$$\theta = E[f(X)]$$

$$\hat{\theta} = \frac{1}{n} \sum f(X_i)$$

$$Var(f(X_i)) = \sigma^2$$

Antithetic Random Numbers / Pairs

X_1, X_2

$$\theta = E[X]$$

$$\hat{\theta} = \frac{X_1 + X_2}{2} \qquad Var(X_i) = \sigma^2$$

$$Var(\hat{\theta}) = \frac{1}{4}(Var(X_1) + Var(X_2) + 2\,Cov(X_1, X_2))$$

$$= \frac{1}{2}\sigma^2 + \frac{1}{2} Cov(X_1, X_2)$$

$$X_1 = h(U_{11}, \ldots, U_{m1})$$

h is a monotone function

$$\tilde{X}_1 = h(1-U_{11}, \ldots, 1-U_{n1})$$

$$Cov(X_1, \tilde{X}_1) < 0$$

$$Var\left(\frac{X_1 + \tilde{X}_1}{2}\right) < Var\left(\frac{X_1 + X_2}{2}\right)$$

2) Control Variates

3) Conditioning

4) Stratification

5) Post-Stratification

Control Variates

$\theta = E[X]$

sps that as an artifact of simulating X, we also get some random variable Y

$E[Y] = \mu_Y$ known

$$E[X + c\cdot(Y - \mu_Y)]$$

$$= E[X] + c\cdot E[Y - \mu_Y]$$

$$= E[X]$$

$$W_i = X_i + c\cdot(Y_i - \mu_Y)$$

$$E\left[\frac{1}{n}\sum W_i\right] = E[X] = \theta$$

$$Var(W) = Var(X) + c^2 Var(Y) + 2c\cdot Cov(X,Y) = f(c)$$

$$f'(c) = 2c\,Var(Y) + 2\,Cov(X,Y) \stackrel{\Delta}{=} 0$$

$$c^* = -\frac{Cov(X,Y)}{Var(Y)}$$

global minimum

convex

$$f''(c) = 2\,Var(Y) > 0$$

in practice, we just use the sample covariance for $Cov(X,Y)$

oftentimes fine use sample variance of Y

how much of a variance reduction do we actually get

$$Var(W) = Var(X) + \left(-\frac{Cov(X,Y)}{Var(Y)}\right)^2 Var(Y)$$

$$+ 2\left(-\frac{Cov(X,Y)}{Var(Y)}\right) \cdot Cov(X,Y)$$

$$= Var(X) + \cancel{\frac{Cov(X,Y)^2}{Var(Y)}} - \cancel{2}\frac{Cov(X,Y)^2}{Var(Y)}$$

$$c = -\frac{Cov(X,Y)}{Var(Y)}$$

Ex $\theta = E[e^U]$

U as a control variate

$E[U] = 0.5 \quad Var(U) = \frac{1}{12}$

$E[e^U] = e - 1$

$Var(e^U) = .242$

$$Cov(U, e^U) = E[Ue^U] - E[U]E[e^U]$$

$$= \int_0^1 xe^x dx - \frac{(e-1)}{2}$$

$$= 1 - \frac{e-1}{2} = .14086$$

$$c^* = -\frac{.14086}{1/12}$$

$$\mathrm{Var}\left(e^U + c^* \cdot (U - 0.5)\right)$$

$$= .242 - \frac{(.14086)^2}{(12)}$$

$$= .004$$

<u>Ex</u> Queuing system

X = total time in system for all arrivals before T

some potential control variates

(1) number of arrivals before T

(2) mean interarrival time before T]

3 mean service time before T]

$\exp(\lambda_1)$

$\exp(\lambda_2)$

$(X_1, I_1, S_1), \ldots,$

(X_n, I_n, S_n)

$$W_i = X_i + c_1 \cdot (I_i - 1/\lambda_1)$$
$$+ c_2 \cdot (S_i - 1/\lambda_2)$$

$$c_1 = -\frac{\mathrm{Cov}(I, X)}{\mathrm{Var}(I)}$$

$$c_2 = -\frac{Cov(S, X)}{Var(S)}$$

$\Longrightarrow$ can use multiple control variates

eg.

$$W = X + \sum_{j=1}^{m} c_j \cdot (Y_j - \mu_j)$$

$$E[Y_i] = \mu_i$$

if Y_i ind.

$$c_i = -\frac{Cov(X, Y_i)}{Var(Y_i)}$$

what if not ind?

$$\underset{c_1, \dots, c_n}{\arg\min} \; E\left[\left(X - \sum_{j=1}^{n} c_j (Y_j - \mu_j) - \theta\right)^2\right]$$

$\Longrightarrow$ multiple regression where X is the dependent var, (response)

$Y_1, \dots, Y_m$ independent variable (factors)

$$c = (-1)\,\beta$$

Conditioning

$$\theta = E[X].$$

simulations of X, we also get some random quantity Y

$$E[X \mid Y] = f(Y)$$

$$f(y) = E[X \mid Y = y]$$

$$E[E[X \mid Y]] = E[X] = \theta$$

$$(X_1, Y_1), \ldots, (X_n, Y_n)$$

$$W_i = E[X \mid Y_i]$$

$$E\left[\frac{1}{n}\sum W_i\right] = E\left[\frac{1}{n}\sum E[X \mid Y_i]\right]$$

$$= E[X] = \theta$$

$$\mathrm{Var}(E[X \mid Y]) \leq \mathrm{Var}(X)$$

$$\mathrm{Var}(X \mid Y)$$

$$= E\left[\left(X - E[X \mid Y]\right)^2 \mid Y\right]$$

$$\mathrm{Var}(X) = \mathrm{Var}(E[X \mid Y])$$
$$+ E[\mathrm{Var}(X \mid Y)]$$

$$\mathrm{Var}(E[X \mid Y])$$

$$= \mathrm{Var}(X) - \underbrace{E[\mathrm{Var}(X \mid Y)]}$$

$$\leq \mathrm{Var}(X)$$

$\mathrm{Var}(X \mid Y)$ to be

large $\to$ ideal

$\underline{\text{Ex}}$ shop where random arrivals,

service times, drop-outs,

spend

X = total revenue $= \sum_{i=1}^{N} X_i$

N = # of customers who actually make a purchase

$$E[X \mid N] = E[X_i] \cdot N$$

$\underline{\text{Ex}}$ Single-server queue

$X_i \sim \exp(\lambda_1)$

$S_i \sim \exp(\lambda_2)$

$W_i \Rightarrow$ time in system for ith arrival

total time in system for the 1st N arrivals

$$W = \sum_{i=1}^{N} W_i$$

Y_i = # of people in queue when person i arrives

$$E[W] = \sum_{i=1}^{N} E[W_i]$$

$$= \sum_{i=1}^{N} E[W_i \mid S_i]$$

$$= \sum_{i=1}^{N} \frac{S_i + 1}{\lambda_2}$$

<u>Ex</u> Finite capacity queue model

M

L is the number of lost customers

HPP(λ)

$E[L]$

T_c = time system spends at capacity

$$E[L \mid T_c] = \lambda \cdot T_c$$

When do I want to use control variate vs conditioning?

)) variate

control va

① jointly sample (X, Y)

know $E[Y]$

② jointly sample (X, Y) conditiony

know $E[X|Y]$