Sample answers

Session 4

Q1 An urn contains 20 balls numbered 1,2,…,20. Select 5 balls at random, without replacement. Let X be the largest number among selected balls.
Determine its p. m. f. and the probability that at least one of the selected numbers is 15 or more.

The possible values are $5, \ldots, 20$. To determine the p. m. f., note that we have $\binom{20}{5}$ outcomes, and, then,

$$P(X = i) = \frac{\binom{i-1}{4}}{\binom{20}{5}}.$$

Finally,

$$P(\text{at least one number 15 or more}) = P(X \geq 15) = \sum_{i=15}^{20} P(X = i) = 1 - \frac{\binom{14}{5}}{\binom{20}{5}}.$$

Q2 Let X be the number shown on a rolled fair die. Compute $EX, E(X^2)$, and $Var(X)$.

This is a standard example of a discrete uniform random variable and

$$EX = \frac{1+2+\ldots+6}{6} = \frac{7}{2},$$
$$EX^2 = \frac{1+2^2+\ldots+6^2}{6} = \frac{91}{6},$$
$$\mathrm{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

Q3 A car dealership sells 0, 1, or 2 luxury cars on any day. When selling a car, the dealer also tries to persuade the customer to buy an extended warranty for the car. Let X denote the number of luxury cars sold in a given day, and let Y denote the number of extended warranties sold.

$$
\begin{aligned}
P[X = 0, Y = 0] &= 1/6 \\
P[X = 1, Y = 0] &= 1/12 \\
P[X = 1, Y = 1] &= 1/6 \\
P[X = 2, Y = 0] &= 1/12 \\
P[X = 2, Y = 1] &= 1/3 \\
P[X = 2, Y = 2] &= 1/6
\end{aligned}
$$

Calculate the variance of X.

Based on the joint p.m.f of (X, Y), the p.m.f of X is

a	0	1	2
$P(X = a)$	$P(X = 0) = P(X = 0, Y = 0) = 1/6$	$P(X = 1) = P(X = 1, Y = 0) + P(X = 1, Y = 1) = \frac{1}{12} + \frac{1}{6} = \frac{3}{12}$	$P(X = 2) = P(X = 2, Y = 0) + P(X = 2, Y = 1) + P(X = 2, Y = 2) = \frac{1}{12} + \frac{1}{3} + \frac{1}{6} = \frac{7}{12}$

The expected value of X is

$E(X) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{3}{12} + 2 \cdot \frac{7}{12} = \frac{17}{12}$

Now we need to compute $E(X^2)$

$$E(X^2) = 0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{3}{12} + 2^2 \cdot \frac{7}{12} = \frac{31}{12}$$

So, the variance of X is

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = \frac{31}{12} - \left(\frac{17}{12}\right)^2 = \frac{83}{144}$$

EX1 An urn contains 11 balls, 3 white, 3 red, and 5 blue balls. Take out 3 balls at random, without replacement. You win \$1 for each red ball you select and lose \$1 for each white ball you select. Determine the p. m. f. of X, the amount you win.

The number of outcomes is $\binom{11}{3}$. X can have values -3, -2, -1, 0, 1, 2, and 3. Let us start with 0. This can occur with one ball of each color or with 3 blue balls:

$$P(X=0) = \frac{3 \cdot 3 \cdot 5 + \binom{5}{3}}{\binom{11}{3}} = \frac{55}{165}.$$

To get $X = 1$, we can have 2 red and 1 white, or 1 red and 2 blue:

$$P(X=1) = P(X=-1) = \frac{\binom{3}{2}\binom{3}{1} + \binom{3}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{39}{165}.$$

The probability that $X = -1$ is the same because of symmetry between the roles that the red and the white balls play. Next, to get $X = 2$ we must have 2 red balls and 1 blue:

$$P(X=-2) = P(X=2) = \frac{\binom{3}{2}\binom{5}{1}}{\binom{11}{3}} = \frac{15}{165}.$$

Finally, a single outcome (3 red balls) produces $X = 3$:

$$P(X=-3) = P(X=3) = \frac{1}{\binom{11}{3}} = \frac{1}{165}.$$

EX2 Rolling a fair die twice. Let the random variable X be the maximum of the two numbers obtained, and let Y be the modulus of their difference (that is, the value of Y is the larger number minus the smaller number).
(a) Write down the joint p.m.f. of (X, Y).
(b) Write down the p.m.f. of X, and calculate its expected value and its variance.
(c) Write down the p.m.f. of Y, and calculate its expected value and its variance.
(d) Are the random variables X and Y independent?

Solution (a)

		Y					
		0	1	2	3	4	5
	1	$\frac{1}{36}$	0	0	0	0	0
	2	$\frac{1}{36}$	$\frac{2}{36}$	0	0	0	0
	3	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	0	0	0
X	4	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	0	0
	5	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	0
	6	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$

The best way to produce this is to write out a 6×6 table giving all possible values for the two throws, work out for each cell what the values of X and Y are, and then count the number of occurrences of each pair. For example: $X = 5$, $Y = 2$ can occur in two ways: the numbers thrown must be $(5, 3)$ or $(3, 5)$.

(b) Take row sums:

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Hence in the usual way

$$E(X) = \frac{161}{36}, \qquad \mathrm{Var}(X) = \frac{2555}{1296}.$$

(c) Take column sums:

y	0	1	2	3	4	5
$P(Y=y)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

and so

$$E(Y) = \frac{35}{18}, \qquad \mathrm{Var}(Y) = \frac{665}{324}.$$

(d) No: e.g. $P(X=1, Y=2) = 0$ but $P(X=1) \cdot P(Y=2) = \frac{8}{1296}$.

EX3 Let X and Y be discrete random variables with joint probability function

$$p(x,y)=\begin{cases}\dfrac{1}{21}, & \text{for } x=0,1,\ldots,5 \text{ and } y=0,\ldots,x\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the variance of Y.

x \ y	0	1	2	3	4	5
0	$\frac{1}{21}$	0	0	0	0	0
1	$\frac{1}{21}$	$\frac{1}{21}$	0	0	0	0
2	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	0	0	0
3	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	0	0
4	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	
5	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$

$$p(y)=\begin{cases}6/21 & y=0\\ 5/21 & y=1\\ 4/21 & y=2\\ 3/21 & y=3\\ 2/21 & y=4\\ 1/21 & y=5\end{cases}$$

$E(Y) = 0*6/21 + 1*5/21 + 2*4/21 + 3*3/21 + 4*2/21 + 5*1/21 = 1.67$

$E(Y^2) = 0*6/21 + 1*5/21 + 4*4/21 + 9*3/21 + 16*2/21 + 25*1/21 = 5$

$Var(Y) = 5 - (1.67)^2 = 2.22$