Taller transformada fourier

① a) $F\{e^{-a|t|}\}, \quad a \in \mathbb{R}^+$

$$F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Definimos valor absoluto

$$|t| = \begin{cases} t & si \quad t \geq 0 \\ -t & si \quad t < 0 \end{cases}$$

$$= \int_{-\infty}^{0} e^{-a(t)}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a - j\omega t)}\, dt + \int_{0}^{\infty} e^{(-a - j\omega)t}\, dt$$

$$= \frac{1}{(a - j\omega)} e^{(a - j\omega)} \Big|_{0}^{\infty} + \frac{1}{(-a - j\omega)} e^{(-a - j\omega)t} \Big|_{0}^{\infty}$$

$$= \left[\frac{e^{(a-j\omega)(0)}}{a-j\omega} - \cancel{\frac{e^{(a-j\omega)(-\infty)}}{a-j\omega}}^{\,0} \right] + \left[\cancel{\frac{e^{(-a-j\omega)(\infty)}}{-a-j\omega}} - \dots \right.$$

$$\left. \dots \frac{e^{(-a-j\omega)(0)}}{-a-j\omega} \right]$$

$$= \frac{1}{a-j\omega} - \frac{1}{-a-j\omega} = \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$X(\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

b) $F\{\cos(\omega_c t)\}$; $\omega_c \in \mathbb{R}$

$$= \int_{-\infty}^{\infty} \cos(\omega_c t)\, e^{-j\omega t}\, dt$$

$$\therefore \cos(\omega_c t) = \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

$$X(\omega) = \frac{1}{2} \int_{-\infty}^{\infty} \left(e^{j\omega_c t} + e^{-j\omega_c t} \right) e^{-j\omega t}\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{j\omega_c t} e^{-j\omega t} \, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j\omega_c t} e^{-j\omega t} \, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{j(\omega_c - \omega)t} \, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{j(-\omega_c - \omega)t} \, dt$$

usando propiedades de delta de Dirac

$$\therefore \int_{-\infty}^{\infty} e^{-j\alpha t} \, dt = 2\pi \delta(\alpha)$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-j(\omega - \omega_c)t} \, dt + \int_{-\infty}^{\infty} e^{-j(\omega + \omega_c)t} \, dt \right]$$

aplicamos

$$= \frac{1}{2} \left[2\pi \delta(\omega - \omega_c) + 2\pi \delta(\omega + \omega_c) \right.$$

$$\boxed{X(\omega) = \pi \left[\delta(\omega - \omega_c) + \delta(\omega + \omega_c) \right]}$$

c) $F\{\operatorname{sen}(w_c t)\}$; $w_c \in \mathbb{R}$

$$= \int_{-\infty}^{\infty} \operatorname{sen}(w_c t)\, e^{-jwt}\, dt$$

$$\therefore \operatorname{sen}(w_c t) = \frac{1}{2j}\left(e^{jw_c t} - e^{-jw_c t}\right)$$

$$= \frac{1}{2j}\int_{-\infty}^{\infty} \left(e^{jw_c t} - e^{-jw_c t}\right) e^{-jwt}\, dt$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{jw_c t} e^{-jwt}\, dt - \int_{-\infty}^{\infty} e^{-jw_c t} e^{-jwt}\, dt\right]$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{j(w_c - w)t}\, dt - \int_{-\infty}^{\infty} e^{-j(w_c + w)t}\, dt\right]$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{-j(w - w_c)}\, dt - \int_{-\infty}^{\infty} e^{-j(w + w_c)t}\, dt\right]$$

aplicando dlta d dsrac

$$= \frac{1}{2j}\left[2\pi\,\delta(\omega - \omega_c) - 2\pi\,\delta(\omega + \omega_c) \right]$$

$$X(\omega) = \frac{1}{j}\,\frac{j}{j}\left[\pi\,\delta(\omega - \omega_c) - \pi\,\delta(\omega + \omega_c) \right]$$

$$\boxed{X(\omega) = j\pi\left[\delta(\omega + \omega_c) - \delta(\omega - \omega_c) \right]}$$

d) $\mathcal{F}\{ f(t)\cos(\omega_c t) \}$, $\omega_c \in \mathbb{R}$; $f(t) \in \mathbb{R}$ &

$$X(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega_c t)\, e^{-j\omega t}\, dt$$

$$\therefore \cos(\omega_c t) = \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

$$X(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} f(t)\left(e^{j\omega_c t} + e^{-j\omega_c t} \right) e^{-j\omega t}\, dt$$

$$X(\omega) = \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t)\, e^{j(\omega_c - \omega)}\, dt + \int_{-\infty}^{\infty} f(t)\, e^{j(\omega_c + \omega)}\, dt \right]$$

$$X(\omega) = \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t)\, e^{-j(\omega-\omega_c)t}\, dt + \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega+\omega_c)t}\, dt\right]$$

$\omega' = \omega - \omega_c$, $\omega'' = \omega + \omega_c$

$$X(\omega) = \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega' t}\, dt + \int_{-\infty}^{\infty} f(t)\, e^{-j\omega'' t}\, dt\right]$$

$$X(\omega) = \frac{1}{2}\left[F\{f(t)\} + F\{f(t)\}\right]$$

$$X(\omega) = \frac{1}{2}\left[F(\omega') + F(\omega'')\right]$$

$$\boxed{X(\omega) = \frac{1}{2}\left[F(\omega-\omega_c) + F(\omega+\omega_c)\right]}$$

e) $F\{e^{-a|t|^2}\}$ $a \in \mathbb{R}^+$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|^2} \cdot e^{-j\omega t}\, dt$$

$$|t| = \begin{cases} t & \text{Si } t \geq 0 \\ -t & \text{Si } t < 0 \end{cases}$$

$$X(w) = \int_{-\infty}^{0} e^{-a(t)^2} e^{-jwt}\, dt + \int_{0}^{\infty} e^{-at^2} e^{-jwt}\, dt$$

$$X(w) = \int_{-\infty}^{0} e^{-at^2} e^{-jwt}\, dt + \int_{0}^{\infty} e^{-at^2} e^{-jwt}\, dt$$

$$X(w) = \int_{-\infty}^{\infty} e^{-at^2} e^{-jwt})\, dt$$

$$X(w) = \int_{-\infty}^{\infty} e^{(-at^2 - \frac{jwt)}{a}}\, dt$$

Completamos cuadrados

$$-a\left(t^2 + \frac{jwt}{a}\right) \Rightarrow -a\left[\left(t + \frac{jw}{2a}\right)^2 - \left(\frac{jw}{2a}\right)^2\right]$$

$$\Rightarrow -a\left(t + \frac{jw}{2a}\right)^2 + a\left(\frac{jw}{2a}\right)^2 \Rightarrow -a\left(t + \frac{jw}{2a}\right)^2$$

$$+ a\,\frac{j^2 w^2}{4a^2}$$

$\rightarrow -a\left(t+\frac{j\omega}{2a}\right)^2 - \frac{\omega^2}{4a}$ volviendo a la integral

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a\left[t+\frac{j\omega}{2a}\right]^2} e^{\frac{-\omega^2}{4a}}\, dt$$

$u = t + \frac{j\omega}{2a}$

$du = dt$

$$X(\omega) = e^{-\frac{\omega^2}{4a}} \underbrace{\int_{-\infty}^{\infty} e^{-au^2}\, du}_{} \rightarrow \text{integral de gauss}$$

$$X(\omega) = e^{\frac{-\omega^2}{4a}} \sqrt{\frac{\pi}{a}} \Rightarrow \boxed{X(\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}}$$

F) $F\{A\,rect_d(t)\}$; $A, d \in \mathbb{R}$

$$X(\omega) = \int_{-\infty}^{\infty} A\,rect_d(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = A \int_{-\frac{d}{2}}^{d/2} e^{-j\omega t}\, dt$$

$$X(\omega) = A\left(\frac{e^{-j\omega t}}{-j\omega}\right)\Big]_{-d/2}^{d/2}$$

$$X(\omega) = A\left[\frac{e^{-j\omega d/2}}{-j\omega} - \frac{e^{-j\omega(-d/2)}}{-j\omega}\right]$$

$$X(\omega) = \frac{-A}{j\omega}\left[e^{-j\omega d/2} - e^{j\omega d/2}\right]$$

$$X(\omega) = \frac{2A}{\omega}\left[\frac{e^{j\omega d/2} - e^{-j\omega d/2}}{2j}\right]$$

$$X(\omega) = \frac{2A}{\omega}\sin\left(\frac{\omega d}{2}\right)$$

$$X(\omega) = \frac{d/2}{d/2}\,\frac{2A}{\omega}\sin\left(\frac{\omega d}{2}\right)$$

$$\therefore \sin\left(\frac{\omega d}{2}\right) = \operatorname{sinc}\left(\frac{\omega d}{2}\right)$$

$$\boxed{X(\omega) = Ad\,\operatorname{sinc}\left(\frac{\omega d}{2}\right)}$$

② Aplique las propiedades de la transformada de Fourier para resolver.

a) $F\left\{ e^{-j\omega_1 t} \cos(\omega_c t) \right\}$, $\omega_1, \omega_c \in \mathbb{R}$

$$F\{x(t)\cdot y(t)\} = \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

$$X(\omega) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{-j\omega_1 t} e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} \cos(\omega_c t) \ldots \right.$$

$$\left. e^{-j\omega t}\, dt \right]$$

$$X(\omega) = \frac{1}{2\pi}\left[F\{e^{-j\omega_1 t}\} * F\{\cos(\omega_c t)\}\right]$$

$$X(\omega) = \frac{1}{2\pi}\left[2\pi\, \delta(\omega-\omega_1) * \pi\left(\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right)\right]$$

$$X(\omega) = \frac{\cancel{2\pi} \cdot \pi}{\cancel{2\pi}}\left[\delta(\omega-\omega_1) * \left(\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right)\right]$$

$$\boxed{X(\omega) = \pi \left[\delta(\omega - \omega_2 - \omega_c) + \delta(\omega + \omega_2 - \omega_0) \right]}$$

b) $F\left\{ a(t) \cos^2(\omega_c t) \right\}, \quad \omega \in \mathbb{R}$

$\therefore \cos^2(\omega_c t) = \frac{1}{2} + \frac{1}{2} \cos(2\omega_c t)$

Entonces

$$X(\omega) = F\left\{ a(t) \times \left(\frac{1}{2} + \frac{1}{2} \cos(2\omega_c t) \right) \right\}$$

$$F\left\{ \frac{1}{2} a(t) + \frac{1}{2} a(t) \cos(2\omega_c t) \right\}$$

propiedad de linealidad

$$\frac{1}{2} F\left\{ a(t) \right\} + \frac{1}{2} F\left\{ a(t) \cos(2\omega_c t) \right\}$$

Primero

$$F\left\{ a(t) \right\} \Rightarrow \boxed{a(\omega) = \pi \delta(\omega) + \frac{1}{j\omega}}$$

Segundo

$$F\left\{ u(t)\cos(2w_c t)\right\} = \frac{1}{2\pi}\left[F\{u(t)\} * F\{\cdots \cos(2w_c t)\}\right]$$

aplicando propiedad de convolucion con delta dirac

$$= \frac{1}{\cancel{2}}\left\{ \cancel{\pi}\left[u(w-2w_c) + u(w+2w_c)\right]\right\}$$

$$= \frac{1}{2}\left[u(w-2w_c) + u(w+2w_c)\right]$$

Entonces

$$F\left\{ u(t)\cos^2(w_c t)\right\} = \frac{1}{2}u(w) + \frac{1}{2}\left(\frac{1}{2}\left[u(w-2w_c) + u(w+2w_c)\right]\right)$$

$$= \frac{\pi \delta(\omega)}{2} + \frac{1}{2j\omega} + \frac{1}{4}\left[\pi\,\delta(\omega - 2\omega_c) + \frac{1}{j(\omega - 2\omega_c)}\right.$$

$$\left. + \pi\delta(\omega + 2\omega_c) + \frac{1}{j(\omega + 2\omega_c)}\right]$$

c) $F^{-1}\left\{\frac{7}{\omega^2 + 6\omega + 45} * \frac{10}{(8 + j\omega/3)^2}\right\} = z(t)$

$$F^{-1}\left\{X(\omega) * Y(\omega)\right\} = 2\pi\; x(t) \cdot y(t)$$

entonces

$$F^{-1}\left\{\frac{7}{\omega^2 + 6\omega + 45}\right\}$$ Completamos cuadrados

$$\omega^2 + 6\omega + 45 = (\omega + 3)^2 - 3^2 + 45 = (\omega + 3)^2 + 6^2$$

$$F^{-1}\left\{\frac{7}{(\omega + 3)^2 + 6^2}\right\} = \frac{7}{6} \quad F^{-1}\left\{\frac{6}{(\omega - 3)^2 + 6^2}\right\}$$

aplicamos formula

$$F^{-1}\left\{\frac{b}{(\omega+a)^2+b^2}\right\} = \pi e^{-at}\sin(bt)u(t)$$

$$x(t) = \frac{1}{6}\pi e^{-3t}\sin(6t)u(t)$$

ahora

$$F^{-1}\left\{\frac{10}{(8+j\omega/3)^2}\right\} = F^{-1}\left\{\frac{10}{\left(\frac{24+j\omega}{3}\right)^2}\right\}$$

$$= F^{-1}\left\{\frac{10}{\left(\frac{24+j\omega}{9}\right)^2}\right.$$

$$F^{-1}\left\{\frac{90}{(24+j\omega)^2}\right\}$$

aplicamos formula

$$F^{-1}\left\{\frac{1}{(j\omega+a)^2}\right\} = t e^{-at}u(t))$$

$$y(t) = 90t e^{-24t}u(t)$$

ahora entonces

$$z(t) = 2\pi\left[\frac{7}{6}\pi e^{-3t} \operatorname{sen}(6t)u(t)\right]\left[90te^{-2\pi t}u(t)\right]$$

$$\boxed{z(t) = 2\pi\, u(t)\left[\frac{7}{6}\pi e^{-3t} \operatorname{sen}(6t) \cdot 90t\, e^{-2\pi t}\right]}$$