1: Homogenous Poisson Process

Poisson Process Basics

A Poisson Process process with parameter $\lambda > 0$ can be thought of as a random function $N(t)$, which counts the number of events that have occurred up through time t, if the following properties are satisfied:

1. $N(0) = 0$
2. $N(t_1) - N(t_2)$ is independent of $N(s_1) - N(s_2)$ if $[t_1, t_2]$ and $[s_1, s_2]$ are disjoint.
3. $N(t) - N(s) \sim Poisson(\lambda(t - s))$

There is a dual aspect to this, which is the sequence of X_1, X_2 of inter-arrival times, that is, the time between the i-th and $(i - 1)$-th events.

We note that by seeing:

$$P(X_1 > t) = P(N(t) = 0) = \exp(-\lambda t)$$

which is the survival function (1 - the cdf) of an $\exp(\lambda)$ distribution. We can thereby conclude that X_1 follows an exponential distribution.

$$P(X_2 > t | X_1 = s) = P(N(t + s) - N(s) = 0) = \exp(\lambda t)$$

so X_2 follows an exponential distribution that is independent of X_1, and so on.

Proposition The interarrival times $X_1, ..., X_n$ are independent and identically distributed exponential random variables with parameter λ.

Likewise, the arrival times (of the ith event) $S_i = \sum_{i=1}^{n} X_i$ will follow a $Gamma(n, \lambda)$ distribution.

Simulating A Poisson Process (Method 1)

In order to get complete information about a Poisson process up until some time T, we need only know the following:

1. $N(T)$, The value at time T
2. The sequence of arrival times $S_1, ..., S_{N(T)}$

We can do this by successively generating inter-arrival times until we get an event time that is after T ; and returning the collection of event times and the terminal value:

1. Let t =0, i=1
2. $\Delta t \sim \exp(\lambda)$
3. if $t + \Delta t > T$ return N, S
4. Else $t = t + \Delta t$, $N = N + 1$, $S_i = t$, i=i+1, go to 2

Implementation

(1) Implement the algorithm from the previous section. Return N(T) as integer, and $S_1, ..., S_{N(T)}$ as an numpy array.

(2) Write a function that takes as inputs: T, $S_1, ..., S_{N(T)}$ and $t < T$, and return the value of the poisson process at time t. Raise an exception if the user requests t > T.

(3) Generate 10000 simulations with $\lambda = 1.5$ and $T = 5$.Calculate the means and covariance the covariance matrix of N(1) - N(0), N(2) - N(1), N(3) - N(2), N(4) - N(3), N(5) - N(4) etc

Simulating From a Poisson Process (cont'd)

The previous method, however, can be somewhat unwieldy, it will require on average generating $\lambda * T$ exponential random variables, that is, we need to generate that many uniforms and evaluate that many logs (the logs being the critical part).

Let's see if we can exploit some properties of the Poisson Process to sample in a more efficient manner.

Suppose we first generate $N(T)$, and observe that $N(T) = k$. What is the distribution of $N(t)$ $(t < T)$ conditional on this event?

$$P(N(t) = x | N(T) = k) = \frac{P(N(t) = x, N(T) - N(t) = k - x}{N(T) = k} \tag{1}$$

$$= \frac{\frac{\exp(-\lambda t)(\lambda t)^x}{x!} \frac{\exp(-\lambda(T-t))(\lambda(T-t))^{k-x}}{(k-x)!}}{\frac{\exp(-\lambda T)(\lambda T)^k}{k!}} \tag{2}$$

$$= \binom{k}{x} \left(\frac{t}{T}\right)^x \left(\frac{T-t}{T}\right)^{k-x} \tag{3}$$

for $x = 0, 1, ..., k$

Which is the pmf of a binomial distribution with parameters k and $p = t/T$.

The outcome $N(t) = x | N(T) = k$ is equivalent to have x event times out k be less than t; from the equivalence with the binomial we can see that each of event time in this case is independent and has probability of being less than or equal to t of $\frac{t}{T}$, that is, they are Uniform(0,T).

This suggests the following algorithm:

1. Generate $N = N(T) \sim Poisson(\lambda T)$
2. Generate the unordered event times $S_1 = U_1 T$, ..., $S_N = U_N T$
3. Sort S_1,...,S_N and return N, S_{(1)}, ..., S_{(N)}

Implementation (part 2)

1. Implement the algorithm from the previous section so that it returns N(T) and numpy array S given lambda and T
2. Copy your evaluation function from the previous section over to this notebook
3. Generate 10000 simulations. For k = 6, 7, 8, 9; plot the empirical pmf of N(t) | N(T) = k for t = 1, 4. Draw a line graph of the theoretical pmf of the corresponding binomial distribution.