$N(t)$ = # of events that occur before time t

① $N(0) = 0$

② $N(t_2) - N(t_1) \perp\!\!\!\perp N(s_2) - N(s_1)$

if $(t_1, t_2) \cap (s_1, s_2) = \phi$

③ $N(t) - N(s) \sim \text{Poiss}(\lambda(t-s))$

$N \sim$ Poisson Proc (λ)

$X_i \sim \exp(\lambda)$

how to simulate? up until T

want to generate the arrival times

① $t = 0$ $S = [\,]$

Interarrival method

② $\Delta t \sim \exp(\lambda)$

③ if $t + \Delta t > T$

return S

else:

$t = t + \Delta t$

S.append(t)

go back to step ②

Order Statistics method / Ordered Statistics

1. $N(T) \sim \text{Poiss}(\lambda T)$
2. given $N(T) = k$

   $Y_1, \ldots, Y_k$ unordered arrival times iid $\text{Unif}(0,T)$

$\Longrightarrow$ 1. $N(T) \sim \text{Poiss}(\lambda T)$

2. $Y_1 = U_1 \cdot T$

   $\vdots$

   $Y_{N(T)} = U_{N(T)} \cdot T$

   arrival times are sorted $Y_{(1)}, \ldots, Y_{(N(T))}$

Nonhomogeneous Poisson Proc.

$\lambda(t) \geq 0 \quad t \geq 0$

1. $N(0) = 0$
2. independent increments if non-overlapping

   $N(t_2) - N(t_1) \perp\!\!\!\perp N(s_2) - N(s_1)$ if $(t_1, t_2) \cap (s_1, s_2)$

$= \emptyset$

③ $N(t_2) - N(t_1)$
$\sim \text{Poiss}\left(\int_{t_1}^{t_2} \lambda(s)\, ds \right)$

$\Lambda(t) = E[N(t)] = \int_0^t \lambda(s)\, ds$

M is a homogeneos poisson process with intensity γ

$p(t) : \mathbb{R}^+ \to [0,1]$

$\underbrace{Y_1, \ldots, Y_n, \ldots}$ arrivel times associated w/ M

$p(Y_1) \Rightarrow$ probability of keeping Y_1

$p(Y_2) \Rightarrow$ probability of keeping Y_2

...

$\Rightarrow$ nohomogenos poisson process with intensity
$\lambda(t) = \gamma \cdot p(t)$

$\Rightarrow$ flip it around ..
want to sample from $NHP(\lambda(t))$

$$\lambda(t) \le \lambda_{max} \quad \text{on} \quad 0 \le t \le T$$

sample

① HPP(λ_{max})

② keep / throw out arrival times $p(t) = \dfrac{\lambda(t)}{\lambda_{max}}$

$$\Rightarrow \text{NHPP} \qquad \frac{\lambda(t)}{\cancel{\lambda_{max}}} \cdot \cancel{\lambda_{max}}$$

(Thinning method)

$\lambda(t) = 3+t$

Ordered Statistics

distribution of unordered arrival times

$N(T) = k$

has cdf

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{\Lambda(x)}{\Lambda(T)}, & 0 < x < T \\ 1, & x \ge T \end{cases}$$

and density

$$f(x) = \frac{\lambda(x)}{\Lambda(T)}$$

statistics method

ordered statistics

① $N(T) \sim Poiss\left(\Lambda(T) = \int_0^T \lambda(s)\,ds\right)$

② given $N(T) = k$

$X_1, \dots, X_k \overset{iid}{\sim}$ w/ cdf $F(t) = \dfrac{\Lambda(t)}{\Lambda(T)}$

using inverse transform or rejection sampling

③ sort $X_1, \dots, X_k$ from smallest to largest and return

Inverse Method

NHP $\Lambda(t)$ N up until T

$\Lambda^{-1}(t)$ exists

$M \sim HPP(1)$ from $[0, \Lambda(T)]$

$t_1, \dots, t_k$ arrivals I get for M

$\Longrightarrow$ then the arrivals for N on T

$\Lambda^{-1}(t_1), \dots, \Lambda^{-1}(t_k)$

Ex Want to sample from a

NHPP

$\lambda(t) = 3t^2$

$$\Lambda(t) = \int_0^t 3s^2 \, ds$$

$$= t^3$$

$\Lambda^{-1}(t) = t^{1/3}$ $T = 2$

sample from $M \sim HP(1)$

$\hookrightarrow [0, T^3]$

$= [0, 8]$

$\Rightarrow$ arrivals for M $[1, 3, 5, 7]$

$\Rightarrow$ arrivals for N $[1, 3^{1/3}, 5^{1/3}, 7^{1/3}]$