Sample answers

Session 2

Q1 Are B and C independent? (**coin tossing**)

The sample space $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$B = \{HHH, HHT, TTH, TTT\}$ $\quad P(B) = \frac{|B|}{|S|} = \frac{4}{8} = \frac{1}{2}$

$C = \{HHH, HTH, THH, TTH\}$ $\quad P(C) = \frac{|C|}{|S|} = \frac{4}{8} = \frac{1}{2}$

$\therefore B \cap C = \{TTH, HHH\}$ $\quad P(B \cap C) = \frac{|B \cap C|}{|S|} = \frac{2}{8} = \frac{1}{4}$

$P(B) \cdot P(C) = P(B \cap C) \Leftrightarrow$ B and C are independent

Q2 Please give the proof of Proposition 1.14.

A, B, C mutually independent

$\Rightarrow$ A and $B \cap C$ are independent (by the definition of mutually independent)

$\therefore P(A \cap (B \cap C)) = P(A) \cdot P(B \cap C)$

$$
\begin{aligned}
P(A \cap (B \cup C)) &= P((A \cap B) \cup (A \cap C)) \\
&= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) \quad \text{(Proposition 1.7 session 1)} \\
&\cancel{= P(A) \cdot P(B)} \\
&= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \\
&= P(A) \cdot P(B) + P(A) \cdot P(C) - P(A) \cdot P(B) \cdot P(C) \quad \text{(mutually independent)} \\
&= P(A) \cdot (P(B) + P(C) - P(B) \cdot P(C)) \\
&= P(A) \cdot (P(B) + P(C) - P(B \cap C)) \quad \text{(mutually independent)} \\
&= P(A) \cdot P(B \cup C) \quad \text{(Proposition 1.7 session 1)}
\end{aligned}
$$

$\Rightarrow$ A and $B \cup C$ are independent.

EX1 A couple is planning to have a family. They decide to stop having children either when they have two boys or when they have four children. Suppose that they are successful in their plan.

(a) Write down the sample space.

(b) Assume that, each time that they have a child, the probability that it is a boy is 1/2, independent of all other times. Find P(E) and P(F) where E = "there are at least two girls", F = "there are more girls than boys".

Using B and G to present having a boy and a girl, respectively.
We have the sample space S:

(a) $\mathcal{S} = \{BB, BGB, GBB, BGGB, GBGB, GGBB, BGGG, GBGG, GGBG, GGGB, GGGG\}.$

Not equally likely

Remark that we need to consider the order of the baby born.

Then, we just pick out those elements from the sample space to get the two events/sets that required

(b) $E = \{BGGB, GBGB, GGBB, BGGG, GBGG, GGBG, GGGB, GGGG\}$,
$F = \{BGGG, GBGG, GGBG, GGGB, GGGG\}$.

Now we have $P(BB) = 1/4$, $P(BGB) = 1/8$, $P(BGGB) = 1/16$, and similarly for the other outcomes. So $P(E) = 8/16 = 1/2$, $P(F) = 5/16$.

EX2 Pick an integer in [1; 1000] at random. Compute the probability that it is divisible neither by 12 nor by 15. Pick an integer in [1; 1000] at random. Compute the probability that it is divisible neither by 12 nor by 15.

The sample space consists of the 1000 integers between 1 and 1000 and let A_r be the subset consisting of integers divisible by r. The cardinality of A_r is $\lfloor 1000/r \rfloor$. Another simple fact is that $A_r \cap A_s = A_{\text{lcm}(r,s)}$, where lcm stands for the least common multiple. Our probability equals

$$\begin{aligned}
1 - P(A_{12} \cup A_{15}) &= 1 - P(A_{12}) - P(A_{15}) + P(A_{12} \cap A_{15}) \\
&= 1 - P(A_{12}) - P(A_{15}) + P(A_{60}) \\
&= 1 - \frac{83}{1000} - \frac{66}{1000} + \frac{16}{1000} = 0.867.
\end{aligned}$$

EX3 A group of 20 Scandinavians consists of 7 Swedes, 3 Finns, and 10 Norwegians. A committee of five people is chosen at random from this group. What is the probability that at least one of the three nations is not represented on the committee?

Let A_1 = the event that Swedes are not represented, A_2 = the event that Finns are not represented, and A_3 = the event that Norwegians are not represented.

$$
\begin{aligned}
P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\
&\quad + P(A_1 \cap A_2 \cap A_3) \\
&= \frac{1}{\binom{20}{5}} \left[\binom{13}{5} + \binom{17}{5} + \binom{10}{5} - \binom{10}{5} - 0 - \binom{7}{5} + 0 \right]
\end{aligned}
$$

EX4 A group of 3 Norwegians, 4 Swedes, and 5 Finns is seated at random around a table. Compute the probability that at least one of the three groups ends up sitting together.

Define $A_N = \{\text{Norwegians sit together}\}$ and similarly A_S, A_F. We have

$$P(A_N) = \frac{3! \cdot 9!}{11!},\ P(A_S) = \frac{4! \cdot 8!}{11!},\ P(A_F) = \frac{5! \cdot 7!}{11!},$$
$$P(A_N \cap A_S) = \frac{3! \cdot 4! \cdot 6!}{11!},\ P(A_N \cap A_F) = \frac{3! \cdot 5! \cdot 5!}{11!},\ P(A_S \cap A_F) = \frac{4! \cdot 5! \cdot 4!}{11!},$$
$$P(A_N \cap A_S \cap A_F) = \frac{3! \cdot 4! \cdot 5! \cdot 2!}{11!}.$$

Therefore,

$$P(A_N \cup A_S \cup A_F) = \frac{3! \cdot 9! + 4! \cdot 8! + 5! \cdot 7! - 3! \cdot 4! \cdot 6! - 3! \cdot 5! \cdot 5! - 4! \cdot 5! \cdot 4! + 3! \cdot 4! \cdot 5! \cdot 2!}{11!}.$$

EX5 You are given $P(A \cup B) = 0.7$ and $P(A \cup B') = 0.9$, Calculate $P(A)$.

First note

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P\left[A \cup B^c\right] = P[A] + P\left[B^c\right] - P\left[A \cap B^c\right]$$

Then add these two equations to get

$$P[A \cup B] + P\left[A \cup B^c\right] = 2P[A] + \left(P[B] + P\left[B^c\right]\right) - \left(P[A \cap B] + P\left[A \cap B^c\right]\right)$$

$$0.7 + 0.9 = 2P[A] + 1 - P\left[(A \cap B) \cup \left(A \cap B^c\right)\right]$$

$$1.6 = 2P[A] + 1 - P[A]$$

$$P[A] = 0.6$$

EX6 An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same colour is 0.44. Calculate the number of blue balls in the second urn.

For $i = 1,2$, let R_i = event that a red ball is drawn from urn i and let B_i = event that a blue ball is drawn from urn i. Then, if x is the number of blue balls in urn 2,

$$0.44 = \Pr[(R_1 \cap R_2) \cup (B_1 \cap B_2)] = \Pr[R_1 \cap R_2] + \Pr[B_1 \cap B_2]$$

$$= \Pr[R_1]\Pr[R_2] + \Pr[B_1]\Pr[B_2]$$

$$= \frac{4}{10}\left(\frac{16}{x+16}\right) + \frac{6}{10}\left(\frac{x}{x+16}\right)$$

Therefore,

$$2.2 = \frac{32}{x+16} + \frac{3x}{x+16} = \frac{3x+32}{x+16}$$

$$2.2x + 35.2 = 3x + 32$$

$$0.8x = 3.2$$

$$x = 4$$