Inverse Transform for cont. random vars

① $U \sim Unif(0,1)$

② $X = F^{-1}(U)$

Ex

$$F(x) = \begin{cases} 0, & x \le 0 \\ x^n & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$$

$u = x^n$

$x = u^{1/n}$

① generate $U \sim Unif(0,1)$

② return $U^{1/n}$

Ex $X \sim exp(\lambda)$

$$F(x) = \begin{cases} 0, & x < 0 \end{cases}$$

$$\Bigg| \; 1-\exp(-\lambda x), \quad x \geq 0$$

$$u = 1-\exp(-\lambda x)$$

$$x = -\frac{\log(1-u)}{\lambda}$$

① $U \sim \text{Unif}(0,1)$

② return $-\dfrac{\log(1-U)}{\lambda} \overset{D}{=} -\dfrac{\log(U)}{\lambda}$

<u>Ex</u> $X \sim \text{Rayleigh}(\sigma) \quad \sigma > 0$

$$f(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathbb{1}_{\{x>0\}}$$

$$F(x) = \int_0^x \frac{y}{\sigma^2} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

$$\Big|^{y=x}$$

$$= -e^{-y^2/2\sigma^2}\Big|_{y=0}$$

$$= 1 - e^{-x^2/2\sigma^2}$$

$$u = 1 - e^{-x^2/2\sigma^2}$$

$$e^{-x^2/2\sigma^2} = 1 - u$$

$$x^2 = -\log(1-u)\cdot 2\sigma^2$$

$$x = \sqrt{-2\sigma^2 \log(1-u)}$$

<u>Ex</u> Weibull dist'n k, λ

$$f(x) = k\lambda^k x^{k-1} \exp\left(-(\lambda x)^k\right) 1_{\{x>0\}}$$

$$F(x) = \int_0^x k\lambda^k y^{k-1} \exp\left(-(\lambda y)^k\right) dy$$

$$= x$$

$$= -\exp\left(-(\lambda y)^k\right)\Big|_{y=0}^{y=x}$$

$$= 1 - \exp\left(-(\lambda x)^k\right)$$

$$u = 1 - \exp\left(-(\lambda x)^k\right)$$

$$(\lambda x)^k = -\log(1-u)$$

$$x = \frac{\left(-\log(1-u)\right)^{1/k}}{\lambda}$$

$$Y \sim \exp(\lambda^k) = -\frac{\log(1-U)}{\lambda^k}$$

$$X = (Y)^{1/k}$$

<u>Ex</u> Pareto (α)

$$f(x) = \frac{\alpha}{x^{\alpha+1}} \mathbb{1}_{\{x>1\}}$$

$$F(x) = \int_1^x \frac{\alpha}{y^{\alpha+1}}\, dy$$

$$= \left[-\frac{1}{y^{\alpha}} \right]_{y=1}^{y=x}$$

$$= 1 - \frac{1}{x^{\alpha}} = 1 - x^{-\alpha}$$

$$u = 1 - x^{-\alpha}$$

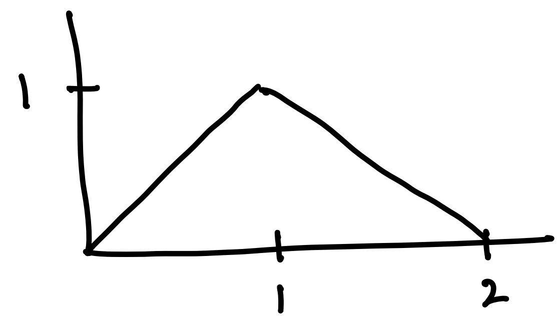
$$x^{-\alpha} = 1 - u$$

$$x^{\alpha} = \frac{1}{1-u}$$

$$x = \left(\frac{1}{1-u}\right)^{1/\alpha}$$

Ex Triangular dist'n

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ (2-x), & 1 < x \le 2 \\ 0, & o/w \end{cases}$$



$$F(x) = \begin{cases} \frac{x^2}{2}, & 0 \le x \le 1 \\ 1 - \frac{(2-x)^2}{2} & 1 < x \le 2 \end{cases}$$

for $1 \le x \le 2$

x

$$F(x) = \int_0 f(y)\,dy$$

$$= \int_0^1 y\,dy + \int_1^x (2-y)\,dy$$

$$= \frac{1}{2} - \frac{1}{2}(2-y)^2 \Big|_1^x$$

$$= 1 - \frac{(2-x)^2}{2}$$

if $U < 0.5$

$$\Rightarrow U = \frac{x^2}{2}$$

$$x = \sqrt{2U}$$

$$0.5 \leq U \leq 1$$

$$U = 1 - \frac{(2-x)^2}{2}$$

$$x = 2 - \sqrt{2(1-U)}$$

<u>Ex</u> Gamma dist'n doesn't have a closed form cdf nor does it have a closed form inverse cdf

even if we have a closed form cdf, we may not have a closed form for inverse cdf

① given $U \sim Unif(0,1)$

② use a search algorithm to find x where $F(x) = U$

$$F(x) - U = 0$$

Rejection Sampling

want to sample from density f

$\exists g$ s/t

$$f(x) \le c \cdot g(x)$$

for some $c > 1$

efficiently sample from g

Rejection algorithm

① $Y \sim g$

② $U \sim \text{Unif}(0,1)$

③ If $U \leq \dfrac{f(Y)}{cg(Y)}$ return Y

otherwise go back to step

<u>Thm</u> ① output has density $f(x)$

② # of iterations is geometric with mean c

$$f(x) \leq cg(x)$$

$$\frac{f(x)}{g(x)} \leq c = \max_{x: g(x)>0} \frac{f(x)}{g(x)}$$

Ex Suppose

$$f(x) = 4x^3 \mathbb{1}_{\{0<x<1\}}$$

$$g(x) = \mathbb{1}_{\{0<x<1\}}$$

↳ density of Unif$(0,1)$

$$c = \max_{0<x<1} \frac{f(x)}{g(x)} = \max_{0<x<1} \frac{4x^3}{1} = 4$$

① $U_1 \sim \text{Unif}(0,1)$

② $U_2 \sim \text{Unif}(0,1)$

③ If $U_2 \le \frac{f(U_1)}{4g(U_1)} = \frac{4U_1^3}{4 \cdot 1}$

$$= U_1^3$$

return U_1

o/w go back to step 1

Ex $f(x) = 20x(1-x)^3, \; 0<x<1$

$g(x) = 1 \; \{0<x<1\}$

$$c = \max_{0<x<1} \frac{f(x)}{g(x)} = \underbrace{20x(1-x)^3}_{h(x)}$$

$$h'(x) = 20\left((1-x)^3 - 3x(1-x)^2\right) \overset{\Delta}{=} 0$$

$$(1-x)^3 - 3x(1-x)^2 = 0$$

$$(1-x) - 3x = 0$$

$$4x = 1$$

$x = 1/4$

$$h\left(\tfrac{1}{4}\right) = 20\left(\tfrac{1}{4}\right)\left(3/4\right)^3$$
$$= \frac{135}{64}$$

$$\frac{f(x)}{cg(x)} = \frac{256}{27}\, x(1-x)^3$$

(1) generate U_1, U_2

(2) If $U_2 \leq \frac{256}{27} U_1 (1-U_1)^3$

return U_1

o/w go back to step (1)

how to sample from a normal distribution

$Z \sim N(0,1)$

$Y = |Z|$

if I multiply Y by (-1) w/ prob 1/2

by (1) w/ prob 1/2

$\Longrightarrow$ this will have $N(0,1)$

$$f_{|Z|}(x) = \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathbb{1}_{\{x>0\}}$$

$$g(x) = \exp(-x) \mathbb{1}_{\{x>0\}}$$

exp(1) proposal

$$\frac{f(x)}{g(x)} = \frac{2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} + x\right)$$

$$\Longrightarrow \text{maximize} \quad -\frac{x^2}{2} + x$$

$$-x + 1 \overset{\Delta}{=} 0$$

$$x = 1$$

$$c = \frac{f(1)}{g(1)} = \frac{2}{\sqrt{2\pi}} \exp\left(\frac{1}{2}\right)$$

$$= \sqrt{\frac{2e}{\pi}}$$

$$\frac{f(x)}{c\,g(x)} = \exp\left(-\frac{1}{2}(x-1)^2\right)$$

U

1. $Y \sim \exp(1)$
2. $U_1 \sim \text{Unif}(0,1)$
3. If $U_1 \leq \exp\left(-\frac{1}{2}(Y-1)^2\right)$

   $X = Y$

   o/w go back to 1

4. $U_2 \sim \text{Unif}(0,1)$

   If $U_2 \leq 0.5$

   let $Z = -X$

   Else

   let $Z = X$

   return Z