$$\int_0^1 f(x)\,dx = \theta$$

$U \sim Unif(0,1)$

$$= E[f(U)]$$

$U_1, \dots, U_n \sim Unif(0,1)$

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} f(U_i)$$

$$\int_{h(0)=a}^{h(1)=b} f(x)\,dx = \left[\int_0^1 \underbrace{f(h(x))\,h'(x)}\,dx\right] \quad U \sim Unif(0,1)$$

$h(0) = a$

$h(1) = b$

$$= E[f(h(U))\,h'(U)]$$

$U_1, \dots, U_n \sim Unif(0,1)$

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} f(h(U_i))\,h'(U_i)$$

if

$h(x) = a + (b-a)x$

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} f(a + (b-a)U_i)(b-a)$$

<u>Case 2:</u> $\theta = \int_{-\infty}^{b} f(x)\,dx$

choose different h

$h(0) = -\infty$

$h(1) = b$

$h(x) = 1 - \frac{1}{x} + b$

$h'(x) = \frac{1}{x^2}$

$$\int_{-\infty}^{b} f(x)\,dx = \int_0^1 \underbrace{f\left(1 - \frac{1}{x} + b\right)\left(\frac{1}{x^2}\right)} dx$$

$$= E\left[f\left(1 - \frac{1}{U} + b\right)\left(\frac{1}{U^2}\right)\right]$$

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} f\left(1 - \frac{1}{U_i} + b\right) \cdot \frac{1}{U_i^2}$$

<u>Case 3:</u> $\int_a^{\infty} f(x)\,dx$ w.r.t $h(0) = a$, $h(1) = \infty$

$h(x) = \frac{1}{1-x} - 1 + a$

$h'(x) = \frac{1}{(1-x)^2}$

$$= \int_0^1 f\left(\frac{1}{1-x} - 1 + a\right) \cdot \frac{1}{(1-x)^2}\,dx$$

$U \sim Unif(0,1)$

$$= E\left[f\left(\frac{1}{1-U} - 1 + a\right) \cdot \frac{1}{(1-U)^2}\right]$$

$$= E\left[f\left(\frac{1}{U} - 1 + a\right) \cdot \frac{1}{(U)^2}\right]$$

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{U_i} - 1 + a\right) \cdot \frac{1}{U_i^2}$$

<u>Case 4:</u> $\int_{-\infty}^{\infty} f(x)\,dx$

$$h(x) = \frac{1}{1-x} - \frac{1}{x}$$

$$h'(x) = \frac{1}{(1-x)^2} + \frac{1}{x^2}$$

$\underline{Ex}$ $\int_0^\infty e^{-x}\,dx$ this is (a,∞) case with $a=0$

use:

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{U_i} - 1 + a\right) \cdot \frac{1}{U_i^2}$$

$$= \frac{1}{n}\sum_{i=1}^{n} \exp\left(-\left(\frac{1}{U_i} - 1\right)\right) \cdot \frac{1}{U_i^2}$$

Integration in higher dimensions

$U_1, \dots, U_d \overset{iid}{\sim} Unif(0,1)$

joint density

$$f(x_1, \dots, x_d) = \mathbb{1}\{0 < x_1 < 1\} \cdot \mathbb{1}\{0 < x_2 < 1\} \cdot \dots \cdot \mathbb{1}\{0 < x_d < 1\}$$

$$= \prod_{i=1}^{d} \mathbb{1}\{0 < x_i < 1\}$$

$$E[g(U_1, \dots, U_d)] = \int_0^1 \dots \int_0^1 g(x_1, \dots, x_d)\, d\underline{x_d \dots x_1}$$

$$= \int_I g(\vec{x})\, d\vec{x}$$

$$I = [0,1] \times [0,1] \times \dots \times [0,1]$$

$$\theta = \int_I g(\vec{x})\, d\vec{x}$$

$$= E\left[g(U_1, \dots, U_d) \right]$$

$U_{11}, \dots, U_{1n}$

$U_{21}, \dots, U_{2n}$ all iid Unif $(0,1)$

$\vdots$

$U_{d1}, \dots, U_{dn}$

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} g(U_{1i}, U_{2i}, \dots, U_{di})$$

<u>Integration by substitution in higher-dimensions</u>

$x \in \mathbb{R}^d$ $\quad g : \mathbb{R}^d \to \mathbb{R}^d$

$g_i : \mathbb{R}^d \to 1$

$$g = \begin{pmatrix} g_1(\vec{x}) \\ \vdots \\ g_d(\vec{x}) \end{pmatrix}$$

<u>Ex</u> $g(x,y) = \begin{pmatrix} x^2 y \\ x+y \end{pmatrix}$

$J_g : \mathbb{R}^d \to \mathbb{R}^{d \times d}$

$$J_g = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_d} \\ \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_d} \\ \vdots & & \vdots \end{bmatrix}$$

$$\begin{bmatrix} \vdots & \vdots \\ \frac{\partial g_d}{\partial x_1} & \frac{\partial g_d}{\partial x_d} \end{bmatrix}$$

$$\int_{h(a)}^{h(b)} f(x)\,dx = \int_a^b f(h(x))\,h'(x)\,dx$$

$$\int_{g(A)} f(\vec{x})\,d\vec{x} = \int_A \underbrace{f(g(x))}\cdot \underbrace{|\det J_g(x)|}\,dx$$

<u>Ex</u> (polar coordinates)

$$\iint_D e^{x^2+y^2}\,dy\,dx$$

$x = r\cos\theta$

$y = r\sin\theta$

$$g(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

D = unit disk

$$J_g(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$|\det J_g| = (\cos\theta)(r\cos\theta) - (-r\sin\theta \cdot \sin\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$\int_0^{2\pi}\int_0^1 e^{r^2} \cdot r \, dr \, d\theta$$

Let's suppose we're interested in integrating some function

$$\int_E f(x_1, \dots, x_d) \, d\vec{x}$$

$$g(I) = E$$

$$= \int_I \underbrace{f(g(x)) \, |\det J_g|}_{} \, d\vec{x}$$

if define

$$h(x_1, \dots, x_d) = f(g(x_1, \dots, x_d)) \, |\det J_g|$$

$$= E[h(U_1, \dots, U_d)]$$

Ex

$$\iint_D e^{x^2+y^2}\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^1 e^{r^2} r\, dr\, d\theta$$

$$= \int_0^1\int_0^1 (e^{r^2} r)(2\pi)\, dr\, d\theta$$

$U_1, \dots, U_n$

$$\frac{1}{n}\sum_{i=1}^{n} e^{U_i^2} \cdot U_i (2\pi)$$

Ex

$$\int_E f(x_1, \dots, x_d)\, d\vec{x}$$

$$E = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_d, b_d]$$

$$g(x_1, \dots, x_d) = \begin{pmatrix} a_1 + (b_1 - a_1) x_1 \\ a_2 + (b_2 - a_2) x_2 \\ \vdots \\ a_d + (b_d - a_d) x_d \end{pmatrix}$$

$$J_g = \begin{bmatrix} (b_1 - a_1) & 0 & \cdots & \\ 0 & (b_2 - a_2) & 0 & \cdots \\ & & \ddots & 0 \\ 0 & & & (b_d - a_d) \end{bmatrix}$$

$$|J_g| = \prod_{i=1}^{n} (b_i - a_i)$$

$$\int_E f(x_1, \dots, x_d) \, d\vec{x}$$

$$= \int_I f\left(a_1 + (b_1 - a_1)x_1, \ldots, a_d + (b_d - a_d)x_d\right) \cdot \prod_{i=1}^{d} (b_i - a_i)\, d\vec{x}$$

$$= E\left[f\left(a_1 + (b_1 - a_1)U_1, \ldots, a_d + (b_d - a_d)U_d\right) \cdot \prod_{i=1}^{d} (b_i - a_i) \right]$$

$U_{11}, \ldots, U_{1n}$

$U_{21}, \ldots, U_{2n}$ all iid $\mathrm{Unif}(0,1)$

$\vdots$

$U_{d1}, \ldots, U_{dn}$

$$\hat{\Theta} = \frac{1}{n} \sum_{i=1}^{n} f\left(a_1 + (b_1 - a_1)U_{1i}, \ldots, a_d + (b_d - a_d)U_{di}\right) \cdot \prod_{i=1}^{d} (b_i - a_i)$$