Lecture 1: Intro, Linear DE, Modeling Input-Response 2/6/19

Example of DE:

$y'(x) = f(x, y(x))$ 1st order (only one derivative)

$\vec{y}'(t) = \vec{f}(t, \vec{y}(t))$ System of ordinary DE (ODE)

trajectories, celestial mechanics

→ motivation for inventing calculus

Notations:

$y = y(t)$, $\dot{y} = \frac{dy}{dt}$ • → means a function of time

↑ dependent variable

$\ddot{y} = \frac{d^2}{dt^2} y$, ..., $y^{(n)} = \frac{d^n}{dt^n} y$

Star of the show:

$\dot{y} = y$, $y(0) = 1$, $\boxed{y(t) = e^t}$

circular/sinusodal functions

Also featuring:

$x(t) = \sin(t)$ or $\cos(t)$ solving $\ddot{x} = x$

METHODS:

- analytic (formulas): exponential, sin, cos ← effective for linear equations
- numerical ← have to use this for non-linear equations
- qualitative (pictures of trajectories)
- fourier analysis, linear algebra

Some equations don't have simple formulas

Fourier Analysis

- vibration + sound waves split into sines and cosines.
- you can apply this to study... heat!
  (heat + diffusion are everywhere)
- light!! ... quantum mechanics!!

Heat equation: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

t (time); x (space); u (temperature)

Schrödinger equation: $i\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$

difference... complex number

Fourier analysis changed what we mean by a function.

Analogy:

Decimal system : Numbers
Fourier Analysis : Functions

Breaking down signals into frequencies

Linear DE

Example 1 (warm-up)

$\dot{y} = 3y$ what is $y(3)$?

Separation of variables

$\frac{dy}{dt} = 3y \quad \int \frac{dy}{y} = \int 3\,dt \quad \ln y = 3t + c$

$y = e^{3t+c} = Ae^{3t} \quad (A = e^{c} > 0)$

Answer = ??

The general solution is ~~Ae~~ $y = Ae^{3t}$, so we don't know what $y(3)$ is.

Good news! One basic solution $y=e^{3t}$ determines a whole family of solutions

Example 2

$$\dot{y} = ay \quad y(0) = y_0$$

Solution: $y(t) = y_0 e^{at}$

☆ "most important". Just remember it...
It solves the most simple differential equations

Best method: knowing the answer

Exponential growth $a=k>0$ $e^{kt} \nearrow$
Exponential decay $a=-k<0$ $e^{-kt} \searrow$
$(k>0)$ use parameters that have a sign when using exponentials.

An ODE is linear if it is linear in the variables $y, \dot{y}$, etc

Example 3

$$\underbrace{e^t}\ddot{y} + \underbrace{6}\dot{y} + \underbrace{8}y = b(t)$$

coefficients

linear inhomogeneous

linear in $y, \dot{y}, \ddot{y}$

homogeneous means $b(t)=0$
inhomogeneous, just put anything on the right hand side

☆ Standard linear 1st order form:

$$\boxed{\dot{y} + p(t)y = q(t)}$$

(Solve on Fri.)

Modeling

give p & q some personality

"you don't actually care about the answer (difference between professionals like me + diledantes like you)"

- conceptual errors... if you give your variables personality + care about the outcome, then you'll care about the answer.

Banking example

$X = x(t)$ bank balance \$U.S.

t time in years

$x(0) = 200 \cdot$ initial balance

compound montly @ 3% interest/year

$\Delta t = 1/12$ = one month

$$\Delta x = x(t+\Delta t) - x(t) = \underbrace{\Delta t \tfrac{3}{100}}_{\frac{1}{12}\text{yr}\ \ \frac{3}{100}/\text{yr}} x(t)$$

← units must balance

$$\boxed{\frac{\Delta x}{\Delta t} = \frac{3}{100} x}$$

Continuous Compounding means $\Delta t \to 0$

$$\boxed{\frac{dx}{dt} = \frac{3}{100} x}$$

initial condition

$X(0) = 200$

Montly deposits: \$10/month

$$\underbrace{\frac{dx}{dt}}_{\$/\text{yr}} = \underbrace{\frac{3}{100}}_{1/\text{yr}} \underbrace{x}_{(\$)} + \underbrace{q(t)}_{\$/\text{yr}}$$

$q(t) = \$120/\text{yr}$

Lecture 2 1st Order Linear ODE 2/8/19

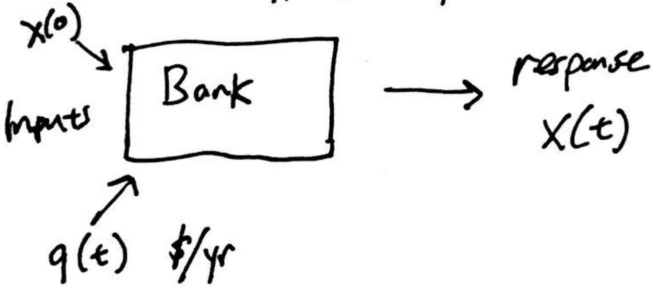
Topics:

Input-response, superposition, linear combination, variation of parameters.

Recall:

$\dot{x} - rx = q(t)$ $r = 3\%/yr$ $x_0 = 200$ $q = 120$ \$/yr.

Model $\dot{x} - rx = q$



Superposition Principle

$x_1(0) = 200$ $x_2(0) = 0$

$q_1(t) = 0$ $q_2(t) = 120$ \$/yr

$\Rightarrow x_1(t) + x_2(t) = x(t)$

Superposition works for all linear differential equations.

- All $\boxed{\text{linear}}$ ODE $r \to r(t)$
- for any merger

☆ Spoiled by non-linearity (less predictable, no clear rules on when to merge)

$$r(x) = \begin{cases} 3\% & x > 0 \text{ (saving)} \\ 6\% & x < 0 \text{ (borrowing)} \end{cases}$$

$x_1(0) = 1000$ $x_2(0) = -800$ $x_1(t) + x_2(t) < x(t)$

$q_1(t) = 0$ $q_2(t) = 120$ \$/yr Best to merge at time $= 0$

$\boxed{\text{Superposition}}$ (r = constant) break tricky problems up...

$\dot{x} - rx = -8 + 3e^{rt}$

splits into 3 parts

$x(t) = -8x_1(t) + 3x_2(t) + ce^{rt}$

$x = -8x_1 + 3x_2 + x_h$

$$\begin{cases} 0. & \dot{x} - rx = 0 & x_h & \text{homogeneous } x_h = Ce^{rt} \\ 1. & \dot{x}_1 - rx_1 = 1 & & x_1 = -1/r \\ 2. & \dot{x}_2 - rx_2 = e^{rt} & & x_2 = te^{rt} \end{cases}$$

linear combinations

If a, b, c are constants (scalars)

input: $a + be^{rt}$ linear combo of $1, e^{rt}$

response: $x(t) = a x_1(t) + b x_2(t) + c e^{rt}$,

a linear combination of x_1, x_2, and e^{rt}

To Solve 1:

$\dot{x}_1 - r x_1 = 1$ good guess: $x_1 = A$ const.

derivative of constant $= 0$ $\Rightarrow 0 + (-rA) = 1$

$\Rightarrow A = -\frac{1}{r}$

To Solve 2:

we'll use variation of parameters (for today)

$*)\ \dot{x} - rx = e^{rt}$

Step 1: Find solution to homogeneous ~~sol~~ equation.

$x_h = ce^{rt}$ set it to 0

Step 2: Seek solution x to $(*)$ in the form

$x = u(t)e^{rt}$ (c is replaced by $u(t)$) parameter c replaced by function $u(t)$

$$(ue^{rt})' - r(ue^{rt}) = e^{rt}$$

$$\dot{u}e^{rt} + \underbrace{ure^{rt} - rue^{rt}}_{0} = e^{rt} \Rightarrow \dot{u}e^{rt} = e^{rt}$$

$$\dot{u} = 1$$

$\dot{u} = 1 \Rightarrow u(t) = t + c$ ← we're looking for x, not u. need to plug it back in!

* here's where most make mistakes *

$x = e^{rt}(t + c) = te^{rt} + ce^{rt}$

↑ c...

$\underbrace{x_2(t) = te^{rt}}_{\text{particular solution to problem}}$ dropped ce^{rt}...? (simplest choice)

General procedure to solve inhomogeneous problem

$$P_1(t)\dot{y} + P_0(t)y = g(t)$$

Find all y_h, homogeneous solutions:

$$P_1 \dot{y}_h + P_2 y_h = 0$$

Find any single y_p, particular solution to the homogeneous problem

Add to get general solution: $\boxed{y = y_p + y_h}$

In general

Step 1 find y_h

Step 2: find any y_p particular solution

general: $y = y_p + y_h = \underbrace{-8\left(\frac{-1}{r}\right) + 3te^{rt}}_{\text{particular solution}} + \underbrace{Ce^{rt}}_{\text{homogeneous solutions}}$

Variation of parameter in general:

(*) $\dot{x} + px = g$

Step 1: Find homogeneous solution $(g = 0)$

$\dot{x} + px = 0 \rightarrow$ separation of variables

$$\frac{dx}{dt} = -p\,x$$

$$\frac{dx}{x} = -p\,dt$$

Minus sign! mistake! watch out!

$$\ln x = -\int p\,dt$$

$$\boxed{x = e^{-\int p\,dt}}$$

Step 2:

seek solution to x in form $x = ux_h$

$$(ux_h)' + pux_h = g$$

$$\dot{u}x_h + \underbrace{u\dot{x}_h + pux_h}_{u(\dot{x}_h + px_h) = 0} = g$$

$\Rightarrow \dot{u}x_h = g \quad \dot{u} = g/x_h$

double check

if it doesn't cancel,

you made mistake in Step 1

Lecture 3: Complex Numbers

2/11/19

- $z = a+bi$, $i^2=-1$ $\quad \bar{z} = a-bi$; $|z| = \sqrt{a^2+b^2}$
- cartesian vs. polar form
- euler's formula

Recall Superposition

$\dot{x}_1 + px_1 = q_1$
$\dot{x}_2 + px_2 = q_2$ $\Rightarrow x + px = q$ $\quad x = ax_1 + bx_2$
$q = aq_1 + bq_2$

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Superposition means multiply

$a[\dot{x}_1 + px_1 = q_1]$
$b[\dot{x}_2 + px_2 = q_2]$

~~Mu~~ and add to obtain

$$\frac{d}{dt}(ax_1 + bx_2) + p(ax_1 + bx_2) = aq_1 + bq_2$$

$\dot{x} = q \quad x = ax_1 + bx_2 \quad q = aq_1 + bq_2$

Why complex numbers? $\quad$ building blocks for solutions to 2nd order equations

$x = \begin{matrix} \sin t \\ \cos t \end{matrix} \quad \ddot{x} = -x$

Complex numbers <u>simplify</u> all aspects of sine and cosine

Cx # $z = a+bi$; a, b real #s $\quad (i^2=-1)$

addition: $(1+3i) + (2+i) = 3+4i$

multiplication: $(1+3i)(2+i) = 2+6i+i+3i^2 = 2+7i-3 = -1+7i$

division: $\frac{1}{(1+3i)} \times \frac{(1-3i)}{(1-3i)} = \frac{1-3i}{1^2-(3i)^2} = \frac{1-3i}{1+9} = \frac{1-3i}{10} = \frac{1}{10} - \frac{3}{10}i \quad a = \frac{1}{10} \quad b = \frac{-3}{10}$

$\bar{z} = a-bi$ $\quad (a+bi)(a-bi) = a^2-(bi)^2 = a^2+b^2$

complex conjugate

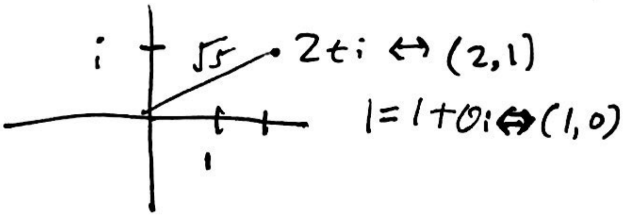
$|z|$ absolute value of z

$|z| = \sqrt{a^2+b^2}$; $|a+0i| = \sqrt{a^2+0^2}$

$|a| = \sqrt{a^2}$

Cartesian Form of z

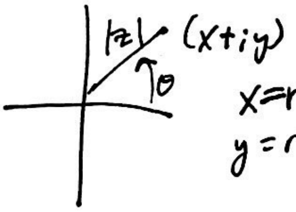
$z = a+bi \leftrightarrow (a,b)$ in 2D plane



$|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$ length of 2+i
(vector from 0 to (2,1))

if $z = 2+i$, $\bar{z} = 2-i$

Polar Form $z = x+iy$



$x = r\cos\theta$
$y = r\sin\theta$ polar coordinates

$z = x+iy =$ Polar form
$= r\cos\theta + i r\sin\theta$
$= r(\cos\theta + i\sin\theta)$

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ ← euler's formula

80% mathematicians descendent from Euler

polar form: $z = x+iy = \boxed{re^{i\theta}}$

Why was Euler right?

- the power series match using $i^2=-1$
- $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ law of exponents covers all trig identities!
- $\boxed{\frac{d}{dt}e^{it} = ie^{it},\ e^{i0}=1}$ matches $\frac{d}{dt}e^{rt} = re^{rt}, e^{r0}=1$

Proof of ③

$\frac{d}{dt}e^{it} = \frac{d}{dt}(\cos t + i\sin t)$
$= -\sin t + i\cos t$

$ie^{it} = i(\cos t + i\sin t)$
$= -\sin t + i\cos t$

same

$(\cos t, \sin t)$ traces at a circle.

Initial Condition

$e^{i0} = \cos 0 + i\sin 0$
$= 1 + i\cdot 0$
$= 1$ ✓

$z\bar{z} = (2+i)(2-i)$
$= 2^2+1^2 = 5$ $z\bar{z} = |z|^2$

Circular motion is very important

Polar Form & Multiplication

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

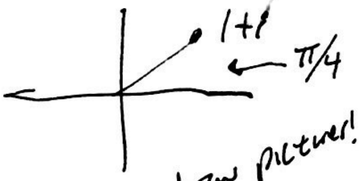
lengths multiply

angles add

$(1+i)^{10} = ?$ $\quad ((1+i)^2)^5$ works (random girl's idea)

- convert to polar form
  (draw picture)

$1+i = \sqrt{2}\, e^{i\pi/4}$

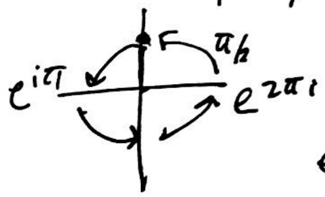


draw picture!

$$(1+i)^{10} = (\sqrt{2}\, e^{i\pi/4})^{10}$$

$$= \sqrt{2}^{10} e^{i\frac{10\pi}{4}} \quad \Leftarrow \text{really easy}$$

$$= 2^5 \boxed{e^{i\,5\pi/2}}$$



$e^{\frac{5\pi i}{2}} = e^{2\pi i} \cdot e^{\pi i/2}$

$= 1 \cdot i = i$

- convert to polar form
- apply law of exponents
- return to Cartesian form

Recitation 2 $\qquad$ 2/12

separable 1st order $\frac{dy}{dx} = h(y)\, g(x)$

to solve: $\frac{dy}{h(y)} = g(x)\,dx$

$$F(y) = \int \frac{dy}{h(y)} = \int g(x)\,dx + C = G(x) + C$$

- Separation of variables

1st order linear inhomogeneous ODE in standard form

$$\frac{dy}{dx} + p(x)y = g(x)$$

$g(x)=0$ then homogeneous

Solve these kinds of eqs. $\boxed{\text{Variation of Parameters}}$

Lecture 4 Complex exponential functions 2/13/19

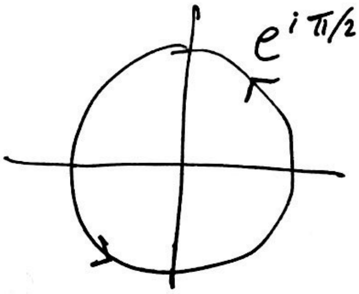
- roots of unity
- $e^{(a+ib)t} = e^{at}(\cos bt + i\sin bt)$
- antiderivatives

Circular motion

$$e^{it} = \cos t + i\sin t$$

$$|e^{it}| = 1$$

$$|\cos t + i\sin t| = \sqrt{\cos^2 + \sin^2 t} = 1$$



$e^{i\pi/2} = i$ $90°$

$1 = e^{i0}$

$$e^{i\pi/2} = \underbrace{\cos\tfrac{\pi}{2}}_{0} + i\underbrace{\sin\tfrac{\pi}{2}}_{1} = 1$$

$e^{\pi i} = -1$

$$\boxed{e^{2\pi i} = 1}$$

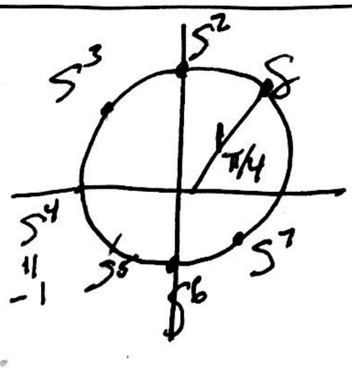
useful fact about circles: they come back to the same place

Roots of $z^n = 1$ $(z = \sqrt[n]{1})$

$n = 8$ $(e^{i\theta})^8 = 1 = e^{2\pi i k}$, $k = 0, \pm 1, \pm 2, \dots$

$8\theta = 2\pi k$ $\qquad z = e^{\pi i k/4} = \zeta^k$

$\theta = \pi k/4$ $\qquad K = 0, \pm 1, \pm 2 \dots$



$$\zeta = e^{i\pi/4}$$

$$= \frac{1+i}{\sqrt{2}} \quad \left(\text{don't change to } \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$$

wasted time

$$\zeta^8 = 1$$

Cartesian form

$\bar{\zeta} = \zeta^7 = \frac{1-i}{\sqrt{2}}$

8 roots of $z^8 = 1$

conjugate pairs $a \pm ib$

$1, \frac{1\pm i}{\sqrt{2}}, \pm i, \frac{-1\pm i}{\sqrt{2}}, -1$

$\zeta^0, \zeta, \bar{\zeta}$ $\quad \zeta^2, \bar{\zeta}^2 \quad \zeta^3, \bar{\zeta}^3, \zeta^4$

Fundamental theorem of algebra

every polynomial factor into linear...

multiplication by $re^{i\theta}$

$r > 0$ expands/shrinks

factor $e^{i\theta}$ rotates

i.e. $\zeta = e^{i\pi/4}$ rotates by $\pi/4$ (45°)

think of circle!

$(1+i)^{10} = (\sqrt{2}\, e^{i\pi/4})^{10} \dots = 32i$

$\zeta = e^{i\pi/4} \quad \zeta^8 = 1$

He sees circle!

$\boxed{\text{Solve: } w^{10} = 32i \text{ new problem}}$ ← 10 roots

one answer $w = 1+i$?

$(re^{i\theta})^{10} = 32i = 32\, e^{i(\frac{\pi}{2} + 2\pi k)}$

$\Rightarrow r^{10} = 32 \quad r = \sqrt{2}$

$10\theta = \frac{\pi}{2} + 2\pi k \quad \Rightarrow \theta = \frac{\pi}{20} + \pi \cancel{k/2}$

now rotating 10th roots 8th roots

$k = 0, \pm 1$ ... there are 10 of them

Complex Exponential

By definition

$$\frac{d}{dt} e^{zt} = z e^{zt}, \quad e^{z0} = 1$$

$$\boxed{e^z e^w = e^{z+w}}$$

$$\boxed{\left| e^{(a+bi)t} \right| = e^{at}}$$

Euler: $z = a+bi$

$$e^{(a+bi)t} = e^{at} e^{bti}$$

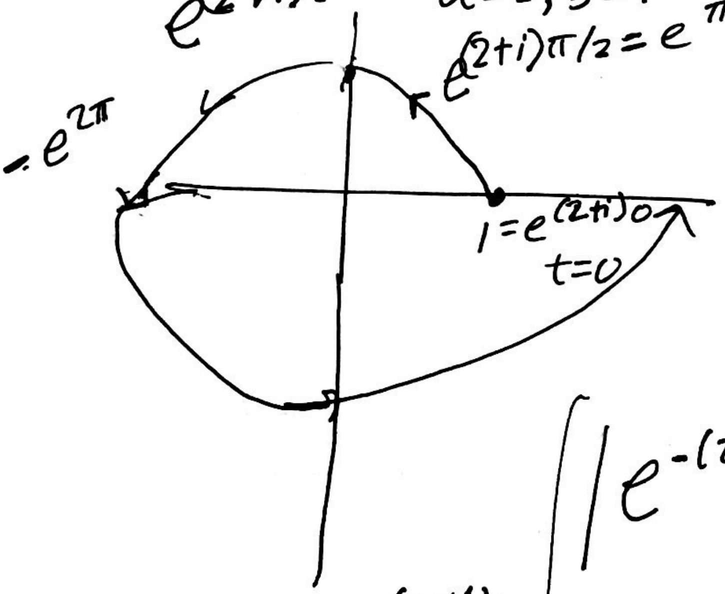
$$\boxed{e^{at}(\sin bt + i \sin bt)}$$

Picture

$e^{(2+i)t}$

$a=2,\ b=1$

$e^{(2+i)\pi/2} = e^{\pi} i$



$$e^{(2+i)\pi/2} = e^{\pi + \pi/2\, i} = e^{\pi} i$$

not a circle.

Spiral.

expanding

Counterclockwise.

$$\left[\left| e^{-(2+i)t} \right| = e^{-2t}\right]$$

shrinking

Family tree

$e^{(a+bi)t}$

parent → e^{ibt} e^{at}

$\cos bt$ $\sin bt$ ← twins

$$e^{-it} = \cos t - i \sin t$$

positive θ = counter clockwise

goes clockwise

integration by parts

$$\int e^{2t} \cos t \, dt =$$

projection of higher dimensional problem

will ~~note~~ integrate this next time →

$$\int e^{2t} (\cos t + i \sin t)\, dt$$

Lecture 5 : 2nd Order Differential Equations 2/15/19

Complex reinforcement, Spring-mass-dashpot, characteristic polynomial
anti-derivative - complex replacement

OH Monday 7:15-8:15 2-255

Complex replacement

$$\int e^{2t} \cos t \, dt =$$

$$\int e^{2t} (\cos t + i \sin t) \, dt = \int e^{(2+i)t} \, dt = Re \int e^{(2+i)t} \, dt$$

$$u(t) = e^{2t} \cos t \qquad v(t) = e^{2t} \sin t$$

$$\int (u + iv) \, dt = \underbrace{\int u \, dt}_{\text{real part}} + i \int v \, dt$$

$$\overbrace{e^{2t+it}} = e^{2t} e^{it}$$

$$\int e^{(2+i)t} \, dt = \frac{1}{2+i} e^{(2+i)t} \quad (+C)$$

need to pick out real part! Convert to a+bi

$$= \frac{2-i}{4+1} e^{2t} (\cos t + i \sin t)$$

don't need

real part $\Rightarrow \dfrac{e^{2t}}{5} (2 \cos t + \sin t) + i (\quad)$

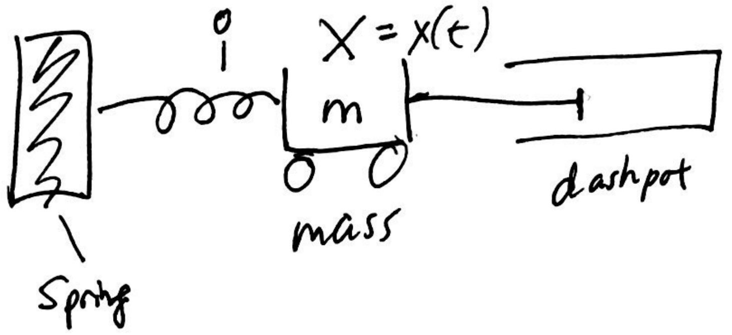
$$\int e^{2t} \cos t \, dt = \frac{e^{2t}}{5} (2 \cos t + \sin t)$$

General pattern:

$$\boxed{\int e^{at} \cos bt \, dt = \frac{e^{at}}{a^2+b^2} (a \cos bt + b \sin bt) + C}$$

$a=2$
$b=1$

Spring - mass - dashpot model



$X = 0$ is the equilibrium position of the spring (fully relaxed)

$X(t)$ is the displacement $\pm$ from equilibrium

dashpot ← resists motion in each direction. Replacement for friction

metaphor: friction (dampening), mass, potential to oscillate

$F = ma$ $\quad ma = F = F_{spring} + F_{dashpot}$

$= m\ddot{x} = -kx - b\dot{x}$

restoring force $\quad$ dampening

we want $(k > 0)$

$(b > 0)$

$m\ddot{x} + b\dot{x} + kx = 0$

homogeneous const. coeff. DE 2nd order

$m = k = 1 \quad \ddot{x} + x = 0 \quad$ cost & sin t to solve

~~gene~~ $x(0) = C_1 \quad \dot{x}(0) = C_2$

Ex 2 $\quad m = 1,\ b = 6,\ k = 5$

$\ddot{x} + 6\dot{x} + 5x = 0$

Main Idea Try $x(t) = e^{rt}$

$r^2 e^{rt} + 6re^{rt} + 5e^{rt} = 0$

$\boxed{r^2 + 6r + 5 = 0}$ characteristic equation

$(r+5)(r+1) = 0$ roots $= r = -5, -1$

solutions

$x_1(t) = e^{-5t}$

$x_2(t) = e^{-t}$

general solution

$x(t) = C_1 e^{-5t} + C_2 e^{-t}$

add initial conditions

$x(0)=1 \quad \dot{x}(0)=0$

$\dot{x}(t) = -5c_1 e^{-5t} - c_2 e^{-t}$

$1 = x(0) = c_1 + c_2 \qquad 0 = \dot{x}(0) = -5c_1 - c_2$

$1 = -4c_1 \qquad c_1 = -\frac{1}{4} \quad c_2 = 5/4$

~~x(t) = ...~~ $\quad x(t) = -\frac{1}{4} e^{-5t} + \frac{5}{4} e^{-1}$

Characteristic equation

if $x = e^{rt}$ solves $m\ddot{x} + b\dot{x} + kx = 0$

then

$mr^2 + r + k$.

Theorem Every equation $\ddot{x} + p_1(t)\dot{x} + p_0(t)x = 0$

two independent solutions $x_1(t)$ and $x_2(t)$

The general solution is

$x(t) = c_1 x_1(t) + c_2 x_2(t)$

and two initial values $x(0)$ and $\dot{x}(0)$ determine c_1 and c_2

Definition of independent

two functions $x_1(t)$ and $x_2(t)$ are linearly independent if neither is a multiple of the other.

If one is a multiple of the other, they are called linear dependent

Ex1

$\ddot{x} + x = 0 \quad r^2 + 1 = 0 \quad r = \pm i \qquad c_1, c_2, c_x$ #

solutions: $e^{it}, e^{-it} \quad \int c_1 e^{it} - c_2 e^{t} \, dt$

Scanned with CamScanner

$e^{it} = \underbrace{\cos t} + \underbrace{i \sin t}$, $e^{-it} = \cos t - i \sin t$

Both real & imaginary parts solve the original function.

Random thing that just clicked 18.100B Analysis along the real line... concerned with real problems

EX 3 Complex roots

$\ddot{x} + 6\dot{x} + 10x = 0$

☆ notice the negative was dropped here!

$r^2 + 6r + 10 = 0$

$(r+3+i)(r+3-i) = 0$ roots: $-3 \pm i \leftarrow -3 \pm i$

or r

$(r+3)^2 + 1 = 0$

$\pm\sqrt{-1} - 3$

$e^{(-3+i)t} = e^{-3t}(\cos t + \sin t)$

General solution

$x(t) = e^{-3t}(c_1 \cos t + c_2 \sin t)$

EX 4 $\ddot{x} + 6\dot{x} + 9x = 0$

$r^2 + 6r + 9 = 0$ $(r+3)^2 = 0$

e^{-3t}

missing solution

te^{-3t}

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Lecture 6: Sinusoids Functions 2/19/19

- rectangular vs. polar form
- amplitude and phase
- damped sinusoids

Review

$\ddot{x} + kx = 0 \qquad k > 0$

$r^2 + k = 0 \qquad$ Characteristic equation

$r = \pm i\omega \qquad \omega^2 = k \qquad \omega > 0$

basic: $e^{i\omega t}, e^{-i\omega t}$

$e^{i\omega t} = u + iv = x$

$\ddot{x} + kx = 0 \qquad (u+iv)^{\cdot\cdot} + k(u+iv) = 0$

$(\ddot{u} + ku) + i(\ddot{v} + kv) = 0 + 0i$

$\ddot{u} + ku = 0$ and $\ddot{v} + kv = 0$

k is real ⇒ used fact that eqn has real coefficients

$\boxed{\text{General Solution to } \ddot{x} + \omega^2 x = 0 \text{ is } a\cos(\omega t) + b\sin(\omega t)}$

aka rectangular form

2nd way to write the solution to $\ddot{x} + \omega^2 x = 0$
(polar form or amplitude phase form)...

$\boxed{A\cos(\omega t - \varphi)}$ A = amplitude φ = phase (radians) ω = angular frequency (radians/time)

Relationship between rectangular & polar form using complex numbers

$\theta = \omega t$

$$a\cos\theta + b\sin\theta = \text{Real}\left((a-bi)(\cos\theta + i\sin\theta)\right)$$
$$= \text{Re}\left((Ae^{-i\varphi})e^{i\theta}\right)$$
$$= A\cos(\theta - \varphi)$$

$\varphi = \arg(a + bi)$

$\boxed{A = \sqrt{a^2 + b^2}}$
$= |a + bi|$

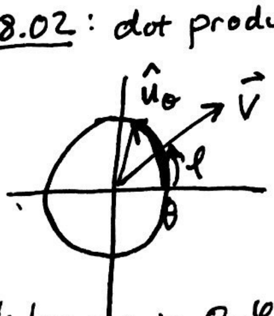
2nd proof of ~~a sin~~ $a\cos\theta + b\sin\theta = A\cos(\theta - \varphi)$

18.02: dot product

$$\parallel$$
$$\langle a, b\rangle \cdot \langle \cos\theta, \sin\theta\rangle$$
$$= |\vec{V}||\hat{u}_\theta|\cos(\theta - \varphi)$$

$\vec{V} = \langle a, b\rangle = (a, b)$

$\hat{u}_\theta = \langle \cos\theta, \sin\theta\rangle$



angle between is $\theta - \varphi$

Suppose $|\vec{V}| = A = 1$ $\quad \vec{V} = \hat{V}$ (V is a unit vector)

interpret $\hat{u}_\theta \cdot \hat{V} = a\cos\theta + b\sin\theta$

tracks circular motion with respect to a new axis $\hat{V}$.

(w.r.t. $\hat{i} = \langle 1, 0\rangle$, gives $\cos\theta$

w.r.t. $\hat{j} = \langle 0, 1\rangle$, gives $\sin\theta$)

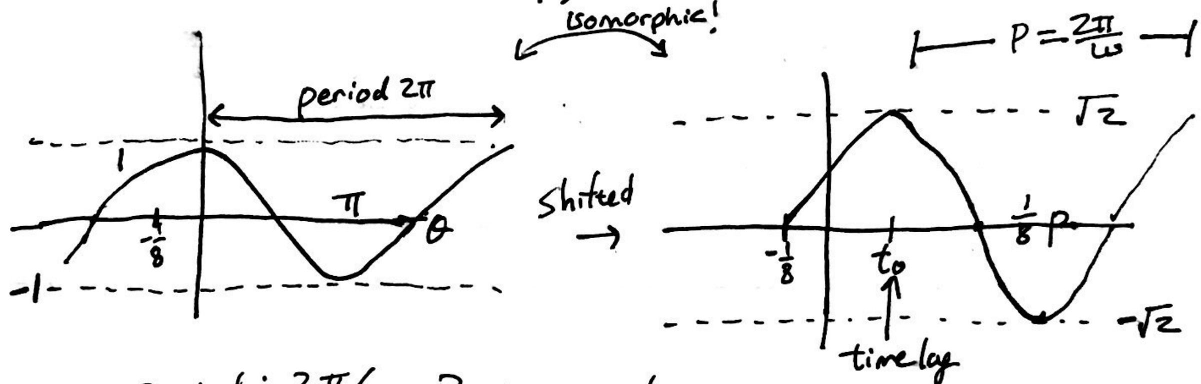
Problems:

1) Find $\cos\omega t + \sin\omega t$ in amplitude-phase form

$A = \sqrt{2}$ $\quad \theta = \omega t$ $\qquad A = \sqrt{1^2+1^2} = \sqrt{2}$

$\sqrt{2}\cos(\omega t - \varphi)$ $\qquad \varphi = \pi/4$ $\qquad \varphi = 45° = \frac{\pi}{4}$

answer: $\sqrt{2}\cos(\omega t - \frac{\pi}{4})$



period: $2\pi/\omega = P$. time period

period 2π, phase $\pi/4 \Rightarrow$ phase shift $= \frac{\pi/4}{2\pi} = \frac{1}{8}$ of period

time shift (or time lag)

$\omega t - \frac{\pi}{4} = \omega(t - t_0)$ $\quad \varphi = \omega t_0$ or $\boxed{t_0 = \varphi/\omega}$ timelag

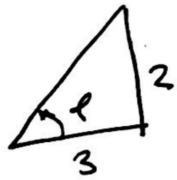
Compare time lag with the period

$t_0/P = \frac{\varphi/\omega}{2\pi/\omega} = \frac{\varphi}{2\pi} = \frac{1}{8}$ in our case

2) $\mathrm{Re}\,\frac{e^{4it}}{3+2i} = \mathrm{Re}\left(\frac{3-2i}{3^2+2^2}(\cos 4t + i\sin 4t)\right) = \boxed{\frac{1}{13}(3\cos 4t + 2\sin 4t)}$

sinusoidal $= \mathrm{Re}\left(\frac{e^{4it}}{\sqrt{13}\,e^{i\varphi}}\right) = \boxed{\frac{1}{\sqrt{13}}\cos(4t-\varphi)}$

convert to polar form:



$\varphi = \arg(3+2i)$

$m\ddot{x} + b\dot{x} + kx = 0 \qquad b>0$

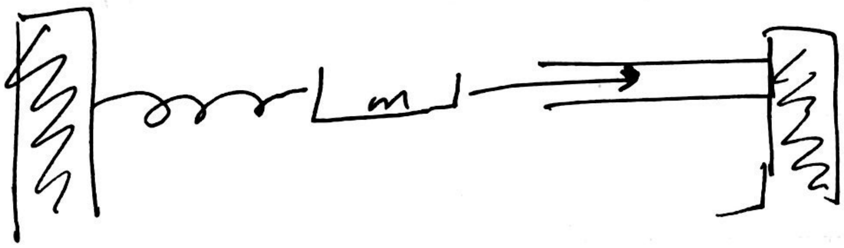
Example 1 (next time)

$$\left.\begin{aligned} &\ddot{x} + 6\dot{x} + 10x = 0 \\ &r^2 + 6r + 10 = 0 \\ &(r+3)^2 + 1 = 0 \\ &\underline{r = -3+i} \end{aligned}\right\} \quad \begin{aligned} &e^{-3t}(a\cos t + b\sin t) \\ &\boxed{\underbrace{Ae^{-3t}}_{\text{decaying amplitude}}\cos(t-\varphi)} \end{aligned}$$

only difference: decaying amplitude

Lecture 7: Damped Harmonic Oscillator 2/20/19

- underdamped = oscillatory
- idea frequency ω_0; real-life frequency ω
- critically damped - overdamped. $T = T_{relax}$



Spring-mass-dashpot

$$m\ddot{x} + b\dot{x} + kx = 0$$

$m > 0$
$b > 0$
$k > 0$

$mr^2 + br + k = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

Case 0 no damping $(b=0)$

$mr^2 + k = 0$ $\quad \omega_0^2 = k/m$ (baseline)

$r^2 = -k/m$ $\quad \omega_0$ ideal frequency

$r = \pm i\omega_0$ $\quad$ natural frequency ω_n n=natural

roots $\pm i\omega_0$

$A\cos(\omega_0 t - \varphi)$

ω_0 angular frequency / circular frequency radians/seconds

$\omega_0 = 2\pi\nu_0$ $\quad \nu_0$ frequency cycles/sec = Hz

440 Hz A
220 Hz A
middle C 260 Hz

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Case 1 $b>0$, $b^2<4mk$

Ex. $\ddot{x}+6\dot{x}+10x=0$

$r^2+6r+10=0$

complete the square $r^2+6r+9+1=0$

$(r+3)^2+1=0$

$(r+3)^2=-1$

$r+3=\pm i$

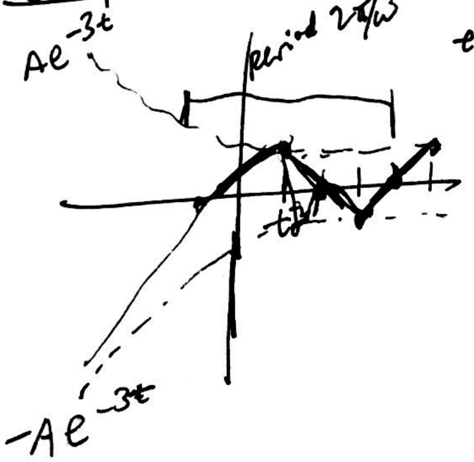
$\boxed{r=-3\pm i}$

$\omega_0=\sqrt{10}$

$A\cos(t-t_0)$ $\omega=1$ $p=3$

$Ae^{-3t}\cos(t-t_0)$ $\boxed{\text{roots: } -p\pm i\omega}$

Graph



exp. profile $\pm e^{-3t}$

$mr^2+br+k=0$ $r^2+\frac{br}{m}+\frac{k}{m}=0$ $\omega_0^2=k/m$ $\omega_0>0$

$\underbrace{(r+p)^2}_{r^2+2pr+p^2}+\underbrace{\omega_0^2-p^2}_{\text{complex roots when positive}}=0$ $p=b/2m$

complex roots $\Leftrightarrow$ $\omega_0^2-p^2>0$

$\boxed{p<\omega_0}$

underdamped.

Definition $\omega=\sqrt{\omega_0^2-p^2}$

Roots

$r=-p\pm i\omega$

same picture.

solution to $m\ddot{x}+b\dot{x}+x=0$ for $b^2<4mk$ are

$Ae^{-pt}\cos(\omega t-\phi)$

damped oscillations

roots are complex. $\underbrace{\sqrt{b^2-4ac}}_{\text{needs to be negative.}}$

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How different is ω from ω_0?
It depends on the values of m, b, k

$\omega_0 = 2\pi(260)$ radians/sec from cycles/sec

$$\ddot{x} + 2p\dot{x} + \omega_0^2 x = 0$$

p = how much softer sound gets (relaxation time)

Definition: relaxation time

$$T = 1/p$$

In e^{-pt} decreases by $\frac{1}{e}$ in time T

$$A e^{-t/T} \cos(2\pi\nu(t-t_0))$$

T = 4 seconds by listening to a sound diminishing

$$\ddot{x} + \frac{2}{4\text{ seconds}}\dot{x} + \left(\frac{2\pi\cdot 260}{\text{sec}}\right)^2 x = 0$$

all $\frac{\text{millimeters}}{\text{sec}^2}$ (units check out)

$\frac{d^2}{dt^2}$ $\frac{1}{\text{sec}^2}$ $\frac{d}{dt}$ $\frac{1}{\text{sec}}$ $\frac{1}{\text{sec}^2}$

$$\ddot{x} + \frac{1}{2}\dot{x} + \cancel{10^6}\, 3\cdot 10^6 x = 0$$

$$\boxed{\omega_0^2 = \omega^2 + p^2} = (-p + i\omega)^2 = (-p + i\omega)(-p - i\omega)$$

$\approx 10^8 = \omega^2$ $+ \frac{1}{4}$ within 10^{-6} in ratio

$0 < p < \omega_0$ if p increases $\to 0$, then ω decreases

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critically damped: $p = \omega_0$

repeated roots.

$r^2 + 2\omega_0 r + p$

$(r+\omega_0)^2 = 0$ solution $C_1 t e^{-\omega_0 t} + C_2 e^{-\omega_0 t}$

$\omega_0 > 0$

Finally the overdamped case: distinct real roots

$(r-r_1)(r-r_2) = r^2 + \frac{b}{m} r + \frac{k}{m}$

$\text{roots} = -\frac{b}{2m} \pm \sqrt{(b/m)^2 - 4k/m}$

roots must be negative

$$e^{(-4+2i)t} = e^{-4t} e^{2it}$$

$$e^{-4t}(\cos(2t) + \sin(2t))$$

$$\Rightarrow C_1 e^{-4t}\cos(2t) + C_2 e^{-4t}\sin(2t)$$

Lecture 8: Higher order DE: counting solutions 2/22/19

- characteristic polynomial $P(r)$
- Operators $D = d/dt$, $P(D)$
- Solving $P(D)x = 0$
- Vector spaces: basis and dimension

Example: $\dddot{x} - 7\dot{x} - 6x = 0$

Plug in $x = e^{rt}$

$r^3 - 7r - 6 = 0$ characteristic equation

$(r+1)(r+2)(r-3)$

roots $r = -1, -2, 3$

general solution (homogeneous) $x(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{3t}$

Characteristic polynomial

$p(r) = r^3 - 7r - 6$

$\boxed{D = \frac{d}{dt}}$ "operator"

$Dx = \frac{d}{dt} x = \dot{x}$

$D^2 x = \left(\frac{d}{dt}\right)^2 x = \ddot{x}$

$\frac{d^2}{(dt)^2}$

$P(D)x$

$= (D^3 - 7D - 6)x = \dddot{x} - 7\dot{x} - 6x$

using operator $D = \frac{d}{dt}$

new notation to describe what differentiation is.

$De^{rt} = re^{rt}$ $D^2 e^{rt} = r^2 e^{rt}$ $\boxed{P(D)e^{rt} = p(r)e^{rt}}$

Conclusion

To solve $P(D)x = 0$

try $x = e^{rt}$

$p(D)e^{rt} = p(r)e^{rt} = 0$

Solution to $P(D)x = 0$

include all e^{st} with s as a root of $p(r) = 0$

In Ex 1 $r = -1, -2, 3$

work.

Ex 2 $(D+1)(D^2+6D+10)x = 0$

$P(r) = (r+1)(r^2+6r+10)$

roots: -1, $-3 \pm i$ $\qquad r^2+6r+10 = (r+3)^2+1$

General Solutions

$$C_1 e^{-t} + C_2 e^{-3t}\cos t + C_3 e^{-3t}\sin t$$

$$= \mathrm{span}(e^{-t}, e^{-3t}\cos t, e^{-3t}\sin t)$$

Vector spaces

goal: describe dimension, basis

Span

the span of $f_1(t), f_2(t) \ldots f_n(t)$ is the set of all linear combinations:

$$c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t)$$

For example:

$\mathrm{span}(1, t, t^2)$ = all polynomials of degree ≤ 2

is all quadratic polynomials

Vector Spaces

- a set with two operations, addition and scalar multiplication with axioms
  - closure under addition
    - for v and w, $v+w$ is in space
  - closure under scalar multiplication
    - for any v and real number a, av is in the space
  - the space contains a zero vector:
    - $v+0 = v$

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$\vec{v}, \vec{w}$, $\vec{v}+\vec{w}$ vector addition $c\vec{v}$ Scalar multiplication $\vec{0}+\vec{v}=\vec{v}$ -

hidden issue: redundancy - have we counted too many?

Linear Independence

A list of vectors is dependent if one can be written as a linear combination of the others

A list of vectors that is linearly independent is one that is not dependent.

- None of the vectors can be omitted. All are essential to the span

- Lower solution space

~~THEOREM~~ A basis of a vector space is a list of independent vectors that span the whole space.

THEOREM The number of vectors in a basis is always the same

basis not unique

is unique!

We can define the dimension of a vector space as the number of vectors in a basis

$1, t, t^2$ is a basis for poly degree ≤ 2

but

$1+t, 1-t, t^2$ is too

Existence + uniqueness for linear differential equation

$$x^{(n)} + p_{n-1}(t)\, x^{(n-1)} + \ldots + p_1(t)\dot{x}_1 + p_0(t)\, x$$

$= q(t) \leftarrow$ inhomogeneous term in ok

If, $p_{n-1}, \cdots, p_0, q$ are continuous functions of t

Then there is exactly one solution to $(*)$ with initial conditions $x(0) = a_0$, $\dot{x}(0) = a_1$, $\cdots$, $x^{(n-1)} t_0 = a_{n+1}$

$a_0, \ldots, a_{n-1} \rightsquigarrow C_1, \ldots, C_n$

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Lecture 9 : Inhomogeneous DE $P(D)x = g(t)$ 2/25/19

- Superposition : $y = y_p + y_h$
- $D = d/dt$ $P(D)\,e^{rt} = P(r)e^{rt}$
- Exponential Response Formula (ERF)
- complex inputs, ERF

$L = p_n(t)D^n + \ldots p_1(t)D + p_0(t)$
(linear operator)
General solution to $Ly = g(t)$ is $y = y_p + y_h$

Step 1: Solve $Ly_h = 0$ (homogeneous)
need all solutions

Step 2: find one particular solution $Ly_p = g$

Then add up. general solution $y = y_p + y_h$

PROOF (superposition)

$Ly = L(y_p + y_h) = Ly_p + Ly_h = g + 0$ ✓

Take any solution $\tilde{y}$, $L\tilde{y} = g$ $L(\tilde{y} - y_p) = L\tilde{y} - Ly_p = g - g = 0$ ✓ $\tilde{y} - y_p = y_h$

Ex 1 $(D^4 + D^3 + 6)x = e^{2t}$ $P(r) = r^4 + r^3 + 6$

plug in $x = Ae^{2t}$ and solve for A

$$\boxed{P(D)\,e^{rt} = p(r)e^{rt}}$$

Exponential Response formula $P(r) \neq 0$

$$\boxed{P(D)\,\frac{e^{rt}}{p(r)} = e^{rt}}$$

ERF

$$\boxed{x = \frac{e^{rt}}{P(r)} \text{ solves } P(D)x = e^{rt} \quad \text{provided } P(r) \neq 0}$$

NOT ON EXAM

(but responsible for knowing how to deal w/ complex arithmetic/exponentials)

Example 1

$P(2) = 2^4 + 2^3 + 6 = 16 + 8 + 6$
$= 30$

Hence a particular solution is $x_p(t) = e^{2t}/30$

(x_h requires roots of $P(r) = 0$)

Example 2

$(D^4 + D^3 + 6)x = 100 = 100e^{0t}$

$x_p(t) = \frac{100e^{0t}}{P(0)}$ $P(0) = 6$
$= \frac{100}{6}$

Compare with plugging in $x = A$
$(D^4 + D^3 + 6)A = 100$
$6A = 100 \Rightarrow A = \frac{100}{6}$

Example 3

$(D^4 + D^3 + 6)z = e^{2it}$

$P(2i) = (2i)^4 + (2i)^3 + 6 = 16 - 8i + 6 = 22 - 8i$ on exam

Hence a particular solution is

$z_p = \frac{e^{2it}}{P(2i)} = \frac{e^{2it}}{22 - 8i}$ ← how complex exponentials arise

(if $P(r) = 0$)

When denominator is 0 (what happens to ERF)

$P(D)e^{rt} = P(r)e^{rt}$ family of formulas → r can vary

differentiate w/ respect to r $\frac{\partial}{\partial r}\left[P(D)e^{rt} = P(r)e^{rt}\right]$ t fixed

$= P(D)(te^{rt}) = P'(r)e^{rt} + \underbrace{P(r)te^{rt}}_{0}$

$\boxed{P(D)\frac{te^{rt}}{P'(r)} = e^{rt} \quad = \text{ERF}'}$ only if $P(r) = 0$, $P'(r) \neq 0$

Lecture 10: Complex Replacement

3/1/19

$P(D) = x = \cos(\omega t)$
$\downarrow$
$P(D) = e^{i\omega t}$

amplitude gain, phase lag
stability, transcience
steady state respons, complex gain

$P(D)(x + iy) = \cos(\omega t) + i\sin(\omega t)$

$z = x + iy$

$p(D)x = \cos(\omega t)$; $P(D)y = \sin(\omega t)$

twins

Ex 1 $\dot{x} + 10x = \cos(5t)$

$i\,[\dot{y} + 10y = \sin(5t)]$ > twins

add up $(\dot{x} + i\dot{y}) + 10(x + iy) = \cos(5t) + i\sin(5t)$

$\dot{z} + 10z = e^{5it}$ $\quad z_p = \dfrac{e^{5it}}{P(5i)} = \dfrac{e^{5it}}{10+5i}$

$P(r) = r + 10$

$x_p = \mathrm{Re}(z_p)$ p(particular solution)

$= \mathrm{Re}\,\dfrac{e^{5it}}{10+5i} = A\cos(5t - \phi)$ $\quad A = \left|\dfrac{1}{10+5i}\right| = \dfrac{1}{\sqrt{125}} = \dfrac{1}{5\sqrt{5}}$

$\phi = \arg(10+5i)$ $\quad \tan^{-1}\left(\frac{1}{2}\right)$

General Solution

$x(t) = \underbrace{\dfrac{1}{\sqrt{125}}\cos(5t - \phi)}_{\text{particular } x_p} + \underbrace{Ce^{-10t}}_{x_h}$

$= \dfrac{1}{\sqrt{125}}\cos(5t - \phi) + Ce^{-10t}$

A system is called <u>stable</u> if every homogeneous solution $x_h(t) \to 0$ as $t \to \infty$

<u>Transience</u> means influence of initial conditions $\to 0$ as $t \to \infty$

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Steady response

$$x_p(t) = \frac{1}{\sqrt{125}} \cos(5t - \phi)$$

valid when t is "large"

always are you get by ERF...

Theorem. Every pure sinusoidal input $a\cos(\omega t)$ into a stable constant coeff. ODE yields a purely sinusoidal steady response. $A\cos(\omega t - \phi)$ ← same ω

Amplitude gain $g = \left|\frac{A}{a}\right|$ $\left(= 1/\sqrt{125}\right)$

phase lag ϕ $a\cos(5t) \to A\cos(5t - \phi)$ is a relative quantity

Ex. 2 $\dot{x} + 10x = a\cos(\omega t)$ ← input

amplitude gain depends on the frequency

$P(r) = r + 10$ $\quad P(i\omega) = 10 + i\omega$

$$z_p = \frac{ae^{i\omega t}}{P(i\omega)} = \frac{ae^{i\omega t}}{10 + i\omega}$$

big picture: complex gain $= G$

$$G = \frac{\text{complex } \cancel{\text{response}} \text{ steady response}}{\text{input: } ae^{i\omega t}}$$

$$= \frac{ae^{i\omega t}/10 + i\omega}{ae^{i\omega t}} = \frac{1}{10 + i\omega}$$

Note $g = |G| = \frac{1}{|10 + i\omega|} = \frac{1}{\sqrt{100 + \omega^2}}$

$\phi = \arg G$
$= \arg(10 + i\omega)$

Ex 3

TIDES. (every 12 hours) Cycle

lag time 2 hours

$\frac{2 \text{ hours}}{12 \text{ hours}} = \frac{1}{6}$ } how long it takes ... (?)

model: $x(t)$ water level in the inlet (Boston Harbor)
$y(t)$ sea level outside in ocean

$$\dot{x}(t) = k(y - x) \qquad k > 0 \text{ coupling constant}$$

Assume $y(t) = \cos(\omega t)$

$\nu = \frac{1}{12}$ cycle/hr, nu $\qquad \omega = 2\pi\nu$ radians/hr

amplitude 1 input

$$\dot{x} + kx = ky(t)$$

Figure out right way of coupling!!!

input is $y(t)$, not $ky(t)$

$$\dot{z} + kz = ke^{i\omega t} \qquad |e^{i\omega t}| = 1 \text{ unit size}$$

input is $y(t)$ NOT $ky(t)$!

$$z_p = \frac{ke^{i\omega t}}{i\omega + k} \quad \text{steady response}$$

important to discuss how fast it dissipates

$$\text{complex gain} = \frac{\text{steady response}}{\text{input}} = \frac{ke^{i\omega t}/(k+i\omega)}{\cancel{k}\, e^{i\omega t}} = \boxed{\frac{k}{k+i\omega}} \quad \text{no units}$$

there's no k here!

real gain
amplitude gain) $= |G| = \frac{k}{\sqrt{k^2+w^2}}$, phase ~~lag~~)

$= -\arg(G)$

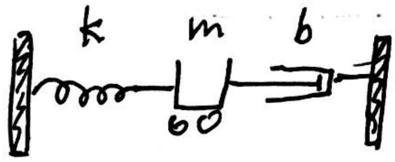
$= \arg(k+iw)$

Lecture 12: Applications: 2nd order ODE

3/4/19

- frequency response
- resonance
- rlc circuits

midterm median: 87
A ≥ 90, B ≥ 75, C ≥ 65



the walls can move

$y_1(t)$ $\quad x \quad$ $y_2(t)$

↑ inputs

$m\ddot{x} = k(y_1 - x) + b(\dot{y}_2 - \dot{x})$

$ma = F$

$m\ddot{x} + b\dot{x} + kx = ky_1 + b\dot{y}_2$

$P(D)x = Q_1(D)y_1 + Q_2(D)y_2$

$P(D) = mD^2 + bD + k$

$Q_1(D) = k$

$Q_2(D) = bD$

equation now:

$$\boxed{P(D)x = Q_1(D)y_1 + Q_2(D)y_2}$$

focusing on the right parameters so you don't get lost.

Blizzard of parameters

m, b, k

inputs: initial conditions, $x(0)$, $\dot{x}(0)$ } related linearly to the response $x(t)$

$y_1(t)$, $y_2(t)$

$y_1(t) = \overset{a=1}{\cancel{a}} \cos(\omega t)$

Deal with inputs one at a time.

unit inputs

rest initial ~~position~~ cond.

$x(0) = \dot{x}(0) = 0$

$y_2(t) = 0$

(nonlinear phenom... gain as a function of frequency)

$P(D)x = Q(D)y_1 \quad (y_2 = 0)$

$y_1 = \cos(\omega t)$

$z_1 = e^{i\omega t}$

frequency response

$$\frac{\text{steady response}}{\text{input}} \rightarrow \frac{\frac{Q_1(i\omega)e^{i\omega t}}{P(i\omega)}}{e^{i\omega t}} = \frac{Q_1(i\omega)}{P(i\omega)} \leftarrow \text{complex gain}$$

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In our example

$$\frac{Q_1(i\omega)}{P(i\omega)} = \frac{k}{m(-\omega)^2+bi\omega+k} = G$$

amplitude gain $|G| = g$

$$\frac{k}{\sqrt{(k-m\omega^2)^2+b^2\omega^2}}$$

Ex. 1 Recall (divide by m)

$$\ddot{x} + \frac{2}{T}\dot{x} + \omega_0^2 x = \omega_0^2 y_1$$

$\frac{2}{T} = \frac{b}{m}$, $\omega_0^2 = \frac{k}{m}$ T relaxation time, ω_0 natural angular frequency

Plug in some values: $T = \infty$ (no damping), $\omega_0 = 100$ radians/sec

rumble

Try inputs $\omega = 99, 100, 101$

$$\ddot{x} + \omega_0^2 x = \omega_0^2 \underbrace{\cos(\omega t)}_{y_1 \text{ unit amplitude}}$$

$P(r) = r^2 + \omega_0^2$

input $= e^{i\omega t}$ response: $\frac{\omega_0^2 e^{i\omega t}}{(i\omega)^2+\omega_0^2} = \frac{\omega_0^2}{\omega_0^2-\omega^2} e^{i\omega t} \rightarrow$ Real part $\boxed{\underbrace{\frac{\omega_0^2}{\omega_0^2-\omega^2}}_{\text{gain}} \cos(\omega t)}$

$\omega_0 = 100$, $\omega = 99$

$$\text{gain} = \left|\frac{100^2}{100^2-99^2}\right| = \left|\frac{100^2}{(100-99)(100+99)}\right| = \left|\frac{10000}{199}\right| \approx 50$$

extremely unexpected, somewhat disturbing

near resonanse

no units for gain.

101 is similar

plug in 100 ... paradox:

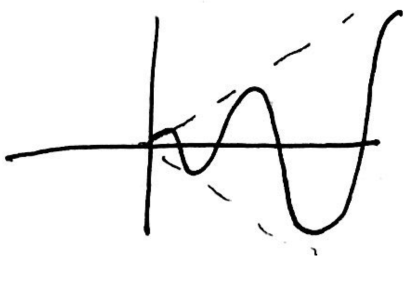
input: $\omega = 100 = \omega_0$ ... denominator is 0

$P(i\omega) = -\omega^2 + \omega_0$ satisfies $P(100i) = 0$ when $\omega_0 = 100$

Instead, we need ERF: input: $e^{i\omega_0 t}$ response: $\frac{\omega_0^2\, t e^{i\omega_0 t}}{P'(i\omega_0)}$ $\frac{100^2 t\, e^{100it}}{200i}$, $e^{100it} = \cos(100t) + i\sin(100t)$

$P'(r) = 2r$ Real part: $\boxed{(50t)\sin 100t}$

$\frac{1}{2}\omega_0 t \sin(\omega_0 t)$



opening out

pure resonance

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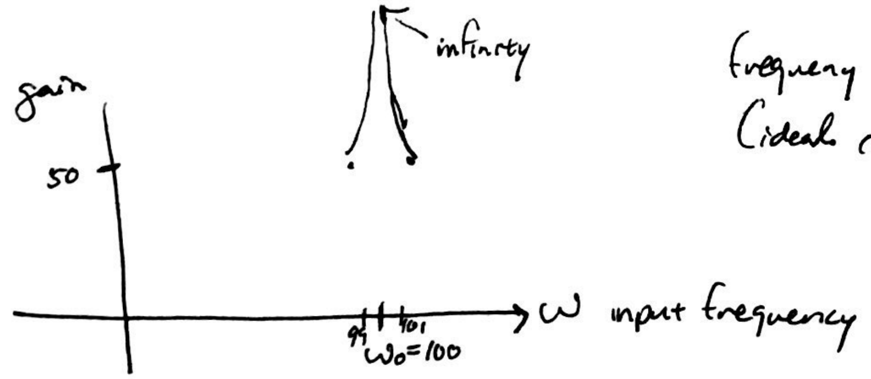
gain

infinity

frequeny-response graph (ideal case & no damping)



damped resonance

take $T = 1$ sec. instead of $T = \infty$

Example 2 $\quad \omega_0 = 100, \ T = 1$ $\quad$ ☆ some dampening

$$\ddot{x} + 2\dot{x} + 100^2 x = 100^2 y(t)$$

$$y(t) = \cos(\omega t) \longrightarrow \tilde{y}(t) = e^{i\omega t}$$

$$\omega_0 = 100$$

response

$$\frac{\omega_0^2 e^{i\omega_0 t}}{P(i\omega_0)} = \frac{\omega_0^2 e^{i\omega_0 t}}{2i\omega_0} \quad P(i\omega_0) = -\omega_0^2 + 2i\omega_0 + \omega_0^2$$
$$= 2i\omega_0$$

real part $\quad \dfrac{\omega_0^2}{2\omega_0} \sin(\omega_0 t)$ nearly maximum response

$T = 1$

$$\frac{\omega_0^2}{|(i\omega)^2 + \frac{2}{T} i\omega + \omega_0^2|}$$

maximize it

By definition ω_r resonance is the value with maximum responce

$$\boxed{\omega_r^2 = \omega_0^2 - \frac{2}{T^2} = \omega_1^2 - \frac{1}{T^2}}$$

ω_1 pseudo frequency

Recitation 3/5/19

$\ddot{x} + 2\dot{x} + 3x = \cos(2t)$ Real $(e^{i2t}) = \cos(2t)$

$\ddot{\tilde{x}} + 2\dot{\tilde{x}} + 3\tilde{x} = e^{i2t}$ Real $\tilde{x} = x$

method of complex replacement

$\tilde{x}(\underbrace{D^2+2D+3}_{P(D)}) = e^{i2t}$ $\frac{e^{i2t}}{P(2i)} = \frac{e^{i2t}}{-1+4i}$ ⇐ need to take real part to get x

$\frac{e^{i2t}}{-1+4i}$ $\theta = \tan^{-1}(4)$ $A = \sqrt{1+16} = \sqrt{17}$ $\frac{e^{i2t}}{\sqrt{17}e^{\tan^{-1}(4)i}} = \frac{1}{\sqrt{17}}e^{(2t-\tan^{-1}(4))i}$

$= \frac{1}{\sqrt{17}}\left(\cos(2t - \tan^{-1}(-4)) + i\sin(2t - \tan^{-1}(-4))\right)$

real part: $\frac{1}{\sqrt{17}}\cos(2t - \tan^{-1}(-4))$

Conjugate way:

$\frac{e^{i2t}}{(-1+4i)} \times \frac{(-1-4i)}{(-1-4i)} = \frac{-\cos 2t + 4\sin 2t + i(-4\cos 2t - \sin 2t)}{17}$

real part: $\frac{-\cos 2t + 4\sin 2t}{17}$

If sin instead of cos, we just take the imaginary part instead.

just saying, doing this the polar way is so much easier.

$\ddot{x} - \dot{x} - 2x = \cos(-t) + \sin(-t)$

If angles different, must use superposition.

method superposition $\begin{cases} \ddot{x}_1 - \dot{x}_1 - 2x_1 = \cos(-t) \\ \ddot{x}_2 - \dot{x}_2 - 2x_2 = \sin(-t) \end{cases}$

$x = x_1 + x_2$

another way: $\cos(-t) + \sin(-t)$ $Re((1-i)e^{-it})$

$\cos(-t) + \sin(-t) + i(-\cos(-t) - \sin(-t))$

$\ddot{\tilde{x}} + \dot{\tilde{x}} - 2\tilde{x} = (1-i)e^{-it}$

$Re(\tilde{x}) = x$

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$$\tilde{x} = (1-i)\,\frac{e^{-it}}{P(-i)}$$

$$(D^2-D-2)\tilde{x}$$
$$\overset{\shortparallel}{P(D)}$$

$$= (1-i)\,\frac{e^{-it}}{-3+i}$$

$$= \frac{1-i}{-3+i}\,e^{-it}$$

now we want to take the real part

$$= \frac{(1-i)(-3-i)}{(-3+i)(-3+i)}\,e^{-it} = \frac{-2+2i}{10}\,e^{-it} = \frac{-1+i}{5}\left(\cos(-t)+i\sin(-t)\right)$$

take real part:

$$\frac{-\cos 2t - \sin 2t}{5}$$

Lecture 13 Intro to Systems of ODE 3/6/19

- RLC Circuits
- matrix multiplication
- companion matrix: 2x2 case

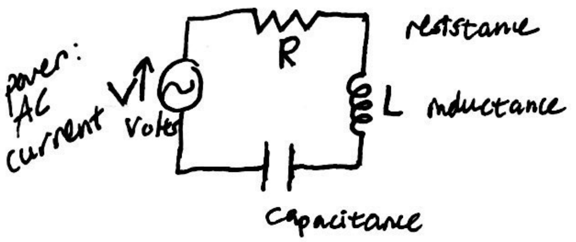
Correction on problem set

multiply $(x\frac{d}{dx}-2)y_s = rx^2$ by $s-2$ before differentiating with respect to s

$\Rightarrow x=e^t$, $x^s=e^{st}$ ERF!

RLC Circuits

used to be used as analog computers. To solve differential eqns.



blizzard of parameters + blizzard of variables.

I current ampere

not like a battery because it alternates.

$V_R = IR$ volt = ohm × amp

$V_C = \frac{1}{C}Q$ volt = $\frac{\text{Coulombs}}{\text{farads}}$

(remember, $CV=Q$?)

$Q=Q(t)$ charge stored at the capacitor.

$V_L = L\frac{dI}{dt}$ volt = henry × $\frac{\text{amp}}{\text{sec}}$

Kirchhoff's voltage law for loops:

$V_R + V_L + V_C = V(t)$

$I = \frac{dQ}{dt}$ $\frac{\text{coulombs}}{\text{sec}}$ = amp

this circuit has 5 variables: I, Q, V_R, V_L, V_C

there are 5 equations → balanced

eliminate 3 var, 3 eqns...

$V_R+V_L+V_C=V(t) \;=\; RI + L\dot{I} + \frac{Q}{C} = V(t)$ ← input

$I=\dot{Q}$

} two equations, two unknowns.

$\dot{Q}=I$

$L\dot{I} = -RI - \frac{Q}{C} + V(t)$

$\dot{Q} = 0\cdot Q + 1\cdot I + 0\cdot V(t)$

$\dot{I} = \frac{-1}{LC}\cdot Q - \frac{R}{L}I + \frac{1}{L}\cdot V(t)$

↑ Qs ↑ Is ↑ inputs

- mention variables in correct order.
- divide by L!

First simplification: write as column vector:

$$\begin{pmatrix}\dot{Q}\\ \dot{I}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{pmatrix}\begin{pmatrix}Q\\ I\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{L}\end{pmatrix}V(t)$$

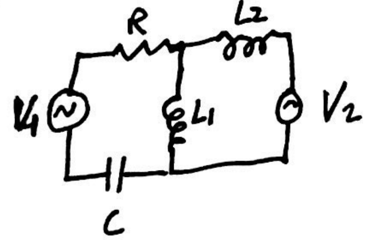
2x1 2x2 matrix 2x1 vector 1x1 scalar Matrix multiplication examples of matrix multiplication:

$(a \; b)\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$

$(a \; b \; c)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$

$$\begin{pmatrix} \dot{Q} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}\begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ 1/L \end{pmatrix} V(t)$$

On problem set 4, give:



reduces to $\begin{pmatrix} \dot{Q} \\ I_{L_1} \\ I_{L_2} \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} Q \\ I_{L_1} \\ I_{L_2} \end{pmatrix} + \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$

3x1 3x3 3x1 3x2 2x1

In this case, there are 9 eqns + unknowns → reduce to 3.

Big savings: (abbreviation of the notation)

get rid of the blizzard of variables

$\vec{x} = \begin{pmatrix} Q \\ I \end{pmatrix}$ $A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ 1/L \end{pmatrix}$

$$\boxed{\dot{\vec{x}} = A\vec{x} + By = A\vec{x} + \vec{q}(t)}$$

Euler: $\dot{y} = iy$ $\dot{\vec{x}} = A\vec{x}$ matrices with weird multiplication rules.

main goal: understand how to divide with matrices

$\underbrace{(AB)C = A(BC)}$ Associative law!

reason why using this notation is good. Cancels an incredible amount of arithmetic.

1st order system of ODE $\longleftrightarrow$ 2nd order scalar ODE

$\xrightarrow{\text{}}$ elimination

one step further: eliminate Q.

1 eqn, 1 unknown (scalar) 2nd order.

$R\dot{I} + L\ddot{I} + \frac{1}{C}\dot{Q} = \dot{V} \iff \boxed{L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}}$

The way it used is backwards.

companion matrix:
associate 2nd order with 1st order system.

<u>Companion matrix</u>

$\ddot{y} + a_1\dot{y} + a_0 y = 0$ can create a new variable $\boxed{\begin{array}{c} v = \dot{y} \\ \text{new variable} \end{array}}$ y

$\dot{y} = v$

$\dot{v} = \ddot{y} = -a_1\dot{y} - a_0 y$

$= -a_1 v - a_0 y$

now need to write in matrix notation.

$\dot{y} = 0y + 1 \cdot v$
$\dot{v} = -a_0 y + (-a_1 v)$

$$\begin{pmatrix} \dot{y} \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}}_{\boxed{\text{Companion matrix}}} \begin{pmatrix} y \\ v \end{pmatrix}$$

(to 2nd order equation)

We know that complex numbers are going to come up... See this next time.

For some reason, I have a feeling that this lecture is very important.
Pay close attention to following related topics...

Lecture 14: 2x2 Systems: Solving $\dot{\vec{x}} = A\vec{x}$. 3/8/19

- eigenvalues, eigenvectors
- characteristic polynomials, trace, determinant

Ex. 0

$\dot{x} = 3x \qquad x(t) = C_1 e^{3t}$ ← general solution

$\dot{y} = 10y \qquad x(t) = C_2 e^{10t}$

Trial function e^{rt}, ce^{rt} ← plugging in things

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ uncoupled system

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ off-diagonal terms are 0

The solutions in vector notation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C_1 e^{3t} \\ C_2 e^{10t} \end{pmatrix} = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} e^{3t} + \begin{pmatrix} 0 \\ C_2 \end{pmatrix} e^{10t}$$

$$= \vec{v}_1 e^{3t} + \vec{V}_2 e^{10t}$$

same thing, but previous constants are now vectors.

Right idea: We have a trial solution of the form $\vec{v}e^{\lambda t}$ λ lambda

Ex. 1 $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$

$\dot{\vec{x}} = A\vec{x}$

$\begin{cases} \dot{x} = -3x + y \\ \dot{y} = x - 3y \end{cases}$

seek solution in the form $\vec{v}e^{\lambda t} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda t} = \vec{x}$

$\dot{\vec{x}} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \lambda \cancel{e^{\lambda t}} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cancel{e^{\lambda t}}$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \lambda = \begin{pmatrix} -3v_1 + v_2 \\ v_1 - 3v_2 \end{pmatrix}$

$\lambda v_1 = -3v_1 + v_2$ 2 equations, 3 unknowns λ is nonlinear, linear in v_1 and v_2

$\lambda v_2 = v_1 - 3v_2$ λ, v_1, v_2

Fix λ:

$(-3-\lambda)v_1 + v_2 = 0$

$v_1 + (-3-\lambda)v_2 = 0$

$\Leftrightarrow \begin{pmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

theory of linear systems – when can you solve?

$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{M}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if $\det(M) \neq 0$, exactly 1 soln. boring solution: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

equivalent to matrix being invertible.

$\det(M)$
$\Downarrow$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $ad-bc$ if 0, matrix called ~~sing~~ singular
$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ else called nonsingular.

When $\det(M) = 0$, then there are non-trivial solutions.

In Ex. 1, $\det\left(\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix}\right) = (-3-\lambda)(-3-\lambda) - 1\cdot 1$

$= 9 + 6\lambda + \lambda^2 - 1 = \underline{\lambda^2 + 6\lambda + 8} = 0$

characteristic polynomial of the matrix.

roots of characteristic polynomial: ~~characte~~
are the values of lambda for which
you get non-zero solutions.

$(\lambda + 2)(\lambda + 4) = 0$ $\lambda = -2, -4$ $\Leftarrow$ roots

$\lambda = -2$ plug in:

$\begin{pmatrix} -3-(-2) & 1 \\ 1 & -3-(-2) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ Can solve by inspection.

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Ex. 1 continued

$\vec{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$ solves $\dot{\vec{x}} = A\vec{x}$ with $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$

$\lambda = -4$

$\begin{pmatrix} -3-(-4) & 1 \\ 1 & -3-(-4) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ works

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$ $\vec{x}_2(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$

need to match correct eigenvalue with eigenvector

Now take linear combinations:
general solution is

$$\boxed{\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}}$$

$-2, -4$ are called eigenvalues

vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are called eigenvectors.

one more place where we can get stuck.

General Case

to solve $\dot{\vec{x}} = A\vec{x}$
we try $\vec{v}e^{\lambda t}$

$\dot{\vec{x}} = \vec{v}\lambda \cancel{e^{\lambda t}} = A\vec{x} = A\vec{v}\cancel{e^{\lambda t}}$

$\vec{v}\lambda = A\vec{v}$

$\boxed{A\vec{v} = \lambda\vec{v}}$

λ = eigenvalue of A

$\vec{v} \neq 0 \to$ is an eigenvector of A

put $\vec{v}$ on left

$A\vec{v} - \lambda\vec{v} = \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(A-\lambda)\vec{v} = \vec{0}$ $\longrightarrow$ test for interesting solution! if det = 0

$\det\left(\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} - \lambda\right) = ??$

look at what succeeded in example

$\Downarrow$

same thing as $\left[\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\lambda\right]\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\leftarrow$ identity matrix

$\vec{v}(A - \lambda I) = \vec{0}$ characteristic equation

In summary:

to solve $\dot{\vec{x}} = A\vec{x}$

solve the characteristic eqn.

$|A - \lambda I| = 0$ $\{\det(A - \lambda I)\} = 0$

roots λ_1, λ_2

To solve each root, solve

$(A - \lambda_i I)\vec{v_i} = \vec{0}$

general solution $x(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$

characteristic polynomial

$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc$

$= \lambda^2 - \underbrace{(a+d)}\lambda + \underbrace{ad-bc}$

trace $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d$ $\quad$ $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$\boxed{\lambda^2 - \text{trace}(A)\lambda + \det(A) \Leftarrow \text{characteristic polynomial}}$

shortcut for EX. 1 and any example