Sample answers

Session 7 Q1&Q2

Proposition 4.4 *Let X and Y be discrete random variables. Then X and Y are independent if and only if, for any values a_i and b_j of X and Y respectively, we have*

$$P(X = a_i \mid Y = b_j) = P(X = a_i).$$

Note that Proposition 4.4 holds only if for any b_j the probability $P(Y = b_j) > 0$

Q1. How to proof?

Q1.

"$\Rightarrow$" giving X and Y are independent, we would prove $\forall a_i, b_j$

having $P(X=a_i \mid Y=b_j) = P(X=a_i)$

X and Y independent

$\Rightarrow \forall a_i, b_j$, we have

$$P(X=a_i, Y=b_j) = P(X=a_i)\cdot P(X=b_j)$$

$$\Rightarrow P(X=a_i \mid Y=b_j)P(Y=b_j) = P(X=a_i)\cdot P(\overset{Y}{\cancel{X}}=b_j) \quad \text{(Bayesian)}$$

$$\Rightarrow P(X=a_i \mid Y=b_j) = P(X=a_i) \quad (\because P(Y=b_j)>0)$$

"$\Leftarrow$" giving $\forall a_i, b_j$ $P(X=a_i \mid Y=b_j)=P(X=a_i)$, we would prove X and Y are independent

$$\forall a_i, b_j \quad P(X=a_i \mid Y=b_j)=P(X=a_i)$$

$$\Rightarrow P(X=a_i \mid Y=b_j)\cdot P(Y=b_j)=P(X=a_i)\cdot P(Y=b_j)$$

$$\Rightarrow P(X=a_i, Y=b_j)=P(X=a_i)\cdot P(Y=b_j) \quad \text{(Bayesian)}$$

$\Rightarrow$ X and Y are independent (by definition)

Q2 An urn has 2 red, 5 white, and 3 green balls. Select 3 balls at random without replacement and let X be the number of red balls and Y the number of white balls. Determine
(a) joint p. m. f. of (X, Y),
(b) marginal p. m. f.'s, (c) $P(X \geq Y)$, and (d) $P(X = 2 | X \geq Y)$.

The joint p. m. f. is given by $P(X = x, Y = y)$ for all possible x and y. In our case, x can be $0, 1$, or 2 and y can be $0, 1, 2$, or 3. The values can be given by the formula

$$P(X = x, Y = y) = \frac{\binom{2}{x}\binom{5}{y}\binom{3}{3-x-y}}{\binom{10}{3}},$$

where we use the convention that $\binom{a}{b} = 0$ if $b > a$, or in the table:

$y \backslash x$	0	1	2	$P(Y = y)$
0	$1/120$	$2 \cdot 3/120$	$3/120$	$10/120$
1	$5 \cdot 3/120$	$2 \cdot 5 \cdot 3/120$	$5/120$	$50/120$
2	$10 \cdot 3/120$	$10 \cdot 2/120$	0	$50/120$
3	$10/120$	0	0	$10/120$
$P(X = x)$	$56/120$	$56/120$	$8/120$	1

The last row and column entries are the respective column and row sums and, therefore, determine the marginal p. m. f.'s. To answer (c) we merely add the relevant probabilities,

$$P(X \geq Y) = \frac{1+6+3+30+5}{120} = \frac{3}{8},$$

and, to answer (d), we compute

$$\frac{P(X=2, X \geq Y)}{P(X \geq Y)} = \frac{\frac{8}{120}}{\frac{3}{8}} = \frac{8}{45}.$$