Find an optimal decision

$f(n) = E[Y|n] + \varepsilon_n \qquad \varepsilon_n \sim N(0, \sigma_n^2)$

maximize the correlation between the various ε_n

minimize the variance $f(n_1) - f(n_2)$

n_1 is better n_2

$f(n_1) - f(n_2) < 0$

reduce the variance of the various error terms ε_n

① Antithetic Sampling

$\theta = E[X] \qquad \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$\underbrace{X_i = h(U_{i1}, \ldots, U_{in})}$

Ex Single-server queue with exp inter-arrival and service times

X is the total in system for the 1st 100 arrivals

X_1 & X_2

$Var\left(\frac{X_1 + X_2}{2}\right) = \frac{1}{4}\left(Var(X_1) + Var(X_2)\right.$

$$+ 2\,\mathrm{Cov}(X_1, X_2))$$

$\Rightarrow$ minimize the variance by maximizing the magnitude of $\mathrm{Cov}(X_1, X_2)$ and having it be negative

$Y \sim \exp(\lambda)$

$$Y = -\frac{\log(1-U)}{\lambda} \overset{D}{\sim} -\frac{\log(U)}{\lambda} = \tilde{Y}$$

$$X_i = h(U_{i1}, \dots, U_{in})$$

$$\tilde{X}_i = h(1-U_{i1}, \dots, 1-U_{in})$$

$\Rightarrow$ X_i & $\tilde{X}_i$ are negatively correlated

so

$$\mathrm{Var}\left(\frac{X_i + \tilde{X}_i}{2}\right) < \mathrm{Var}\left(\frac{X_1 + X_2}{2}\right)$$

if $X_1 \perp\!\!\!\perp X_2$

this is actually generally true if $h(u_1, \dots, u_n)$ is a monotone function in each of its arguments

<u>Ex</u> $U \sim \mathrm{Unif}(0,1)$

$$\theta = E[e^U] = \int_0^1 e^x\,dx = e-1$$

$$Var(e^U) = E[e^{2U}] - E[e^U]^2$$

$$= \int_0^1 e^{2x}\,dx - (e-1)^2$$

$$= \frac{e^2-1}{2} - (e-1)^2 = \underline{.242}$$

U_1, U_2 ind.

$$Var\left(\frac{e^{U_1}+e^{U_2}}{2}\right) = \frac{Var(e^U)}{2} = .121 \,]$$

antithetic pair:

$$\frac{e^{U_1}+e^{1-U_1}}{2}$$

$$Var\left(\frac{e^{U_1}+e^{1-U_1}}{2}\right) = \frac{1}{4}\left(\overbrace{Var(e^{U_1})} + \overbrace{Var(e^{1-U_1})} + 2\,Cov(e^{U_1}, e^{1-U_1})\right)$$

$$= \frac{1}{2} Var(e^U) + \frac{1}{2} Cov(e^{U_1}, e^{1-U_1})$$

$$= .0039 \,]$$

$$\mathrm{Cov}\left(e^{U_1}, e^{1-U_1}\right) =$$

$$E\left[\underbrace{e^{U_1} \cdot e^{1-U_1}}\right] - E\left[e^{U_1}\right] E\left[e^{1-U_1}\right]$$

$$= E[e] - (e-1)^2 = -.2342$$

$\Rightarrow$ variance reduction of about 97%

$$Y \sim \exp(\lambda)$$

$$= \frac{-\log(U)}{\lambda}$$

$$\log(U) = -\lambda Y$$

$$U = e^{-\lambda Y}$$

$$1-U = 1 - e^{-\lambda Y}$$

$$\tilde{Y} = -\log(1 - e^{-\lambda Y})$$

$$\frac{v}{\lambda}$$

single server queue

→ h is increasing function
wrt to service times
decreasing function
wrt to interarrival times