$X \sim MVN(\mu, \Sigma)$

$\mu \in \mathbb{R}^d$

$\Sigma \in \mathbb{R}^{d \times d}$ positive-definite sym matrix

$E[X] = \mu$

$Var(X) = \Sigma$

$X \sim MVN(0, I)$

X is a vector of iid $N(0,1)$ r.vs.

$X \sim MVN(\mu, \Sigma)$

$$\underbrace{a + AX \sim MVN(a + A\mu, A \Sigma A^T)}$$

$\Sigma = L L^T$

$X \sim MVN(0, I)$

$\mu + LX \sim MVN(\mu, \Sigma)$

Copula is the joint cdf of

$(U_1, \ldots, U_d)$ $U_i \sim Unif(0,1)$

but not necessarily

ind.

$X = (X_1, \ldots, X_d)$ $F_1, \ldots, F_d$

copula of X is the joint cdf of

$(F_1(X_1), \ldots, F_d(X_d))$

Gaussian copula correlation matrix

$\Sigma \Rightarrow \Sigma_{ii} = 1$

copula of $MVN(0, \Sigma)$

$X = (X_1, \ldots, X_d)$

$Y = (\Phi(X_1), \ldots, \Phi(X_d))$

sample fr. a random vector X
w/ marginals given by $F_1, \ldots, F_d$
Gaussian copula w/ Σ (unit diagonal)

① $Y \sim MVN(0, \Sigma)$

② $U = (\Phi(Y_1), \ldots, \Phi(Y_d))$

③ $X = (F_1^{-1}(\Phi(Y_1)), \ldots, F_d^{-1}(\Phi(Y_d)))$

$X_1 = (X_{11}, \ldots, X_{1d})$

$\vdots$

$X_n = (X_{n1}, \ldots, X_{nd})$

fit models for marginals
$\hat{F}_1, \ldots, \hat{F}_d$

$U_i = (\hat{F}_1(X_{i1}), \ldots, \hat{F}_d(X_{id}))$

$$Y_i = \left(\Phi^{-1}(\hat{F}_1(X_{i1})), \ldots, \Phi^{-1}(\hat{F}(X_{id})) \right)$$

---

t-copula can be used to capture more extreme tail dependencies

$$Z \sim N(0,1) \qquad T = \frac{Z}{\sqrt{Y/r}} \sim t_r$$

$$Y \sim \chi^2_r$$

$$X \sim MVN(0, \Sigma)$$

$$Y \sim \chi^2_r$$

$$T = \frac{X}{\sqrt{Y/r}} \sim T_{\Sigma, r}$$

① $X \sim MVN(0, \Sigma)$ $\quad \Sigma_{ii} = 1$

$Y \sim \chi^2_v$

$$T = \frac{X}{\sqrt{Y/v}}$$

$F_{T,r}$ = cdf of a t-dist with r

②

$$U = (F_{T,r}(T_1), \ldots, F_{T,r}(T_d))$$

③ $G_1, \ldots, G_d$

$$B = (G_1^{-1}(F_{T,v}(T_1)), \ldots, G_d^{-1}(F_{T,v}(T_d)))$$

Markov Chain Monte Carlo

Def Time homogeneous Markov Chain is a collection of random variable

$X_1, \dots, X_t, \dots$

that have a common support S

where conditional distribution

$$p(x_t \mid x_{t-1}, \dots, x_1) \quad \text{for all } t$$

$$= \underbrace{p(x_t \mid x_{t-1})}_{p(x \mid y)}$$

Given a Markov Chain

$$p(x_t | x_{t-1})$$

$P_t \leftarrow$ marginal distn of X_t

$\hookrightarrow$ this depends only on P_1

$$P_t(x_t) = \int p(x_t | x_{t-1}) P_{t-1}(x_{t-1}) \, dx_{t-1}$$

$$= \int \cdots \int p(x_t | x_{t-1}) p(x_{t-1} | x_{t-2}) \cdots p(x_2 | x_1) P_1(x_1) \, dx_1 \, dx_2 \cdots dx_t$$

<u>Def</u> a distribution P^* is stationary wrt to a transition kernel $p(x|y)$ if

$$p^*(x) = \int p(x|y)\, p^*(y)\, dy$$

$\Longrightarrow$ If I have Markov Chain w/ $p(x|y)$, and I initialize w/ p^* that is stationary w.r.t. $p(x|y)$,

then the marginal of X_t is p^* for every t.

$\underline{p_1 = p^*}$

$$p_2(x) = \int p(x|y)\, p_1(y)\, dy$$

$$= \int p(x|y)\, p^*(y)\, dy$$

$$= p^*(x)$$

Under some technical conditions,

given $p(x|y)$, $\exists$ unique p^* stationary.

If p^* exists, regardless of what we use as p_1

$$\lim_{t \to \infty} p_t = p^*$$

Idea: inverse of the above

want to sample from f

find a Markov Chain whose stationary distribution is f

sample X_1 however I want

generate a large # of transitions,

$X_t \sim f$ for t large

Metropolis - Hastings

a sufficient condition for f to be stationary wrt $p(x|y)$

$$\forall x,y \quad p(x|y)\, f(y) = p(y|x)\, f(x)$$

$$P_2(x) = \int p(x|y)\, f(y)\, dy$$

$$= \int p(y|x)\, f(x)\, dy$$

$$= f(x) \int p(y|x)\, dy$$

$$= f(x)$$

$$\frac{p(x|y)}{p(y|x)} = \frac{f(x)}{f(y)}$$

Split into 2 steps:

① generate proposal from $X_t = y$

$\theta \sim h(x|y)$

② given θ, accept or reject

w/ $a(\theta = x|y)$

$U \sim \text{Unif}(0,1)$

If $U \leq a(\theta|y)$

$X_{t+1} = \theta$

O/w $\quad X_{t+1} = y$

$$p(x|y) = h(x|y)\, a(x|y) \quad x \neq y$$

$$p(x|x) = \int h(y|x)\,(1 - a(y|x))\, dy$$

can re-write this:

$$\frac{p(x|y)}{p(y|x)} = \frac{f(x)}{f(y)}$$

as

$$\frac{h(x|y)\, a(x|y)}{h(y|x)\, a(y|x)} = \frac{f(x)}{f(y)}$$

assume we have a proposal kernel

$$\frac{a(x|y)}{a(y|x)} = \frac{f(x)\, h(y|x)}{f(y)\, h(x|y)}$$

$$a(x|y) = \min\left(1, \frac{f(x)\, h(y|x)}{f(y)\, h(x|y)}\right)$$

$$f(x)\, h(y|x) < f(y)\, h(x|y)$$

① $X_1 \sim$ whatever $\quad t=1$

② $\theta_{t+1} \sim h(\ \mid X_t)$

$U \sim \text{Unif}(0,1)$

if $U \le a(\theta_{t+1} \mid X_t)$

then $X_{t+1} = \theta_{t+1}$

o/w $X_{t+1} = X_t$

$t = t+1$

repeat until I have generated enough transitions