Sample answers

Session 3

Q1 Toss a coin 10 times. If you know (a) that exactly 7 Heads are tossed, (b) that at least 7 Heads are tossed, what are the probabilities that your first toss is Heads given (a) and (b), respectively?

For (a),

$$P(\text{first toss H}|\text{exactly 7 H's}) = \frac{\binom{9}{6} \cdot \frac{1}{2^{10}}}{\binom{10}{7} \cdot \frac{1}{2^{10}}} = \frac{7}{10}.$$

Why is this not surprising? Conditioned on 7 Heads, they are equally likely to occur on any given 7 tosses. If you choose 7 tosses out of 10 at random, the first toss is included in your choice with probability $\frac{7}{10}$.

For (b), the answer is, after canceling $\frac{1}{2^{10}}$,

$$\frac{\binom{9}{6} + \binom{9}{7} + \binom{9}{8} + \binom{9}{9}}{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}} = \frac{65}{88} \approx 0.7386.$$

Q2 2% of the population have a certain blood disease in a serious form; 10% have it in a mild form; and 88% don't have it at all. A new blood test is developed; the probability of testing positive is 9/10 if the subject has the serious form, 6/10 if the subject has the mild form, and 1/10 if the subject doesn't have the disease. I have just tested positive. What is the probability that I have this serious form of the disease?

Solution

Let A_1 be 'has disease in serious form', A_2 be 'has disease in mild form', and A_3 be 'doesn't have disease'.
Let B be 'test positive'. We need to compute $P(A_1|B)$
We are given that A_1, A_2, A_3 form a partition and

$$P(A_1) = 0.02 \qquad P(A_2) = 0.1 \qquad P(A_3) = 0.88$$
$$P(B \mid A_1) = 0.9 \quad P(B \mid A_2) = 0.6 \quad P(B \mid A_3) = 0.1$$

Thus, by the Theorem of Total Probability,

$$P(B) = 0.9 \times 0.02 + 0.6 \times 0.1 + 0.1 \times 0.88 = 0.166,$$

and then by Bayes' Theorem,

$$P(A_1 \mid B) = \frac{P(B \mid A_1)P(A_1)}{P(B)} = \frac{0.9 \times 0.02}{0.166} = 0.108$$

EX1 Toss two fair coins, blindfolded. Somebody tells you that you tossed at least one Heads. What is the probability that both tosses are Heads?

Here $A = \{\text{both H}\}$, $B = \{\text{at least one H}\}$, and

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\text{both H})}{P(\text{at least one H})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

EX2 Flip a fair coin. If you toss Heads, roll 1 die. If you toss Tails, roll 2 dice. Compute the probability that you roll exactly one 6.

Here you condition on the outcome of the coin toss, which could be Heads (event F) or Tails (event F^c). If $A = \{\text{exactly one } 6\}$, then $P(A|F) = \frac{1}{6}$, $P(A|F^c) = \frac{2 \cdot 5}{36}$, $P(F) = P(F^c) = \frac{1}{2}$ and so

$$P(A) = P(F)P(A|F) + P(F^c)P(A|F^c) = \frac{2}{9}.$$

EX3 A factory has three machines, M_1, M_2 and M_3, that produce items (say, lightbulbs). It is impossible to tell which machine produced a particular item, but some are defective. Here are the known numbers:

machine	proportion of items made	prob. any made item is defective
M_1	0.2	0.001
M_2	0.3	0.002
M_3	0.5	0.003

You pick an item, test it, and find it is defective. What is the probability that it was made by machine M_2?

Let D be the event that an item is defective and let M_i also denote the event that the item was made by machine i. Then, $P(D|M_1) = 0.001$, $P(D|M_2) = 0.002$, $P(D|M_3) = 0.003$, $P(M_1) = 0.2$, $P(M_2) = 0.3$, $P(M_3) = 0.5$, and so

$$P(M_2|D) = \frac{0.002 \cdot 0.3}{0.001 \cdot 0.2 + 0.002 \cdot 0.3 + 0.003 \cdot 0.5} \approx 0.26.$$

EX4 An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.
(ii) 70% of the customers insure more than one car.
(iii) 20% of the customers insure a sports car.
(iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

Let M = event that customer insures more than one car and S = event that customer insurers a sports car. Then applying DeMorgan's Law, compute the desired probability as:

$$\Pr\left(M^c \cap S^c\right) = \Pr\left[\left(M \cup S\right)^c\right] = 1 - \Pr\left(M \cup S\right) = 1 - \left[\Pr\left(M\right) + \Pr\left(S\right) - \Pr\left(M \cap S\right)\right]$$

$$= 1 - \Pr\left(M\right) - \Pr\left(S\right) + \Pr\left(S \middle| M\right)\Pr\left(M\right) = 1 - 0.70 - 0.20 + \left(0.15\right)\left(0.70\right) = 0.205$$

EX5 The Land of Nod lies in the monsoon zone, and has just two seasons, Wet and Dry. TheWet season lasts for 1/3 of the year, and the Dry season for 2/3 of the year. During the Wet season, the probability that it is raining is 3/4; during the Dry season, the probability that it is raining is 1/6.

(a) I visit the capital city, Oneirabad, on a random day of the year. What is the probability that it is raining when I arrive?

(b) I visit Oneirabad on a random day, and it is raining when I arrive. Given this information, what is the probability that my visit is during the Wet season?

(c) I visit Oneirabad on a random day, and it is raining when I arrive. Given this information, what is the probability that it will be raining when I return to Oneirabad in a year's time?

Solution (a) Let W be the event 'it is the wet season', D the event 'it is the dry season', and R the event 'it is raining when I arrive'. We are given that $P(W) = 1/3$, $P(D) = 2/3$, $P(R \mid W) = 3/4$, $P(R \mid D) = 1/6$. By the ToTP,

$$\begin{aligned} P(R) &= P(R \mid W)P(W) + P(R \mid D)P(D) \\ &= (3/4) \cdot (1/3) + (1/6) \cdot (2/3) = 13/36. \end{aligned}$$

(b) By Bayes' Theorem,

$$P(W \mid R) = \frac{P(R \mid W)P(W)}{P(R)} = \frac{(3/4)\cdot(1/3)}{13/36} = \frac{9}{13}.$$

(c) Let R' be the event 'it is raining in a year's time'. The information we are given is that $P(R \cap R' \mid W) = P(R \mid W)P(R' \mid W)$ and similarly for D. Thus

$$\begin{aligned} P(R \cap R') &= P(R \cap R' \mid W)P(W) + P(R \cap R' \mid D)P(D) \\ &= (3/4)^2 \cdot (1/3) + (1/6)^2 \cdot (2/3) = \frac{89}{432}, \end{aligned}$$

and so

$$P(R' \mid R) = \frac{P(R \cap R')}{P(R)} = \frac{89/432}{13/36} = \frac{89}{156}.$$

EX6 An actuary is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only this risk factor (and no others). For any two of the three factors, the probability is 0.12 that she has exactly these two risk factors (but not the other). The probability that a woman has all three risk factors, given that she has A and B, is 1/3. Calculate the probability that a woman has none of the three risk factors, given that she does not have risk factor A.

Let x be the probability of having all three risk factors.

$$\frac{1}{3} = P[A \cap B \cap C \mid A \cap B] = \frac{P[A \cap B \cap C]}{P[A \cap B]} = \frac{x}{x + 0.12}$$

It follows that

$$x = \frac{1}{3}(x + 0.12) = \frac{1}{3}x + 0.04$$

$$\frac{2}{3}x = 0.04$$

$$x = 0.06$$

Now we want to find

$$P\left[(A \cup B \cup C)^c \mid A^c\right] = \frac{P\left[(A \cup B \cup C)^c\right]}{P\left[A^c\right]}$$

$$= \frac{1 - P[A \cup B \cup C]}{1 - P[A]}$$

$$= \frac{1 - 3(0.10) - 3(0.12) - 0.06}{1 - 0.10 - 2(0.12) - 0.06}$$

$$= \frac{0.28}{0.60} = 0.467$$

Ex7 Many casinos allow you to bet eve
money on the following game. Two dic
are rolled and the sum S is observed.
If $S \in \{7,11\}$, you win immediately. If $S \in \{2,3,12\}$, you lose immediately.
If $S \in \{4,5,6,8,9,10\}$, the pair of dice is rolled repeatedly until one of the following happens:

- S repeats, in which case you win.
- 7 appears, in which case you lose.

What is the winning probability?

1. You win on the first roll with probability $\frac{8}{36}$.
2. Otherwise,

- you roll a 4 (probability $\frac{3}{36}$), then win with probability $\frac{\frac{3}{36}}{\frac{3}{36}+\frac{6}{36}} = \frac{3}{3+6} = \frac{1}{3}$;
- you roll a 5 (probability $\frac{4}{36}$), then win with probability $\frac{4}{4+6} = \frac{2}{5}$;
- you roll a 6 (probability $\frac{5}{36}$), then win with probability $\frac{5}{5+6} = \frac{5}{11}$;
- you roll a 8 (probability $\frac{5}{36}$), then win with probability $\frac{5}{5+6} = \frac{5}{11}$;
- you roll a 9 (probability $\frac{4}{36}$), then win with probability $\frac{4}{4+6} = \frac{2}{5}$;
- you roll a 10 (probability $\frac{3}{36}$), then win with probability $\frac{3}{3+6} = \frac{1}{3}$.

Using Bayes' formula,

$$P(\text{win}) = \frac{8}{36} + 2\left(\frac{3}{36}\cdot\frac{1}{3} + \frac{4}{36}\cdot\frac{2}{5} + \frac{5}{36}\cdot\frac{5}{11}\right) = \frac{244}{495} \approx 0.4929,$$

a decent game by casino standards.