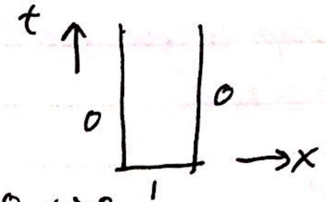
PDE : the heat Equation 4/26/19

Monday office hour moved to WED May 1st 7-9pm room 2-255

- fourier's solution
- analogy with $\dot{\vec{u}} = A\vec{u}$
- completeness of fourier series
- scaling of heat-diffusion

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le \pi$$



$u(x,t) \approx \frac{4}{\pi} e^{-t} \sin x$

initial condition $u(x,0) = 1$

boundary conditions $u(0,t) = u(\pi,t) = 0$ $\underline{t>0}$

$(v''(x) = \lambda v(x))$

Special solution (normal modes)

$u(x,t) = e^{\lambda_n t} v_n(x)$ $\quad v_n(x) = \sin(nx)$

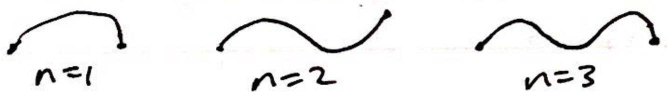
$\lambda_n = -n^2 \quad = e^{-n^2 t} \sin(nx)$

$v''(x) = \lambda v(x)$ $\quad v(0) = v(\pi) = 0$
(boundary value problem)
λ eigenvalue solution $v(x)$ is an eigenfunction

$\lambda = -\omega^2$ $\quad \omega$ frequency
↑ eigenvalue $\quad \lambda$ eigenvalue

☆ Not the notation of recitation 19

k angular wave number
$\leftrightarrow \omega$
$\sin(\omega x) \leftrightarrow \sin(kx)$



Fourier's solution: use superposition & his building blocks

$$u(x,t) = \sum_{1}^{\infty} b_n e^{-n^2 t} \sin(nx) \longrightarrow \text{solves } \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad t>0$$

and $u(0,t) = 0$; $u(\pi,t) = 0$

<u>Last condition</u> = initial value

$$1 = u(x,0) = \sum_{1}^{\infty} b_n e^{-n^2 \cdot 0} \sin(nx) = \sum_{1}^{\infty} b_n \sin(nx)$$

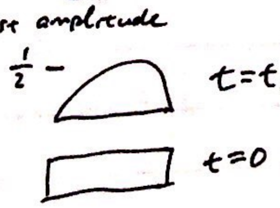
$$1 = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right) \qquad 0<x<\pi$$

Answer: $u(x,t) = \frac{4}{\pi}\left(e^{-t}\sin x + \frac{1}{3} e^{-9t} \sin(3x) + \dots\right)$

$b_1 = \frac{4}{\pi}$ $\quad b_2 = 0$ $\quad b_3 = \frac{4}{3\pi}$

largest amplitude

For example: $\frac{4}{\pi} e^{-t_1} = 1/2$



the second largest amplitude:

$\frac{4}{3\pi} e^{-9t_1} \approx 10^{-4}$

Analogy with $\dot{\vec{u}} = A\vec{u}$ $(\vec{u}(0) = \vec{u}_0)$ $\quad A\vec{v}_n = \lambda_n \vec{v}$

Suppose $A\vec{v}_n = \lambda_n \vec{v}_n$ $\quad \longleftrightarrow \quad \frac{\partial^2}{\partial x^2} V_n(x) = -n^2 V_n(x)$

$V_n(x) = \sin(nx)$

solves $\vec{u}(t) = \sum b_n e^{\lambda_n t} \vec{v}_n$

The very last step is the initial condition

$\vec{u}_0 = \vec{u}(0) = \sum b_n \vec{v}_n$

Finding b_n from $\vec{u}_0$ is a linear algebra problem $\left(\vec{v}_1 | \dots | \vec{v}_n\right)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \vec{u}_0$

Easiest case is when $\vec{v}_1, \dots, \vec{v}_n$ are orthogonal

$$\boxed{\vec{v}_n^T \vec{u}_0} = \vec{v}_n^T \sum b_m \vec{v}_m = \sum b_m \vec{v}_n^T \vec{v}_m = \boxed{b_n \vec{v}_n^T \vec{v}_n} + 0 + 0 + 0 + \dots$$

$$\boxed{b_n = \frac{\vec{v}_n^T \vec{u}_0}{\vec{v}_n^T \vec{v}_n}} \qquad b_n = \frac{2}{\pi} \int_0^{\pi} \overset{u\,dx}{u(x,0) \sin(nx) dx} = \frac{\langle u_0, \sin nx \rangle}{\langle \sin(nx), \sin(nx) \rangle}$$

<u>Completeness</u> Fourier <u>claimed</u> that he could get every function from $\sin(nx)$, $n = 1, 2, 3, \dots$

on $0 < x < \pi$

argued about for ~100 yrs. proved in 1900. Fejér.

yes established for continuous functions

This comes up a lot in quantum mechanics

<u>Scaling</u> of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\quad u(x,t) \to u(x/L, t/L^2)$

$0 < x < \pi \qquad 0 < x < L\pi$

$L\pi = 10\text{cm}$ $\quad 2L\pi = 20\text{cm}$

e^{-t/L^2} $\quad L = 1, e^{-t}$ $\quad e^{-t/(2L)^2} = e^{-t/4L^2}$

microscopic measurement

if the length of the bar is doubled, ~~the~~ cooling rate is slower by a factor of 4.

<u>Microscopic level</u>

$u(k/N, t) = u_k(t)$

$k = 1$

$0 \quad 1$

$h = 1/N$

$a = \frac{1}{h^2}$

$$\dot{u}_k = a(u_{k+1} - u_k) + a(u_{k-1} - u_k)$$

$$= \frac{1}{h^2}(u(x+h) + u(x-h) - 2u(x))$$

$X = k/N \quad x + h = \frac{k+1}{N} \quad x - h = \frac{k-1}{N}$

$$\frac{\partial}{\partial t} u(x,t) = \lim_{h\to 0} \frac{1}{h^2}\left(u(x+h) + u(x-h) - 2u(x)\right)$$

$$= \lim_{h\to 0} \frac{\left(\frac{\partial u}{\partial h}(x+h,t) + \frac{\partial u}{\partial h}(x-h,t)\right)}{2h}$$

$$= \lim_{h\to 0} \frac{\frac{\partial^2}{\partial h^2}\left(u(x+h,t) + u(x-h,t)\right)}{2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

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4/29 Heat and Waves

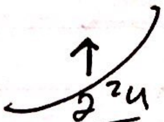
- qualitative explanations
- insulated ends
- wave eqn: standing waves

qualitative ~~idea~~ idea of heat transfer starts with understanding the equilibrium

temperature + constant ~~balance~~ balance $\longleftrightarrow \frac{\partial^2 u}{\partial x^2} = 0$

trend toward equilibrium

$\downarrow \frac{\partial^2 y}{\partial x^2} < 0$



$\uparrow \frac{\partial^2 u}{\partial x^2} > 0$

Simplest differentiable equation that reflects this behavior is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$ linear proportional

Diffusion eqn is the same

- spread of salt in water
- brownian motion 1905
- stock prices 1905
- light passing through clouds

Recall: $u(x,t)$ temperature on $0 \le x \le \pi$

$u(0,t) = u(\pi,t) = 0$

$$u(x,t) = \sum_1^{\infty} b_n e^{-n^2 t} \sin(nx),$$

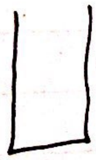
$$b_n = \frac{2}{\pi} \int_0^{\pi} u_0(x) \sin(nx)\,dx$$

0 0

$u_0(x)$ initial temp at $t=0$

Insulated ends: zero flux across $x=0$ and $x=\pi$

$$\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(\pi,t)$$ NEW



$\frac{\partial u}{\partial x} = 0$ $\rightarrow \frac{\partial u}{\partial x} = 0$

Apply Separation

$u(x,t) = v(x)w(t)$ $\quad v''(x) = \lambda v(x) \quad \dot{w} = \lambda w$

Also no nonzero solutions when $\lambda > 0$

and $v'(0) = v'(\pi) = 0$ new solution

$\lambda = 0 \quad v(x) = 1$ solves $v'' = 0 \quad v'(0) = v'(\pi) = 0$

$\lambda < 0$ choose $\omega > 0$ $\quad \lambda = -\omega^2$.

$v(x) = a\cos(\omega x) + b\sin(\omega x)$ $\quad$ solve $v'' = -\omega^2 v$

$v'(x) = -a\omega\sin(\omega x) + b\omega\cos(\omega x)$

$v'(0) = 0 \Rightarrow 0 + b\omega\cos 0 = 0 \Rightarrow b = 0$

Try $v(x) = \cos(\omega x)$ (set $a=1$)

$v'(x) = -\omega\sin(\omega x)$

$v'(0) = 0$ ✓ | $v'(\pi) = 0 \iff \sin(\omega\pi) = 0 \iff \omega = 1, 2, 3 \ldots$

$$u(x,t) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n e^{-n^2 t}\cos(nx) \longleftrightarrow a_n = \frac{2}{\pi}\int_0^{\pi} u_0(x)\cos(nx)\,dx$$

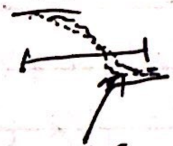
Main Conclusion

Zero temperature ends

$\rightarrow$ 0 equilibrium $\quad v=1$

1st mode $\sin(x)e^{-t}$

Insulated case



$\rightarrow$ average temperature in the equilibrium

cos

1st non constant mode is $\cos(x)e^{-t}$

Conservation law:

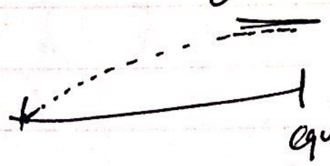
average temperature is conserved

$$\frac{1}{\pi}\int_0^{\pi} u_0(x)\,dx = \frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} u(x,t)\,dx \quad \text{for all } t$$

average does not change with t

Exercise: what happens with mixed boundary condition

0 end $= 0$ $\quad$ other $=$ insulation



equilibrium $= 0$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

vibrating string $\leftrightarrow$ $u =$ displacement

sound: $u =$ air pressure

$c =$ speed of sound.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

fixed ends $u(0,t) = u(\pi,t) = 0 \quad \forall t > 0$

$u(x,t) = v(x)\, g(t)$

$v\ddot{g} = c v'' g \qquad \frac{\ddot{g}}{g} = c^2 \frac{v''}{v}$

$v''(x) = \lambda v(x)$

$\ddot{g}(t) = \lambda c^2 g(t)$

$v(0) = v(\pi) = 0$

We already know that the solution v are $\boxed{v_n(x) \sin(nx) \quad \lambda = -n^2}$

Hence $\ddot{g}_n = -n^2 c^2 g_n \qquad g_n(t)\,?$

$g_n = A_n \cos(\omega_n t + \varphi_n) \qquad \boxed{\omega_n = nc}$

General solution

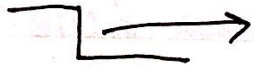
$$u(x,t) = \sum_{n=1}^{\infty} A_n \underbrace{\boxed{\cos(nct + \varphi_n) \sin(nx)}}_{\text{standing wave}}$$

we hear the frequency $nc \cdot 2\pi$ Hz

we see a vibration with shape $\sin(nx)$

~~Stdy~~ Standing Wave, Traveling wave

$u(x,t) = f(x - ct)$



Or $= f(x + ct)$ towards the left $\qquad$ Towards the right

$\underbrace{\hspace{6em}}_{\text{speed is } c}$

Lecture Midterm 3 Review 5/1/19

0-Z 26-100 ...
Office hrs 7-9 2-255 Wednes.

- numerical solutions / Euler's method) Find $\vec{x}, \vec{t}$)
- Fundamental matrix, variation of parameters, exponential matrix
- decoupling problems 1: ($A\vec{x}=\vec{b}$) $P(0)x = f(t)$ ← periodic f
- decoupling 2: ($\dot{\vec{u}} = A\vec{u} + \vec{q}(t)$) second approach to this via decoupling

↔ PDE HEAT EQUATION

Solving $\dot{\vec{x}} = A(t)\vec{x}$ (homogeneous case $\vec{b}(t)=0$)

$\underline{X}(t) = (\vec{x}_1(t) / \vec{x}_2(t))$ Fundamental matrix whose columns are independent solutions.

2x2 case

$\dot{\underline{X}} = A(t)\underline{X}(t)$

variation of parameters method $\Rightarrow \vec{x}(t) = \underline{X}(t)\int \underline{X}(t)^{-1} b(t)\,dt$

1) write it in its definite form.

Formulas on practice test will be on exam!

$$\vec{x}(t) = \underline{X}(t)\left(\int_0^t \underline{X}(s)^{-1}\vec{b}(s)\,ds + \vec{c}\right)$$

$\vec{x}(0) \to$ integral is 0 $\int_0^0$

solves $\dot{\vec{x}} = A(t)\vec{x}(t) + \vec{b}(t)$; $\vec{x}(0)=\vec{x}_0$

$$= \underline{X}(0)(0+\vec{c}) \Rightarrow \vec{c} = \underline{X}(0)^{-1}\vec{x}_0$$

Only case in which we can compute $\underline{X}(t)$ is when A is constant (matrix)

Constant coefficient case

he doesn't want us to do a lot of arithmetic.

ask about this variation of parameters formula

Formula: $\vec{x}(t) = \begin{pmatrix} e^{2t} & 1 \\ -e^{2t} & 1 \end{pmatrix}\int_0^t \frac{1}{2e^{2s}}\begin{pmatrix} 1 & -1 \\ e^{2s} & e^{2s} \end{pmatrix}\begin{pmatrix} s \\ s^2 \end{pmatrix} ds + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$; $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\lambda = 2$ $\lambda = 0$

$e^{0t} = 1$ $e^{2(0)} = 1$

$$\underline{X}(t) = \left(e^{\lambda_1 t}\vec{v}_1 \,|\, e^{\lambda_2 t}\vec{v}_2\right) = \begin{pmatrix} e^{2t} & 1 \\ -e^{2t} & 1 \end{pmatrix}$$

$\underline{X}(t)^{-1} = \frac{1}{2e^{2t}}\begin{pmatrix} 1 & -1 \\ e^{2t} & e^{2t} \end{pmatrix}$ Solve: $\dot{\vec{x}} = A\vec{x} + \begin{pmatrix} t \\ t^2 \end{pmatrix}$; $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

one other method: Exponential matrix.
Special fundamental matrix
(but details could be tedious to implement)

$$e^{At} = \Psi(t)\Psi(0)^{-1} \quad \begin{pmatrix} e^{2t} & 1 \\ -e^{2t} & 1 \end{pmatrix}\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \qquad S = (\vec{v}_1 | \vec{v}_2)$$

$$= S e^{Dt} S^{-1} \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & \cancel{e^{0t}}\ 1 \end{pmatrix}\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$= I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots \leftarrow$$ used for deficient (incomplete) eigenvalues (not enough eigenvalues)

$$\underbrace{\vec{x}(t) = \int_0^t e^{A(t-s)} \vec{b}(s)\, ds + e^{At}\vec{x}_0}$$

→ same formula as $\Psi(t)\left(\int_0^t \Psi(s)^{-1}\vec{b}(s)\, ds + \vec{c}\right)$

$e^{A(0)} = I$ extra property so inverses are easy to compute

complexity theory check it algorithm.

→ simplest possible way the formula could be.

$e^{it} = \cos(t) + i\sin(t)$ hides complexity

e^{At} hides all the complexity of $\dot{\vec{x}} = A\vec{x} + \vec{b}(t)$

$\vec{x}(t) = \sum A_n \cos(nt + \varphi_n)$

$A_n \geq 0$

decoupling 1:

$$A\vec{x}=\vec{b} \qquad A\vec{v}_n=\lambda_n\vec{v}_n$$

$$\vec{b}=\sum b_n\vec{v}_n \Rightarrow \vec{x}=\sum\frac{b_n}{\lambda_n}\vec{v}_n$$

Example 1

$$\ddot{x}+100^2x=100^2\sum b_n\sin(nt)$$

$V_n(t)=\sin(nt)$

$P(D)=D^2+100^2$

$P(D)V_n=\underbrace{((-n)^2+100^2)}_{\lambda_n}V_n$

Start w/ sums of multiples of V_n,
answer = multiple of vns

$$x(t)=100^2\sum\frac{b_n}{-n^2+100^2}\sin(nt)$$

$-n^2+100^2 \longrightarrow P(D)$

+ (homogeneous solution)

odd functions w/ period 2π

1) does depend

2) also a homogeneous solution.

also if denominator is 0 doesn't work ← nonperiodic solution

nullspace/column space is whole thing (???) ASK DURING OFFICE HOURS

In real life denom. never 0?

Example 2

$$\ddot{x}+\dot{x}+100^2x=100^2\sum b_n\sin(nt)$$

coupling

$A\vec{x}=\vec{y}=\sum b_n\vec{v}_n$

$$\vec{x}(t)=\sum A_n\cos(nt+\varphi_n)=\sum A_n\sin(nt+\varphi_n+\pi/2)$$

~~plan~~ times phase shifted represents combo ~~of~~ sins/coss.

we only care about the amplitude

$$A_n=\frac{100^2|b_n|}{|-n^2+in+100^2|}$$

⇐ what are important parts of the signal

$y(t)=\sum b_n\sin(nt)$

compare Fourier coefficients

$P(D)=D^2+D+100^2$

$P(in)$

$$|ERF|=\left|\frac{1}{P(in)}\right|\cdot 100^2|b_n|$$

response

decoupling 2: $\dot{\vec{u}} = A\vec{u} + q(t)$, $\vec{u}(0)=\vec{u}_0$ $A=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

not split problem. Extra feature:

$\underbrace{\text{Matrix is symmetric} \quad A^T = A}$

$\underbrace{\vec{V}_1 = \begin{pmatrix}1\\-1\end{pmatrix} \quad \vec{V}_2 \begin{pmatrix}1\\1\end{pmatrix}}_{\text{perpendicular to each other!}}$

important for Farier coefficient formula

$\dot{C}_1(t) = V_1^T \dot{\vec{u}} = \vec{V}_1^T A\vec{u} + \vec{V}_1^T \vec{q}(t)$, $\vec{V}_1^T \vec{u}(0) = \ldots$

$(1,-1)$

$$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = 2\begin{pmatrix}1\\-1\end{pmatrix} \longrightarrow A\vec{V}_1 = 2\vec{V}_1$$

$$(1 \quad -1)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = 2\,(1 -1) \rightarrow V_1^T A = 2V_1^T$$

$C_1(t) = \vec{V}_1^T \vec{u}(t)$

$= (1 \ -1)\begin{pmatrix}u_1\\u_2\end{pmatrix}$

$= u_1 - u_2$

$\dot{C}_1 = 2C_1 + t + t^2$

$C_1(0) = (1 \ -1)\begin{pmatrix}0\\1\end{pmatrix}$ $= -1$ $\vec{x}_0$

$V_1^T q(t) \quad (1 \quad 1)\begin{pmatrix}t\\t^2\end{pmatrix} = t + t^2$

$\boxed{C_1(0) = 1}$

Equation & initial conditions have nothing to do w/ rest of equation

☆ do this problem from exercises

Heat equation is same thing

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t) \qquad u(x,0) = u_0(x)$$

if you have initial conditions + boundary conditions

$u(0,t) = u(\pi,t) = 0$

each part breaks into pieces

$u(x,t) = \sum w_n(t)\sin(nx)$; $g(x,t) = \sum g_n(t)\sin(nx)$

$u_0(x) = \sum b_n \sin(nx)$

~~decoupt~~ decoupled equation

$$\boxed{\dot{w}_n = -n^2 w_n + g_n(t); \quad w_n(0) = b_n}$$

pde gets broken into normal modes

$\dot{\vec{u}} = A\vec{u}$

$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} u$

$\{u(x,t)\ u(x_1,t), u(x_2,t)\}$

$0 < x < \pi$

$u(x,t) \cdot \vec{u} = \begin{pmatrix} u(x_1,t) \\ u(x_2,t) \\ u(x_3,t) \\ \vdots \\ u(x_N,t) \end{pmatrix}$

$N=100$

$\boxed{\dot{u}_2 = a(u_1 - u_2) + a(u_3 - u_2)}$

Newton's law of cooling

tridiagonal matrix

$\dot{\vec{u}} = \begin{pmatrix} -2a & & & \\ a & -2a & a & 0\ 0\ 0 \\ & & & -2a \end{pmatrix}$

$a(u_{k+1} + u_{k-1} - 2u_k)$ — 2nd difference

decades

$\boxed{A\vec{v}_n = \lambda_n \vec{v}_n}$

$e^{\lambda_n t} V_n$

$u(1), u(2), u(3)$

1st difference $u(k+1) - u(k) = V(k)$

2nd difference

$V(k+1) - V(k) = u(k+2) - u(k+1) - (u(k+1) - u(k))$

$= u(k+2) + u(k) - 2u(k+1)$

boundary conditions

$\Downarrow$

eigenfunctions

$A\vec{V}_n = \lambda_n \vec{V}_n$

$\ddot{\vec{u}} = A\vec{u} + \vec{q}(t)$

$u(x,t) = \sum b_n(t)\sin(nx)$

$q(x,t) = \sum q_n(t) \sin(nx)$

known

$\frac{\partial u}{\partial t} = \sum \dot{b}_n(t)\sin(nx)$

$\frac{\partial^2 u}{\partial x^2} = \sum b_n(t)(-n^2)\sin(nx)$

$\sum \dot{b}_n(t)\sin(nx) = \sum -n^2 b_n(t)\sin(nx)$

$+ \sum q_n \cancel{(t)} \sin(nx)$

$\dot{b}_n = -n^2 b_n + q_n(t)$

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q(x,t)$

$\sin(nx) = v_n(x)$

v_n

$A \leftrightarrow \frac{\partial^2}{\partial x^2} \quad v_n''(x) = -n^2 v_n(x)$

$\lambda_n = -n^2$

$\dot{b}_n = -n^2 b_n$

$\frac{\partial u}{\partial t} = \sum \dot{b}_n(t)\sin(nx)$

$\frac{\partial^2 u}{\partial x^2} = \sum b_n(t)(-n^2)\sin(nx)$

$\dot{b}_n = -n^2 b_n + q_n(t)$

$\dot{b}_{100} = -100^2 b_{100} + q_{100}(t)$

$\sum \dot{b}_n(t)\sin(nx) = \sum -n^2 b_n \sin(nx)$

$+ \sum q_n(t)\sin(nx)$

$\lambda_n = -n^2$

$\sin(nx) = v_n(x)$ boundary condition

$u(0,t) = u(\pi,t) = 0$

$\frac{\partial^2}{\partial x^2} v_n(x) = v_n''(x) = \lambda_n v_n(x)$

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Nonlinear ODE:

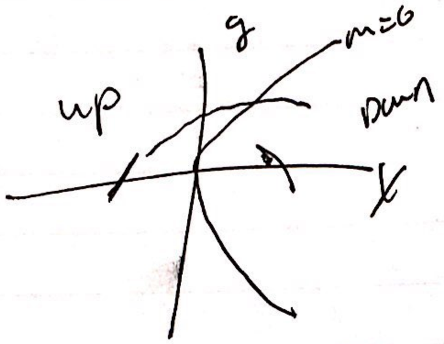
$y'(x) = f(x, y(x))$

Slope field/isoclines
nullclines
existence/uniqueness
"fences"

Qualitative/Graphical methods

Example 1 $y'(x) = y(x)^2 - x$ linear $y'(x) = a(x)y(x) + b(x)$

no formulas to solutions (ever)



sol'ns $y = y(x)$ ⇐ solution curves

Slope field

$f(x,y) = y'$ (slope)

$(1,0) = (x_0, y_0)$ $f(1,0) = 0^2 - 1 = -1$

$(0,0) \Rightarrow 0 = f(0,0)$ $f(x,y) = y^2 - x$

$(-1,0) \to 1 = f(-1,0)$

NULLCLINES

$f(x,y) = 0$

slope $= 0$

Existence and uniqueness.

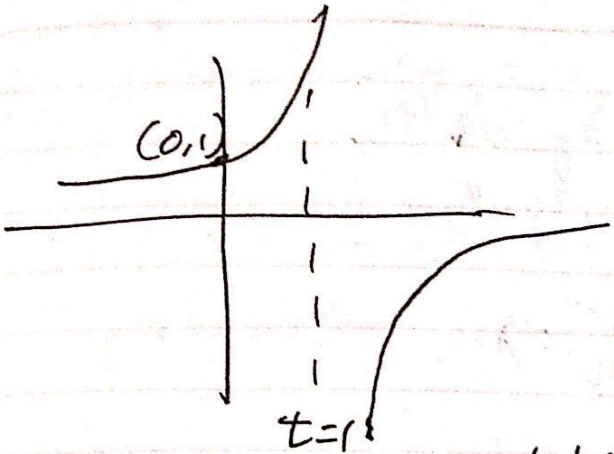
Theorem If $f(x,y)$ is continuous near (x_0, y_0) then there is exactly one solution curve through (x_0, y_0) → $y' = f(x,y)$

Example 2:

$\dot{y} = y^2$ $\frac{dy}{dt} = y^2$ (separable

$\frac{dy}{y^2} = dt \Rightarrow -y^{-1} = t + C \Rightarrow y = \frac{-1}{t+c}$

$y(2) = \cancel{\frac{1}{1-2}} = -1$ formulas give the wrong answer

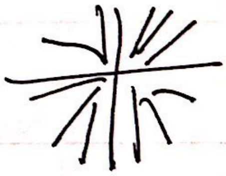


through every point, there's a solution curve.

whether equations of fluid dynamics have 3D?

Example 3

$y' = y/x$ $\quad \frac{dy}{y} = \frac{dx}{x} \Rightarrow \ln y = \ln x + C$.



$y = Ax \quad A = e^c$

also $A = -e^c$

$A = 0$ work.

$f(x,y) = y/x \Rightarrow$ discontinuous (undefined for $x=0$)

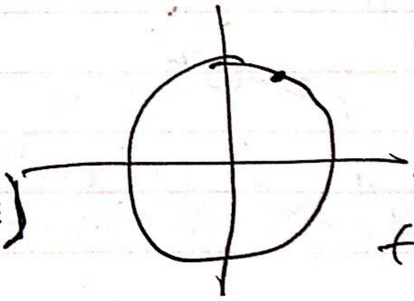
y axis causes problems

no solution through $(0,y_0)$ $y_0 \neq 0$ & many solns that tend to $(0,0)$

Example 4

$y' = -x/y$

$y\,dy = x\,dx$ $\quad \frac{1}{2}(x^2+y^2) = c$ (constant)



$y=0$ is a place where $f(x,y)$ is undefined

Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial^2 x}$$

Boundary conditions

$u(0,t) = 0 = u(\pi, t)$

Initial conditions

$$\begin{cases} u(x,0) = 1 \\ \frac{\partial u}{\partial t}(x,0) = x \end{cases}$$

$u(x,t) = y(t) v(t)$

$\ddot{y} v = c^2 y v''$

$$\frac{\ddot{y}}{c^2 y} = \frac{v''}{v} = \lambda$$

Boundary conditions

$u(0,t) = 0$

$y(t) v(0) \to v(0) = 0$

$u(\pi, t) = 0 \to v(\pi) = 0$

$\parallel$

$y(t) v(\pi)$

find what:

$\frac{v''}{v} = \lambda$ w/ $v(0) = v(\pi) = 0$

$\lambda > 0$ nope

$\lambda = 0$ nope

$\lambda < 0$

$\lambda = -w^2$

$\frac{v''}{v} = w^2$ $v'' + w^2 v = 0$

$\ldots$ $v(x) = c_1 \cos(wx) + c_2 \sin(wx)$

$0 = v(0) = c_1$

$v(x) = \sin(wx)$

$v(\pi) = 0$; $\sin(\pi) = 0$

$\Rightarrow w = 0, 1, 2, 3$

$\lambda = 0, -1.$

$\lambda = -1, -4, -9, -16$

$$\frac{\ddot{y}}{c^2 y} = -n^2 \qquad \ddot{y} + c^2 n^2 y = 0$$

$$y(t) = a\cos(nt) + b\sin(nt)$$

$$y(t) = \cos(nt) \qquad y(t) = \sin(nt)$$

$$u(x,t) = y(t)\,v(x)$$

$$\ddot{y}\,v = c^2 y\, v''$$

- $\cos(cnt)\sin(nx)$
- $\sin(cnt)\sin(nx)$

$$\frac{\ddot{V}}{c^2 y} = \frac{v''}{v} = \lambda$$

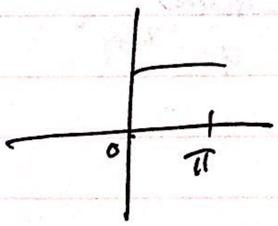
general solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(cnt)\sin(nx) + \sum_{n=1}^{\infty} b_n \sin(cnt)\sin(nx)$$

Now use the initial condition:

$$u(x,0) = 1 = \sum_{n=1}^{\infty} a_n \sin(nx)$$

a_n are the numbers that express 1 as a sine series



$$a_n = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$$u(x,t) = \sum a_n \cos(cnt)\sin(nx) + \sum b_n \sin(cnt)\sin(nx)$$

$$\frac{\partial u}{\partial t} \Rightarrow \frac{\partial u}{\partial t} = \sum -a_n cn \sin(cnt)\sin(nx) + \sum b_n cn \cos(cnt)\sin(nx)$$

$$x = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \boxed{b_n cn} \sin(nx)$$

$$= -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin(nx) \qquad b_n cn = \frac{-2(-1)^n}{n} \qquad b_n = \frac{(-2)(-1)^n}{cn^2}$$

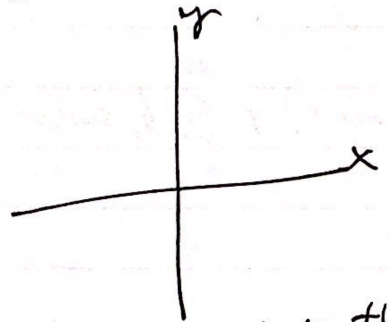
$$\frac{4}{\pi}\sum_{\substack{n=1\\ n\ \text{odd}}}^{\infty} \frac{\cos(cnt)+\sin(nx)}{n}. \quad -\frac{2}{c}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin(cnt)\sin(nx)$$

wave equation has two sums

Isoclines:

$y' = y - x^3 + xy$

can't find ~~exact~~ solution



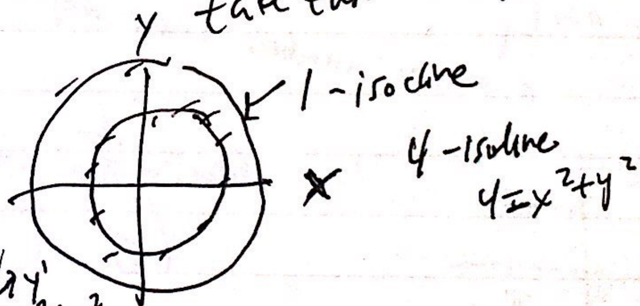
Isocline is a curve in the (x,y) plane where y' is constant

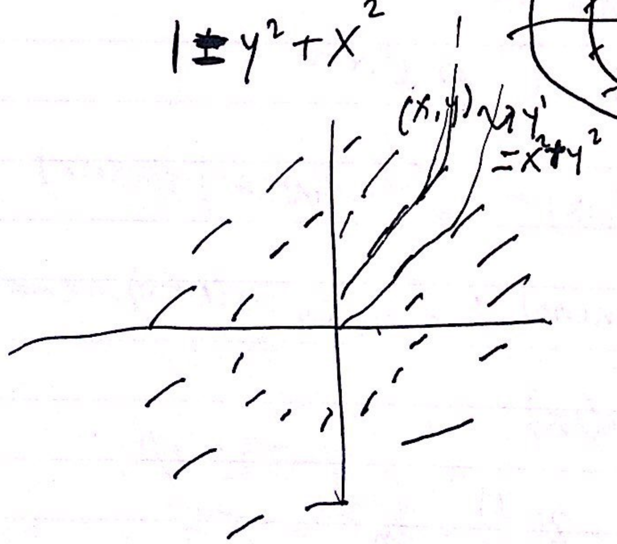
If $y' = 2 = y^2 - x^3 + xy$ — hard to plot

take easier example:

$y' = y^2 + x^2$

$1 = y^2 + x^2$





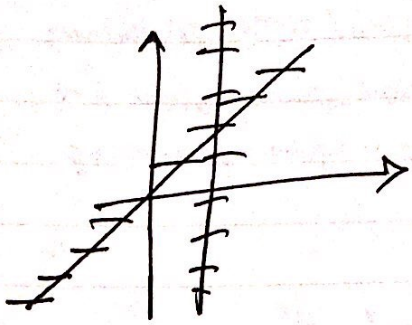
$y' = y(x-1)(x-y)$

$0 = y(x-1)(x-y)$

$y=0$

$x-1=0 \rightarrow x=1$

$x-y=0 \rightarrow x=y$



Isoclines: lines are drawn where slopes are the same. You can use these shapes to estimate solutions.

Autonomous Equations

5/8/19

$\dot{x} = f(x)$

- isoclines / time invariance
- critical points / phase line
- logistic equation/harvesting (bifurcation diagram)

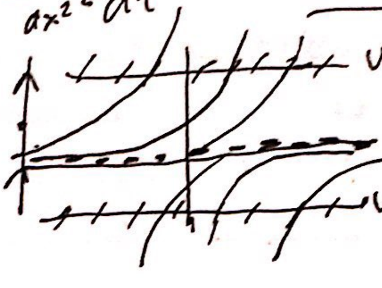
Example 1

$\dot{x} = x^2 \quad \frac{dx}{dt} = x^2 \quad \frac{dx}{dx^2} = dt$

Won't use formulas for solutions

isoclines $\boxed{f(x) = m}$

$f(x) = x^2 \quad x^2 = m$



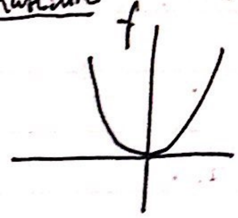
Time invariance:

$x(t)$ a solution implies $x(t-t_0)$ is a solution

Autonomous := coeff. independent of t

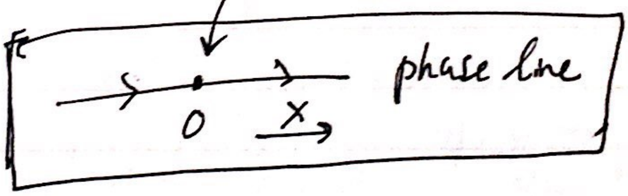
In contrast with $\dot{x}(t) = a(t)x^2 + b(t)x + c(t)$

phaseline



$f(x) = x^2$

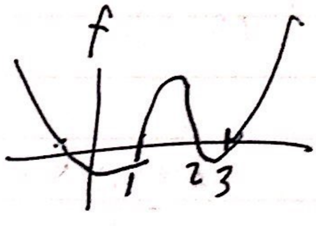
Semistable critical point



$f(x) = 0$ critical point

$(\dot{x} = 0)$

$f(x) = (x-1)^2(x-2)(x-3)$

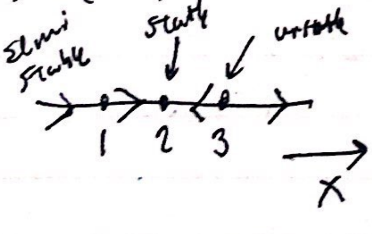


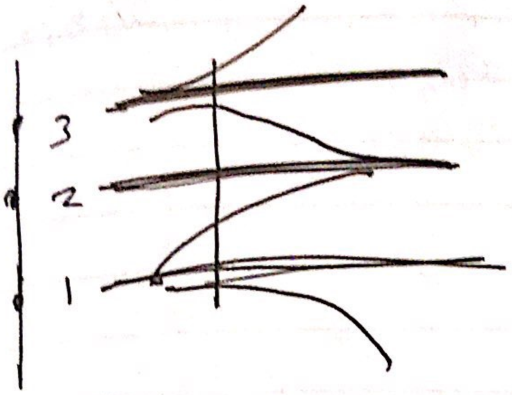
$x \to \pm\infty$

$\approx x^4$

Example 2

$\dot{x} = (x-1)^2(x-2)(x-3)$





the "domain of attraction" of 2 is $1<x<3$ (as $t\to\infty$)

Example 3

Logistic equation $\dot{x} = ax - bx^2$ $\quad a>0$, $b>0$

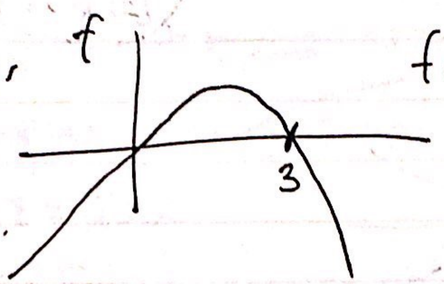
$= k(x)\,x$

$k(x) = (a-bx)$

$a=3$ $\quad b=1$

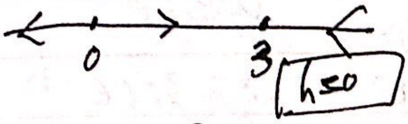
x units 1000 frogs kilofrogs

$\dot{x} = 3x - x^2 = (3-x)x$



$f(x) = (3-x)x$

$x \to \pm\infty \approx -x^2$

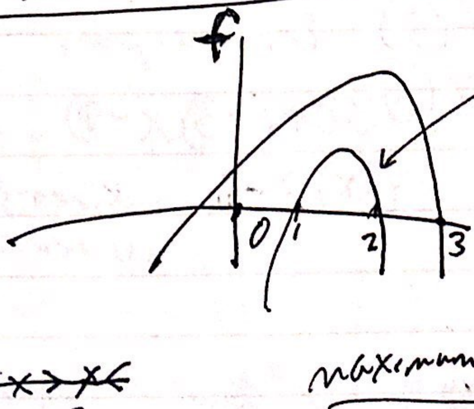


Example 4 Harvesting $\dot{x} = 3x - x^2 - h$

h = harvesting rate $\quad t$(month) $\quad h = \#$frog/month

$f(x) = 3x - x^2 \quad f(3/2) = \frac{9}{4}$

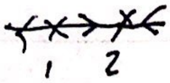
phase line for each h



$h=2$

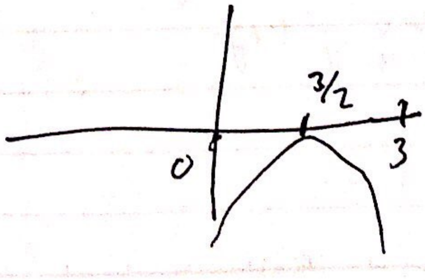
$3x - x^2 - 2 = 0$

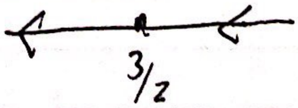
@ $x = 1, 2$



maximum harvest rate

$h = 9/4$





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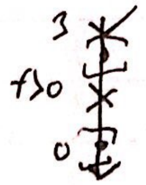
Lecture : 2x2 Nonlinear Systems 5/10/19

- 1D linear approximation $\dot{x}=f(x,y)$
- critical points: sources, sinks, saddles $\dot{y}=g(x,y)$
- 2D linear approximation: Jacobian matrix

1 Dimensional linear approximation

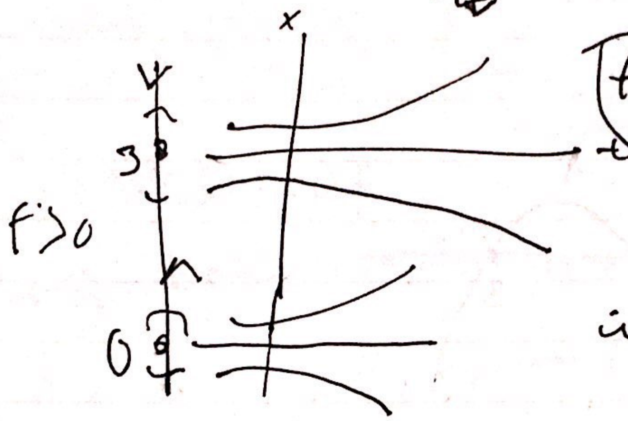
$\dot{x}=3x-x^2$

$\boxed{f(x)=3x-x^2}$



$\dot{x}=3x-x^2$ when $x\approx 0$

$\dot{x}\approx 3x$ $\quad x^2 << x$



$f(x)=0 \iff x=0,3$

$x(t)\approx ce^{3t}$ works well as $t\to\infty$

$u=x-3$ $\quad u\approx 0$ $\quad \dot{u}=-3u-u^2$

$\dot{u}=\dot{x}=3x-x^2$
$=(3-x)x$
$=-u(u+3)$
$=-3u-u^2$

$u\approx ce^{-3t}$

$x\approx 3+ce^{-3t}$ as $t\to+\infty$

$f(x)=3x-x^2 \quad f'(x)=3-2x$
$=f(3)+f'(3)(x-3)$

linear approx

$f'(3)=3-2\cdot 3=-3$

$\dot{u}=-3u \dots u=ce^{-3t}$ etc.

$f(3)$ critical point

$f(x)\approx 0+(-3)(x-3)$

$f(x)\approx -3u$ $\quad x$ near 3, u near 0

In 2D

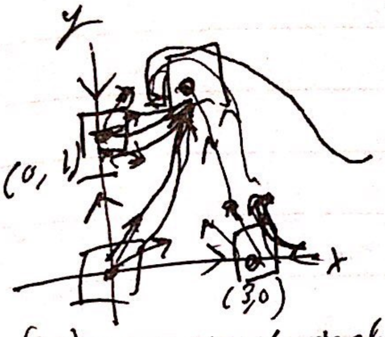
Example coupled system
$$\begin{cases}\dot{x}=3x-x^2-xy\\ \dot{y}=y-y^2+xy\end{cases}$$

x # deer
y # wolves

predator-prey model

Linearization near $(x,y)=(0,0)$ is $\dot{x}=3x$, $\dot{y}=y$ $\qquad \dot{\vec{x}}=\begin{pmatrix}3&0\\0&1\end{pmatrix}\vec{x}$ $\qquad \vec{x}=\begin{pmatrix}x\\y\end{pmatrix}$

Source



not physical $y<0$

$\boxed{y=0 \quad \dot{x}=3x-x^3}$

$\boxed{x=0 \quad \dot{y}=y-y^2}$

$y(1-y)$

left out possibility that this escapes to infinity

What is the fate of (x,y) as $t\to\infty$?

To answer, ① we find the critical point

② we linearize at each critical point $\boxed{\text{if } x(0)>0,\ y(0)>0}$

Step 1 Set $f(x,y)=0$, $g(x,y)=0$ $\qquad \dot{\vec{x}}=\begin{pmatrix}0\\0\end{pmatrix}$ goes no where (stationary point)

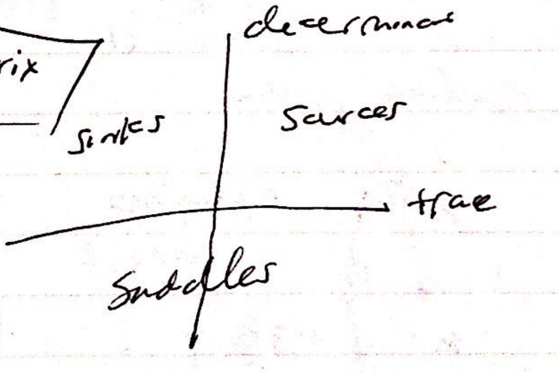
$0=3x-x^2-xy=x(3-x-y)$

$0=y-y^2+xy=y(1-y+x)$ $\Rightarrow$ 4 cases $(0,1),(0,0),(3,0)$

Lastly: $x+y=3$, $y-x=1$ $\qquad (x,y)=(2,1)$

Analogue of $f'(x)$ for

$\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}f(x,y)\\g(x,y)\end{pmatrix} \Rightarrow$ Jacobian matrix



$$J=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix} \qquad \begin{pmatrix}x\\y\end{pmatrix}$$

Example: $J=\begin{pmatrix}3-2x-y & -x\\ y & 1-2y+x\end{pmatrix}$

$f=3x-x^2-xy$

$g=y-y^2+xy$

Source $J(0,0)=\begin{pmatrix}3&0\\0&1\end{pmatrix}$

Saddle $J(3,0)=\begin{pmatrix}-3&-3\\0&4\end{pmatrix}$

Saddle $J(0,1)=\begin{pmatrix}2&0\\1&-1\end{pmatrix}$

$J(1,2)=\begin{pmatrix}-1&-1\\2&-2\end{pmatrix}$

Sink

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Lecture 2x2 Nonlinear Systems 5/13/19

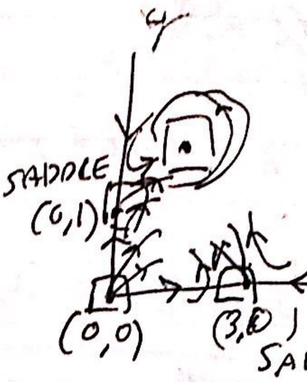
recap

∘ pendulum - phase plane! ↗ what are the meaningful variables

Conservation of energy

Monday before exam, or new room tbd

$$\dot{x} = 3x - x^2 + xy \qquad x = \#\text{ deer } (1000\text{s})$$
$$\dot{y} = y - y^2 - xy \qquad y = \#\text{ wolves } (100\text{s})$$



$x \geq 4$

$\dot{x} < 0$

$\dot{x} = 3x - x^2 - xy$

$\leq 3x - x^2$

$x \leq 4$

Critical pts: (0,0) (3,0) (0,1) (1,2)

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 3-2x-y & -x \\ y & 1-2y+x \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \qquad J(3,0) = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

$$J(0,1) = \begin{pmatrix} -3 & -3 \\ 0 & 4 \end{pmatrix} \qquad J(1,2) = \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$$

At critical point (1,2)

$$J(1,2) = \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$$

trace $= -3$ det $= 4$

$\lambda^2 + 3\lambda + 4 = 0 \qquad \lambda = \frac{-3 \pm \sqrt{9 - 4 \cdot 4}}{2}$

eigenvalues are complex. $\lambda = \frac{-3}{2} + i\omega$

Spiral sink

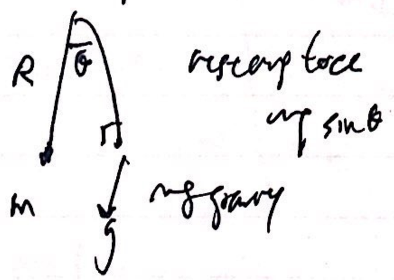
(stable spiral)

see if clockwise or counter-clockwise

$$\begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\dot{y} > 0$ counterclockwise

Nonlinear pendulum



good when θ is small

$\sin\theta \approx \theta$

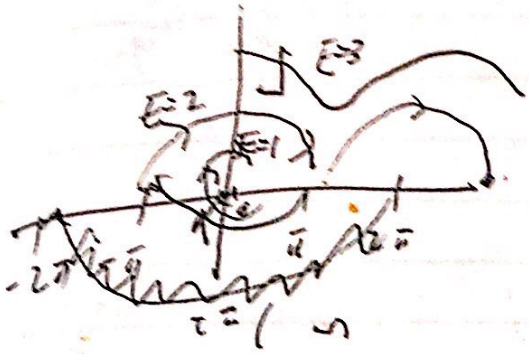
Introduce the companion system

new variable $\omega = \dot{\theta}$ angular velocity

$$\begin{cases} \dot{\theta} = \omega \\ mR\dot{\omega} = -mg\sin\theta \end{cases}$$

$\dot{\theta} = \omega$ $\quad$ $\ddot{\theta} = -k\theta$

~~[illegible]~~ $\dot{\omega} = -k\theta$ $\quad$ $k>0$ $\quad$ periodic motion



phase diagram.

ω in ... position

new variable:

<u>Energy</u> controls the nonlinear picture

$$E = \text{kinetic} + \text{potential} = \frac{1}{2}m(R\dot{\theta})^2 + mgR(1-\cos\theta)$$

force of gravity

$$E(\theta,\omega) = \frac{1}{2}\omega^2 + 2(1-\cos\theta)$$

$m=1,\ g=2,\ R=1$

$\frac{d}{dt}E = \dot{E} = 0$ (conserved!)

$$\dot{E} = \omega\dot{\omega} + 2(\sin\theta)\dot{\theta}$$
$$= \omega(-2\sin\theta) + (2\sin\theta)\omega = 0$$