$\Rightarrow$ want to sample from f

ability to sample from g

$f(x) \leq c\,g(x)$ *

then the following works to get samples from f

① $Y \sim g$

② $U \sim \text{Unif}(0,1)$

③ $U \leq \dfrac{f(Y)}{c\,g(Y)} \Rightarrow$ return Y

o/w go back to step ①

$$c = \max_{x : g(x) > 0} \frac{f(x)}{g(x)}$$

$\Rightarrow$ distribution of the # of samples we need to get an accepting geo rv w/ exp c

$Y \sim g(x)$

prob acceptance $\cdot c$

Ex $f(x) = \begin{cases} x, & 0 < x < 1 \\ (2-x), & 1 \le x < 2 \\ 0, & o/w \end{cases}$

use proposal $g(x) = \frac{1}{2} \mathbb{1}_{\{0 < x < 2\}} \to \text{Unif}(0,2)$

sample by $U_1 \sim \text{Unif}(0,1)$

$Y = 2 \cdot U_1$

$$c = \max_{x: g(x) > 0} \frac{f(x)}{g(x)} = \max_{0<x<2} \frac{f(x)}{1/2} = 2$$

1. $U_1 \sim \text{Unif}(0,1)$, $Y = 2U_1$
2. $U_2 \sim \text{Unif}(0,1)$
3. If $U_2 \le \frac{f(Y)}{2 \cdot 1/2} = f(Y)$ then return Y

o/w go back to step 1

Ex $X \sim \text{Gamma}(\alpha, \lambda)$]

$$f(x) = \frac{\lambda^{\alpha} x^{\alpha-1} \exp(-\lambda x)}{\Gamma(\alpha)} \mathbb{1}_{\{x > 0\}}$$

$E[X] = \frac{\alpha}{\lambda}$

proposal density $\quad Y \sim \exp(\beta)$

$$g(x) = \beta \exp(-\beta x) \mathbb{1}_{\{x>0\}}$$

$$h(x) = \frac{f(x)}{g(x)} = \underbrace{\frac{\lambda^{\alpha}}{\beta \Gamma(\alpha)}} x^{\alpha-1} \cdot e^{-\lambda x + \beta x} \quad (x>0)$$

$\Rightarrow$ want to see for which combination of α, β, λ this is bounded and then what c is

$\Rightarrow$ proportional to

$$x^{\alpha-1} \exp(-(\lambda-\beta)x)$$

at $x=0$ if $\alpha < 1$

$\Rightarrow$ blows up, can't use exponential proposal

if $\alpha \geq 1$ no singularity at $x=0$

as $x \to \infty$ $\exp(-(\lambda-\beta)x)$ dominates

$\lambda - \beta > 0$

$$\boxed{\begin{array}{l} (1)\ \alpha \geq 1 \\ (2)\ \lambda-\beta > 0 \end{array}}$$

$$h(x) = \frac{f(x)}{g(x)} = \frac{\lambda^{\alpha}}{\beta\Gamma(\alpha)} x^{\alpha-1} \cdot e^{-\lambda x + \beta x}$$

$$h'(x) = \frac{\lambda^{\alpha}}{\beta\Gamma(\alpha)} \Big((\alpha-1) x^{\alpha-2} e^{-\lambda x + \beta x} + (\beta-\lambda) x^{\alpha-1} e^{-\lambda x + \beta x} \Big) \overset{\Delta}{=} 0$$

$$(\alpha-1) x^{\alpha-2} + (\beta-\lambda) x^{\alpha-1} = 0$$

$$x = \frac{\alpha-1}{\lambda-\beta}$$

$$h\left(\frac{\alpha-1}{\lambda-\beta}\right) = \frac{\lambda^{\alpha}}{\beta\Gamma(\alpha)} \left(\frac{\alpha-1}{\lambda-\beta}\right)^{\alpha-1} e^{1-\alpha} = c$$

$$p(x) = \sum w_i \cdot p_i(x) \qquad w_i \geq 0, \quad \sum w_i = 1$$

$$f(x) = \sum w_i \cdot f_i(x)$$

$\Rightarrow$ can just do traditional composition method

Ex Car insurance

$\lambda_i \sim \exp(\beta)$ gamma (α, β)

$X_i \mid \lambda_i \sim \text{Poisson}(\lambda_i)$

$$p(x) = \int p(x \mid \lambda) \cdot f_\lambda(\lambda)\, d\lambda$$

$$= \int \frac{e^{-\lambda} \lambda^x}{x!} \cdot \beta \exp(-\beta\lambda)\, d\lambda$$

$\Longrightarrow$ continuous mixture

outcomes are $f(x \mid \theta)$

$\theta \sim g(\theta)$

n samples $\theta_1, \ldots, \theta_n \sim g(\theta)$

$X_1 \sim f(x \mid \theta_1)$

$X_2 \sim f(x \mid \theta_2)$

$\vdots$

$X_n \sim f(x \mid \theta_n)$