Problem 1.

Consider the autonomous dynamical system

$$\dot{x} = f(x, t),$$

with initial set $\Theta \subseteq \mathrm{val}(X)$, and let $V : \mathrm{val}(X) \to \mathbb{R}$ be a Lyapunov function for the system.

Let S be a sublevel set of V, defined as

$$S = \{x \in \mathrm{val}(X) \mid V(x) \leq c\}$$

for some constant $c \in \mathbb{R}$. Suppose $\Theta \subseteq S$. We will show that S is an inductive invariant of the system.

By assumption, $\Theta \subseteq S$, so the initial set is included in the candidate invariant. Thus, the initialization condition is satisfied.

Since V is a Lyapunov function, it satisfies:

$$\frac{d}{dt}V(x(t)) = \nabla V(x) \cdot f(x, t) \leq 0.$$

This implies that $V(x(t))$ is non-increasing along any trajectory of the system. In particular, if $x(0) \in S$, then $V(x(0)) \leq c$, and for all $t \geq 0$,

$$V(x(t)) \leq V(x(0)) \leq c \Rightarrow x(t) \in S.$$

Hence, every trajectory starting in S remains in S for all future time.

Since $\Theta \subseteq S$ and S is forward invariant under the dynamics, S is an inductive invariant of the system.

$$\boxed{S \text{ is an inductive invariant.}}$$

Problem 2.

(a) Convexity for Linear Systems

Claim: For the linear autonomous system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n,$$

if the initial set $\Theta \subseteq \mathbb{R}^n$ is convex, then the set of reachable states at time t, denoted by $\text{Reach}(\Theta, [t, t])$, is also convex.

Proof:

The solution to the linear system at time t with initial condition $x(0) = x_0$ is:

$$x(t) = e^{At} x_0.$$

Therefore, the reachable set at time t is:

$$\text{Reach}(\Theta, [t, t]) = \{e^{At} x_0 \mid x_0 \in \Theta\}.$$

Let $x_1, x_2 \in \Theta$. Since Θ is convex, for any $\lambda \in [0, 1]$, the point

$$x_\lambda = \lambda x_1 + (1 - \lambda) x_2 \in \Theta.$$

Then the corresponding reachable point is:

$$e^{At} x_\lambda = e^{At}(\lambda x_1 + (1 - \lambda) x_2) = \lambda e^{At} x_1 + (1 - \lambda) e^{At} x_2.$$

This is a convex combination of two reachable points, so it belongs to $\text{Reach}(\Theta, [t, t])$.

Conclusion: The reachable set is closed under convex combinations, so it is convex.

(b) Nonlinearity Can Break Convexity

Example: Consider the nonlinear system in $\mathbb{R}^2$:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1^3.$$

This is the well-known Duffing oscillator. Let's choose the initial set:

$$\Theta = \{x \in \mathbb{R}^2 \mid x_1 \in [-1, 1], x_2 = 0\},$$

which is a convex line segment along the x_1-axis.

Now consider the evolution of two initial points: - $x^{(1)}(0) = [-1, 0]^T$, - $x^{(2)}(0) = [1, 0]^T$.

Because of the nonlinearity in the restoring force $-x_1 + x_1^3$, these two trajectories will curve in opposite directions in phase space. Their solutions $x^{(1)}(t)$ and $x^{(2)}(t)$ at some small $t > 0$ will lie on opposite sides of the origin in the x_2-direction.

Now consider the midpoint of the initial conditions:

$$x^{(3)}(0) = \frac{1}{2} x^{(1)}(0) + \frac{1}{2} x^{(2)}(0) = [0, 0]^T.$$

The origin is an unstable equilibrium point of the system. So at time $t > 0$, the state $x^{(3)}(t)$ remains at the origin, while $x^{(1)}(t)$ and $x^{(2)}(t)$ move away from it.

Hence,

$$x^{(3)}(t) \neq \frac{1}{2} x^{(1)}(t) + \frac{1}{2} x^{(2)}(t),$$

and the convex combination of two reachable states does not necessarily correspond to a reachable state. Thus, the reachable set is not convex.

Conclusion: Nonlinear dynamics can distort trajectories so that the image of a convex initial set under the flow of the system is no longer convex.

Problem 3.

1. **AF** f_1 **(infinitely often):**
$$AF\, f_1 \equiv \neg EG\, \neg f_1$$

2. **AG** f_1 **(invariance):**
$$AG\, f_1 \equiv \neg E\, true\, U\, \neg f_1$$

3. **AFAG** f_1 **(stabilization):**
$$AF AG\, f_1 \equiv \neg EG\, EF\, \neg f_1$$

4. **A** f_1 **U** f_2**:**
$$A(f_1\, U\; f_2) \equiv \neg E(\neg f_2\, U\, (\neg f_1 \wedge \neg f_2)) \wedge \neg EG\, \neg f_2$$

Problem 4.

CTL Automaton for AFAG

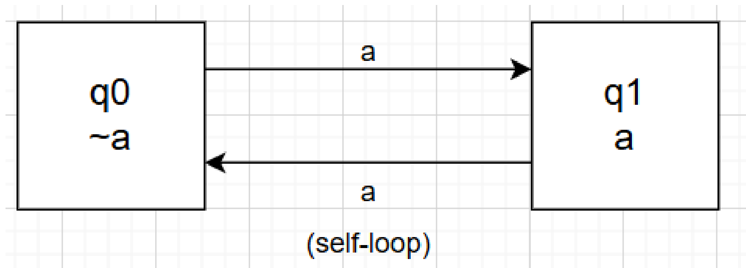

Figure 1: AF'AG

CTL Automaton for AF

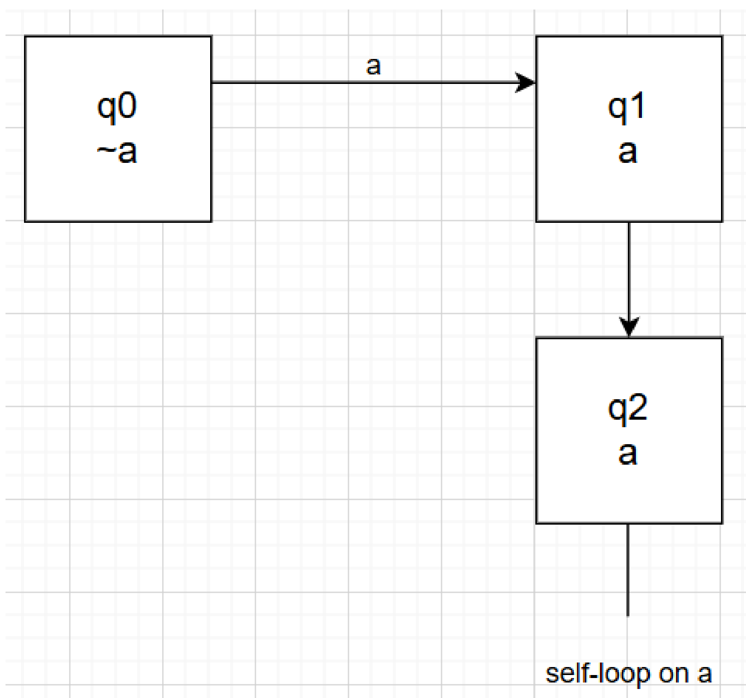

Figure 2: AF

Problem 5.

We are given an automaton $\mathcal{A} = \langle Q, Q_0, T, L \rangle$ with:

- States $Q = \{s_0, s_1, s_2, s_3, s_4\}$
- Initial states $Q_0 = \{s_0, s_3\}$
- Atomic propositions: $AP = \{a, b\}$
- State labels (as seen in the diagram):
  - $L(s_0) = \emptyset$
  - $L(s_1) = \{a\}$
  - $L(s_2) = \{a, b\}$
  - $L(s_3) = \{b\}$
  - $L(s_4) = \{b\}$

We are asked to evaluate two CTL formulas:
Formula 1: $\varphi_1 = A(a\, U\, b) \vee EX(AG\, b)$

(a) First, we analyze $A(a\, U\, b)$. This formula states that on *all* paths, eventually b must hold, and until then, a must hold.

- $s_1 \to s_2$: this path satisfies $a\, U\, b$.
- s_2: both a and b hold at once — this trivially satisfies $a\, U\, b$.
- s_0: does not satisfy the formula, since $L(s_0) = \emptyset$ — a does not hold initially.
- s_3: a never holds; hence, the formula fails.
- s_4: similar to s_3, only b holds.

Therefore, the states satisfying $A(a\, U\, b)$ are:

$$\{s_1, s_2\}$$

(b) Now consider $EX(AG\, b)$ — "there exists a next state where b holds globally on all paths."

- $AG\, b$ holds in s_3 and s_4 because they loop on themselves and are labeled with $\{b\}$.
- s_0 has a transition to s_4, so $s_0 \models EX(AG\, b)$.
- $s_2 \to s_3$: so s_2 also satisfies the formula.

Thus, the states satisfying $EX(AG\, b)$ are:

$$\{s_0, s_2\}$$

(c) Taking the union (since it's a disjunction), the states satisfying φ_1 are:

$$\{s_0, s_1, s_2\}$$

(d) Since one of the initial states (s_0) satisfies the formula, we conclude that the automaton $\mathcal{A} \models \varphi_1$.

Formula 2: $\varphi_2 = AG\,A(a\,U\,b)$
This formula says: *on all paths, globally, it must hold that $A(a\,U\,b)$ is true from every reachable state.*

- From s_0, we already saw that $s_0 \not\models A(a\,U\,b)$, so the formula fails immediately.
- From s_3, a never holds — so $s_3 \not\models A(a\,U\,b)$ either.
- Therefore, *no path from the initial states maintains the invariant* required by φ_2.

Hence, no state satisfies φ_2, and:

$$\mathcal{A} \not\models \varphi_2$$

Final Answer Summary

- $\varphi_1 = A(a\,U\,b) \vee EX(AG\,b)$:
  - Satisfying states: $\{s_0, s_1, s_2\}$
  - $\mathcal{A} \models \varphi_1$: **Yes**
- $\varphi_2 = AG\,A(a\,U\,b)$:
  - Satisfying states: $\emptyset$
  - $\mathcal{A} \models \varphi_2$: **No**