Introduction to Zero Knowledge Proofs

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Outline

NP Language

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- Language $\mathcal{L}_\mathcal{R} = \{x : \exists w \text{ s.t.} (x, w) \in \mathcal{R}\}$ is induced from relation $\mathcal{R}$
- $\mathcal{L}_\mathcal{R}$ is in NP iff there is a deterministic polynomial time algorithm f such that $f(x, w) = 1 \Leftrightarrow (x, w) \in \mathcal{R}$ (we say f decides $\mathcal{R}$)

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Denote an execution of this protocol by $\langle \mathcal{P}, \mathcal{V} \rangle(x)$, and $\mathrm{tr}\langle \mathcal{P}, \mathcal{V} \rangle(x)$ is the transcript of the execution, which is the collection of all interaction messages

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  3. The verifier checks $\mathrm{ECDSAVerify}(\sigma, \mathrm{pk}, \mathrm{m}) = 1$

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- Denote by $\langle\mathcal{A}\|\mathcal{E},\mathcal{V}\rangle(x) \to (w, b)$ the protocol where $\mathcal{A}$ interacts with $\mathcal{V}$ and in the meantime $\mathcal{E}$ has access to the internal state of $\mathcal{A}$. At the end, $\mathcal{E}$ outputs w and the verifier outputs b

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Knowledge Soundness

For every adversary $\mathcal{A}$, there exists an extractor $\mathcal{E}_{\mathcal{A}}$, such that $\Pr[\langle \mathcal{A}\|\mathcal{E}_{\mathcal{A}}, \mathcal{V}\rangle(x) \rightarrow (w, b) : b = 1 \wedge (x, w) \notin \mathcal{R}] < \varepsilon$

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Formally: for any verifier $\mathcal{V}$, there exists a simulator $\mathcal{S}$, which on input a valid statement x, can sample the verifier view, i.e. the distribution of $\mathcal{S}(x)$ is indifferentiable from that of $(r, \mathrm{tr}\langle\mathcal{P},\mathcal{V}\rangle(x))$

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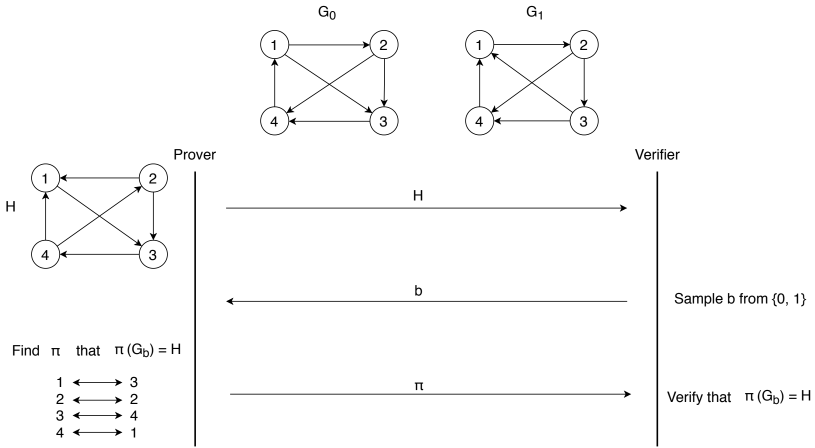
Zero-knowledge: for any verifier $\mathcal{V}$, the simulator $\mathcal{S}$ does whatever $\mathcal{V}$ does, and in the last step directly sets $b' = b$

Example: Graph Isomorphism



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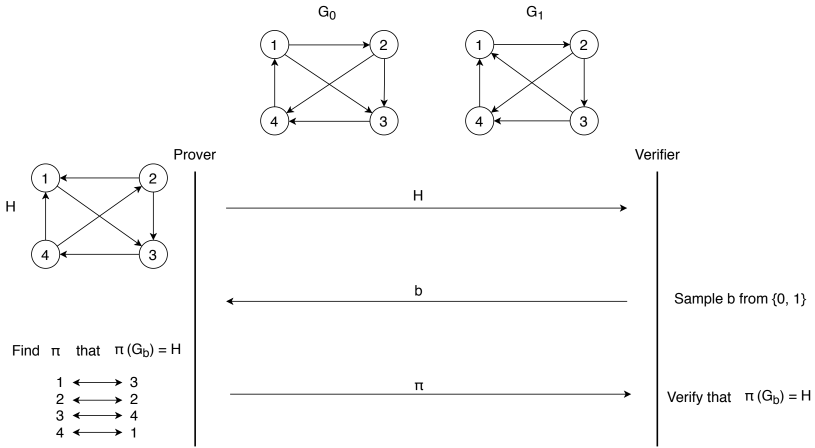

Zero-Knowledge: $\mathcal{S}$ samples the view (H, b, π) as follows:

1. Uniformly sample permutation π and bit b
2. Compute $H = \pi(G_b)$, output (H, b, π)

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4. $\mathcal{E}$ outputs $\pi = \pi_1^{-1} \circ \pi_0$

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Conclusion: NIZK only exists for easy problems.

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- Trusted Third Party (TTP)

Q/A