8.03 PSet 2 — Key concept: These are damped harmonic oscillators ⇒ damped slows down motion — 12/23/19

Problem 1

displacement $s(t)$ from equilibrium of pen ... obeys eq'n for damped harmonic oscillators

$\ddot{s}(t) + \gamma\dot{s}(t) + \omega_0^2 s(t) = 0$

a) Find ~~time evolution~~ find $z(t)$ of displacement if pen is critically damped & $s(0)=0$ & $\dot{s}(0)=V_0$. Does s change sign before equilibrium position at $s=0$

$\frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \Rightarrow \Big] \Rightarrow e^{-\frac{\gamma}{2}t}$

set sqrt $=0$ $\quad \frac{\gamma^2}{4} = \omega_0^2 \Rightarrow \omega_0 = \frac{\gamma}{2} \quad \gamma = 2\omega_0$

$z(t) = Ae^{-\frac{\gamma}{2}t}$ $\quad H^2 + \gamma H + \omega_0^2 = 0 = H^2 + 2\omega_0 + \omega_0^2 = 0 = (H+\omega_0)^2$

$(\ddot{s} + 2\omega_0\dot{s} + \omega_0^2 s)e^{\omega_0 t} = 0$

$= \frac{d^2}{dt^2}(Xe^{\omega_0 t}) = (\frac{dx}{dt}e^{\omega_0 t} + X\omega_0 e^{\omega_0 t}) = \frac{dx^2}{dt^2}(e^{\omega_0 t}) + \frac{dx}{dt}(\omega_0 e^{\omega_0 t}) + \omega_0 \frac{dx}{dt}e^{\omega_0 t} + X\omega_0^2 e^{\omega_0 t} = \frac{dx^2}{dt^2} + (2\omega_0 e^{\omega_0 t})\frac{dx}{dt} + \omega_0^2 x$

found in 18.03 where duplicate roots: $e^{-t}, te^{-t}, t^2e^{-t}$ etc.

$Xe^{\omega_0 t} = A + Bt \Rightarrow X = (A+Bt)e^{-\omega_0 t}$

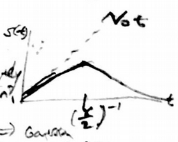
$s(0) = 0 = (A + B \cdot 0)e^0 \Rightarrow A = 0$

$\dot{s}(0) = V_0 = Be^{-\omega_0 t} + (A+Bt)(-\omega_0)e^{-\omega_0 t} \Rightarrow B = V_0$

$s = V_0 t\, e^{-\omega_0 t} = \boxed{V_0 t\, e^{-\frac{\gamma}{2}t}}$

Note $s(t) > 0 \ \forall t$ ⇒ no overshoot returning to $s=0$

V_0 change sign before settling to $s=0$



approximately a Gaussian! $\int e^{-x^2}dx \Rightarrow$ Gaussian

b) Find the response of an overdamped pen to initial conditions $s(0) = s_0$ & $\dot{s}(0) = 0$

overdamped: $\frac{\gamma^2}{4} - \omega_0^2 > 0$

Soln: $z(t) = Ae^{-T_1 t} + Be^{-T_2 t}$ where $T_{1,2} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$

$z(0) = A + B = s_0$ $\quad \dot{z}(t) = -T_1 Ae^{-T_1 t} - T_2 Be^{-T_2 t}$

$\dot{z}(0) = 0 = -T_1 A - T_2 B = 0 \quad -T_1 A = T_2 B$

$A - \frac{T_1}{T_2}A = s_0 \quad B = -\frac{T_1}{T_2}A$

$\Rightarrow A = \frac{s_0}{1 - \frac{T_1}{T_2}} \quad B = -\frac{T_1}{T_2}\frac{s_0}{1-\frac{T_1}{T_2}}$

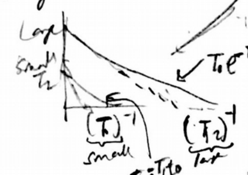
$\boxed{z(t) = \frac{s_0}{1 - \frac{T_1}{T_2}}\left(e^{-T_1 t} - \frac{T_1}{T_2}e^{-T_2 t}\right)}$

c) For ω_0 does s overshoot its equilibrium position at $s=0$?

if $s(t) = 0$, $A(e^{-T_1 t} - \frac{T_1}{T_2}e^{-T_2 t}) \Rightarrow \frac{T_1}{T_2}e^{-T_2 t} = e^{-T_1 t}$

$T_1 e^{-T_2 t} = T_2 e^{-T_1 t}$

assume $T_1 > T_2$ — don't cross... no overshoot.



Problem 2 Simple Electrical Circuit

$C = 1\,\mu F \quad L = 1\,mH \quad R = 10\,\Omega \quad$ if $V_c(0) = 10V = V_0$ & $i(0) = 500\,mA$

Find $V_c(t)$ for $t > 0$

$V_L + V_C + V_R = 0 \Rightarrow L\frac{di}{dt} + \frac{1}{C}\int i\,dt + iR = 0 \quad i = -\frac{dQ}{dt} = -C\frac{dV}{dt}$

$CV = Q$; current flows opposite to capacitor

$LC\frac{d\dot{V}_c}{dt^2} + \frac{1}{C}(C V_c) + C\frac{dV_c}{dt}R = 0$

$\Rightarrow LC\frac{d^2V_c}{dt^2} + V_c + C\frac{dV_c}{dt}R = 0 = e^{\sigma t}$

$\ddot{V}_c + \frac{R}{L}\dot{V}_c + \frac{1}{LC}V_c = 0$ here we always want + signs

For each element: $V_L = L\frac{di}{dt}$, $V_R = -iR$, $V_C = \frac{1}{C}\int i\,dt$; $+V_L - V_R - V_C = 0$ $\quad (L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0)$

$\omega_0 = \frac{1}{\sqrt{LC}}$ $\quad D^2 + \frac{R}{L}D + \frac{1}{LC} \Rightarrow$ roots $= \frac{-\frac{R}{L} \pm \sqrt{(\frac{R}{L})^2 - \frac{4}{LC}}}{2}$ $\quad s^2 + \frac{R}{L}s + \frac{1}{LC} = e$

$= \frac{-R}{2L} \pm \sqrt{(\frac{R}{L})^2/4 - \omega_0^2}$

roots $= -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \frac{1}{LC}} \Rightarrow e^{-\frac{\gamma}{2}t}e^{\pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}t}$ $\quad i = \frac{dQ}{dt} = -C\frac{dV}{dt}$

$\Rightarrow e^{-\frac{\gamma}{2}t}\left(C_1\cos(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}) + C_2\sin(\sqrt{\omega_0^2 - \frac{\gamma^2}{4}})\right)$

$V_c(t) = Ae^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi)$ where $\omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

$V_c(0) = V_d\cos(\phi) = 10 \Rightarrow$ call A V_0

Conventions: resonance freq. $\omega_0 = \frac{1}{LC}$; damped resonant freq. $\omega_d = \sqrt{\omega_0 - \frac{\gamma^2}{4}}$

$i = -C\frac{dV}{dt} = -CV_0\left(e^{-\frac{\gamma t}{2}}\cos(\omega_d t + \phi)\right)'$ $\quad \omega_0 = \frac{1}{\sqrt{LC}} = (10^{-3} \times 10^{-6})^{-1/2} = \sqrt{10} \times 10^4$

$i(t) = CV_0\left(-\frac{\gamma}{2}e^{-\frac{\gamma t}{2}}\cos(\omega_d t + \phi) - e^{-\frac{\gamma t}{2}}\omega_d\sin(\omega_d t + \phi)\right)$ $\quad \gamma = \frac{R}{L} = \frac{10}{10^{-3}} = 10^4$

$i(0) = CV_0(\frac{\gamma}{2}\cos\phi + \omega_d\sin(\phi)) = 0.5 \Rightarrow \underbrace{\frac{C\gamma}{2}}_{\frac{10^{-2}}{2}}\underbrace{V_0\cos\phi}_{10} + (C\omega_d)V_0\sin\phi = 0.5$ $\quad = \frac{10}{\cos\phi}$

$0.05 + C\omega_d\, 10\tan\phi = 0.5$ solve for ϕ.

8.03 soln manual is wrong fyi!

Now $V_0 = \frac{10}{\cos\phi}$ full eq'n $\boxed{V_c(t) = V_0 e^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi)}$ — this is ω_d not ω_0 $\quad \omega_d = \sqrt{\omega_0 - \frac{\gamma^2}{4}}$

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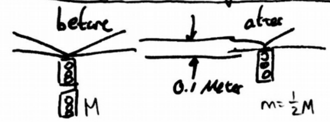
Question 3: ~~Oscillating~~ Traffic Light



traffic signal oscillates.
damped mass-spring oscillator
wire → spring
amplitude disturbances took 4 seconds to damp out
lower half of signal broke. New equilibrium position

a) Undamped natural freq. (in Hertz) after separation.

$y=0$, 0.1 m: $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ $\quad \omega_0 = \sqrt{\frac{k}{m}}$ $\quad \Delta F = k\Delta x$

$\rightarrow F = Mg$ $\quad Mg = k\,0.1$ $\quad k = \frac{Mg}{0.1}$

$\rightarrow F = \frac{M}{2}g$, $m = \frac{M}{2}$

$\omega_0 = \sqrt{\frac{Mg}{0.1\,m}} = \sqrt{\frac{g}{0.1}} = \sqrt{98.1}$

$\nu = \frac{\omega_0}{2\pi} = \frac{\sqrt{98}}{2\pi} = 1.58\,Hz$

b) dissipation is in suspension system. What is the decay time of traffic light amplitude oscillation after separation?

$\gamma \propto \frac{1}{m}$ $\quad 2m \rightarrow 4s$ $\quad m \rightarrow 2s$

$\gamma = \frac{1}{m}$ because $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}e^{i\omega t}$

c) Analytic expression for $y(t)$ after separation. Assume light was stationary ($\dot{y}=0$) at prev equilibrium position at time of separation $t=0$.

$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0$ $\quad y(t) = Ae^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi)$

$\frac{\gamma}{2} = \frac{1}{2s} = \frac{1}{2}\,s^{-1}$ → decay time $\Rightarrow e^{-\frac{\gamma}{2}t}$ $\quad \omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \sqrt{98 - \left(\frac{1}{2}\right)^2} = 9.887\,s^{-1}$

$y(0) = -0.1 = A\cos(\phi)$

$\dot{y}(t) = A\,{-\frac{\gamma}{2}}\,e^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi) + A\omega_d(-\sin(\omega_d t + \phi)) = 0$

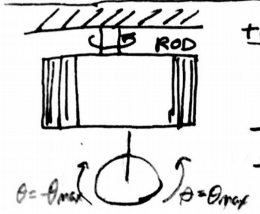
$\dot{y}(0) = -A\frac{\gamma}{2}\cos(\phi) - A\omega_d\sin(\phi) = 0$

$-\frac{\gamma}{2\omega_d} = \tan\phi = \frac{-1\,s^{-1}}{2(9.887\,s^{-1})}$ $\quad = A\frac{\gamma}{2}e^{-\frac{\gamma}{2}t}\cos(\omega_d t + \phi) - A\omega_d\sin(\omega_d t + \phi)$

$\Rightarrow \phi = -2.89$ $\quad A = \frac{-0.1}{\cos\phi} \approx \frac{-0.1\,m}{\cos(2.89)} \approx -0.1\,m$

$\Rightarrow \boxed{y(t) = (-0.1)e^{-\frac{1}{2}t}\cos(9.887t + 2.89)}$

Problem 4: Torsional Oscillator as a Measurement Device



torsional oscillator
- hollow cylindrical spool hung by thin beryllium copper torsion rod from stationary ~~pt.~~ support
- Mylar film wound on spool
- avg spacing of 500 Å between layers of Mylar ⇒ allows gas to flow between sheets of film.
- frequency of oscillator: 1300 Hz, $\omega = 2.5\times10^5$

a) Find relation between fractional frequency shift ($d\nu/\nu$) & fractional shift in mass of spool dM/M. "fractional" just proportion $\nu = \frac{\omega}{2\pi}$

angular with damping: $\Rightarrow I\frac{d^2\theta}{dt^2} + C\frac{d\theta}{dt} + k\theta = \tau(t)$ $\quad C \Rightarrow$ angular damping constant

$\ddot{\theta} + \frac{C}{I}\dot{\theta} + \frac{k}{I}\theta = \frac{\tau(t)}{I}$ $\quad I = \frac{1}{2}MR^2$ $\quad \tau = -k\theta = I\ddot{\theta}$ $\quad \vec{L} = I\vec{\theta}$, $\vec{\tau} = \frac{d\vec{L}}{dt}$

w/o damping: $I\ddot{\theta} = -k\theta$ $\quad \omega^2 = \frac{k}{I}$ $\quad \Rightarrow \nu = \frac{\omega}{2\pi} \propto \sqrt{\frac{k}{I}} \propto \frac{1}{\sqrt{M}}$

$\nu = AM^{-1/2}$, $\quad \frac{d\nu}{dM} = \left(-\frac{1}{2}\right)AM^{-3/2} = \left(-\frac{1}{2}\right)\nu M^{-1}$ $\Rightarrow \boxed{\frac{d\nu}{\nu} = -\frac{1}{2}\frac{dM}{M}}$

b) mass of Mylar dominates mass of spool. Calculate change in # absorbed helium atoms / cm^2 resolvable. Helium atomic weight = 4 $\quad$ mass of proton $= 1.67\times10^{-24}$ g

density of one layer helium atom $\approx 10^{14}$ atoms/cm^2

What fraction of a single layer does this resolution correspond to? part 2 question

that can be resolved. Fractional changes to frequency of oscillator $10^{-3}/Q$ resolvable

$\left|\frac{d\nu}{\nu}\right| = \left|\frac{10^{-3}}{Q}\right| = \left|\frac{10^{-3}}{2.5\times10^5}\right| = 4\times10^{-9}$ $\quad \left|\frac{dM}{M}\right| = \left|-2\frac{d\nu}{\nu}\right| = 8\times10^{-9}$

one turn of mylar, h, atoms from both sides, mylar

$M_{mylar} = 2\pi r h t \rho$ ← mass density of mylar

$M_{atoms} = 2(2\pi r h)m\sigma$ ← surface density $\frac{atoms}{cm^2}$ (two sides; mass of one atom)

$\left|\frac{dM}{M}\right| = \frac{dM_{atoms}}{M_{mylar}} = \frac{2(2\pi r h)m\,d\sigma}{2\pi r h t \rho} = \frac{2m\,d\sigma}{t\rho} = \frac{2(1.67\times10^{-24}\times4)}{(2.5\times10^{-4})\times1.42}\,d\sigma$ atoms, cm

$= 3.76\times10^{-20}d\sigma = 8\times10^{-9} \Rightarrow d\sigma = 2\times10^{11}$ (Atoms/cm^2)

Fraction of monolayer $= \frac{d\sigma}{\sigma} = \frac{2\times10^{11}}{10^{14}} = 2\times10^{-3}$

unit checking: $\underbrace{\frac{2(g \times \text{atomic weight})}{cm \times \frac{g}{cm^3} = \frac{g}{cm^2}}}_{cm^2\ \text{atomic weight}} \times \frac{atoms}{cm^2}$ $\quad \times \frac{atoms}{cm^2} \Rightarrow mass? = \frac{dM}{M}$

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problem 5: Addition of Harmonic Variables

a) - linear system driven by F_1 oscillates w/ amplitude 2.000cm
- driven by another force F_2 at same frequency, amplitude is 1.414 cm
- driven by both forces at same time: $F = F_1 + F_2$, amplitude is 1.414 cm

$\Rightarrow$ what is the phase difference between F_1 & F_2?

irrelevant to problem but critical concept

$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}$ guess soln $z(t) = Ae^{i(\omega t + \delta)}$ $\boxed{\text{background}}$

$P(D) = D^2 + \gamma D + \omega_0^2$ $P(i\omega) = -\omega^2 + i\gamma\omega + \omega_0^2$ $\omega \neq \omega_0$ is the drive frequency not natural freq.

$(-\omega^2 + i\gamma\omega + \omega_0^2) A e^{i(\omega t + \delta)} = \frac{F_0}{m} e^{i\omega t}$

divide both sides by $e^{i(\omega t + \delta)}$ $\Rightarrow (\omega_0^2 - \omega^2)A + i\gamma\omega A = \frac{F_0}{m} e^{-i\delta}$ phase argument

$Ae^{i\delta} \Rightarrow$ $\sqrt{(\omega_0 - \omega)^2 + (\gamma\omega)^2}$ $\gamma\omega$ $\omega_0^2 - \omega^2$ $\Rightarrow A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0 - \omega)^2 + (\gamma\omega)^2}}$

$\tan\delta = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$

$F = F_1 + F_2 \Rightarrow X = X_1 + X_2$

$\sqrt{2}$ $\sqrt{2}$ $F_1 + F_2$ F_2 δ 2 F_1 A

$\delta = \boxed{180 - 45 = 135°}$

b) Magnitudes same as part A, but now frequencies of F_1 & F_2 are $\nu_1 = 100\,Hz$ & $\nu_2 = 102\,Hz$
ν_1 & ν_2 not near natural frequencies of system.

Make sketch of system to sum of two forces $F = F_1 + F_2$

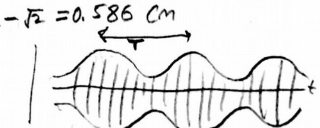
Sketch amplitude vs. time after steady-state.

Give max. & min. amplitudes & peak to peak separation

$X = X_1 + X_2$

max $\quad X_1 \; 2 \quad \sqrt{2} \; X_2 \longrightarrow 2 + \sqrt{2} = 3.414$ cm $\quad$ amplitude vs. time after steady-state

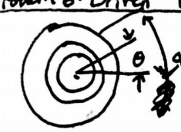
min $\quad X_1 \; 2 \quad X_2 \; \sqrt{2} \longrightarrow 2 - \sqrt{2} = 0.586$ cm



peak-to-peak separation

$T = \frac{1}{\nu} = \frac{1}{2}$ $\qquad$ $T = \frac{1}{|\nu_1 - \nu_2|} = \frac{1}{2} = 0.5\,s$

Problem 6: Driven Torsional Oscillator



- torsional oscillator w/ moment of inertia I driven by viscous drag between it & outer concentric ring
- restoring torque: $-K\theta$ torque due to friction: $-b\frac{d}{dt}(\theta - \alpha)$
b: constant α: angular pos. of outer ring
- outer ring driven in steady state such $\underline{\alpha(t) = \alpha_0 \cos(\omega t)}$

a) Write down dfq governing angular position $\theta(t)$ of oscillator. is α $\dot{\theta}$?

$I\ddot{\theta}(t) + b\dot{\theta}(t) + K\theta$ $\qquad I\ddot{\theta} = -K\theta - b\frac{d}{dt}(\theta - \alpha)$

$I\ddot{\theta} + b\dot{\theta} + K\theta = b\dot{\alpha}$ $\qquad = -K\theta - b\dot{\theta} + b\dot{\alpha}$ $\qquad \alpha = \alpha_0 \cos(\omega t)$

$\boxed{\ddot{\theta} + \frac{b}{I}\dot{\theta} + \frac{K}{I}\theta = \frac{b}{I}\dot{\alpha} = \frac{b}{I}\frac{d}{dt}(\alpha_0 \cos(\omega t))}$

b) The angular displacement has form: $\theta(t) = \theta_0(\omega)\cos(\omega t - \delta(\omega))$

Find $\theta_0(\omega)/\alpha_0$ & make sketch of result.

$\theta(t) = \theta_0(\omega)\cos(\omega t - \delta(\omega))$

$\theta(t) = Re[\theta_0(\omega) e^{i(\omega t - \delta(\omega))}]$ $\qquad \alpha = Re[\alpha_0 e^{i\omega t}]$

$\ddot{\theta} + \frac{b}{I}\dot{\theta} + \frac{K}{I}\theta = \frac{b}{I}\frac{d}{dt}[Re\, \alpha_0 e^{i\omega t}] = Re[\frac{b}{I}\alpha_0 i\omega e^{i\omega t}] = Re[\gamma\alpha_0 i\omega e^{i\omega t}]$

$P(D) = D^2 + \frac{b}{I}D + \frac{K}{I}$ $\quad p(i\omega) = -\omega^2 + \gamma i\omega + \frac{K}{I}$ ω_0^2

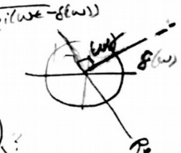
$(-\omega^2 + i\gamma\omega + \omega_0^2)\theta_0(\omega) e^{i(\omega t - \delta(\omega))} = i\gamma\alpha_0\omega e^{i\omega t}$

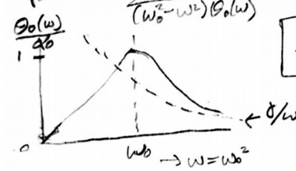
$e^{i(\omega t - \delta(\omega))}$ $\qquad e^{i(\omega t - \delta(\omega))}$

$\Rightarrow (-\omega^2 + i\gamma\omega + \omega_0^2)\theta_0(\omega) = i\omega\gamma\alpha_0 e^{i\delta(\omega)}$

$= \sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\,\theta_0$ $\quad \gamma\omega\theta_0(\omega)$ $\quad (\omega_0^2 - \omega^2)\theta_0(\omega)$ $\qquad \frac{\theta_0(\omega)}{\alpha_0} = \frac{|i\omega\gamma e^{i\delta(\omega)}|}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$

$\boxed{\frac{\theta_0(\omega)}{\alpha_0} = \frac{\omega\gamma}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}}$ $\qquad$ explanation





$\omega\gamma$ $\quad \omega\gamma\cos(\delta(\omega))$ $\quad \omega\gamma\sin(\delta(\omega))$

$\Leftarrow$ $e^{i\delta(\omega)} = \cos(\delta(\omega)) + i\sin(\delta(\omega))$
$ie^{i\delta(\omega)} = i\cos(\delta(\omega)) - \sin(\delta(\omega))$

c) Find analytical expression for phase angle $\delta(\omega)$ & sketch result.

$\frac{-\omega\gamma\alpha_0 \sin\theta}{\omega\gamma\alpha_0 \cos\theta} = \frac{(\omega_0^2 - \omega^2)\theta_0(\omega)}{\omega\gamma\theta_0(\omega)}$ $\qquad$ Im: $\omega\gamma\theta_0(\omega) = \omega\gamma\alpha_0\cos\delta$

Re: $(\omega_0^2 - \omega^2)\theta_0(\omega) = -\omega\gamma\alpha_0\sin\delta$

$\Rightarrow \tan\delta = -\frac{\omega_0^2 - \omega^2}{\omega\gamma}$