2: Nonhomogeneous Poisson Process

Nonhomogeneous Poisson Process Basics

We say a process $N(t)$ is a nonhomogeneous Poisson Process with intensity $\lambda(t) > 0$ for $t \geq 0$ if:

1. N(0) = 0
2. Non-overlapping increments are independent
3. $N(t_2) - N(t_1) \sim Poisson(\int_{t_1}^{t_2} \lambda(s)ds)$

An equivalent way of formulating condition three (we are hand-waving a bit here) is as follows:

1. $\lim_{h \to 0} P(N(t+h) - N(t) = 1)/h = \lambda(t)$
2. $\lim_{h \to 0} P(N(t+h) - N(t) > 1)/h = 0$

Thinning Method

Proposition Suppose $M(t) \sim PP(\lambda)$. If there is some function $0 < p(t) < 1$ for $t \geq 0$. If N(t) is a counting process, where each event in $M(t)$ is independently accepted with probability $p(t), then N(t) is a non-homogeneous Poisson Process with $\lambda(t) = \lambda p(t)$.

"Proof" N(0) = 0 and independent increments follow directly from the construction.

$$\lim_{h \to 0} \frac{P(N(t+h) - N(t) = 1)}{h} = \frac{P(M(t+h) - M(t) = 1, accepted)}{h} = \lambda p(t) = \lambda(t)$$

We can similarly show that the probability of having two or more events goes to zero as h goes to zero.

This method suggests an algorithm for generating random variables from a nonhomogeneous Poisson Process up until time T:

1. Find $\lambda = \max_t \lambda(t)$, let $p(t) = \frac{\lambda(t)}{\lambda}$
2. Generate the arrivals of $M(t) \sim PP(\lambda) : t_1, ..., t_{M(t)}$
3. Keep each one with probability $\frac{\lambda(t_1)}{\lambda}, ..., \frac{\lambda(t_{M(t)})}{\lambda}$

Thinning Method Implementation

Implement the thinning Method for a non-homogeneous poisson process, given a function $\lambda(t)$ with maximum value λ_{max}.

Generate 10,000 paths from a nonhomogeneous poisson process with intensity $lambda(t) = -(t-1)^2 + 2$ for $0 \leq t \leq 2$. Use scipy.optimize.minimize_scalar to find the maximum of this function.

Over the grid t = [.01, .02, ..., 1.0], plot mean number of events that have occured by that time over the set of paths. Compare to to the theoretical integral of $\lambda(t)$, $-t^3/3 + t^2 + t$

Order Statistic Method

We can do something similar to the order statistic method we used for the regular Poisson process; however, this time we will be sampling from a different distribution than the uniform (accounting for the fact that the intensity of the nonhomogeneous poisson process is not constant).

Let us briefly introduce some notation, the expected number of arrivals by time T can be expressed as:

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

.

If we recall, given two independent poisson random variables, X and Y with means λ_1 and λ_2 respectively:

$$X|X+Y=k \sim Bin(k, \frac{\lambda_1}{\lambda_1+\lambda_2})$$

This means that, for $t < T$

$$N(t)|N(T) = N(t) + (N(T) - N(t)) = k \sim Bin(k, \frac{\Lambda(t)}{\Lambda(T)})$$

Which means, more specifically:

$$P(N(t)=i|N(T)=k) = \binom{k}{i}(\frac{\Lambda(t)}{\Lambda(T)})^i(1-\frac{\Lambda(t)}{\Lambda(T)})^{k-i}$$

, which implies that the conditional cdf of the (unordered) arrival times is: $\frac{\Lambda(t)}{\Lambda(T)}$ for $0 < t < T$, and therefore the pdf is: $\frac{\lambda(t)}{\Lambda(T)}$ for $0 < t < T$.

This gives us the following algorithm:

1. Calculate $\Lambda(T)$
2. Generate $N(T) \sim Poisson(\Lambda(T))$
3. generate $X_1, ..., X_{N(T)} \sim \frac{\lambda(t)}{\Lambda(T)}$
4. return the sorted arrival times $X_{(1)}, \ldots, X_{(N(T))}$

Order Statistic Method

Implement the order statistic method, following the below scaffold for the intensity function:

$\lambda(t) = 42te^{-t^2}$ for $0 \leq t \leq 6$.

Inversion Method

Before we begin, we make one quick note about the distribution of inter-arrival times for a non-homogenous Poisson Process. If t_i is the i_th arrival time, then:

$$P(t_i > T|t_{i=1} = t) = P(N(T) - N(t) = 0) = \exp(-\int_t^T \lambda(s)ds)$$

Proposition Let $N(t) \sim NPP(\lambda(t))$ with mean function $\Lambda(t) = \int_0^t \lambda(s)ds$. Suppose Λ is strictly increasing (i.e. \lambda is never zero). If $t_1, ..., t_n$ are the arrival times (before some ending time T_{\end}, then $\Lambda(t_1), ..., \Lambda(t_n)$ are the arrival times of a Poisson Process $M(t)$ with intensity $\lambda = 1$.

Proof (informal) Since Λ is strictly increasing, an inverse function Λ^{-1} exists. Then, if $t_1, ..., t_n$ are the arrival times of a Poisson Process $M(t)$ with intensity, $\Lambda^{-1}(t_1), ..., \Lambda^{-1}(t_n)$ are the arrival arrival times of the corresponding nonhomogeneous Poisson process.

If we recall, for M(t), $P(t_i > T|t_{i-1} > t) = \exp(-(T - t))$.

Then,

$$\begin{aligned} P(\Lambda^{-1}(t_i) > T|\Lambda^{-1}(t_{i-1}) = t) &= P(t_i > \Lambda(T)|t_{i-1} = \Lambda(t)) \\ &= \exp(-(\Lambda(T) - \Lambda(t)) \\ &= \exp(-\int_t^T \lambda(s)ds) \end{aligned}$$

Which is exactly the distribution of the interrarrival times of the nonhomogeneous poisson process of interest.

This suggests the following algorithm:

1. Given $\Lambda(t)$, find the inverse function $\Lambda^{-1}(t)$
2. Generate the arrival times $s_1, ..., s_n$ of $M(t) \sim PP(1)$ from $0 \leq t \leq \Lambda(T)$
3. Return the arrival times $t_1 = \Lambda^{-1}(s_1), ..., t_n = \Lambda^{-1}(s_n)$

Inversion Method

Implement the inversion method given generic mean function $\Lambda(t)$.

Generate some paths using $\Lambda(t) = 1 - exp(-t^2)$ from $0 \leq t \leq 10$.

Generate 10,000 paths from a nonhomogeneous poisson process with mean function $\Lambda(t) = 1 - exp(-t^2)$ from $0 \leq t \leq 10$.

Over the grid t = [.01, .02, ..., 1.0], plot mean number of events that have occured by that time over the set of paths. Compare to the theoretical mean function.