Quantum Physics

At Any Cost

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To the Great and Beautiful Nation
Which Gave Me Everything and More:
A Shelter, Opportunities and Inspiration,
And The Desire To Explore

Contents

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My friends, discussing the book with you was both illuminating and fun.

Finally, special acknowledgment must be given to my son, Daniel, for his help with fixing colors in many figures.

Yury Deshko
Weehawken, New Jersey
2024

Preface

This book is the result of lectures delivered to curious, motivated, and studious high schoolers. The lectures ran during the years 2019-2024 in various formats, but mostly in class during a three week summer school organized by Columbia University Pre-College Programs. Additionally, the same lectures were taught remotely to selected students of Ukrainian Physics and Mathematics Lyceum.

The material has been designed to be accessible to people with solid background in high-school algebra and physics (mostly mechanics). Several years of teaching to a relatively diverse set of students proved that nearly all material can be efficiently absorbed by most, provided diligent work is done on exercises and problem. The last fact confirms a well-known truism: *No real learning occurs without practice.*

Exercises are essential part of this book. They are carefully selected to help readers get better understanding of the material and they are also fully solved. The difficulty of the exercises varies from simple to quite challenging.

This book *is not a standard textbook*. It differs from many excellent introductions into Quantum Physics in that it lacks the breadth and rigor of the latter. However, this book serves a special purpose: It tries to act as the *bridge* between elementary and popular books and the more challenging college-level textbooks.

If a picture is worth a thousand words, then a formula is worth a couple of hundred words. This book contains pictures and formulas aplenty. Hopefully, the readers for whom this book is intended will enjoy both.

Some sections are marked with an asterisk, for example **Transposition***. Those sections contain material that is either optional or a bit more advanced that usual. These sections can be skipped without significant impact on the main message of the book.

At Any Cost

The subtitle of this book has been inspired by the letter from Max Karl Ernst Ludwig Planck to an American physicist Robert Williams Wood. Describing his desperate attempts to explain the experimental results on the electromagnetic radiation from hot materials, Max Planck wrote[1] (italics are mine):

> ***Max Planck to Robert Wood***
> A theoretical interpretation therefore had to be found *at any cost*, no matter how high. It was clear to me that classical physics could offer no solution to this problem, and would have meant that all energy would eventually transfer from matter to radiation. ...This approach was opened to me by maintaining the two laws of thermodynamics. The two laws, it seems to me, must be upheld under all circumstances. For the rest, I was ready to sacrifice every one of my previous convictions about physical laws. ...[One] finds that the continuous loss of energy into radiation can be prevented by assuming that energy is forced at the outset to remain together in certain quanta. This was purely a formal assumption and I really did not give it much thought except that *no matter what the cost, I must bring about a positive result.*

Trying to provide a theoretical explanation at any cost, Max Planck introduced the idea of energy quanta, initiating the development of quantum ideas and becoming "the father of quantum physics."

[1] Source!

1. Introduction

This quantum business is so incredibly important and difficult that everyone should busy himself with it.

A. Einstein in a letter to his friend Jakob Laub in 1908, as quoted by A. Wheeler in "The Mystery and The Message Of The Quantum"

Abstract In this chapter.

QUANTUM PHYSICS IS A CENTURY-OLD BRANCH OF PHYSICS. ITS SUCCESS is unparalleled and yet quantum physics is unfinished in one sense: There is no clear and widely adopted consensus on what some of quantum ideas "really mean."

The progression of the topics from numbers to tensors can be viewed as follows:

$$\begin{array}{lclcl} \text{Numbers} & \rightarrow & \text{Vectors} & \rightarrow & \text{Tensors.} \\ \text{Tensors}^{(0)} & \rightarrow & \text{Tensors}^{(1)} & \rightarrow & \text{Tensors}^{(2+)}\,. \end{array}$$

Here the superscript in parentheses indicates the rank of the tensor[1].

As we move from numbers to tensors, the level of abstraction increases. To a significant degree, the difficulty of understanding tensors is due to high level of abstraction used in the definition of tensors as mathematical objects. Abstraction is the price we pay for more powerful and versatile tools. But more powerful tools are needed as scientists address more and more advanced problems.

1.1 What Is Quantum Physics?

In October of 1912, Albert Einstein wrote in a letter to his physicist

[1] Don't worry if the concept of *rank* seems unclear right now – it will be explained in due time.

1.2 Brief Historical Context

In October of 1912, Albert Einstein wrote in a letter to his physicist

1.3 Who Needs Quantum Physics?

In October of 1912, Albert Einstein wrote in a letter to his physicist friend Arnold Sommerfeld:

> ***Example of mybio environment***
> I am now exclusively occupied with the problem of gravitation theory and hope, with the help of a local mathematician friend, to overcome all the difficulties. One thing is certain, however, that never in my life have I been quite so tormented. A great respect for mathematics has been instilled within me, the subtler aspects of which, in my stupidity, I regarded until now as a pure luxury. Against this problem [of gravitation] the original problem of the theory of relativity is child's play.

In the period from 1905 to 1916 Einstein was feverishly working on the General Theory of Relativity – the next best theory of gravity since Newton. The mathematics of general relativity is based on the calculus of tensors, created by Italian mathematicians Ricci-Curbastro and Levi-Civita roughly a decade before Einstein started working on the problem of gravity.

1.4 Why is Quantum Physics Hard?

Now what are tensors more rigorously? Can we give a short definition to this concept? Let us take a look at several examples and see whether they shed sufficient light. The definitions given below differ from each other, but they simply convey *the same idea in different ways.* sectionChallenges To illustrate the concepts of functions, operators, their structures and properties, we will be using schematics like the one

1.4.1 Language

To illustrate the concepts of functions, operators, their structures and properties, we will be using schematics like the one

The Encyclopedia of Mathematics [2] provides the following definition:

Definition 1.1 ☞ *Example of mydef environment*

Tensor on a vector space V over a field k is an element t of the vector space

$$T^{p,q}(V) = (\otimes^p V) \otimes (\otimes^q V^*),$$

where $V^* = \mathrm{Hom}(V, k)$ is the dual space of V.

To understand this defintion we first need to understand what *vector space* is, what *field* is, what *dual* means, and what is going on with superscripts and circles (e.g., in $\otimes^q$).

1.5 Quantum Versus Classical

Sometimes to illustrate mathematical concepts and *relations between them*, we will use diagrams. Diagrams are helpful in highlighting some general features of *mathematical structures*.

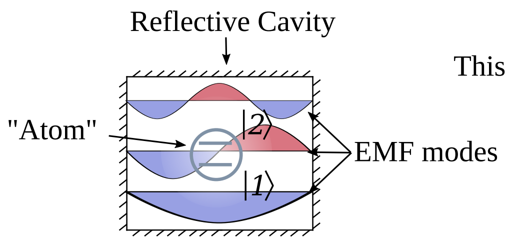

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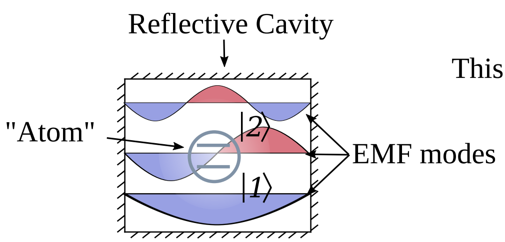
Fig. 1.1: Diagrams are used to graphically represent sets of objects and relationships between them. Arrows can connect (map) elements of one set with another. Such mappings may have names: **mlg** returns mileage for a given car, **clr** – color, and **smk** determines whether two cars are of the same make.

A particular property of a car-point can then be represented using an arrow that connects the car-point to another point in the relevant set. We say that such an arrow *maps* points of one set into another set. The Figure1.1(b) shows three maps: **mlg** gives the mileage for each car from

[2] `https://encyclopediaofmath.org/wiki/Tensor_on_a_vector_space`

Fig. 1.2: Schematics can be used to represent functions, operators, their compositions and structure.



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the set Λ, **clr** gives the color for each car, and **smk** compares whether two cars have the same make.

Exercise 1.1
Extend the diagram from the Figure 1.1(b), adding a set of different car makes (e.g., Ford, Toyota, Fiat, etc.) Come up with a mapping from this set into the Boolean set B.

1.6 Quantum Puzzle

To illustrate the concepts of functions, operators, their structures and properties, we will be using schematics like the one shown in the Figure 1.2.

A simple schematic element is represented as a box with inputs and outputs. A box can have a name (label) which describes what the function does to its input. The number of inputs and outputs can vary depending on the complexity of a function.

Chapter Highlights

- *Natural evolution of mathematical objects from numbers, through vectors, leads to tensors.*
- *Each successive tier of mathematical object in the progression "numbers, vectors, tensors" is more abstract and more powerful.*
- *Numbers, vectors, and tensors are all conceptually connected.*

2. Physics

NUMBERS are powerful mathematical objects. They are used to solve an endless list of problems that involve *quantities*. As mathematics and sciences progressed, natural numbers evolved into whole numbers, then into rational numbers and beyond.[1]

> ***Prerequisite Knowledge***
> To fully understand the material of this chapter, readers should be comfortable with the following concepts:
> - State
> - Dynamical equations

2.1 Goals and Methods

Physics is a *human* activity pursuing the following major goals: *Describe*, *explain*, and *predict* phenomena comprising the observed world.

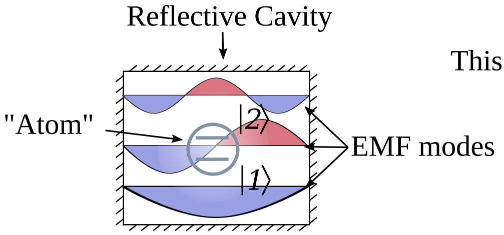
results can be applied in a wide range of fields. In part, the universality of mathematics stems from the *general* and *abstract* nature of mathematical concepts. Let us illustrate this using an example.

An astute farmer notices that 49 sacks of grains can be arranged in a square with each side having 7 sacks (see the Figure 2.1). When one sack is used up, the remaining 48 sacks can be arranged as a rectangle 6 by 8 sacks.

> **Exercise 2.1**
> Think how you would represent the generalized relations of the types

[1] A superb account of this process is given in the book *"Number: The Language of Science"* by Tobias Dantzig.

Fig. 2.1: 49 objects can be arranged in a square 7x7. 48 objects can be arranged as a rectangle of 6x8.



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given in the Figure ?? at the level of sets? What kind of diagrams would you draw?

2.2 Common Sense

Mathematics is a remarkably effective and universal discipline, its methods and results can be applied in a wide range of fields.

2.2.1 Detached Observer

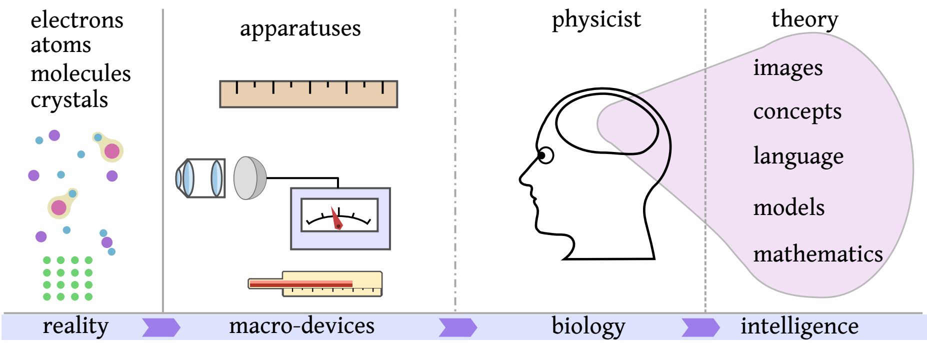

Fig. 2.2: Observer in classical view of the world is detached, separated from the "true" reality.

2.3 Deterministic Evolution

The completeness of a state is a very strong constraint. Not only it means "everything there is to know at a given moment", but also "know state now – know state always." The latter is an expression of *determinism*: the knowledge of a system is completely determined once the state and its law of evolution are known.

However, state by itself is not sufficient to satisfy the latter requirement, it must be supplemented by the so called *dynamical equations*. These equations are specific to a physical system and encapsulate the laws that govern internal interactions. EXAMPLE?

Denoting the mathematical representation of the state as ξ, the evolution of the state between the moments of time $T = t$ and $T = \tau$ may be written as a functional dependence:

$$\xi_\tau = U_{\tau,t}\,\xi_t\,.$$

For $\tau > t$ we determine the future state, while for $\tau < t$ we determine the state in the past (relative to the moment t).

Example

For circular motion the state is the angle $\xi = \phi$, and the evolution is given by a simple formula

$$\phi_\tau = U_{\tau,t}\,\phi_t = \omega(\tau - t) + \phi_t\,.$$

Notice that in this case the evolution function depends on the time *difference* $\tau - t$ and not on each moment of time separately:

$$U_{\tau,t} = U_{\tau-t}\,.$$

It must be emphasized again, that the final state $\xi_f = \xi_\tau$ is determined by two factors: the initial state $\xi_i = \xi_t$ and the laws of physics encoded in the evolution function $U_{\tau,t}$.

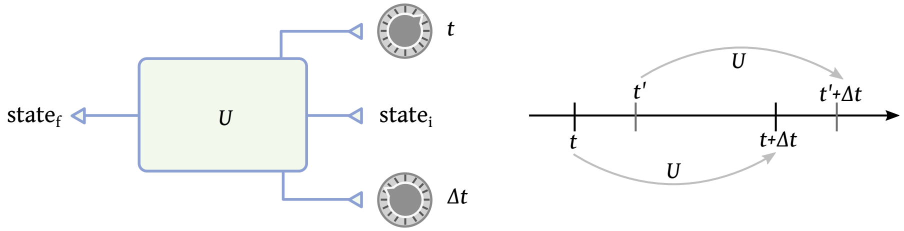

Fig. 2.3: Evolution operator transforms an initial state into the final state in time Δt.

The laws of physics are timeless[2], as illustrated by the Coulomb's law of interaction between charges q and Q at a distance r apart: $F_C = kqQ/r^2$. The timeless nature of the physical laws requires that the same initial state ξ_i evolves into the same final state ξ_f regardless of when the evolution starts and as long as the time interval between the beginning and the end of evolution is the same. Mathematically this is expressed as

[2]Technical term is *time-translation invariant.*

follows:

$$U_{\tau,t}\,\xi_i = U_{\tau',t'}\,\xi_i$$

for *any* initial state ξ_i, as long as $\tau' - t' = \tau - t = \Delta t$.

Thus, for *any* values of t and t', we have

$$U_{t+\Delta t,t} = U_{t'+\Delta t,t'}\,.$$

This equation says that the evolution function U becomes insensitive to the values t and t', and only depends on Δt – the time interval between the beginning and the end of evolution. Therefore, we can write the following connection between the states at different moments:

$$\xi_{t+\Delta t} = U_{\Delta t}\,\xi_t\,.$$

This connection holds *for any moment of time* t and time interval Δt.

In physics the states are represented using numbers, vectors, functions, and similar mathematical objects. Common to all of these types of objects is a very basic property of "additivity" and "scalability". That is, one can – at least formally – add and subtract states, as well as multiply them by numbers. For example, for any two states ξ_1 and ξ_2, one can write equations like

$$\xi_3 = 2\xi_1 + 3\xi_2 \qquad \text{or} \qquad \Delta\xi = \xi_2 - \xi_1\,.$$

Depending on a particular representation of the state, the evolution function U might be a "usual" function, an operator, or something else entirely. Regardless of what the exact *type* of U is, its job is always the same – map initial state ξ_i at time t into the final state ξ_f at time $t + \Delta t$.

For $\Delta t = 0$ the evolution function U must be a simple *identity* function:

$$U_0 = I\,.$$

Furthermore, for a continuous evolution, it is necessary for small changes in time δt to produce small changes in the state $\delta\xi$:

$$\xi_{t+\delta t} = U_{\delta t}\xi_t = \xi_t + \delta\xi\,.$$

For a continuous evolution of the state, the evolution function U must be continuous. This implies that for small time intervals it produces small changes:

$$U_{\delta t} \approx I + \delta U = I + G\delta t\,,$$

where G is called the *generator* of state evolution. The meaning of the generator is clear from its definition – it specifies how fast the state evolution happens: $G = \partial_t U$.

In terms of the generator, the evolution equation can be written using the relations

$$\delta\xi = \xi_{t+\delta t} - \xi_t = (I + G\delta t)\,\xi_t - \xi_t = G\delta t\xi_t\,.$$

Finally, dividing both sides by δt and using the ∂-notation, we arrive at the Schrodinger-type of equation for the *continuous* state evolution:

$$\partial_t\xi = G\,\xi_t\,. \tag{2.1}$$

The equation (2.1) is a general form of state evolution equations used in physics. It appears in many cases where the dynamics of a system is described as a *continuous deterministic evolution.*

> ***Deterministic Evolution Equations***
> Equations similar to (2.1) can be found in many physical theories. In quantum theory it is Schrodinger equation, which can be written as follows:
> $$\partial_t|\Psi\rangle = -i\hat{H}\,|\Psi\rangle\,.$$

2.4 Classical and Quantum

Mathematics is a remarkably effective and universal discipline, its methods and results can be applied in a wide range of fields.

2.5 State

State is a very important concept in physics. It means *complete* but *minimal* knowledge about possible behavior of a given system. By *minimal* knowledge we mean that if position of particle x is known, there is no need to know x^3 or any other one-to-one function of position. We only need to know and keep track of the *essential* information.

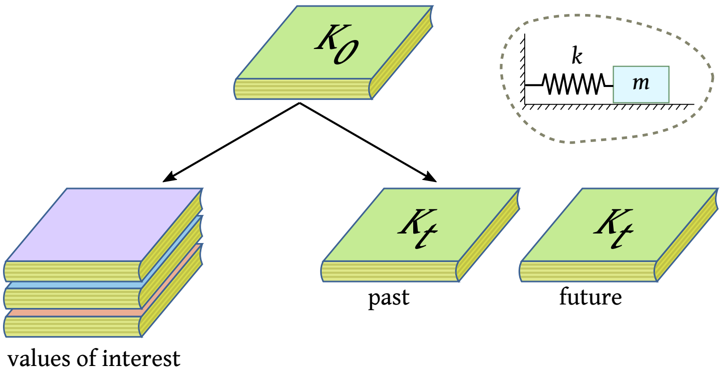

Fig. 2.4: State is a minimal and complete knowledge about a physical system.

2.6 Measurement

Measurement is the source of our knowledge about the world. This is true for both classical and quantum physics.

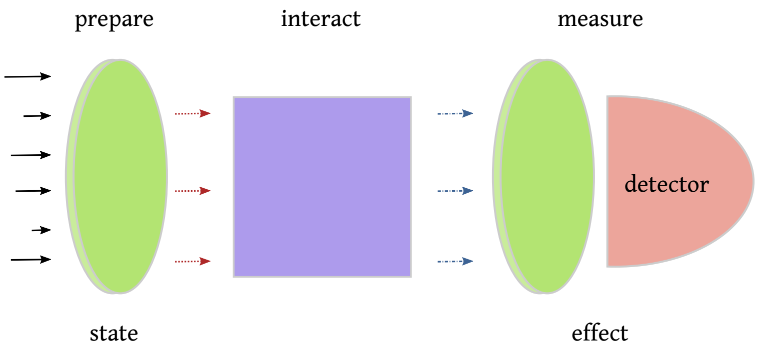

Fig. 2.5: Three stages of measurement process: Preparation of a system in a certain state, followed by the interaction of the system with external system, ending with the measurement which extract the information.

2.7 Atoms

Classical physics predicts a continuous decay of unstable configuration of charges. What is observed is a spontaneous decay of stable configuration of charges. Quantum physics elegantly explains the latter.

2.8 Particles

Classical physics predicts a continuous decay of unstable configuration of charges. What is observed is a spontaneous decay of stable configuration of charges. Quantum physics elegantly explains the latter.

2.9 Polarization and Spin

Mathematics is a remarkably effective and universal discipline, its methods and results can be applied in a wide range of fields.

Chapter Highlights

- *The power of mathematical concepts and methods increases with the level of abstraction.*
- *Learning new concepts often involves learning new terminology. The latter can create an artificial mental barrier.*
- *"Usual" numbers form a mathematical structure. The structure is revealed through various relations that exist between numbers.*
- *Relations between numbers are expressed using the concept of functions and operations (e.g., addition). Each operation is characterized by its arity – the number of arguments it accepts as an input.*

3. Mathematics

In the previous chapter we learned about numbers and various relations between them. As a particular class of relations we discussed functions. We introduced *binary* and *unary* functions and different ways functions can be combined (*composed*) to produce new functions. We also learned that functions can be represented in various ways and that none of those different representations defines the concept of function completely. Each representation of a function highlighted some important aspect of it.

Prerequisite Knowledge

To fully understand the material of this chapter, readers should be comfortable with the following concepts:

- State
- Dynamical equations

3.1 Arrows

To arrive at the idea of vectors we will start with simple geometrical objects – arrows in a plane, as illustrated in the Figure 3.1.

Symbolically, we will denote vectors by placing an arrow over letters:

$$\vec{a}\,,\vec{b}\,,\vec{c}\,,\ldots\,,\vec{\alpha}\,,\vec{\beta}\,.$$

3.1.1 Dirac Notation

3.2 Scalar Product

To arrive at the idea of vectors we will start with simple geometrical objects – arrows in a plane, as illustrated in the Figure 3.1.

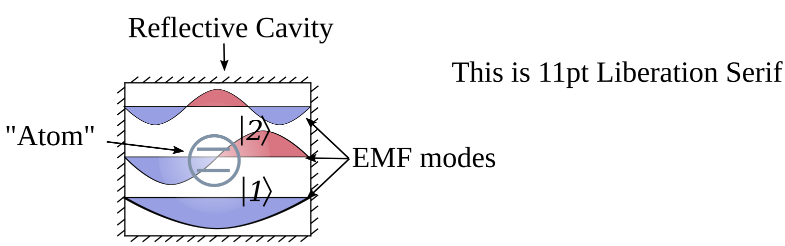

Fig. 3.1: Set of arrows starting at the same origin point O. All imaginable arrows taken as one set form the arrow space $\overset{\Rightarrow}{A}$.

3.3 Operators

To arrive at the idea of vectors we will start with simple geometrical objects – arrows in a plane.

$$\langle\phi|\phi\rangle$$

and

$$|\phi\rangle\langle\phi|\,.$$

3.4 Spaces

To arrive at the idea of vectors we will start with simple geometrical objects – arrows in a plane.

$$\langle\phi|\phi\rangle$$

and

$$|\phi\rangle\langle\phi|\,.$$

Chapter Highlights

- *Arrows in a plane provide a simple model for vectors.*
- *Arrows can be manipulated in ways analogous to numbers: Two arrows be added, an arrow can be "scaled" (stretched or compressed). Arrows form an algebra.*
- *Basis is an extremely important concept. Basis is a set of objects (arrows) that can be used to "build" all other similar objects (arrows). At the same time, basis can not be used to build itself – basis arrows are independent.*

4. Classical Physics

I think we may ultimately reach the stage when it is possible to set up quantum theory without any reference to classical theory, just as we already have reached the stage where we can set up the Einstein gravitational theory without any reference to the Newtonian theory. But from the point of view of teaching students, I think one would always have to proceed by stages – not expect too much from them, teach them first the elementary theories and gradually develop their minds; and that will always involve working from the classical theory first.

P. A. M. Dirac, Lectures on Quantum Field Theory, Belfer Graduate School of Science, Yeshiva University, New York, 1966, p.43.

The concept of *operators* extends the idea of functions. An unary numeric function f takes some numeric value x as an input and produces another numeric value y:

$$f\,x = y \quad \text{or} \quad x \xrightarrow{f} y\,.$$

In mathematical jargon, f *maps* x into y.

☑ *Prerequisite Knowledge*
To fully understand the material of this chapter, readers should be comfortable with the following concepts:

- State
- Dynamical equations

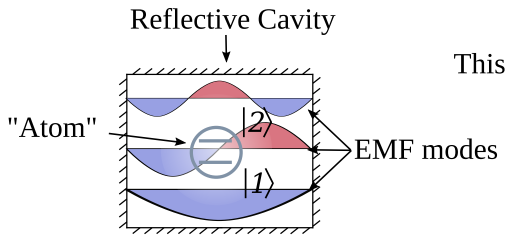

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Fig. 4.1: Operators extend the idea of functions. (a) An unary function f can be applied to a number x to produce another number y. (b) An unary operator $\widehat{F}$ can be applied to a vector $\vec{a}$ to yield another vector $\vec{b}$.

4.1 System

An action of an operator F on arrows can be represented symbolically as an equation:

$$F\vec{a} = \vec{b}\,.$$

Often a "hat" is placed on top of an operator[1], to emphasize that it is different from numeric function:

$$\boxed{\widehat{F}\vec{a} = \vec{b}\,.}$$

Simple Operators

It is easy to come up with examples of operators:

- Unit operator (or *identity* operator), such that

$$\widehat{I}\vec{a} = \vec{a}\,.$$

- "Zeroing" operator that maps every vector into a zero vector:

$$\widehat{0}\vec{a} = \vec{0}\,.$$

[1]In Quantum Mechanics, for example.

To fully describe an operator, we must describe how it acts *on every* arrow.

Examples

Let us take a closer look at a couple of operators. While studying these examples we must keep in mind that the relations between components are *specific to basis* and will change if we change the basis. The question of how exactly the relation between components changes will be addressed later in Section **??** for the simplest types of operators.

Matrix

Here is an example of matrix:

$$\widehat{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} .$$

Similar approach can be used to find the components of any linear operator.

4.2 Oscillator

An action of an operator F on arrows can be represented symbolically as an equation.

4.3 State

An action of an operator F on arrows can be represented symbolically as an equation.

4.4 Dynamics

An action of an operator F on arrows can be represented symbolically as an equation.

4.5 Hamiltonian

An action of an operator F on arrows can be represented symbolically as an equation.

4.6 Lagrangian

An action of an operator F on arrows can be represented symbolically as an equation.

4.7 Field

An action of an operator F on arrows can be represented symbolically as an equation.

4.8 Ideal Versus Real

An action of an operator F on arrows can be represented symbolically as an equation.

Chapter Highlights

- *Operators extends the idea of functions.*
- *Numeric functions (e.g.,* $\sin x$*) act on numbers and yield other numbers. Operators may act on vectors to yield other vectors or numbers.*
- *Linear operators represent the simplest and yet powerful class of operators on vectors.*
- *Linear operators can be represented graphically or symbolically.*

5. Quantum Physics

The first type of operators – and corresponding tensors – that we encountered has a simple type:

$$\widehat{L}\,\vec{a} = \vec{b}\,.$$

It is a linear unary function mapping vectors into vectors.

Prerequisite Knowledge
To fully understand the material of this chapter, readers should be comfortable with the following concepts:

- State
- Dynamical equations

5.1 Quantum System

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$|\sigma\rangle\langle\sigma|\,\vec{a}\,\vec{b} = x\,.$$

5.2 Quantum State

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$|\sigma\rangle\langle\sigma|\,\vec{a}\,\vec{b} = x\,.$$

5.2.1 States Overlap

$$\langle\psi|\phi\rangle.$$

5.3 Quantum Dynamics

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$|\sigma\rangle\langle\sigma|\vec{a}\,\vec{b} = x\,.$$

5.4 Quantum Hamiltonian

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$|\sigma\rangle\langle\sigma|\vec{a}\,\vec{b} = x\,.$$

5.5 Quantum Oscillator

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$|\sigma\rangle\langle\sigma|\vec{a}\,\vec{b} = x\,.$$

5.6 Quantum Bit

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$|\sigma\rangle\langle\,|\vec{a}\,\vec{b} = x\,.$$

5.6.1 Physical Realization of Qubits

Recall that harmonic oscillator is any physical system with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{kx^2}{2}\,.$$

Many concrete physical systems can be described using this Hamiltonian and thus provide specific *realizations* of the oscillator model. Similarly, many concrete physical systems realize the idea of a qubit.

5.7 Interacting Qubits

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$|\sigma\rangle\langle\sigma|\vec{a}\,\vec{b} = x\,.$$

5.7.1 Computational Basis

$$|\Upsilon\rangle_1 = |0\rangle|0\rangle,\ |\Upsilon\rangle_2 = |0\rangle|1\rangle,\ |\Upsilon\rangle_3 = |1\rangle|0\rangle,\ |\Upsilon\rangle_4 = |1\rangle|1\rangle\,.$$

Q: Are there other states, which are also basis and product? Smth like

$$|\Xi\rangle = |+\rangle|+\rangle\,.$$

5.7.2 Bell States

$$|\Phi\rangle^+\,,\quad |\Phi\rangle^-\,,\quad |\Psi\rangle^+\,,\quad |\Psi\rangle^-\,.$$

5.7.3 GHZ State

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$|\sigma\rangle\langle\sigma|\,\vec{a}\,\vec{b} = x\,.$$

5.8 Quantum Field

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$|\sigma\rangle\langle\sigma|\,\vec{a}\,\vec{b} = x\,.$$

We are looking for a binary operator $\widehat{\sigma}$ that yields a number based on two vectors:

$$\widehat{\sigma}\,\vec{a}\,\vec{b} = x\,.$$

We will call this operator $\widehat{\sigma}$ *dol*-operator[1], based on the key letters of the phrase "degree of overlap".

> ❗ ***Reminder***
> When we say that an operator $\widehat{\Gamma}$ is given or known, we mean that we know how it acts on *any vector* $\vec{a}$:
>
> $$\widehat{\Gamma}\,\vec{a} = x_a\,.$$

[1]This is not a standard terminology.

Array of equations:

$$\widehat{\Gamma}_1 \vec{e}_1 = 1 \tag{5.1}$$
$$\widehat{\Gamma}_1 \vec{e}_2 = 0 \tag{5.2}$$
$$\widehat{\Gamma}_1 \vec{e}_3 = 0 \tag{5.3}$$
$$\ldots \tag{5.4}$$

Chapter Highlights

- *Two vectors can be compared for similarity by calculating the "degree of overlap". The longer two vectors are and the closer their mutual direction – the greater the overlap is.*
- *Degree of overlap can be described by a binary linear operator* $\widehat{\sigma}$. *This operator is closely related to the concept of scalar product of two vectors.*
- *When scalar product (or, equivalently, degree of overlap) is defined for vectors, each vector receives a "special relative" – conjugate vector – that lives in different vector space, called conjugate or dual space.*
- *When the degree-of-overlap operator* $\widehat{\sigma}$ *is partially applied, the result is a unary linear operator that yields a number for every input vector. Importantly, such an operator is also a vector, albeit not an arrow-like vector.*

6. Applications

WE are now ready to appreciate how tensors are used in “real life”. In this chapter we will encounter examples of tensors that are used in mathematics, physics, and engineering.

Prerequisite Knowledge

To fully understand the material of this chapter, readers should be comfortable with the following concepts:

- State
- Dynamical equations

6.1 Hydrogen-like Atoms

$$|\alpha\rangle\langle\beta|$$

$$E_n = -\frac{E_i}{n^2} \, .$$

6.2 Quantum Dots

$$|\alpha\rangle\langle\beta|$$

$$E_n = -\frac{E_i}{n^2} \, .$$

6.3 Spontaneous Emission

$$|\alpha\rangle\langle\beta|$$

$$E_n = -\frac{E_i}{n^2} \, .$$

6.4 Stimulated Emission

$$|\alpha\rangle\langle\beta|$$

$$E_n = -\frac{E_i}{n^2}.$$

6.5 Lasers

$$|\alpha\rangle\langle\beta|$$

$$E_n = -\frac{E_i}{n^2}.$$

6.6 Photoeffect

$$|\alpha\rangle\langle\beta|$$

$$E_n = -\frac{E_i}{n^2}.$$

6.7 Conductors

$$|\alpha\rangle\langle\beta|$$

$$E_n = -\frac{E_i}{n^2}.$$

6.8 Entanglement

$$|\alpha\rangle\langle\beta|$$

$$E_n = -\frac{E_i}{n^2}.$$

δ-Notation

When a quantity x changes by a tiny amount, we will denote the change using small Greek letter δ (delta) as follows:

$$\boxed{\delta x \text{ - tiny change of } x.}$$

A convenient way to write all components of a second rank tensor is to use table-like structure called *matrix*:

$$F^{\mu\nu} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix}.$$

In the matrix, the first index μ of $F^{\mu\nu}$ corresponds to the row, while the second index ν corresponds to the column. Both rows and columns are enumerated from 0 to 3.

Using matrix form, we can write the electromagnetic tensor in terms of the electric and magnetic fields:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\mathcal{E}^1 & -\mathcal{E}^2 & -\mathcal{E}^3 \\ \mathcal{E}^1 & 0 & -\mathcal{B}^3 & \mathcal{B}^2 \\ \mathcal{E}^2 & \mathcal{B}^3 & 0 & -\mathcal{B}^1 \\ \mathcal{E}^3 & -\mathcal{B}^2 & \mathcal{B}^1 & 0 \end{pmatrix}.$$

Chapter Highlights

- *Tensors find application in various areas of science and math.*
- *Geometrical properties of surfaces and spaces can be described using metric tensor.*
- *Physical properties of solids are often anisotropic – depend on the direction of applied "force". Such properties are best described by various tensors: stress tensor, mobility tensor, piezoelectric tensor, and others.*
- *At the fundamental level electric and magnetic fields are united in a single physical object – electromagnetic field. Electromagnetic field is described by an antisymmetric tensor of the second rank.*

7. Implications

We are now ready to appreciate the implications of quantum physics.

Prerequisite Knowledge

To fully understand the material of this chapter, readers should be comfortable with the following concepts:

- State
- Dynamical equations

Discuss Mermins papers. Wheeler's ideas.

δ-Notation

When a quantity x changes by a tiny amount, we will denote the change using small Greek letter δ (delta) as follows:

$$\boxed{\delta x \text{ - tiny change of } x.}$$

A convenient way to write all components of a second rank tensor is to use table-like structure called *matrix*.

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- *Tensors find application in various areas of science and math.*
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8. Appendix

We are now ready to appreciate the implications of quantum physics.

8.1 Physics

When a quantity x changes by a tiny amount, we will denote the change using small Greek letter δ (delta) as follows:

$$\boxed{\delta x \text{ - tiny change of } x.}$$

A convenient way to write all components of a second rank tensor is to use table-like structure called *matrix*.

8.1.1 Notation

K and E_k – Kinetic energy of a system.
Π and E_p – Potential energy of a system.
E – Total mechanical energy ($E = E_K + E_P$) written in terms of velocity v and position x.
H – Hamiltonian of a system: $H = K + \Pi$. Differs from E because kinetic energy written in terms of *momentum* p instead of velocity.
L – Lagrangian (Lagrange function) of a system: $L = E_K - E_p$. It is the "imbalance" of energies.
Δx – Change of a value of a variable x.
δx – "Tiny" change of a value of a variable x.
∂ – Rate of change.
∂_t – Rate of change with respect to time.
∂_x – Rate of change with respect to variable x (e.g. position).
$\partial_t f$ – Rate of change of f with respect to t.

It means exactly the following

$$\partial_t f = \frac{\delta f}{\delta t} = \frac{f(t+\delta t) - f(t)}{\delta t}\,.$$

$\vec{\xi}$ – State of a system in Hamiltonian dynamics. It is a vector with components $\vec{\xi} = (x, p)$.
$\hat{J}$– Operation (operator) of rotation by 90 degrees.
$\hat{R}(\theta)$ – Operation (operator) of rotation by θ.
h – Quantum of action (Planck's constant). In SI units its numerical value is $h = 6.626 \times 10^{-34}(J \cdot s)$.
$\hbar$ – "Reduced Planck's constant". A convenience notation for often used combination $\hbar = h/(2\pi)$.
A – Action.
Ψ – Quantum state.
$|\Psi\rangle$ – Quantum state vector.
ϕ, θ – Angle variables.
ω – Angular speed (also angular velocity). Often it has the following meaning: $\omega = \partial_t \theta$.
$\vec{e}_1, \vec{e}_2$ – Basis vectors. Usually they have unit length and point in mutually perpendicular directions.
z – Arbitrary *numeric* variable, $\vec{z}$ – arbitrary *vector* variable, $\hat{z}$ – arbitrary *operator*.
$\overset{\circ}{A}$ – Angstrom, a unit of length in the world of atoms. $1\overset{\circ}{A} = 10^{-9}(m)$. Hydrogen atom is about $1\overset{\circ}{A}$ in diameter.
c – Speed of light in vacuum.
ν – Frequency of oscillations measured as the number of oscillations per second, in Hz.

8.1.2 Constants

Below is the list of various physical constants used in these notes.
$q_e = 1.6 \times 10^{-19}\,(C)$ – Charge quantum (charge of an electron).
$m_e = 9.1 \times 10^{-31}\,(kg)$ – rest-energy (aka mass) of an electron.
$k = \frac{1}{4\pi\epsilon_0} = 9 \times 10^9\,(N \cdot m^2/C^2)$ – Coulomb constant – force between two unit charges 1 meter apart.
10^{-9} s = 1 nanosecond – the unit of time in atomic world. It is a "heartbeat of atoms".
$1\,(eV) = q_e\,(J)$ – 1 electron-volt. It is the kinetic energy an electron

would acquire when accelerated by a simply 1V battery. A tiny value.
$m_e c^2 / q_e = 0.5\, MeV$ – rest-energy of an electron measured in electron-volts. Roughly speaking, we will need half a million 1-volt batteries to accelerate an electron to make its kinetic energy comparable to its rest-energy.
$k = 100\,(N/m)$ is a spring constant of a spring that stretches by 0.1 of a meter when 1 kilogram mass is attached to it.

8.2 Mathematics

When a quantity x changes by a tiny amount, we will denote the change using small Greek letter δ (delta) as follows:

$$\boxed{\delta x \text{ - tiny change of } x.}$$

A convenient way to write all components of a second rank tensor is to use table-like structure called *matrix*.

8.2.1 Greek Alphabet

$\mathrm{A}\,\alpha$	alpha	$\mathrm{B}\,\beta$	beta
$\Gamma\,\gamma$	gamma	$\Delta\,\delta$	delta
$\mathrm{E}\,\epsilon$	epsilon	$\mathrm{Z}\,\zeta$	zeta
$\mathrm{H}\,\eta$	eta	$\Theta\,\theta$	theta
$\mathrm{I}\,\iota$	iota	$\mathrm{K}\,\kappa$	kappa
$\Lambda\,\lambda$	lambda	$\mathrm{M}\,\mu$	mu
$\mathrm{N}\,\nu$	nu	$\Xi\,\xi$	xi
$\mathrm{O}\,\mathrm{o}$	omicron	$\Pi\,\pi$	pi
$\mathrm{P}\,\rho$	rho	$\Sigma\,\sigma$	sigma
$\mathrm{T}\,\tau$	tau	$\Upsilon\,\upsilon$	upsilon
$\Phi\,\phi$	phi	$\mathrm{X}\,\chi$	chi
$\Psi\,\psi$	psi	$\Omega\,\omega$	omega

Table 8.1: Greek Alphabet

In mathematics most often we use θ and ϕ for angles. Sometimes α and β are also used. Occasionally ψ is used to denote angle.

In physics λ is used to denote the wavelength of light, ν – frequency in Hertz (periods of oscillations per second), ω – angular speed (number of radians of rotation per second).

The symbols Ψ and Φ are usually used to denote quantum state vectors.

9. Solutions

Exercise 1.1

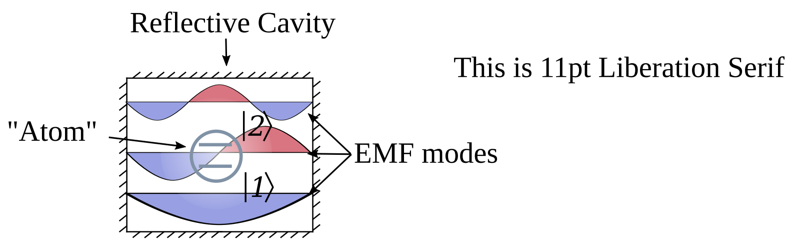

Fig. 9.1: The set M contains all possible makes of cars: Ford, Toyota, etc.

The diagram in the Figure 9.1 shows the set M – the set of all possible makes of cars. A mapping **trk** returns $true$ if a given car maker produces trucks.

Exercise 2.1

Any binary function can be viewed as a unary function if two inputs are replaced by a single input of a *pair of numbers*. Similarly for a function with two outputs. This idea is illustrated in the Figure 9.2(a): The function **swp** is viewed as a unary function which swaps the numbers in an *ordered pair*:

$$\textbf{swp}\ (n, m) = (m, n)\,.$$

Given the set $\mathbb{Z}$ of whole numbers, we can create the set of all possible *ordered pairs* (n, m). This set can be denoted as follows:

$$(\mathbb{Z}, \mathbb{Z}) \ \text{ or } \ \mathbb{Z} \times \mathbb{Z}\,.$$

The latter notation is standard in mathematics, but the former way of writing is

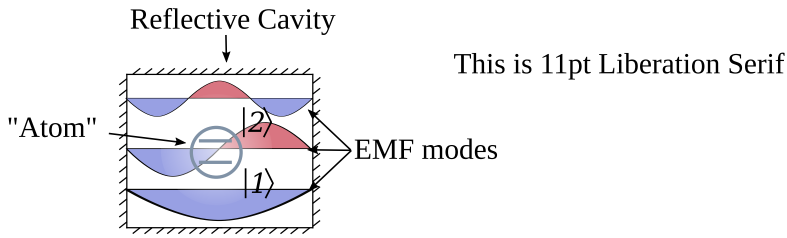

Fig. 9.2: (a) Two inputs (outputs) of a function can be replaced with a single input of a ***pair*** of numbers, turning a binary function into a unary one. (b) That.

also acceptable. We can similarly denote the set of all *ordered triples*:

$$(\mathbb{Z}, \mathbb{Z}, \mathbb{Z}) \text{ or } \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\,.$$

With the notation introduced above, the action of functions with multiple inputs or outputs can be depicted on the level of sets. The Figure 9.2(b) shows how this works for the functions **swp** and **max** .

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