$\theta = E[X]$

Y

X, Y are not independent

(1) Control Variate

(X, Y) have to know the expectation of Y

$W = X + c \cdot (Y - \mu_Y)$

$c = -\dfrac{Cov(X, Y)}{Var(Y)}$ ← typically estimated from simulations

← can estimate from data

$E[Y_i] = \mu_i$

$(X, Y_1, \ldots, Y_d)$

$W = X + \sum_{j=1}^{d} c_j \cdot (Y_j - \mu_j)$

if $Y_1, \ldots, Y_d$ all independent

$c_j = -\dfrac{Cov(X, Y_j)}{Var(Y_j)}$

$\Longrightarrow$ if not independent

$$\mathrm{Var}(W) = \mathrm{Var}(X) - \frac{\mathrm{Cov}(X,Y)^2}{\mathrm{Var}(Y)}$$

② Conditioning

maybe $E[Y]$

$E[X|Y]$

$$E[E[X|Y]] = E[X]$$

$Y_1, \dots, Y_n$

$$E\left[\frac{1}{n}\sum_{i=1}^{n} E[X|Y_i]\right] = E[X]$$

$$\mathrm{Var}(X) = \mathrm{Var}(E[X|Y]) + E[\mathrm{Var}(X|Y)]$$

$$\mathrm{Var}(E[X|Y]) = \mathrm{Var}(X) - E[\mathrm{Var}(X|Y)] \leq \mathrm{Var}(X)$$

$$\mathrm{Var}(X|Y) = E[(X - E[X|Y])^2 | Y]$$
$$= E[X^2|Y] - E[X|Y]^2$$

$\rightarrow$ if you can use Y both for conditioning and as a control variate, conditioning will give you a better reduction of variance, but can get a better reduction in variance by using both!

$$W = E[X|Y] + c \cdot (Y - \mu_Y)$$

$$c = -\frac{\mathrm{Cov}(E[X|Y], Y)}{\mathrm{Var}(Y)}$$

Ex (control variates)

single server queue

X = total tim in system
for first N arrivals

③ Stratification
(Stratified Sampling)

$\theta = E[X]$

Y is a discrete random
with w/ support

$y_1, \dots, y_k$

$p_i = P(Y = y_i)$]

can sample
$X \mid Y = y_i$ for $y_1, \dots, y_k$

$X_1, \dots, X_n$

$Var(\overline{X}) = \frac{1}{n} Var(X)$

$\Rightarrow$ generate $n \cdot p_i$ samples
$X \mid Y = y_i$ $i = 1, \dots, k$

$\overline{X}_i \Rightarrow$ sample mean of those samples

$$S' = \sum_{i=1}^{k} \overline{X}_i \cdot p_i$$

$$E[S'] = \sum_{i=1}^{k} p_i \cdot E[X \mid Y = y_i]$$

$$= E\big[E[X \mid Y]\big] = E[X]$$

$$Var(\overline{X}_i) = \frac{Var(X \mid Y = y_i)}{n p_i}$$

$\uparrow$ np_i samples $X \mid Y = y_i$

$$Var(S') = Var\left(\sum_{i=1}^{k} p_i \overline{X}_i\right)$$

$$= \sum_{i=1}^{k} p_i^2 \, Var(\overline{X}_i)$$

$$= \sum_{i=1}^{k} \frac{p_i^{\cancel{2}}}{n \cancel{p_i}} \cdot Var(X \mid Y = y_i)$$

$$= \frac{1}{n} \underbrace{\sum_{i=1}^{k} p_i \, Var(X \mid Y = y_i)}$$

$$= \frac{1}{n} E[Var(X \mid Y)]$$

$$Var(X) = E[Var(X \mid Y)] + Var(E[X \mid Y])$$

$$Var(\bar{X}) = \frac{1}{n} Var(X) \geq \frac{1}{n} E[Var(X \mid Y)]$$

$$= Var(S)$$

<u>Note:</u> (1) auxiliary variable discrete, but doesn't have finite support

N poisson (λ) Y finite support version

$Y = N$ if $N < k$

$Y = k$ if $N \geq k$

$P(Y=i)$ $i = 0, \ldots, k-1$

$$= \frac{e^{-\lambda}\lambda^{i}}{i!}$$

$$P(Y=k) = 1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda}\lambda^{i}}{i!}$$

$X \mid N = i$, $i = 0, 1, \ldots, k-1$

$\underbrace{X \mid N \geq k}$

Ex single-server queue

X = total time in system for arrivals before T

N = # of arrivals

generate samples

$X \mid N = 0 \implies 0$

$X \mid N = 1$

$X \mid N = 2$

$\vdots$

$X \mid N = k-1$

$X \mid N \geq k$

Post stratification

Y discrete, finite support,
known pmf

$(X_1, Y_1), \ldots, (X_n, Y_n)$

$\overline{X}_i$ = sample mean of
all samples $Y = y_i$

$$S = \sum_{i=1}^{k} p_i \overline{X_i}$$

Variance Reduction

Is there a auxiliary variable?

NO

Is $X = h(U_1, \dots, U_m)$ with monotonic?

antithetic

Yes

Do we know the distribution of Y?

NO

DO WE KNOW $E[X|Y]$

Yes

conditioning

NO

Do we know THE $E[Y]$

CONTROL

Yes

Can we sample from $X|Y$

Yes

NO

stratification

post

VARIATE