Calculus
Exercise 2 Solution

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A.

1. $y = f(x) = 4x - 3x^2, \quad (2, -4)$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{4(x+h) - 3(x+h)^2 - 4x + 3x^2}{h} \\
&= \lim_{h \to 0} \frac{4x + 4h - 3x^2 - 6xh - 3h^2 - 4x + 3x^2}{h} \\
&= \lim_{h \to 0} \frac{h(4 - 6x - 3h)}{h} \\
&= 4 - 6x \\
\therefore f'(2) &= -8 \\
\therefore \quad y + 4 &= -8(x-2) \\
\text{Or, } y &= -8x + 12
\end{aligned}$$

2. (a) $y = 3 + 4x^2 - 2x^3$

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{3 + 4(a+h)^2 - 2(a+h)^3 - 3 - 4a^2 + 2a^3}{h} \\
&= \lim_{h \to 0} \frac{4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2 - 6ah^2 - 2h^3 - 4a^2 + 2x^3}{h} \\
&= \lim_{h \to 0} \left(8a + 4h - 6a^2 - 6ah - 2h^2\right) \\
&= 8a - 6a^2
\end{aligned}$$

   (b) $\because f'(1) = 2, f'(2) = -8$
   $\therefore$ tangent line at $(1, 5)$: $\quad y - 5 = 2(x-1),$ or $y = 2x + 3$
   tangent line at $(2, 3)$: $y - 3 = -8(x-2),$ or $y = -8x + 19$

3. $y = 40t - 16t^2, \quad t = 2$

$$\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{40(2+h) - 16(2+h)^2 - 40(2) + 16(2)^2}{h} \\
&= \lim_{h \to 0} \frac{80 + 40h - 16(4) - 16(4h) - 16h^2 - 80 + 16(2)^2}{h} \\
&= \lim_{h \to 0} (40 - 64 - 16h) \\
&= -24 (\text{ft/s})
\end{aligned}$$

4. $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$

5. $f(t) = \dfrac{2t+1}{t+3}$

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\frac{2(a+h)+1}{(a+h)+3} - \frac{2a+1}{a+3}}{h} \\
&= \lim_{h\to 0} \frac{(2a+2h+1)(a+3)-(2a+1)(a+h+3)}{(a+h+3)(a+3)h} \\
&= \lim_{h\to 0} \frac{5h}{(a+h+3)(a+3)h} \\
&= \frac{5}{(a+3)^2}
\end{aligned}$$

6. (a) $\displaystyle\lim_{h\to 0}\frac{(1+h)^{10}-1}{h} = \lim_{h\to 0}\frac{(1+h)^{10}-1^{10}}{h} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$
   $\therefore f(x) = x^{10}, a = 1$

   (b) $\displaystyle\lim_{x\to 5}\frac{2^x-32}{x-5} = \lim_{x\to 5}\frac{2^x-2^5}{x-5} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$
   $\therefore f(x) = 2^x, a = 5$

7. $C(x) = 5000 + 10x + 0.05x^2$

   (a) i. $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105)-C(100)}{105-100} = \dfrac{101.25}{5} = \$20.25\text{/unit}$

   ii. $\dfrac{\Delta c}{\Delta x} = \dfrac{c(101)-c(100)}{101-100} = \dfrac{20.05}{1} = \$20.05\text{/unit}$

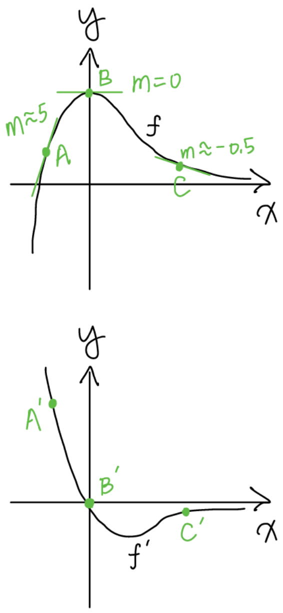
   (b)

$$\begin{aligned}
C'(100) &= \lim_{h\to 0}\frac{C(100+h)-C(100)}{h} \\
&= \lim_{h\to 0}\frac{5000+10(100+h)+0.05(100+h)^2-5000-10(100)-0.05(100)^2}{h} \\
&= \lim_{h\to 0}(20+0.05h) \\
&= \$20\text{/unit}
\end{aligned}$$

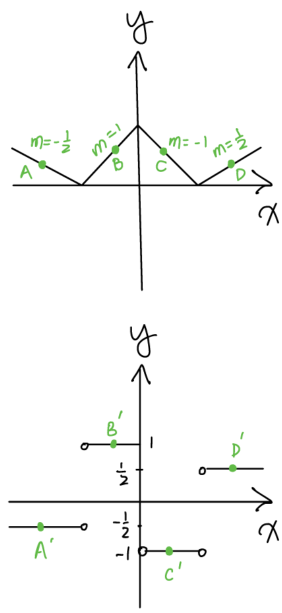
8. (a) $f'(-3) = -0.2$
   (b) $f'(-2) = 0$
   (c) $f'(-1) = 1$
   (d) $f'(0) = 2$
   (e) $f'(1) = 1$
   (f) $f'(2) = 0$
   (g) $f'(3) = -0.2$

9. Solutions:

(a)



(b)



10. The instantaneous rate of change of percentage of capacity with respect to elapsed time in hour.

11. (a) $f(x) = \frac{1}{2}x - \frac{1}{3}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{2}(x+h) - \frac{1}{3} - \frac{1}{2}x + \frac{1}{3}}{h} = \lim_{h \to 0} \frac{\frac{1}{2}h}{h} = \frac{1}{2}$$

Domain: $f(x) : x \in R$; $\quad f'(x) : x \in R$

(b) $G(t) = \dfrac{1-2t}{3+t}$

$$\begin{aligned} G'(t) &= \lim_{h \to 0} \frac{\frac{1-2(t+h)}{3+(t+h)} - \frac{1-2t}{3+t}}{h} \\ &= \lim_{h \to 0} \frac{(3+t)(1-2t-2h) - (1-2t)(3+t+h)}{(3+t)(3+t+h)h} \\ &- \lim_{h \to 0} \frac{-7h}{(3+t)(3+t+h)h} \\ &= -\frac{7}{(3+t)^2} \end{aligned}$$

Domain: $G(t) : t \neq 3$, $\quad G'(t) : t \neq 3$

12. (a) At $x = -4$, a corner
At $x = 0$, a discontinuity
At $x = 2.5$, a Vertical Tangent

(b) At $x = -1$, a vertical Tangent
At $x = 4$, a corner

13. Let $a : f(x), b : g(x), c : h(x)$
Note that $g'(x) = 0$ while $f(x) = 0$
And $h'(x) \to 0$ while $g(x) \to 0$
$\therefore \quad f(x) = g'(x)$ and $g(x) = h'(x)$
That is, c is the position function, b is the velocity function, and a is the acceleration function.

14. (a) $f'(x) = 0$

(b) $f'(t) = -\dfrac{2}{3}$

(c) $f'(x) = 3x^2 - 4$

(d) $\because g(x) = x^2(1-2x) = x^2 - 2x^3$
$\therefore g'(x) = 2x - 6x$

(e) $g'(t) = -\dfrac{3}{2}t^{-7/4}$

(f) $\because A(s) = -\dfrac{12}{s^5} = -12s^{-5}$
$\therefore A'(s) = 60s^{-6} = \dfrac{60}{s^6}$

(g) $\because S(p) = \sqrt{p} - p = p^{1/2} - p$
$\therefore S'(p) = \dfrac{1}{2}p^{-1/2} - 1$

(h) $\because R(a) = (3a+1)^2 = 9a^2 + 6a + 1$
$\therefore R'(a) = 18a + 6$

(i) $\because y = \dfrac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$
$\therefore y' = \dfrac{3}{2}x^{1/2} + 2x^{-1/2} - \dfrac{3}{2}x^{-3/2}$

(j) $f'(t) = \dfrac{2(2+\sqrt{t}) - \frac{1}{2}t^{-1/2} \cdot 2t}{(2+\sqrt{t})^2} = \dfrac{4 + 2\sqrt{t} - \sqrt{t}}{(2+\sqrt{t})^2} = \dfrac{4+\sqrt{t}}{(2+\sqrt{t})^2}$

15.
$$\because \frac{dy}{dx} = \frac{2(x+1) - 1 \cdot 2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$
$$\therefore \left.\frac{dy}{dx}\right|_{x=1} = \frac{2}{(1+1)^2} = \frac{1}{2}$$
$$y - 1 = \frac{1}{2}(x-1)$$
Tangent line: $y = \dfrac{1}{2}x + \dfrac{1}{2}$

16.
$$\because \frac{dy}{dx} = \frac{3\left(x^2+1\right) - 2x(3x+1)}{\left(x^2+1\right)^2}$$
$$\left.\frac{dy}{dx}\right|_{x=1} = -\frac{1}{2}$$
$\therefore$ Tangent line: $y - 2 = -\dfrac{1}{2}(x-1)$ or $y = -\dfrac{1}{2}x + \dfrac{5}{2}$

Normal line: $y - 2 = 2(x-1)$ or $y = 2x$

17. $f(x) = \dfrac{x^2}{1+2x}$

$$\begin{aligned}
f'(x) &= \frac{2x(1+2x) - 2\cdot x^2}{(1+2x)^2} \\
&= \frac{2x + 4x^2 - 2x^2}{1+4x+4x^2} \\
&= \frac{2x^2+2x}{4x^2+4x+1} \\
f''(x) &= \frac{(4x+2)\left(4x^2+4x+1\right) - (8x+4)\left(2x^2+2x\right)}{\left(4x^2+4x+1\right)^2} \\
&= \frac{(4x+2)\left(4x^2+4x+1-4x^2-4x\right)}{(2x+1)^4} \\
&= \frac{2(2x+1)}{(2x+1)^4} = \frac{2}{(2x+1)^3}
\end{aligned}$$

18. (a) $(fg)'(5) = f'(5)g(5) + g'(5)f(5) = 6\times(-3) + 2\times 1 = -16$

(b) $(f/g)'(5) = \dfrac{f'(5)g(5) - g'(5)f(5)}{[g(5)]^2} = \dfrac{6\times(-3) - 2\times 1}{(-3)^2} = -\dfrac{20}{9}$

(c) $(g/f)'(5) = \dfrac{g'(5)f(5) - f'(5)g(5)}{[f(5)]^2} = \dfrac{2\times 1 - 6\times(-3)}{1^2} = 20$

19. (a) $f(x) = 3x^2 - 2\cos x$
$f'(x) = 6x + 2\sin x$

(b) $y = \sec\theta\tan\theta$
$\dfrac{dy}{d\theta} = \sec\theta\tan^2\theta + \sec^3\theta$

(c) $y = \dfrac{x}{2-\tan x}$
$\dfrac{dy}{dx} = \dfrac{(2-\tan x) - \left(-\sec^2 x\right)x}{(2-\tan x)^2} = \dfrac{2-\tan x + x\sec^2 x}{(2-\tan x)^2}$

(d) $y = \dfrac{t\sin t}{1+t}$
$\dfrac{dy}{dt} = \dfrac{(\sin t + t\cos t)(1+t) - t\sin t}{(1+t)^2} = \dfrac{\sin t + t\cos t + t^2\cos t}{(1+t)^2}$

20. $y = \sec x, \quad \left(\frac{\pi}{3}, 2\right)$

$$\frac{dy}{dx} = \sec x\tan x$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = \sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = 2\cdot\sqrt{3}$$

$\therefore$ Tangent line: $y - 2 = 2\sqrt{3}\left(x - \dfrac{\pi}{3}\right)$

Or $y = 2\sqrt{3}x + 2 - \dfrac{2}{3}\sqrt{3}\pi$

21. $y = 2x\sin x, \quad \left(\frac{\pi}{2}, \pi\right)$

$$\begin{aligned}
\frac{dy}{dx} &= 2\sin x + 2x\cos x \\
\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} &= 2\sin\left(\frac{\pi}{2}\right) + 2\cdot\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) \\
&= 2 + \pi\cdot 0 = 2 \\
\therefore \text{Tangent line: } y - \pi &= 2\left(x - \frac{\pi}{2}\right) \\
\text{Or } y &= 2x
\end{aligned}$$

22. $H(\theta) = \theta\sin\theta$
$H'(\theta) = \sin\theta + \theta\cos\theta$
$H''(\theta) = \cos\theta + \cos\theta - \theta\sin\theta = 2\cos\theta - \theta\sin\theta$

B.

1. (a) $y' = -\sin\left(a^3 + x^3\right)\cdot 3x^2 = -3x^2\sin\left(a^3 + x^3\right)$

   (b) $y' = 3\left(\frac{x^2+1}{x^2-1}\right)^2\cdot\left[\frac{2x\left(x^2-1\right) - 2x\left(x^2+1\right)}{\left(x^2-1\right)^2}\right] = \frac{3\left(x^2+1\right)^2\cdot 2x(-2)}{\left(x^2-1\right)^4} = \frac{-12x\left(x^2+1\right)^2}{\left(x^2-1\right)^4}$

   (c)

$$\begin{aligned}
\because F(z) &= \sqrt{\frac{z-1}{z+1}} = \left(\frac{z-1}{z+1}\right)^{\frac{1}{2}} \\
\therefore F'(z) &= \frac{1}{2}\left(\frac{z-1}{z+1}\right)^{-\frac{1}{2}}\cdot\left[\frac{1\cdot(z+1) - 1\cdot(z-1)}{(z+1)^2}\right] \\
&= \frac{1}{2}\cdot\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}}\cdot\frac{2}{(z+1)^2} \\
&= \frac{\sqrt{z+1}}{\sqrt{z-1}(z+1)^2} \quad \text{or} \quad \frac{\sqrt{z^2-1}}{(z-1)(z+1)^2}
\end{aligned}$$

   (d) $y' = \cos(\tan 2x)\sec^2(2x)\cdot 2 = 2\cos(\tan 2x)\cdot\sec^2(2x)$

   (e) $y' = 3\left[x^2 + (1-3x)^5\right]^2\cdot\left[2x + 5(1-3x)^4\cdot(-3)\right] = 3\left[x^2 + (1-3x)^5\right]^2\cdot\left[2x - 15\left(1-3x\right)^7\right]$

   (f) $y' = -\sin\sqrt{\sin(\tan\pi x)}\cdot\frac{1}{2}[\sin(\tan\pi x)]^{-\frac{1}{2}}\cdot\cos(\tan\pi x)\cdot\sec^2(\pi x)\cdot\pi$

2. $H'(t) = \sec^2(3t)\cdot 3 = 3\sec^2(3t)$
$H''(t) = 6\sec(3t)\cdot\sec(3t)\tan(3t)\cdot 3 = 18\sec^2(3t)\tan(3t)$

3. $y = \sin(\sin x), \quad (\pi, 0)$

$$\begin{aligned}
\frac{dy}{dx} &= \cos(\sin\pi)\cdot\cos x \\
\left.\frac{dy}{dx}\right|_{x=\pi} &= \cos(0)\cdot(-1) = 1\cdot(-1) = -1 \\
\therefore \text{Tangent line: } y - 0 &= -1(x-\pi) \\
\text{Or} \quad y &= -x + \pi
\end{aligned}$$

4. $F'(5) = f'(g(5)) \cdot g'(5) = f'(-2) \cdot 6 = 4 \cdot 6 = 24$

5. (a) $u'(1) = f'(g(1)) \cdot g'(1) = f'(3) \cdot (-3) = -\frac{1}{4} \cdot (-3) = \frac{3}{4}$

   (b) $v'(1) = g'(f(1)) \cdot f'(1) = g'(2) \cdot (2)$
   $\because g'(2)$ does not exists,
   $\therefore v'(1)$ is undefined.

   (c) $w'(1) = g'(g(1)) \cdot g'(1) = g'(3) \cdot (-3) = \frac{2}{3} \cdot (-3) = -2$

6.
$$\begin{aligned} r'(1) &= f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1) \\ &= f'(g(2)) \cdot g'(2) \cdot 4 \\ &= f'(3) \cdot 5 \cdot 4 \\ &= 6 \cdot 5 \cdot 4 \\ &= 120 \end{aligned}$$

C.

1. (a)
$$\begin{aligned} \because \frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(1) \\ 3x^2 + 3y^2 \cdot \frac{dy}{dx} &= 0 \\ \therefore \frac{dy}{dx} &= \frac{-3x^2}{3y^2} = -\frac{x^2}{y^2} \end{aligned}$$

   (b)
$$\begin{aligned} \because \frac{d}{dx}\left[x^4(x+y)\right] &= \frac{d}{dx}\left[y^2(3x-y)\right] \\ 4x^3(x+y) + \left(1 + 1 \cdot \frac{dy}{dx}\right) \cdot x^4 &= 2y \cdot \frac{dy}{dx}(3x-y) + \left(3 - 1 \cdot \frac{dy}{dx}\right) y^2 \\ 4x^4 + 4x^3y + x^4 + x^4\frac{dy}{dx} &= 6xy\frac{dy}{dx} - 2y^2\frac{dy}{dx} + 3y^2 - y^2\frac{dy}{dx} \\ x^4\frac{dy}{dx} + 3y^2\frac{dy}{dx} - 6xy\frac{dy}{dx} &= 3y^2 - 5x^4 - 4x^3y \\ \therefore \frac{dy}{dx} &= \frac{3y^2 - 5x^4 - 4x^3y}{3y^2 + x^4 - 6xy} \end{aligned}$$

   (c)
$$\begin{aligned} \because \frac{d}{dx}(4\cos x \sin y) &= \frac{d}{dx}(1) \\ -4\sin x \sin y + 4\cos x \cos y \cdot \frac{dy}{dx} &= 0 \\ \therefore \frac{dy}{dx} &= \frac{\sin x \sin y}{\cos x \cos y} = \tan x \tan y \end{aligned}$$

(d)

$$
\begin{aligned}
\because \frac{d}{dx}\left[(xy)^{1/2}\right] &= \frac{d}{dx}(1+x^2y) \\
\frac{1}{2}(xy)^{-1/2}\cdot\left(1\cdot y+1\cdot\frac{dy}{dx}\cdot x\right) &= 2xy+\frac{dy}{dx}\cdot x^2 \\
\frac{1}{2}x^{-1/2}y^{1/2}+\frac{1}{2}x^{1/2}y^{-1/2}\cdot\frac{dy}{dx} &= 2xy+x^2\cdot\frac{dy}{dx} \\
\therefore \frac{dy}{dx} &= \frac{2xy-\frac{1}{2}x^{-1/2}\cdot\frac{1}{2}}{\frac{1}{2}x^{1/2}y^{-1/2}-x^2} \quad \text{or} \quad \frac{4x^{3/2}y^{3/2}-y}{x-2x^{5/2}y^{1/2}}
\end{aligned}
$$

2.

$$
\begin{aligned}
\because \{f(x)+x^2[f(x)]^3\}' &= (10)' \\
f'(x)+2x[f(x)]^3+x^2\cdot 3[f(x)]^2\cdot f'(x) &= 0 \\
f'(x) &= \frac{-2x[f(x)]^3}{1+3x^2[f(x)]^2} \\
\therefore f'(1) &= \frac{-2\cdot 1\cdot (2)^3}{1+3(1)^2\cdot(2)^2} = -\frac{16}{13}
\end{aligned}
$$

D.

1. (a)

$$
\begin{aligned}
v(t) = h'(t) &= 24.5-9.8t \\
\therefore v(2) &= 4.9(\text{m/s}) \\
v(4) &= -14.7(\text{m/s})
\end{aligned}
$$

(b)

$$
\begin{aligned}
v(t) = 24.5-9.8t &= 0 \\
t &= 2.5(\text{s})
\end{aligned}
$$

(c)

$$
h(2.5) = 32.625(\text{m})
$$

(d)

$$
\begin{aligned}
h(t) = 2+24.5t-4.9t^2 &= 0 \\
t &= \frac{-24.5\pm\sqrt{24.5^2-4\cdot(-4.9)\cdot(2)}}{2(-4.9)} \\
\therefore t_1 \approx -0.08 \text{ (reject)}, &\quad t_2 \approx 5.08(\text{s})
\end{aligned}
$$

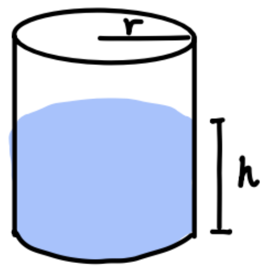
(e)

$$
v(5.08) \approx -25.3(\text{m/s})
$$

2. Given: $r = 5(\mathrm{m}), \dfrac{dV}{dt} = 3(\mathrm{m}^3/\mathrm{min})$

   Find: $\dfrac{dh}{dt} = ?$

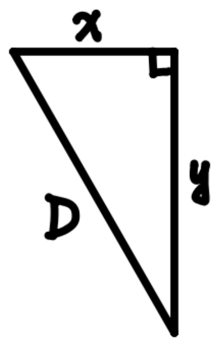
$$\begin{aligned} V &= \pi r^2 h \\ \frac{dV}{dt} &= \pi r^2 \cdot \frac{dh}{dt} \\ 3 &= \pi \cdot 25 \cdot \frac{dh}{dt} \\ \therefore \frac{dh}{dt} &= \frac{3}{25\pi} \approx 0.038(\mathrm{m}/\mathrm{min}) \end{aligned}$$



3. Given: $\dfrac{dy}{dt} = 60(\mathrm{mi}/\mathrm{h}), \dfrac{dx}{dt} = 25(\mathrm{mi}/\mathrm{h})$

   Find: $\left.\dfrac{dD}{dt}\right|_{t=2} = ?$

$$\begin{aligned} \because D^2 &= x^2 + y^2 \\ y(2) &= 120 \\ x(2) &= 50 \\ D(2) &= \sqrt{50^2 + 120^2} = 130 \\ \text{Then, } \because 2D \cdot \frac{dD}{dt} &= 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} \\ 2(130) \cdot \frac{dD}{dt} &= 2(50)(25) + 2(60)(120) \\ \therefore \frac{dD}{dt} &= 65(\mathrm{mi}/\mathrm{h}) \end{aligned}$$



4. Given: $\dfrac{dV}{dt} = 0.2(\mathrm{m}^3/\mathrm{min})$

   Find: $\left.\dfrac{dh}{dt}\right|_{h=0.3} = ?$

$$\begin{aligned} V &= (t+b)h \times \frac{1}{2} \times 10 \\ &= (b + 2x + b) \cdot 5h \\ &= 10h(b+x) \\ &= 10h(b + \frac{1}{2}h) \qquad (\because \frac{h}{x} = \frac{0.5}{0.25} = 2) \\ &= 10bh + 5h^2 \\ \text{Then, } \because \frac{dV}{dt} &= 10b \cdot \frac{dh}{dt} + 10h \cdot \frac{dh}{dt} \\ 0.2 &= 10(0.3) \cdot \frac{dh}{dt} + 10(0.3) \cdot \frac{dh}{dt} \\ \therefore \frac{dh}{dt} &= \frac{1}{30} \text{ or } 0.03\ (\mathrm{m}/\mathrm{min}) \end{aligned}$$