Linear Algebra Review

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ő Vector Spaces

A *vector space* (over $\mathbb{R}$) is a set $\mathcal{V}$ with two operations, + and * that satisfy for

$u, v, w \in \mathcal{V}$ and $a, b \in \mathbb{R}$.

- $u + v \in \mathcal{V}$
- There exists $0 \in \mathcal{V}$ such that $0 + v = v$
- There exists $-v \in \mathcal{V}$ such that $v + -v = 0$
- $u + v = v + u$
- $(u + v) + w = u + (v + w)$
- $a * v \in \mathcal{V}$
- $1 * v = v$
- $a * (b * v) = (ab) * v$
- $a * (u + v) = a * u + a * v$
- $(a + b) * v = a * v + b * v$

ő *n*-Dimensional Reals

An example of a vector space is the *n*-dimensional reals, $\mathbb{R}^n$.

In this space the vectors are of the form $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ with $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Addition (+) and scalar multiplication (*) are defined elementwise, i.e.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} \quad \text{and} \quad c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}.$$

ő Inner Product

Let $\mathcal{V}$ be a vector space over $\mathbb{R}$, then $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is an *inner product* if:

- $\langle au, v \rangle = a\langle u, v \rangle$
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $\langle u, v \rangle = \langle v, u \rangle$
- $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

Example: The dot product in $\mathbb{R}^n$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \ldots a_n b_n$$

ő Norms

Let $\mathcal{V}$ be a vector space over $\mathbb{R}$, $||\cdot||:\mathcal{V}\rightarrow\mathbb{R}$ is a *norm* if:

- $||v|| \geq 0$ with equality if and only if $v = 0$.
- $||a * v|| = |a|\ ||v||$
- $||u + v|| \leq ||u|| + ||v||$, this is known as the *triangle inequality*.

ő Common Norms for $\mathbb{R}^n$

Let v be a vector in $\mathbb{R}^n$, then some common norms are:

- $||v||_1 = |v_1| + |v_2| + \cdots + |v_n|$
- $||v||_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
- $||v||_p = (|v_1|^p + |v_2|^p + \cdots + |v_n|^p)^{1/p}$
- $||v||_\infty = \max_{i=1,\ldots,n} \{|v_i|\}$

Unit Vectors

Let $\mathcal{V}$ be a vector space with a norm $||\cdot||$.

We call a vector a *unit vector* if $||v|| = 1$.

ő Spans

Let *U* be a set of vectors in a vector space, $\mathcal{V}$, the *span* of *U* is the set of all linear combinations of vectors in *U*.

ő Linear Dependence/Independence

Let $v_1, v_2, \ldots, v_n \in \mathcal{V}$ be a collection of vectors in a vector space, $\mathcal{V}$.

We say the collection is *linearly dependent* if there exists a vector, $v_j \neq 0$, such that:

$$v_j = \sum_{i \neq j} a_i v_i \ .$$

If a collection is not linearly dependent, then we say it is *linearly independent*.

ő Basis of a Vector Space

Let $\mathcal{B}$ be a set of vectors from a vector space $\mathcal{V}$.

We say that $\mathcal{B}$ forms a *basis* of $\mathcal{V}$ if $\mathcal{B}$ is linearly independent and $\operatorname{span}(\mathcal{B}) = \mathcal{V}$.

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Example: A basis for $\mathbb{R}^n$ is:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

This is called the *standard basis* or *natural basis*

Orthogonal Vectors

Let $u, v \in \mathbb{R}^n$, $u, v \neq 0$. We say that u and v are *orthogonal* vectors if:

$$u \cdot v = 0$$

Orthonormal Basis

A basis, $\mathcal{B}$, is *orthonormal* if it consists of unit vectors that are orthogonal to one another.

ő Matrices

An $m \times n$ matrix *A* is an array of numbers with m rows and n columns.

Examples: $\begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

<u>Important Matrices</u>

The *zero matrix* is a matrix where each entry is zero.

The *identity matrix* is a matrix where each diagonal element is one and the rest are zero. The second matrix above is the 3 x 3 identity matrix. This is denoted as I

ő Matrix Addition

Matrix addition is done elementwise.

For example if the i, j entry of *A* is $a_{i,j}$ and the i, j of *B* is $b_{i,j}$ then the i, j of *A* + *B* is $a_{i,j} + b_{i,j}$.

ő Matrix-Matrix Multiplication

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then we can define *matrix multiplication* like so:

If $C = AB$, then C is an $m \times p$ matrix with

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + a_{i,3}b_{3,j} + \cdots + a_{i,n}b_{n,j}$$

as its i, j entry.

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1*0+2*2 & 1*1+2*1 \\ 3*0+1*2 & 3*1+1*1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & 4 \end{pmatrix}$$

ő Matrix-Vector Multiplication

If you treat a vector like a matrix you can use the rules of matrix-matrix multiplication we defined in the previous slide to multiply a vector by a matrix.

ő Trace of a Matrix

Let A be an $n \times n$ matrix, the *trace* of A is the sum of its diagonal elements, i.e.
$\mathrm{Tr}(A) = a_{1,1} + a_{2,2} + \cdots + a_{n,n}$.

ő Transpose

Let A be an $m \times n$ matrix. We define the *transpose* of A to be the matrix whose i, j entry is the j, i entry of A. This is denoted as A^T .

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \qquad A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

If A = A^T we say that A is a *symmetric* matrix.

ő Inverse

Let A be an $n \times n$ matrix. If there is a matrix B so that $AB = BA = I$, then we call B the *inverse* of A. We would denote B as A^{-1}.

If a square matrix has an inverse, we say it is *invertible* or *non-singular*. If a square matrix does not have an inverse, it is a *singular* matrix.

Example:

If $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$.

ő Singular Matrices

A matrix is singular if either:

- The vectors corresponding to the columns of the matrix are linearly dependent or
- The vectors corresponding to the rows of the matrix are linearly dependent.

Rank of a Matrix

The *rank* of a matrix is:

- The size of largest set of linearly independent columns or equivalently
- The size of largest set of linearly independent rows

An n x n square matrix is said to be of *full rank* if its rank is equal to n.

ő Eigenvalues and Eigenvectors

An $n \times n$ matrix A has an *eigenvalue*, λ, if there exists a vector $x \neq 0$ such that $Ax = \lambda x$. The vector x is known as an *eigenvector* for λ.

ő Singular Values

Let A be an $m \times n$ matrix. The *singular values* of A are the square roots of the eigenvalues of $A^T A$. These are typically denoted with a σ.

If A has rank r, then A has r singular values (counting duplicates).

ő Singular Value Decomposition

Let A be an $m \times n$ matrix with rank r .

The *singular value decomposition* of A is a way to factor A into a product of three matrices:

$$A = U\Sigma V^T$$, where:

- U is an $m \times m$ matrix with orthogonal rows and orthogonal columns (known as an orthogonal matrix),
- V is an $n \times n$ orthogonal matrix and
- Σ is an $m \times n$ matrix whose first r diagonal entries correspond to the singular values of A in descending order and whose remaining entries are zero.

The columns of U are known as *left singular vectors* and the columns of V^T are known as the *right singular vectors*.

ő Properties of Symmetric Matrices

Let A be an $n \times n$ symmetric matrix, then:

- A has real eigenvalues
- XAX^T is also a symmetric matrix for any matrix, X for which the multiplication is possible

ő Positive Definite Matrices

An $n \times n$ matrix A is *positive definite* if for any $x \in \mathbb{R}^n$, $x \neq 0$ it holds that $x^T A x > 0$.

Positive definite matrices have the following properties:

- They have full rank,
- Their eigenvalues are strictly positive and
- They are symmetric

ő Further Notes on Matrices

For additional notes on matrices see the Matrix Cookbook from the University of Waterloo, https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf.