Sample answers

Session 6

Exercises

EX1 An archer shoots an arrow at a target. The distance of the arrow from the centre of the target is a random variable X whose p.d.f. is given by

$$f_X(x) = \begin{cases} (3+2x-x^2)/9 & \text{if } x \leq 3, \\ 0 & \text{if } x > 3. \end{cases}$$

The archer's score is determined as follows:

Distance	$X < 0.5$	$0.5 \leq X < 1$	$1 \leq X < 1.5$	$1.5 \leq X < 2$	$X \geq 2$
Score	10	7	4	1	0

Construct the probability mass function for the archer's score, and find the archer's expected score.

Solution First we work out the probability of the arrow being in each of the given bands:

$$\begin{aligned} P(X < 0.5) = F_X(0.5) - F_X(0) &= \int_0^{0.5} \frac{3+2x-x^2}{9}\,\mathrm{d}x \\ &= \left[\frac{9x+3x^2-x^3}{27}\right]_0^{1/2} \\ &= \frac{41}{216}. \end{aligned}$$

Similarly we find that $P(0.5 \leq X < 1) = 47/216$, $P(1 \leq X < 1.5) = 47/216$, $P(1.5 \leq X < 2) = 41/216$, and $P(X \geq 2) = 40/216$. So the p.m.f. fot the archer's score S is

s	0	1	4	7	10
$P(S=s)$	$\frac{40}{216}$	$\frac{41}{216}$	$\frac{47}{216}$	$\frac{47}{216}$	$\frac{41}{216}$

Hence

$$E(S) = \frac{41 + 47\cdot 4 + 47\cdot 7 + 41\cdot 10}{216} = \frac{121}{27}.$$

EX2 Let T be the lifetime in years of new bus engines. Suppose that T is continuous with probability density function

$$f_T(x) = \begin{cases} 0 & \text{for } x < 1 \\ \dfrac{d}{x^3} & \text{for } x > 1 \end{cases}$$

for some constant d.
(a) Find the value of d.
(b) Find the mean and median of T.
(c) Suppose that 240 new bus engines are installed at the same time, and that their lifetimes are independent. By making an appropriate approximation, find the probability that at most 10 of the engines last for 4 years or more.

Solution (a) The integral of $f_T(x)$, over the support of T, must be 1. That is,

$$\begin{aligned} 1 &= \int_1^{\infty} \frac{d}{x^3}\,\mathrm{d}x \\ &= \left[\frac{-d}{2x^2}\right]_1^{\infty} \\ &= d/2, \end{aligned}$$

so $d = 2$.

(b) The c.d.f. of T is obtained by integrating the p.d.f.; that is, it is

$$F_T(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1 - \dfrac{1}{x^2} & \text{for } x > 1 \end{cases}$$

The mean of T is

$$\int_1^\infty x f_T(x)\,\mathrm{d}x = \int_1^\infty \frac{2}{x^2}\,\mathrm{d}x = 2.$$

The median is the value m such that $F_T(m) = 1/2$. That is, $1 - 1/m^2 = 1/2$, or $m = \sqrt{2}$.

(c) The probability that an engine lasts for four years or more is

$$1 - F_T(4) = 1 - \left(1 - \left(\frac{1}{4}\right)^2\right) = \frac{1}{16}.$$

So, if 240 engines are installed, the number which last for four years or more is a binomial random variable $X \sim \mathrm{Bin}(240, 1/16)$, with expected value $240 \times (1/16) = 15$ and variance $240 \times (1/16) \times (15/16) = 225/16$.

We approximate X by $Y \sim N(15, (15/4)^2)$. Using the continuity correction, $P(X \leq 10) \approx P(Y \leq 10.5)$.

Now, if $Z = (Y - 15)/(15/4)$, then $Z \sim N(0, 1)$, and

$$\begin{aligned} P(Y \leq 10.5) &= P(Z \leq -1.2) \\ &= 1 - P(Z \leq 1.2) \\ &= 0.1151 \end{aligned}$$

using the table of the standard normal distribution.

Note that we start with the continuous random variable T, move to the discrete random variable X, and then move on to the continuous random variables Y and Z, where finally Z is standard normal and so is in the tables.

EX3 Assume that a lightbulb lasts on average 100 hours. Assuming exponential distribution, compute the probability that it lasts more than 200 hours and the probability that it lasts less than 50 hours.

Let X be the waiting time for the bulb to burn out. Then, X is Exponential with $\lambda = \frac{1}{100}$ and

$$P(X \geq 200) = e^{-2} \approx 0.1353,$$

$$P(X \leq 50) = 1 - e^{-\frac{1}{2}} \approx 0.3935.$$

EX4 How many times do you need to toss a fair coin to get at least 100 heads with probability at least 90%?

Let n be the number of tosses that we are looking for. For S_n, which is Binomial$(n, \frac{1}{2})$, we need to find n so that

$$P(S_n \geq 100) \approx 0.9.$$

We will use below that $n > 200$, as the probability would be approximately $\frac{1}{2}$ for $n = 200$ (see the previous example). Here is the computation:

$$
\begin{aligned}
P\left(\frac{S_n - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}} \geq \frac{100 - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}}\right) &\approx P\left(Z \geq \frac{100 - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}}\right) \\
&= P\left(Z \geq \frac{200 - n}{\sqrt{n}}\right) \\
&= P\left(Z \geq -\left(\frac{n - 200}{\sqrt{n}}\right)\right) \\
&= P\left(Z \leq \frac{n - 200}{\sqrt{n}}\right) \\
&= \Phi\left(\frac{n - 200}{\sqrt{n}}\right) \\
&= 0.9
\end{aligned}
$$

Now, according to the tables, $\Phi(1.28) \approx 0.9$, thus we need to solve $\frac{n-200}{\sqrt{n}} = 1.28$, that is,

$$n - 1.28\sqrt{n} - 200 = 0.$$

This is a quadratic equation in $\sqrt{n}$, with the only positive solution

$$\sqrt{n} = \frac{1.28 + \sqrt{1.28^2 + 800}}{2}.$$

Rounding up the number n we get from above, we conclude that $n = 219$. (In fact, the probability of getting at most 99 heads changes from about 0.1108 to about 0.0990 as n changes from 217 to 218.)

EX5 The lifetime of a machine part has a continuous distribution on the interval $(0, 40)$ with probability density function f, where $f\ (x)$ is proportional to $(10\ +\ x)^{-2}$. Calculate the probability that the lifetime of the machine part is less than five.

Let the random variable X be the future lifetime of a machine part. We know that the density of X has the form $f(x) = C(10 + x)^{-2}$ for $0 < x < 40$ (and it is equal to zero otherwise). First, determine the proportionality constant C from the condition $\int_0^{40} f(x)dx = 1$:

$$1 = \int_0^{40} f(x)dx = -C(10+x)^{-1} \Big|_0^{40} = \frac{2}{25}C$$

so that $C = 25/2 = 12.5$. Then,

$$P(X < 5) = \int_0^5 12.5(10+x)^{-2}\,dx = -12.5(10+x)^{-1} \Big|_0^5 = -0.8333 + 1.25 = 0.4167.$$

EX6 The working lifetime, in years, of a particular model of bread maker is normally distributed with mean 10 and variance 4. Calculate the 12th percentile of the working lifetime, in years.

Let X be normal with mean 10 and variance 4. Let Z have the standard normal distribution. Let p = 12th percentile. Then

$$0.12 = P(X \le p) = P\left(\frac{X-10}{2} \le \frac{p-10}{2}\right) = P\left(Z \le \frac{p-10}{2}\right).$$

From the tables, $P(Z \le -1.175) = 0.12$. Therefore,

$$\frac{p-10}{2} = -1.175;\ p-10 = -2.35;\ p = 7.65.$$