① we want to sample Unif(0,1)

$$U_n = \frac{x_n}{m}$$

$x_n \Rightarrow$ pseudo-random number w/ support $0, 1, \ldots, m-1$

② $x_0 =$ seed

$\vdots$

$$x_n = a x_{n-1} + b \mod m$$

$a = 3$

$m = 10$

$x_0 = 4$

$x_1 = 2$

$x_2 = 6$

$x_3 = 8$

$x_4 = 4$

going forward $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$

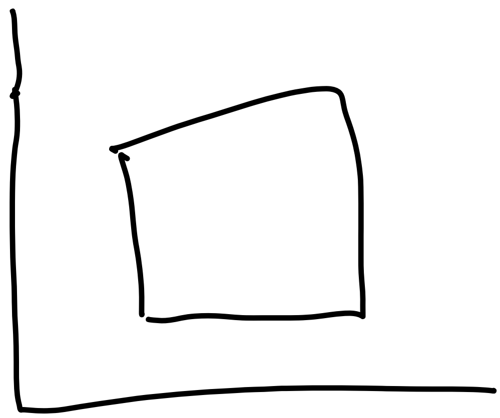
<u>Monte Carlo Method</u>

$$\int_a^b f(x)\,dx = \text{area under the curve}$$

$$\int_c^d \int_c^b f(x,y)\,dy\,dx$$

in 1-dimension

in 2-d



Random Variable

X cdf $F: \mathbb{R} \to [0,1]$

$$F(x) = P(X \le x)$$

pdf f

$$P(a < X \le b) = \int_a^b f(x)\,dx$$

$$P(X \le b) = \int_{-\infty}^{b} f(x)\,dx$$

$$f(x) = F'(x)$$

$$E[X] = \int x f(x)\,dx \rightarrow \mu$$

$$E[g(x)] = \int g(x) \cdot f(x)\,dx$$

$$Var(X) = E[(X - E[X])^2]$$

$$= E[X^2] - E[X]^2 \rightarrow \sigma^2$$

Law of Large #s

$X_1, \ldots, X_n$ iid $E[X_i] = \mu$

$Var(X_i) = \sigma^2$

$$\overline{X_n} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$\lim_{n \to \infty} \overline{X_n} = \mu$$

<u>Central Limit Theorem</u>

$X_1, \ldots, X_n$ iid μ, σ^2

$$\overline{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\int_c^d f(h(x))\, h'(x)\,dx = \int_{h(c)}^{h(d)} f(y)\,dy$$

<u>Ex</u> $\displaystyle\int_0^{\sqrt{\pi}} \sin(x^2)\, 2x\,dx$

π

$$= \int_0^{\pi} \sin(x)\,dx = -\cos(x) \Big|_0^{\pi} = 2$$

Unif (0,1)

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{o/w} \end{cases}$$

$$= \mathbb{1}\{0 \leq x \leq 1\}(x)$$

$$E[U] = \int_0^1 x\,dx \quad \left(= \int_{-\infty}^{\infty} x \cdot \mathbb{1}\{0 \leq x \leq 1\}\,dx \right)$$

$$E[g(U)] = \int_0^1 g(x)\,dx$$

$$\theta = \underbrace{\int_0^1 g(x)\,dx}$$

$$= E[g(U)]$$

(incidentally)

$U_1, \ldots, U_n \sim \text{Unif}(0,1)$

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} g(U_i) \xrightarrow[n \to \infty]{} \theta$$

how can I quantify
my error term?

confidence interval (95% CI)

S^2 is the sample variance

$$= \frac{1}{n-1} \sum_{i=1}^{n} \left(g(U_i) - \hat{\theta} \right)^2$$

$(1-\alpha)100\%$ CI

$$\hat{\theta} \pm Z_{\alpha/2} \cdot \frac{S}{\sqrt{n}}$$

what happens if we need to instead
integrate

$$\left[\theta = \int_a^b f(x)\,dx \right]$$

$$\int_c^d \underbrace{f(h(x))\,h'(x)}\,dx = \int_{h(c)}^{h(d)} f(y)\,dy = \int_a^b f(y)\,dy$$

start on the right hand side and use

$c = 0$ $\quad h(0) = a$

$d = 1$ $\quad h(1) = b$

what is a function s/t

$h(0) = a$

$h(1) = b$

$h(x) = a + (b-a)x$

$h'(x) = (b-a)$

$$\int_a^b f(x)\,dx = \int_0^1 f(a + (b-a)x)(b-a)\,dx$$

$$= E\left[f(a + (b-a)U)(b-a)\right]$$

$U_1, \dots, U_n \sim \mathrm{Unif}(0,1)$

$$\hat{\Theta} = \frac{1}{n}\sum_{i=1}^{n} f(a + (b-a)U_i)(b-a)$$

$a = 3$

$b = 7$

$f(x) = x^2$

$$= \frac{1}{n}\sum_{i=1}^{n} (3 + 4 \cdot U_i)^2 (4)$$