Sample answers

Session 4

Q1 An urn contains 20 balls numbered 1,2,…,20. Select 5 balls at random, without replacement. Let X be the largest number among selected balls.
Determine its p. m. f. and the probability that at least one of the selected numbers is 15 or more.

The possible values are $5, \ldots, 20$. To determine the p. m. f., note that we have $\binom{20}{5}$ outcomes, and, then,

$$P(X = i) = \frac{\binom{i-1}{4}}{\binom{20}{5}}.$$

Finally,

$$P(\text{at least one number 15 or more}) = P(X \geq 15) = \sum_{i=15}^{20} P(X = i) = 1 - \frac{\binom{14}{5}}{\binom{20}{5}}.$$

Q2 Let X be the number shown on a rolled fair die. Compute $EX, E(X^2)$, and $Var(X)$.

This is a standard example of a discrete uniform random variable and

$$EX = \frac{1+2+\ldots+6}{6} = \frac{7}{2},$$

$$EX^2 = \frac{1+2^2+\ldots+6^2}{6} = \frac{91}{6},$$

$$\mathrm{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

EX1 An urn contains 11 balls, 3 white, 3 red, and 5 blue balls. Take out 3 balls at random, without replacement. You win \$1 for each red ball you select and lose \$1 for each white ball you select. Determine the p. m. f. of X, the amount you win.

The number of outcomes is $\binom{11}{3}$. X can have values -3, -2, -1, 0, 1, 2, and 3. Let us start with 0. This can occur with one ball of each color or with 3 blue balls:

$$P(X=0) = \frac{3 \cdot 3 \cdot 5 + \binom{5}{3}}{\binom{11}{3}} = \frac{55}{165}.$$

To get $X = 1$, we can have 2 red and 1 white, or 1 red and 2 blue:

$$P(X=1) = P(X=-1) = \frac{\binom{3}{2}\binom{3}{1} + \binom{3}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{39}{165}.$$

The probability that $X = -1$ is the same because of symmetry between the roles that the red and the white balls play. Next, to get $X = 2$ we must have 2 red balls and 1 blue:

$$P(X=-2) = P(X=2) = \frac{\binom{3}{2}\binom{5}{1}}{\binom{11}{3}} = \frac{15}{165}.$$

Finally, a single outcome (3 red balls) produces $X = 3$:

$$P(X=-3) = P(X=3) = \frac{1}{\binom{11}{3}} = \frac{1}{165}.$$