$X \sim p \leftarrow$ want

$Y \sim q \leftarrow$ can sample

$$c = \max_{x:\, q(x) > 0} \frac{p(x)}{q(x)}$$

$p(x) \leq c \cdot q(x)$ for all x

$\Rightarrow$ (note this implies if $q(x) = 0$
$p(x) = 0$, so
$S_x \subseteq S_Y$)

① $Y \sim q(x)$

② $U \sim \text{Unif}(0,1)$

③ If $U \leq \dfrac{p(Y)}{c\, q(Y)} \Rightarrow$ accept Y

if not, then go back to step ①

$\Rightarrow$ # of iterates = geo r.v. w/ expectation $= c$

when choosing proposal dist'n, we prefer c smaller (all else is equal)

Ex

$$p(x) = \begin{cases} 0.25, & x = 1 \\ 0.15, & x = 2 \\ 0.22, & x = 3 \\ 0.18, & x = 4 \\ 0.2, & x = 5 \end{cases}$$

$$g(x) = 0.2 \text{ if } x = 1, 2, 3, 4, 5$$

$$c = \max_x \frac{p(x)}{g(x)} = \frac{0.25}{0.2} = 1.25$$

① $U_1 \sim \text{Unif}(0,1)$ $Y = \lceil 5U_1 \rceil$

② $U_2 \sim \text{Unif}(0,1)$

③ If $U_2 \leq \dfrac{p(Y)}{1.25 \cdot g(Y)}$ return Y

repeat ①, ②

Ex $X \sim$ geometric random variable - 1 (θ)

$S_x = \{0, 1, 2, 3, \dots\}$

$p(x) = (1-\theta)^x \theta, \quad x = 0, 1, 2, \dots$

$$q(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$c = \max \frac{p(x)}{q(x)} = \frac{(1-\theta)^x \theta}{\frac{e^{-\lambda}\lambda^x}{x!}}$$

$$= \left(\frac{\theta}{e^{-\lambda}}\right) \underbrace{\left(\frac{1-\theta}{\lambda}\right)^x \cdot x!}$$

for some x large enough

$$x \left(\frac{1-\theta}{\lambda}\right) > 1$$

so obviously also > 1
for all x bigger than that

Composition approach

Mixture distn is a convex combination of other distributions $p_1, \ldots, p_k$

$$p(x) = \sum_{i=1}^{k} \lambda_i \cdot p_i(x) \qquad \sum_{i=1}^{k} \lambda_i = 1 \qquad \lambda_i \geq 0$$

# of accidents:

w/ prob 0.9, nice weather

Poiss(1)

w/ prob 0.05, rain

Poiss(3)

0.04, snow

Poiss(5)

0.01, sleet

Poiss(11)

$$p(x) = (0.9)\left(\frac{e^{-1}}{1}\right) \Big\}$$

$$x!$$

$$+ (0.05)\left(\frac{e^{-3}3^x}{x!}\right)\}$$

$$+ (0.04)\left(\frac{e^{-5}5^x}{x!}\right)\}$$

$$+ (0.01)\left(\frac{e^{-11}11^x}{x!}\right)\}$$

1. sample weather (mixing distn)
   $U \sim Unif(0,1)$
   $0 \le U \le 0.90 \Rightarrow$ good weath
   $0.9 < U \le 0.95 \Rightarrow$ rain etc.

2. given the weather, sample from the corresponding distribution in the mixtn
   if good weather $\Rightarrow$ poisson (1)
   rain $\Rightarrow$ poisson (3)

<u>Composition method</u>

$n \qquad\qquad k \quad .$

$$p(x) = \sum_{i=1}^{n} \lambda_i \, p_i(x) \qquad \sum_{i=1} \lambda_i = 1$$

$$\lambda_i \geq 0$$

① generate $U \sim \text{Unif}(0,1)$

if $U \leq \lambda_1 \Rightarrow I = 1$

if $\lambda_1 < U \leq \lambda_1 + \lambda_2 \Rightarrow I = 2$

$\vdots$

② given I

generate sample from p_I

$x = 0, 1, 2, \ldots$

$$p(x) = \frac{1}{3} e^{-1}/x! + (2/3)^{x+1}\left(\frac{1}{3}\right)$$

$$= \left(\frac{1}{3}\right)\left(\frac{e^{-1}}{x!}\right)$$

$\uparrow$ Poiss(1)

$$+ \left(\frac{2}{3}\right)\underbrace{\left(\frac{2}{3}\right)^{x}\left(\frac{1}{3}\right)}$$

Continuous Random Variables

Inverse Transform Method

$$F^{-1}(u) = \min\{x : F(x) \geq u\}$$
$$= \min\{x : F(x) = u\}$$

__Prop__ $U \sim \text{Unif}(0,1)$

F is strictly increasing and continuous.

$F^{-1}(U)$ has cdf F.

__Pf__ $Y \doteq F^{-1}(U)$

$$F_Y(x) = P(Y \leq x)$$
$$= P(F^{-1}(U) \leq x)$$
$$= P(F(F^{-1}(U)) \leq F(x))$$
$$= P(U \leq F(x))$$

$= F(x)$

$\Longrightarrow F^{-1}(U)$ has cdf F

Inverse Transform method

want to sample w/ cdf F of cont. r.v.

1. $U \sim \text{Unif}(0,1)$
2. return $F^{-1}(U)$

__Ex__ Suppose
$$F(x) = \begin{cases} 0, & x \le 0 \\ x^n, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$$

$u = x^n$

$u^{1/n} = x$

$F^{-1}(u) = u^{1/n}$

<u>Ex</u> $X \sim \exp(\lambda)$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - \exp(-\lambda x), & x \geq 0 \end{cases}$$

$$u = 1 - \exp(-\lambda x)$$

$$\exp(-\lambda x) = 1 - u$$

$$x = -\frac{\log(1-u)}{\lambda}$$

① U nif $(0,1)$

② return $-\frac{\log(1-U)}{\lambda}$ or

$\left[-\frac{\log(U)}{\lambda} \right]$

($1-U$ & U have same dist'n)

Ex Gamma(r, λ)

$$f(x) = \frac{\lambda e^{-\lambda x} \cdot (\lambda x)^{r-1}}{\Gamma(r)}$$

$$F(x) = \int_0^x \frac{\lambda e^{-\lambda y} (\lambda y)^{r-1}}{\Gamma(r)} \, dy$$

suppose r is an integer

$\implies$ how to generate a gamma(r, λ)

$U_1, \ldots, U_r \sim \text{Unif}(0,1)$

$$X = \sum_{i=1}^{r} - \frac{\log(U_i)}{\lambda}$$

$$= -\frac{1}{\lambda} \sum_{i=1}^{r} \log(U_i)$$

$$= -\frac{1}{\lambda} \log\left(\prod_{i=1}^{r} U_i\right)$$