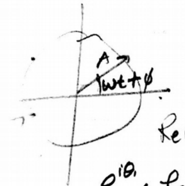
8.03 Pset 1   Due 1 ✓ 2   3   4 ✓   12/22/19

Problem Set.

$E(t) = E_0\cos(\omega t) + E_0\cos(\omega t+\delta) + E_0\cos(\omega t + 2\delta)$

a) Make drawing in complex plane of sum as addition of three complex vectors

$E_0\cos(\omega t)$
$=$



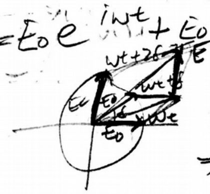
$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$E_0\cos(\theta)$
$= E_0 e^{i\omega t}$?

$Re(Ae^{i(\omega t+\phi)})$

$e^{i\theta_1}\cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

$E(t) = E_0 e^{i\omega t} + E_0 e^{i(\omega t+\delta)} + E_0 e^{i(\omega t + 2\delta)}$



Plot the locus of the positions of sum vectors as δ varies from 0 to 2π for time $t=0$

→ complex #s can be represented on an Argand diagram

can consider complex #s as pts or vectors.

modulus: distance of pt. from origin.

$w = \frac{6}{5} + \frac{8}{5}i$

$|w| = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{8}{5}\right)^2}$
$= \sqrt{4} = 2$
distance of pt from origin

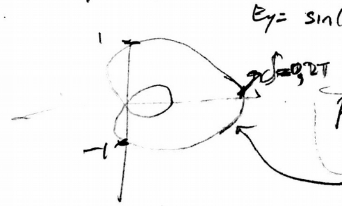
locus of pt. z → all pts z which satisfy condition

$X = r\cos\theta$
$y = r\sin\theta$
rectangular coordinates

plot (E_x, E_y) → $E_x = 1 + \cos\delta + \cos(2\delta)$
$E_y = \sin(\delta) + \sin(2\delta)$

plot (E_x, E_y)
as δ varies from $0 \to 2\pi$

Assume E_0



ParametricPlot[{1+Cos[δ]+Cos[2·δ], Sin[δ]+Sin[2δ]},
{δ, 0, 2π}]

wolfram alpha to produce

b) Find amplitude of sum vectors by summing each vector described as magnitude multiplied by a phase

Sum ⇒ 3 terms.

Make plot of δ as δ varies from 0 to 2π

Amplitude → sum vector: magnitude × phase.

$Ae^{i(\omega t+\delta)}$

magnitude × phase

$E(t) = E_0\cos(\omega t) + E_0\cos(\omega t+\delta) + E_0\cos(\omega t+2\delta)$
$= Re\left(E_0\left(e^{i\omega t} + e^{i(\omega t+\delta)} + e^{i(\omega t+2\delta)}\right)\right)$
$= E_0 e^{i(\omega t+\delta)}\left(e^{-i\delta} + 1 + e^{i\delta}\right)$
$= (1 + 2\cos\delta)$
$\to = \boxed{E_0 e^{i(\omega t+\delta)}(1+2\cos\delta)}$

$\boxed{\begin{array}{l}\text{common to remember}\\ \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\\ \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}\end{array}}$

different way of doing it

$E_0^{tot} = \sqrt{E_0^2\left(e^{i\omega t} + e^{i(\omega t+\delta)} + e^{i(\omega t+2\delta)}\right)\left(e^{-i\omega t} + e^{-i(\omega t+\delta)} + e^{-i(\omega t+2\delta)}\right)}$

magnitude $= \sqrt{E^2}$

at $t=0$

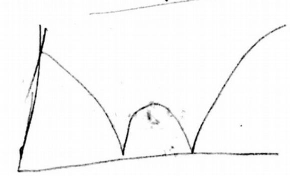
$\Rightarrow = \sqrt{E_0^2(1+e^{i\delta}+e^{i2\delta})(1+e^{-i\delta}+e^{-i2\delta})}$

$= E_0\sqrt{1+e^{-i\delta}+e^{-i2\delta}+e^{i\delta}+1+e^{-i\delta}+e^{i2\delta}+e^{i\delta}+1}$

$= E_0\sqrt{3+2e^{-i\delta}+2e^{i\delta}+e^{-i2\delta}+e^{i2\delta}}$

$= E_0\sqrt{3+2(e^{-i\delta}+e^{i\delta})+e^{-i2\delta}+e^{i2\delta}} = \boxed{E_0\sqrt{3+2(e^{-i\delta}+e^{i\delta}) + 2\cos(2\delta)}}$

$= E_0\sqrt{3+4\cos(\delta)+2\cos(2\delta)}$ ⇒ δ from $0 \to 2\pi$



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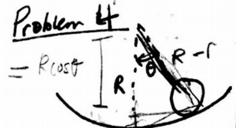
c) The power in a harmonic signal ∝ square of amplitude of the signal. Make sketch of relative power $\frac{P(\delta)}{P(0)}$ as δ varies from 0 to 2π

$P \propto E_0^2 (3 + 4\cos(\delta) + 2\cos(2\delta))$

make plot of $\frac{P(\delta)}{P(0)}$

$P(0) = (3+4+2) = 9$

Problem 4



ball of radius r & mass M moving ~~[illegible]~~ under influence of gravity rolls back & forth

$r < R$

$\boxed{\text{Show for small displacements } \theta(t) \text{ is harmonic \& find frequency of oscillation} \quad \omega \text{ is frequency of oscillation}}$

$h = (R-r) - (R-r)\cos\theta = R(1-\cos\theta)$

$V(\theta) \; mg(R-r)(1-\cos\theta)$

$\approx mg(R-r)(1-(1-\frac{\theta^2}{2}))$

$\approx mg(R-r)\frac{\theta^2}{2}$

$\frac{dx^2}{dt} = \omega^2 x$

Small angle approximations (memorize!)

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$\tan\theta \approx \theta$

$mv = \frac{mgh}{2}$

potential energy

kinetic energy

$KE = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm}\omega^2$

$= \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \frac{V_{cm}^2}{r}$

$= \frac{1}{2} V_{cm}^2 (M + \frac{I_{cm}}{r^2})$

$I_{cm} =$

$d\phi = \frac{d(X_{cm})}{r}$

$\Rightarrow \omega = \frac{\dot{X}_{cm}}{r} = \frac{V_{cm}}{r}$

I_{cm} (given uniform) $= \frac{2}{5} M r^2$

$V_{cm} = (R-r)\dot\theta$

$KE = \frac{1}{2}(M + \frac{2}{5}M)(R-r)^2\dot\theta^2 = \frac{7}{10} M (R-r)^2 \dot\theta^2$

$\frac{7}{5}M$

$E = V(\theta) + KE(\theta) = mg(R-r)\frac{\theta^2}{2} + \frac{7}{10} M (R-r)^2 \dot\theta^2$

$mg(R-r)\theta\dot\theta + \frac{7}{10} M (R-r)^2 2\dot\theta\ddot\theta = 0$

$\frac{dE}{dt} = 0$ energy conserved

$\Rightarrow -Mg(R-r)\theta\dot\theta = \frac{7}{5} M (R-r)^2 \dot\theta\ddot\theta$

$-g\theta = \frac{7}{5}(R-r)\ddot\theta$

$\ddot\theta + \frac{5g}{7(R-r)}\theta = 0$

$\ddot\theta = \frac{-5g}{7(R-r)}\theta$

$\boxed{\omega = \sqrt{\frac{5g}{7(R-r)}}}$

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~~diatomic molecule~~

- relative distance r(t) executes periodic oscillations
- If potential energy of molecule as function of r given by V(r), time dependence of r(t) is identical to particle of reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ moving in potential V(r)
- analytic expression: Morse potential $V(r) = B(1 - e^{-\beta(r-r_0)})^2 - B$

B: depth of well r_0: equilibrium separation

β: how rapidly energy rises as one moves away from equilibrium

a) Find frequency of small classical vibrations about min. of Morse potential in terms of B, m, and β

$V(r) = B(1 - 2e^{-\beta(r-r_0)} + e^{-2\beta(r-r_0)}) - B$

$= B - 2Be^{-\beta(r-r_0)} + Be^{-2\beta(r-r_0)} - B$

$= Be^{-2\beta(r-r_0)} - 2Be^{-\beta(r-r_0)}$

$V(r)|_{r=r_0} = B - 2B = -B$

$\frac{dV}{dr} = B(-2\beta)e^{-2\beta(r-r_0)} - 2B(-\beta)e^{-\beta(r-r_0)}$

$\frac{dV}{dr} = 2B\beta e^{-\beta(r-r_0)} - 2B\beta e^{-2\beta(r-r_0)} = 0$

$\frac{dV}{dr}\Big|_{r=r_0} \Rightarrow 2B\beta - 2B\beta = 0$

find w... defined as r_0 + B

$\frac{d^2V}{dr^2} = 2B\beta(-\beta)e^{-\beta(r-r_0)} - 2B\beta(-2\beta)e^{-2\beta(r-r_0)}$

$= -2B\beta^2 e^{-\beta(r-r_0)} + 4B\beta^2 e^{-2\beta(r-r_0)}$

$\frac{d^2V}{dr^2}\Big|_{r=r_0} = 4B\beta^2 - 2B\beta^2 = 2B\beta^2$

I guess $\frac{d^2V}{dr^2}$ is k...?

$\Rightarrow V(r) = -B + \frac{1}{2}(2B\beta^2)(r-r_0)^2$

Taylor expansion

$V(r) = V(r_0) + V'(r_0)(r-r_0) + \frac{1}{2}V''(r_0)(r-r_0)^2$

$= -B + 0 + \frac{1}{2}(2B\beta^2)(r-r_0)^2$

"Find frequency about the minimum: Taylor expansion around min, then use spring constant" $k = 2B\beta^2$

$m = \frac{1}{2}$ mass of diatomic molecule

$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{2B\beta^2}{\frac{1}{2}m}} = 2\beta\sqrt{\frac{B}{m}}$

b) For molecule H_2 $\beta = 1.93$ angstrom^{-1} $r_0 = 0.74$ angstrom $B = 4.8\,eV$
mass of hydrogen is 1.67×10^{-24} g. What is the frequency of small vibrations?
observed vibrational frequency is 1.32×10^{14} Hz.

$1\,eV = 1.6\times10^{-12}$ ergs
$1\,Å^{-1} = 10^{8}\,cm^{-1}$

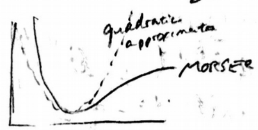
erg: unit of energy equal to 10^{-7} J.

$$\omega = 2\beta\sqrt{\frac{B}{m}} = 2(1.93\,Å^{-1})\sqrt{\frac{4.8\,eV}{1.67\times10^{-24}\,g}}$$

$$= 2(1.93\times10^{8}\,cm^{-1})\sqrt{\frac{4.8\times(1.6\times10^{-12})\,erg}{1.67\times10^{-24}\,g}}$$

H_2: $1/s$ ...

Convert to Hz: $\nu = \frac{\omega}{2\pi} = \frac{1}{\pi}(1.93\times10^{8}\,cm^{-1})\sqrt{\frac{4.8\times(1.6\times10^{-12})\,erg}{1.67\times10^{-24}\,g}} = 1.32\times10^{14}\,Hz$



Problem 2

Find real representation to displacement, velocity, and acceleration in form:
$\eta(t) = a\cos(\omega t) + b\sin(\omega t)$

a) $X(t) = Ae^{j(\omega t - \frac{\pi}{4})} = A\cos(\omega t - \frac{\pi}{4}) =$
$\hookrightarrow = (A\cos\phi)\cos\omega t - (A\sin\phi)\sin\omega t$
$= A\cos(-\frac{\pi}{4})\cos\omega t - (A\sin(-\frac{\pi}{4}))\sin\omega t$

$\cos(-\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ $\quad \sin(-\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$ $\quad \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$= A\frac{\sqrt{2}}{2}\cos(\omega t) + A\frac{\sqrt{2}}{2}\sin(\omega t)$

$X(t) = \boxed{\frac{A}{\sqrt{2}}\cos(\omega t) + \frac{A}{\sqrt{2}}\sin(\omega t)}$

$\boxed{\dot{X}(t) = \frac{A}{\sqrt{2}}\omega\cos(\omega t) - \frac{A}{\sqrt{2}}\omega\sin(\omega t)}$ $\quad \boxed{\ddot{X}(t) = -\frac{A}{\sqrt{2}}\omega^2\cos(\omega t) - \frac{A}{\sqrt{2}}\omega^2\sin(\omega t)}$

b) $X(t) = Be^{j(\omega t + \frac{\pi}{2})} = B\cos(\frac{\pi}{2})\cos\omega t - B\sin(\frac{\pi}{2})\sin\omega t$
$\dot{x}(t) = 0 + B\sin\omega t$ $\quad \dot{x}(t) = \frac{B}{\omega}\cos(\omega t)$ $\quad X = \frac{B}{\omega^2}\sin(\omega t)$

c) $\dot{X}(t) = \frac{C}{1+j\omega t}e^{j(\omega t)}$
$e^{j(\omega t)} = \cos(\omega t) + i\sin(\omega t)$

$Re\left[\frac{C(1-i\omega t)}{1+\omega^2t^2}(\cos(\omega t) + i\sin(\omega t))\right] = \boxed{\frac{C\cos(\omega t)}{1+(\omega t)^2} + \frac{C\omega t\sin(\omega t)}{1+(\omega t)^2}} = \dot{X}(t)$

$\ddot{X}(t) = j\omega\dot{X}(t) = \frac{C(1-i\omega t)i\omega}{1+\omega^2t^2}e^{j(\omega t)} = \frac{i\omega C + C\omega^2 t}{1+\omega^2t^2}(\cos(\omega t) + i\sin(\omega t))$

$\boxed{\ddot{X}(t) = \frac{-\omega C\sin(\omega t)}{1+\omega^2t^2} + \frac{C\omega^2 t\cos(\omega t)}{1+\omega^2t^2}}$

$X(t) = \frac{1}{j\omega}\dot{X}(t) = \frac{C(1-i\omega t)}{i\omega(1+\omega^2t^2)}e^{j(\omega t)}$
$= \left(\frac{C/i\omega}{1+\omega^2t^2} - \frac{Ci\omega t/i\omega}{1+\omega^2t^2}\right)(\cos(\omega t) + i\sin(\omega t)) = \boxed{\frac{C\sin(\omega t)}{\omega(1+\omega^2t^2)} - \frac{Ct\cos(\omega t)}{1+\omega^2t^2}}$