# STMC HKOI Notes

Useful mathematics

## 1 Summation and Products

### 1.1 Summation ($\Sigma$ Notation)

Let $a_i$ be a sequence. Then the summation of $a_i$ from $i = m$ to $i = n$ is defined as:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-2} + a_{n-1} + a_n \tag{1}$$

For example:

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + \cdots + n - 1 + n$$

$$\sum_{i=1}^{10} \frac{i(i-1)}{i+2} = \frac{1(1-1)}{1+2} + \frac{2(2-1)}{2+2} + \frac{3(3-1)}{3+2} + \cdots + \frac{9(9-1)}{9+2} + \frac{10(10-1)}{10+2}$$

### 1.2 Product ($\Pi$ Notation)

Similarly, let $a_i$ be a sequence. Then the product of $a_i$ from $i = m$ to $i = n$ is defined as:

$$\prod_{i=m}^{n} a_i = a_m \times a_{m+1} \times \cdots \times a_{n-2} \times a_{n-1} \times a_n \tag{2}$$

For example:

$$\prod_{i=1}^{10} \frac{i}{i+1} = \left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right)\cdots\left(\frac{9}{9+1}\right)\left(\frac{10}{10+1}\right)$$

## 1.3 Useful results

### 1.3.1

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2} \tag{3}$$

Consider an inertial reference frame (i.e not accelerating) which will be denoted $S_0$, and a accelerating reference frame, $S$ that has an acceleration of $A$.

**Note 1. Capital Letters refer to the accelerating reference frame *S* while lowercase letters refer to the inertial reference frame $S_0$**

Picture a moving reference frame, $S$, moving relative to $S_0$. Imagine in the the moving reference frame that a ball with mass, $m$ is being thrown. In order to consider the motion of the ball, the motion must be first considered in the inertial reference frame.

$$F = m\ddot{r_0} \tag{4}$$

Where $r_0$ is the ball's position relative to $S_0$.

Now, by considering the motion of the ball in the accelerating frame, the ball position relative to $S$ is $R$. (It's velocity is $\dot{R}$. Thus, relating $R$ to $r_0$, we have:

$$\dot{r_0} = \dot{R} + V \tag{5}$$

Newton's second law for the inertial reference frame by differentiate and multiplying by mass is:

$$F_{\text{inertial}} = -mA = -m\ddot{R} \tag{6}$$

## 1.4 The Tides

**The Tidal Force**

$$F_{tide} = -GM_m m(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}) \tag{7}$$

Where:

$$\begin{aligned} G &= \text{Gravitational Constant} \\ d &= \text{Object's Position Relative to Moon} \\ d_0 &= \text{Earth's Center Relative to the moon} \\ M_m &= \text{Mass of the moon} \end{aligned}$$

### 1.5 The Angular Velocity Vector

The rest of the notes and the chapter will over reference frames that are rotating with respect to the inertial reference frame, so angular velocity has to be used.

**Definition 1. Euler's Theorem** - The most general motion of any body relative to a fixed point $O$ is a rotation about some axis through $O$ To specify this rotation about a given point O, we only have to give the direction of the axis and the rate of rotation, or angular velocity $\omega$. Because this has a magnitude and direction, it is an obvious choice to write this rotation vector as $\omega$, the angular velocity vector. That is:

$$\omega = \omega \mathbf{u} \tag{8}$$

Where **u** is the unit vector

**Vector Velocity**
The velocity at any point, $P$ (position, $r$) is given by:

$$v = \omega \ x \ r \tag{9}$$

### Addition of Angular Velocities

One can add angular velocities just like linear velocities. If body 3 is rotating at angular velocity $\omega_{32}$ relative to frame 2, and frame 2 is rotating at angular velocity $\omega_{21}$ relative to frame 1, then body 3 is rotating relative to frame 1 at angular velocity:

$$\omega_{31} = \omega_{32} + \omega_{21} \tag{10}$$

### 1.6 Time Derivatives in Rotating Frames

If frame S has a angular velocity, $\Omega$ relative to $S_0$ then the time derivative of a single vector **Q** as seen in the two frames are related by:

$$(\frac{d\mathbf{Q}}{dt})_{S_0} = (\frac{d\mathbf{Q}}{dt})_S \ + \Omega \ x \ \mathbf{Q} \tag{11}$$

### 1.7 Netwon's Second Law in a Rotating Frame

A particle in an inertial reference frame, $S_0$ obeys Newton's second law as we are use to:

$$m\frac{d^2r}{dt^2} = F \tag{12}$$

Using the results from equation 8, the time derivative for a rotating frame with reference to an inertial frame can be given by:

$$(\frac{dr}{dt})_{S_0} = (\frac{dr}{dt})_s \ + \Omega \ x \ r \tag{13}$$

By differentiation, Newton's second law becomes:

$$m\ddot{r} = F + 2m\dot{r}\ x\ \Omega\ + m(\Omega\ x\ r)\ x\ \Omega \tag{14}$$

Where $F$ is the sum of all forces in the inertial reference frame.

## 1.8 The Centrifugal Force

This is an inertial force in a rotating reference frame

$$F_{\text{cf}} = m(\Omega\ x\ r)\ x\ \Omega \tag{15}$$

**Free-Fall Acceleration (Non-Vertical Gravity)**

$$F_{\text{eff}} = F_{\text{grav}} + F_{\text{cf}} = mg_0 + m\Omega^2 R\sin(\theta)\hat{\rho} \tag{16}$$

The acceleration due to the Centrifugal force is simply

$$\begin{aligned} g &= g_0 + \Omega^2 R\sin(\theta)\hat{\rho} \\ g_{\text{rad}} &= g_0 - \Omega^2 R\sin^2(\theta) \\ g_{\text{tan}} &= \Omega^2 R\sin(\theta)\cos(\theta) \end{aligned} \tag{17}$$

The angle between g and its radial direction is:

$$\alpha \approx \frac{g_{\text{tan}}}{g_{\text{rad}}} \tag{18}$$

The maxium value at ($\theta = 45$):

$$\alpha_{\text{max}} = \frac{\Omega^2 R}{2g_0} \tag{19}$$

## 1.9 Coriolis Force

The Coriolis Force is another inertial force in a rotating reference frame that an object experiences when it is moving.

$$F_{\text{cor}} = 2m\dot{r}\ x\ \Omega = 2mv\ x\ \Omega \tag{20}$$

The maximum acceleration, $a$ that the Coriolis force could produce acting by itself with $v$ perpendicular to $\Omega$ is:

$$a_{\text{max}} = 2v\Omega \tag{21}$$

**Direction of the Coriolis Force**
The Direction of the Coriolis force us always perpendicular to the velocity of the object (hence equation 17), and is given by the right hand rule.

### 1.10 Free Fall and the Coriolis Force

$$m\ddot{r} = mg_0 + F_{\text{cf}} + F_{\text{cor}} \tag{22}$$

### 1.11 The Foucault Pendulum

See chapter 9, Page 354. There is no need to recopy what is in the book here.