X

$S_x = \{x_1 < x_2 < x_3 < \dots\}$

$p(x) = P(X = x)$ $\quad p_i = p(x_i)$

$F(x) = P(X \leq x)$ $\quad F_i = F(x_i)$

$F_0 = 0$

$U \sim \text{Unif}(0,1)$

return x_j if

$F_{j-1} < U \leq F_j$

$P(X = x_j) = P(F_{j-1} < U \leq F_j)$

$= F_j - F_{j-1} = p_j$

$g(x) : \mathbb{R} \to \mathbb{R}$

$g(x) = y$

$g^{-1}(y) = g^{-1}(g(x))$

$= x$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1/2, & 0 \le x < 1 \\ 3/4, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

$$F^{-1}(u) = \min\{x : F(x) \ge u\}$$

$\Rightarrow$ inverse transform can be written $F^{-1}(U)$

$X \sim geo(q)$ $\quad S_x = \{1, 2, 3, \ldots\}$

$$F(n) = 1 - (1-q)^n$$

$$u = 1 - (1-q)^n$$

$$(1-q)^n = 1 - u$$

$$n = \left\lceil \frac{\log(1-u)}{\quad} \right.$$

'log (1-q) '

<u>Rejection Sampling</u>

We want to sample from a dist'n that S_x and pmf p

$$S_x = \{x_1 < x_2 < \dots\} \qquad p(x_i) = p_i$$

Suppose dist'n with support $S_Y \supseteq S_x$ and it has pmf q

$\Longrightarrow$ can sample from q

suppose $c > 1$ s/t $p(x) \leq c \cdot q(x)$ for all x

$$\frac{p(x)}{q(x)} \leq c \quad \text{if } q(x) > 0$$

① $Y \sim q$

② $U \sim \text{Unif}(0,1)$

$p(Y)$

③ If $U \le \frac{p(Y)}{cq(Y)}$

return Y

o/w go back to step ①

$\Longrightarrow$ this algorithm returns random numbers that have pmf p

the number of iterations to return a number is a geometric random variable with mean c

$S_x = \{x_1, \dots, x_n\}$

$$h(x) = \frac{p(x)}{q(x)} \le c$$

$$c = \max_{x_1,\dots,x_n} \frac{p(x_i)}{q(x_i)}$$

$$x^* = \operatorname{argmax} h(x)$$

$$h(x^*) = c$$

for each y

$$P(\text{keeping}) = \frac{p(y)}{q(y)} \Big/ \frac{p(x^*)}{q(x^*)}$$

$X \to$ output of algorithm

$Y \to$ proposal

$$p_j = P\left(Y = x_j \,\middle|\, U \le \frac{p(Y)}{c\,q(Y)}\right)$$

$$= \frac{\overset{(1)}{P\left(Y = x_j,\ U \le \frac{p(Y)}{c\,q(Y)}\right)}}{\underbrace{P\left(U \le \frac{p(Y)}{c\,q(Y)}\right)}_{(2)}}$$

$$(1)\quad P(Y = x_j) \cdot P\left(U \le \frac{p(\overset{x_j}{Y})}{c \cdot q(\underset{x_j}{Y})} \,\middle|\, Y = x_j\right)$$

$$= q_j \cdot \frac{p_j}{c \cdot q_j} = \frac{p_j}{c}$$

② $P\left(U \le \frac{p(Y)}{c\,q(Y)}\right)$

$$= \sum_{x_i \in \mathcal{X}_Y} P\left(U \le \frac{p(x_i)}{c\,q(x_i)} \,\Big|\, Y = x_i\right) \cdot P(Y = x_i)$$

$$= \sum \cancel{q_i} \cdot \frac{p_i}{c \cdot \cancel{q_i}} = 1/c$$

$$① / ② = \frac{p_j}{c} \Big/ 1/c = p_j$$

<u>Ex</u> $S_x = \{1, 2, 3, 4, 5\}$

$p(1) = 0.25$

$p(2) = 0.15$

$p(3) = 0.22$

$p(3) = 0.$ [illegible]

$p(4) = 0.18$

$p(5) = 0.2$

g discrete uniform on $1, \ldots, 5$

$$g(i) = 0.2$$

$$c = \max_{x \in S_x} \frac{p(x)}{g(x)} = \max \frac{p(x)}{0.2}$$

$$= \frac{1}{0.2} \cdot \max(p(x))$$

$$= \frac{0.25}{0.2} = 1.25$$

1. $U_1 \sim \text{Unif}(0,1)$

   $Y = \lceil 5U_1 \rceil$

2. $U_2 \sim \text{Unif}(0,1)$

3. If $U_2 \leq \dfrac{p(Y)}{g(Y) \cdot 1.25}$

$$= \frac{p(Y)}{(0.2)(1.25)}$$

return Y

o/w go back to step 1

$U_1 = 0.9$

$Y = 5$

$U_2 = 0.99$

is $0.99 \le \frac{p(5)}{(0.2)(1.25)} = \frac{\cancel{0.2}\ 1}{\cancel{(0.2)}(1.25)}$

0.8 no!

$U_1 = 0.6$

$Y = 3$

$U_2 = 0.1$

is $0.1 \le \frac{p(3)}{(0.2)(1.25)}$

$$\left(\frac{0.22}{0.2}\right) / 1.25)$$

$$1.1 / 1.25$$