\ p\ w$ (if p is valid in all worlds then p is obligatory in all worlds)
34. In all worlds w, $O\ p \rightarrow_M$ possible_deontic p (ought implies can: If p is obligatory, then p is at least deontically possible)

I then formalized the concept of stealing with small propositions. I formalized arguments from Kant's categorical imperative that stealing being universal is contradictory. This allowed me to prove that stealing is not permissible.

35. In all worlds w, $\sim_M(\forall a, a$ steals in $w)$ (stealing being universal is contradictory)
36. In all worlds w, stealing is not morally permissible

Fitch's Paradox of Knowability (Fitch.v)

I then went back to my original S5 system to formalize Fitch's Paradox of Knowability, which uses a similar modal logic to S5. However, rather than the box and diamond operators, Fitch uses K (known) and L (possible to be known). I encoded $K\ p$ as $\Box p$ and $L\ p$ and $\diamond \Box p$ as a direct extension of S5. This allowed me to show a contradiction in the following proposition. Finding the contradiction took me longer to figure out but I managed to the Law of the Excluded Middle from the Logic lecture to reach a contradiction with $\Box p$ in world w'.

37. Contradiction in the statement $\big(\forall p, \forall w, (p \rightarrow_M L\ p)\big) \rightarrow \big(\forall p, \forall w, (p \rightarrow_M K\ p)\big)$ which says all truths are knowable implies all truths are known

First Ontological Argument: Spinoza (Spinoza.v)

I decided to first formalize Spinoza's ontological argument, specifically using Spinoza's proof by contradiction argument, as I felt it was the most natural extension of S5. Spinoza used terms such as substance, essence, and existence which I built on top of S5. I used the following version of Spinoza's argument, from this journal.

38. (Axiom) A substance cannot be produced by another substance (Spinoza also takes this as an axiom)
39. (Axiom) Everything that exists has some cause
40. Based on #36 and #37, in all worlds w, for all x, if x is a substance in w then x was self-caused in w (if a substance exists, it was self-caused)
41. (Axiom) Self-caused substances necessarily exist

42. (Axiom) God is a substance
43. In all worlds w, for all x, if x is a substance in w then the essence of x involves existence
44. (Axiom) If something can be conceived as not existing then its essence does not involve existence (Spinoza takes this as an axiom)
45. Assume that God can be conceived as not existing. Based on #44, God's essence does not involve existence
46. Based on #42, #43, God's essence does involve existence
47. The proposition that God exists in all worlds w, $\Box E(G)$, follows from a contradiction of #45 and #46

I initially could not finish the proof because Spinoza does not make clear whether existence is a precondition for self-cause or if it is a result of self-cause. I found some contention online about this point, especially in this paper and on this discussion forum. This is due to Spinoza's argument claiming that substances, if they exist, have essences that "involve their existence." Since God is a substance, God also falls under that criterion, but nowhere does it stipulate that God exists as a precondition. As a result, if I take existence as a requirement for self-cause, then existence would be a precondition for the results of self-cause, which makes it impossible to complete the proof without assuming that God exists in at least one world. In that case, I had assumed that God existed in at least one world in order to make the stronger claim that God's essence involves existence, $\Box E(G)$. However, if I do not treat existence as a requirement for self-cause, existence is still guaranteed as the essence of a substance guarantees existence.

Second Ontological Argument: Gödel (Godel.v)

The last step of this project was proving Gödel's ontological argument. I used this specific version of Gödel's argument from this paper, already written in modal logic, shown below. This was written in quantified modal logic KB but since S5 is a stronger system, the arguments carried over.

A1	Either a property or its negation is positive, but not both:	$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
A2	A property necessarily implied by a positive property is positive:	$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
T1	Positive properties are possibly exemplified:	$\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$
D1	A *God-like* being possesses all positive properties:	$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
A3	The property of being God-like is positive:	$P(G)$
C	Possibly, God exists:	$\Diamond\exists x G(x)$
A4	Positive properties are necessarily positive:	$\forall\phi[P(\phi) \rightarrow \Box\, P(\phi)]$
D2	An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:	$\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
T2	Being God-like is an essence of any God-like being:	$\forall x[G(x) \rightarrow G\ ess.\ x]$
D3	*Necessary existence* of an individual is the necessary exemplification of all its essences:	$NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$
A5	Necessary existence is a positive property:	$P(NE)$
T3	Necessarily, God exists:	$\Box\exists x G(x)$

Figure 1: Scott's version of Gödel's ontological argument [12].

I first defined Properties (which are simply functions from objects to modal propositions) and added a parameter which allows for a Property to be Positive.

48. (Axiom) A1
49. (Axiom) A2
50. T1

To prove T1, since I could not see a way to magically get an object to prove the $exists$ clause, I proved it similarly to Spinoza's argument with a proof by contradiction. The most obvious premise to contradict, is A1, which states that either X is a positive property or $\sim X$ is but not both. I assumed that a positive property X is never exemplified in any world accessible from w. This means that in all accessible worlds, no object has property X. I can use Axiom A2 to show that $\sim X$ is a positive property also. This directly contradicts A1 so it must be the case that X is exemplified in a world accessible from w.

51. D1: $GodLike$
52. (Axiom) A3
53. C1
54. (Axiom) A4
55. D2: $Essence$
56. T2

Clearing the first requirement for $Essence$ is straightforward. To clear the second requirement for $Essence$, I first showed that ψ is positive in the current world w. Then I can apply Axiom A4 to get that ψ is necessarily positive (meaning it is positive in every world). The rest of the proof followed from that result and from using the definition of $GodLike$.

57. D3: $NecessaryExistence$
58. (Axiom) A5
59. T3

T3 follows from first using C1 to show that God possibly exists, meaning there is an object g in a certain world such that $GodLike\ g$. T2 tells us that $GodLike$ is an essence of g. By definition of $GodLike$, g has all positive properties in w. By A5, $NecessaryExistence$ is a positive property. Thus, g exemplifies $NecessaryExistence$. By definition of $NecessaryExistence$, we have necessary exemplification of all of g's essences, including $GodLike$, which means $GodLike$ holds in every world. By S5 symmetry of the relation on worlds, we can finish the proof.

Sources

- https://courses.umass.edu/phil511-gmh/pdf/C08-2016.pdf
- https://plato.stanford.edu/entries/identity-indiscernible/
- https://plato.stanford.edu/entries/logic-deontic/
- https://www.jstor.org/stable/40230656?seq=1
- https://plato.stanford.edu/entries/kant-moral/
- https://www.jstor.org/stable/40041109
- https://en.wikipedia.org/wiki/Fitch's_paradox_of_knowability
- https://www.researchgate.net/publication/259145246_Godel's_God_in_IsabelleHOL