Planar Graphs:

A graph G= (V, E) is said to be planar if it can be drawn in the plane so that no two edges of G intersect at a point other than a vertex. Such a drawing of a planar graph is called a planar embedding of the graph. For example, K_4 is planar since it has a planar embedding as shown in figure 1.8.1.

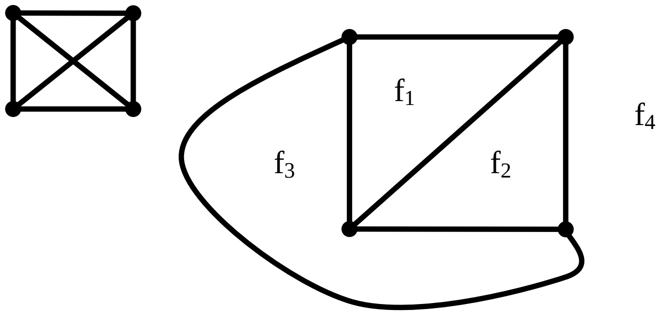
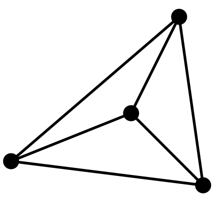


Figure 1.8.1: K_4 and a planar embedding of K_4.

In the above planar embedding of K_4, one edge is drawn as a curve whereas all edges are drawn as straight lines. Is it possible for a planar graph to have a planar embedding in which each edge is drawn as a straight line? The answer is yes as shown by Fary. We just state the result and omit the proof as the proof is complicated.

Theorem 1.7.1: Every planar graph admits a planar embedding in which each edge is drawn as a straight line segment.

For example, the following is such a planar embedding of K_4.



A planar embedding of K_4 in which each edge is drawn as a line segment.

Consider a planar embedding of a planar graph in a piece of paper. If we cut the paper along the edges of the planar embedding by a sharp knife, then the paper will be divided into pieces that are called the **regions or faces** of the planar graph with respect to the planar embedding. Formally, **a region or face** of a planar graph with respect to a planar embedding is an area of the plane that is bounded by edges and not further divided into sub areas. For example, the planar graph K_4 has four faces with respect to the planar embedding drawn in figure 1.8.1. The number of edges in the boundary of a face is called the length of that face. It can be seen from Figure 1.8.1 that each face is of length three of K_4. Since a planar embedding of a planar graph is drawn in the plane and the plane is unbounded, every planar embedding of a planar graph has an unbounded face. The face f_4 in the planar embedding of k_4 in Figure 1.7.1 is the unbounded face.

A planar graph can have more than one planar embeddings. For example, we have seen already two planar embeddings of K_4. The number of faces in each planar embedding may be different. However, this is not the case. It was shown by Euler's that the number of faces in any two planar embeddings of a planar graphs are same and related to the number of vertices and number of edges. So, once we know that a graph G is planar, and then we may say that G has such and such number of faces without referring to any planar embedding.

Theorem 1.8.1: (Euler Formula) For a connected planar graph G = (V, E) with n vertices, m edges and f faces, n - m + f = 2.

Proof: We prove the theorem by induction on m, the number of edges of G. If m=0, then n=1 as G is connected. So, f=1. Thus, n-m+f=1-0+1=2. So, the theorem is true foe m=0. Assume that, the theorem is true for all connected planar graphs with k or fewer edges. Let G be a connected planar graph with k+1 edges. If G has a vertex of degree 1, say x, then let G'=G-x. It is easy to see that G' is a connected planar graphs with k edges. Again, number of vertices of G' is one less than the number of vertices of G and number of faces of G' is equal to the number of vertices of G. So, by induction hypothesis, (n-1)-k + f = 2, where n and f are the number of vertices and faces of G, respectively. So, n-(k+1)-f=2. So, the theorem is true for G in this case. Next assume that G has no vertex of degree 1. Since, G is connected, $\delta(G) \geq 2$. So, by Theorem 1. 2.1, G has a cycle. Let e be any edge in some cycle in G. Let G'=G-e. Now G' is a connected planar graph and has same number of vertices as G and one less edges than G. The cycle containing the edge e determines a face in every planar embedding of G. Now, that cycle will be missing in G'. So, G' has f-1 faces and k edges. So, by induction hypothesis, n-k+f-1=2. This implies, n-(k+1)+f=2. So, the theorem is true for G in this case as well. So, by induction principle, the theorem is true.□

To show that a graph is planar, one has to produce a planar embedding of the graph. However, to show that a graph is non planar one has to show that either the graph satisfies a property that is not satisfied by any planar graph , or out of all possible diagrams of G, no one is a planar embedding. For large graphs it is not feasible show that a graph to be non planar by showing that none of the diagrams of the graph is a planar embedding. So, next we will see some properties of planar graphs and use these to show some graphs to be non planar.

A planar graph G =(V,E) is said to be a maximal planar graph if G + uv is non planar for every $u,v \in V$ with $uv \notin E$. So, every face is of length three in any planar embedding of a maximal planar graph. K_4 is a maximal planar graph which can be seen easily. In fact, a planar graph G is a maximal planar graph if and only if each face is of length three in any planar embedding of G.

Corollary 1.8.2: The number of edges in a maximal planar graph is 3n-6.

Proof: Let G be a maximal planar graph of order n, size m and has f faces. Note that G must be connected. So, by Euler's formula, n-m+f=2. Since, G is a maximal planar graph, each face is of size three in any planar embedding of G. So, the number of edges of G is 3f. But each edge is shared by exactly two faces, we have 2m=3f. So, m=n+f-2=n+(2/3)m-2. So, m=3n-6. □

Corollary 1.8.3: Let G be a planar graph of order n and size m. Then, $m \leq 3n-6$.
Proof: If G is not maximal planar, then keep on joining nonadjacent vertices of G so that the graph G' obtained from G by successively adding edges is maximal planar. Let m' be the size of G'. Then, by corollary 1.8.2, m'=3n-6. Since, $m \leq m'$, $m \leq 3n-6$. □

Corollary 1.8.4: There is a vertex of degree at most five in any planar graph G.
Proof: Let G be any planar graph having n vertices and m edges. If possible, suppose that G has no vertex of degree at most 5. So, each vertex of G is of degree 6 or more. Then, $\sum_{v \in V(G)} d(v) \geq 6n$. However, $\sum_{v \in V(G)} d(v) = 2m \leq 6n-12$ as G is a planar graph. So, $6n \leq 6n-12$, which is a contradiction. So, G must have at least one vertex of degree at most five. □

Corollary 1.8.5: The number of edges in a planar bipartite graph of order n is at most 2n-4.
Proof: Let G be a planar bipartite graph with n vertices and m edges. Consider a planar embedding of G. Since, G is bipartite, G has no cycle of length three. So, each face in the planar embedding contains at least four edges. Let N denote the number obtained by summing, over all faces, the number of edges on the boundary of a face. Then, $N \geq 4f$. Since each edge lies on the boundary of at most two faces, that is, $N \leq 2m$. So, $2m \geq N \geq 4f = 4(m-n+2) = 4m-4n+8$. So, $m \leq 2n-4$. □

Next, we show two important non-planar graphs.
Corollary 1.8.6: K_5 is non planar.
Proof: The proof is by contradiction. If possible, suppose that K_5 is planar. So, number edges in K_5 must be at most 9 according to corollary 1.8.2. However K_5 has 10 edges. This is a contradiction. So, K_5 is non-planar. □

Corollary 1.8.7: $K_{3,3}$ is non planar.

Proof: The proof is by contradiction. If possible, suppose that $K_{3,3}$ is planar. Now, $K_{3,3}$ is a bipartite graph with 6 vertices and 9 edges. So, by corollary 1.8.5, $K_{3,3}$ must have at most 12-4=8 edges. However, $K_{3,3}$ has 9 edges. This is a contradiction. So, $K_{3,3}$ is non planar. □

Let G=(V,E) be a graph. Let uv be an edge of G. The **subdivision of the edge** uv means replacing the edge uv by a two edges ux and xv by introducing a new vertex x of degree 2 in the graph. A graph G_1 is said to be a **subdivision** of G_2 if G_1 can be obtained from G_2 by a sequence of subdivisions of edges. Figure 1.9.3 contains four graphs G_1, G_2, G_3 and G_4. The graphs G_1 and G_2 are subdivisions of G_3, where as G_4 is not the subdivision of G_3. A graph G_1 is said to be **homeomorphic** to a graph G_2 if G_1 and G_2 are subdivisions of the same graph G_3. The graphs G_1 and G_2 of Figure 1.8.3 are hemeomorphic as they are subdivisions of the same graph G_3 of Figure 1.8.3.

If G_1 is planar, then any subdivision of G_1 is also planar. So, if G_1 and G_2 are homeomorphic, then G_1 and G_2 are both planar or both non planar. Also note that all the sub graphs of a planar are planar. Hence, once a graph contains a non planar subgraph, then the graph must be non planar. We have already seen than K_5 and $K_{3,3}$ are non planar graphs. So, any graph homeomorphic to K_5 or $K_{3,3}$ us also non planar. So, a planar graph can not contain a subgraph homeomorphic to K_5 or $K_{3,3}$. The converse of this is also true which was shown by Kurotoski. However, the proof is involved and hence we omit this. Below, we formally state a characterization of planar graph which is known as **Kurotoski's Theorem**.

Theorem 1.9.8: (Kurotoski's theorem) A graph G is planar if and only if it does not any subgraph homeomorphic to K_5 or $K_{3,3}$.

We illustrate the theorem by showing that the Peterson's graph is non planar. Note that, Peterson's graph satisfies all the necessary conditions of planar graphs discussed earlier. Figure 1.9.4 contains the Petersen graph and a subgraph of Petersen graph that is a subdivision of $K_{3,3}$ and hence is homeomorphic to $K_{3,3}$. So, by Theorem 1.9.8, the Petersen graph is non planar.

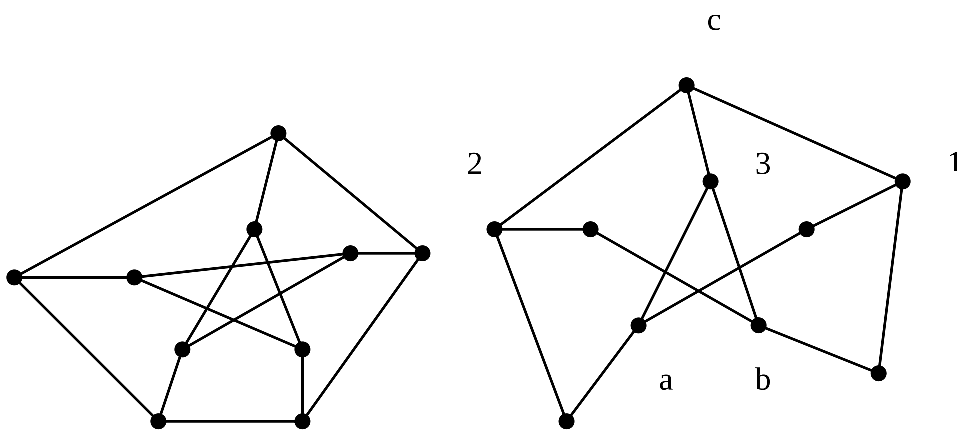


Figure 1.9.4: Petersen graph and a subgraph homeomorphic to $K_{3,3}$.

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