DESCENT OF PSEUDOCOHERENT AND PERFECT COMPLEXES ON SHEAFY AND LOCALLY TATE ADIC SPACES

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Abstract. These are the notes for my talk on Andreychev's master thesis [And21] on descent theorem for several categories of (quasi)coherent modules on sheafy and locally Tate adic spaces, on 30th October 2024 (part 1). If time permits (though actually not), I will present a little bit of his nuclear-continuous K-theory for sheafy and locally Tate adic spaces [And23] (part 2).

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Part 1. Talk on 30th October 2024

1. Descent theorems

1.1. **Algebraic case.** Let us recall the descent of certain quasi-coherent modules in the algebraic case, by which I mean for schemes.

1.1.1. For any (ordinary) ring A, we denote by $\mathscr{D}(A)$ the (unbounded) derived ∞-category of A-modules (whose homotopy category is the usual derived category $D(A)$ of A-modules).

Recall that, while the (Čech) descent of abelian categories can be formulated in terms of presheaves with values in $\mathrm{Cat}_{(2,1)}$, the $(2,1)$-category of (1-)categories (where 2-morphisms only take account of the natural transformations that are natural equivalences), the descent of derived categories however should be formulated in terms of higher categories, so that we need the ∞-categorical enhancement $\mathscr{D}(A)$ of $D(A)$. Whence, we will put ourselves into the framework of (pre)sheaves with values in Cat_∞, the ∞-category of ∞-categories.

1.1.2. **Definition** (Pseudocoherent and perfect complexes)**.** Let A be an (ordinary) ring and $M \in \mathscr{D}(A)$.

(i) We say that M is a *pseudocoherent complex* if there is a quasi-isomorphism

$$M \simeq (\cdots \to P_{n-2} \to P_{n-1} \to P_n \to 0)$$

where all P_i's are finite projective A-modules (or equivalently, all P_i's are compact projective objects in the heart $\mathscr{D}(A)^{\heartsuit} \simeq \mathrm{Mod}_A$). We denote by

$$\mathrm{PCoh}_A \subset \mathscr{D}(A)$$

the full subcategory of pseudocoherent complexes; more particularly, if the above resolution of M exists, we say M *has tor-amplitude* $\leq n$, and denote by

$$\mathrm{PCoh}_A^{\leq n} \subset \mathscr{D}(A)$$

the full ∞-subcategory of such complexes.

(ii) We say that M is a *perfect complex* if there is a quasi-isomorphism

$$M \simeq (0 \to P_a \to \cdots \to P_b \to 0)$$

where all P_i's are finite projective A-modules. We denote by

$$\mathrm{Perf}(A) \subset \mathscr{D}(A)$$

the full ∞-subcategory of perfect complexes; more particularly, if the above resolution of M exists, we say M *has tor-amplitude in* $[a, b]$, and denote by

$$\mathrm{Perf}^{[a,b]}(A) \subset \mathscr{D}(A)$$

the full ∞-subcategory of such complexes.

1.1.3. **Remark.** Let A be an (ordinary) ring. The tor-amplitude information of a complex in $\mathscr{D}(A)$ is related to its concentration degrees, and to the existence of its resolution with terms chosen from any *given* family of compact projective generators of Mod_A (e.g. $\mathcal{P}$ is set to be the set of all finite *free* A-modules).

(i) We have

$$\mathrm{PCoh}_A^{\leq n} = \mathrm{PCoh}_A \cap \mathscr{D}^{\leq n}(A) \subset \mathscr{D}(A).$$

And any $M \in \mathrm{PCoh}_A^{\leq n}$ admits a resolution as above but with P_i's chosen from any *given* family $\mathcal{P}$.

(ii) We have
$$\mathrm{Perf}^{[a,b]}(A) \subset \mathrm{Perf}(A) \cap \mathscr{D}^{[a,b]}(A),$$
but in general this is not an equality; for example, in $\mathscr{D}(Z)$, the complex $\mathbf{Z}/2 \simeq (\mathbf{Z} \xrightarrow{2\cdot} \mathbf{Z}) \in \mathscr{D}(\mathbf{Z})$ has tor-amplitude in $[-1, 0]$ but not in $[0, 0]$, even though $\mathbf{Z}/2$ is concentrated in degree 0. Moreover, not all $M \in \mathrm{Perf}^{[a,b]}(A)$ admit a resolution as above with P_i's chosen from a *given* family $\mathcal{P}$ of compact projective generators; for example, $\mathbf{Z}/2 \in \mathscr{D}(\mathbf{Z}/6)$ is finite projective, so belongs to $\mathrm{Perf}^{[0,0]}(\mathbf{Z}/6)$, but it is not quasi-isomorphic to any complex $P[0]$ with P being a finite free $\mathbf{Z}/6$-module; even worse, $\mathbf{Z}/2$ does not admit any finite resolution with terms being finite free $\mathbf{Z}/6$-modules.

1.1.4. **Theorem.** *The presheaves on affine schemes with values in* Cat_∞
$$U = \operatorname{Spec} A \mapsto \mathrm{PCoh}_A, \mathrm{PCoh}_A^{\le n}, \mathrm{Perf}(A), \mathrm{Perf}^{[a,b]}(A)$$
all satisfy fpqc (hyper)descent. In particular, these extend respectively to associated sheaves of ∞-categories on all schemes.

Very brief sketch of idea of the proof. The proof can be divided into two steps:

(i) The first step is to prove the descent of the much more general derived category $\mathscr{D}(A)$.

Theorem (Lurie)**.** *The presheaf on affine schemes with values in* Cat_∞
$$U = \operatorname{Spec} A \mapsto \mathscr{D}(A)$$
satisfies fpqc (hyper)descent.

In particular, at the abelian level, one recovers the classical descent of quasi-coherent modules:

Theorem (Grothendieck)**.** *The presheaf on affine schemes with values in* $\mathrm{Cat}_{(2,1)}$
$$U = \operatorname{Spec} A \mapsto \mathrm{QCoh}(U) = \mathrm{Mod}_A$$
satisfies fpqc descent.

(ii) The second step is then to prove that the conditions cutting out the desired full subcategories are descendable, i.e. they can localise and can be checked locally.

□

1.1.5. **Definition.** Let A be an (ordinary) ring. We denote by

$$\mathrm{PCoh}_A^0 := \mathrm{PCoh}_A \cap \mathrm{Mod}_A \subset \mathrm{Mod}_A$$

the full subcategory of *pseudocoherent A-modules*, and denote by

$$\mathrm{FinProj}_A \simeq \mathrm{Perf}^{[0,0]}(A) \subset \mathrm{Perf}(A) \cap \mathrm{Mod}_A \subset \mathrm{Mod}_A$$

the full subcategory of *finite projective A-modules.*

1.1.6. **Corollary.** *The presheaves on affine schemes with values in* $\mathrm{Cat}_{(2,1)}$

$$U = \operatorname{Spec} A \mapsto \mathrm{PCoh}_A^0, \mathrm{FinProj}_A$$

all satisfy fpqc descent.

1.1.7. **Corollary.** *For any (ordinary) ring A, the functor*

$$\mathrm{FinProj}_A \to \mathrm{VB}(\operatorname{Spec} A), \quad M \to \widetilde{M}$$

is an equivalence of categories, with one quasi-inverse given by the functor taking global sections $\Gamma(\operatorname{Spec} A, -)$.

Here, for any scheme X, we denote by $\mathrm{VB}(X)$ the category of vector bundles over X.

1.1.8**. Application: Nisnevich descent of the non-connective K-theory.** The non-connective K-theory functor $\mathbf{K}(-) : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{idem}} \to \mathcal{S}\mathrm{p}$ being a *localising invariant* (i.e. sending Verdier sequences to fiber sequences) and $\mathrm{Perf}(-)$ satisfying fpqc whence *a fortiori* Nisnevich descent and verifying an open-closed excision Verdier sequence, the functor $\mathbf{K}(\mathrm{Perf}(-))$ satisfies Nisnevich descent on schemes.

1.2. **Analytic geometry, I: Huber pairs.**

1.2.1**. Basic geometric objects: overview.** In analytic geometry, one considers analytic functions (i.e. those that can be expressed by a formal power series) that are well-defined over a domain (i.e. convergence of formal power series). There are several ways to make sense of it in the nonarchimedean world.

(i) Let K be a nonarchimedean field which we assume, for simplicity, to be non-trivially valued. Tate started the rigid-analytic geometry by requiring the ring of functions $\mathcal{O}(\mathbf{B}_K^1)$ of the closed unit ball over K to be the ring $K\langle T\rangle$ consisting of convergent power series, i.e. formal power series $\sum_{i\in\mathbf{N}} a_iT^i$ with coefficients $a_i \in K$ converging to 0. This ring is (strongly) Noetherian, and it is naturally equipped with the topology induced from that of K. The corresponding geometric object $\mathbf{B}^1_K$ (in parallel with schemes) is the maximal spectrum of $K\langle T\rangle$ with certain so-called *G-topology* where coverings are *admissible coverings*. In general, we consider the *K-affinoid algebras*, which are nonzero quotients of $K\langle T_1,\dots,T_n\rangle$, and the associated geometric object is the maximal spectrum $\operatorname{Sp} A$ with the G-topology.

(ii) This maximal spectrum $\operatorname{Sp} A$ is missing points, just as maximal spectrum of an algebra miss points corresponding to prime ideals that are not maximal. To have better understanding of the geometry of $\operatorname{Sp} A$, one needs to pass to finer spectra. There are two mainstreams: Berkovich spaces and Huber's adic spaces.

(iii) Berkovich came up with the idea that one should look at (rank 1) valuations on residue fields at points of the prime spectrum to fill up the missing points. It turns out that the points are well filled up, so that we obtain the Berkovich spectrum $\mathcal{M}(A)$, which, equipped with again certain G-topology, is a compact Hausdorff topological space.

(iv) However, on Berkovich spaces, the coverings are still *admissible coverings*, but not all open coverings, which remains an unsatisfying flaw. For example, on the rigid-analytic unit disk as well as the Berkovich closed unit disk $\mathbf{B}^1_K = \operatorname{Sp} K\langle T\rangle$, the family of rational open immersions

$$\left(\coprod_{n\geq 1}\{|T|\leq |\pi|^{1/p^n}\}\right)\amalg\{|T|=1\}\to\mathbf{B}^1_K$$

is jointly surjective but is not an admissible covering. To remedy the non-admissibility, one may take Huber's point of view, to see that there exists a rank 2 valuation η^- on $K\langle T\rangle$ such that $|T|_{\eta^-}<1$ but $|T|_{\eta^-}\geq|\pi|^{1/p^n}$ for any $n\geq 1$, so that this rank 2 point is in fact missed by the above open coverings. More concreteley, such $|\cdot|_{\eta^-}$ takes values in the totally ordered abelian group

$$\Gamma=(\mathbf{R}_{>0},\times)\times(\mathbf{R}_{>0},\times)$$

with the alphabetical order

$$(a,b)\leq(a',b')\overset{\text{def}}{\Leftrightarrow}(a<a')\vee(a=a'\wedge b\leq b'),$$

and can be represented by

$$|\cdot|_{\eta^-}:K\langle T\rangle\to\Gamma\cup\{0\},\quad a\mapsto(|a|_K,1),\quad T\mapsto(1,\frac{1}{2}).$$

(v) By adding higher rank valuations to the Berkovich spectrum, Huber succeeded in defining the *adic spectrum* $\operatorname{Spa}(A,A^\circ)$ in place of $\operatorname{Sp} A$ or $\mathcal{M}(A)$, whose coverings are simply given by open coverings (which can in turn be refined by rational open coverings). Although $\operatorname{Spa}(A,A^\circ)$ is not Hausdorff any more, but it is a spectral space and the good properties of open coverings will allow us to employ very algebraic methods to study the analytic geometry, just as we can use algebraic methods to study algebraic geometry/schemes.

We will use the language of Huber's adic spaces. This notion generalises to a much bigger class of objects than K-affinoid algebras. Let us introduce them now.

1.2.2. **Definition.** A *(complete) Huber ring* is a topological ring A such that, there exists an open subring A_0 and a finitely generated ideal I of A_0 such that the induced topology on A_0 is the I-adic topology (and A_0 is I-adically complete).

We call such subring $A_0 \subset A$ a *subring of definition* and $I \subset A_0$ an *ideal of definition*, and we call the pair (A_0, I) a *pair of definition* of A.

1.2.3**.** Given a Huber ring A, there are

(i) a notion of *power bounded elements*; we denote by A° the subset (in fact an open subring) of all power bounded elements of A;
(ii) a notion of *topologically unipotent elements*, and we denote by $A^{\circ\circ}$ the subset (in fact an open ideal of A°) of all topologically unipotent elements of A.

1.2.4. **Example.** Any ordinary ring R with discrete topology is a Huber ring, with a pair of definition $(R, 0)$. We have $R^\circ = R$ and $R^{\circ\circ} = 0$.

1.2.5. **Example.** Let K be a (non-trivially valued) nonarchimedean field with a pseudouniformiser $\pi \in K^\times \cap \mathfrak{m}_K = \mathfrak{m}_K \backslash \{0\}$. Then any K-affinoid algebra A is a Huber ring. If we write $A = K\langle T_1, \ldots, T_n\rangle / I$, then the image of the pair $(\mathcal{O}_K\langle T_1, \ldots, T_n\rangle, (\pi))$ is a pair of definition of A. We have

$$K\langle T_1, \ldots, T_n\rangle^\circ = \mathcal{O}_K\langle T_1, \ldots, T_n\rangle, \quad K\langle T_1, \ldots, T_n\rangle^{\circ\circ} = \mathfrak{m}_K\langle T_1, \ldots, T_n\rangle.$$

The Huber ring A itself is not enough for constructing adic spaces, we need some auxiliary data A^+ to specify a "reference of boundedness", i.e. with respect to which "lattice" of A we want to talk about boundedness. It will be an analogue of $\mathbf{Z}_p$-lattices in a $\mathbf{Q}_p$-Banach spaces.

1.2.6. **Definition.** A *(complete) Huber pair* (A, A^+) consists of a (complete) Huber ring A and an integrally closed open subring $A^+ \subset A^\circ$.

1.2.7. **Example.** Examples of complete Huber pairs:

(i) discrete Huber pairs (R, R^+), where R is an ordinary ring with discrete topology and $R^+ \subset R$ is any integrally closed subring of R^+; for example, $(\mathbf{Z}[T], \mathbf{Z})$, $(\mathbf{Z}[T], \mathbf{Z}[T])$, $(\mathbf{Z}[T^{\pm 1}], \mathbf{Z}[T^{-1}])$;

(ii) $(K\langle T_1,\dots,T_n\rangle, \mathcal{O}_K\langle T_1,\dots,T_n\rangle)$;
(iii) $(K\left\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\right\rangle, \mathcal{O}_K\left\langle T_1^{1/p^\infty},\dots,T_n^{1/p^\infty}\right\rangle)$, which is non-Noetherian;

The "integrally closed" assumption is not so essential. One may also take a subset $S \subset A^\circ$ to form a pair (A, S), then the theory of adic spaces as well as analytic rings that we will introduce does not distinguish (A, S) from $(A, \overline{S \cup A^{\circ\circ}})$ where $\overline{S \cup A^{\circ\circ}}$ is the integral closure of the subring generated by $S \cup A^{\circ\circ}$, or equivalently the smallest integrally closed open subring generated by S. For this reason, we may write in the second argument of a Huber pair a subset that is not integrally closed (even nor a subring), and call such (A, S) a *pre-Huber pair*.

1.2.8. **Example.** Examples of pre-Huber pairs:

(i) $(R, \mathbf{Z})$, where R is an ordinary ring with discrete topology;
(ii) $(K\langle T_1,\dots,T_n\rangle, \mathcal{O}_K)$, whose associated Huber pair is $(K\langle T_1,\dots,T_n\rangle, \mathcal{O}_K+\mathfrak{m}_K\langle T_1,\dots,T_n\rangle)$

1.3. **Analytic geometry, II: affinoid adic spaces.** Next, we are going to explain the notion of *sheafy* and *locally Tate* Huber pairs. In order to make sense of it, firstly we need some knowledge of geometric objects attached to Huber pairs, which are *affinoid adic spaces*.

1.3.1. **Definition.** Let (A, A^+) be a Huber pair. The *adic spectrum* of it or *affinoid adic space* attached to it is the set of valuations satisfying two conditions:

$$\operatorname{Spa}(A, A^+) := \left\{ |\cdot|_v : A \to \Gamma_v \cap \{0\} \;\middle|\; |A^{\circ\circ}|_v < 1, \left|A^+\right|_v \leq 1 \right\}_{/\simeq}.$$

Here, a valuation $|\cdot|_v$ satisfies the first condition (or equivalently $|I|_v < 1$) if and only if it is continuous with respect to the topology of A and the order topology (by requiring $\{a \in \Gamma \mid a < \gamma\}$ and $\{a \in \Gamma \mid a > \gamma\}$ to be open for all $\gamma \in \Gamma$, or equivalently, if $\Gamma \neq \{1\}$ is not the trivial group, requiring $\{a \in \Gamma \mid a \leq \gamma\}$ to be open and closed for all $\gamma \in \Gamma$), in which case we call $|\cdot|_v$ a *continuous valuation*. This first condition does not concern the "reference of boundedness" datum A^+.

The second condition, however, is where this datum A^+ speaks. It picks out the valuations that are bounded on what should be bounded, i.e. bounded on A^+. Admitting the first condition, it is easy to see that the condition $|S|_v \leq 1$ is insensitive to two operations: taking the subring generated by $S \cup A^\circ$, and taking the integral closure of S; this reflects our saying that (A, S) and $(A, \overline{S \cup A^{\circ\circ}})$ are undistinguishable (in geometry).

Two valuations on A are said to be *equivalent* if they induce the same partial order relation on A.

1.3.2. **Remark.** There is a map

$$\mathrm{Spa}(A, A^+) \to \mathrm{Spec}\, A, \quad |\cdot|_v \mapsto \ker(|\cdot|_v) = \{f \in A \mid |f|_v = 0\}.$$

The fibre over a point $\mathfrak{p} \in \mathrm{Spec}\, A$ consists of valuations (of any rank) on the residue field $\kappa(\mathfrak{p})$ such that the image of A^+ has valuations ≤ 1. However, certain fibres may be empty.

1.3.3. **Definition.** Let (A, A^+) be a Huber pair. We define the *analytic topology* on $\mathrm{Spa}(A, A^+)$ as the coarsest topology such that the map

$$\mathrm{ev}_f : \mathrm{Spa}(A, A^+) \to \prod_{v \in \mathrm{Spa}(A, A^+)} (\Gamma_v \cup \{0\}), \quad v \mapsto |f|_v$$

is continuous for any $f \in A$, where we put the product topology of valuation topologies on the right hand side.

The topology on affine schemes has a basis given by principal open subsets. Similarly, the topology on affinoid adic spaces has a basis given by *rational open subsets*.

1.3.4. **Definition.** Let (A, A^+) be a Huber pair. A *rational open subsets* of $\mathrm{Spa}(A, A^+)$ is a subset of the form

$$\mathrm{Spa}(A, A^+)\left(\frac{f_1, \ldots, f_n}{g}\right) = \left\{|\cdot|_v \in \mathrm{Spa}(A, A^+) \;\Big|\; |f_i|_v \leq |g|_v \neq 0, \ {}^\forall i = 1, \ldots, n\right\}$$

where $f_1, \ldots, f_n, g$ generate an open ideal of A.

It easy to see from definition of the analytic topology on affinoid adic spaces that the rational open subsets form a basis (the key point being that any finite intersection of rational open subsets is still a rational open subset), and that they are an analogue of principal open subsets of affine schemes.

1.3.5. **Remark.** Recall that a principal open subset $D(f)$ of an affine scheme $\mathrm{Spec}\, R$ is obtained by requiring f to be non-vanishing. Similarly, a rational open subset is cut out by two steps:

(i) requiring g to be non-vanishing,
(ii) then requiring the elements $\frac{f_1}{g}, \ldots, \frac{f_n}{g}$ to have valuation ≤ 1.

And on the level of Huber pairs (A, A^+), this translates to:

(i) Inverting g in the first argument A, to get $A[\frac{1}{g}]$;
(ii) Adjoining $\frac{f_1}{g}, \ldots, \frac{f_n}{g}$ to the second argument A^+, to get $A^+[\frac{f_1}{g}, \ldots, \frac{f_n}{g}]$.

This will be the intuitive slogan of the localisation of Huber pairs.

One instance of this slogan is illustrated by the following proposition:

1.3.6. **Proposition.** *Let (A, A^+) be a Huber pair and $U \subset \mathrm{Spa}(A, A^+)$ be a rational open subset defined by $f_1, \ldots, f_n, g$ (generating an open ideal of A). There exists a universal Huber pair (A_U, A_U^+) initial among all maps of Huber pairs $f^\sharp : (A, A^+) \to (B, B^+)$ such that the induced map (of topological spaces) $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors through the subset U. More concretely, (A_U, A_U^+) can be described as the completion of the Huber pair $(A[\frac{1}{g}], A^+[\frac{f_1}{g}, \ldots, \frac{f_n}{g}])$.*

Here, the *completion* of the Huber pair $(B, B^+) := (A[\frac{1}{g}], A^+[\frac{f_1}{g}, \ldots, \frac{f_n}{g}])$ goes as follows: given any pair of definition (A_0, I) of A, we

(i) form the pair of definition $(A_0[\frac{f_1}{g}, \ldots, \frac{f_n}{g}], IA_0[\frac{f_1}{g}, \ldots, \frac{f_n}{g}])$ of (B, B^+),
(ii) then I-adically complete it,
(iii) then tensor it by $- \otimes_{A_0[\frac{f_1}{g}, \ldots, \frac{f_n}{g}]} B$ to get $\widehat{B}$,
(iv) then take the closure of the image of B^+ in $\widehat{B}$,

to finally get the completion

$$(A_U, A_U^+) = (\widehat{B}, \widehat{B}^+) =: \left(A\left\langle \frac{1}{g} \right\rangle, A^+\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \right\rangle\right).$$

By universality of (A_U, A_U^+), the resulting complete Huber pair $(A\left\langle \frac{1}{g} \right\rangle, A^+\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \right\rangle)$ does not depend on the choice of $f_1, \ldots, f_n, g$ (generating an open ideal of A) that defines the rational subset U.

1.3.7. **Proposition.** *Let (A, A^+) be a Huber pair and $U \subset \mathrm{Spa}(A, A^+)$ be a rational open subset. Then the map $(A, A^+) \to (A_U, A_U^+)$ induces a homeomorphism*

$$\mathrm{Spa}(A_U, A_U^+) \xrightarrow{\simeq} U.$$

Therefore, we can say that $(A, A^+) \to (A_U, A_U^+)$ is a *rational localisation.* Moreover, if $\{U_i\}_i$ is a *rational open covering of* $\mathrm{Spa}(A, A^+)$, i.e. an open covering by rational open subsets, then we say that $\{(A_{U_i}, A_{U_i}^+)\}_i$ is a *rational open covering of the Huber pair* (A, A^+).

1.3.8. **Example.** Consider the discrete Huber pair $(\mathbf{Z}[T], \mathbf{Z})$. We have a rational open covering

$$\mathrm{Spa}(\mathbf{Z}[T], \mathbf{Z}[T]) \amalg \mathrm{Spa}(\mathbf{Z}[T^{\pm 1}], \mathbf{Z}[T^{-1}]) \to \mathrm{Spa}(\mathbf{Z}[T], \mathbf{Z}). \tag{1.3.8.1}$$

We have the following intuitive interpretation of this covering:

(i) $\mathrm{Spa}(\mathbf{Z}[T],\mathbf{Z})=\mathbf{A}^1_{\mathbf{Z}}$ is the adic affine line over $\mathbf{Z}$.

(ii) $\mathrm{Spa}(\mathbf{Z}[T],\mathbf{Z}[T])=\{|T|\leq 1\}=\mathbf{B}^1_{\mathbf{Z}}$ is the adic closed unit disk over $\mathbf{Z}$, which is a rational open subset.

(iii) $\mathrm{Spa}(\mathbf{Z}[T^{\pm 1}],\mathbf{Z}[T^{-1}])=\{|T|\geq 1\}=\iota(\mathbf{B}^1_{\mathbf{Z}}\backslash\{0\})$ is the adic punctured closed unit disk over $\mathbf{Z}$ centered at ∞ (where ι denotes the usual involution $T\mapsto T^{-1}$ of the projective line), which is also a rational open subset.

1.3.9. **Example.** Let K be a (non-discretely valued) non archimedean field. Consider the discrete pre-Huber pair $(K\langle T\rangle,\mathcal{O}_K)$, with associated discrete Huber pair $(K\langle T\rangle,\mathcal{O}_K+\mathfrak{m}_K\langle T\rangle)$. It fits into the base change diagram

$$\begin{array}{ccc}\mathrm{Spa}(K\langle T\rangle,\mathcal{O}_K) & \longrightarrow & \mathrm{Spa}(\mathbf{Z}[T],\mathbf{Z})\\ \downarrow & & \downarrow\\ \mathrm{Spa}(K,\mathcal{O}_K) & \longrightarrow & \mathrm{Spa}(\mathbf{Z},\mathbf{Z})\end{array}$$

We have a rational open covering

$$\mathrm{Spa}(K\langle T\rangle,\mathcal{O}_K\langle T\rangle)\amalg\mathrm{Spa}(K\left\langle T^{\pm 1}\right\rangle,\mathcal{O}_K\left\langle T^{-1}\right\rangle)\to\mathrm{Spa}(K\langle T\rangle,\mathcal{O}_K),$$

or in terms of Huber pairs

$$\mathrm{Spa}(K\langle T\rangle,\mathcal{O}_K\langle T\rangle)\amalg\mathrm{Spa}(K\left\langle T^{\pm 1}\right\rangle,\mathcal{O}_K\left\langle T^{-1}\right\rangle+\mathfrak{m}_K\langle T\rangle)\to\mathrm{Spa}(K\langle T\rangle,\mathcal{O}_K+\mathfrak{m}_K\langle T\rangle).$$

We have the following intuitive interpretation of this covering:

(i) $\mathrm{Spa}(K\langle T\rangle,\mathcal{O}_K)=\overline{\mathbf{B}}^1_K$ is the adic compactified closed unit disc over K.

(ii) $\mathrm{Spa}(K\langle T\rangle,\mathcal{O}_K\langle T\rangle)=\{|T|\leq 1\}=\mathbf{B}^1_K$ is the adic closed unit disk over K, which is a rational open subset.

(iii) $\mathrm{Spa}(K\langle T^{\pm 1}\rangle,\mathcal{O}_K\langle T^{-1}\rangle)=\{|T|\geq 1\}=\overline{\mathbf{B}}^1_K\cap\iota(\mathbf{B}^1_K)$ is the adic outerly-compactified torus over K (where ι denotes the usual involution $T\mapsto T^{-1}$ of the projective line), which is also a rational open subset.

1.3.10. **Example.** Let (A,A^+) be a Huber pair and $f\in A$. Let us now consider the map

$$(\mathbf{Z}[T],\mathbf{Z})\to(A,A^+),\quad T\mapsto f,$$

along which the rational open covering (1.3.8.1) is base changed to the rational open covering

$$\{|f|\leq 1\}\cup\{|f|\geq 1\},$$

or in terms of pre-Huber pairs

$$\mathrm{Spa}(A\langle f\rangle, A^+\langle f\rangle) \amalg \mathrm{Spa}(A\left\langle \frac{1}{f}\right\rangle, A^+\left\langle \frac{1}{f}\right\rangle) \to \mathrm{Spa}(A, A^+).$$

Now it is time to explain the term *locally Tate*.

1.3.11. **Definition** (Locally Tate Huber pairs)**.** Let (A, A^+) be a (complete) Huber pair.

(i) The Huber ring A or the Huber pair (A, A^+) is called *Tate* if A admits a topologically nilpotent unit, i.e. if there exists an element $\varpi \in A^{\circ\circ} \cap A^\times$.

(ii) (A, A^+) is called *locally Tate* (or *analytic* following Kedlaya's naming) if it becomes Tate after passing to a rational open covering, i.e. it admits a rational open covering $\{(A_i, A_i^+)\}_i$ with all (A_i, A_i^+)'s being Tate.

1.3.12. **Remark** (Kedlaya, [Ked17, Lemma 1.1.3])**.** A (complete) Huber pair (A, A^+) is locally Tate if and only if $A^{\circ\circ}$ generates the unit ideal of A. So this notion depends only on A.

1.3.13. **Example.** The following Huber rings are Tate:

(i) Any K-affinoid algebra with K non-trivially valued nonarchimedean field;

(ii) Any perfectoid ring;

(iii) The ring of continuous functions $C(S, A)$ where S is a profinite set and A is a Tate Huber ring.

Next, we explain the term *sheafy*.

1.3.14. **Definition** (Sheafy Huber pairs)**.** Let (A, A^+) be a Huber pair and $X := \mathrm{Spa}(A, A^+)$ be the associated affinoid adic space.

(i) We define the structural presheaf on X by sending any open subset $U \subset X$ to

$$\mathcal{O}_X(U) = \varprojlim_{V \subset U} A_V.$$

where V runs through all rational open subsets of X contained in U.

(ii) We say that (A, A^+) is sheafy if $\mathcal{O}_X$ is a sheaf (of sets/rings) for the analytic topology on X.

1.3.15. **Remark** (Kedlaya, [Ked17, Remark 1.6.9])**.** Whether a locally Tate (complete) Huber pair (A, A^+) is sheafy depends only on A.

1.3.16. **Example.** Although exotic non-sheafy examples exist, in practice, every Huber pair that we will encounter is sheafy:

(i) If R is discrete, then (R, R^+) is sheafy.
(ii) If A is Tate and strongly Noetherian (e.g. a K-affinoid algebra with K non-trivially valued nonarchimedean field), then (A, A^+) is sheafy.
(iii) If A is a *sousperfectoid* ring (i.e. there exists a continuous A-linear morphism from A to a (rational) perfectoid ring B that splits in the category of topological A-modules), then (A, A^+) is sheafy, and more generally $(C(S, A), C(S, A^+))$ is sheafy.

1.4. **Analytic case.**

1.4.1. **Theorem.** *Let* (A, A^+) *be a sheafy complete Huber pair. Then the presheaves on rational open subsets of* $\mathrm{Spa}(A, A^+)$ *with values in* Cat_∞

$$U \mapsto \mathrm{PCoh}_{A_U},\ \mathrm{PCoh}_{A_U}^{\leq n},\ \mathrm{Perf}(A_U),\ \mathrm{Perf}^{[a,b]}(A_U)$$

all satisfy analytic descent.

1.4.2. **Remark.** In the original article, there is a "locally Tate" assumption. But it turns out to be unnecessary except for the descent of stably uniform pseudocoherent modules PCoh_A^0, see the remark (3.0.2).

1.4.3. **Corollary.** *For any sheafy complete Huber pair* (A, A^+)*, the functor*

$$\mathrm{FinProj}_A \to \mathrm{VB}_{\mathrm{an}}(\mathrm{Spa}(A, A^+)), \quad M \mapsto (\widetilde{M} : U \mapsto M \otimes_A A_U)$$

is an equivalence of categories, with one quasi-inverse given by the functor taking global sections $\Gamma(\mathrm{Spa}(A, A^+), -)$.

1.4.4. **Remark.** The étale site over a sheafy adic space, e.g. for simplicity $\mathrm{Spa}(A, A^+)$, is generated by rational open coverings of (A, A^+) and finite étale coverings of A. For a finite étale covering $A \to B$, we have

$$\mathrm{Perf}(B) = \mathrm{Mod}_B(\mathrm{Perf}(A)),$$

so $\mathrm{Perf}(-)$ satisfies finite étale descent. Altogether, $\mathrm{Perf}(-)$ satisfies étale descent, suggesting that $\mathbf{K}(\mathrm{Perf}(-))$ satisfies étale descent on $\mathrm{Spa}(A, A^+)$. However, the thus defined K-theory does not admit an open-closed (excision) fiber sequence. From this perspective, the better substitute for $\mathrm{Perf}(-)$ would be the $\mathrm{Nuc}(-)$ that we will encounter later during the proof.

We will define $\mathrm{Nuc}(-)$ in the future, but let us record the following properties of it:

(i) Let A be a complete Huber ring, then $\mathrm{Nuc}((A, A^+)_{\blacksquare})$ is independent of A^+. We denote it by $\mathrm{Nuc}(A)$. It is a *dualisable category*, or equivalently, a dualisable object in $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$.

(ii) Let A, B be Tate Huber rings, and $A \to B$ a finite étale map. Then we have

$$\mathrm{Nuc}(B) \simeq \mathrm{Mod}_B(\mathrm{Nuc}(A)) \simeq \mathrm{Nuc}(A) \otimes_{\mathcal{C}\mathrm{ond}_A, \mathrm{Perf}(A)} \mathrm{Perf}(B) \simeq \mathrm{Nuc}(A) \otimes_{\mathcal{C}\mathrm{ond}_A, \mathscr{D}(A)} \mathscr{D}(B).$$

By [Mat16, Corollary 3.42], the natural map $\mathrm{Nuc}(A) \to \varprojlim_{n\in\Delta} \mathrm{Nuc}(B^{\otimes_A (n+1)})$ is an equivalence in Cat_∞ (or equivalently in $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$, since the faithful embedding $\mathcal{P}\mathrm{r}^L \hookrightarrow \mathrm{Cat}_\infty$ preserves limits by [Lur09, Proposition 5.5.3.13]).

(iii) The étale topology on an adic space is generated by the analytic topology and finite étale maps. Hence, we obtain a sheaf $\mathrm{Nuc}(-)$ satisfying étale descent on sheafy and locally Tate adic spaces. It also satisfies excision as promised.

(iv) An appropriate K-theory machine $\mathbf{K}_{\mathrm{cont}} : \mathrm{Pr}^{\mathrm{dual}}_{\mathrm{st}} \to \mathcal{S}\mathrm{p}$, which is required to be a *localising invariant*, will produce a *nuclear-continuous K-theory* $\mathbf{K}^{\mathrm{nuc}}(-) := \mathbf{K}_{\mathrm{cont}}(\mathrm{Nuc}(-))$ satisfying Nisnevich descent. The subtlety in the definition of $\mathbf{K}_{\mathrm{cont}}(-)$ will be addressed in the supplementary subsection 6.2.

1.4.5. **State of art.** Let us review related previous results on descent of $\mathrm{Perf}(A)$ and PCoh_A in analytic geometry.

(i) Flat descent of Coh_A for K-affinoid algebras was proved by Bosch-Görtz-Gabber.

(ii) Descent of $\mathrm{FinProj}_A$ was proved by Kedlaya-Liu in the (sheafy and) Tate case and later generalised by Kedlaya in the (sheafy and) locally Tate case. Their method is quite direct, involving properties of topological modules.

(iii) Let $\pi \in K$ be a (nonzero) pseudouniformiser. Let $\mathcal{A}\mathrm{lg}^\flat_{\mathcal{O}_K}$ denote the category of $\mathcal{O}_K$-algebras that are π-torsion-free and π-adically complete. Drinfeld proved the π-completely faithfully flat descent of $\mathrm{VB}(R[\frac{1}{\pi}])$ for $R \in \mathcal{A}\mathrm{lg}^\flat_{\mathcal{O}_K}$. Akhil Mathew generalised this to $\mathrm{PCoh}^{(\leq n)}_{R[\frac{1}{\pi}]}$ and $\mathrm{Perf}^{([a,b])}(R[\frac{1}{\pi}])$ [Mat22].

(iv) In the same article, Akhil Mathew proved flat hyperdescent of $\mathrm{PCoh}^{(\leq n)}_A$ and $\mathrm{Perf}^{([a,b])}(A)$ for K-affinoid algebras. The key is that the faithfully flatness of a map between such objects $A \to A'$ implies universal descendability of maps of integral models in $\mathcal{A}\mathrm{lg}^\flat_{\mathcal{O}_K}$ (see [Mat22, Proposition 6.14]).

1.4.6. **Difficulties.** Compared with the proof in the algebraic case, the difficulty is to find suitable replacement of the category $\mathscr{D}(A)$ of "quasi-coherent complexes" in analytic geometry. Since the rational localisation involves completion process and is thus finer that Zariski localisation, one must consider certain category of topological A-modules to capture the information lost in this process in order to get descent (in fact, $\mathscr{D}(A)$ does not satisfy analytic descent on $\mathrm{Spa}(A, A^+)$).

However, the naive choice of topological A-modules will not work, since they do not even form an abelian category (for example, consider the map $\widehat{\bigoplus}_{\mathbf{N}}\mathbf{Q}_p \to \widehat{\bigoplus}_{\mathbf{N}}\mathbf{Q}_p$ given by multiplcation by n on the n-th factor, its image and coimage do not coincide). Even worse, there is no canonical (complete) topological A-module structure on finitely presented A-modules when A is non-Noetherian[1].

Lemma. *There exist a (commutative unital) ring R and two elements $x, t \in R$, such that*

(i) R is x-adically complete;

(ii) R/t is not x-adically complete.

Proof. See the discussion in [Sta18, Section 05JD]. To recall the construction there, let

$$R = k[t, z_1, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, x]/(z_i t - x^i w_i, z_i w_j)$$

with x and t be the evident elements of R. Indeed, the element

$$f = \sum_{i=1}^{\infty} w_i x^i$$

belongs to the ideal (t, x^n) because $x^m w_m \in (t)$ for all m; but one can show that $f \notin (t)$. In fact, we show this as follows.

If $f = tg$, then g has the (unique) expansion $g = \sum_{i=l} g_l x^l$ with $g_l \in k[t, z_l, w_l]$ and g_l not involving tz_l or $z_l w_l$. Since $tz_i z_j = 0$ we may as well assume that none of the g_l have terms involving the products $z_i z_j$. Examining the process to get tg in canonical form we see the following: Given any term cm of g_l where $c \in k$ and m is a monomial in t, z_i, w_j and we make the following replacement

- if the monomial m does not involve any z_i, then ctm is a term of f_l, and
- if the monomial m does involve a z_i then it is equal to $m = z_i$ and we see that cw_i is term of f_{l+i}.

Since g_0 is a polynomial only finitely many of the variables z_i occur in it. Pick n such that z_n does not occur in g_0. Then the rules above show that w_n does not occur in f_n which is a contradiction. It follows that $R^\wedge/(t)$ is not complete, see [Sta18, Lemma 031A]. □

[1]We know that for a Noetherian ring R complete with respect to an ideal I (which is necessarily finitely generated), all the finitely generated R-modules (which are automatically finitely presented) are I-adically complete; so it makes sense to consider the (abelian) category of finite generated R-modules that are I-adically complete.

Facing these difficulties, Akhil Mathew's trick is to consider, for $A = R[\frac{1}{\pi}]$ with $R \in \mathcal{A}\mathrm{lg}^{\flat}_{\mathcal{O}_K}$, certain category $\mathcal{M}(A)$ between the desired category for descent and the usual $\mathscr{D}(A)$:

$$\mathrm{PCoh}_A \hookrightarrow \mathcal{M}(A) \to \mathscr{D}(A),$$

where $\mathcal{M}(A)$ is roughly speaking the left-completed π-isogeny category of bounded above derived π-complete complexes in $\mathscr{D}(R)$, such that:

- The first arrow is fully faithful embedding;
- The second arrow is t-exact, conservative;
- $\mathcal{M}(R[\frac{1}{\pi}])$ satisfies π-completely faithfully flat descent on $\mathcal{A}\mathrm{lg}^{\flat}_{\mathcal{O}_K}$ (by monadicity theorem of Barr-Beck-Lurie).

Despite the capacity of proving strong *flat* descent results, his approach has a drawback that the intermediate descendable category $\mathcal{M}(A)$ is not closed under taking colimits (even the simplest countable direct sums), which should be a crucial property of the candidate of the category of "(topological) quasi-coherent complexes" over (A, A^+).

Nevertheless, one should remark that the use of derived π-completeness suggests an algebraic way of defining topological module structures.

We will introduce a precise analogue of "quasi-coherent complexes" over (A, A^+) in the next section, using Clausen-Scholze's notion of *analytic rings*.

2. "Quasi-coherent modules" on affinoid adic spaces, following Clausen-Scholze

Our candidate for the ∞-category of "quasi-coherent modules" over an affinoid adic space $\mathrm{Spa}(A, A^+)$ will be the full subcategory $\mathscr{D}((A, A^+)_{\blacksquare}) \subset \mathscr{D}(\underline{A})$ consisting of $(A, A^+)_{\blacksquare}$-modules. This section is devoted to explaining what these modules are.

2.1. **Condensed mathematics.**

For a topological ring A, the category of usual topological modules is often pathological: coimage and image may not coincide. But our belief is that one should be to do homological algebra of *topological* modules! Indeed, it is plausible to do so with *condensed* modules (in the language of condensed mathematics).

2.1.1. **Definition.** Let $\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}$ be the category of profinite sets with finitary topology, and EDS be its subcategory of *extremally disconnected sets* (i.e. any surjective map from a profinite set to it splits).

2.1.2. **Definition.** The terminology *condensed* will refer to looking at sheaves on the site $\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}$ or EDS. (The idea is that the condensed structure records the topological information on algebraic objects by testing with continuous functions from profinite sets.)

(i) We define the *category of condensed sets* as

$$\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et}) := \mathcal{S}\mathrm{hv}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}, \mathcal{S}\mathrm{et}) \simeq \mathcal{S}\mathrm{hv}(\mathrm{EDS}, \mathcal{S}\mathrm{et}).$$

(ii) We define the *category of condensed abelian groups*

$$\mathcal{C}\mathrm{ond}(\mathcal{A}\mathrm{b}) := \mathcal{S}\mathrm{hv}(\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}, \mathcal{A}\mathrm{b}) \simeq \mathcal{S}\mathrm{hv}(\mathrm{EDS}, \mathcal{A}\mathrm{b}),$$

which is also the category of abelian group objects among condensed sets.

(iii) Similarly, we have the notion of *condensed monoids/groups/rings*, etc.

(iv) Let A be a topological space, we define its associated condensed set as

$$\underline{A} : \mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}, \quad S \mapsto C(S, A),$$

where $C(S, -)$ denotes the set of continuous functions from S to a topological space. If A is a topological group/abelian group/monoid/rings, etc., then so is $\underline{A}$.

(v) Let A be a topological ring, we define the *category of condensed* $\underline{A}$*-modules* as the category of $\underline{A}$-module objects among condensed sets

$$\mathrm{Mod}_{\underline{A}} := \mathrm{Mod}_{\underline{A}}(\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})) = \mathrm{Mod}_{\underline{A}}(\mathcal{C}\mathrm{ond}(\mathcal{A}\mathrm{b})).$$

In particular, we have $\mathcal{C}\mathrm{ond}(\mathcal{A}\mathrm{b}) = \mathrm{Mod}_{\underline{\mathbf{Z}}}$.

We record the following theorem on the structure of $C(S, \mathbf{Z})$ for future reference:

2.1.3. **Theorem** (Specker, Nöbeling)**.** *For any profinite set S, the abelian group $C(S, \mathbf{Z})$ is a free abelian group.*

2.1.4**. Notation.** For any profinite set S, we will often denote by I a set such that $C(S, \mathbf{Z}) \simeq \bigoplus_I \mathbf{Z}$. Note that the choice of I is not canonical but this non-canonicality will be of little importance.

Condensed modules have really nice homological algebraic behaviours:

2.1.5. **Proposition.** *Let A be a topological ring A.*

(i) The category $\mathrm{Mod}_{\underline{A}}$ *is a Grothendieck abelian category, satisfying the axioms:*
- *(AB3) All colimits exist;*
- *(AB3*) All limits exist;*

- *(AB4) All direct sums are exact;*
- *(AB4*) All products are exact;*
- *(AB5) All filtered colimits are exact;*
- *(AB6) For any index set J and filtered categories $I_j, j \in J$ with functors*

$$I_j \to \mathrm{Mod}_{\underline{A}}, \quad i \mapsto M_i,$$

the natural map

$$\varinjlim_{(i_j \in I_j)_j} \prod_{j \in J} M_{i_j} \to \prod_{j \in J} \varinjlim_{(i_j \in I_j)} M_{i_j}$$

is an isomorphism.

(ii) Moreover, $\mathrm{Mod}_{\underline{A}}$ *is generated by a family of compact projective objects; more precisely,*

- *for any $S \in \mathrm{EDS}$, the free $\underline{A}$-module over S:*

$$\underline{A}[S] = \begin{pmatrix} \mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}^{\mathrm{op}} & \to & \mathcal{S}\mathrm{et} \\ T & \mapsto & C(T, A)[C(T, S)] \end{pmatrix}^{\text{sheafify}}$$

is a compact projective object in $\mathrm{Mod}_{\underline{A}}$*;*

- *the family*

$$\mathcal{P}_{\underline{A}} := \{\underline{A}[S], S \in \mathrm{EDS}\}$$

generates the category $\mathrm{Mod}_{\underline{A}}$.

(iii) The category $\mathrm{Mod}_{\underline{A}}$ *is equipped with a canonical structure of symmetric monoidal category* $(\mathrm{Mod}_{\underline{A}}, \otimes_{\underline{A}}, \underline{A})$*, which is closed, with canonical internal Hom object functor* $\underline{\mathrm{Hom}}_{\underline{A}}(-,-)$.

2.1.6. **Example.** Here is an easy compuation:

$$(R)\,\mathrm{Hom}_{\underline{A}}(\underline{A}[S], \underline{A}) = \mathrm{Hom}_{\mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{et})}(\underline{S}, \underline{A}) = \underline{A}(S) = C(S, A),$$

where the last equality use the definition of $\underline{A}$.

The proposition holds for condensed modules over condensed rings (not necessarily coming from topological rings).

2.1.7. **Definition.** We define the derived ∞-category of $\underline{A}$-modules as the derived ∞-category of the abelian category $\mathrm{Mod}_{\underline{A}}$

$$\mathscr{D}(\underline{A}) := \mathscr{D}(\mathrm{Mod}_{\underline{A}}).$$

It is equipped with a canonical structure of symmetric monoidal category $(\mathscr{D}(\underline{A}), \otimes^L_{\underline{A}}, \underline{A}[0])$, which is closed, with canonical internal Hom object functor $R\underline{\mathrm{Hom}}_{\underline{A}}(-,-)$.

2.1.8. **Remark.** THe notation is not confusing: the bifunctors $-\otimes^L_{\underline{A}}-$ and $R\underline{\mathrm{Hom}}_{\underline{A}}(-,-)$ are indeed the respective derived functor of the bifunctors $-\otimes^L_{\underline{A}}-$ and $R\underline{\mathrm{Hom}}_{\underline{A}}(-,-)$ on $\mathrm{Mod}_{\underline{A}}$.

2.1.9. **Remark.** The condensed module theory (as well as its derived version) enjoys very nice base change properties.

However, the category $\mathrm{Mod}_{\underline{A}}$ (and $\mathscr{D}(\underline{A})$) still does not give a satisfying answer to "quasi-coherent modules" on $\mathrm{Spa}(A, A^+)$, (partly) because the $\underline{A}$-module structure does not take account of the "reference of boundedness" or "lattice" A^+ inside A.

2.2. **Solid abelian groups.** This subsection provides a preliminary example of analytic rings, and can be skipped if the readers wish to proceed directly to analytic rings associated with (complete) Huber pairs.

2.2.1. **Definition.** For any profinite set S with pro-finite presentation $S = \varprojlim_i S_i$, we define the *solidification of* $\mathbf{Z}[S] \in \mathcal{C}\mathrm{ond}(\mathcal{A}\mathrm{b})$ as the condensed abelian group

$$\mathbf{Z}_\blacksquare[S] := \varprojlim_i \mathbf{Z}[S_i] \in \mathcal{C}\mathrm{ond}(\mathcal{A}\mathrm{b}),$$

which receives a canonical natural map from $\mathbf{Z}[S]$.

2.2.2. **Definition.** We define *solid (condensed) abelian groups* and *solid complexes (of condensed abelian groups)* as follows:

(i) A condensed abelian group $M \in \mathcal{C}\mathrm{ond}(\mathcal{A}\mathrm{b})$ is *solid* if for all profinite sets S (or equivalently just for $S \in \mathrm{EDS}$), the natural map

$$\mathrm{Hom}(\mathbf{Z}_\blacksquare[S], M) \to \mathrm{Hom}(\mathbf{Z}[S], M) = M(S)$$

is an isomorphism. We denote by

$$\mathrm{Mod}_{\mathbf{Z}_\blacksquare} = \mathrm{Mod}_{(\mathbf{Z},\mathbf{Z})_\blacksquare} = \mathcal{S}\mathrm{olid} \subset \mathrm{Mod}_{\underline{\mathbf{Z}}} = \mathcal{C}\mathrm{ond}(\mathcal{A}\mathrm{b})$$

the full subcategory of solid (condensed) abelian groups.

(ii) A complex $M \in \mathscr{D}(\mathcal{C}\mathrm{ond}(\mathcal{A}\mathrm{b}))$ of condensed abelian groups is *solid* if for all profinite sets S (or equivalently just for $S \in \mathrm{EDS}$), the natural map

$$R\operatorname{Hom}(\mathbf{Z}_{\blacksquare}[S], M) \to R\operatorname{Hom}(\mathbf{Z}[S], M) = R\Gamma(S, M)$$

is a quasi-isomorphism. We denote by

$$\mathscr{D}(\mathbf{Z}_{\blacksquare}) = \mathscr{D}((\mathbf{Z}, \mathbf{Z})_{\blacksquare}) = \mathscr{D}(\mathcal{S}\mathrm{olid}) \subset \mathscr{D}(\underline{\mathbf{Z}}) = \mathscr{D}(\mathcal{C}\mathrm{ond}(\mathcal{A}\mathrm{b}))$$

the full ∞-subcategory of solid complexes (of condensed abelian groups).

2.2.3. **Proposition.** *The categories* $\mathrm{Mod}_{\mathbf{Z}_{\blacksquare}}$ *and* $\mathscr{D}(\mathbf{Z}_{\blacksquare})$ *have the following properties:*

(i) The subcategory $\mathrm{Mod}_{\mathbf{Z}_{\blacksquare}} \subset \mathrm{Mod}_{\underline{\mathbf{Z}}}$ *is stable under taking limits, colimits, extensions, internal Hom objects* $\underline{\mathrm{Hom}}_{\underline{\mathbf{Z}}}(M, -)$ *against* $M \in \mathrm{Mod}_{\underline{\mathbf{Z}}}$. *The fully faithful embedding admits a left adjoint* $(-)^{\blacksquare/\mathbf{Z}}$ *called* solidification, *which is the unique colimit-preserving extension of*

$$\mathbf{Z}[S] \mapsto \mathbf{Z}_{\blacksquare}[S].$$

Moreover, $\mathrm{Mod}_{\mathbf{Z}_{\blacksquare}}$ *has a family of compact projective objects* $\mathbf{Z}_{\blacksquare}[S] \simeq \prod_I \mathbf{Z}$*; and it is equipped with a canonical symmetric monoidal structure* $(\mathrm{Mod}_{\mathbf{Z}_{\blacksquare}}, - \otimes_{\mathbf{Z}_{\blacksquare}} -, \underline{\mathbf{Z}})$*, which is closed, with associated internal Hom object functor* $\mathrm{Hom}_{\underline{\mathbf{Z}}}(-,-)$.

(ii) The subcategory $\mathscr{D}(\mathbf{Z}_{\blacksquare}) \subset \mathscr{D}(\underline{\mathbf{Z}})$ *is stable under taking limits, colimits, extensions, internal Hom objects* $R\underline{\mathrm{Hom}}(M, -)$ *against* $M \in \mathrm{Mod}_{\underline{\mathbf{Z}}}$. *The fully faithful embedding admits a left adjoint* $(-)^{L\blacksquare/\mathbf{Z}}$ *called* derived solidification, *which is the unique colimit-preserving extension of*

$$\mathbf{Z}[S] \mapsto \mathbf{Z}_{\blacksquare}[S].$$

Moreover, $\mathscr{D}(\mathbf{Z}_{\blacksquare})$ *has a family of compact projective objects* $\mathbf{Z}_{\blacksquare}[S] \simeq \prod_I \mathbf{Z}$ *concentrated in degree* 0*; and it is equipped with a canonical symmetric monoidal structure* $(\mathscr{D}(\mathbf{Z}_{\blacksquare}), - \otimes^L_{\mathbf{Z}_{\blacksquare}} -, \underline{\mathbf{Z}})$*, which is closed, with associated internal Hom object functor* $R\operatorname{Hom}_{\underline{\mathbf{Z}}}(-,-)$.

(iii) A complex $M \in \mathscr{D}(\underline{\mathbf{Z}})$ *is solid if and only if* $H^i(M) \in \mathrm{Mod}_{\underline{\mathbf{Z}}}$ *is solid. The functor* $\mathscr{D}(\mathrm{Mod}_{\mathbf{Z}_{\blacksquare}}) \to \mathscr{D}(\underline{\mathbf{Z}})$ *is fully faithful, and its essential image is identified with* $\mathscr{D}(\mathbf{Z}_{\blacksquare})$.

(iv) We have

$$- \otimes_{\mathbf{Z}_{\blacksquare}} - = (- \otimes_{\underline{\mathbf{Z}}} -)^{\blacksquare/\mathbf{Z}}, \quad - \otimes^L_{\mathbf{Z}_{\blacksquare}} - = (- \otimes^L_{\underline{\mathbf{Z}}} -)^{L\blacksquare/\mathbf{Z}}.$$

(v) The (bi)functor $- \otimes^L_{\mathbf{Z}_{\blacksquare}} -$ *is the left derived functor of* $- \otimes_{\mathbf{Z}_{\blacksquare}} -$ *on either factor.*

2.2.4. **Example.** Since $\underline{\mathbf{Z}} = \mathbf{Z}[*]$ is solid, and solidness is stable under limits and colimits, we have that:

(i) For any abelian group M with descrete topology, $\underline{M} \in \mathrm{Mod}_{\mathbf{Z}_\blacksquare}$.

(ii) The condensed ring

$$\mathbf{Z}[T] = \varinjlim_i \mathbf{Z}[T]/T^i \in \mathrm{Mod}_{\mathbf{Z}_\blacksquare}$$

is solid.

(iii) The condensed ring

$$\mathbf{Z}[[T]] = \varprojlim_i \mathbf{Z}[T]/T^i \in \mathrm{Mod}_{\mathbf{Z}_\blacksquare}$$

is solid.

(iv) The map

$$\mathbf{Z}[\mathbf{N} \cup \{\infty\}]/\mathbf{Z}[\{\infty\}] \to \mathbf{Z}[[T]] \simeq \varprojlim_i \mathbf{Z}[T]/T^i$$

exhibits the unit of the solidification-forgetful adjunction.

(v) Evaluating $T = p$, one gets

$$\mathbf{Z}_p = \mathbf{Z}[[T]]/(T-p) \in \mathrm{Mod}_{\mathbf{Z}_\blacksquare}.$$

(vi) The condensed ring $\underline{\mathbf{R}}$ is not solid, because

$$\underline{R}^{\blacksquare/\mathbf{Z}} = 0.$$

(vii) The usual condensed tensor product $\mathbf{Z}_p \otimes_{\underline{\mathbf{Z}}} \mathbf{Z}_l$ is nonzero, but the solid tensor product

$$\mathbf{Z}_p \otimes_{\mathbf{Z}_\blacksquare} \mathbf{Z}_l = \begin{cases} \mathbf{Z}_p & l = p \\ 0 & l \neq p \end{cases}$$

takes account of mixed characteristics and separates them.

2.3. **Analytic rings.**

2.3.1. Roughly, an analytic ring is a pair $(\mathcal{A}, \mathcal{M}_{\mathcal{A}})$, where $\mathcal{A}$ is some condensed ring, and $\mathcal{M}_{\mathcal{A}}$ is a measure that tells you which series in an $\mathcal{A}$-module are summable.

Typically here, we have $(A, \mathcal{M}) = (A, A^+)_\blacksquare$, which is induced by a Huber ring (A, A^+). We will put ourselves into this setting.

Let A be a topological ring and $S \subset A$ be a subset (without any topology). We want to complete A-modules M with respect to elements in S. Classically, it is making $M \to M_f^\wedge := \varprojlim_n M/f^n$ isomorphisms for all $f \in S$. If S is a finite set, then it is just a sequence (though the order of doing this may cause issues) of f-adic completion (possibly derived completion); if S is infinite, then it might be more complicated, but in fact, it is easy and would be certain localisation of A-modules with respect to the full subcategories of S-complete A-modules. However, this kind of completion is not good for us, because what we wanted was in fact a "Banach-type completion", i.e. completion of a bounded unit ball A^+ with respect to elements of S. For example, if A is a $\mathbf{Q}_p$-Banach algebra, we certainly want to complete A-modules M with a $\mathbf{Z}_p$-lattice M^+ in the way that M^+ completes to $(M^+)_p^\wedge$, so $M \mapsto (M^+)_p^\wedge[\frac{1}{p}]$ but not $M \mapsto M_p^\wedge = 0$.

The solution will be given by $(A,S)_\blacksquare$-solid modules. We fix here our notation without diving into their full definitions.

2.3.2. **Definition.** Let A be a complete Huber ring and $S \subset A$ be a subset (without any topology).

(i) We denote by $\mathrm{Mod}_{(A,S)_\blacksquare} \subset \mathrm{Mod}_{\underline{A}}$ the full subcategory of $(A,S)_\blacksquare$-solid modules.
(ii) We denote by $\mathscr{D}((A,S)_\blacksquare) \subset \mathscr{D}(\underline{A})$ the full ∞-subcategory of $(A,S)_\blacksquare$-solid complexes of $\underline{A}$-modules.

2.3.3. **Remark.** We will often face the situation where $(A,S) = (A,A^+)$ is a Huber pair. In fact, it turns out that replacing S by $\overline{S \cup A^{\circ\circ}}$ will not affect the $(A,S)_\blacksquare$-solidness, so we will always assume S to be an integrally closed open subring $A^+ \subset A$.

In this case, the $(A,A^+)_\blacksquare$-solidness of an $\underline{A}$-module M an be thought of as "completeness with respect to certain $\underline{A^+}$-lattice".

2.3.4. **Example.** We have $(\mathbf{Q}_p, \mathbf{Z})_\blacksquare \simeq (\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare$.

We record some good properties of $(A,A^+)_\blacksquare$-modules, which are *almost* the same as those of $\mathrm{Mod}_{\mathbf{Z}_\blacksquare}$.

2.3.5. **Proposition.** *The categories* $\mathrm{Mod}_{(A,A^+)_\blacksquare}$ *and* $\mathscr{D}((A,A^+)_\blacksquare)$ *have the following properties:*

(i) The subcategory $\mathrm{Mod}_{(A,A^+)_\blacksquare} \subset \mathrm{Mod}_{\underline{A}}$ *is stable under taking limits, colimits, extensions, internal Hom objects* $\underline{\mathrm{Hom}}_{\underline{A}}(M,-)$ *against* $M \in \mathrm{Mod}_{\underline{A}}$. *The fully faithful*

embedding admits a left adjoint $(-)^{\blacksquare/A^+}$ *called* solidification*, which is the unique colimit-preserving extension of*

$$\underline{A}[S] \mapsto (A, A^+)_{\blacksquare}[S].$$

Moreover, $\mathrm{Mod}_{(A,A^+)_{\blacksquare}}$ *has a family of compact projective objects* $(A, A^+)_{\blacksquare}[S]$*; and it is equipped with a canonical symmetric monoidal structure* $(\mathrm{Mod}_{(A,A^+)_{\blacksquare}}, - \otimes_{(A,A^+)_{\blacksquare}} -, \underline{A})$*, which is closed, with associated internal Hom object functor* $\mathrm{Hom}_{\underline{A}}(-,-)$*.*

(ii) *The subcategory* $\mathscr{D}((A, A^+)_{\blacksquare}) \subset \mathscr{D}(\underline{A})$ *is stable under taking limits, colimits, extensions, internal Hom objects* $R\underline{\mathrm{Hom}}_{\underline{A}}(M, -)$ *against* $M \in \mathrm{Mod}_{\underline{A}}$*. The fully faithful embedding admits a left adjoint* $(-)^{L\blacksquare/A^+}$ *called* derived solidification*, which is the unique colimit-preserving extension of*

$$\underline{A}[S] \mapsto (A, A^+)_{\blacksquare}[S].$$

Moreover, $\mathscr{D}((A, A^+)_{\blacksquare})$ *has a family of compact projective objects* $(A, A^+)_{\blacksquare}[S]$ *concentrated in degree* 0*; and it is equipped with a canonical symmetric monoidal structure* $(\mathscr{D}((A, A^+)_{\blacksquare}), - \otimes^L_{(A,A^+)_{\blacksquare}} -, \underline{A})$*, which is closed, with associated internal Hom object functor* $R\,\mathrm{Hom}_{\underline{A}}(-,-)$*.*

(iii) *A complex* $M \in \mathscr{D}(\underline{A})$ *is solid if and only if* $H^i(M) \in \mathrm{Mod}_{\underline{A}}$ *is solid. The functor* $\mathscr{D}(\mathrm{Mod}_{(A,A^+)_{\blacksquare}}) \to \mathscr{D}(\underline{A})$ *is fully faithful, and its essential image is identified with* $\mathscr{D}((A, A^+)_{\blacksquare})$*.*

(iv) *We have*

$$- \otimes_{(A,A^+)_{\blacksquare}} - = (- \otimes_{\underline{A}} -)^{\blacksquare/A^+}, \quad - \otimes^L_{(A,A^+)_{\blacksquare}} - = (- \otimes^L_{\underline{A}} -)^{L\blacksquare/A^+}.$$

(v) *The (bi)functor* $- \otimes^L_{(A,A^+)_{\blacksquare}} -$ *is the left derived functor of* $- \otimes_{(A,A^+)_{\blacksquare}} -$ *on either factor (reason:* $(\underline{A}[S])^{\blacksquare/A^+} \to (\underline{A}[S])^{L\blacksquare/A^+}$ *is an isomorphism, i.e.* $(\underline{A}[S])^{L\blacksquare/A^+}$ *is concentrated in degree* 0*).*

(vi) *We have* $\underline{A} \in \mathrm{Mod}_{(A,A^+)_{\blacksquare}}$*.*

2.3.6. **Remark.** When $R = \mathbf{Z}$ with discrete topology, this recovers the categories $\mathrm{Mod}_{\mathbf{Z}_{\blacksquare}}$ and $\mathscr{D}(\mathbf{Z}_{\blacksquare})$ of the previous subsection.

What is left now is the expression of

$$(A, A^+)_{\blacksquare}[S] = (\underline{A}[S])^{\blacksquare/A^+} \xrightarrow{\simeq} (\underline{A}[S])^{L\blacksquare/A^+}.$$

Before that, let us see some examples, which illustrate the very counter-intuitive property that “solidification/completion” preserves all colimits.

2.3.7. **Example.** A counter-intuitive example would be the *preservation of colimits of the "solidification/completion" functor* $(-)^{(L)\blacksquare/\mathbf{Z}_p^+}$ (for $(A, A^+) = (\mathbf{Q}_p, \mathbf{Z}_p)$); this implies that

$$\bigoplus_{\mathbf{N}} \underline{\mathbf{Q}_p} \simeq \underline{\bigoplus_{\mathbf{N}} \mathbf{Q}_p} \in \mathrm{Mod}_{(\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare}.$$

Which means that its $(\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare$-solidification is *not* the $\mathbf{Q}_p$-Banach space

$$\widehat{\bigoplus}_{\mathbf{N}} \underline{\mathbf{Q}_p} \simeq \underline{\widehat{\bigoplus}_{\mathbf{N}} \mathbf{Q}_p} \in \mathrm{Mod}_{(\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare}.$$

This contradicts our naive intuition that $\bigoplus_{\mathbf{N}} \mathbf{Q}_p$ should be p-adically completed to $\widehat{\bigoplus}_{\mathbf{N}} \mathbf{Q}_p$.

In fact, we should keep the following intuition:

- The $(\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare$-solidification does not change $\underline{\mathbf{Q}_p} = \underline{\mathbf{Q}_p}[*]$ as well as its direct sums.
- The $(\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare$-solidification is an operation that takes place only on compact projective *generators* $\underline{\mathbf{Q}_p}[S]$, which are solidified to

  $$(\underline{\mathbf{Q}_p}[S])^{L\blacksquare/\mathbf{Z}_p} \simeq (\prod_I \underline{\mathbf{Z}_p})[\frac{1}{p}].$$

  This provides a "formal" substitute in *solid* p-adic functional analysis to $\mathbf{Q}_p$-Banach spaces in *classical* p-adic functional analysis.

For this reason of dissimilarity, we will abandon the word "completion" in "solidification/completion", in order to mark the solidification as an independent (yet analogous) operation from the completion.

Let us come back to the expression of $(A, A^+)_\blacksquare[S]$. They are made so that the $(A, A^+)_\blacksquare$-solidness can be checked using individual (or equivalently, finitely many) elements $f \in A^+$.

2.3.8. **Definition.** Firstly, we work with *discrete* Huber pairs.

(i) For any pair (R, R) where R is a discrete finitely generated $\mathbf{Z}$-algebra, we define

$$R_\blacksquare[S] := (R, R)_\blacksquare[S] := \varprojlim_i \underline{R}[S_i]$$

for any extremally disconnected set $S = \varprojlim_i S_i$.

(ii) For any pair (R, R) where R is arbitrary discrete Z-algebra, keeping in mind solidity should be checked using individual elements, thus using only finitely many elements, we define

$$R_\blacksquare[S] := (R, R)_\blacksquare[S] := \varinjlim_{R' \to R} (\underline{R'})_\blacksquare[S_i]$$

where R' runs over all finitely generated sub-$\mathbf{Z}$-algebras of R.

(iii) For any discrete Huber pairs (R, R^+), we define

$$(R, R^+)_\blacksquare[S] := (\underline{R}[S])^{(L)\blacksquare/R^+} = \underline{R} \otimes_{\underline{R^+}} (R^+)_\blacksquare[S].$$

The last identity can be clarified by the following computation: for any (discrete) R-module M, we have

$$(M[S])^{L\blacksquare/R^+} \simeq \underline{M}^{(L)\blacksquare/R^+} \simeq \underline{M} \otimes^{(L)}_{\underline{R^+}} (R^+)_\blacksquare[S]$$

2.3.9. **Proposition.** *The structure $(R, R^+)_\blacksquare$ defines a* complete analytic ring*.*

For any (topological) ring A, we denote by A^δ the topological ring A endowed with the discrete topology.

2.3.10. **Definition.** Now, we turn to general complete Huber pairs (A, A^+):

(v) For any complete Huber pair (A, A^+), we define

$$(A, A^+)_\blacksquare[S] := (\underline{A}[S])^{(L)\blacksquare/A^{+\delta}} \not\simeq \underline{A} \otimes_{\underline{R^+}} (R^+)_\blacksquare[S].$$

The last non-isomorphism is due to the in general non-discrete topology on A.

2.3.11. **Proposition.** *Let (A, A^+) be a complete Huber pair. Then*

$$(A, A^+)_\blacksquare[S] = \varinjlim_{B \subset A^+} (\underline{A}[S])^{(L)\blacksquare/B^\delta} = \varinjlim_{B \subset A^+} \varinjlim_{M \subset A} (\underline{M}[S])^{(L)\blacksquare/B^\delta} \simeq \varinjlim_{B \subset A^+} \varinjlim_{M \subset A} \prod_I \underline{M}$$

where B runs through all finitely generated (over $\mathbf{Z}$) subalgebras of A^+, and $M = \varprojlim_i M_i$ is an open subgroup of A that is a prodiscrete B-module with discrete B-module modulo I^n quotients M_i (one can check that B is always contained in some ring of definition A_0 with an ideal of definition I).

The $(A, A^+)_\blacksquare$-solidness can indeed be checked with respect to individual elements of A^+:

2.3.12. **Proposition.** *Let (A, A^+) be a complete Huber pair and $M \in \mathscr{D}(\underline{A})$. Then the following assertions are equivalent:*

(i) M is $(A, A^+)_\blacksquare$-solid.
(ii) M is $(A, R)_\blacksquare$-solid for any finitely generated (over $\mathbf{Z}$) subalgebra of A^+.
(iii) M is $(A, \{f\})_\blacksquare$-solid for any $f \in A^+$.
(iv) M is $\mathbf{Z}[T]_\blacksquare$-solid via the map $\underline{\mathbf{Z}[T]} \to \underline{A}$, $T \mapsto f$ for any $f \in A^+$.

2.3.13. **Example.** Let $M = \varprojlim_i M_i$ be a prodiscrete abelian group. Denote $M[T] := M \otimes_{\underline{\mathbf{Z}}} \underline{\mathbf{Z}[T]} \in \mathrm{Mod}_{\underline{\mathbf{Z}[T]}}$ where T is a variable (to be distinguished from $M[S] = M \otimes_{\underline{\mathbf{Z}}} \mathbf{Z}[S] \in \mathrm{Mod}_{\mathbf{Z}}$ where S is a profinite set). Then

$$(M[T])^{L\blacksquare/\mathbf{Z}} \simeq M[T]$$
$$(M[T])^{L\blacksquare/\mathbf{Z}[T]} \simeq M\langle T\rangle = \varprojlim_i M_i[T].$$

2.3.14. **Corollary.** *A map of complete Huber pairs $(A, A^+) \to (B, B^+)$ induces a map of analytic rings $(A, A^+)_\blacksquare \to (B, B^+)_\blacksquare$ is a map of complete analytic rings, i.e. $\underline{B} \in \mathrm{Mod}_{(A,A^+)_\blacksquare}$.*

2.4. Pushout and steadiness.

2.4.1. **Remark.** Pushout of analytic rings makes more sense with the larger class of *analytic animated rings* $(\mathcal{A}, \mathcal{M}_\mathcal{A})$.

Let us point out the principal difference when dealing with analytic *animated* rings:

(i) The definition involves the *derived* $(\mathcal{A}, \mathcal{M}_\mathcal{A})$-solidification
$$- \otimes^L_\mathcal{A} (\mathcal{A}, \mathcal{M}_\mathcal{A}) := (-)^{L\blacksquare/\mathcal{M}_\mathcal{A}} : \mathscr{D}_{\geq 0}(\mathcal{A}) \to \mathscr{D}_{\geq 0}(\mathcal{A}, \mathcal{M}_\mathcal{A}).$$
We denote by $(\mathcal{A}, \mathcal{M}_\mathcal{A})[S]$ the image of $\mathcal{A}[S]$; it is connective, but it may not be concentrated in degree 0 in general.

(ii) For a good notion of commutativity of analytic animated rings, one needs the Frobenius map $\mathcal{A} \to \mathcal{A}/^L p$ to induce a map of analytic animated rings $(\mathcal{A}, \mathcal{M}_\mathcal{A}) \to (\mathcal{A}/^L p, \mathcal{M}/^L p)$. This is the case for example when $(\mathcal{A}, \mathcal{M}_\mathcal{A})[S]$ is concentrated in degree 0, or when $(\mathcal{A}, \mathcal{M}_\mathcal{A})$ is over $\mathbf{Z}_\blacksquare$; so it is in practice always the case.

(iii) The ∞-subcategory $\mathscr{D}(\mathcal{A}, \mathcal{M}_\mathcal{A}) \subset \mathscr{D}(\mathcal{A})$ is in general not equivalent to the derived category of the abelian category $\mathscr{D}(\mathcal{A}, \mathcal{M}_\mathcal{A})^\heartsuit$. Nevertheless, we still have that $M \in \mathscr{D}_{\geq 0}(\mathcal{A})$ lies in $\mathscr{D}_{\geq 0}(\mathcal{A}, \mathcal{M}_\mathcal{A})$ if and only if all $H^i(M)$'s lie in $\mathcal{D}(\mathcal{A}, \mathcal{M}_\mathcal{A})^\heartsuit$.

2.4.2. **Definition.** We denote the pushout of analytic animated rings along maps $(\mathcal{B}, \mathcal{M}_\mathcal{B}) \leftarrow (\mathcal{A}, \mathcal{M}_\mathcal{A}) \to (\mathcal{C}, \mathcal{M}_\mathcal{C})$ by
$$(\mathcal{E}, \mathcal{M}_\mathcal{E}) := (\mathcal{B}, \mathcal{M}_\mathcal{B}) \otimes^L_{(\mathcal{A}, \mathcal{M}_\mathcal{A})} (\mathcal{C}, \mathcal{M}_\mathcal{C}).$$

It is not true that $(\mathcal{B}, \mathcal{M}_\mathcal{B})$-solidification and $(\mathcal{C}, \mathcal{M}_\mathcal{C})$-solidification commute with each other. Hence in general, the $(\mathcal{E}, \mathcal{M}_\mathcal{E})$-solidification is an (infinite) iterated colimit of interwined $(\mathcal{B}, \mathcal{M}_\mathcal{B})$-solidification and $(\mathcal{C}, \mathcal{M}_\mathcal{C})$-solidification. If two solidification commute, we are happy.

2.4.3. **Definition.** A map of analytic animated rings $(\mathcal{A}, \mathcal{M}_{\mathcal{A}}) \to (\mathcal{B}, \mathcal{M}_{\mathcal{B}})$ is called *steady* if for any map of analytic animated rings $(\mathcal{C}, \mathcal{M}_{\mathcal{C}})$, the following equivalent assertions hold:

(i) For any $(\mathcal{C}, \mathcal{M}_{\mathcal{C}})$-solid complex M, the complex $M \otimes^L_{(\mathcal{A},\mathcal{M}_{\mathcal{A}})} (\mathcal{B}, \mathcal{M}_{\mathcal{B}})$ is already $(\mathcal{C}, \mathcal{M}_{\mathcal{C}})$-solid.

(ii) For any $(\mathcal{C}, \mathcal{M}_{\mathcal{C}})$-solid complex M, the natural map

$$M \otimes^L_{(\mathcal{A},\mathcal{M}_{\mathcal{A}})} (\mathcal{B}, \mathcal{M}_{\mathcal{B}}) \to M \otimes^L_{(\mathcal{C},\mathcal{M}_{\mathcal{C}})} (\mathcal{E}, \mathcal{M}_{\mathcal{E}})$$

is a quasi-isomorphism.

The notion is related to a morphism of Huber pairs $f : (A, A^+) \to (B, B^+)$ being *adic* (i.e. there exist a pair of definition (A_0, I) of A and a ring of definition $B_0 \subset B$ such that $f(A_0) \subset B_0$ and $(B_0, f(I)B_0)$ is a pair of definition of B).

2.4.4. **Lemma.** *If both maps of analytic animated rings $(\mathcal{A}, \mathcal{M}_{\mathcal{A}}) \to (\mathcal{B}, \mathcal{M}_{\mathcal{B}})$ and $(\mathcal{A}, \mathcal{M}_{\mathcal{A}}) \to (\mathcal{C}, \mathcal{M}_{\mathcal{C}})$ are steady, then there are natural equivalences*

$$(-\otimes^L_{(\mathcal{A},\mathcal{M}_{\mathcal{A}})}(\mathcal{B}, \mathcal{M}_{\mathcal{B}}))\otimes^L_{(\mathcal{A},\mathcal{M}_{\mathcal{A}})}(\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \simeq -\otimes^L_{(\mathcal{A},\mathcal{M}_{\mathcal{A}})}(\mathcal{E}, \mathcal{M}_{\mathcal{E}}) \simeq (- \simeq \otimes^L_{(\mathcal{A},\mathcal{M}_{\mathcal{A}})}(\mathcal{C}, \mathcal{M}_{\mathcal{C}}))\otimes^L_{(\mathcal{A},\mathcal{M}_{\mathcal{A}})}(\mathcal{B}, \mathcal{M}_{\mathcal{B}})$$

of functors $\mathscr{D}(\mathcal{E}) \to \mathscr{D}(\mathcal{E})$ that factor through $\mathscr{D}(\mathcal{E}, \mathcal{M}_{\mathcal{E}}) \subset \mathscr{D}(\mathcal{E})$.

Steady maps have good properties:

2.4.5. **Proposition.** *Steadiness is stable under base change, composition and all colimits.*

2.4.6. **Example.** All maps (and their compositions) in the following diagram are steady:

$$\begin{tikzcd}
(\mathbf{Z}[T^{\pm 1}], \mathbf{Z}[T])_{\blacksquare} \arrow[r] & (\mathbf{Z}[T^{\pm 1}], \mathbf{Z}[T^{\pm 1}])_{\blacksquare} & (\mathbf{Z}[T^{\pm 1}], \mathbf{Z}[T^{-1}])_{\blacksquare} \arrow[l] \\
\mathbf{Z}[T]_{\blacksquare} \arrow[u] & (\mathbf{Z}[T], \mathbf{Z})_{\blacksquare} \arrow[l] \arrow[r] \arrow[u] \arrow[ul] \arrow[ur] & (\mathbf{Z}[T^{\pm 1}], \mathbf{Z})_{\blacksquare} \arrow[u] \\
 & \mathbf{Z}_{\blacksquare} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

which can be translated intuitively as

$$\begin{array}{ccccc}
\{0 \neq |T| \leq 1\} & \supset & \{|T| = 1\} & \subset & \{|T| \geq 1\} \\
\cap & \subsetneq & \cap & \supsetneq & \cap \\
\{|T| \leq 1\} & \subset & \text{Affine line} & \supset & \{|T| \neq 0\} \\
& \searrow & \downarrow & \swarrow & \\
& & \text{Base point.} & &
\end{array}$$

Rational localisations can be deduced from this diagram by finite steps of base change and composition.

2.4.7. **Relation with rational localisation.** Let (A, A^+) be a complete Huber pair and let $f, g \in A$ generating an open ideal. Consider the pushout

$$(\mathcal{B}, \mathcal{M}_{\mathcal{B}}) := \Big((A, A^+)_\blacksquare \otimes^L_{(\mathbf{Z}[T], Z)_\blacksquare} (\mathbf{Z}[U^{\pm 1}], \mathbf{Z})\Big) \otimes^L_{(\mathbf{Z}[T], \mathbf{Z})_\blacksquare} \mathbf{Z}[T]_\blacksquare$$

induced by the maps

- "inverting g": $Z[U] \to A, \quad U \mapsto g$, and
- "requiring $\left|\frac{f}{g}\right| \leq 1$": $\mathbf{Z}[T] \to (A, A^+)_\blacksquare \otimes^L_{(\mathbf{Z}[U], Z)_\blacksquare} (\mathbf{Z}[U^{\pm 1}], \mathbf{Z}), \quad T \mapsto f \otimes U^{-1}$.

Computation shows that the $(\mathcal{B}, \mathcal{M}_{\mathcal{B}})$-solidification of $\underline{A}$ is isomorphic to

$$\underline{A\langle T\rangle}/^L(gT - f).$$

Therefore, if the multiplication map by $A\langle T\rangle \overset{gT-f}{\to} A\langle T\rangle$ is a closed embedding (e.g. when (A, A^+) is a complete Huber pair that is *discrete* or *sheafy and locally Tate*, or such that f, g *generate the unit ideal of* A), then we obtain

$$\Big((A, A^+)_\blacksquare \otimes^L_{(\mathbf{Z}[U], Z)_\blacksquare} (\mathbf{Z}[U^{\pm 1}], \mathbf{Z})\Big) \otimes^L_{(\mathbf{Z}[T], \mathbf{Z})_\blacksquare} \mathbf{Z}[T]_\blacksquare \simeq \left(A\left\langle \frac{f}{g}\right\rangle, A^+\left\langle \frac{f}{g}\right\rangle\right)_\blacksquare.$$

This is [And21, Proposition 4.11].

In particular, we obtain:

2.4.8. **Corollary.** *Let (A, A^+) be a discrete or sheafy and locally Tate (complete) Huber pair and $U \subset \mathrm{Spa}(A, A^+)$ be a rational open subset of the form $\{1 \leq |f|\}$ or $\{|f| \leq 1\}$. Then the induced map of analytic rings $(A, A^+)_\blacksquare \to (A_U, A_U^+)_\blacksquare$ is steady.*

3. Analytic descent of "quasi-coherent modules" on sheafy locally Tate adic spaces

We are going to prove:

3.0.1. **Theorem.** *Let (A, A^+) be a sheafy complete Huber pair. Then the presheaf on rational open subsets of* $\mathrm{Spa}(A, A^+)$ *with values in* Cat_∞

$$U \mapsto \mathscr{D}((A_U, A_U^+)_\blacksquare)$$

satisfies analytic descent.

3.0.2. **Remark.** We remark that, going through the original proof of Andreychev, we find that the "locally Tate" assumption is not necessary, so it has been removed from the above statement. Since this descent is the only place where this assumption could be used, this will imply that "locally Tate" assumption is not necessary neither for the theorem (1.4.1) (except for the analytic descent of stably uniform pseudocoherent modules PCoh_A^0, which we are not discussing in this talk, due to open mapping theorem issues), cf. its final proof.

Zongze Liu has informed us that Kedlaya in his recent ongoing course (Fall 2024) has also proved it in this form, cf. combination of [Ked24, Theorem 14.1.15, Theorem 14.4.1, Proposition 13.3.2].

3.1. **Comparison with the proof in algebraic case.** Let us recall the proof of the following theorem:

3.1.1. **Theorem** (Lurie)**.** *The presheaf on affine schemes with values in* Cat_∞

$$U = \mathrm{Spec}\, A \mapsto \mathscr{D}(A)$$

satisfies fpqc (hyper)descent.

Proof. Observe first that

(a) *Any Zariski affine open covering of can be refined by a finite composition of coverings of the form $D(f) \cup D(1 - f)$. So we are reduced to this covering.*

We denote $X = \operatorname{Spec} A$, $U = D(f)$ and $V = D(1-f)$. This step does not use the special form of the affine opens U and V. We need to prove that the diagram

$$\begin{tikzcd}
\mathscr{D}(X) \arrow[r, "L_U"] \arrow[d, "L_V"] & \mathscr{D}(U) \arrow[d, "L_V"] \\
\mathscr{D}(V) \arrow[r] & \mathscr{D}(U \cap V)
\end{tikzcd}$$

is Cartesian in Cat_∞. Here, we have used the facts that

(b) *L_U and L_V are left adjoints to fully faithful embeddings.*

(c) *L_U sends $\mathscr{D}(V)$ into $\mathscr{D}(U \cap V)$, and L_V sends $\mathscr{D}(U)$ into $\mathscr{D}(U \cap V)$; and $L_U \circ L_V \simeq L_{U\cap V} \simeq L_V \circ L_U$.*

Using again the point (b), the Cartesianness is equivalent to proving the equivalence of the functor

$$F : \mathscr{D}(X) \to \mathscr{D}(U) \times_{\mathscr{D}(U\cap V)} \mathscr{D}(V), \quad M \mapsto (L_U M, L_V M, L_V(L_U M) \overset{\iota}{\simeq} L_U(L_V M)).$$

This functor admits a right adjoint

$$G : \mathscr{D}(U) \times_{\mathscr{D}(U\cap V)} \mathscr{D}(V) \to \mathscr{D}(X), \quad (M_U, M_V, L_V M_U \overset{\iota}{\simeq} L_U M_V) \mapsto M_U \times_{L_V M_U \overset{\iota}{\simeq} L_U M_V} M_V.$$

Let us prove that the unit $\mathrm{id}_{\mathscr{D}(X)} \to GF$ is an equivalence. Observe that

(d) *If $U \amalg V \to X$ is a Zariski covering, then the functor $\mathscr{D}(X) \to \mathscr{D}(U) \times \mathscr{D}(V)$ is conservative.*

Then it is enough to check equivalence after localising (L_U, L_V), which is then very obvious. The proof that the counit $FG \to \mathrm{id}$ is an equivalence is similar. □

Though (b)–(d) are true in general, it is enough to know them for the special covering as in (a) in order to prove the theorem.

Let us come back to the analytic case. We will proceed samely as above, but replacing (a)–(d) with appropriate analytic analogues.

We have already two of them:

- The (b) is easy: L_U in the analytic geometry would be

$$L_U := - \otimes^L_{(A,A^+)_\blacksquare} (A_U, A_U^+)_\blacksquare = (- \otimes^L_{\underline{A}} \underline{A_U})^{L\blacksquare/A_U^+},$$

  which is the left adjoint to the fully faithful inclusion $\mathscr{D}((A_U, A_U^+)_\blacksquare) \to \mathscr{D}((A, A^+)_\blacksquare)$ (2.3.5).

- The (c) is also clear (though non trivially), from the steadiness of rational localisation of *discrete* or *sheafy* (complete) Huber pairs (2.4.8).

3.2. **Laurent coverings.** We provide an analytic analogue of the (a):

3.2.1. **Theorem** (Tate, Bosch-Güntzer-Remmert; Huber, Gabber-Ramero)**.** *Let* (A, A^+) *be a complete Huber pair. Then any analytic open covering of* $\mathrm{Spa}(A, A^+)$ *can be refined by a finite composition of affinoid coverings of the following types:*

- *Simple balanced covering:* $X(\frac{1}{f}) \amalg X(\frac{1}{1-f}) \to X$.
- *Simple Laurent covering:* $X(\frac{1}{f}) \amalg X(\frac{f}{1}) \to X$.

If moreoever A is Tate, then the second type alone will suffice.

In fact, for adic spaces, Huber proved the refinement by Laurent coverings, [Hub94, Lemma 2.6], and an argument of Gabber-Ramero shows that these are further refined by simple balanced and simple Laurent ones, see [Ked17, Lemma 1.6.12].

3.3. **Discrete Huber pairs.** We provide an analogue of (d) for discrete Huber pairs:

3.3.1. **Proposition.** *The functor*

$$\mathscr{D}((\mathbf{Z}[T], \mathbf{Z})_{\blacksquare}) \to \mathscr{D}(\mathbf{Z}[T]_{\blacksquare}) \times \mathscr{D}((\mathbf{Z}[T^{\pm 1}], \mathbf{Z}[T^{-1}])_{\blacksquare})$$

associated with the simple Laurent covering (1.3.8.1) is conservative.

Sketch of proof. One can write the functor as induced by two functors $F_{|T|\leq 1}$ and $F_{|T|\geq 1}$ by projecting to two factors. By some important computations in [CS19], we obtain

$$\ker F_{|T|\leq 1} = \mathrm{Mod}_{\mathbf{Z}((T^{-1}))}(\mathscr{D}((\mathbf{Z}[T], \mathbf{Z})_{\blacksquare})), \quad \ker F_{|T|\geq 1} = \mathrm{Mod}_{\mathbf{Z}[[T]]}(\mathscr{D}((\mathbf{Z}[T], \mathbf{Z})_{\blacksquare})).$$

Hence the kernel of the functor in question is $\mathrm{Mod}_{\mathbf{Z}((T^{-1}))\otimes^{L}_{(\mathbf{Z}[T],\mathbf{Z})_{\blacksquare}}\mathbf{Z}[[T]]}(\mathscr{D}((\mathbf{Z}[T], \mathbf{Z})_{\blacksquare}))$; however,

$$\mathbf{Z}((T^{-1})) \otimes_{(\mathbf{Z}[T],\mathbf{Z})_{\blacksquare}} \mathbf{Z}[[T]] = \mathbf{Z}((T^{-1})) \otimes^{(L)}_{\mathbf{Z}[T]} \mathbf{Z}[[T]] = 0,$$

because if T is a topologically nilpotent unit such that T^{-1} is also topologically nilpotent, then $1 = (T \cdot T^{-1})^n \to 0$ as $n \to +\infty$. Therefore, the last kernel is zero, saying the conservativity. $\square$

3.3.2. **Proposition.** *The functor*

$$\mathscr{D}((\mathbf{Z}[T], \mathbf{Z})_{\blacksquare}) \to \mathscr{D}((\mathbf{Z}[T^{\pm 1}], \mathbf{Z}[T^{-1}])_{\blacksquare}) \times \mathscr{D}((\mathbf{Z}[T^{\pm 1}], \mathbf{Z}[T^{-1}])_{\blacksquare})$$

is conservative.

Proof. The proof is similar. But this time, one uses that $\mathbf{Z}[[1-T]] \otimes^{(L)}_{(\mathbf{Z}[T],\mathbf{Z})_\blacksquare} \mathbf{Z}[[T]] = 0$. $\square$

3.4. **Proof of analytic descent of "quasi-coherent modules".** To conclude the proof, we only need to prove the following analogue of (d):

3.4.1. **Proposition.** *Let (A, A^+) be a (complete) Huber pair and $f \in A$. Then the functors*

$$\mathscr{D}((A, A^+)_\blacksquare) \to \mathscr{D}((A\left\langle \frac{1}{f} \right\rangle, A^+\left\langle \frac{1}{f} \right\rangle)_\blacksquare) \times \mathscr{D}((A\left\langle \frac{f}{1} \right\rangle, A^+\left\langle \frac{f}{1} \right\rangle)_\blacksquare)$$

$$\mathscr{D}((A, A^+)_\blacksquare) \to \mathscr{D}((A\left\langle \frac{1}{f} \right\rangle, A^+\left\langle \frac{1}{f} \right\rangle)_\blacksquare) \times \mathscr{D}((A\left\langle \frac{1}{1-f} \right\rangle, A^+\left\langle \frac{1}{1-f} \right\rangle)_\blacksquare)$$

are conservative.

Proof. We consider only the first functor; the proof is similar for the second. Consider the following commutative diagram of funtors:

$$\begin{tikzcd}
\mathscr{D}((A, A^+)_\blacksquare) \arrow[r] \arrow[d] & \mathscr{D}((A\langle \frac{1}{f} \rangle, A^+\langle \frac{1}{f} \rangle)_\blacksquare) \times \mathscr{D}((A\langle \frac{f}{1} \rangle, A^+\langle \frac{f}{1} \rangle)_\blacksquare) \arrow[d] \\
\mathscr{D}((\mathbf{Z}[T], \mathbf{Z})_\blacksquare) \arrow[r] & \mathscr{D}(\mathbf{Z}[T]_\blacksquare) \times \mathscr{D}((\mathbf{Z}[T^{\pm 1}], \mathbf{Z}[T^{-1}])_\blacksquare)
\end{tikzcd}$$

induced by the map $(\mathbf{Z}[T], \mathbf{Z}) \to (A, A^+)$, $T \mapsto f$, where vertical maps are forgetful functors. Now, the left vertical map is conservative (as well as the right vertical one), and the lower horizontal arrow is convervative too; hence so is the upper one. $\square$

Now we can proceed as in the proof in algebraic case to conclude the proof of the descent theorem (3.0.1) of $\mathscr{D}((A, A^+)_\blacksquare)$.

4. Cut out some descendable full subcategory of "quasi-coherent modules"

Our goal will be proving analytic descent of the full subcategory of dualisable modules $\mathscr{D}((A, A^+)_\blacksquare)^{\mathrm{dual}} \subset \mathscr{D}((A, A^+)_\blacksquare)$ satisfies analytic descent. However, the definition of dualisability will not be good enough to verify descent properties. Hence, to cut out the dualisability condition, one needs some auxiliary descendable properties of "quasi-coherent modules", which may be of other independent interests.

4.1. **Dualisable objects.** Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal ∞-category. For example, it can be $(\mathscr{D}((A, A^+)_\blacksquare), - \otimes^L_{(A,A^+)_\blacksquare} -, \underline{A})$ for complete Huber pairs (A, A^+).

4.1.1. **Definition** (Dualisable objects)**.** Let $\mathcal{C}$ be as above. An object $M \in (\mathcal{C}, \otimes, 1)$ is called *dualisable* if there exists a *dual* $M' \in \mathcal{C}$, an evaluation map $\mathrm{ev}_M : M' \otimes M \to 1$ and a coevaluation map $\mathrm{coev}_M : 1 \to M \otimes M'$ such that

$$M \simeq 1 \otimes M \overset{\mathrm{coev}_M \otimes \mathrm{id}}{\to} M \otimes M' \otimes M \overset{\mathrm{id} \otimes \mathrm{ev}_M}{\to} M \otimes 1 \simeq M$$

$$M' \simeq M' \otimes 1 \overset{\mathrm{id} \otimes \mathrm{coev}_M}{\to} M' \otimes M \otimes M' \overset{\mathrm{ev}_M \otimes \mathrm{id}}{\to} 1 \otimes M' \simeq M'$$

compose naturally to identity morphisms respectively. We denote by $\mathcal{C}^{\mathrm{dual}}$ the full subcategory of dualisable objects of $\mathcal{C}$.

Dualisability is preserved by symmetric monoidal functors.

4.1.2. **Lemma.** *If $(\mathcal{C}, \otimes, 1)$ is closed (so that $-\otimes-$ right adjoints to internal Hom $\underline{\mathrm{Hom}}(-,-)$), then for any $M \in \mathcal{C}^{\mathrm{dual}}$ (with dual M'), we have an isomorphism natural in $N \in \mathcal{C}$*

$$M' \otimes N \simeq \underline{\mathrm{Hom}}(M, N).$$

In particular, we have an isomorphism $M' \simeq \underline{\mathrm{Hom}}(M, 1) =: M^\vee$.

It is unclear whether $\mathscr{D}((A, A^+)_\blacksquare)^{\mathrm{dual}}$ satisfies analytic descent, since the conditions in the definition is seemingly not quite descendable.

4.2. **(Pseudo)compact objects.** Let $\mathcal{C}$ be a *stable* infinity category with colimits, let $R\,\mathrm{Hom}(-,-)$ denote the mapping *spectrum* in $\mathcal{C}$.

4.2.1. **Definition** (Compact objects)**.** Let $\mathcal{C}$ be as above. An object $M \in \mathcal{C}$ is called *compact* if $R\,\mathrm{Hom}(M, -) : \mathcal{C} \to \mathcal{S}\mathrm{p}$ commutes with filtered colimits, or equivalently [Lur16, Proposition 1.4.4.1 (2)], commutes with direct sums. We denote by $\mathcal{C}^\omega$ the full subcategory of compact objects of $\mathcal{C}$ (the ω stands for "ω-filtered" colimits).

4.2.2. **Lemma.** *A complex $M \in \mathscr{D}((A, A^+)_\blacksquare)$ is compact if and only if M is a retract of a finite complex with terms being compact projective genetors $(A, A^+)_\blacksquare[S_i]$ with profinite set S_i at the degree i.*

This holds in general for any compactly generated $\mathcal{C}$ in place of $\mathscr{D}((A, A^+)_\blacksquare)$.

Compactness is a descendable property:

4.2.3. **Proposition.** *Let* (A, A^+) *be a (complete) Huber pair and* $\{U_i \to \mathrm{Spa}(A, A^+)\}_i$ *be a rational open covering. An object* $M \in \mathscr{D}((A, A^+)_\blacksquare)$ *is compact if and only if all* $M \otimes^L_{(A,A^+)_\blacksquare} (A_U, A_U^+)_\blacksquare \in \mathrm{Spa}(A_U, A_U^+)$ *are compact.*

Similarly, we may consider the pseudocompactness, which is a weaker notion than compactness. But now we have to take account of the t-structure on $\mathcal{C}$, for example the canonical t-structure on $\mathscr{D}((A, A^+)_\blacksquare)$.

4.2.4. **Definition** (Pseudocompact objects)**.** Let $\mathcal{C}$ be as above (with t-structure). An object $M \in \mathcal{C}$ is called *pseudocompact* if for any $n \in \mathbf{Z}$, $R\,\mathrm{Hom}(M, -) : \mathcal{C}^{\geq n} \to \mathcal{S}\mathrm{p}$ commutes with filtered colimits, or equivalently [Lur16, Proposition 1.4.4.1 (2)], for any $n \in \mathbf{Z}$[2], $R\,\mathrm{Hom}(M, -) : \mathcal{C}^{\geq n} \to \mathcal{S}\mathrm{p}$ commutes with direct sums. We denote by $\mathcal{C}^{\mathrm{pc}}$ the full subcategory of pseudocompact objects of $\mathcal{C}$.

4.2.5. **Lemma.** *A complex* $M \in \mathscr{D}((A, A^+)_\blacksquare)$ *is pseudocompact if and only if* M *is admits a bounded above resolution with terms being compact projective generators* $(A, A^+)_\blacksquare[S_i]$ *with profinite set* S_i *at the degree* i*.*

This holds in general for any compactly generated $\mathcal{C}$ in place of $\mathscr{D}((A, A^+)_\blacksquare)$.

Pseudocompactness is also a descendable property:

4.2.6. **Proposition.** *Let* (A, A^+) *be a (complete) Huber pair and* $\{U_i \to \mathrm{Spa}(A, A^+)\}_i$ *be a rational open covering. An object* $M \in \mathscr{D}((A, A^+)_\blacksquare)$ *is pseudocompact if and only if all* $M \otimes^L_{(A,A^+)_\blacksquare} (A_U, A_U^+)_\blacksquare \in \mathrm{Spa}(A_U, A_U^+)$ *are pseudocompact.*

4.3. **Nuclear objects.** Let $(\mathcal{C}, \otimes, 1)$ be a closed symmetric mononidal ∞-category (*resp.* stable ∞-category). For example, it can be $(\mathscr{D}((A, A^+)_\blacksquare), - \otimes^L_{(A,A^+)_\blacksquare} -, \underline{A})$ for complete Huber pairs (A, A^+).

4.3.1**. Shorthand notation.** Let $P \in \mathcal{C}$. We define the object in $\mathcal{C}$

$$P^\vee := \underline{\mathrm{Hom}}(P, 1)$$

and the mapping space (*resp.* mapping spectrum)

$$P(*) := R\,\mathrm{Hom}(1, P).$$

There is a canonical evaluation map $\mathrm{ev}_P : P^\vee \otimes P \to 1$.

[2]One must repeat this, since for fixed $n \in \mathbf{Z}$, the two conditions may not be equivalent.

4.3.2. **Definition** (Trace-class maps and Nuclear objects)**.** Let $\mathcal{C}$ be as above.

(i) A map $f : P \to Q$ in $\mathcal{C}$ is called *trace-class* or *nuclear* if it lies in the image of the natural map

$$\pi_0(P^\vee \otimes Q)(*) \to \pi_0 R\,\mathrm{Hom}(P, Q),$$

or equivalently, if there exists a map $\phi : 1 \to P^\vee \otimes Q$, such that the composed map

$$P \overset{\mathrm{id}\otimes\phi}{\to} P \otimes P^\vee \otimes Q \overset{\mathrm{ev}_P\otimes\mathrm{id}}{\to} Q$$

is equivalent to f.

(ii) Suppose that $\mathcal{C}$ is compactly generated by a family $\mathcal{P}$ (of compact generators). An object $N \in \mathcal{C}$ is called *nuclear* if for any $P \in \mathcal{P}$ (or equivalently, for any compact object P), the natural map of spaces (*resp.* spectra)

$$(P^\vee \otimes N)(*) \to R\,\mathrm{Hom}(P, N)$$

is an isomorphism. We denote by $\mathcal{C}^{\mathrm{nuc}}$ the full subcategory of nuclear objects of $\mathcal{C}$.

4.3.3. **Proposition.** *Suppose that $\mathcal{C}$ is compactly generated by a family $\mathcal{P}$. If $N \in \mathcal{C}^{\mathrm{nuc}}$, then for any $M \in \mathcal{C}$ and any compact object P, the natural map*

$$\underline{\mathrm{Hom}}(P, M) \otimes N \to \underline{\mathrm{Hom}}(P, M \otimes N)$$

is an isomorphism.

When $\mathcal{C} = \mathscr{D}((A, A^+)_\blacksquare)$, we write

$$\mathrm{Nuc}((A, A^+)_\blacksquare) := \mathscr{D}((A, A^+)_\blacksquare)^{\mathrm{nuc}}.$$

4.3.4. **Remark.** It turns out that $\mathrm{Nuc}((A, A^+)_\blacksquare)$ depends only on A, but not on A^+, for any (complete) Huber ring A [And23, Korollar 3.18].

Nuclearity is sort of an "orthogonal" concept to (pseudo)compactness.

4.3.5. **Example.** We have

$$(\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare[S] \simeq (\prod_I \underline{\mathbf{Z}_p})[\frac{1}{p}] \notin \mathrm{Nuc}((\mathbf{Q}_p, \mathbf{Z}_p)),$$

while

$$(\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare[S]^\vee \simeq C(S, \mathbf{Q}_p) \in \mathrm{Nuc}((\mathbf{Q}_p, \mathbf{Z}_p)).$$

The ∞-subcategory $\mathrm{Nuc}((\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare) \subset \mathscr{D}((\mathbf{Q}_p, \mathbf{Z}_p)_\blacksquare)$ is stable under colimits and generated under colimits by the $\mathbf{Q}_p$-Banach spaces $C(S, \mathbf{Q}_p)$, and contains for example $\prod_I \underline{\mathbf{Q}_p}$ and $\mathbf{Q}_p$-Fréchet spaces.

4.3.6. **Proposition.** *Let (A, A^+) be a (complete) Huber pair and $\{U_i \to \mathrm{Spa}(A, A^+)\}_i$ be a rational open covering. An object $M \in \mathscr{D}((A, A^+)_\blacksquare)$ is nuclear if and only if all $M \otimes^L_{(A,A^+)_\blacksquare} (A_U, A_U^+)_\blacksquare \in \mathrm{Spa}(A_U, A_U^+)$ are nuclear.*

4.4. **Proof of analytic descent of $\mathscr{D}((A, A^+)_\blacksquare)^{\mathrm{dual}}$.** Now we claim that dualisability satisfies descent, by observing an abstract nonsense:

4.4.1. **Proposition.** *Let $\mathcal{C}$ be an ∞-category (such that dualisability, compactness and nuclearity are well-defined). Let $M \in \mathcal{C}$.*

(i) If M is compact and nuclear, then M is dualisable.

(ii) The converse is true if the unit object $1 \in \mathcal{C}$ is compact.

Proof. (i) Since M is nuclear and M is compact, we can set $P = M$ in the definition of nuclearity so as to obtain

$$(M^\vee \otimes M)(*) \xrightarrow{\simeq} R\,\mathrm{Hom}(M, M).$$

In particular, id_M is trace-class and is induced by some $\phi : 1 \to M^\vee \otimes M$. Then the canonical evaluation map $\mathrm{ev}_M : M^\vee \otimes M \to 1$ together with the coevaluation map $\mathrm{coev}_M := \phi$ makes M dualisable with dual $M^\vee$.

(ii) Conversely, suppose that $1 \in \mathcal{C}$ is compact. For any dualisable $M \in \mathcal{C}$ with dual M', we have

$$\begin{array}{ccccccc}
R\,\mathrm{Hom}(M, \bigoplus_i N_i) & \simeq & R\,\mathrm{Hom}(1, \underline{\mathrm{Hom}}(M, (\bigoplus_i N_i))) & \overset{(4.1.2)}{\simeq} & R\,\mathrm{Hom}(1, M' \otimes (\bigoplus_i N_i)) & & \\
 & & & & \wr\!\!\mid & 1 \in \mathcal{C} \text{ is compact} \\
\bigoplus_i R\,\mathrm{Hom}(M, N_i) & \simeq & \bigoplus_i R\,\mathrm{Hom}(1, \underline{\mathrm{Hom}}(M, N_i)) & \overset{(4.1.2)}{\simeq} & \bigoplus_i R\,\mathrm{Hom}(1, M' \otimes N_i) & &
\end{array}$$

proving the compactness of M. On the other hand, still by (4.1.2), we obtain

$$M^\vee \otimes N \simeq M' \otimes N \simeq \underline{\mathrm{Hom}}(M, N)$$

for any $N \in \mathcal{C}$; in particular for $N = M$, we see that id_M is trace-class. Then

$$M = \varinjlim(M \overset{\mathrm{id}_M}{\to} M \overset{\mathrm{id}_M}{\to} \cdots).$$

Using the following lemma, one sees that for any compact object $P \in \mathcal{C}$, the diagonal morphisms induce

$$\begin{aligned} P^\vee \otimes M &= \varinjlim(P^\vee \otimes M \to P^\vee \otimes M \to \cdots) \\ &= \varinjlim(\underline{\mathrm{Hom}}(P,M) \to \underline{\mathrm{Hom}}(P,M) \to \cdots) \simeq \underline{\mathrm{Hom}}(P,M), \end{aligned}$$

whence the nuclearity of M. □

4.4.2. **Lemma.** *Let $f : P \to Q$ be a trace-class map in $\mathcal{C}$. For any $M \in \mathcal{C}$, there exists a dashed arrow fitting into the following commutative diagram*

$$\begin{tikzcd} M^\vee \otimes P \arrow[r] \arrow[d] & M^\vee \otimes Q \arrow[d] \\ \underline{\mathrm{Hom}}(M,P) \arrow[r] \arrow[ur, dashed, "\delta"] & \underline{\mathrm{Hom}}(M,Q). \end{tikzcd}$$

Proof. Suppose f is induced by $\phi : 1 \to P^\vee \otimes Q$. Then

$$\underline{\mathrm{Hom}}(M,P) \xrightarrow{\underline{\mathrm{Hom}}(M,\mathrm{id}\otimes\phi)} \underline{\mathrm{Hom}}(M, P \otimes P^\vee \otimes Q) \xrightarrow{\underline{\mathrm{Hom}}(M,\mathrm{ev}_P\otimes\mathrm{id})} \underline{\mathrm{Hom}}(M,Q)$$

compose to $\underline{\mathrm{Hom}}(M,f)$. But this fits into the commutative diagram:

$$\begin{tikzcd} \underline{\mathrm{Hom}}(M,P) \arrow[r, "\underline{\mathrm{Hom}}(M,\mathrm{id}\otimes\phi)"] \arrow[dr, "\mathrm{id}\otimes\phi"'] \arrow[drr, dashed, bend right, "\delta"'] & \underline{\mathrm{Hom}}(M, P \otimes P^\vee \otimes Q) \arrow[r, "\underline{\mathrm{Hom}}(M,\mathrm{ev}_P\otimes\mathrm{id})"] & \underline{\mathrm{Hom}}(M,Q) \\ & \underline{\mathrm{Hom}}(M,P) \otimes P^\vee \otimes Q \arrow[u] \arrow[r, "\mathrm{ev}\otimes\mathrm{id}"] & M^\vee \otimes Q \arrow[u] \end{tikzcd}$$

where δ denotes the obvious composition. □

So the descendability of nuclearity and compactness implies that dualisability is also a descendable property:

4.4.3. **Proposition.** *Let (A, A^+) be a (complete) Huber pair and $\{U_i \to \mathrm{Spa}(A, A^+)\}_i$ be a rational open covering. An object $M \in \mathscr{D}((A, A^+)_\blacksquare)$ is dualisable if and only if all $M \otimes^L_{(A,A^+)_\blacksquare} (A_U, A_U^+)_\blacksquare \in \mathrm{Spa}(A_U, A_U^+)$ are dualisable.*

5. Final touch: get discreteness

For now, we have no idea of the distance between the combination of full subcategories of $\mathscr{D}((A, A^+)_\blacksquare)$ and the categories $\mathrm{Perf}(A)$ and PCoh_A, which are thought of as discrete modules.

5.1. **Discrete modules.** First, we embed $\mathscr{D}(A)$ into $\mathscr{D}((A, A^+)_\blacksquare)$. The objects of its essential image are said to be *discrete modules*.

5.1.1. **Definition.** Let A be a topological ring. We define the *condensification* functor

$$\mathcal{C}\mathrm{ond}_A : \mathscr{D}(A) \to \mathscr{D}(\underline{A}), \quad M^\bullet \mapsto \underline{M^\bullet} \otimes^L_{\underline{A^\delta}} \underline{A},$$

where we put discrete topology on complexes $M^\bullet \in \mathscr{D}(A)$.

The functor $\mathcal{C}\mathrm{ond}_A$ is the composite of two functors:

(i) $\mathscr{D}(A) \hookrightarrow \mathscr{D}(\underline{A^\delta})$, $M^\bullet \to \underline{M^\bullet}$, which is fully faithful, exact, and preserves filtered colimits;

(ii) $\mathscr{D}(\underline{A^\delta}) \to \mathscr{D}(\underline{A})$, $N \mapsto N \otimes^L_{\underline{A^\delta}} \underline{A}$, which is simply the derived base change.

5.1.2. **Definition** (Discrete objects)**.** Let A be a topological ring. An object $M \in \mathscr{D}(\underline{A})$ is called *(relatively) discrete* if $M \simeq \mathcal{C}\mathrm{ond}_A(M_0)$ for some $M_0 \in \mathscr{D}(A)$.

5.1.3. **Lemma.** *Let A be a topological ring and $i \in \mathbf{Z}$.*

(i) The functor $\mathcal{C}\mathrm{ond}_A$ is fully faithful, exact, symmetric monoidal, and preserves filtered colimits. And for $M \in \mathscr{D}(A)$, if $H^i(\mathcal{C}\mathrm{ond}_A(M)) = 0$, then $H^i(M) = 0$ (tor-amplitude control).

Moreover, assume that A is a complete Huber ring.

(ii) The functor $\mathcal{C}\mathrm{ond}_A$ lands in $\mathscr{D}((A, A^+)_\blacksquare) \subset \mathscr{D}(\underline{A})$.

(iii) Assume that A is moreover locally Tate. If $M \in \mathscr{D}(A)$ is quasi-isomorphic to a complex of finite free A-modules and verifies $H^i(M) = 0$, then $H^i(\mathcal{C}\mathrm{ond}_A(M)) = 0$.

Indeed, (ii) follows because by completeness of the Huber ring A, $\underline{A} \in \mathscr{D}((A, A^+)_\blacksquare)$, which is stable under colimits; (iii) follows from the open mapping theorem for complete and first countable topological A-modules.

5.1.4. **Remark.** The (ii) can be strengthened: if A is a complete Huber ring, then the functor $\mathcal{C}\mathrm{ond}_A$ lands in $\mathrm{Nuc}((A, A^+)_\blacksquare) \subset \mathscr{D}(\underline{A})$. Indeed, this subcategory is stable under colimits, and $\underline{A} = \underline{A}[*] \in \mathscr{D}((A, A^+)_\blacksquare)$.

Thus, discreteness implies nuclearity in $\mathscr{D}((A, A^+)_\blacksquare)$.

5.1.5. **Fact.** For $M \in \mathscr{D}(A)$ a classcial complex, the following are equivalent:

(i) M is dualisable in $\mathscr{D}(A)$;
(ii) M is compact in $\mathscr{D}(A)$;
(iii) $M \in \mathrm{Perf}(A)$.

Similarly, the following are also equivalent:

(i) M is pseudocompact in $\mathscr{D}(A)$;
(ii) $M \in \mathrm{PCoh}_A$.

5.1.6. **Corollary.** *Let (A, A^+) be a complete Huber pair.*

(i) An object $M \in \mathscr{D}((A, A^+)_\blacksquare)$ is discrete and dualisable if and only if $M = \mathcal{C}\mathrm{ond}_A(M_0)$ for some dualisable $M_0 \in \mathscr{D}(A)$.
(ii) An object $M \in \mathscr{D}((A, A^+)_\blacksquare)$ is discrete and pseudocompact if and only if $M = \mathcal{C}\mathrm{ond}_A(M_0)$ for some pseudocoherent $M_0 \in \mathscr{D}(A)$.

Proof. (i) The "if" direction is clear since $\mathcal{C}\mathrm{ond}_A$ is symmetric monoidal.

For the "only if" direction, write $M = \mathcal{C}\mathrm{ond}_A(M_0)$ for $M_0 \in \mathscr{D}(A)$. Notice that M is compact; so by fully faithfulness and colimit preservation property of $\mathcal{C}\mathrm{ond}_A$, M_0 is compact in $\mathscr{D}(A)$, hence M_0 is dualisable in $\mathscr{D}(A)$ by the fact above.

(ii) The "if" part can be seen by writing pseudocoherent complexes as a bounded above complex of finite free A-modules. The "only if" part is similar to that of (i). ☐

5.2. **Get discreteness from descendable properties.** Discreteness is not a descendable property, as $\mathscr{D}(A)$ does not satisfy analytic descent.

However, we have the following upshot:

5.2.1. **Theorem.** *Let (A, A^+) be a complete Huber pair. Let $M \in \mathscr{D}((A, A^+)_\blacksquare)$ be pseudocompact and nuclear. Then M is discrete.*

Sketch of proof. The proof consists of two steps: the first is abstract, while the second is essential.

Step 1: Given a family of compact projective generators $\mathcal{P} \subset \mathscr{D}((A, A^+)_\blacksquare)$ such that for any $P \in \mathcal{P}$, the object $P^\vee = (R)\,\mathrm{Hom}(P, 1)$ is concentrated in degree 0; for example, can choose $\mathcal{P} = \{(A, A^+)_\blacksquare[S],\ S \text{ profinite sets}\}$, since $((A, A^+)_\blacksquare[S])^\vee \simeq \underline{C(S, A)}$ sitting in degree O. Any pseudocompact and nuclear M can be written as a filtered colimt

$$M = \varinjlim_n M_n$$

such that:

(a) All M_n are dualisable and are successive extension of objects of the form

$$\mathrm{cone}(1 - f : P \to P)$$

where $P \in \mathcal{P}$ and $f : \mathcal{P} \to \mathcal{P}$ is a trace-class map.

(b) We have

$$\mathrm{cone}(M_n \to M) \in \mathscr{D}^{\leq -n}((A, A^+)_\blacksquare).$$

Let us now prove this. By shifting around, we may assume M lies in $\mathscr{D}^{\leq 0}((A, A^+)_\blacksquare)$ and takes the form

$$M \simeq (\cdots \to P_2 \to P_1 \to P_0 \to 0)$$

with $P_i \in \mathcal{P}$. Consider the map

$$\iota : P_0[0] \to M$$

induced by the map $\mathrm{id} : P_0 \to P_0$ in degree 0. We have (commutative diagram of) maps

$$\begin{tikzcd}
\mathrm{Hom}(P_0[0], M) & (P_0^\vee \otimes M)(*) \arrow[l, "\simeq"'] \\
\mathrm{Hom}(P_0[0], P_0[0]) \arrow[u] & (P_0^\vee \otimes P_0)(*) \arrow[l] \arrow[u, two heads]
\end{tikzcd}$$

where the upper map is by nuclearity of M, and the surjectivity of the right map follows from the assumption that $P_0^\vee$ lies in $\mathscr{D}^{\leq 0}$. Thus, there exists a trace-class map $f : P_0 \to P_0$ such that ι factors as

$$\iota : P_0[0] \xrightarrow{f} P_0[0] \xrightarrow{\iota} M,$$

which induces a map

$$M_1 := \mathrm{cone}(1 - f : P_0 \to P_0) \xrightarrow{\iota} M$$

whose cone lies in $\mathscr{D}^{\leq -1}((A, A^+)_\blacksquare)$. Therefore, we can iterate this process to find all the desired M_n's.

Step 2: By Step 1 and since the condensification functor $\mathcal{C}\mathrm{ond}_A$ preserves filtered colimits, it is then enough to prove that $\mathrm{cone}(1 - f : (A, A^+)_\blacksquare[S] \to (A, A^+)_\blacksquare[S])$ is discrete for any profinite set S and any trace-class map $f : (A, A^+)_\blacksquare[S] \to (A, A^+)_\blacksquare[S]$.

For notational convenience and without loss of generality, we consider the case where $(A, A^+) = (\mathbf{Q}_p, \mathbf{Z}_p)$ and S is a *light* profinite set (i.e. S is a countable projective limit of finite sets, so in particular $C(S, \mathbf{Z}) \simeq \bigoplus_I \mathbf{Z}$ for some countable set I).

Being trace-class, the map f is of the form

$$\begin{aligned} f : (\prod_I \underline{\mathbf{Z}_p})[\frac{1}{p}] &\to (\prod_I \underline{\mathbf{Z}_p})[\frac{1}{p}] \\ m &\mapsto \sum_i f_i(m) \otimes y_i \end{aligned}$$

with all $y_i \in p^{-N}\mathbf{Z}_p$ for some uniform $N \in \mathbf{N}$ and $f_i \in C(S, \mathbf{Q}_p) = \widehat{\bigoplus}_I \mathbf{Q}_p$ converging to 0. One can rescale so that $y_i \in \mathbf{Z}_p$ for all $i \in I$.

Since $f_i \to 0$, there exists a finite subset $I_0 \subset I$ such that $f_i \in C(S, p\mathbf{Z}_p)$ for any $i \in I \backslash I_0$. By decomposing $\prod_I \underline{\mathbf{Z}_p}$ as $\prod_{I_0} \underline{\mathbf{Z}_p} \times \prod_{I \backslash I_0} \underline{\mathbf{Z}_p}$, one can "represent" f by the matrix

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where for example we set $F_{11} = f|_{I_0}$ and $F_{22} = f|_{I \backslash I_0}$. Hence $1 - f$ is "represented" by the matrix

$$\begin{pmatrix} 1 - F_{11} & -F_{12} \\ -F_{21} & 1 - F_{22} \end{pmatrix}.$$

By our choice of I_0 such that $f_i \in C(S, p\mathbf{Z}_p)$ for any $i \in I \backslash I_0$, we see that $1 - F_{22}$ is invertible on $(\prod_{I \backslash I_0} \underline{\mathbf{Z}_p})[\frac{1}{p}]$, because we can write down the formal inverse power series and check its convergence. Therefore, its cone is quasi-isomorphic to

$$\operatorname{cone}\left(1 - F_{11} : (\prod_{I_0} \underline{\mathbf{Z}_p})[\frac{1}{p}] \to (\prod_{I_0} \underline{\mathbf{Z}_p})[\frac{1}{p}]\right),$$

which is a two-term complex of finite free $\mathbf{Q}_p$-modules whence discrete. □

5.3. Proof of analytic descent of $\mathrm{Perf}(A)$ and PCoh_A.

Proof of (1.4.1). The tor-amplitude controlness of $\mathcal{C}\mathrm{ond}_A$ (5.1.3) for locally Tate complete Huber pairs (A, A^+) reduces us to considering only the cases of $\mathrm{Perf}(A)$ and PCoh_A.

For $\mathrm{Perf}(A)$**:** By corollary (5.1.6, i), we have an equivalence

$$\mathrm{Perf}(A) \simeq \mathscr{D}(A)^{\mathrm{dual}} \xrightarrow{\simeq} \mathscr{D}((A, A^+)_{\blacksquare})^{\mathrm{dual}} \cap \mathscr{D}(A).$$

Since dualisability in $\mathscr{D}((A,A^+)_\blacksquare)$ (equivalent to compactness and nuclearity) implies discreteness by theorem (5.2.1), we have $\mathscr{D}((A,A^+)_\blacksquare)^{\mathrm{dual}} \subset \mathscr{D}(A)$. So we have

$$\mathrm{Perf}(A) \xrightarrow{\simeq} \mathscr{D}((A,A^+)_\blacksquare)^{\mathrm{dual}},$$

which satisfies analytic descent.

For PCoh_A**:** Any discrete module is nuclear in $\mathscr{D}((A,A^+)_\blacksquare)$, so we have an equivalence

$$\mathrm{PCoh}_A \to \mathscr{D}((A,A^+))^{\mathrm{pc}} \cap \mathscr{D}(A) = \mathscr{D}((A,A^+))^{\mathrm{pc}} \cap \mathscr{D}(A) \cap \mathrm{Nuc}((A,A^+)_\blacksquare)$$

by corollary (5.1.6, ii). By theorem (5.2.1), pseudocompactness and nuclearity implies discreteness, hence $\mathscr{D}((A,A^+))^{\mathrm{pc}} \cap \mathrm{Nuc}((A,A^+)_\blacksquare) \subset \mathscr{D}(A)$. Therefore we obtain

$$\mathrm{PCoh}_A \xrightarrow{\simeq} \mathscr{D}((A,A^+))^{\mathrm{pc}} \cap \mathrm{Nuc}((A,A^+)_\blacksquare),$$

which satisfies analytic descent. □

Part 2. **Complements**

6. Analytic K-theory

This section concerns the analytic K-theory, about which unfortunately I had no time to discuss during my talk.

6.1. **From algebraic to analytic K-theory.**

6.1.1**.** The *non-connective*[3] algerbaic K-theory of a scheme X is defined as

$$\mathbf{K}(X) := \mathbf{K}(\mathscr{D}(X)) := \mathbf{K}(\mathscr{D}(X)^\omega) = \mathbf{K}(\mathrm{Perf}(X)) \in \mathcal{S}\mathrm{p},$$

which fits into a more general theory, the Robert Thomason's (non-connective) K-theory $\mathbf{K}(\mathcal{C})$ of any compactly generated ∞-category $\mathcal{C}$ (which we think of as a "large" category, while idempotent complete stable ∞-categories are "small"); The compact generation is important here, as this type of large categories can be reconstructed from small ones, and

[3]Recall that the *connective* K-theory $K(-)$ is *not* a localising invariant on $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{idem}}$, because the Verdier quotient in this category is the usual Verdier quotient composed with the idempotent completion (i.e. the so-called *Karoubian closure*), and this last operation changes K_0 but not $K_{\geq 1}$; in fact, $K_0(\mathcal{C}) \to K_0(\mathcal{C}^{\mathrm{idem}})$ is injective (and isomorphism if and only if $\mathcal{C}$ is idempotent complete) and $K_i(\mathcal{C}) \xrightarrow{\simeq} K_i(\mathcal{C}^{\mathrm{idem}})$ for any $i \geq 1$. On the contrary, the non-connective K-theory $\mathbf{K}(-)$ *is* a localising invariant on $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{idem}}$ (with values in $\mathcal{S}\mathrm{p}$). So the better-behaved K-theory should be the non-connective one, which is the only one that we will consider. The two are related by $K(-) \to \Omega^\infty \mathbf{K}(-)$ which is an equivalence on connected covers.

$\mathbf{K}(\mathcal{C})$ is defined as $\mathbf{K}(\mathcal{C}^\omega)$. The presheaf $X \mapsto \mathbf{K}(X)$ on schemes satisfies Nisnevich descent since $\mathscr{D}(X)$ (and hence $\mathrm{Perf}(X) = \mathscr{D}(X)^\omega$) does so (it even satisfies fpqc descent) and verifies open-closed excision sequence in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{idem}}$.

6.1.2. Now we would like to pass from algebraic to analytic K-theory. There are at least two approaches:

(i) One way is perhaps via formal models. However, the K-theory of perfect complexes on formal schemes is *not* a reasonable invariant, because the restriction functor to an open subscheme is not a localisation in the categorical sense: for example, the functor $\mathscr{D}(\mathbf{Z}[[x,y]]) \to \mathscr{D}(\mathbf{Z}[[x,x^{-1},y]])$ is not induced by the localisation "inverting x^{-1}" (since $\mathbf{Z}[x][[y]][x^{-1}] \not\simeq \mathbf{Z}[x,x^{-1}][[y]]$), there should be involved the completion with respect to the element x^{-1}. Some topological information, e.g. completeness with respect to the lattice A^+, is missing in the usual (discrete) derived category of quasi-coherent sheaves.

(ii) Another idea is to consider the presheaf $\mathrm{Spa}(A,A^+) \mapsto K(\mathrm{Perf}(A))$ on complete Huber pairs that are locally Tate. Although this satisfies analytic descent, the Nisnevich descent fails.

Thus our slogans:

- *We should keep the topological information provided by A^+ in a Huber pair (A,A^+)!*
- *We need Nisnevich excision for our potential replacement of* $\mathrm{Spa}(A,A^+) \mapsto \mathrm{Perf}(A)$ *(or even $\mathscr{D}(A)$)!*

In order to record the topological information, the derived category $\mathscr{D}(A)$ of "quasi-coherent modules" has two good analogues in the analytic setting, namely $\mathscr{D}(A,A^+)_{\blacksquare}$ and its full subcategory $\mathrm{Nuc}(A,A^+) = \mathrm{Nuc}(A)$ (we write this last equality to stress the fact that the information carried by A^+ is lost in the category $\mathrm{Nuc}(A,A^+)$ although it is incorporated into the process of defining $\mathrm{Nuc}(A,A^+)$). Our first attempt is then to define

$$\mathbf{K}(\mathrm{Spa}(A,A^+)) \overset{?}{:=} \mathbf{K}(\mathscr{D}(A,A^+)_{\blacksquare})$$

or

$$\mathbf{K}(\mathrm{Spa}(A,A^+)) \overset{?}{:=} \mathbf{K}(\mathrm{Nuc}(A)).$$

However, there are two problems with them:

(i) The bigger category $\mathscr{D}(A,A^+)_{\blacksquare}$ is compactly generated, but its subcategory of compact objects $\mathscr{D}(A,A^+)_{\blacksquare}^{\omega}$ is too big to have nonzero $\mathbf{K}(-)$.

(ii) The less big category $\mathrm{Nuc}(A)$ is *not* compactly generated, hence we cannot apply Thomason's K-theory $\mathbf{K}(-)$ to it (recall that the latter is only defined for compactly generated categories).

To solve these problems, it would be better to work in between (i) and (ii).

6.2. **Issues explained.** We elaborate a little bit on the issues that have appeared in the previous subsection.

6.2.1. **Why does not $\mathbf{K}(\mathrm{Perf}(A))$ satisfies Nisnevich excision?** Of course, $\mathrm{Perf}(A)$ satisfies analytic descent and finite étale descent, so it satisfies Nisnevich descent. However, the problem comes when applying $\mathbf{K}(-)$: there is no open-closed excision *Verdier localising* sequence (in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{idem}}$) for $\mathrm{Perf}(-)$ on (even locally Tate) adic spaces, so that we do not obtain Nisnevich descent after applying $\mathbf{K}(-)$.

Recall that the algebraic open-closed excision Verdier sequence is proved by looking at $\mathscr{D}(A)$ which satsifies descent and admits an open-closed excision sequence. In the analytic case, we should look at $\mathscr{D}((A, A^+)_\blacksquare)$; but we are not able to recover such sequence for $\mathrm{Perf}(-)$ in this way.

Solution: Nevertheless, by doing similarly, one can obtain an open-closed excision Verdier sequence for $\mathrm{Nuc}(-)$ [And23, Lemma 5.11].

6.2.2. **Why is $\mathscr{D}(A, A^+)^\omega_\blacksquare$ too big?** It contains all $(A, A^+)_\blacksquare[S]$, hence admits countable products, so by Eilenberg's swindle, namely using the Hilbert's Hotel, we have

$$\mathbf{K}(\mathscr{D}(A, A^+)^\omega_\blacksquare) = *.$$

Solution: Consider a less big category than $\mathscr{D}((A, A^+)_\blacksquare)$.

6.2.3. **Why is not $\mathrm{Nuc}(A)$ compactly generated?** We have seen in the proof of theorem (1.4.1) that

$$\mathrm{Perf}(A) = \mathscr{D}(A)^\omega = \mathscr{D}(A)^{\mathrm{dual}} \xrightarrow{\simeq} \mathscr{D}((A, A^+)_\blacksquare)^{\mathrm{dual}} = \mathscr{D}((A, A^+)_\blacksquare)^\omega \cap \mathrm{Nuc}(A) \subset \mathrm{Nuc}(A)^\omega.$$

But we also have containment $\mathscr{D}((A, A^+)_\blacksquare)^\omega \cap \mathrm{Nuc}(A) \supset \mathrm{Nuc}(A)^\omega$ as can be seen via the right adjointable map $\mathrm{Nuc}(A) \to \mathscr{D}(A, A^+)$, whose right adjoint in $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$ is $(-)^{\mathrm{tr}}$ (6.3.2); so it preserves colimits by definition of morphisms in $\mathcal{P}\mathrm{r}^L$. Hence we have equivalences:

$$\mathrm{Perf}(A) = \mathscr{D}(A)^\omega = \mathscr{D}(A)^{\mathrm{dual}} \xrightarrow{\simeq} \mathscr{D}((A, A^+)_\blacksquare)^{\mathrm{dual}} = \mathscr{D}((A, A^+)_\blacksquare)^\omega \cap \mathrm{Nuc}(A) = \mathrm{Nuc}(A)^\omega,$$

whence the following commutative diagram:

$$\begin{array}{ccccc}
\mathscr{D}(A) & \hookrightarrow & \mathrm{Nuc}(A) & \hookrightarrow & \mathscr{D}((A,A^+)_\blacksquare) \\
\cup & & \cup & & \cup \\
\mathrm{Perf}(A)=\mathscr{D}(A)^\omega & \xrightarrow{\simeq} & \mathrm{Nuc}(A)^\omega & \longrightarrow & \mathscr{D}((A,A^+)_\blacksquare)^\omega.
\end{array}$$

In particular, $\mathrm{Nuc}(A)^\omega$ compactly generates only the full subcategory $\mathscr{D}(A)\subset \mathrm{Nuc}(A)$.

Solution/Question: Can one extend the non-connective K-theory $\mathbf{K}(-)$ beyond compactly generated categories? Yes! One can extend it to *dualisable* categories.

6.3. **Dualisable ∞-categories.**

6.3.1. **Definition** (Lurie)**.** A presentable stable ∞-category $\mathcal{C}$ is *dualisable* if it is a dualisable object in the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$ endowed with Lurie's tensor product, or equivalently, if it is a retract in $\mathcal{P}\mathrm{r}^L_{(\mathrm{st})}$ of compact generated ∞-category.

A canonical choice of this retract is the "unusual Yoneda embedding" with cone the *Calkin category of* $\mathcal{C}$

$$\mathcal{C}\overset{\hat{y}}{\hookrightarrow}\mathrm{Ind}(\mathcal{C}^{\omega_1})\to \mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C}),$$

whose right adjoints (in $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$) are

$$\mathcal{C}\overset{\mathrm{colim}}{\longleftarrow}\mathrm{Ind}(\mathcal{C}^{\omega_1})\hookleftarrow \mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C}).$$

Moreover, it can be shown that $\mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C})$ is compactly generated; its full subcategory of compact objects is denoted by $\mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C})^\omega$.

6.3.2. **Theorem.** *Let (A,A^+) be a (complete) Huber pair. Then the ∞-category $\mathrm{Nuc}(A)$ is dualisable. More precisely, the fully faithful embedding $\mathrm{Nuc}(A)\subset\mathscr{D}((A,A^+)_\blacksquare)$ admits an explicit retract in $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$ (i.e. a retract in Cat_∞ that preserves colimits) given by*

$$\begin{aligned}(-)^{\mathrm{tr}}:\mathscr{D}((A,A^+)_\blacksquare)\to\mathrm{Nuc}(A),\quad M\mapsto M^{\mathrm{tr}} &:= \varinjlim_{\substack{P\in\mathscr{D}((A,A^+)_\blacksquare)^\omega\\ \underline{A}\to P\otimes^L_{(A,A^+)_\blacksquare}M}} P^\vee \\ &= \varinjlim_{\substack{S\in\mathrm{EDS}\\ \underline{A}\to(A,A^+)_\blacksquare[S]\otimes^{(L)}_{(A,A^+)_\blacksquare}M}} \underline{C(S,A)}.\end{aligned}$$

In particular, $\mathrm{Nuc}(A)$ *is a dualisable object in* $\mathcal{P}\mathrm{r}^L_{\mathrm{st},/\mathrm{Perf}(A)}$.

In other words, via identification $\mathscr{D}((A,A^+)_\blacksquare) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathscr{D}((A,A^+)_\blacksquare)^\omega, \mathcal{S}\mathrm{p})$, the object M^{tr} is identified with the functor

$$P \mapsto (P^\vee \otimes^L_{(A,A^+)_\blacksquare} M)(*).$$

6.4. **Nuclear-continuous K-theory.**

6.4.1. Before introducing Efimov's K-theory, let us fix some notations. Let $\mathcal{P}\mathrm{r}^{L,\mathrm{dual}}_{\mathrm{st}}$ be the subcategory of $\mathcal{P}\mathrm{r}^{L}_{\mathrm{st}}$ consisting of dualisable objects; the morphisms are right adjointable ones (in $\mathcal{P}\mathrm{r}^{L,\mathrm{dual}}_{\mathrm{st}}$), i.e. the morphisms whose right adjoint preserves colimits; hence, there is a faithful but not full embedding $\mathcal{P}\mathrm{r}^{L,\mathrm{dual}}_{\mathrm{st}} \to \mathcal{P}\mathrm{r}^{L}_{\mathrm{st}}$. Let $\mathrm{Cat}^{\mathrm{perf}}_\infty$ be the category of small idempotent complete stable ∞-categories; the morphisms are exact morphisms; there is a fully faithful embedding $\mathrm{Cat}^{\mathrm{perf}}_\infty \to \mathcal{P}\mathrm{r}^{L,\mathrm{dual}}_{\mathrm{st}}$ sending $\mathcal{C} \mapsto \mathrm{Ind}(\mathcal{C})$; its essential image is the full subcategory $\mathcal{P}\mathrm{r}^{L,\mathrm{cg}}_{\mathrm{st}} \subset \mathcal{P}\mathrm{r}^{L,\mathrm{dual}}_{\mathrm{st}}$ consisting of compactly generated categories.

6.4.2. **Efimov's K-theory.** According to Lurie, the dualisable categories are precisely retracts in $\mathcal{P}\mathrm{r}^{L}_{\mathrm{st}}$ of compactly generated categories.

With the idea that $\mathbf{K}(-)$ extends to the larger class of dualisable categories by sending Verdier sequence to fiber sequence, and recalling the following canonical Verdier sequence in $\mathcal{P}\mathrm{r}^{L}_{\mathrm{st}}$

$$\mathcal{C} \overset{\hat{y}}{\hookrightarrow} \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C}),$$

Efimov defines the K-theory for dualisable categories $\mathcal{C}$ as

$$\begin{aligned}\mathbf{K}^{\mathrm{Ef}}(\mathcal{C}) &:= \mathrm{fib}(\mathbf{K}(\mathrm{Ind}(\mathcal{C}^{\omega_1})) \to \mathbf{K}(\mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C})))\\ &\simeq \mathrm{fib}(\mathbf{K}(\mathcal{C}^{\omega_1}) \to \mathbf{K}(\mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C})^\omega))\\ &\simeq \mathrm{fib}(* \to \mathbf{K}(\mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C})^\omega))\\ &\simeq \Omega\mathbf{K}(\mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C})^\omega).\end{aligned}$$

Here $\mathbf{K}(\mathcal{C}^{\omega_1}) = *$ by the Hilbert's Hotel arguement; the same holds if one replace ω_1 by any other (uncountable) regular cardinal $\kappa > \omega$.

One checks easily that if $\mathcal{C}$ is compactly generated, then

$$\mathbf{K}^{\mathrm{Ef}}(\mathcal{C}) = \Omega\mathbf{K}(\mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C})^\omega) \simeq \mathbf{K}(\mathcal{C}^\omega),$$

so Efimov's K-theory $\mathbf{K}^{\mathrm{Ef}}$ indeed extends $\mathbf{K}(-)$.

6.4.3. **Remark.** The procedure $\mathbf{K} \mapsto \mathbf{K}^{\mathrm{Ef}}$ can be done for any localising invariant $F \in \mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathscr{V})$, and associates with it a (unique) localising invariant $F_{\mathrm{cont}} \in \mathrm{Fun}(\mathcal{P}\mathrm{r}_{\mathrm{st}}^{L,\mathrm{dual}}, \mathscr{V})$ such that $F_{\mathrm{cont}} \circ \mathrm{Ind} = F$:

Theorem (Efimov). *Let $\mathscr{V}$ be a stable ∞-category (of coefficients). Then the natural map*

$$\mathrm{Fun}(\mathcal{P}\mathrm{r}_{\mathrm{st}}^{L,\mathrm{dual}}, \mathscr{V}) \to \mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathscr{V}), \quad G \mapsto G \circ \mathrm{Ind}$$

induces an equivalence between the full subcategories of localising invariants (i.e. functors preserving final objects and sending Verdier localising sequences to fiber sequences). The inverse is given by the continuous extension map

$$(F_{\mathrm{cont}} : \mathcal{C} \mapsto \Omega F(\mathcal{C}\mathrm{alk}_{\omega_1}(\mathcal{C})^\omega)) \leftarrow\!\shortmid F$$

constructed via the arguments of the above discussion (Hilbert's Hotel, etc.).

The stability of $\mathscr{V}$ may not be essential, and can be replaced by any "minimal" condition that guarantee the well-posedness of the theorem.

6.4.4. **Definition** (Nuclear-continuous K-theory)**.** For any (complete) Huber pair (A, A^+), the ∞-category $\mathrm{Nuc}(A)$ is dualisable, so we can define

$$\mathbf{K}^{\mathrm{nuc}}(A) := \mathbf{K}^{\mathrm{Ef}}(\mathrm{Nuc}(A)), \quad K^{\mathrm{nuc}}(A) := \Omega^\infty \mathbf{K}^{\mathrm{nuc}}(A),$$

This extends to arbitrary sheafy and locally Tate adic spaces X, since $\mathbf{K}^{\mathrm{nuc}}(-)$ satisfies analytic descent on sheafy and locally Tate affinoids.

Combined with Nisnevich descent of $\mathrm{Nuc}(-)$ and the open-closed excision Verdier sequence for $\mathrm{Nuc}(-)$, this implies that $\mathbf{K}^{\mathrm{nuc}}(-)$ satisfies Nisnevich descent.

6.5. **Continuous K-theory.**

6.5.1. **Definition.** Let (A_0, I) be an *adic pair*, i.e. $I \subset A_0$ is a finitely generated ideal of the ring A_0, which is endowed with I-adic topology for which it is complete. In this situation, one defines the *(non-connective) continuous K-theory* $\mathbf{K}_{\mathrm{cont}}(A_0)$ (à la Morrow) without appealing to the condensed mathematics, namely as

$$\mathbf{K}_{\mathrm{cont}}(A_0) := \varprojlim_n \mathbf{K}(A_0/I^n),$$

which underlies the pro-spectrum "$\varprojlim_n$" $\mathbf{K}(A_0/I^n)$ or the condensed spectrum

$$\underline{\mathbf{K}_{\mathrm{cont}}}(A_0) := \varprojlim_n \underline{\mathbf{K}(A_0/I^n)} \in \mathcal{C}\mathrm{ond}(\mathcal{S}\mathrm{p}).$$

This definition plays a central role in the classical approach to the K-theory for rigid-analytic spaces.

If moreover the ideal I is *weakly pro-regular* (which is always the case in rigid-analytic geometry and more generally if (A_0, I) is a pair of definition of a complete Huber ring that is Tate), Andreychev's nuclear-continuous K-theory and the classical definition agree by the following theorem due to Efimov.

6.5.2. **Theorem** (Efimov's Continuity Theorem, cf. report here)**.** *Let (A_0, I) be an adic pair, where the finitely generated ideal of definition $I \subset A_0$ is weakly pro-regular. Then the natural maps*

$$\mathbf{K}^{\mathrm{nuc}}(A_0) \to \mathbf{K}_{\mathrm{cont}}(A_0/I^n) \simeq \mathbf{K}(A_0/I^n), \quad n \geq 0$$

induce isomosphisms in the ∞-category $\mathcal{S}\mathrm{p}$

$$\mathbf{K}^{\mathrm{nuc}}(A_0) \xrightarrow{\simeq} \mathbf{K}_{\mathrm{cont}}(A_0) = \varprojlim_n \mathbf{K}(A_0/I^n).$$

6.5.3. **Definition.** For any Tate complete Huber ring A with a pair of definition (A_0, I) and a pseudouniformiser $\varpi \in A^\times \cap A_0^{\circ\circ}$ (so that $A = A_0[\frac{1}{\varpi}]$), one defines the continuous K-theory $\mathbf{K}_{\mathrm{cont}}(A)$ as the pushout

(6.5.3.1)
$$\begin{array}{ccc} \mathbf{K}(A_0) & \longrightarrow & \mathbf{K}(A) \\ \downarrow & & \downarrow \\ \mathbf{K}_{\mathrm{cont}}(A_0) & \longrightarrow & \mathbf{K}_{\mathrm{cont}}(A) \end{array}$$

within the (stable) ∞-category $\mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p})$.

It is well-defined (i.e. does not depend on the choice of (A_0, I) for the given A) up to weak equivalence (i.e. becoming an equivalence when applying the functor $\mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p}) \to \mathrm{pro}^{\mathrm{light}}(\mathcal{S}\mathrm{p}^+)$) [KST19, Proposition 5.4].

The hereby defined continuous K-theory of Tate rings is then globalised into rigid-analytic spaces by the so-called *pro-cdh descent*. The obvious conceptual problem of this approach, namely the fact that all constructions are somewhat *ad hoc* (one needs to check well-definedness and functoriality), means that the proofs are difficult to grasp in the *non-Noetherian* case. In particular, it is not clear how to prove the descent for general analytical adic spaces (e.g. perfectoid spaces). However, with Andreychev's more abstract definition, we can prove the optimal *descent theorem.*

For this, we show the following consequence of Efimov's Continuity Theorem (6.5.2):

6.5.4. **Theorem.** *For Tate complete Huber rings, the two definitions of analytic K-theory, namely the continuous one and the nuclear-continuous one, match naturally, i.e. there is a natural isomorphism of spectra*

$$\mathbf{K}_{\mathrm{cont}}(A) \xrightarrow{\simeq} \mathbf{K}^{\mathrm{nuc}}(A).$$

Proof. Indeed, observe the following diagram:

(6.5.4.1)
$$\begin{tikzcd} \mathrm{Tor}(\varpi^\infty) \arrow[r] \arrow[d] & \mathscr{D}(A_0) \arrow[r, "L"] \arrow[d] & \mathscr{D}(A_0[\frac{1}{\varpi}]) \arrow[d] \\ \mathrm{Tor}^{\mathrm{nuc}}(\varpi^\infty) \arrow[r] & \mathrm{Nuc}(A_0) \arrow[r, "L'"] & \mathrm{Nuc}(A_0[\frac{1}{\varpi}]). \end{tikzcd}$$

Here L and L' denote the canonical localisation functors and $\mathrm{Tor}(\varpi^\infty)$ and $\mathrm{Tor}^{\mathrm{nuc}}(\varpi^\infty)$ denote their kernels. Then the function $\mathrm{Tor}(\varpi^\infty) \to \mathrm{Tor}^{\mathrm{nuc}}(\varpi^\infty)$ is an equivalence according to [And23, Satz 4.11], hence the diagram

(6.5.4.2)
$$\begin{tikzcd} \mathbf{K}(A_0) \arrow[r, "L"] \arrow[d] & \mathbf{K}(A_0[\frac{1}{\varpi}]) \arrow[d] \\ \mathbf{K}^{\mathrm{nuc}}(A_0) \arrow[r, "L'"] & \mathbf{K}^{\mathrm{nuc}}(A_0[\frac{1}{\varpi}]) \end{tikzcd}$$

is a pullpack-pushout square in the stable ∞-category $\mathcal{S}\mathrm{p}$. By Efimov's Continuity Theorem (6.5.2), we have $\mathbf{K}^{\mathrm{nuc}}(A_0) \xrightarrow{\simeq} \mathbf{K}_{\mathrm{cont}}(A_0)$, hence we get $\mathbf{K}_{\mathrm{cont}}(A) \xrightarrow{\simeq} \mathbf{K}^{\mathrm{nuc}}(A)$. $\square$

6.6. **Condensed nuclear-continuous K-theory.** It is possible to upgrade the isomorphism of spectra (6.5.4) to an isomorphism of condensed spectra. For this, we first upgrade $\mathbf{K}^{\mathrm{nuc}}(A)$ to a condensed spectra $\underline{\mathbf{K}^{\mathrm{nuc}}}(A)$.

6.6.1**. Construction.** Let (A, A^+) be a (complete) Huber pair. Recall that for any profinite set S, the object $\mathrm{Spa}(A, A^+) \times \underline{S}$ is the well-defined adic space associated with the (complete) Huber pair $(C(S, A), C(S, A^+))$. Consider the presheaf

$$\begin{aligned} \underline{\mathbf{K}^{\mathrm{nuc}}}(A) : \mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in} &\to \mathcal{S}\mathrm{p} \\ S &\mapsto \mathbf{K}^{\mathrm{nuc}}(C(S, A)) \end{aligned}$$

Of course, it sends finite disjoint union to products, so defines a sheaf on EDS.

6.6.2. **Theorem.** *Let (A_0, I) be an adic pair with $I \subset A_0$ weakly pro-regular ideal. The presheaf $\underline{\mathbf{K}^{\mathrm{nuc}}}(A_0)$ is a sheaf on $\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}$. More precisely, for any profinite set S and any hypercovering $S_\bullet \to S$ by extremally disconnected sets, we have a natural isomorphism*

$$\mathbf{K}^{\mathrm{nuc}}(C(S, A_0)) \xrightarrow{\simeq} \varprojlim_{\Delta} \mathbf{K}^{\mathrm{nuc}}(C(S_\bullet, A_0)).$$

Proof. (Indicated by Clausen) We have a $\mathbf{Z}_{\blacksquare}$-solid algebra $\underline{A_0} = \varprojlim_n \underline{A_0/I^n}$. Since the natural map

$$\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathbf{Z}[S_\bullet] \to \mathbf{Z}[S]$$

becomes an equivalence after $\mathbf{Z}_{\blacksquare}$-solidification [CS19, Proposition 5.6], after taking (external) $R\,\mathrm{Hom}(-, \underline{A_0})$ we obtain a cosimplicial resolution (of rings)

$$C(S, A_0) \xrightarrow{\simeq} \lim_{\Delta} C(S_\bullet, A_0).$$

By Efimov's Continuity Theorem (6.5.2), there is a natural isomorphism of spectra

$$\mathbf{K}^{\mathrm{nuc}}(C(S, A_0)) \xrightarrow{\simeq} \varprojlim_{n} \mathbf{K}(C(S, A_0/I^n)).$$

We claim that

$$\mathbf{K}(C(S, A_0/I^n)) \xrightarrow{\simeq} \lim_{\Delta} \mathbf{K}(C(S_\bullet, A_0/I^n)), \quad n \in \mathbf{N}.$$

Indeed, by writing the hypercovering $S_\bullet \to S$ as a cofiltered limit of hypercoverings $S_{\bullet,j} \to S_j$ (indexed by $j \in J$) of finite sets by finite sets, which in particular splits, we have

$$\mathbf{K}(C(S_j, A_0/I^n)) \xrightarrow{\simeq} \lim_{\Delta} \mathbf{K}(C(S_{\bullet,j}, A_0/I^n)).$$

Now take the filtered colimit with respect to $j \in J^{\mathrm{op}}$: as the algebraic K-theory $\mathbf{K}(-)$ commutes with filtered colimits of rings and $C(S, A_0/I^n) = \varinjlim_j C(S_j, A_0/I^n)$, similarly termwisely $C(S_\bullet, A_0/I^n) = \varinjlim_j C(S_{\bullet,j}, A_0/I^n)$, we obtain

$$\begin{aligned}
\mathbf{K}(C(S, A_0/I^n)) \simeq \varinjlim_{j} \mathbf{K}(C(S_j, A_0/I^n)) &\xrightarrow{\simeq} \varinjlim_{j} \lim_{\Delta} \mathbf{K}(C(S_{\bullet,j}, A_0/I^n)) \\
&\xrightarrow{\simeq} \lim_{\Delta} \varinjlim_{j} \mathbf{K}(C(S_{\bullet,j}, A_0/I^n)) \\
&\simeq \lim_{\Delta} \mathbf{K}(C(S_\bullet, A_0/I^n))
\end{aligned}$$

The second to last isomorphism can be seen by comparing the *convergent* spectral sequences

$$\begin{aligned}
E_{1,(j)}^{p,q} = \pi_{-q}(\mathbf{K}(C(S_{p,j}, A_0/I^n))) &\Rightarrow \pi_{-(p+q)}(\mathbf{K}(C(S_j, A_0/I^n))) \\
E_1^{p,q} = \pi_{-q}(\mathbf{K}(C(S_p, A_0/I^n))) &\Rightarrow \pi_{-(p+q)}(\mathbf{K}(C(S, A_0/I^n))).
\end{aligned}$$

Indeed, $\pi_{-(p+q)}$ commutes with filtered colimits, and

$$\pi_{-q}\mathbf{K}(C(T, A_0/I^n)) \simeq \mathrm{Map}(T, \pi_{-q}\mathbf{K}(A_0/I^n)) \simeq C(T, \mathbf{Z}) \otimes_{\mathbf{Z}} \pi_{-q}\mathbf{K}(A_0/I^n)$$

for any finite set T, so the the above spectral sequences degenerates already at the page E_1 by exactness of the complexes

$$0 \to C(S_j, \mathbf{Z}) \to C(S_{1,j}, \mathbf{Z}) \to C(S_{2,j}, \mathbf{Z}) \to \cdots, \quad j \in J.$$

$\square$

6.6.3. **Corollary.** *For any Tate complete Huber ring A, we have a natural morphism*

$$\mathbf{K}^{\mathrm{nuc}}(C(S, A)) \xrightarrow{\simeq} \varprojlim_{\Delta} \mathbf{K}^{\mathrm{nuc}}(C(S_\bullet, A_0)),$$

in particular, $\underline{\mathbf{K}^{\mathrm{nuc}}}(A)$ is a sheaf on $\mathcal{P}\mathrm{ro}\mathcal{F}\mathrm{in}$.

Proof. This follows from the theorem as in the proof of (6.5.4), using the diagram (6.5.4.2). $\square$

6.6.4. **Theorem.** *For any Tate complete Huber ring A, there is a natural morphism of condensed spectra*

$$\underline{\mathbf{K}_{\mathrm{cont}}}(A) \xrightarrow{\simeq} \underline{\mathbf{K}^{\mathrm{nuc}}}(A).$$

Proof. For any profinite set S, the topological ring $C(S, A)$ is still a Tate complete Huber ring, so we may apply theorem (6.5.4) to conclude. $\square$

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