Introduction to Zero Knowledge Proofs

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Outline

1. NP Language
2. Proof Systems
3. Zero Knowledge
4. NIZK

NP Language

NP Language

What is NP language?

- Relation $\mathcal{R} = \{(x, w)\}$ is a set of instance-witness pairs
- Language $\mathcal{L}_{\mathcal{R}} = \{x : \exists w \text{ s.t.} (x, w) \in \mathcal{R}\}$ is induced from relation $\mathcal{R}$
- $\mathcal{L}_{\mathcal{R}}$ is in NP iff there is a deterministic polynomial time algorithm f such that $f(x, w) = 1 \Leftrightarrow (x, w) \in \mathcal{R}$ (we say f decides $\mathcal{R}$)

Proof Systems

Proof Systems

A proof system is an interactive protocol where the prover $\mathcal{P}$ tries to convince the verifier $\mathcal{V}$ a statement x is true

- **Completeness:** if statement x is true, then the verifier $\mathcal{V}$ accepts (outputs 1) with probability at least $1 - \eta$
- **Soundness:** if statement x is false, then for any prover $\mathcal{P}$, the verifier $\mathcal{V}$ accepts with probability less than ε (called soundness error)

Denote an execution of this protocol by $\langle \mathcal{P}, \mathcal{V} \rangle(x)$, and $\mathrm{tr}\langle \mathcal{P}, \mathcal{V} \rangle(x)$ is the transcript of the execution, which is the collection of all interaction messages

Proof Systems

Examples

- For every NP language $\mathcal{L}$, the statement $x \in \mathcal{L}$ has a trivial proof system:
  1. The prover sends w to the verifier
  2. By definition of NP, the verifier checks $f(x, w) = 1$
  3. Perfect completeness and perfect soundness
- Given an ECDSA public key pk, the prover proves that it learns the secret key sk
  1. The verifier samples a message m and sends to prover
  2. The prover generates a signature σ and sends to verifier
  3. The verifier checks $\mathrm{ECDSAVerify}(\sigma, \mathrm{pk}, \mathrm{m}) = 1$

Soundness

Soundness has several variations

- **Computational**: holds only for bounded adversary
- **Knowledge**: requires that the prover knows the witness

Proof systems with different soundness have different names

	Standard	**Knowledge**
Statistical	Proof	Proof of Knowledge
Computational	Argument	Argument of Knowledge

Remark

Soundness is a property of the verifier.

Soundness

As cryptographer, when we construct a proof system and say it has knowledge soundness, we must prove it.

- But how to prove that an adversary $\mathcal{A}$ knows something?
- Generally, we cannot prove this using game-based strategy, that means solving a hard problem when $\mathcal{A}$ doesn't know

Instead, we use Extractor to formalize the notion of knowing

- Assume $\mathcal{A}$ and $\mathcal{E}$ are two algorithms, let $\mathcal{A} \| \mathcal{E}$ denote the algorithm where $\mathcal{A}$ and $\mathcal{E}$ execute simultaneously and $\mathcal{E}$ has white-box access to the internal state of $\mathcal{A}$
- Denote by $\langle \mathcal{A} \| \mathcal{E}, \mathcal{V} \rangle(x) \to (w, b)$ the protocol where $\mathcal{A}$ interacts with $\mathcal{V}$ and in the meantime $\mathcal{E}$ has access to the internal state of $\mathcal{A}$. At the end, $\mathcal{E}$ outputs w and the verifier outputs b

Knowledge Soundness

For every adversary $\mathcal{A}$, there exists an extractor $\mathcal{E}_{\mathcal{A}}$, such that $\Pr[\langle \mathcal{A} \| \mathcal{E}_{\mathcal{A}}, \mathcal{V} \rangle(x) \to (w, b) : b = 1 \wedge (x, w) \notin \mathcal{R}] < \varepsilon$

Soundness

Another way to define the extractor

- Denote by $\mathcal{E}^{\langle \mathcal{A}, \mathcal{V} \rangle (x)}$ an algorithm which has black-box access to the protocol $\langle \mathcal{A}, \mathcal{V} \rangle (x)$, which means:
  - $\mathcal{E}$ can read all the messages during interaction
  - $\mathcal{E}$ can rewind the protocol back to any point during the execution, and reexecute the protocol from that point

Knowledge Soundness (Witness-extended emulation)

For every adversary $\mathcal{A}$, there exists an extractor $\mathcal{E}_{\mathcal{A}}$, such that
$\Pr[\mathcal{E}_{\mathcal{A}}^{\langle \mathcal{A}, \mathcal{V} \rangle (x)} \to w : \langle \mathcal{A}, \mathcal{V} \rangle (x) \to 1 \wedge (x, w) \notin \mathcal{R}] < \varepsilon$

Zero Knowledge

Zero Knowledge

Zero-Knowledge Proofs are proof systems that also have

- **Zero-Knowledgeness:** if statement x is true, then the verifier cannot get any information from its view (except the correctness of x)
- The view of the verifier consists of: randomness r and the transcript $\mathrm{tr}\langle\mathcal{P},\mathcal{V}\rangle(x)$

Formally: for any verifier $\mathcal{V}$, there exists a simulator $\mathcal{S}$, which on input a valid statement x, can sample the verifier view, i.e. the distribution of $\mathcal{S}(x)$ is indifferentiable from that of $(r, \mathrm{tr}\langle\mathcal{P},\mathcal{V}\rangle(x))$

Zero Knowledge

ZK has several variations

- **Statistical**: statistical difference $\mathrm{SD}(\mathcal{S}(x), (r, \mathrm{tr}\langle\mathcal{P}, \mathcal{V}\rangle(x)))$ is negligible (zero for perfect ZK)
- **Computational**: for any P.P.T. differentiator $\mathcal{D}$, $|\Pr[D(\mathcal{S}(x)) = 1] - \Pr[D((r, \mathrm{tr}\langle\mathcal{P}, \mathcal{V}\rangle(x))) = 1]|$ is negligible
- **Honest Verifier**: assumes that the verifier follows the protocol (but may be curious, i.e. try to learn some information from the view)

Remark

ZK is a property of the prover.

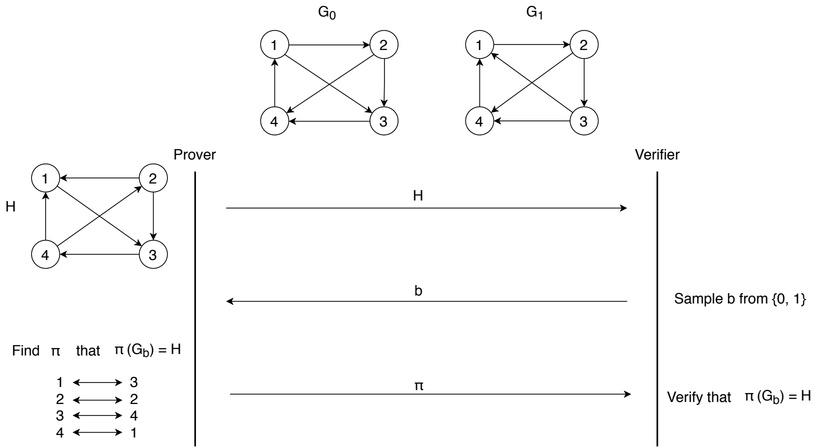
Example: Color Balls

Prove to a blindfold verifier that two balls have different colors without revealing the colors

1. The verifier takes each ball in one hand, and shows the prover
2. The verifier puts the hands behind its back, samples a bit $b \in \{0, 1\}$ in its mind. If $b = 1$, the verifier switches the balls, otherwise it does nothing
3. The verifier shows the balls to the prover, and the prover guesses b'
4. the verifier accepts iff $b' = b$

Zero-knowledge: for any verifier $\mathcal{V}$, the simulator $\mathcal{S}$ does whatever $\mathcal{V}$ does, and in the last step directly sets $b' = b$

Example: Graph Isomorphism



Zero-Knowledge: $\mathcal{S}$ samples the view (H, b, π) as follows:

1. Uniformly sample permutation π and bit b
2. Compute $H = \pi(G_b)$, output (H, b, π)

Example: Graph Isomorphism

Knowledge Soundness: construct the following extractor $\mathcal{E}$, which has black-box control of the protocol execution, and can read the transcript

1. $\mathcal{E}$ observes a full execution and records the transcript (H, b, π_b)
2. $\mathcal{E}$ keeps rewinding the execution back to the point exactly before the verifier samples b, until it observes the verifier sampling $b' \neq b$
3. $\mathcal{E}$ lets the execution proceed and obtains $(b', \pi_{b'})$
4. $\mathcal{E}$ outputs $\pi = \pi_1^{-1} \circ \pi_0$

NIZK

Non-Interactive Zero Knowledge

Non-interactive proof system consists of a single proof π from prover to verifier

You may imagine NIZK works as follows

- $\pi \leftarrow \mathcal{P}(x, w)$
- $0/1 \leftarrow \mathcal{V}(x, \pi)$

Question: does π contain any knowledge?

- Zero-knowledgeness says no, anyone can easily generate it
- Soundness says, if x is hard to decide, as a certificate to its validity, π is also hard to compute

Conclusion: NIZK only exists for easy problems.

Non-Interactive Zero Knowledge

NIZK is only possible in Common Reference String (CRS) model

- Structured Reference String (SRS)
- Uniform Random String (URS)

Additionally, we need at least one of

- Random Oracle (RO) model
- Trusted Third Party (TTP)

Q/A