Chapter 5 Quadratic Forms

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Quadratic Forms

A *quadratic function* $f : R \to R$ has the form $f(x) = a \cdot x^2$. Generalization of this notion to two variables is the *quadratic form*

$$Q(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2.$$

Here each term has degree 2 (the sum of exponents is 2 for all summands).

A quadratic form of three variables looks as

$$\begin{aligned} f(x_1, x_2, x_3) = & \\ & a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \\ & a_{21}x_2x_1 + a_{22}x_2^2 + a_{23}x_2x_3 + \\ & a_{31}x_1x_3 + a_{32}x_3x_2 + a_{33}x_3^2. \end{aligned}$$

A general quadratic form of n variables is a real-valued function $Q : R^n \to R$ of the form

$$\begin{array}{rccccccc} Q(x_1, x_2, ..., x_n) = & a_{11}x_1^2 & + & a_{12}x_1x_2 & + & ... & + & a_{1n}x_1x_n + \\ & a_{21}x_2x_1 & + & a_{22}x_2^2 & + & ... & + & a_{2n}x_2x_n + \\ & ... & ... & ... & ... & & & \\ & a_{n1}x_nx_1 & + & a_{n2}x_nx_2 & + & ... & + & a_{nn}x_n^2 \end{array}$$

In short $Q(x_1, x_2, ..., x_n) = \sum_{i,j}^n a_{ij}x_ix_j$.

As we see a quadratic form is determined by the matrix

$$A = \begin{pmatrix} a_{11} \; ... \; a_{1n} \\ .... \\ a_{n1} \; ... \; a_{nn} \end{pmatrix}.$$

Quadratic Forms

Matrix Representation of Quadratic Forms

Let $Q(x_1, x_2, ..., x_n) = \sum_{i,j}^{n} a_{ij}x_ix_j$ be a quadratic form with matrix A. Easy to see that

$$Q(x_1, ..., x_n) = (x_1, ..., x_n) \cdot \begin{pmatrix} a_{11} \; ... \; a_{1n} \\ .... \\ a_{n1} \; ... \; a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ .. \\ x_n \end{pmatrix}.$$

The matrix A is called the **matrix of the quadratic form**.

Equivalently $Q(x) = x^T \cdot A \cdot x$.

Example. The quadratic form $Q(x_1, x_2, x_3) = 5x_1^2 - 10x_1x_2 + x_2^2$ whose symmetric matrix is $A = \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}$ is the product of three matrices

$$(x_1, x_2, x_3) \cdot \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Example: Quadratic Form

Consider the following square matrix **A**:

$$\mathbf{A} = \begin{bmatrix} -3 & 5 \\ 4 & -2 \end{bmatrix}$$

We can compute the quadratic form by using the vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then,

$$\begin{aligned} \mathbf{x}^\intercal \mathbf{A} \mathbf{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -3x_1 + 5x_2 \\ 4x_1 - 2x_2 \end{bmatrix} \\ &= x_1(-3x_1 + 5x_2) + x_2(4x_1 - 2x_2) \\ &= -3x_1^2 + 5x_1x_2 + 4x_1x_2 - 2x_2^2 \\ &= -3x_1^2 + 9x_1x_2 - 2x_2^2 \end{aligned}$$

By looking at the exponents in the final expression, you can see why this is called a quadratic form or transformation of **A**.

Examples: Finding the Matrix of Quadratic Forms

For example, let's find the matrix of the quadratic form:

$$ax_1^2 + bx_1x_2 + cx_2^2 \Rightarrow \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}$$

$$-5x_1^2 + 8x_1x_2 + 9x_2^2 \Rightarrow \begin{bmatrix} -5 & 4 \\ 4 & 9 \end{bmatrix}$$

$$3x_1^2 + -4x_1x_2 + 6x_2^2 \Rightarrow \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

$$8x_1^2 + 7x_2^2 + -3x_3^2 + -6x_1x_2 + 4x_1x_3 + -2x_2x_3 \Rightarrow \begin{bmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

And with this knowledge, we can reverse this process by computing $x^T Ax$

Conic Section Properties

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Conic Section	Characteristic
Circle	$A = C \neq 0$
Ellipse	$A \neq C$, $AC > 0$
Parabola	Either $A = 0$ *or* $C = 0$, but not both
Hyperbola	$AC < 0$

Example: Transforming Quadratic Forms

Example 1: Transform the quadratic form $f = {x_1}^2 + 2x_1x_2 + 2x_1\,x_3 + 2{x_2}^2 + 4x_2\,x_3 + {x_3}^2$ into one with no cross-product term.

Example 2: Transform the quadratic form $f = 2x_1x_2 + 2x_1\,x_3 - 6\,x_2\,x_3$ into one with no cross-product term.

Orthogonally diagonalizable

Definition: An $n \times n$ matrix A is called orthogonally diagonalizable if there is an orthogonal matrix P such that $P^{-1}AP = P^T AP$ is diagonal.

Theorem 1 (Principle Axes Theorem): Suppose $\boldsymbol{A}$ is an $n \times n$ **symmetric matrix**. Then there exists an orthogonal matrix P and a diagonal matrix B such that $P^T AP = B$. That is, every symmetric matrix is **orthogonally diagonalizable**.

Matrix congruence

Two square matrices A and B over a field are called **congruent** if there exists an **invertible matrix $\boldsymbol{P}$** over the same field such that

$$P^T AP = B$$

where " T " denotes the matrix transpose.

Principal Axis Theorem

This leads us directly to the *Principal Axis Theorem*, which states that if we let A be a square $n \times n$ matrix, then there is an **orthogonal change of variable**, $\vec{x} = \vec{P}$, that transforms the quadratic form $x^T Ax$ into a quadratic form $y^T Dy$ with no cross-product term. And the columns of P are called the *principal axes* and determine the axis of rotation.

Steps for Making a Change of Variables

Don't fear. Here are the steps for making a change of variables:

1. **Make the matrix A** of the quadratic form like we did for the examples above.
2. **Find eigenvalues** by solving the characteristic equation.
3. **Identify each eigenspace** (i.e., find the eigenvectors that correspond to each eigenvalue)
4. **Check for orthogonality**, and if necessary, apply the Gram-Schmidt process
5. **Normalize** to create matrix P that is orthonormal
6. **Write P and D matrices** and use them to create the transformed quadratic equation without cross-product terms and identify the quadratic form.

Example: Transforming Quadratic Forms

Transform the quadratic form $Q(x) = 8x_1^2 + 6x_1x_2$ into one with no cross-product term.

First, we will find the matrix of the quadratic form like we did for the examples above.

$$Q(x) = 8x_1^2 + 6x_1x_2$$

$$\Rightarrow Q(x) = 8x_1^2 + 6x_1x_2 + 0x_2^2$$

$$\Rightarrow \begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix}$$

Next, we will find our eigenvalues.

$$\det \begin{bmatrix} 8-\lambda & 3 \\ & \\ 3 & 0-\lambda \end{bmatrix} = 0$$

$$(8-\lambda)(0-\lambda) - (3)(3) = 0$$

$$\lambda^2 - 8\lambda - 9 = 0$$

$$(\lambda - 9)(\lambda + 1) = 0$$

$$\lambda = 9, -1$$

Example: Transforming Quadratic Forms

Now, we will find the basis for the eigenspace associated with each eigenvector.

$$\lambda = 9 \Rightarrow \begin{bmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\lambda = -1 \Rightarrow \begin{bmatrix} 9 & 3 & 0 \\ 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \text{ or } \quad \vec{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Okay, so now that we have our two eigenvectors, we must verify that they are orthogonal.

$$\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{v_1} \underbrace{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{v_2} \quad v_1 \cdot v_2 = 0$$

Example: Transforming Quadratic Forms

Now, it is time to normalize our vectors to create an orthonormal P matrix.

$$\|v_1\| = \sqrt{(3)^2 + (1)^2} = \sqrt{10}$$

$$\Rightarrow u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$\|v_2\| = \sqrt{(-1)^2 + (3)^2} = \sqrt{10}$$

$$\Rightarrow u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$P = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}, \quad D = \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix}$$

Example: Transforming Quadratic Forms

And remembering that $A = PDP^{-1} = PDP^T$ from our previous lesson on symmetric matrices, we can verify our work as follows, knowing that the columns of P are our principal axes.

$$\begin{bmatrix} 8 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

Lastly, we write our new quadratic form and identify the conic section.

$$x^T Ax = y^T Dy$$

$$\Rightarrow \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= 9y_1y_1 - y_2y_2 \Rightarrow \underbrace{9y_1^2 - y_2^2}_{\text{Hyperbola}}$$

And that's all we have to do to transform the quadratic form!

Short summary

Theorem *(The Principal Axes Theorem) Every quadratic form f can be diagonalized. More specifically, if $f(x) = x^T Ax$ is a quadratic form in $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then there exists an orthogonal matrix Q such that*

$$f(x) = x^T Ax = \lambda_1 y_1^2 + ... + \lambda_n y_n^2$$

where $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = Q^T \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\lambda_1, ..., \lambda_n$ are the eigenvalues of the matrix A.

Example: Transforming Quadratic Forms

Example 1: Find an orthogonal transformation $x = Cy$ to transform the quadratic form

$$f = 17{x_1}^2 + 14\,{x_2}^2 + 14{x_3}^2 - 4x_1x_2 - 4x_1\,x_3 - 8x_2\,x_3$$

into one with no cross-product term.

Example 2: Transform the conic section $3x^2 + 2xy + 3y^2 - 8 = 0$

into one with no cross-product term.

Homework

Find an orthogonal transformation to convert the following quadratic form into one with no cross-product term.

(1) $f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_2x_3$;

(2) $f = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$;

(3) $f = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$.

Homework

(1)

$$P=(p_1,p_2,p_3)=\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

$$f=y_1^2+2y_2^2+3y_3^2$$

(2)

$$P=(p_1,p_2,p_3)=\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix},$$

$$f=-y_1^2-y_2^2+2y_3^2.$$

(3)

$$P=\begin{pmatrix} -\frac{2\sqrt{5}}{5} & \frac{2\sqrt{5}}{15} & -\frac{1}{3} \\ \frac{\sqrt{5}}{5} & \frac{4\sqrt{5}}{15} & -\frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3} \end{pmatrix},$$

$$f=y_1^2+y_2^2+10y_3^2$$

Definiteness of Quadratic Forms

A quadratic form of one variable is just a quadratic function $Q(x) = a \cdot x^2$.

If $a > 0$ then $Q(x) > 0$ for each nonzero x.

If $a < 0$ then $Q(x) < 0$ for each nonzero x.

So the sign of the coefficient a determines the sign of *one variable* quadratic form.

The notion of *definiteness* described bellow generalizes this phenomenon for multivariable quadratic forms.

Definiteness of Quadratic Forms

Generic Examples

The quadratic form $Q(x,y) = x^2 + y^2$ is *positive* for *all* nonzero (that is $(x,y) \neq (0,0)$) arguments (x,y). Such forms are called *positive definite.*

The quadratic form $Q(x,y) = -x^2 - y^2$ is *negative* for *all* nonzero arguments (x,y). Such forms are called *negative definite.*

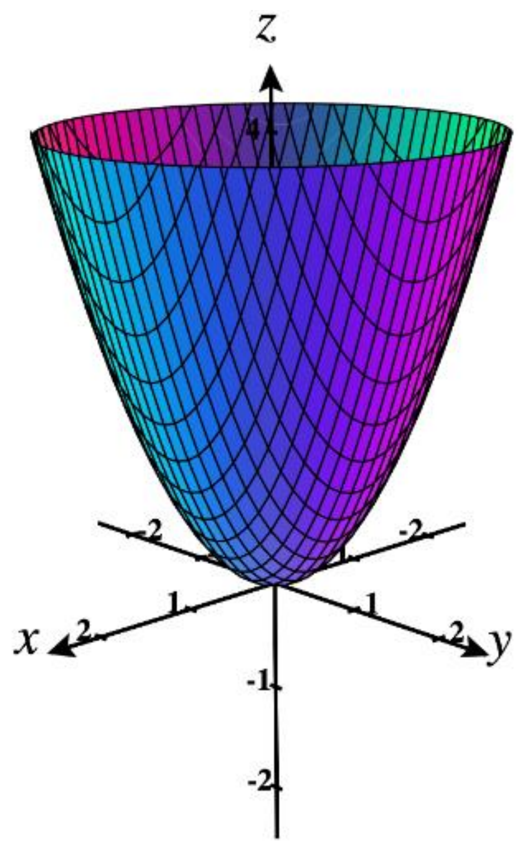
The quadratic form $Q(x,y) = (x-y)^2$ is *nonnegative.* This means that $Q(x,y) = (x-y)^2$ is either positive or zero for nonzero arguments. Such forms are called *positive semidefinite.*

The quadratic form $Q(x,y) = -(x-y)^2$ is *nonpositive.* This means that $Q(x,y) = (x-y)^2$ is either negative or zero for nonzero arguments. Such forms are called *negative semidefinite.*

The quadratic form $Q(x,y) = x^2 - y^2$ is called *indefinite* since it can take both positive and negative values, for example $Q(3,1) = 9 - 1 = 8 > 0$, $Q(1,3) = 1 - 9 = -8 < 0$.

Definiteness of Quadratic Forms

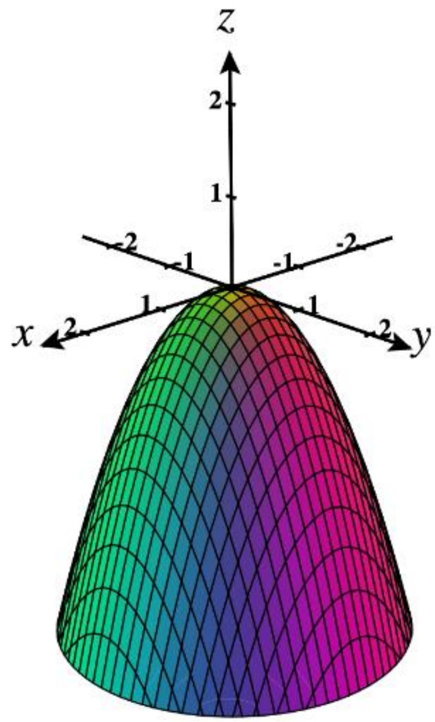
Example. The quadratic form $p(x, y) = x^2 + y^2$ is positive definite. Its graph is pictured below.



The graphs of other positive definite quadratic forms on $\mathbb{R}^2$ look similar, though they may be stretched in various directions. Notice that for a positive definite quadratic form, there is always a strict minimum at the origin.

Definiteness of Quadratic Forms

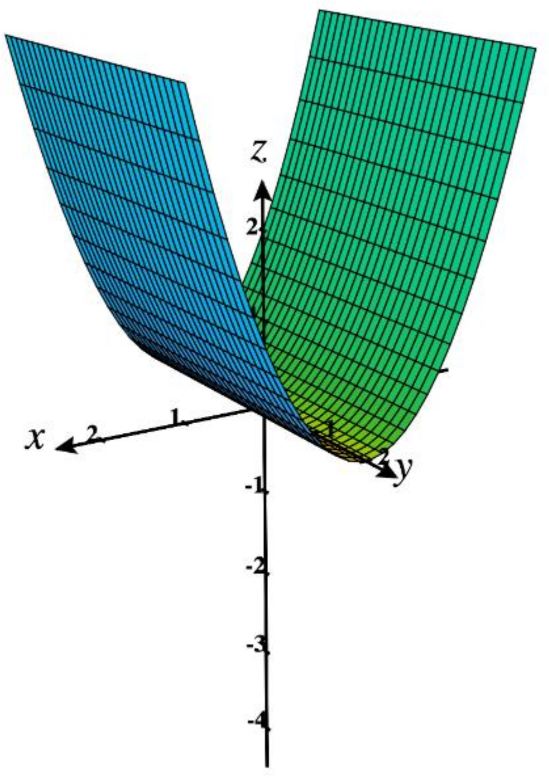
The quadratic form $p(x, y) = -x^2 - y^2$ is negative definite. Its graph is pictured below.



The graphs of other negative definite quadratic forms look similar, though they may be stretched in various directions. Notice that for a negative definite quadratic form, there is always a strict maximum at the origin.

Definiteness of Quadratic Forms

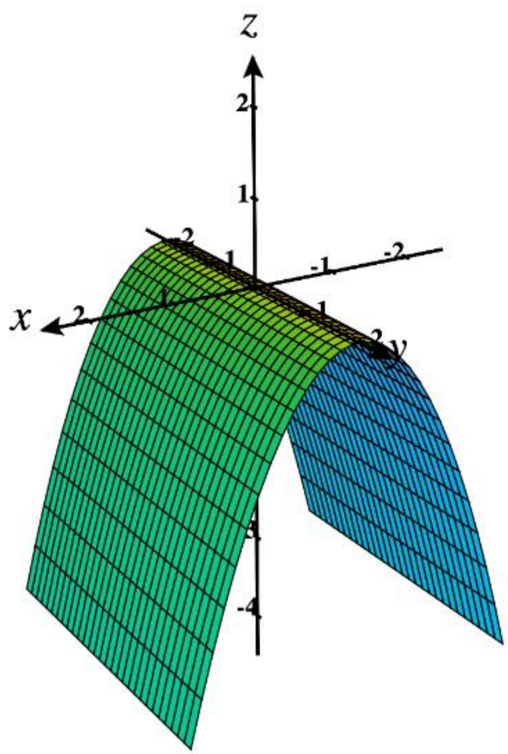
The quadratic form $p(x, y) = x^2$ is positive semi-definite. Its graph is pictured below.



The graphs of other positive semi-definite quadratic forms look similar, though they may be stretched in various directions.

Definiteness of Quadratic Forms

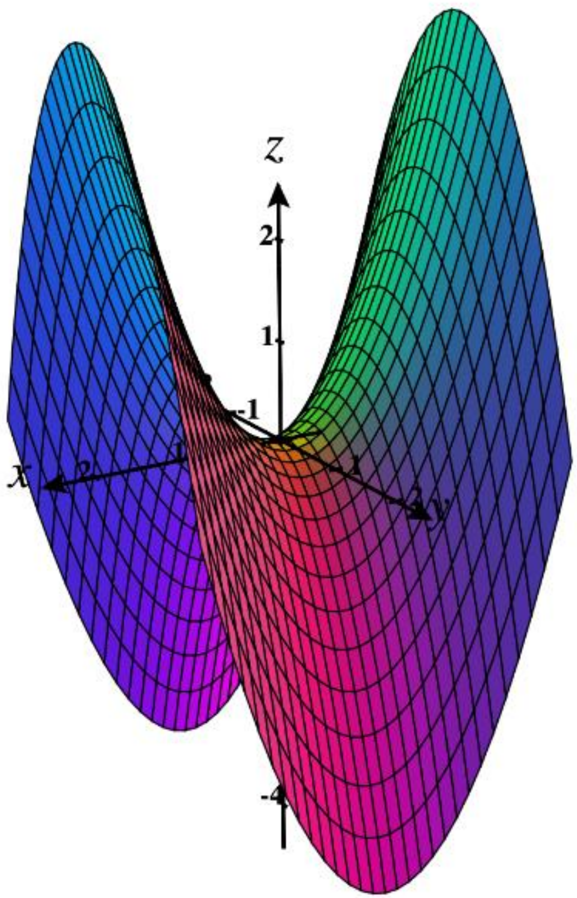
The quadratic form $p(x, y) = -x^2$ is negative semi-definite. Its graph is pictured below.



The graphs of other negative semi-definite quadratic forms look similar, though they may be stretched in various directions.

Definiteness of Quadratic Forms

The quadratic form $p(x, y) = x^2 - y^2$ is indefinite. Its graph is pictured below.



The graphs of other indefinite quadratic forms look similar, though they may be stretched in various directions. Notice the behavior of the graph around the origin; because of its shape, this is called a *saddle point*.

Definiteness of Quadratic Forms

Definition. A quadratic form $Q(x) = x^T \cdot A \cdot x$ (equivalently *a symmetric matrix* A) is
(a) **positive definite** if $Q(x) > 0$ for all $x \neq 0 \in R^n$;
(b) **positive semidefinite** if $Q(x) \geq 0$ for all $x \neq 0 \in R^n$;
(c) **negative definite** if $Q(x) < 0$ for all $x \neq 0 \in R^n$;
(d) **negative semidefinite** if $Q(x) \leq 0$ for all $x \neq 0 \in R^n$;
(e) **indefinite** if $Q(x) > 0$ for some x and $Q(x) < 0$ for some other x.

Definiteness of 2 Variable Quadratic Form

Let $Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 = (x_1, x_2) \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a 2 variable quadratic form.

Here $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is the symmetric matrix of the quadratic form. The determinant $\begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2$ is called *discriminant* of Q.

Easy to see that

$$ax_1^2 + 2bx_1x_2 + cx_2^2 = a(x_1 + \frac{b}{a}x_2)^2 + \frac{ac - b^2}{a}x_2^2.$$

Let us use the notation $D_1 = a$, $D_2 = ac - b^2$. Actually D_1 and D_2 are *leading principal minors* of A. Note that there exists one more principal (non leading) minor (of degree 1) $D_1' = c$.

Then

$$Q(x_1, x_2) = D_1(x_1 + \frac{b}{a}x_2)^2 + \frac{D_2}{D_1}x_2^2.$$

Definiteness of 2 Variable Quadratic Form

From this expression we obtain:

1. If $D_1 > 0$ and $D_2 > 0$ then the form is of $x^2 + y^2$ type, so it is *positive definite*;

2. If $D_1 < 0$ and $D_2 > 0$ then the form is of $-x^2 - y^2$ type, so it is *negative definite*;

3. If $D_1 > 0$ and $D_2 < 0$ then the form is of $x^2 - y^2$ type, so it is *indefinite*;

   If $D_1 < 0$ and $D_2 < 0$ then the form is of $-x^2 + y^2$ type, so it is also *indefinite*;

Thus if $D_2 < 0$ then the form is indefinite.

Definiteness of 2 Variable Quadratic Form

Semidefiniteness depends not only on leading principal minors $D_1, \ D_2$ but also on *all* principal minors, in this case on $D_1' = c$ too.

4. If $D_1 \geq 0, \ D_1' \geq 0$ and $D_2 \geq 0$ then the form is *positive semidefinite.*

Note that only $D_1 \geq 0$ and $D_2 \geq 0$ is not enough, the additional condition $D_1' \geq 0$ here is absolutely necessary: consider the form $Q(x_1, x_2) = -x_2^2$ with $a = 0, \ b = 0, \ c = -1$, here $D_1 = a \geq 0, \ D_2 = ac - b^2 \geq 0$, nevertheless the form is not positive semidiefinite.

5. If $D_1 \leq 0, \ D_1' \leq 0$ and $D_2 \geq 0$ then the form is *negative semidefinite.*

Note that only $D_1 \leq 0$ and $D_2 \geq 0$ is not enough,the additional condition $D_1' \leq 0$ again is absolutely necessary: consider the form $Q(x_1, x_2) = x_2^2$ with $a = 0, \ b = 0, \ c = 1$, here $D_1 = a \leq 0, \ D_2 = ac - b^2 \geq 0$, nevertheless the form is not negative semidiefinite.

Definiteness of 3 Variable Quadratic Form

Let us start with the following
Example. $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$. The symmetric matrix of this quadratic form is

$$\begin{pmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{pmatrix}.$$

The leading principal minors of this matrix are

$$|D_1| = \begin{vmatrix} 1 \end{vmatrix} = 1, \quad |D_2| = \begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = -2, \quad |D_3| = \begin{vmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{vmatrix} = -18.$$

Definiteness of 3 Variable Quadratic Form

Now look:

$$
\begin{aligned}
&Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3 = \\
&x_1^2 - 4x_1x_2 + 8x_1x_3 + 2x_2^2 - 7x_3^2 = x_1^2 - 4x_1(x_2 - 2x_3) + 2x_2^2 - 7x_3^2 = \\
&[x_1^2 - 4x_1(x_2 - 2x_3) + 4(x_2 - 2x_3) - 4(x_2 - 2x_3)] + 2x_2^2 - 7x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2x_2^2 - 16x_2x_3 - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2(x_2^2 - 8x_2x_3) - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2[x_2^2 - 8x_2x_3 + 16x_3^2 - 16x_3^2] - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2[x_2 - 4x_3]^2 - 16x_3^2) - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2[x_2 - 4x_3]^2 + 32x_3^2 - 23x_3^2 = \\
&[x_1 - 2x_2 + 4x_3]^2 - 2[x_2 - 4x_3]^2 + 9x_3^2 = \\
&|D_1| l_1^2 + \frac{|D_2|}{|D_1|} l_2^2 + \frac{|D_3|}{|D_2|} l_3^2
\end{aligned}
$$

where

$$
\begin{aligned}
l_1 &= x_1 \quad -2x_2 \quad +4x_3, \\
l_2 &= \qquad\quad\; x_2 \quad -4x_3, \\
l_3 &= \qquad\qquad\qquad\;\; x_3.
\end{aligned}
$$

Definiteness of 3 Variable Quadratic Form

That is (l_1, l_2, l_3) are linear combinations of (x_1, x_2, x_3). More precisely

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where

$$P = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

is a nonsingular matrix (changing variables).

Now turn to general 3 variable quadratic form

$$Q(x_1, x_2, x_3) = (x_1, x_2, x_3) \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The following three determinants

$$|D_1| = \begin{vmatrix} a_{11} \end{vmatrix}, \quad |D_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad |D_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

are *leading principal minors.*

Definiteness of 3 Variable Quadratic Form

It is possible to show that, as in 2 variable case, if $|D_1| \neq 0, \ |D_2| \neq 0$, then

$$Q(x_1, x_2, x_3) = |D_1| l_1^2 + \frac{|D_2|}{|D_1|} l_2^2 + \frac{|D_3|}{|D_2|} l_3^2$$

where l_1, l_2, l_3 are some linear combinations of x_1, x_2, x_3 (this is called **Lagrange's reduction**).

This implies the following criteria:

1. The form is positive definite iff $|D_1| > 0$, $|D_2| > 0$, $|D_3| > 0$, that is all principal minors are positive.

2. The form is negative definite iff $|D_1| < 0$, $|D_2| > 0$, $|D_3| < 0$, that is principal minors alternate in sign starting with negative one.

Definiteness of 3 Variable Quadratic Form

Example. Determine the definiteness of the form $Q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$.
Solution. The matrix of our form is

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$

The leading principal minors are

$$|D_1| = 3 > 0, \quad |D_2| = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} > 5, \quad |D_3| = \begin{vmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{vmatrix} = 18 > 0,$$

thus the form is positive definite.

Definiteness of n Variable Quadratic Form

Let $Q(x_1,...,x_n) = (x_1,...,x_n) \cdot \begin{pmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ be an n variable quadratic form.

The following n determinants

$$|D_1| = \begin{vmatrix} a_{11} \end{vmatrix}, \quad |D_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \; ... \; , \; |D_n| = \begin{vmatrix} a_{11} & ... & a_{1n} \\ a_{21} & ... & a_{2n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{vmatrix}$$

are *leading principal minors.*

As in previous cases, it is possible to show that

$$Q(x_1, \; ... \; , x_3) = |D_1| l_1^2 + \frac{|D_2|}{|D_1|} l_2^2 + \; ... \; + \frac{|D_n|}{|D_{n-1}|} l_n^2$$

where $(l_1, l_2, \; ... \; , l_n)$ are linear combinations of $(x_1, x_2, \; ... \; , x_n)$, more precisely

$$\begin{pmatrix} l_1 \\ ... \\ l_n \end{pmatrix} = \begin{pmatrix} p_{11} & ... & p_{1n} \\ . & . & . \\ p_{n1} & ... & p_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$$

where

$$P = \begin{pmatrix} p_{11} & ... & p_{1n} \\ . & . & . \\ p_{n1} & ... & p_{nn} \end{pmatrix}$$

is a nonsingular matrix (changing variables).

Definiteness of n Variable Quadratic Form

Theorem 2 *1. A quadratic form is positive definite if and only if*

$$|D_1| > 0, \quad |D_2| > 0, \quad \dots \quad , |D_n| > 0,$$

that is all principal minors are positive;

2. A quadratic form is negative definite if and only if

$$|D_1| < 0, \quad |D_2| > 0, \quad |D_3| < 0, \quad |D_4| > 0, \quad \dots \quad ,$$

that is principal minors alternate in sign starting with negative one.

3. If some kth order leading principal minor is nonzero but does not fit either of the above two sign patterns, then the form is indefinite.

Definiteness of n Variable Quadratic Form

The situation with semidefiniteness is more complicated, here are involved not only leading principal minors, but *all* principal minors.

Theorem 3 *1. A quadratic form is positive semidefinite if and only if all principal minors are* ≥ 0*;*

2. A quadratic form is negative semidefinite if and only if all principal minors of odd degree are ≤ 0*, and all principal minors of even degree are* ≥ 0.

Definiteness and Eigenvalues

As we know a symmetric $n \times n$ matrix has n real eigenvalues (maybe some multiple).

Theorem 4 *Given a quadratic form* $Q(x) = x^T Ax$ *and let* $\lambda_1, \ldots, \lambda_n$ *be eigenvalues of* A*. Then* $Q(x)$ *is*

- *positive definite iff* $\lambda_i > 0, i = 1, \ldots, n$;
- *negative definite iff* $\lambda_i < 0, i = 1, \ldots, n$;
- *positive semidefinite iff* $\lambda_i \geq 0, i = 1, \ldots, n$;
- *negative semidefinite iff* $\lambda_i \leq 0, i = 1, \ldots, n$;

Short summary

$$Q(x_1, x_2, ..., x_n) = \sum_{i,j}^{n} a_{ij}x_ix_j = (x_1, ..., x_n)\cdot \begin{pmatrix} a_{11} \ ... \ a_{1n} \\ .... \\ a_{n1} \ ... \ a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ .. \\ x_n \end{pmatrix} = x^T \cdot A \cdot x$$

A is symmetric. If not, take its symmetrization $A' = \frac{A+A^T}{2}$.

Definiteness of $Q(x)$:

(a) positive definite if $Q(x) > 0$ for all $x \neq 0 \in R^n$;
(b) positive semidefinite if $Q(x) \geq 0$ for all $x \neq 0 \in R^n$;
(c) negative definite if $Q(x) < 0$ for all $x \neq 0 \in R^n$;
(d) negative semidefinite if $Q(x) \leq 0$ for all $x \neq 0 \in R^n$;
(e) indefinite if $Q(x) > 0$ for some x and $Q(x) < 0$ for some other x.

Definiteness and Optimality

If Q is positive definite then $x = 0$ is global maximum;
If Q is negative definite then $x = 0$ is global minimum.

Short summary

Leading principal minors

$$|D_1| = \begin{vmatrix} a_{11} \end{vmatrix}, \quad |D_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \ \ldots, \ |D_n| = |A|.$$

A quadratic form $Q(x)$ is:

Positive definite iff $|D_1| > 0, \quad |D_2| > 0, \quad \ldots \quad, |D_n| > 0.$

Negative definite iff $|D_1| < 0, \quad |D_2| > 0, \quad |D_3| < 0, \quad |D_4| > 0, \quad \ldots \quad .$

Indefinite iff some nonzero D_k violates above sign patterns.

Positive semidefinite iff all principal minors $M_k \geq 0$.

Negative semidefinite iff all $M_{2k+1} \leq 0$ and $M_{2k} \geq 0$.

Exercises

Example 1. Determine the definiteness of the quadratic form $q(\vec{x}) = x_1^2 + 2x_1x_2 + x_2^2$.

Exercises

Example 1. Determine the definiteness of the quadratic form $q(\vec{x}) = x_1^2 + 2x_1x_2 + x_2^2$.

(*Solution*) This form can be written as

$$q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x},$$

so we'll compute the eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We have

$$\det(A - \lambda I) = \left| \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \right| = (1-\lambda)^2 - 1 = \lambda(\lambda - 2),$$

so the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 0$. Since both eigenvalues are non-negative, q takes on only non-negative values. Moreover, since $\lambda_2 = 0$, q has a nontrivial kernel, and is thus positive semi-definite. $\lozenge$

Exercises

Example 2. For which real numbers k is the quadratic form

$$q(\vec{x}) = kx_1^2 - 6x_1x_2 + kx_2^2$$

positive-definite?

Exercises

(*Solution*) To determine the definiteness of this form we'll need to consider the matrix

$$A = \begin{bmatrix} k & -3 \\ -3 & k \end{bmatrix},$$

whose characterstic polynomial is

$$\det(A - \lambda I) = \left| \begin{bmatrix} k - \lambda & -3 \\ -3 & k - \lambda \end{bmatrix} \right| = (k - \lambda)^2 - 9 = \lambda^2 - 2k\lambda + (k^2 - 9).$$

We can either factor this polynomial as

$$\det(A - \lambda I) = (\lambda - (k + 3))(\lambda - (k - 3))$$

or use the quadratic equation to find its roots:

$$\lambda = \frac{2k \pm \sqrt{4k^2 - 4(k^2 - 9)}}{2} = \frac{2k \pm \sqrt{36}}{2} = k \pm 3.$$

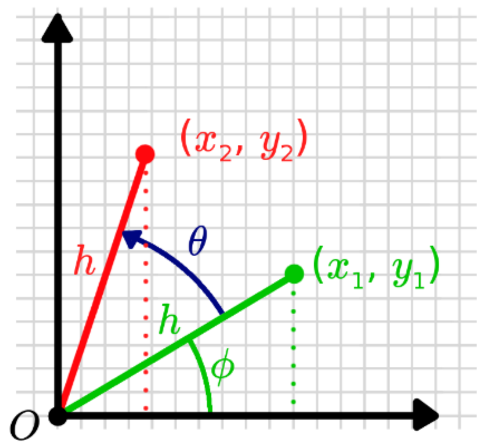
Whichever method we use, we find that $\lambda_1 = k + 3$ and $\lambda_2 = k - 3$. In order for q to be positive-definite, both of these eigenvalues must be positive, and in particular we must have $\lambda_2 > 0$. So $k > 3$ is a necessary and sufficient condition for q to be a positive-definite quadratic form. $\diamondsuit$

Appendix

Rotation Matrix in 2D

The process of rotating an object with respect to an angle in a two-dimensional plane is 2D rotation. We accomplish this rotation with the help of a 2 x 2 rotation matrix that has the standard form as given below:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$



If we want to rotate a vector with the coordinates (x_2, y_2) then we use matrix multiplication to perform the rotation as follows:

$$\begin{aligned} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} h\cos(\phi+\theta) \\ h\sin(\phi+\theta) \end{bmatrix} \\ &= \begin{bmatrix} h\cos(\phi)\cos(\theta) - h\sin(\phi)\sin(\theta) \\ h\sin(\phi)\cos(\theta) + h\cos(\phi)\sin(\theta) \end{bmatrix} \\ &= \begin{bmatrix} x_1\cos(\theta) - y_1\sin(\theta) \\ x_1\sin(\theta) + y_1\cos(\theta) \end{bmatrix} \end{aligned}$$

On solving this equation we get,

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Here, θ is the angle of rotation in the anti-clockwise direction.

Appendix

Identifying a Conic by Eliminating the Cross Product Term

(a) Identify the conic whose equation is $5x^2 - 4xy + 8y^2 - 36 = 0$ by rotating the xy-axes to put the conic in standard position.

(b) Find the angle θ through which you rotated the xy-axes in part (a).

Appendix

Solution (a) The given equation can be written in the matrix form

$$\mathbf{x}^T A\mathbf{x} = 36$$

where

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$$

The characteristic polynomial of A is

$$\begin{vmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 8 \end{vmatrix} = (\lambda - 4)(\lambda - 9)$$

so the eigenvalues are $\lambda = 4$ and $\lambda = 9$. We leave it for you to show that orthonormal bases for the eigenspaces are

$$\lambda = 4: \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \quad \lambda = 9: \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

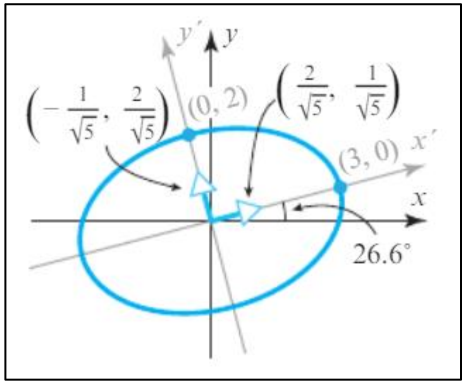
Thus, A is orthogonally diagonalized by

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Appendix

Moreover, it happens by chance that $\det(P) = 1$, so we are assured that the substitution $\mathbf{x} = P\mathbf{x}'$ performs a rotation of axes. It follows from (16) that the equation of the conic in the $x'y'$-coordinate system is

$$[x' \quad y']\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = 36$$



which we can write as

$$4x'^2 + 9y'^2 = 36 \quad \text{or} \quad \frac{x'^2}{9} + \frac{y'^2}{4} = 1$$

We can now see from Table 1 that the conic is an ellipse whose axis has length $2\alpha = 6$ in the x'-direction and length $2\beta = 4$ in the y'-direction.

Solution (b) It follows the transition matrix

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

which implies that

$$\cos\theta = \frac{2}{\sqrt{5}}, \quad \sin\theta = \frac{1}{\sqrt{5}}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{2}$$

Thus, $\theta = \tan^{-1}\frac{1}{2} \approx 26.6°$