Probability & Statistics

Dongjiao Ge

djge@cityu.edu.mo

Session 02

1.8 Stopping rules

You are taking a test….
You are allowed to take the test up to **three** times. If you pass the test, then you don’t need to take it again.The sample space is

$$S = \{p, fp, ffp, fff\},$$

Where f presents fail, p presents pass.

If all outcomes were **equally likely**, then your chance of **eventually passing** the test would be 3/4.

But it is ***unreasonable*** here to assume that all the outcomes are ***equally likely***……

You may be very likely to pass on the first attempt.

Assume that the probability that you pass the test is 0.8. (By Proposition 3 (see session 1), your chance of failing is 0.2.) Let us further assume that, no matter how many times you have failed, your chance of passing at the next attempt is still 0.8. Then we have

$$\begin{aligned} P(p) &= 0.8, \\ P(fp) &= 0.2 \cdot 0.8 = 0.16, \\ P(ffp) &= 0.2^2 \cdot 0.8 = 0.032, \\ P(fff) &= 0.2^3 = 0.008. \end{aligned}$$

'multiplication rule' See next chapter

Thus the probability that you eventually get the certificate is $P(\{p, f\,p, f\,f\,p\}) = 0.8 + 0.16 + 0.032 = 0.992$. Alternatively, you eventually get the certificate unless you fail three times, so the probability is 1−0.008 = 0.992.

A stopping rule is a rule of the type described here, i.e., ***continue the experiment until some specified occurrence happens.*** The experiment may potentially be infinite.

For example, if you toss a coin repeatedly until you obtain heads, the sample space is

$$S = \{H, TH, TTH, TTTH, \dots\}$$

which contains infinite many outcomes.

In the first example, the rule is '*stop if either you pass or you have taken the test three times*'. This ensures that the sample space is *finite*.

Other kinds of stopping rules …
e.g., The number of coin tosses might be determined by some other random process such as the roll of a die; or we might toss a coin until we have obtained heads twice; and so on.

1.9 Questionnaire results

We have a questionnaire like this

1. I have a hat containing 20 balls, 10 red and 10 blue. I draw 10 balls from the hat. I am interested in the event that I draw exactly five red and five blue balls. Do you think that this is more likely if I note the colour of each ball I draw and replace it in the hat, or if I don't replace the balls in the hat after drawing?

More likely with replacement ☐ *More likely without replacement* ☐

2. What colour are your eyes?

Blue ☐ *Brown* ☐ *Green* ☐ *Other* ☐

3. Do you own a mobile phone? *Yes* ☐ *No* ☐

A certain class of students were invited to complete the questionnaire.

After discarding incomplete questionnaires, the results were as follows:

Answer to question	"More likely with replacement"		"More likely without replacement"	
Eyes	Brown	Other	Brown	Other
Mobile phone	35	4	35	9
No mobile phone	10	3	7	1

What can we conclude?

- Half the class thought that, in the experiment with the coloured balls, sampling

with replacement made the result more likely. In fact, it is more likely if we sample without replacement.

- Do eye colour and mobile phone ownership would influence your answer?

If yes, then of the 87 people with brown eyes, half of them (i.e. 43 or 44) would answer "with replacement", whereas in fact 45 did. Also, of the 83 people with mobile phones, we would expect half (that is, 41 or 42) would answer "with replacement", whereas in fact 39 of them did.

  - So perhaps we have demonstrated that people who own mobile phones are slightly smarter than average, whereas people with brown eyes are slightly less smart!

- In fact, we have shown no such thing, since our results refer only to the people who filled out the questionnaire. But they do show that these events are not independent, in a sense we will come too soon.

- On the other hand, since 83 out of 104 people have mobile phones, if we think that phone ownership and eye colour are independent, we would expect that the same fraction 83/104 of the 87 brown-eyed people would have phones, i.e. $(83 \cdot 87)/104 = 69.4$ people. In fact, the number is 70, or as near as we can expect. So indeed it seems that eye colour and phone ownership are more-or-less independent.

1.10 Independence

Two events A and B are said to be independent if

$$P(A \cap B) = P(A) \cdot P(B).$$

Remarks…

- This is the **definition** of independence of events.
- Do ***not*** say that two events are independent if one has no influence on the other; and under no circumstances say that A and B are independent if $A \cap \mathrm{B} = \phi$ (this is the statement that A and B are disjoint, which is quite a different thing!)
- Do ***not*** ever say that $P(A \cap B) = P(A) \cdot P(B)$ unless you have some good reason for assuming that A and B are independent (either because this is given in the question, or it is reasonable to do so).

Back to the questionnaire example….

Suppose that a student is chosen at random from those who filled out the questionnaire.

- A --- the event that this student thought that the event was more likely if we sample with replacement;
- B --- the event that the student has brown eyes;
- C --- the event that the student has a mobile phone.

Then

$$
\begin{aligned}
P(A) &= 52/104 = 0.5, \\
P(B) &= 87/104 = 0.8365, \\
P(C) &= 83/104 = 0.7981.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
P(A \cap B) &= 45/104 = 0.4327, \qquad & P(A) \cdot P(B) &= 0.4183, \\
P(A \cap C) &= 39/104 = 0.375, \qquad & P(A) \cdot P(C) &= 0.3990, \\
P(B \cap C) &= 70/104 = 0.6731, \qquad & P(B) \cdot P(C) &= 0.6676.
\end{aligned}
$$

None of the three pairs is independent, but in a sense B and C ‘come closer’ than either of the others, as we noted.

In practice, if it is the case that event A has no effect on the outcome of event B, then A and B are independent. But this does not apply in the other direction. There might be a very definite connection between A and B, but still, it could happen that $P(A \cap B) = P(A) \cdot P(B)$, so that A and B are independent.

Let see an example .....

Example

If we toss a coin more than once, or roll a die more than once, then you may assume that different tosses or rolls are independent.

More precisely, if we roll a fair six-sided die twice, then the probability of getting 4 on the first throw and 5 on the second is 1/36, since we assume that all 36 combinations of the two throws are equally likely. But $(1/36) = (1/6) \cdot (1/6)$, and the separate probabilities of getting 4 on the first throw and of getting 5 on the second are both equal to 1/6. So the two events are independent. This would work just as well for any other combination.

In general, it is always <u>**OK**</u> to assume that the outcomes of different tosses of a coin, or different throws of a die, <u>*are independent*</u>. This holds even if the examples are not all equally likely.

Let us have a look at another example…

Example

I have two red pens, one green pen, and one blue pen. I choose two pens without replacement.

- A --- the event that I choose exactly one red pen,
- B --- the event that I choose exactly one green pen.

If the pens are called R_1, R_2, G, B, then

$$\begin{aligned} S &= \{R_1R_2, R_1G, R_1B, R_2G, R_2B, GB\}, \\ A &= \{R_1G, R_1B, R_2G, R_2B\}, \\ B &= \{R_1G, R_2G, GB\} \end{aligned}$$

We have

$$P(A) = 4/6 = 2/3, P(B) = 3/6 = 1/2, P(A \cap B) = 2/6 = 1/3 = P(A)P(B)$$

***A* and *B* are independent.**

Suppose that I add a purple pen, and I do the same experiment. We have

$$P(A) = 6/10 = 3/5 \;\; P(B) = 4/10 = 2/5, \; P(A \cap B) = 2/10 = 1/5 \neq P(A)P(B)$$

A and B are no longer independent

It is very difficult to tell whether events are independent or not.

In practice, assume that events are independent only if either you are told to assume it, or the events are the outcomes of different throws of a coin or die. (There is one other case where you can assume independence: this is the result of different draws, with replacement, from a set of objects.)

Example

We toss a fair coin three times and note the results. Each of the eight possible outcomes has probability 1/8.

- A --- the event 'there are more heads than tails'
- B --- the event 'the results of the first two tosses are the same'. Then

  - $A = \{HHH, HHT, HTH, THH\}, P(A) = 1/2,$
  - $B = \{HHH, HHT, TTH, TTT\}, P(B) = 1/2,$
  - $A \cap B = \{HHH, HHT\}, P(A \cap B) = 1/4;$

<u>A and B are independent</u>

However, both A and B clearly involve the results of the first two tosses and it is not possible to make a convincing argument that one of these events has no influence or effect on the other. For example, let C be the event 'heads on the last toss'. Then,

- $C = \{HHH, HTH, THH, TTH\}, P(C) = 1/2,$
- $A \cap C = \{HHH, HTH, THH\}, P(A \cap C) = 3/8;$

<u>A and C are not independent.</u>

Q1 Are B and C independent? (**10 minutes**)

1.11 Mutual independence

In the coin-tossing example above it is possible to have three events A , B and , C so that A and B are independent, B and C are independent, but A and C are not independent.

If all three pairs of events happen to be independent, can we conclude that

$$P(A \cap \mathrm{B} \cap \mathrm{C}) \;=\; P(A) \;\cdot P(B) \;\cdot P(C)?$$

This seems very reasonable; in Axiom 3, we only required all pairs of events to be exclusive in order to justify our conclusion. Unfortunately, it is **<u>not true</u>**. . .

Example

In the coin-tossing example, we have
A --- the event 'first and second tosses have the same result',
B --- the event 'first and third tosses have the same result,
C --- the event 'second and third tosses have the same result'.

We can get $P(A) = P(B) = P(C) = 1/2$, and that the events $A \cap \mathrm{B}, B \cap \mathrm{C}, A \cap \mathrm{C}$, and $A \cap \mathrm{B} \cap \mathrm{C}$ are all equal to $\{HHH, TTT\}$, with probability 1/4. Thus any pair of the three events are independent, but

$$\begin{aligned} P(A \cap B \cap C) &= 1/4, \\ P(A) \cdot P(B) \cdot P(C) &= 1/8. \end{aligned}$$

So A, B, C are not mutually independent.

The correct definition and proposition is:

Let $A_1, \ldots, A_n$ be events. We say that these events are ***mutually independent*** if, given any distinct indices $i_1, i_2, \ldots, i_k$ with $k \geq 1$, the events

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{k-1}} \quad \text{and} \quad A_{i_k}$$

are independent.

In other words, any one of the events is independent of the intersection of any number of the other events in the set.

Proposition 1.11 *Let $A_1, \ldots, A_n$ be mutually independent. Then*

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdots P(A_n).$$

Remarks

- The same ‘physical’ arguments that justify that two events (such as two tosses of a coin, or two throws of a die) are independent, also justify that any number of such events are mutually independent.
- If we toss a fair coin six times, the probability of getting the sequence $HHTHHT$ is $\left(\frac{1}{2}\right)^6 = 1/64$, and the same would apply for any other sequence. In other words, all 64 possible outcomes are equally likely.

1.12 Properties of independence

Proposition 1.12 *If A and B are independent, then A and B' are independent.*

Proof.

A and B independent $\Leftrightarrow P(A \cap \mathrm{B}) = P(A) \cdot P(B)$

From Corollary 4, we know that $P(B') = 1 - P(B)$.

$(A \cap \mathrm{B}) \cap (A \cap B') = \phi$ (no outcome can be both in $B\ and\ B'$), and

$(A \cap \mathrm{B}) \cup (A \cap B') = A$ (since every event in A is either in B or in B');

so by Axiom 3, we have that $P(A) = P(A \cap \mathrm{B}) + P(A \cap \mathrm{B}')$. Thus,

$$\begin{aligned} P(A \cap B') &= P(A) - P(A \cap B) \\ &= P(A) - P(A) \cdot P(B) \\ &\qquad \text{(since } A \text{ and } B \text{ are independent)} \\ &= P(A)(1 - P(B)) \\ &= P(A) \cdot P(B'), \end{aligned}$$

That means A and B’ are independent.

Corollary 1.13 *If A and B are independent, so are A' and B'.*

Proof. (this is not the exact proof) Apply the Proposition twice, first to A and B (to show that A and B' are independent), and then to B' and A (to show that B' and A' are independent).

More generally, if events $A_1, \ldots, A_n$ are *mutually independent*, and we replace some of them by their complements, then the resulting events are *mutually independent*. We have to be a bit careful though. For example, A and A' are not usually independent!

Proposition 1.14 *Let events A, B, C be mutually independent. Then A and* $B \cap C$ *are independent, and A and* $B \cup C$ *are independent.*

Q2 Please give the proof of Proposition 1.14. (**15 minutes**)

Example Consider the example of the test that we looked at earlier. You are allowed up to three attempts to pass the test.

Suppose that your chance of passing the test is 0.8. Suppose also that the events of passing the test on any number of different occasions are mutually independent. Then, by Proposition 1.11, the probability of any sequence of passes and fails is the product of the probabilities of the terms in the sequence. That is,

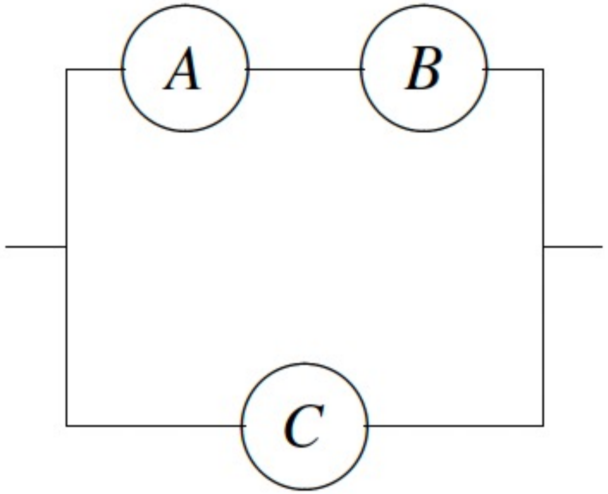
$$P(p) = 0.8,\ P(fp) = (0.2)\cdot(0.8),\ P(ffp) = (0.2)^2\cdot(0.8),\ P(fff) = (0.2)^3,$$

as we claimed in the earlier example.

In other words, *mutual independence* is the condition we need to justify the argument we used in that example.

Example

The electrical apparatus in the diagram works so long as current can flow from left to right. The three components are independent. The probability that component A works is 0.8; the probability that component B works is 0.9; and the probability that component C works is 0.75. Find the probability that the apparatus works.



Assume

A --- the event 'component A works',

B --- the event 'component B works',

C --- the event 'component C works'.

Now the apparatus will work if either A and B are working, or C is working (or possibly both). Thus the event we are interested in is $(A \cap \mathrm{B}) \cup C$.

$$\begin{aligned}
P((A\cap B)\cup C)) &= P(A\cap B)+P(C)-P(A\cap B\cap C)\\
&\qquad \text{(by Inclusion–Exclusion)}\\
&= P(A)\cdot P(B)+P(C)-P(A)\cdot P(B)\cdot P(C)\\
&\qquad \text{(by mutual independence)}\\
&= (0.8)\cdot(0.9)+(0.75)-(0.8)\cdot(0.9)\cdot(0.75)\\
&= 0.93.
\end{aligned}$$

Alternatively, we can consider its <u>*complementary event*</u> --- the apparatus will not work. It happens when both paths are blocked, that is, if C is not working and one of A and B is also not working. Thus, the event that the apparatus does not work is $(A' \cup B') \cap C'$. By the Distributive Law, this is equal to $(A' \cap C') \cup (B' \cap C')$. We have

$$\begin{aligned}
P((A'\cap C')\cup(B'\cap C') &= P(A'\cap C')+P(B'\cap C')-P(A'\cap B'\cap C')\\
&\qquad \text{(by Inclusion–Exclusion)}\\
&= P(A')\cdot P(C')+P(B')\cdot P(C')-P(A')\cdot P(B')\cdot P(C')\\
&\qquad \text{(by mutual independence of } A',B',C')\\
&= (0.2)\cdot(0.25)+(0.1)\cdot(0.25)-(0.2)\cdot(0.1)\cdot(0.25)\\
&= 0.07,
\end{aligned}$$

so the apparatus works with probability $1-0.07 = 0.93$.

Remark

There is a trap here which you should take care to avoid. You might be tempted to say $P(A' \cap C') = (0.2) \cdot (0.25) = 0.05$, and $P(B' \cap C') = (0.1) \cdot (0.25) = 0.025$; and conclude that

$$P((A' \cap C') \cup (B' \cap C')) = 0.05 + 0.025 - (0.05) \cdot (0.025) = 0.07375$$

by the Principle of Inclusion and Exclusion. But this is not correct, since the events $A' \cap C'$ and $B' \cap C'$ are not independent!

Example

We can always assume that successive tosses of a coin are mutually independent, even if it is not a fair coin. Suppose that I have a coin which has probability 0.6 of coming down heads. I toss the coin three times. What are the probabilities of getting three heads, two heads, one head, or no heads?

For three heads, since successive tosses are mutually independent, the probability is $(0.6)^3 = 0.216$. The probability of tails on any toss is $1 - 0.6 = 0.4$. Now the event 'two heads' can occur in three possible ways, as $HHT, HTH, or\ THH$. Each outcome has probability $(0.6) \cdot (0.6) \cdot (0.4) = 0.144$. So the probability of two heads is $3 \cdot (0.144) = 0.432$. Similarly the probability of one head is $3 \cdot (0.6) \cdot (0.4)^2 = 0.288$, and the probability of no heads is $(0.4)^3 = 0.064$.

As a check, we have 0.216+0.432+0.288+0.064 = 1.

Exercises

EX1 A couple is planning to have a family. They decide to stop having children either when they have two boys or when they have four children. Suppose that they are successful in their plan.
(a) Write down the sample space.
(b) Assume that, each time that they have a child, the probability that it is a
boy is 1/2, independent of all other times. Find P(E) and P(F) where
E = “there are at least two girls”, F = “there are more girls than boys”.

EX2 Pick an integer in [1; 1000] at random. Compute the probability that it is divisible neither by 12 nor by 15. Pick an integer in [1; 1000] at random. Compute the probability that it is divisible neither by 12 nor by 15.

EX3 A group of 20 Scandinavians consists of 7 Swedes, 3 Finns, and 10 Norwegians. A committee of five people is chosen at random from this group. What is the probability that at least one of the three nations is not represented on the committee?

EX4 A group of 3 Norwegians, 4 Swedes, and 5 Finns is seated at random around a table. Compute the probability that at least one of the three groups ends up sitting together.

EX5 You are given $P(A \cup B) = 0.7$ and $P(A \cup B') = 0.9$, Calculate $P(A)$.

EX6 An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls.
A single ball is drawn from each urn. The probability that both balls are the same colour is 0.44.
Calculate the number of blue balls in the second urn.