STMC HKOI Training

Mathematical Proofs

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Goal today

- Proof in mathematics
- Implication
- Contrapositive
- Contradiction
- Mathematical Induction

Mathematics is not arithmetic

- What does it mean to be good in mathematics?
- Are mathematicians human-calculators?
- In fact, mathematics is different from arithmetic, the computation of numbers

Example: Sum of odd numbers

Consider the following sum:

$$S = 1 + 3 + 5 + 7 + \cdots + 99$$

What is the value of the sum?

Example: Sum of odd numbers

- Someone who are reluctant to think will simply crunch the numbers and calculate the sum term by term (arithmetic)
- However, a mathematician would look for *patterns* in the sum
- Let's embark on a journy to enhanace our problem solving skills and be more like a thinker, but not a calculator

Example: Sum of odd numbers

- A very useful strategy in problem solving in general is to **list a few special cases**:

n	S
1	1
2	1+3=4
3	1+3+5=9
4	1+3+5+7=16
5	1+3+5+7+9=25

Did you spot a pattern?

Example: Sum of odd numbers

- According to our observation, it seems like the sum of first n odd numbers is n^2
- Therefore we form a **hypothesis**:

Hypothesis:

The sum of first n odd numbers is n^2

- *If* this hypothesis is true, then we can immediately get the answer, because 99 is the 50th odd number, so the answer is simply $50^2 = 2500$
- *But is it?* Before answering the question, let's look at the story of Gauss

Carl Friedrich Gauss

Carl Friedrich Gauss is one of the greatest mathematician in history. Often termed as the ”Prince of Mathematics”, he made exceptional contributions across many fields in mathematics and science.

Today, we shall look at a widely circulated story of his early genius and reflect on important elements of mathematical thinking along the way.

Story about summing from 0 to 100

This version of the story is adapted from NRICH

- When Gauss was still at primary school, his teacher one day told asked his class to add together all the numbers from $1 - 100$, assuming that this task would occupy them for quite a while.
- However, to the shock of his teacher, Gauss quickly found the correct answer: 5050

$$1 + 2 + 3 + \cdots + 100 = 5050$$

Story about summing from 0 to 100

- How did he do it? Eight year old Gauss pointed out that the problem was actually quite simple.
- What he observed is as follows:

$$
\begin{aligned}
S &= 1 + 2 + 3 + \cdots + 98 + 99 + 100 \\
S &= 100 + 99 + 98 + \cdots + 3 + 2 + 1 \\
\therefore 2S &= (1 + 100) + (2 + 99) \cdots + (99 + 2) + (100 + 1) \\
&= (101) + (101) + \cdots + (101) + (101) = 101 \times 100 \\
\therefore S &= 101 \times 50 = 5050
\end{aligned}
$$

Story about summing from 0 to 100

- What did we learn in this story?
- This story tells us that rather than blindly performing mental arithmetic, **a clean solution can be found by exploiting the structure of the problem**
- Furthermore, by investigating the structure, we in turns **gain more insight than headless computation**
- In fact, this is the difference between arithemtic and mathematics

Going back to the sum of odds ...

- Can you now provide an logical argument why the sum of first n odd numbers is n^2?

Going back to the sum of odds …

- Can you now provide an logical argument why the sum of first n odd numbers is n^2?
- Here is a solution

$$
\begin{aligned}
S &= 1 + 3 + 5 + \cdots + 2n - 1 \\
S &= 2n - 1 + 2n - 3 + \cdots + 1 \\
\therefore 2S &= (2n) + (2n) + \cdots + (2n) = 2n^2 \\
\therefore S &= n^2
\end{aligned}
$$

Short summary

Hopefully, through this example, you see that:

- The greatest insight always lies in discovering the *structure* of the problem
- Examples can reveals structure within a problem
- You have to *provide an argument (proof)* to support your hypothesis

Importance of proof

- In the summary above, we have stressed the importance of providing a proof / argument to support your claim
- Curious of you may wonder: Why all the trouble? After all, if something holds for many situation, wouldn't it holds for all situation?
- The short answer is no. **Unless we enumerate all possibilities, things that holds true for large number of cases can fail**

Importance of proof

- To see the point, let's consider a famous example.
- Consider the following sequence by Euler (1774):

$$F_n = n^2 - n + 41$$

- If we list the put $n = 0, 1, 2, 3, \cdots, 39, 40$ to the sequence, we will found that all of them are prime numbers (Try it!)
- However, when $n = 41$ F_{41} is obviously not a prime because $F_{41} = 1681 = 41^2$

Importance of proof

Some terms of F_n:

n	1	2	3	4	5	$\cdots$	37	38	39	40	41
F_n	41	43	47	53	61	$\cdots$	1373	1447	1523	1601	1681

From $n = 1 - 40$, F_n are all primes, but when $n = 41$ it fails. This illustrated something that holds true for a large number of cases won't necessarily holds forever.

Therefore, we need always need to find a **proof** to demonstrate results definitively.

Implication

- Let P,Q be two statements. When say **P implies Q** or **If P then Q**, we mean:

"If P is true, then Q is true"

- For example, when we say *"If the sky is cloudy, the sun is not visible"*. We have:

$$\begin{aligned} P &= \text{The sky is cloudy} \\ Q &= \text{The sun is not visible} \\ P \implies Q &= \text{If the sky is cloudy, then the sun is not visible} \end{aligned}$$

Implication

- Note that when P is false, Q can be either true or false. For example, even when the sky is not cloudy, it doesn't mean that the sun is visible, it can as well be that it's midnight.

$$\text{Sky not cloudy} \not\Longrightarrow \text{Sun is visible}$$

- In a similar vein, $P \implies Q$ **does not imply** $Q \implies P$. A simple counterexampe: If someone is your father, he is a male. But that does not imply if someone is a male, he is your father.

Implication

- Some more examples showing that $P \implies Q$ does not mean $Q \implies P$:
  1. If a shape is a square, it has 4 right angles $\not\Longrightarrow$ If a shape has 4 right angles, it is a square
  2. If a number is divisible by 4, it's divisible by 2 $\not\Longrightarrow$ If a number is divisible by 2, it's divisible by 4
  3. If x is positive, x^2 is positive $\not\Longrightarrow$ If x^2 is positive, x is positive
  4. If you are sad, you cry $\not\Longrightarrow$ If you cry, you are sad
- Hopefully these examples are enough to convince you that proving ”if A then B” and ”if B then A” are different things

Exercise

Let's prove the following statements:

1. Let a, b be two positive integers, if a is odd and b is even, then $a + b$ is odd (Hint: A odd number can alway be written in form of $2n + 1$)
2. If n is odd, then $3n + 5$ is even
3. If n is odd, then $4n^3 + 2n + 1$ is odd
4. If b, c are divisible by a, then bc is divisible by a

Proof by Contradiction

- Now we shall introduce some proving techniques. On of such techniques is called **proof by contradiction**
- Let's consider an example:

Example:

If x^2 is divisible by 2, then x is divisible by 2.

Proof.

Suppose it's wrong and there exist some x_0 so that x_0^2 is even but x_0 is odd. Then since x_0 is odd, $x_0 = 2n + 1$. But that implies $x_0^2 = (2n+1)^2 = 4n^2 + 4n + 1$ $= 2(2n+2) + 1$, which is odd. But x_0^2 cannot be odd and even, so contradiction. $\square$

Proof by Contradiction

- As illustrated in the example, a proof by contradiction works by assuming the statement is wrong, then derive a contradiction, that is, an impossible situation from the assumption, which disprove the assumption and proves the original statement.
- Let's demonstrate that with another example.

Proof by Contradiction

Statement

There exist no integers a, b for which $18a + 6b = 1$

Proof.

Suppose not, that is, there exist some integers a, b so that $18a + 6b = 1$, then divide both side by 6, we have $3a + b = 1/6$, but this is impossible, because the LHS is an integer by RHS is not. So a contradiction is reached. Hence there does not exist integers a, b for which $18a + 6b = 1$ $\square$

Exercise

Here are some exercises:

1. If two integers a, b satisfy $a + b \geq 19$, then $a \geq 10$ or $b \geq 10$
2. If $a^2 - 2a + 7$ is even then a is odd
3. If a is a real number, then $a^2 \geq 0$
4. If $ab = 0$ then either a or b equals to 0
5. If a, b are integers, then $a^2 - 4b - 2 \neq 0$
6. Suppose a, b, c are integers and $a^2 + b^2 = c^2$, then a or b is even

Proof by Contrapositive

- As we have illustrated in previous examples $P \implies Q$ does not mean $Q \implies P$.
- However, we can show that if $P \implies Q$, then not $Q \implies$ not P

Theorem (Contrapositive)

If $P \implies Q$, then not $Q \implies$ not P

Proof.

We shall use contradiction. Suppose $P \implies Q$ is true but somehow not $Q \implies$ not P is false. Then for some P, Q we have Q is false but P is true. But by assumption, $P \implies Q$, so Q must be true, which is a contradiction. $\square$

Proof by Contrapositive

- The result provide the grounding for a method of proof called **proof by contrapositive**
- In short, because $p \rightarrow q$ is equivalent to $\neg q \rightarrow \neg p$, we can prove a statement by it's contrapositive, which is sometimes easier
- Consider the following claim:
- **Claim:** Let $x \in \mathbb{Z}^+$. If $x^2 - 6x + 5$ is even, then x is odd.
- How can we prove this claim?

Proof by Contrapositive

- Instead proving directly, we prove it's contrapositive.
- The contrapositive of the statement is:
- **Contrapositive Claim:** If x is even, then $x^2 - 6x + 5$ is odd
- This is almost trivial to prove, because:

$$x^2 - 6x + 5 = \overbrace{x(x-6)}^{\text{even if x even}} + \overbrace{5}^{\text{odd}}$$

So the sum must be odd

Mathematical induction

- We shall introduce a proving technique which we will use very often in the future
- This is called the **mathematical induction**
- This method is very useful in proving statement of form $P(n)$, with n being positive integers. Examples of such statements are:

1. "$1+2+3+\cdots+n = n(n+1)/2$ for all positive integer n"
2. "For integer $n \geq 12$, there exist some integers a, b such that $n = 4a+5b$"
3. "$3^n - 1$ is a multiple of 2 for $n = 1, 2, 3, \cdots$"

- Let's try to introduce the idea with some example

Mathematical induction

Example

Prove that $P(n) = 1 + 2 + 3 + \cdots + n = n(n+1)/2$ is true for $n \in \mathbb{Z}^+$

Proof.

We will first prove that the statement is true when $n = 1$:
When $n = 1$, LHS=1, RHS=$1(1+1)/2 = 1$. So it's true.
Now **assume** $1 + 2 + 3 + \cdots + k = k(k+1)/2$ for some integer k. We want to show $P(k) \implies P(k+1)$.This is easy, because if we know $P(k)$ is true, then RHS$= 1 + 2 + 3 + \cdots + k + k + 1 = (1 + 2 + 3 + \cdots + k) + k + 1 = k(k+1)/2 + k + 1 = (k+1)(k+2)/2 =$LHS. So if $P(k)$ is true then $P(k+1)$ is true.
But we have proved $P(1)$ is true. So $P(1) \implies P(2)$ and $P(2) \implies P(3)$ and so on.
So we have successfully proved the statement is true for all n $\square$

Mathematical induction

- The example above illustrated the idea of mathematical induction
- In a mathematical induction proof, we first show that the statement holds for some simple cases (e.g. $n = 1$). This is also called the **base case**.
- Then we show that if the statement is true for some integer k, then the statement is also true for the next integer $k + 1$ (i.e. $P(k) \implies P(k+1)$). This is called the **induction step**
- Using the induction step repeatedly, we can show that the statement holds for all natural number n.
- For example, if the base case is $P(1)$, and we show that $P(k) \implies P(k+1)$. Then $P(2)$ is true because $P(1) \implies P(1+1)$. But now $P(3)$ is also true because $P(2) \implies P(2+1)$. Repeating this we can show P holds for all $n \geq 1$

Mathematical induction

Example (Sum of squares):

Prove that:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \geq 1$, n is integer using mathematical induction.

Mathematical induction

Proof.

Base case: When $n = 1$, LHS = $1^2 = 1$. RHS = $1(1+1)(2(1)+1)/6 = 1$. So $P(1)$ is true.

Induction step: Suppose $P(k)$ is true, that is:

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Mathematical induction

Proof.

Then when $P(k+1)$:

$$\begin{aligned}\text{LHS} &= 1^2 + 2^2 + 3^2 + \cdots k^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{\left[k(2k+1) + 6(k+1)\right](k+1)}{6} \\ &= \frac{(k+1)(k+2(2k+3)}{6} = \text{RHS}\end{aligned}$$

So by principle of mathematical induction the statement is true for all $n \geq 1$

$\square$

Mathematical induction

Example (Divisibility)

Show that $3^n - 1$ is divisible by 2 for $n = 1, 2, 3, \cdots$

Proof.

Base case: Trivial, because LHS=$3^1 - 1 = 2$ which is obviously divisible by 2.
Induction step: If $3^k - 1$ is divisible by 2 for some $k \in \mathbb{Z}^+$, then when $n = k + 1$:

$$\begin{aligned} \text{LHS} &= 3^{k+1} - 1 = 3(3^k - 1 + 1) - 1 \\ &= 3(3^k - 1) + 2 = 3(2N) + 2 = 2(3N + 1) \end{aligned}$$

which is obviously divisible by 2 $\square$

Mathematical induction

Example (Checkerboard filling)

Prove that for any checkerboard of size $2^n \times 2^n$ can be tiled with L-shaped tiles in such a way that there is exactly one $1x1$ left anywhere we like in the checkerboard.

Proof.

Refer to `https://www.amsi.org.au/teacher_modules/pdfs/Maths_delivers/Induction5.pdf` for the illustrations and proof. $\square$

Exercise

1. Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = [n(n+1)/2]^2$
2. Show that $a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} = a(1 - r^n)/(1 - r)$
3. Prove that $2^n \geq n$ for $n = 1, 2, 3, \cdots$
4. Prove that any integer $n \geq 2$ is either a prime or can be expressed as a product of primes
5. A country has n city, every city is connected by a one-way road. Show that there is a route that passes through every city.