Def Exponential distribution

$X \sim \exp(\lambda)$

$f(x) = \lambda \exp(-\lambda x) \mathbb{1}_{\{x>0\}}$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1-\exp(-\lambda x), & x \geq 0 \end{cases}$$

$$P(X > x) = 1 - F(x) = \underbrace{\exp(-\lambda x)}$$

$\shortparallel$

$P(X \geq x)$

$E[X] = 1/\lambda \qquad Var(X) = 1/\lambda^2$

$X \sim \exp(\lambda) \qquad a > 0$

$aX \sim \exp\left(\frac{\lambda}{a}\right)$

$\frac{X}{a} \sim \exp(a\lambda)$

$$P(X > s+t \mid X > s) = P(X > t)$$

$$\begin{aligned} &\| \\ &\frac{P(X > s+t,\ X > s)}{P(X>s)} = \frac{P(X > s+t)}{P(X>s)} \\ &= \frac{\exp(-\lambda(s+t))}{\exp(-\lambda(s))} = \exp(-\lambda t) \end{aligned}$$

$$\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$$

<u>Def</u> (homogeneous) Poisson Process

$\lambda > 0$

$\rightarrow$ random step functions that increase by 1 at a discrete set of times

① $N(0) = 0$

② $N(t_2) - N(t_1) \perp\!\!\!\perp N(s_2) - N(s_1)$

$[t_1, t_2] \cap [s_1, s_2]$

(3) $N(t) - N(s) \sim \text{Poiss}(\lambda(t-s))$

$X_1, X_2, \ldots,$ inter-arrival times

$S_1, S_2, \ldots,$ arrival times

$S_n = \sum_{i=1}^{n} X_i$

<u>Prop</u> each $X_i \sim \exp(\lambda)$
all independent

<u>Pf</u> $P(X_1 > t) = P(N(t) = 0)$

$= P(\underbrace{N(t) - N(0)}_{\text{Poiss}(\lambda t)} = 0)$

$= e^{-\lambda t}$

$\hookrightarrow$ 1 - cdf of an $\exp(\lambda)$ evaluate at t

$\Longrightarrow X_1 \sim \exp(\lambda)$

$$P(X_2 > t \mid X_1 = s)$$

$$= P(\underbrace{N(t+s) - N(s)}_{\text{Poiss}(\lambda t)} = 0)$$

$$= \exp(-\lambda t)$$

We want to simulate a Poisson process until time T

$\to$ equivalent to generating the set of arrival times

all we need to do is generate $S_n = \sum_{i=1}^{n} X_i$

until we get m

s/t $S_m > T$

return $S_0, S_1, \dots, S_{m-1}$

① $t = 0$, $S = [\,]$

② $\Delta t \sim \exp(\lambda)$

③ if $t + \Delta t > T$
return S

else
$t = t + \Delta t$
S.append(t)
go back to step ②

$\Longrightarrow$ expected # of exp rvs.
is λT

$X_1 \sim \text{Poiss}(\lambda_1)$
$X_2 \sim \text{Poiss}(\lambda_2)$

$X_1 \mid X_1 + X_2 = n$

$$P(X_1 = x \mid X_1 + X_2 = k)$$

$$= \frac{P(X_1 = x,\ X_1 + X_2 = k)}{P(X_1 + X_2 = k)}$$

$$= \frac{P(X_1 = x,\ X_2 = k - x)}{P(X_1 + X_2 = k)}$$

$$= \frac{\left\{\frac{\cancel{e^{-\lambda_1}} \lambda_1^{x}}{x!}\right\} \cdot \left\{\frac{\cancel{e^{-\lambda_2}} \lambda_2^{(k-x)}}{(k-x)!}\right\}}{\left\{\frac{\cancel{e^{-(\lambda_1+\lambda_2)}} (\lambda_1 + \lambda_2)^{k}}{k!}\right\}}$$

$$= \frac{k!}{x!(k-x)!} \left(\frac{\lambda_1}{(\lambda_1 + \lambda_2)}\right)^{x} \left(\frac{\lambda_2}{(\lambda_1 + \lambda_2)}\right)^{k-x}$$

$$= \binom{k}{x} p^{x} (1-p)^{k-x} \qquad p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$N \sim \mathrm{Poiss}P(\lambda)$

$$P(N(t)=x \mid N(T)=k)$$

$$= \binom{k}{x} \left(\frac{t}{T}\right)^{x} \left(1-\frac{t}{T}\right)^{k-x}$$

$N(t) \sim \text{Poiss}(\lambda t)$ *

$N(T) - N(t) \sim \text{Poiss}(\lambda (T-t))$

$$p = \frac{\cancel{\lambda} t}{\cancel{\lambda t} + \cancel{\lambda}(T-\cancel{t})} = \frac{t}{T}$$

$\Longrightarrow$ unordered arrival times conditional $N(T)=k$

are $\text{Unif}(0,T)$

① $N(T) \sim \text{Poiss}(\lambda T)$

② $U_1, \dots, U_{N(T)} \sim \text{Unif}(0,1)$

return $T \cdot U_1, TU_2, \dots, TU_{N(T)}$