Catherine Zeng pset 2

1) $\omega(k) = \omega(k_0) + (k-k_0)\left(\frac{d\omega}{dk}\right)_{k_0} + \frac{1}{2}(k-k_0)^2\left(\frac{d^2\omega}{dk^2}\right)_{k_0}$

$\Psi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx-\omega t)} dk \propto \int^{\infty} A(k) e^{(i(\omega(k_0)+(k-k_0)(\frac{d\omega}{dk})_{k_0} + \frac{1}{2}(k-k_0)^2(\frac{d^2\omega}{dk^2})_{k_0})}$

$A(k) = \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx = \int_{-\infty}^{\infty} C e^{ik_0 x} e^{-x^2/2L^2} e^{-ikx} dx = \cdot$

$\Rightarrow C e^{-(k-k_0)^2 L^2/2}$

$C = \frac{\sqrt{\pi}}{\sqrt{2L^2}}$ $\quad \Psi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i[kx-\omega(k)t]} \frac{dk}{2\pi}$

$= C \frac{1}{(1+i\hbar t/mL^2)^{1/2}} e^{i(k_0 x - (\hbar k_0^2/2m)t)} e^{-(x-\hbar k_0 t/m)^2/2L^2(1+i\hbar t/mL^2)}$

the wave packet will expand because $\sigma^2(t)$ is related to $\left[\left(\frac{\partial^2\omega}{\partial k^2}\right)\Big|_{k_0}\right]^2$, therefore the sign of $\left(\frac{\partial^2\omega}{\partial k^2}\right)\Big|_{k_0}$ doesn't matter

2) $\omega(k) = \omega_0 + (k-k_0)\left(\frac{d\omega}{dk}\right)_{k_0} + \frac{1}{2}(k-k_0)^2\left(\frac{d^2\omega}{dk^2}\right)_{k_0}$

a. $\boxed{v_g = \frac{\partial\omega}{\partial k}\Big|_{k=k_0}} = \left(\frac{\partial\omega}{\partial k}\right)_{k_0}$

b. $w = \frac{\hbar k^2}{2m}$ $\quad E = \hbar\omega = \frac{p^2}{2m} = \frac{\hbar k^2}{2m}$ $\quad p = \hbar k$

$m^{-1} = \frac{\frac{d\omega}{dk}|_{k_0}}{\hbar k_0} = \boxed{\frac{\frac{d^2\omega}{dk^2}|_{k_0}}{\hbar}}$

c. $\hat{H} = \frac{\hat{p}^2}{2m_{eff}} + \hat{V}$ $\quad \hat{H} = \hbar\omega(\hat{k}) = i\hbar\frac{\partial}{\partial t}$

operators

$\hat{p} = \frac{i\hbar\partial}{\partial x}$

$\hat{k} = \frac{-i\partial}{\partial x}$

particle in free space: $\Psi(x,t) = \int A(k) e^{ikx} e^{-i\omega(k)t} \frac{dk}{2\pi}$

$\Rightarrow \frac{i\hbar\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi$

photon in fiber $\quad \omega(k) = ?$ $\quad i\hbar\frac{\partial}{\partial t}\Psi = \int A(k) e^{ikx} \hbar\omega(k) e^{i\omega t} \frac{dk}{\partial x}$

$\hbar\omega_0 + (k-k_0)\left(\frac{\partial\omega}{\partial k}\right)_{k_0} + \frac{1}{2}(k-k_0)^2\left(\frac{\partial^2\omega}{\partial k^2}\right)_{k_0}$

$\boxed{\hat{H} = i\hbar\frac{\partial\Psi}{\partial t} = \left(\hbar\omega_0 - \hbar k\left(\frac{\partial\omega}{\partial k}\right)_{k_0}\right)\Psi - i\frac{\partial\Psi}{\partial x}\hbar\left(\frac{\partial\omega}{\partial k}\right)_{k_0}}$

3) $F(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad G(k)=\int_{-\infty}^{\infty} g(x)e^{-ikx}dx$

$$\Rightarrow \int_{-\infty}^{\infty} f^*(x)g(x)\,dx=\int_{-\infty}^{\infty} F^*(k)G(k)\frac{dk}{2\pi}$$

a)

$$\cancel{F(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx}$$

$$f^*(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)^* e^{-ikx}dk \qquad g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)e^{ikx}dk$$

$$\int_{-\infty}^{\infty} f^* g(x)\,dx=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k')^* e^{-ik'x}dk'\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} G(k)e^{ikx}dk\right)dx$$

$$=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k')^*G(k)\left(\int_{-\infty}^{\infty} e^{i(k-k')x}dx\right)dk'\,dk \quad \delta(k-k')=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}dx$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k')^*\underbrace{\left[\int_{-\infty}^{\infty} G(k)\,\delta(k-k')\,dk'\right]}_{=G(k)}dk$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(k')G(k)\,dk=\boxed{\int_{-\infty}^{\infty} F^*(k)\,G(k)\frac{dk}{2\pi}}$$

b) $\int_{-\infty}^{\infty}\psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)\,dx$

$$=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hbar k A^*(k')A(k)\underbrace{\left(\int_{-\infty}^{\infty} e^{i\left[\left(kx-\frac{\hbar k^2}{2m}t\right)-\left(k'x-\frac{\hbar (k')^2}{2m}t\right)\right]}dx\right)dk'\,dk}$$

~~this~~ used:

$$-i\hbar\frac{\partial}{\partial x}\Psi(x,t)=\int_{-\infty}^{\infty}\hbar k A(k)e^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}\frac{dk}{2\pi}$$

$$=\boxed{\int_{-\infty}^{\infty} A^*(k)\,\hbar k\, A(k)\frac{dk}{2\pi}}$$

c) $\langle x\rangle=\int_{-\infty}^{\infty}\psi^*(x)\,x\psi(x)\,dx=\int_{-\infty}^{\infty} A^*(k)\,\hat{Q}(k)\,A(k)\frac{dk}{2\pi}$

$$\frac{\int_{-\infty}^{\infty}\psi^*(x)\,x\,\psi(x)\,dx}{\int_{-\infty}^{\infty}\psi^*(x)\psi(x)\,dx}=\frac{\int_{-\infty}^{\infty} A^*(k)\,x\,A(k)\frac{dk}{2\pi}}{\int_{-\infty}^{\infty} A^*(k)\,A(k)\frac{dk}{2\pi}}$$

$$\Rightarrow \boxed{\hat{Q}(k)=x}$$

$$x\Psi(x,t)=\int_{-\infty}^{\infty} x A(k)e^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}\frac{dk}{2\pi}$$

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}A^*(k')\,x\,A(k)\left(e^{i\left(kx-\frac{\hbar k^2}{2m}t\right)-i\left(k'x-\frac{\hbar k'^2}{2m}t\right)}\right.$$

$$=\int_{-\infty}^{\infty} A^*(k)\,x\,A(k)\frac{dk}{2\pi}$$

d) $i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{1}{2}m\omega^2x^2\psi(x,t) \Rightarrow \hat{E} = \hat{P}\ldots$

recast in k-space for $A(k,t)$ for

$\psi(x,t) = \int_{-\infty}^{\infty} A(k,t)\, e^{ikx}\frac{dk}{2\pi}$ $\quad A(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x,t)\, e^{-ikx} dk$

$\Rightarrow \psi^*(x,t) = \int_{-\infty}^{\infty} A^*(k,t)e^{ikx}\frac{dk}{2\pi}$ $\quad A^*(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi^*(x,t)e^{-ikx} dk$

→ I'm not sure where to go next - will work on it over the weekend. Also will probably do some optional problems

4)

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2$$

a) $\frac{d}{dt}\langle\hat{x}\rangle = \frac{d}{dt}\int_{-\infty}^{\infty} x\Psi^*(x,t)\,\Psi(x,t)\,dx = \int_{-\infty}^{\infty}\Psi^*(x,t)\left(\frac{-i\hbar}{m}\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx$

$= \frac{d}{dt}\underbrace{\int_{-\infty}^{\infty} x\,P(x,t)\,dx}_{=\langle x\rangle} \Rightarrow \frac{\int_{-\infty}^{\infty}\Psi^*(x,t)\hat{p}\,\Psi(x,t)\,dx}{m}$ because $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

$\Rightarrow \boxed{\frac{d}{dt}\langle x\rangle = \frac{\langle\hat{p}\rangle}{m}}$

b) $\frac{d}{dt}\langle\hat{p}\rangle = \frac{d}{dt}\int_{-\infty}^{\infty}\Psi^*(x,t)\,\hat{p}\,\Psi(x,t)\,dx = \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx$

$= -i\hbar\left(\int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x}\,dx + \int_{-\infty}^{\infty}\Psi^*\frac{\partial^2\Psi}{\partial t\,\partial x}\,dx\right)$

$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi$

$= \int_{-\infty}^{\infty}\left(i\hbar\frac{\partial\Psi}{\partial t}\right)^*\frac{\partial\Psi}{\partial x}\,dx + \int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial x}\left(i\hbar\frac{\partial\Psi}{\partial t}\right)dx$

applying $\nearrow$

$\Rightarrow = \int_{-\infty}^{\infty}\left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}\right) + V(x)\frac{\partial|\Psi|^2}{\partial x}\right)dx$

$= -\int_{-\infty}^{\infty}\frac{dV(x)}{dx}|\Psi|^2\,dx \boxed{= -\left\langle\frac{dV}{dx}\right\rangle}$

c) $\frac{d}{dt}\langle\hat{H}\rangle = \frac{d}{dt}\int\Psi^*(x,t)\,\hat{H}\,\Psi(x,t)\,dx$

$= \left\langle\cancel{\frac{\partial\hat{H}}{\partial t}}\right\rangle^{0} + \int\frac{d\Psi^*}{dt}\hat{H}\Psi\,dx + \int\Psi^*\hat{H}\frac{\partial\Psi}{\partial t}\,dx$

$\int\Psi^*\hat{H}\frac{\partial\Psi}{\partial t}\,dx = \int\Psi^*\hat{H}\underbrace{\frac{1}{i\hbar}\hat{H}\Psi}\,dx = \frac{1}{i\hbar}\int\Psi^*\hat{H}\hat{H}\Psi\,dx$

this come from

$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi \Rightarrow \frac{\partial\Psi}{\partial t} = \frac{\hat{H}\Psi}{i\hbar}$

$\int\frac{d\Psi^*}{\partial t}\hat{H}\Psi\,dx \quad -i\hbar\frac{\partial\Psi^*}{\partial t} = \hat{H}^*\Psi^* \Rightarrow \frac{\partial\Psi^*}{\partial t} = \frac{\hat{H}^*\Psi^*}{-i\hbar}$

$\frac{1}{-i\hbar}\int\Psi^*\hat{H}\hat{H}\Psi\,dx + \frac{1}{i\hbar}\int\Psi^*\hat{H}\hat{H}\Psi\,dx \boxed{= 0}$

d) $\langle \hat{E} \rangle = \int_{-\infty}^{\infty} \psi^* \frac{(-\hbar^2)}{2m} \frac{\partial^2}{\partial x^2} \psi \, dx = \frac{1}{2m} \int_{-\infty}^{\infty} \psi^* \underbrace{-\hbar^2 \frac{\partial^2}{\partial x^2}} \psi \, dx$

$\hat{p} = -i\hbar \frac{\partial}{\partial x}$

$\hat{p}^2 = \hbar^2 \frac{\partial^2}{x^2}$

$= \frac{1}{2m} \int_{-\infty}^{\infty} \psi^* \hat{p}^2 \frac{\partial^2}{\partial x^2} \psi \, dx$

$\boxed{= \frac{\langle \hat{p}^2 \rangle}{2m}}$

e)

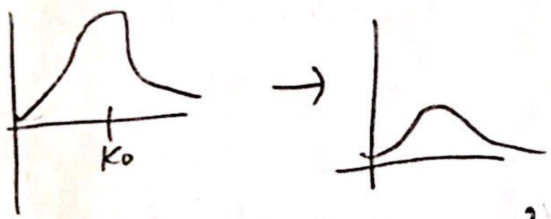
Answers from problem	classical consideration
$\frac{d}{dt} \langle \hat{x} \rangle = \frac{\langle \hat{p} \rangle}{m}$	$\frac{d}{dt} x(t) = \frac{p(t)}{m}$
$\frac{d}{dt} \langle \hat{p} \rangle = -\langle \frac{dV}{dx} \rangle$	$\frac{d}{dt} p(t) = -\frac{dV}{dx}$
$\frac{d}{dt} \langle \hat{H} \rangle = 0$	$\frac{d}{dt} H(t) = 0$
$\frac{d}{dt} \langle \hat{E} \rangle = \frac{\langle \hat{p}^2 \rangle}{2m}$	$\frac{d}{dt}(E) = \frac{(p(t))^2}{2m}$

5) a. $\int_{-\infty}^{\infty} |\psi(x,0)|^2 = 1$ this is from $e^{i(kx - \omega t)}$ $E = \hbar\omega = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m} \Rightarrow \omega = \frac{\hbar k^2}{2m}$

b. $\psi(x,t) = \int_{-\infty}^{\infty} A(k) e^{ikx} e^{-\frac{i\hbar k^2}{2m}t} \frac{dk}{2\pi}$ $A(k) = \int_{-\infty}^{\infty} \psi(x,0) e^{-ikx} dx = \int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{2}}\, e^{ik_0 x} \operatorname{sech}(\alpha x) e^{-ikx} dx$

$\psi(x,t) = \int_{-\infty}^{\infty} \frac{\pi}{\sqrt{2\alpha}} \operatorname{sech}\left(\frac{\pi(k-k_0)}{2\alpha}\right) e^{i(kx - \frac{\hbar k^2}{2m}t)} \frac{dk}{2\pi}$ $= \frac{\pi}{\sqrt{2\alpha}} \operatorname{sech}\left(\frac{\pi(k-k_0)}{2\pi}\right)$

c. $|\psi(x,t)|^2$ when $t \to \infty$



$V_g = \frac{\partial\omega}{\partial k} = \frac{2\hbar k_0}{2m} = \frac{\hbar k_0}{m}$

bucket of marbles, dumping & getting everywhere
Marbles faster goes farther
So the shape stretches.
gets stretched out based on velocity

$(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle \Rightarrow (\Delta x(t))^2 \big|_{t=0} + (\Delta V)^2 t^2$

$P_k(k) = \frac{|A(k)|^2}{2\pi}$ $P_v(V) = \underset{\text{const}}{\overset{\wedge}{=}} P(k) \big|_{k=\frac{mv}{\hbar}}$ $p = mv$ $k = \frac{mv}{\hbar}$

d. the avg velocity doesn't change because there is no force.

e. $[\Delta x(t)]^2 = \langle (x - \langle x \rangle)^2 \rangle \Rightarrow (\Delta x(t))^2 \big|_{t=0} + (\Delta V)^2 t^2$