1) Variation of parameters for scalar equations

V.O.P. Formula:

$$x = \mathbb{X}\int \mathbb{X}^{-1} b\, dt$$

Exam 3, Question 2:

Step 1: Find fundamental matrix

this is already given!

B is a 2x2 matrix such that $e^{Bt} = \begin{pmatrix} 1 & 4t \\ 0 & 1 \end{pmatrix}$. Solve:

$$\dot{x} = Bx + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Finding Inverse:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

if $\det(A) = 0$, NO INVERSE!

Step 2: plug into V.O.P formula

$$e^{Bt} = \mathbb{X} = \begin{pmatrix} 1 & 4t \\ 0 & 1 \end{pmatrix} \qquad \mathbb{X}^{-1} = \frac{1}{(1)(1)-(4t)(0)}\begin{pmatrix} 1 & -4t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4t \\ 0 & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 & 4t \\ 0 & 1 \end{pmatrix}\underbrace{\int \begin{pmatrix} 1 & -4t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} dt}$$

$$= \int \begin{pmatrix} -4t \\ 1 \end{pmatrix} dt = \begin{pmatrix} -2t^2 \\ t \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 & 4t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2t^2 \\ t \end{pmatrix} + \begin{pmatrix} 1 & 4t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 2t^2 \\ t \end{pmatrix} + \begin{pmatrix} C_1 + 4t C_2 \\ C_2 \end{pmatrix}$$

Step 3: Solve for C w/ initial condition

$$x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x(0) = \cancel{\begin{pmatrix} 2(0)^2 \\ 0 \end{pmatrix}}^{0} + \begin{pmatrix} C_1 + \cancel{4(0)C_2}^{0} \\ C_2 \end{pmatrix} \Rightarrow \begin{matrix} C_1 = 0 \\ C_2 = 0 \end{matrix}$$

$$\boxed{\text{Final solution: } x = \begin{bmatrix} 2t^2 \\ t \end{bmatrix}}$$

Worked Example from lms

$$\dot{x} = Ax + r(t) \text{ where } x = \begin{pmatrix} x \\ y \end{pmatrix} \quad A = \begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix} \quad r = \begin{pmatrix} 5e^{-t} \\ 0 \end{pmatrix}$$

Step 1: Find fundamental matrix

$(\lambda^2 - tr(A) + \det(A))$

$$\lambda^2 - (-3-4) + [(-3)(-4) - (3)(2)] = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1) \qquad \lambda = -6, -1$$

$$\lambda_1 = -6 \quad v_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \qquad \lambda_2 = -1 \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

general sol'n: $C_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-6t}\begin{pmatrix} -2 \\ 3 \end{pmatrix} \Rightarrow \mathbb{X} = \begin{pmatrix} e^{-t} & -2e^{-6t} \\ e^{-t} & 3e^{-6t} \end{pmatrix}$ order doesn't matter, btw

Step 2: plug into Variation of Parameters formula

Formula:

$$x = \mathbb{X}\int \mathbb{X}^{-1} b\, dt \qquad \mathbb{X}^{-1} = \frac{1}{\det(\mathbb{X})}\begin{pmatrix} 3e^{-6t} & 2e^{-6t} \\ -e^{-t} & e^{-t} \end{pmatrix}$$

math was done... $\Rightarrow \mathbb{X}^{-1} r = \begin{pmatrix} 3 \\ -e^{5t} \end{pmatrix} \qquad \int \mathbb{X}^{-1} r\, dt = \begin{pmatrix} 3t \\ -\frac{1}{5}e^{5t} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$

$$x = \mathbb{X}\begin{pmatrix} 3t \\ -\frac{1}{5}e^{5t} \end{pmatrix} + \mathbb{X}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Step 3: Find C using initial condition

this problem did not require it, but see example problem above.

2) Complex numbers

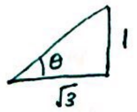
Final practice 1, Question 2:

write $\dfrac{(\sqrt{3}+i)^{25}}{2^{25}}$ in polar form and in rectangular form.

Step 1: convert rectangular forms to polar

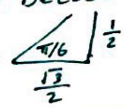
rewrite $(\sqrt{3}+i)$ in polar form : $re^{i\theta}$

$r = \sqrt{(\sqrt{3})^2+1^2} = 2$



$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

(because



$\tan\frac{\pi}{6} = \frac{1/2}{\sqrt{3}/2}$)

$re^{i\theta} = 2e^{i\pi/6}$

Step 2: Solve for final polar form

$$\left(\frac{\cancel{2}e^{i\pi/6}}{\cancel{2}}\right)^{25} = e^{25(\frac{\pi}{6})i} = e^{(24/6)\pi i}\cdot e^{1/6\, i\pi} = \underbrace{e^{4\pi i}}_{\substack{=\cos(4\pi)+i\cancel{\sin(4\pi)} \\ =1}} e^{i\pi/6} = \boxed{e^{i\pi/6}}$$

Step 3: convert back to rectangular

$$e^{i\pi/6} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}+\frac{1}{2}i}$$

Find the complex roots of $z^3-8=0$ in polar form. (Final practice 2, Part I, Question 2)

Step 1: Substitute z for polar form $re^{i\theta}$. Find r.

Let $z = re^{i\theta}$ $\quad z^3 = 8 \Rightarrow (re^{i\theta})^3 = 8$ ~~so~~

$= r^3e^{3i\theta} \Rightarrow r^3=8,\ r=2$ and $e^{3i\theta}=1$

Step 2: Figure out θ.

Solve for $e^{3i\theta} = 1 = \cos(3\theta)+i\sin(3\theta)$

because there is no complex component, $i\sin(3\theta)=0$ and $\cos(3\theta)=1$

$\Rightarrow 3\theta = 2\pi K$ where $K = 0, 1, 2$, etc.

$\theta = \frac{2\pi K}{3} = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, etc.

Step 3: plug back into $re^{i\theta}$

> since we need three roots, we return:
> $2,\ 2e^{i\,2\pi/3},\ 2e^{i\,4\pi/3}$

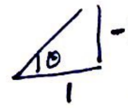
> 5) Input-response models with periodic input, especially using complex replacement, leading to amplitude and phase lag.

Find amplitude, angular frequency, phase lag, time lag of:

$$f(t) = Re\left(\frac{e^{-4it}}{1-i}\right)$$

Step 1: convert rectangular parts to polar

Rewrite $(1-i)$ in $re^{i\theta}$ form: $\quad r = \sqrt{1^2+(-1)^2} = \sqrt{2}$



$\theta = \frac{-\pi}{4}$

$1-i = \sqrt{2}\,e^{(-\frac{\pi}{4})i}$

Step 2: plug back in & write in amplitude-phase form.

$$f(t) = Re\left(\frac{e^{-4it}}{\sqrt{2}e^{(-\frac{\pi}{4})i}}\right) = Re\left(\frac{1}{\sqrt{2}}e^{i\left(-4t-\left(-\frac{\pi}{4}\right)\right)}\right) = \frac{1}{\sqrt{2}}\cos\left(-4t+\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\cos\left(4t-\frac{\pi}{4}\right)$$

> Remember:
> $\cos(t) = \cos(-t)$
> $\sin(t) = -\sin(t)$

Step 3: Find all properties

> Amplitude $= \frac{1}{\sqrt{2}}$ $\quad \omega$ (angular frequency) $= 4$ $\quad$ phase lag $(\phi) = \frac{\pi}{4}$
>
> time lag $= \frac{\phi}{\omega} = \frac{\pi/4}{4} = \frac{\pi}{16}$

3) Input-response models, especially with heat

Jerison gave this form during the review session: $\dot{x} = k(y(t) - x) + q(t)$

- 2 inputs: $y(t)$ and $q(t)$
- expand: $\dot{x} + kx = ky(t) + q(t)$
- Jerison said to set $q(t) = 0$ to solve homogenous sol'n $x(t) = Ce^{-kt}$ (this tells us that $y(t)$ will also be 0!)

Exam 1, problem 5

The temperature ($x = x(t)$) of a pot of water is modeled by:

$$\frac{dx}{dt} = k(T_{ext} - x) + q(t)$$

where T_{ext} is the temperature outside and $q(t)$ is the input of the camp stove.

a) When $T_{ext} = 0$ and stove is turned off ($q(t) = 0$), water takes 10 min. to cool from 20° to 10°. What are $q(t)$ and k?

Step 1: Set up homogenous sol'n

$q(t) = 0 \quad T_{ext} = 0 \quad \frac{dx}{dt} = -kx \Rightarrow x = Ce^{-kt}$

Step 2: solve using initial conditions

plug in $t = 0$ and $t = 10$: $x(0) = 20$, $x(10) = 10$

$Ce^{-k(0)} = 20 \Rightarrow C = 20 \qquad 20e^{-k(10)} = 10 \qquad e^{-k(10)} = \frac{1}{2}$

$-k(10) = \ln(\frac{1}{2}) \qquad k = \frac{-\ln(\frac{1}{2})}{10} = \frac{1}{10}\ln(2)$ per min.

Step 3: make sure units check out

Checking units: $\frac{\text{temp}}{\text{min.}} = \frac{1}{\text{min.}}(\text{temp}) + \frac{\text{temp}}{\text{min.}}$ ✓ checks out

b) Now $T_{ext} = 0$, $k = 1/10$ per min., $q(t) = 10e^{-t/20}$ degrees/min.

$$\dot{x} + \frac{1}{10}x = 10e^{-t/20}$$

You fill the pot with water that's 0°C at $t = 0$. Find the formula for the temperature.

Step 1: Find homogenous sol'n

$x = x_p + x_h \qquad \dot{x} + \frac{1}{10}x = 0 \qquad \dot{x} = -\frac{1}{10}x \Rightarrow x_h = Ce^{-t/10}$

Step 2: Find particular sol'n

$$x_p = \frac{10e^{-t/20}}{P(-\frac{1}{20})} = \frac{10e^{-t/20}}{-\frac{1}{20} + \frac{1}{10}} = \frac{10e^{-t/20}}{\frac{1}{20}} = 200e^{-t/20}$$

Step 3: solve for C, find general sol'n

$x(t) = 200e^{-t/20} + Ce^{-t/10}$

because $x(0) = 0$, $C = -200$

so $\boxed{x(t) = 200e^{-t/20} - 200e^{-t/10}}$

Optional! When does the water boil?

$x(t) = 200(e^{-t/20} - e^{-t/10})$ set $y = e^{-t/20}$ $y^2 = e^{-t/10}$

look for $100 = 200(y - y^2) \Rightarrow y^2 - y + \frac{1}{2} = 0 \qquad y = \frac{1 \pm \sqrt{1-2}}{2}$

because there are complex roots, the water never reaches boiling, no sol'n

Optional problem: Practice Final, Part I, Question 5. I'm not going to cover it because it's not in the $\dot{x} = k(y(t) - x) + q(t)$ form, but it may be good for extra practice.

4) Superposition $x_p + x_h$ and exponential response formula (ERF)

Superposition without ERF example:

Final practice 2; part 1, question 1:

Find the general solution to the differential equation:

$$\dot{y} = t^{-3} - 4t^{-1}y$$

Step 1: find homogenous equation

$$\dot{y} + 4t^{-1}y = t^{-3} \quad \text{homogenous equation: } \dot{y} + 4t^{-1}y = 0$$

$$\frac{dy}{dt} = -\frac{4y}{t} \Rightarrow \int \frac{dy}{y} = \int -\frac{4\,dt}{t} \Rightarrow \ln|y| = -4\ln|t| + C$$

$$y_h = \underbrace{Ct}^{-4} = \frac{C}{t^4}$$

pay attention to how $e^{(-4\ln|t|)} = t^{-4}$

Step 2: solve for particular sol'n

$$y = \underbrace{C(t)}t^{-4}$$

also watch out here. C is a function of time!

$$\dot{y} = -4\,C(t)\,t^{-5} + C'(t)\,t^{-4}$$

$$\left(\cancel{-4\frac{C(t)}{t^5}} + \frac{C'(t)}{t^4}\right) + \cancel{4\frac{1}{t}\frac{C(t)}{t^4}} = t^{-3}$$

$$\int C'(t) = \int (t^4)(t^{-3}) = \int t \Rightarrow C(t) = \frac{t^2}{2}$$

Step 3: put back into $y = y_h + y_p$

$$y_p = \frac{C(t)}{t^4} = \frac{t^2/2}{t^4} = \frac{1}{2t^2}$$

$$\boxed{y = y_h + y_p = \frac{C}{t^4} + \frac{1}{2t^2}}$$

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lms Week 4, 9 Operators and exponential response, 14 Worked example

Find the general solution to:

$$2\ddot{x} + \dot{x} + x = 1 + 2e^t$$

Step 1: Find the homogenous solution

$$P(s) = 2s^2 + s + 1 \qquad \text{roots: } \frac{-1 \pm \sqrt{1^2 - 4(2)(1)}}{2(2)} = \frac{-1 \pm \sqrt{7}\,i}{4}$$

$$\Rightarrow e^{\left(\frac{-1 \pm \sqrt{7}i}{4}\right)t} = e^{-\frac{t}{4}}\, e^{\pm\sqrt{7}/4\, i t}$$

(just take pos. parts!)

$$x_h(t) = e^{-t/4}\left(C_1 \cos\left(\frac{t\sqrt{7}}{4}\right) + C_2 \sin\left(\frac{t\sqrt{7}}{4}\right)\right)$$

$$= Ae^{-t/4}\cos\left(\frac{t\sqrt{7}}{4} - \phi\right)$$

Step 2: Find particular solution

inhomogenous equation is $P(D)x = 1 + 2e^t$ $\Leftarrow$ linear combo. of 1 and e^t

$$P(D)x = 1 = e^{0t} \qquad \frac{1}{P(0)} = 1$$

$$P(D)x = 2e^t \qquad \frac{2e^t}{P(1)} = \frac{e^t}{2}$$

$$x_p = 1 + \frac{1}{2}e^t$$

Step 3: combine for general solution.

General Sol'n, $x_p + x_h = x$:

$$x(t) = 1 + \frac{1}{2}e^t + Ae^{-t/4}\cos\left(\frac{t\sqrt{7}}{4} - \phi\right)$$

$$\left(\text{or } x(t) = 1 + \frac{1}{2}e^t + e^{-t/4}\left(C_1\cos\left(\frac{t\sqrt{7}}{4}\right) + C_2\sin\left(\frac{t\sqrt{7}}{4}\right)\right)\right)$$

Practice Final 2, section II, question I

Find a basis for the space of real solutions $y = y(t)$ for:

$$(D^2+9)(D+1)^3 y = 0 \qquad (D = d/dt)$$

Characteristic polynomial

$p(r) = (r^2+9)(r+1)^3$ has roots $\pm 3i$ (multiplicity 1)

$r = -1$ (multiplicity 3)

$3i \Rightarrow e^{3ti} \Rightarrow X_h(t) = C_1\cos(3t) + C_2\sin(3t)$

$$\boxed{\sin(3t),\ \cos(3t),\ e^{-t},\ te^{-t},\ t^2e^{-t}}$$

6) Solving $Ax=b$: Gaussian elimination and counting dimensions.

Final practice 1, Question 7

Let $A=\begin{pmatrix}1&0&1\\1&1&1\\3&1&3\end{pmatrix}$ $\quad b=\begin{pmatrix}1\\3\\5\end{pmatrix}$ Find general solution to $Ax=b$

Step 1: put in row-eschelon form

Form augmented matrix $\left(\begin{array}{ccc|c}1&0&1&1\\1&1&1&3\\3&1&3&5\end{array}\right)$ ... gaussian elimination gets $\begin{array}{c}\left(\begin{array}{ccc|c}\boxed{1}&0&1&1\\0&\boxed{1}&0&2\\0&0&0&0\end{array}\right)\\ x \quad y \quad z \qquad\end{array}$

note that if b is not given, it's just zeros here

write system of eq'ns:

$x = 1 - z = 1 - s$

$y = 2$

$z = s$

Step 2: write general sol'n as basis.

$$\begin{pmatrix}1-s\\2\\s\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}+s\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

nullspace is 1D

basis of nullspace: $\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ basis of column-space: $\left\{\begin{pmatrix}1\\1\\3\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$

rank: 2 $\quad \left(\underbrace{\text{rank}}_{2} + \underbrace{\text{nullity}}_{1} = \underbrace{\dim A}_{3}\right)$ ...because 3 columns.

Facts to remember:

- volume conserving: $\mathbb{R}^3 \to \mathbb{R}^3$, show that $|\det(A)|=1$
- if $\det(A)\neq 0$, there is NO NULLSPACE
- if $\det(A)=0$, there is ~~[illegible]~~ $\geq$ ONE NULLSPACE
- column space = rank = # pivots ... same thing
- rank + nullity = # dims.

☆ Watch out for diff. between basis of a column space and ☆ basis of a nullspace

- basis of nullspace : set $b=0$, then solve
- basis of column space : find columns of pivots in original matrix

Example where you have to remember to set $b=0$ for basis of nullspace:

Exam 2, Question 5, part C

$$\left(\begin{array}{cccc|c}1&2&3&0&b_1\\0&1&0&\frac{1}{2}&b_2/4\\0&0&0&1&b_3\\0&0&0&0&b_4-b_1-4b_3\end{array}\right)\Rightarrow\begin{array}{c}\left(\begin{array}{cccc|c}1&2&3&0&0\\0&1&0&\frac{1}{2}&0\\0&0&0&1&0\\0&0&0&0&0\end{array}\right)\\ x\quad y\quad z\quad n\qquad\end{array}$$

$x = -2y - 3z = -3s$

$y = -\frac{1}{2}n = 0$

$n = 0$

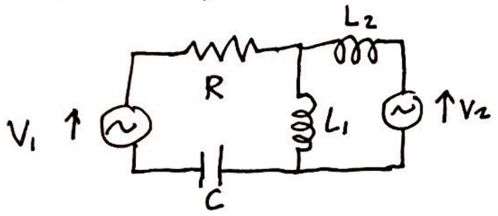
$z = s$

rearrange: $s\begin{pmatrix}-3\\0\\1\\0\end{pmatrix}$

$\Rightarrow$ basis of nullspace: $\left\{\begin{pmatrix}-3\\0\\1\\0\end{pmatrix}\right\}$ one free variable, so one-dimensional nullspace

8) Set up systems of ODEs in terms of matrices and vectors

Pset 4, Question 2: RLC Circuit modeling



Voltage drops:

$V_R = RI_R \quad V_C = \frac{1}{C}Q \quad V_{L_1} = L_1\dot{I}_{L_1} \quad V_{L_2} = L_2\dot{I}_{L_2}$

$V_R + V_{L_1} + V_C = V_1(t) \qquad V_{L_1} + V_{L_2} = V_2(t)$

Kirchhoff: $V_1(t) = V_R + V_{L_1} + V_C \;;\; V_2(t) = V_{L_1} + V_{L_2}$

Use elimination to find equations in matrix form for Q, I_{L_1}, and I_{L_2}:

also, $I_R + I_{L_2} = I_{L_1}$

$I_{L_1} \neq I_C + I_{L_2} \;;\; \dot{Q} = I_C$

$$\begin{pmatrix}\dot{Q}\\ \dot{I}_{L_1}\\ \dot{I}_{L_2}\end{pmatrix} = \begin{pmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\end{pmatrix}\begin{pmatrix}Q\\ I_{L_1}\\ I_{L_2}\end{pmatrix} + \begin{pmatrix}\cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\end{pmatrix}\begin{pmatrix}V_1\\ V_2\end{pmatrix}$$

step 1: figure out $\dot{I}$s

$$V_1(t) = RI_R + L_1\dot{I}_{L_1} + \frac{Q}{C} \Rightarrow \dot{I}_{L_1} = -\frac{R\overbrace{I_R}^{= I_{L_1} - I_{L_2}}}{L_1} - \frac{Q}{CL_1} + \frac{V_1(t)}{L_1} = \boxed{\dot{I}_{L_1} = -\frac{RI_{L_1}}{L_1} + \frac{RI_{L_2}}{L_1} - \frac{Q}{CL_1} + \frac{V_1(t)}{L_1}}$$

$$V_2(t) = V_{L_1} + V_{L_2} = L_1\dot{I}_{L_1} + L_2\dot{I}_{L_2}$$

$$\dot{I}_{L_2} = \frac{V_2(t)}{L_2} - \frac{L_1\dot{I}_{L_1}}{L_2} = \boxed{\frac{V_2(t)}{L_2} + \frac{RI_{L_1}}{L_2} - \frac{RI_{L_2}}{L_2} + \frac{Q}{CL_2} - \frac{V_1(t)}{L_2} = \dot{I}_{L_2}}$$

step 2: figure out $\dot{Q}$

$$\boxed{\dot{Q}_C = I_C = I_{L_1} - I_{L_2}}$$

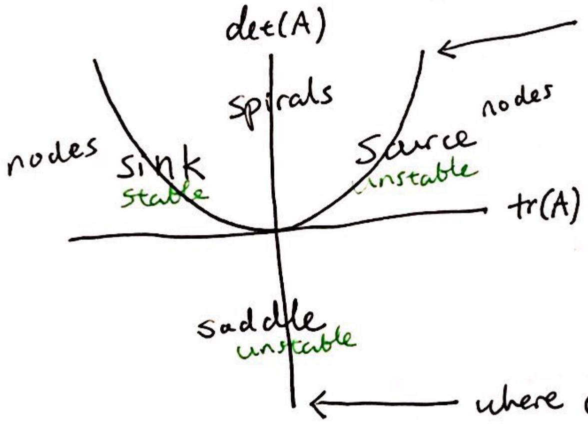
$$\boxed{\begin{pmatrix}\dot{Q}\\ \dot{I}_{L_1}\\ \dot{I}_{L_2}\end{pmatrix} = \begin{pmatrix}0 & 1 & -1\\ -\frac{1}{CL_1} & -\frac{R}{L_1} & \frac{R}{L_1}\\ \frac{1}{CL_2} & \frac{R}{L_2} & \frac{-R}{L_2}\end{pmatrix}\begin{pmatrix}Q\\ I_{L_1}\\ I_{L_2}\end{pmatrix} + \begin{pmatrix}0 & 0\\ \frac{1}{L_1} & 0\\ -\frac{1}{L_2} & \frac{1}{L_2}\end{pmatrix}\begin{pmatrix}V_1\\ V_2\end{pmatrix}}$$

I'm not sure what other systems of ODEs could be set up as matrices especially..

9) Linear phase portraits in 2 dimensions

trace-determinant plane

$\det(A) > \dfrac{[\operatorname{trace}(A)]^2}{4}$



where centers happen

Structure stability:
- if we change coefficients slightly, picture does not change

Stability
$x(t) \to 0$ as $t \to \infty$

Pset 4, question 3, part B.

$A = \begin{pmatrix} 2 & 0 \\ -5 & 10 \end{pmatrix}$ sketch this.

step 1: Find eigenvalues + eigenvectors

$\lambda^2 - \operatorname{tr}(A) + \det(A) = \lambda^2 - 12\lambda + 20 = (x-10)(x-2) = 0$ $\boxed{\lambda = 10, 2}$

trace = 12
det = 20

$\lambda_1 = 10$ $\begin{pmatrix} 2-10=-8 & 0 \\ -5 & 10-10=0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\boxed{\vec{V}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$\lambda_2 = 2$ $\begin{pmatrix} 2-2=0 & 0 \\ -5 & 10-2=8 \end{pmatrix}\begin{pmatrix} 8 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\boxed{\vec{V}_2 = \begin{pmatrix} 8 \\ 5 \end{pmatrix}}$

nodal source, unstable

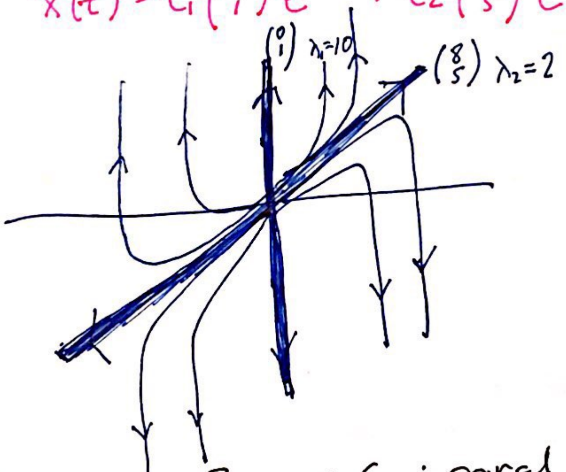
step 2: analyze behavior as $t \to \infty$ and $t \to -\infty$

$\vec{x}(t) = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{10t} + C_2 \begin{pmatrix} 8 \\ 5 \end{pmatrix} e^{2t}$

as $t \to \infty$: $e^{10t} \gg e^{2t}$

as $t \to -\infty$: $e^{2t} \gg e^{10t}$

step 3: graph it.



Pset 4, question 3, part C: spiral

step 1) Figure out if spiral in or out

$A = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}$ $tr = -4$
$\det = 4 - (-9) = 13$

$\lambda^2 + 4\lambda + 13 \to \dfrac{-4 \pm \sqrt{16 - 4(13)}}{2} = \dfrac{-4 \pm \sqrt{16-52}}{2}$

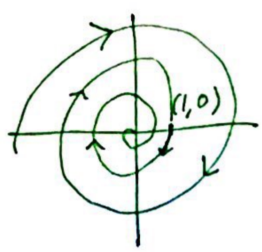
$\boxed{\lambda = -2 \pm 3i}$

spiral inwards

step 2: figure out direction

find velocity vector

$\begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$



Spiral, sink,
~~direc~~ ~~in~~ direction: clockwise

10) Fundamental solutions of linear homogenous solutions

- Jerison talked about the $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$ form, so I assume A is going to be some addition. ☆ note pt. 6 of question below

Midterm 3 Practice, Question 2

a) $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$ $V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Step 1: compute eigenvalues using $AV = \lambda V$

$$AV_1 = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -V_1$$

$$AV_2 = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2V_2$$

$$\boxed{\begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 2 \end{matrix}}$$

Two ways of finding fundamental matrix

Step 2: Use either 1) general sol'n or 2) $e^{At} = Se^{Dt}S^{-1}$

1) Using general sol'n

$$x(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \qquad \underline{X}(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{pmatrix}$$

2) Using e^{At}

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad S^{-1} = \frac{1}{\underset{=1}{\det(S)}}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\underline{X}(t) = e^{At} = Se^{Dt}S^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -e^{-t}+2e^{2t} & -e^{-t}+e^{2t} \\ 2e^{-t}-2e^{2t} & 2e^{-t}-e^{2t} \end{pmatrix}$$

b) Find the fundamental matrix for the system $\dot{x} = (A^2 + A^3)x$

$A^2 + A^3$ has the same eigenvectors as A

$$\lambda_1^2 + \lambda_1^3 = (-1)^3 + (-1)^2 = 0$$
$$\lambda_2^2 + \lambda_2^3 = 2^3 + 2^2 = 12$$

$$\Rightarrow x = c_1 e^{0t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{12t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{X} = \begin{pmatrix} 1 & e^{12t} \\ -2 & -e^{12t} \end{pmatrix}$$

or, if we use the e^{At} method:

$$\underline{X}(t) = e^{(A^2+A^3)t} = Se^{(D^2+D^3)}S^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} e^{0t} & 0 \\ 0 & e^{12t} \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1+2e^{12t} & -1+e^{12t} \\ 2-2e^{12t} & 2-e^{12t} \end{pmatrix}$$

11) Decoupling differential equations

- Jerison mentioned during review session questions to do:
  1) Midterm 3 practice, version 1, question 2 pt. c
  2) Midterm 3, prob 3 orthogonal eigenvectors case — Pay attention to part b!

Practice Midterm 3, version 1, question 2, part C

$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$ $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

From earlier pt. of problem:
$D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$, $S^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$

let $\dot{x} = Ax + \begin{pmatrix} 7e^{-t} \\ 9e^{-t} \end{pmatrix}$ $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Find decoupled equations $u_1(t)$, $u_2(t)$ for $\underbrace{x(t) = u_1(t)\, v_1 + u_2(t)\, v_2}$

note that $x = Su$ because $S = (\vec{v}_1 \mid \vec{v}_2)$ $\Rightarrow u = S^{-1}x$

step 1: replace A with SDS^{-1}

write A as SDS^{-1} $\Rightarrow \dot{x} = SDS^{-1}x + \begin{pmatrix} 7e^{-t} \\ 9e^{-t} \end{pmatrix}$

step 2: multiply by S^{-1}

multiply by S^{-1} $\Rightarrow$ $S^{-1}\dot{x} = DS^{-1}x + S^{-1}\begin{pmatrix} 7e^{-t} \\ 9e^{-t} \end{pmatrix}$

step 3: rewrite using $u = S^{-1}x$

rewrite using $u = S^{-1}x$ property:

$$\dot{u} = Du + S^{-1}\begin{pmatrix} 7e^{-t} \\ 9e^{-t} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 7e^{-t} \\ 9e^{-t} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} -u_1 \\ 2u_2 \end{pmatrix} + \begin{pmatrix} -16e^{-t} \\ 23e^{-t} \end{pmatrix}$$

step 4: solve for initial conditions

Initial condition:

$$u(0) = S^{-1}x(0) = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Final system:

$\dot{u}_1(t) = -u_1(t) - 16e^{-t}$ $u_1(0) = -1$

$\dot{u}_2(t) = 2u_2(t) + 23e^{-t}$ $u_2(0) = 2$

Midterm 3, prob 3

$A = \begin{pmatrix} -11 & 8 \\ 8 & 1 \end{pmatrix}$ orthogonal eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ means $A = A^T$

If $\dot{x} = Ax$ and $x(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, find decoupled eq'ns & initial conditions given by:

$y_1(t) = v_1^T x(t)$, $y_2(t) = v_2^T x(t)$

> Note: Sometimes they will give you $\dot{x} = Ax + b$ instead.
> in that case, $\dot{y}_j = v_j^T(A\vec{x} + b)$
> $= \lambda_j(\vec{v}_j^T\vec{x}) + v_j^T b$
> $= \lambda_j y_j + v_j^T b$
> see prac. midterm 3, question 2 version 3

step 1: derive y, replace $\dot{x}(t)$ with given eq'n

Concept: $\dot{y}_j = \vec{v}_j^T \dot{\vec{x}}(t) = \vec{v}_j^T A\vec{x} = \lambda_j \underbrace{\vec{v}_j^T \vec{x}}_{y} = \lambda_j y_j$

step 2: find eigenvalues

Find eigenvalues:

$A\vec{v}_1 = \begin{pmatrix} -11 & 8 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5\vec{v}_1$

$A\vec{v}_2 = \begin{pmatrix} -11 & 8 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -15\vec{v}_2$

$\boxed{\lambda_1 = 5 \quad \lambda_2 = -15}$

We're done here, but if there was a b term, we would also need to solve for that!!!

Step 3: plug back into eq'n. Non-existant here because not necessary

step 4: plug initial conditions

Initial Conditions:

$y_1(0) = \vec{v}_1^T \vec{x}(0) = (1\ 2)\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 + 8 = \boxed{11}$

$y_2(0) = (-2\ 1)\begin{pmatrix} 3 \\ 4 \end{pmatrix} = -6 + 4 = \boxed{-2}$

Decoupled eq'ns:

$$\boxed{\begin{matrix} \dot{y}_1 = 5y_1 & y_1(0) = 11 \\ \dot{y}_2 = -15y_2 & y_2(0) = -2 \end{matrix}}$$

b) Find C_1 and C_2 so that and solve for y_1, y_2

$$x(t) = C_1 y_1(t) V_1 + C_2 y_2(t) V_2$$

step 1: multiply all sides by $\vec{V}^T$

multiply by $\vec{V_1}^T$

$\vec{V_1}^T \vec{V_2} = 0$ because they are orthogonal.

$$\underbrace{\vec{V_1}^T x(t)}_{= y_1(t)} = C_1 y_1(t) \vec{V_1}^T \vec{V_1} + \cancel{C_2 y_2(t) V_2 \vec{V_1}^T}^{\,0}$$

$$\Rightarrow C_1 \vec{V_1}^T \vec{V_1} = 1 \qquad C_1 = \frac{1}{\vec{V_1}^T \vec{V_1}} = \frac{1}{(1\ 2)\binom{1}{2}} = \boxed{\frac{1}{5}}$$

$$\underbrace{\vec{V_2}^T x(t)}_{= y_2(t)} = \cancel{C_1 y_1(t) \vec{V_2}^T \vec{V_1}}^{\,0} + C_2 y_2(t) \vec{V_2} \vec{V_2}^T$$

$$C_2 = \frac{1}{\vec{V_2} \vec{V_2}^T} = \frac{1}{(-2\ 1)\binom{-2}{1}} = \boxed{\frac{1}{5}}$$

$$\boxed{y_1(t) = 11 e^{5t} \ ; \quad y_2(t) = -2e^{-15t}}$$

$$\boxed{\vec{X}(t) = \frac{1}{5} 11 e^{-5t} \binom{1}{2} + \frac{1}{5}(-2e^{-15t})\binom{-2}{1}}$$

Practice midterm 3, version 3, question 2

$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ orthogonal eigenvectors $V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $V_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $V_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

decoupled: $C_j(t) = V_j^T x(t)$ where $\dot{x} = Ax + b(t)$ $b = \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix}$ $x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Solved: $\dot{C_1} = 2C_1 + 6t \quad C_1(0) = 1$

$\dot{C_2} = -C_2 + t \quad C_2(0) = -1$

$\dot{C_3} = -C_3 - 3t \quad C_3(0) = 1$

Find a formula for $X(t)$ in terms of $C_j(t)$.

this only works because $\vec{V_1}, \vec{V_2}, \vec{V_3}$ is a basis where $\{\vec{V_1}, \vec{V_2}, \vec{V_3}\}$ spans $\mathbb{R}^3$

step 1: write out gen eq'n

$$\Rightarrow \vec{X(t)} = a_1 \vec{V_1} + a_2 \vec{V_2} + a_3 \vec{V_3}$$

step 2: multiply by $\vec{V}^T$

multiply by $\vec{V_1}^T$

$$\underbrace{\vec{V_1}^T \vec{X(t)}}_{= C_1(t)} = a_1 \vec{V_1} \vec{V_1}^T + \cancel{a_2 \vec{V_2} \vec{V_1}^T}^{\,0} + \cancel{a_3 \vec{V_3} \vec{V_1}^T}^{\,0} \qquad a_1 = \frac{C_1(t)}{\vec{V_1}^T \vec{V_1}} = \frac{C_1(t)}{3}$$

do the same for all: $a_2 = \frac{C_2(t)}{\vec{V_2}^T \vec{V_2}} = \frac{C_2(t)}{2}$ $\quad a_3 = \frac{C_3(t)}{\vec{V_3}^T \vec{V_3}} = \frac{C_3(t)}{6}$

step 3: plug back into gen. eq'n

$$\boxed{\text{Final: } \vec{X(t)} = \frac{C_1(t)}{3} \vec{V_1} + \frac{C_2(t)}{2} \vec{V_2} + \frac{C_3(t)}{6} \vec{V_3}}$$

12) Euler's method for scalar equations and systems

- we will be asked for an autonomous 2x2 system

Example from review session:

$A = \begin{pmatrix} 5 & 2 \\ -8 & -3 \end{pmatrix}$ $\quad \dot{\vec{x}} = A\vec{x} = f(x)$ $\quad h = 1/2$ $\quad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \vec{x}(1) = ?$

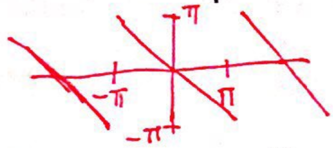
$\vec{x}_{n+1} = \vec{x}_n + hf(\vec{x}_n)$

n	$\vec{x}_n$	$\vec{f}(x_n)$	$h\vec{f}(\vec{x}_n)$
0	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 2 \\ -8 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 5 \\ -8 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -4 \end{pmatrix}$
1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 5/2 \\ -4 \end{pmatrix}$ $\boxed{= \begin{pmatrix} 7/2 \\ -4 \end{pmatrix}}$		

13) Fourier Series

Exam 3 Practice, version 2, question 3

a) Sketch the 2π periodic function $f(t)$ defined by $f(t) = -t$ on $[-\pi, \pi)$ on $[-2\pi, 2\pi)$



b) Give the general expression for the fourier series
What is $a_0/2$?

$$g(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_n \cos(nt) + \sum_{i=1}^{\infty} b_n \sin(nt)$$

$$\underbrace{\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt}_{\text{this is the mean value}} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

c) Find the Fourier coefficient of $f(t)$.

this is a odd function $\Rightarrow a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (-t)\sin(nt)\,dt$$

moved neg. sign to front of integral $\rightarrow$ $u = t$ $\quad dv = \sin(nt)$

$du = dt \quad v = \frac{-\cos(nt)}{n}$

Integration by parts
$$\int u\,dv = uv - \int v\,du$$

$$-\frac{1}{\pi}\left[t\left(\frac{-\cos(nt)}{n}\right)\right]\Bigg|_{-\pi}^{\pi} - \cancel{\int_{-\pi}^{\pi} \frac{-\cos(nt)}{n}\,dt}^{\,0} = -\frac{1}{\pi}\left[\pi\left(-\frac{\cos(n\pi)}{n}\right) - (-\pi)\left(-\frac{\cos(n(-\pi))}{n}\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi\cos(n\pi)}{n} + \frac{\pi\cos(-n\pi)}{n}\right) = \frac{2\cos(n\pi)}{n} = \frac{2(-1)^n}{n}$$

d) $\ddot{x} + \frac{1}{2}x = f(t)$ Find the fourier coefficients of an odd solution $x(t)$

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(nt) \qquad \ddot{x} = -\sum_{n=1}^{\infty} b_n n^2 \sin(nt) \qquad \ddot{x} + \frac{1}{2}x = -\sum_{n=1}^{\infty} b_n n^2 \sin(nt) + \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin(nt)$$

$$= \sum_{n=1}^{\infty} \left(-n^2 + \frac{1}{2}\right) b_n \sin(nt)$$

$$b_n = \frac{2(-1)^n}{n} \Big/ \left(-n^2 + \frac{1}{2}\right) = 2\,\frac{(-1)^n}{n\left(\frac{1}{2} - n^2\right)}$$

↑ comes from part C

this problem is an example of when you find fourier coefficients by comparing two ways of solving for it.

Exam 3 Practice version 1, question 4

Let $f(x)$ be the periodic function with period 4 such that:

$$f(t) = \begin{cases} 0 & \text{if } -2 \le t \le 0 \text{ and} \\ 6 & \text{if } 0 < t < 2 \end{cases}$$

original square wave
$$Sq(t) = \begin{cases} 1 & \text{if } 0 < t < \pi \\ -1 & \text{if } -\pi < t < 0 \end{cases}$$
$$= \frac{4}{\pi} \sum_{\substack{n \ge 1 \\ odd}} \frac{\sin(nt)}{n}$$

a) Find the fourier series for f

1) scale $sq(t)$ to range $(-2, 2)$

$$Sq\left(\frac{\pi}{2}t\right) = \begin{cases} 1 & 0 < \frac{\pi}{2}t < \pi \\ -1 & -\pi < \frac{\pi}{2}t < 0 \end{cases} = \begin{cases} 1 & 0 < t < 2 \\ -1 & -2 < t < 0 \end{cases}$$

2) Multiply by 3 and add 3

$$3 + 3\,Sq\left(\frac{\pi}{2}t\right) = \begin{cases} 6 & \text{if } 0 < t < 2 \\ 0 & \text{if } -2 < t < 0 \end{cases} \qquad f(t) = 3 + 3Sq\left(\frac{\pi}{2}t\right) = 3 + \frac{12}{\pi}\sum_{\substack{n \ge 1 \\ odd}} \frac{\sin\left(n\frac{\pi}{2}t\right)}{n}$$

b) Find a particular sol'n to $\ddot{x} + 10x = f(t)$

pick our sin part: $\ddot{x} + 10x = \sin\left(\frac{n\pi}{2}t\right) = Im\left(e^{\frac{in\pi t}{2}}\right) \qquad P(r) = r^2 + 10$

$$z = \frac{e^{in\pi t/2}}{P\left(\frac{in\pi}{2}\right)} = \frac{1}{10 - \frac{n^2\pi^2}{4}} e^{n\pi t/2} = \frac{4}{40 - n^2\pi^2} e^{in\pi t/2} \qquad x = Im(z) = \frac{4}{40 - n^2\pi^2}\sin\left(\frac{n\pi t}{2}\right)$$

solve for constant part:

$\ddot{x} + 10x = 3 \qquad x(t) = \frac{3}{10}$

Superposition, $f(t)$ is
$$\boxed{\frac{3}{10} + \frac{12}{\pi}\sum_{\substack{n \ge 1 \\ odd}} \frac{1}{n}\,\frac{4}{40 - n^2\pi^2}\sin\left(\frac{n\pi t}{2}\right)}$$

Periodic solutions happen when $P\left(\frac{in\pi}{2}\right) = 0$ (denominator of the ERF part)

14) Partial differential eq'ns: finding ~~the~~ normal modes and applying Fourier coefficient formulas.

• guess: wave equation will be on exam

lmx Week 13, 32 The Wave Equation, 5. Initial Conditions:

Solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with $u(0,t)=0$, $u(L,t)=0$ $\left.\begin{array}{l} u(x,0)=f(x) \\ \frac{\partial u}{\partial t}(x,0)=g(x) \end{array}\right\}$ Initial conditions

1) Separation of Variables

$u(x,t) = X(x)\,\phi(t)$ $\quad X\frac{d^2\phi}{dt^2} = c^2 \phi \frac{\partial^2 u}{\partial x^2}$ $\quad \underbrace{\frac{1}{c^2\phi}\frac{d^2\phi}{dt^2}}_{\text{only a function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{only a function of } x} = \lambda$ (constant)

2) $X(0)=0$ and $X(L)=0$

$\to X = \sin(\alpha x) = 0$ $\quad \to \sin(\alpha L) = 0$ $\quad \alpha L = 0, \pi, 2\pi, 3\pi$, etc $\Rightarrow \alpha = \frac{n\pi}{L}$ where $n = 0,1,2,$ etc

$X = \sin\left(\frac{n\pi x}{L}\right)$ $\quad u = \sin\left(\frac{n\pi x}{L}\right)\phi(t)$

3) $\frac{\partial^2 u}{\partial t^2} = \phi''(t)\sin\left(\frac{n\pi x}{L}\right)$ $\quad \frac{\partial^2 u}{\partial x^2} = -\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi}{L}x\right)\phi(t)$

$$\phi''(t)\cancel{\sin\left(\frac{n\pi x}{L}\right)} = \underbrace{c^2 - \left(\frac{n\pi}{L}\right)^2}\cancel{\sin\left(\frac{n\pi}{L}x\right)}\phi(t)$$

$$\phi''(t) = -\left(\frac{cn\pi}{L}\right)^2\phi(t) \qquad \omega = \frac{cn\pi}{L}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\underbrace{\left[\alpha_n \sin\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right)\right]}_{=\phi(t)}}$$

$$\underbrace{\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx} \quad \boxed{\alpha_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}}_{\text{Initial Conditions}}$$

B_n gotten using: $f(x) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)B_n$ using $\longrightarrow$

α_n gotten using: $g(x) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\alpha_n\frac{n\pi c}{L}$ $\longrightarrow$

Remember that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} \frac{L}{2} & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$

Practice ~~Midterm~~ Final 1, Question 12

$\frac{\partial u}{\partial t} + ku = c\frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) = u(\pi,t) = 0 \quad t>0$

Solutions take form: $u(x,t) = \sum_{n=1}^{\infty} b_n w_n(t)\sin(nx)$

a) Find solution satisfied by w_n and solve it with $w_n(0)=1$

plug $w_n(t)\sin(nx)$ into differential equation:

$$\dot{w}_n \sin(nx) + k w_n(t)\sin(nx) = -cn^2 w_n(t)\sin(nx)$$

rearrange: $\underbrace{(\dot{w}_n(t) + (k+cn^2)w_n(t))}_{=0}\sin(nx) = 0$

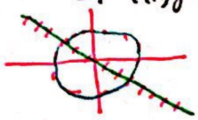
$$\Rightarrow w_n(t) = e^{-(k+cn^2)t}$$

By the way, $-(k+cn^2)$ is just lambda. the form is $e^{\lambda t}$

15) 1D nonlinear solutions: isoclines, critical points and phase line, long term outcomes

- a C-isocline is the set of points in plane where each point has slope C.
  $\Rightarrow f(x,y) = C$

i.e.



Practice Final v.1, Question 5

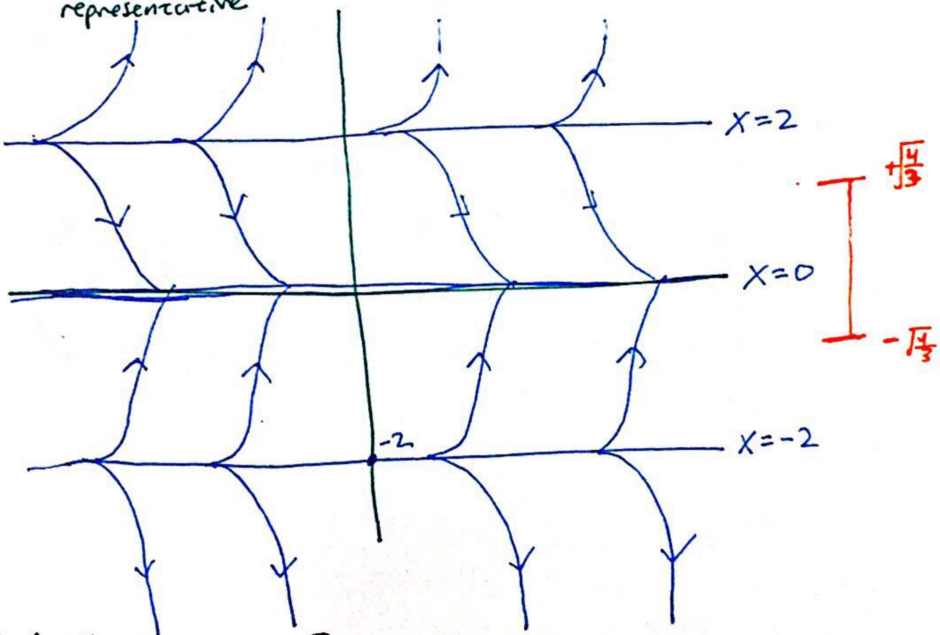
Draw the phase line for: $\dot{x} = x^3 - 4x$

1. solve for critical points:

$f(x) = \dot{x} = x(x-2)(x+2)$ critical pts. at $0, 2, -2$

test points: $f(-1) = +$ $f(-3) = -$ $f(1) = -$ $f(3) = +$

2. Draw representative solutions to the equation in the (t,x) plane



If you want to draw slightly more accurate picture, take 2nd derivative

$\ddot{x} = 3x^2 - 4 = 0$

when $x = \pm\sqrt{\frac{4}{3}}$

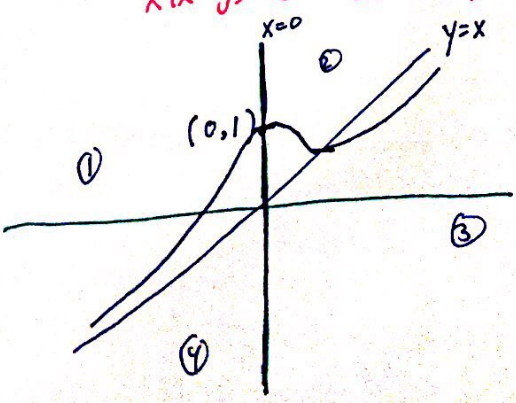
Notice the rate of slope change changes (concavity)

Practice Final V1, Question 13

Draw 0-isoclines of $y' = x(x-y)$ and sketch solution satisfying $y(0) = 1$

Solve for critical points:

$x(x-y) = 0$ when $x = 0$ or $y = x$



test points (one in each space):

① test: $(-1,0)$ $-1(-1) = 2$ $+$

② $(1,2)$ $1(1-2) = -$

③ $(1,0)$ $1(1) = +$

④ $(-1,-2)$ $-1(-1-(-2)) = -$

Does the solution curve cross the positive x-axis?

No. the curve decreases in region 2 but then increases in region 3. (So it never crosses)

16) 2D nonlinear solutions: critical points and phase plane, linearization, and long term outcomes

lms, week 14, 37 Linearization and population models, 15. Big Picture of Deer-Wolf System

Consider prey-predator system:

$$\dot{x} = x(3-x) - xy = f(x,y)$$
$$\dot{y} = y(1-y) + xy = g(x,y)$$

Find Critical points:

$$f = g = 0 \begin{cases} x=0, \ y = 3-x \\ y=0, \ x = y-1 \end{cases} \Rightarrow (0,0), (0,1), (3,0), (1,2)$$

Linearize the system:

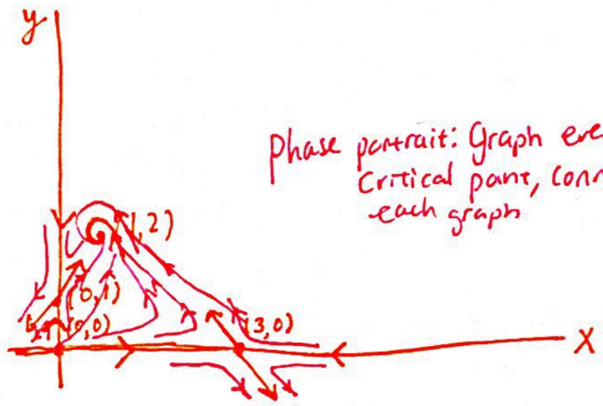
compute Jacobian

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

evaluate at critical points

Critical Pts	(0,0)	(0,1)	(3,0)	(1,2)
J	$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -3 & -3 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$
λ_1, λ_2	3, 1	2, -1	-3, 4	$(-3 \pm i\sqrt{7})/2$
graph	unstable node	saddle (not stable)	saddle unstable	spiral stable

Graph:



Phase portrait: Graph every critical point, connect each graph