1. $\int_0^{2\pi} \cos(mx)\cos(nx)dx$, where m, n are non-negative integers.
2. $\int_0^{2\pi} \cos(mx)\sin(nx)dx$, where m, n are non-negative integers.
3. $\int_0^{2\pi} \sin(mx)\sin(nx)dx$, where m, n are non-negative integers.
4. $\sum_{k=0}^{N-1} \cos(mx_k)\cos(nx_k)$, where m, n are non-negative integers, and $x_k = \frac{2k\pi}{N}$
5. $\sum_{k=0}^{N-1} \cos(mx_k)\sin(nx_k)$, where m, n are non-negative integers, and $x_k = \frac{2k\pi}{N}$
6. $\sum_{k=0}^{N-1} \sin(mx_k)\sin(nx_k)$, where m, n are non-negative integers, and $x_k = \frac{2k\pi}{N}$

$\cos(m+n)x = \cos mx \cos nx - \sin mx \sin nx$
$\cos(m-n)x = \cos mx \cos nx + \sin mx \sin nx$
$\sin(m+n)x = \sin mx \cos nx + \cos mx \sin nx$
$\sin(m-n)x = \sin mx \cos nx - \cos mx \sin nx$

$\cos mx \sin nx = \frac{1}{2}\left[\sin(m+n)x - \sin(m-n)x\right]$

1. $\cos(mx)\cos(nx) =$

$\cos($ $x)$

$\sin($

$\int_0^{2\pi} \cos mx \cos nx \, dx$

$= \int_0^{2\pi} \left[\cos(m+n)x + \cos(m-n)x\right]\frac{1}{2} \, dx$

if $m = n$

$= \int_0^{2\pi} \frac{1}{2}\cos 2mx + \frac{1}{2}\int_0^{2\pi} 1 \, dx$

$= 0 + \frac{2\pi}{2}$

$= \pi$

if $m \neq n$

$\frac{1}{2}\int_0^{2\pi} \cos(m+n)x + \cos(m-n)x \, dx$

$= \frac{1}{2}\int_0^{2\pi} \cos(m+n)x \, dx + \frac{1}{2}\int \cos(m-n)x \, dx$

$= \frac{1}{2} \left.\frac{\sin(m+n)x}{m+n}\right|_0^{2\pi} + \frac{1}{2} \left.\frac{\sin(m-n)x}{m-n}\right|_0^{2\pi}$

$\therefore$

$= 0 + 0$

$= 0$

$\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases}$ #

Where m, n are integers.

$\int_0$ co

2. $\int_0^{2\pi} \cos mx \sin nx \, dx$

$= \frac{1}{2}\int_0^{2\pi} \sin(m+n)x \, dx - \frac{1}{2}\int_0^{2\pi} \sin(m-n)x \, dx$

if $m = n$

$= \frac{1}{2}\int_0^{2\pi} \sin 2nx \, dx - \frac{1}{2}\int_0^{2\pi} \sin 0x \, dx$

$= 0$

if $m \neq n$

$= \frac{1}{2} \left.\frac{-\cos(m+n)x}{m+n}\right|_0^{2\pi} - \frac{1}{2} \left.\frac{-\cos(m-n)x}{m-n}\right|_0^{2\pi}$

$= 0$

$\int_0^{2\pi} \cos mx \sin nx \, dx = 0$ #

Where m, n are integers.

3. $\int_0^{2\pi} \sin mx \sin nx \, dx$

$= \frac{1}{2}\int_0^{2\pi} [\cos(m-n)x - \cos(m+n)x]\, dx$

$\cos^2 \pi k$

if $m=n$

$\frac{1}{2}\int_0^{2\pi} 1\, dx - \frac{1}{2}\int_0^{2\pi} \cos 2mx \, dx$

$= \pi$

if $m \neq n$

$= \frac{1}{2} \frac{\sin(m-n)x}{m-n}\Big|_0^{2\pi} - \frac{1}{2}\frac{\sin(m-n)x}{m-n}\Big|_0^{2\pi}$

$= 0$

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m \neq n. \end{cases}$$

$x_k = \frac{2\pi k}{N}$

4. $\sum_{k=0}^{N-1} \cos mx_k \cos nx_k = \frac{1}{2}\sum_{k=0}^{N-1} \cos(m+n)x_k + \cos(m-n)x_k$

$= \frac{1}{2}\sum_{k=0}^{N-1} \left[\cos(m+n)\frac{2\pi k}{N} + \cos(m-n)\frac{2\pi k}{N}\right]$

From 1.

$$\frac{2\pi}{N}\sum_{k=0}^{N-1} \frac{1}{2}\left[\cos(m+n)\frac{2\pi k}{N} + \cos(m-n)\frac{2\pi k}{N}\right] = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m=n \end{cases}$$

$$\Rightarrow \sum_{k=0}^{N-1} \cos mx_k \cos nx_k = \begin{cases} 0, & \text{if } m \neq n \\ \frac{N}{2}, & \text{if } m=n \end{cases}$$

but if $m=n=\frac{N}{2}$

$\sum_{k=0}^{N-1} \frac{1}{2}\left[\cos(m+n)\frac{2\pi k}{N} + \cos(m-n)\frac{2\pi k}{N}\right]$

$= \sum_{k=0}^{N-1} \frac{1}{2}[1 + 1]$

$= \sum_{k=0}^{N-1} 1$

$= N$ #

if $m=n=N$,

$\sum_{k=0}^{N-1} \frac{1}{2}\left[\cos(m+n)\frac{2k\pi}{N} + \cos(m-n)\frac{2\pi k}{N}\right]$

$= \sum_{k=0}^{N-1} \frac{1}{2}[\cos 4k\pi + \cos 0]$

$= N$.

if $m=n=0$ $\sum_{k=0}^{N-1} \cos mx_k \cos nx_k$

$= N$.

$$\sum_{k=0}^{N-1} \cos mx_k \sin nx_k = \begin{cases} 0, & \text{if } m \neq n \\ \frac{N}{2}, & \text{otherwise.} \\ N, & \text{if } m=n=\frac{N}{2}, \\ & m=n=rN,\ r \text{ is integer} \\ & m=n=0 \end{cases}$$

#

5. $\sum_{K=0}^{N-1} \cos m x_K \sin n x_K \qquad x_K = \frac{2\pi k}{N}$

$$\frac{2\pi}{N}\left[\sum_{K=0}^{N-1} \cos m x_K \sin n x_K\right] = 0. \text{ (From 2.)}$$

$$\Rightarrow \sum_{K=0}^{N-1} \cos m x_K \sin n x_K = 0$$

6. $\sum_{K=0}^{N-1} \sin m x_K \sin n x_K \qquad x_K = \frac{2\pi k}{N}$

From 3,

$$\frac{2\pi}{N}\sum_{K=0}^{N-1} \sin m x_K \sin n x_K = \begin{cases} \pi, & \text{if } m=n \\ 0, & \text{if } m\neq n \end{cases}$$

$$\Rightarrow \sum_{K=0}^{N-1} \sin m x_K \sin n x_K = \begin{cases} \frac{N}{2}, & \text{if } m=n \\ 0. & \text{if } m\neq n. \end{cases}$$