Calculus
Exercise 1 Solution

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A.

1. (a) $f(1) = 3$
   (b) $f(-1) = -0.4$
   (c) $f(0) = 1$ and $f(3) = 1$
   (d) $f(-0.6) \approx 0$
   (e) $D : x \in [-2, 4] \quad R : y \in [-1, 3]$
   (f) $x \in (-2, 1)$

2. (a) No.
   (b) Yes. $D : x \in [-2, 2] \quad R : y \in [-1, 2]$
   (c) Yes. $D : x \in [-3, 2] \quad R : y \in [-3, -2) \cup [-1, 3]$

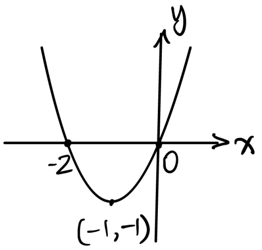
3. $\because f(x) = \frac{1}{x}$
   $\therefore \frac{f(x)-f(a)}{x-a} = \frac{\frac{1}{x}-\frac{1}{a}}{x-a} = \frac{\frac{a-x}{ax}}{x-a} = \frac{-(x-a)}{(x-a)ax} = -\frac{1}{ax}$

4. (a) $D : x \in (-\infty, +\infty)$
   (b) $\begin{cases} 2-\sqrt{p} \geqslant 0 \\ p \geqslant 0 \end{cases} \Rightarrow \begin{cases} \sqrt{p} \leqslant 2 \\ p \geqslant 0 \end{cases} \Rightarrow \begin{cases} p \leqslant 4 \\ p \geqslant 0 \end{cases} \quad \therefore D : p \in [0, 4]$

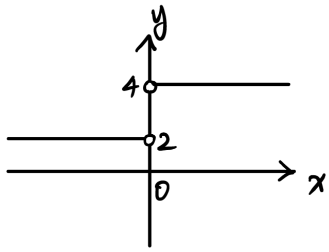
5. (a) $f(t) = 2t + t^2 = t(t+2)$
   $D : x \in (-\infty, +\infty)$
   (b) $G(x) = \frac{3x+|x|}{x} = \begin{cases} 4 \text{ if } x > 0 \\ 2 \text{ if } x < 0 \end{cases}$
   $D : x \in (-\infty, 0) \cup (0, +\infty)$
   (c) $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$
   $D : x \in (-\infty, +\infty)$

6. $f(x) = a(x+1)(x-0)(x-2)$
   $f(1) = 6$
   $\therefore \quad a(1+1)(1-0)(1-2) = 6, \quad a = -3$
   $\therefore \quad y = -3x(x+1)(x-2)$
   $\quad = -3x^3 + 3x^2 + 6x$

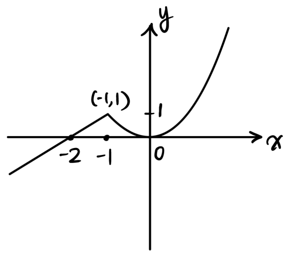
(a)



(b)

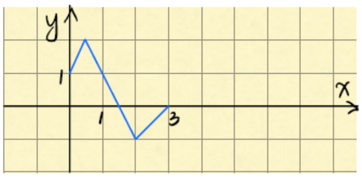


(c)

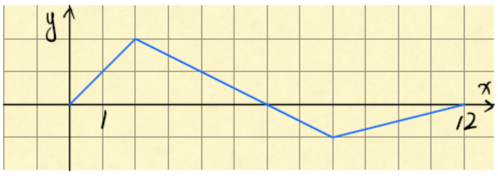


7. Solutions:

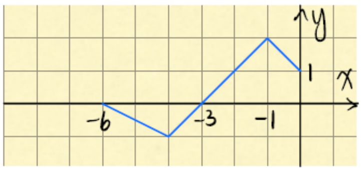
a)



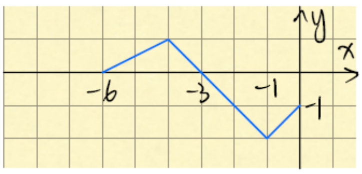
b)



c)



d)



8. $f(x) = x^3 + 2x^2, g(x) = 3x^2 - 1$

   (a) $(f+g)(x) = x^3 + 5x^2 - 1, \quad D : x \in R$

   (b) $(f-g)(x) = x^3 - x^2 + 1, \quad D : x \in R$

   (c) $(fg)(x) = 3x^5 + 6x^4 - x^3 - 2x^2, \quad D : x \in R$

   (d) $(f/g)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}, \quad D : x \in (-\infty, -\dfrac{\sqrt{3}}{3}) \cup (-\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}) \cup (\dfrac{\sqrt{3}}{3}, +\infty)$

9. $$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) \\ &= f\left(g\left(x^2\right)\right) \\ &= f\left(\sin\left(x^2\right)\right) \\ &= 3\sin\left(x^2\right) - 2\end{aligned}$$

10. (a) $\because F(x) = (2x + x^2)^4 = (f \circ g)(x) = f(g(x))$

$\therefore f(x) = x^4, \quad g(x) = 2x + x^2$

(b) $\because F(x) = \dfrac{\sqrt[3]{x}}{1 + \sqrt[3]{x}} = f(g(x))$

$\therefore f(x) = \dfrac{x}{1 + x}, \quad g(x) = \sqrt[3]{x}$

11. (a) -1 (c) DNE (e) 0 (g) 1

(b) -2 (d) 2 (f) DNE (h) 3

12. (a) $\displaystyle\lim_{x \to -3^+} \frac{x+2}{x+3} = -\infty$

(b) $\displaystyle\because \lim_{x \to 1^+} \frac{2-x}{(x-1)^2} = +\infty, \quad \lim_{x \to 1^-} \frac{2-x}{(x-1)^2} = +\infty$

$\displaystyle\therefore \lim_{x \to 1} \frac{2-x}{(x-1)^2} = +\infty$

(c) $\displaystyle\lim_{x \to -2^+} \frac{x-1}{x^2(x+2)} = -\infty$

(d) $\displaystyle\lim_{x \to 2\pi^-} x \csc x = -\infty$

(e) $\displaystyle\lim_{x \to 2^+} \frac{x^2 - 2x - 8}{x^2 - 5x + 6} = \lim_{x \to 2^+} \frac{(x+2)(x-4)}{(x-2)(x-3)} = +\infty$

13. (a) $\displaystyle\lim_{x \to 2}[f(x) + 5g(x)] = \lim_{x \to 2} f(x) + 5 \lim_{x \to 2} g(x) = 4 + 5 \times (-2) = -6$

(b) $\displaystyle\lim_{x \to 2}[g(x)]^3 = \left[\lim_{x \to 2} g(x)\right]^3 = (-2)^3 = -8$

(c) $\displaystyle\lim_{x \to 2} \sqrt{f(x)} = \sqrt{\lim_{x \to 2} f(x)} = \sqrt{4} = 2$

(d) $\displaystyle\lim_{x \to 2} \frac{3f(x)}{g(x)} = \frac{3 \lim_{x \to 2} f(x)}{\lim_{x \to 2} g(x)} = \frac{(3)(4)}{-2} = -6$

(e) $\displaystyle\because \lim_{x \to 2} h(x) = 0,$

$\displaystyle\therefore \lim_{x \to 2} \frac{g(x)}{h(x)}$ DNE

(f) $\displaystyle\lim_{x \to 2} \frac{g(x)h(x)}{f(x)} = \frac{\lim_{x \to 2} g(x) \cdot \lim_{x \to 2} h(x)}{\lim_{x \to 2} f(x)} = \frac{(-2)(0)}{4} = 0$

14. (a) Let $f(x) = \dfrac{x^2 - 5x + 6}{x - 5} = \dfrac{(x-2)(x-3)}{x-5}$

$\displaystyle\because \lim_{x \to 5^+} \frac{(x-2)(x-3)}{x-5} = +\infty, \quad \lim_{x \to 5^-} \frac{(x-2)(x-3)}{x-5} = -\infty$

$\displaystyle\therefore \lim_{x \to 5} f(x)$ DNE.

(b) $\displaystyle\lim_{h \to 0} \frac{(-5+h)^2 - 25}{h} = \lim_{h \to 0} \frac{h^2 - 10h + 25 - 25}{h} = \lim_{h \to 0}(h - 10) = -10$

(c) $\displaystyle\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} \cdot \frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3} = \lim_{h \to 0} \frac{(9+h) - 9}{h(\sqrt{9+h} + 3)}$

$\displaystyle= \lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3} = \frac{1}{\sqrt{9+0} + 3} = \frac{1}{6}$

(d) $\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}\cdot\frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}=\lim_{t\to 0}\frac{(1+t)-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})}$
$=\lim_{t\to 0}\frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})}=\frac{2}{\sqrt{1}+\sqrt{1}}=1$

(e) $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\cdot\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}$
$=\lim_{t\to 0}\frac{1-(1+t)}{t\sqrt{1+t}(1+\sqrt{1+t})}=\frac{-1}{\sqrt{1}(1+\sqrt{1})}=-\frac{1}{2}$

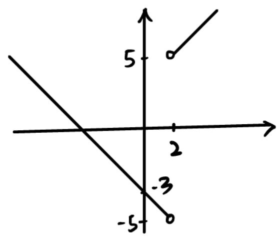
(f) $\because \lim_{x\to 3^+}(2x+|x-3|)=\lim_{x\to 3^+}(2x+x-3)=6$
$\lim_{x\to 3^-}(2x+|x-3|)=\lim_{x\to 3^-}(2x-x+3)=6$
$\therefore \lim_{x\to 3^-}(2x+|x-3|)=6$

15. (a) i. $\lim_{x\to 2^+}g(x)=\lim_{x\to 2^+}\frac{x^2+x-6}{x-2}=\lim_{x\to 2^+}\frac{(x-2)(x+3)}{x-2}=5$

ii. $\lim_{x\to 2^-}g(x)=\lim_{x\to 2^-}\frac{(x-2)(x+3)}{-(x-2)}=-5$

(b) $\because \lim_{x\to 2^+}g(x)\neq\lim_{x\to 2^-}g(x)$
$\therefore \lim_{x\to 2}g(x)$ DNE.

(c) $g(x)=\begin{cases} x+3 & \text{if } x>2\\ -x-3 & \text{if } x<2.\end{cases}$



16. (a) -2

(b) 2

(c) $+\infty$

(d) $-\infty$

(e) H.A. : $y=2, y=-2$
V.A. : $x=1, x=3$

17. (a) $\lim_{t\to+\infty}\frac{\sqrt{t}+t^2}{2t-t^2}\cdot\frac{t^{-2}}{t^{-2}}=\lim_{t\to+\infty}\frac{t^{-3/2}+1}{2t^{-1}-1}=-1$

(b) $\lim_{x\to+\infty}\frac{(2x^2+1)^2}{(x-1)^2(x^2+x)}=\lim_{x\to+\infty}\frac{4x^4+\text{ other terms}}{x^4+\text{ other terms}}=4$

(c) $\lim_{x\to+\infty}\left(\sqrt{9x^2+x}-3x\right)\cdot\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}=\lim_{x\to+\infty}\frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x}\cdot\frac{x^{-1}}{x^{-1}}$
$=\lim_{x\to+\infty}\frac{1}{\sqrt{9+(1/x)}+3}=\frac{1}{\sqrt{9+0}+3}=\frac{1}{6}$

(d) $\lim_{x\to+\infty} \frac{x^4-3x^2+x}{x^3-x+2} \cdot \frac{x^{-3}}{x^{-3}} = \lim_{x\to+\infty} \frac{x-3x^{-1}+x^{-2}}{1-x^{-2}+2x^{-3}} = +\infty$

18. We have $f(x) = \frac{g(x)}{h(x)} = \frac{a_1(x-1)(x+1)}{a_2(x-4)(x+1)}$

(a) $\because \lim_{x\to-1} f(x) = \frac{a_1(x-1)(x+1)}{a_2(x-4)(x+1)} = \frac{-2a_1}{-5a_2} = 2$
$\therefore a_1 = 5a_2$
Thus, $f(0) = \frac{5a_2(-1)}{a_2(-4)} = \frac{5}{4}$

(b) $\lim_{x\to+\infty} f(x) = \frac{a_1}{a_2} = \frac{5a_2}{a_2} = 5$

B.

1. $\because f, g$ are continuous,
$\therefore 3f(x) + f(x)g(x)$ is also continuous.
Thus,
$$\begin{aligned} \lim_{x\to 2}[3f(x) + f(x)g(x)] = 3f(2) + f(2)g(2) &= 36 \\ f(2)[3+g(2)] &= 36 \\ \therefore f(2) = \frac{36}{3+6} &= 4. \end{aligned}$$

2. $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 2+1 = 3$
$\therefore f(2)$ can be redefined as 3 , that is,
$$F(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq -2 \\ 3 & \text{if } x = 2 \end{cases} \quad \text{is continuous at } 2.$$

3. For $f(x) = \begin{cases} cx^2+2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x > 2 \end{cases}$ to be continuous on $(-\infty, +\infty)$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$ must be true. Thus,
$$\begin{aligned} \lim_{x\to 2^+}\left(x^3 - cx\right) &= \lim_{x\to 2^-}\left(cx^2+2x\right) \\ 8 - 2c &= 4c + 4 \\ c &= \frac{2}{3}. \end{aligned}$$