Quantum Physics

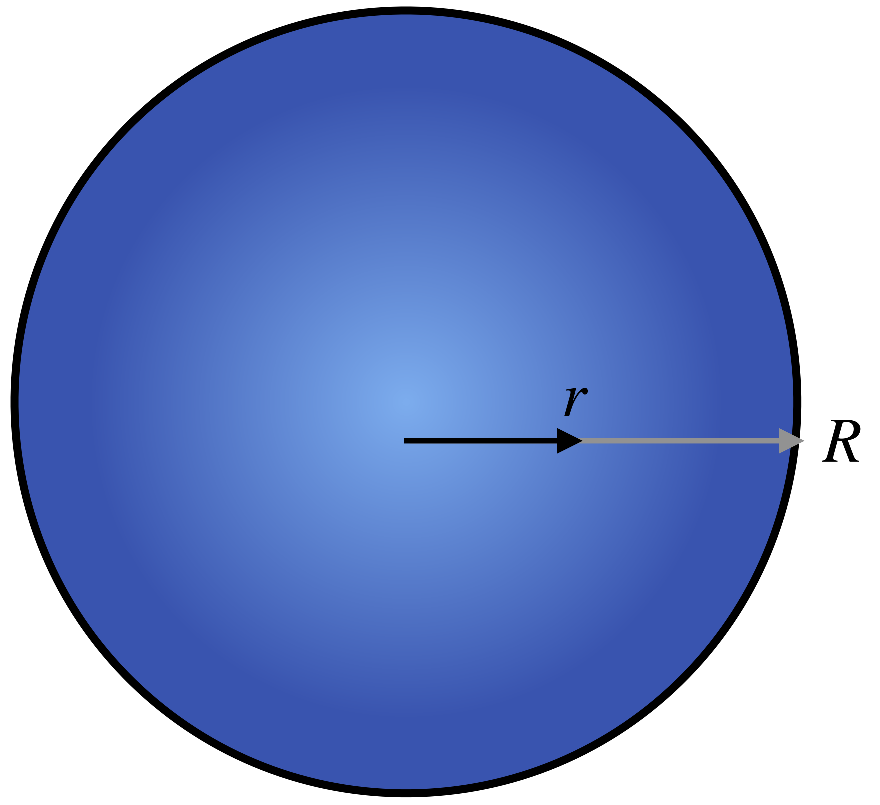
2024

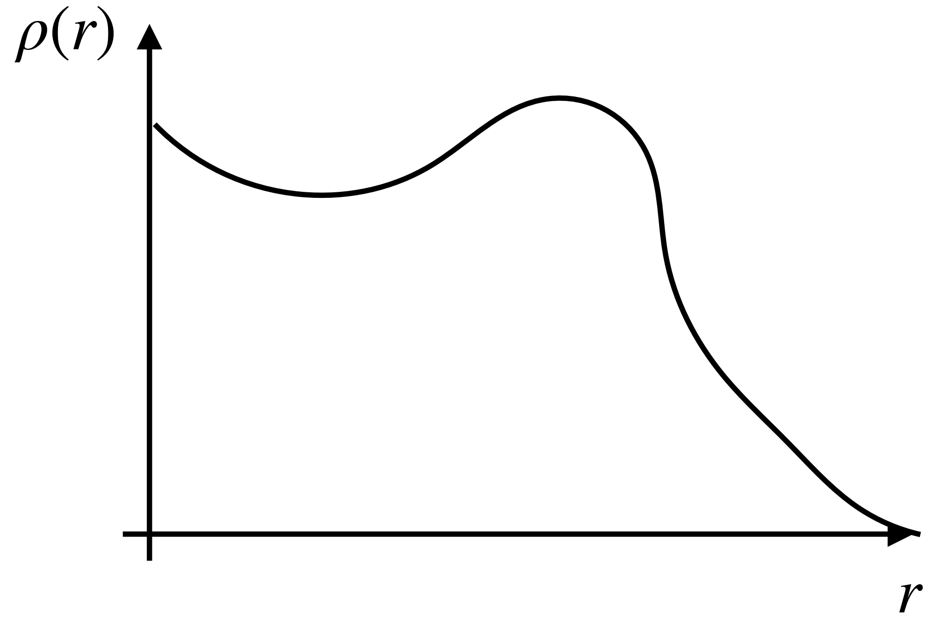
The Theory/Framework Of *Almost* Everything *Today*

Yury Deshko

Functionals

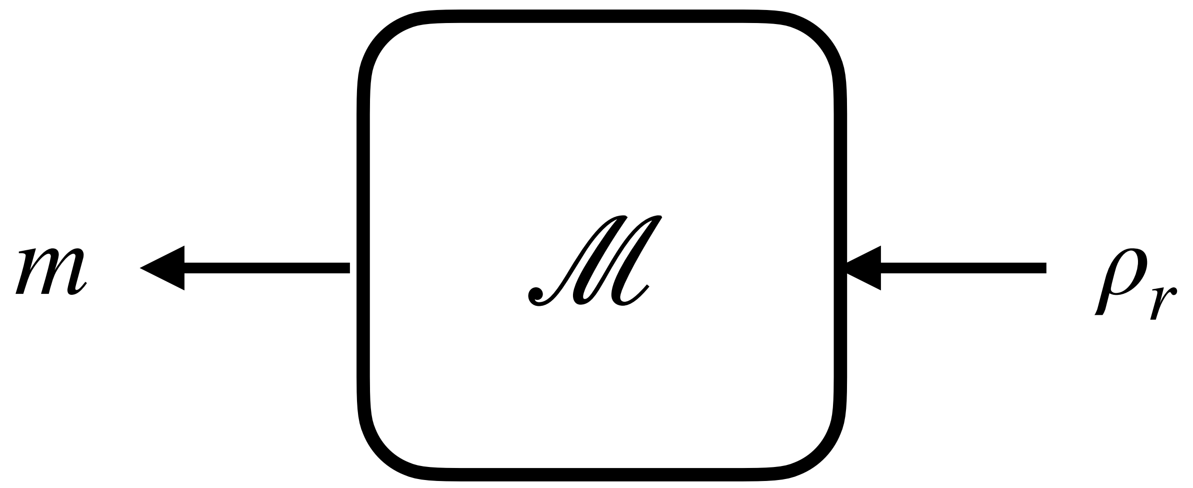
And Their Use in Physics





Density of a star/planet vs the radius from the center

Calculate total mass for a given density distribution

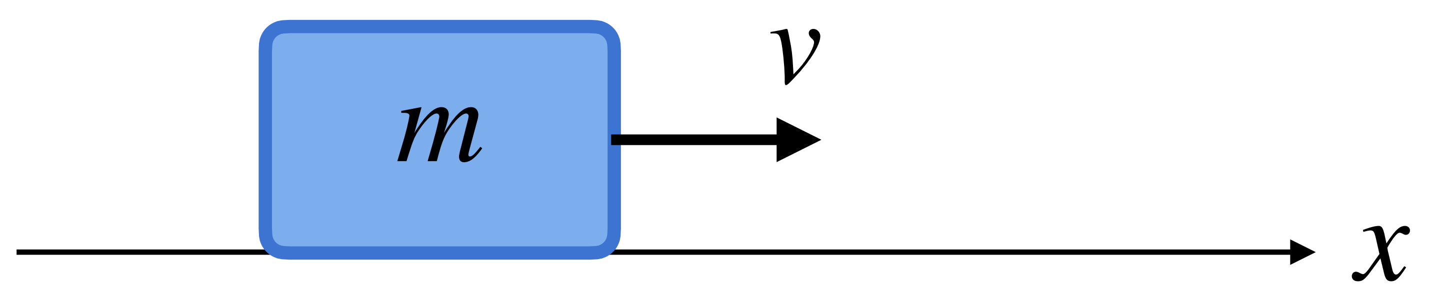


$\mathscr{M}$: **function —> number**

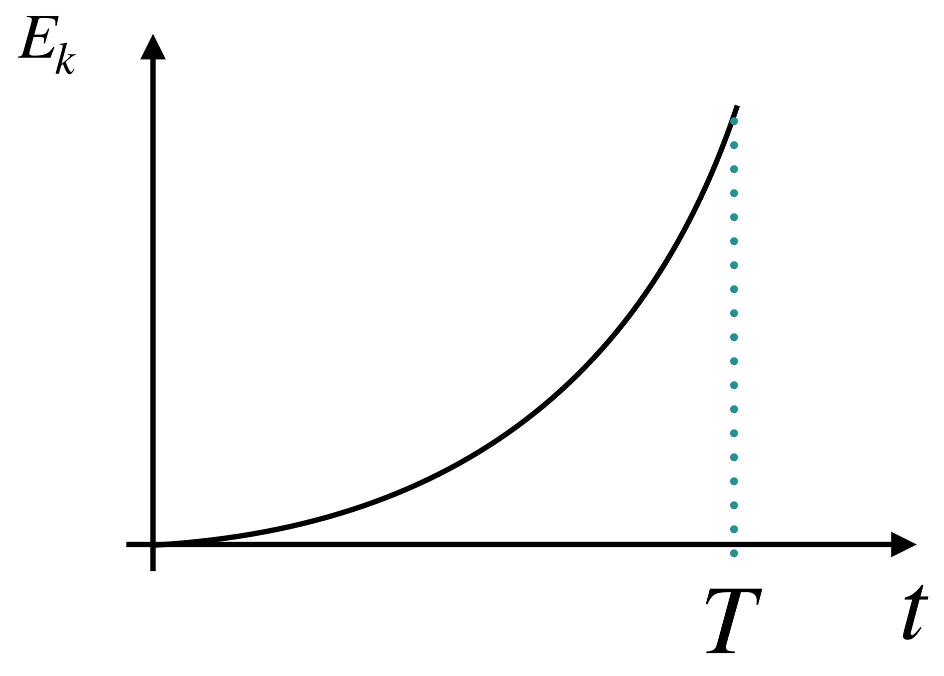
ρ_r $\qquad\qquad$ m

Functionals

And Their Use in Physics



$$x_t \quad \rightarrow \quad v_t = \partial_t x \quad \rightarrow \quad E_k = \frac{mv^2}{2}$$



Kinetic energy vs time

Calculate total "action" — energy*time

$$E_k(T) \cdot T \longleftarrow \boxed{\mathscr{A}} \longleftarrow x_t$$

$\mathscr{A}$: function —> number

$$x_t \qquad\qquad E_k(T) \cdot T$$

Action $A = Et$ is of fundamental importance.
It is quantized, like electric charge $q_e = e$

Exercise: Show that for constant acceleration, $E_k t = px$

Course Overview

Course Structure And Goals

- Part 1 : Mathematical Concepts And Tools
- Part 2 : Classical Physics
- Part 3 : Quantum Physics

We want to understand SchrEq

Operator

$$i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle$$

Vector, but can be also made an operator

Rate of change with respect to time

Operator

Today we will understand $|\Psi\rangle$.

Warm Up

Rate of Change Operator

$$i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle$$

Rate of change with respect to time

$$v_t \longleftarrow \boxed{\partial_t} \longleftarrow x_t$$

Function in — function out

$$10t \longleftarrow \boxed{\partial_t} \longleftarrow 5t^2$$

$$x_t = \frac{at^2}{2}$$

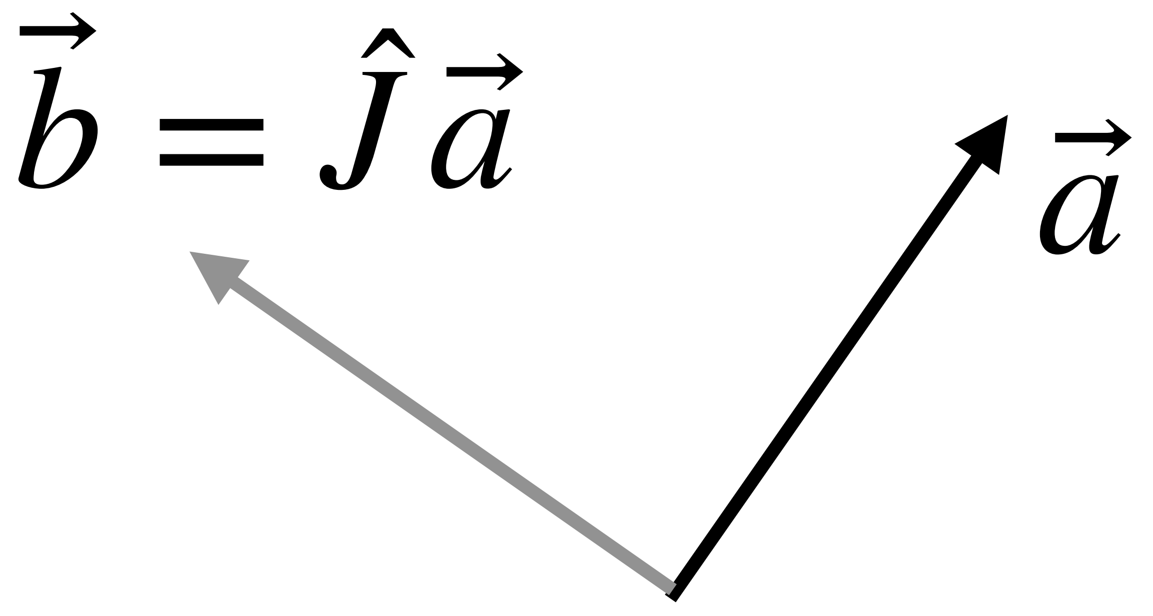
$$\delta x = 5(t+\delta t)^2 - 5t^2 = 5t^2 + 10t\delta t + 5(\delta t)^2 - 5t^2 = 10t\delta t + 5(\delta t)^2$$

$$v_t = \partial_t x = \frac{\delta x}{\delta t} = 10t + 5\delta t \approx 10t$$

$5\delta t$ is an important term, tells us about the error of real-world approximations, e.g. in numerical computations using machines, like computers.

Special Operator

Simple Yet Powerful Orthogonal Transformation



$\hat{J}$ performs <u>counter-clockwise</u> rotation of any arrow.

$$\vec{b} = \hat{J}\vec{a}$$

$$\vec{c} = \hat{J}\vec{b} = \hat{J}(\hat{J}\vec{a}) = (\hat{J}\circ\hat{J})\,\vec{a} = \hat{J}^2\vec{a} = -\vec{a}$$

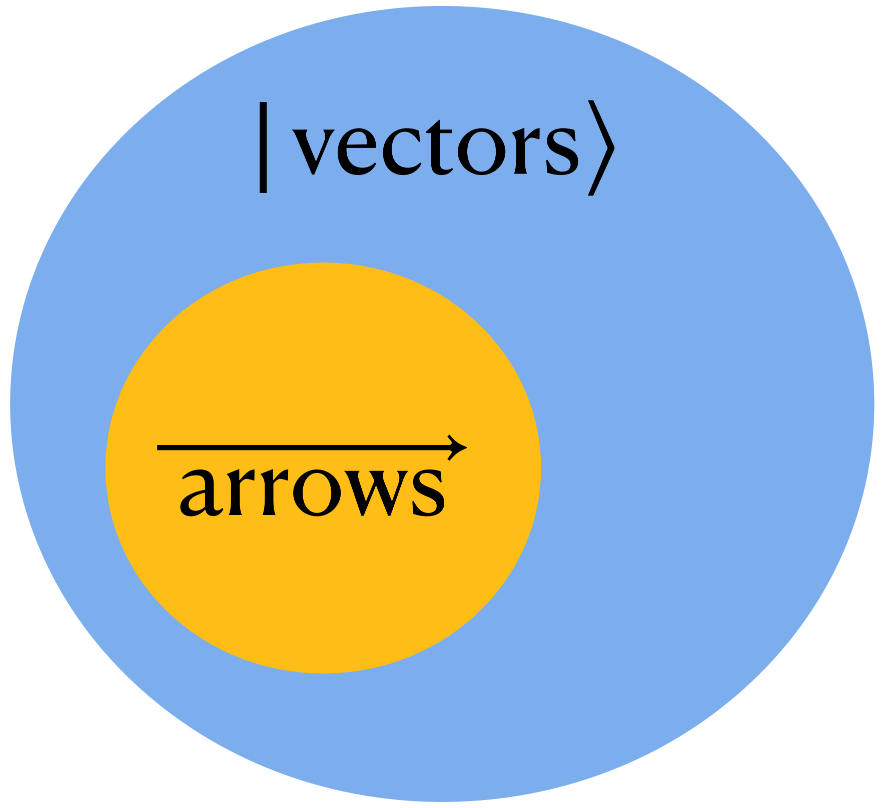
$$\hat{J}^2 = -\hat{I}$$

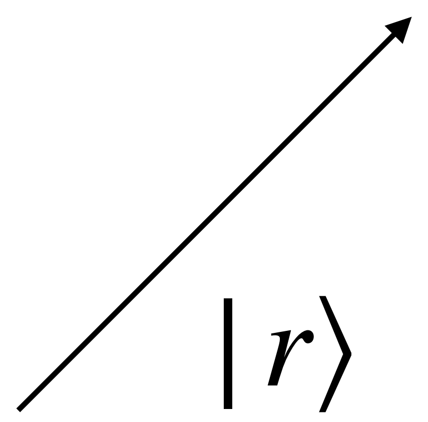
$$\hat{J}^3 = -\hat{J}$$

$$\hat{J}^4 = \hat{I}$$

Dirac Notation

Vectors are Richer Than Just Arrows





$$\vec{r} \quad \leftrightarrow \quad |r\rangle$$

$$\vec{v} \quad \leftrightarrow \quad |v\rangle$$

$$\vec{F} \quad \leftrightarrow \quad |F\rangle$$

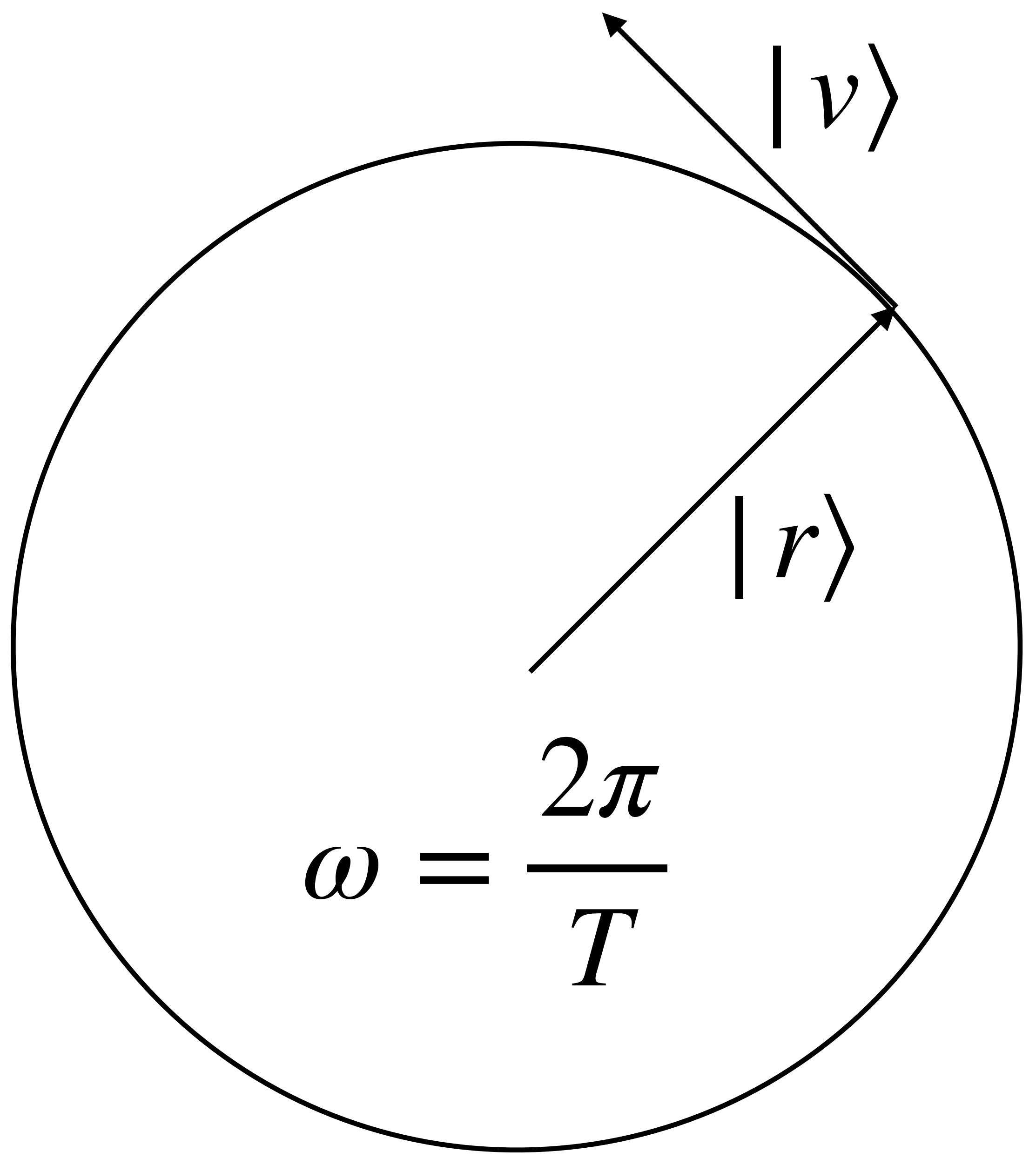
Paul Adrien Maurice Dirac in 1939 introduced modern vector notation for quantum mechanics.

$$\vec{a} \quad \leftrightarrow \quad |a\rangle$$

We can use it in “normal” physics too, if we want. Let’s switch to Dirac notation as early as possible.

Circular Motion

And Schrödinger Equation



Circular motion with constant angular speed ω

$$v = \frac{2\pi r}{T} = \omega r$$

$$t \quad \to \quad t + \delta t$$

$$|r\rangle \quad \to \quad \boxed{|r\rangle + \delta|r\rangle} \qquad\qquad \delta|r\rangle \perp |r\rangle$$

$$|v\rangle = \boxed{\partial_t|r\rangle} \qquad\qquad |v\rangle \perp |r\rangle$$

Take $|r\rangle$, scale it down to unit length (dividing by its length r), then rotate with $\hat{J}$ to make it perpendicular to $|r\rangle$. Finally, scale it up to the length of $|v\rangle$ given by $v = \omega r$

$$|v\rangle = v\hat{J}\left(\frac{1}{r}|r\rangle\right) = \omega\hat{J}|r\rangle$$

$$\partial_t|r\rangle = \omega\hat{J}|r\rangle$$

Now act with $\hat{J}$ on both sides

$$\hat{J}\partial_t|r\rangle = -\omega|r\rangle$$

Compare to SchrEq

$$\boxed{\begin{aligned} \hat{J}\hbar\partial_t|r\rangle &= -\hbar\omega|r\rangle \\ i\hbar\partial_t|\Psi\rangle &= \hat{H}|\Psi\rangle \end{aligned}}$$

Sidenote: Special Equation

Rate of change of *F* is proportional to *F*

$$\hat{J}\hbar\partial_t|r\rangle = -\hbar\omega|r\rangle$$ ← Circular motion

$$i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle$$ ← Quantum "motion"

$$\partial_t|r\rangle = \hat{J}\omega|r\rangle$$

$$\partial_t|\Psi\rangle = -\frac{i}{\hbar}\hat{H}|\Psi\rangle$$

$$\partial_t f = Cf$$

Special case:

$$\partial_t f = f$$

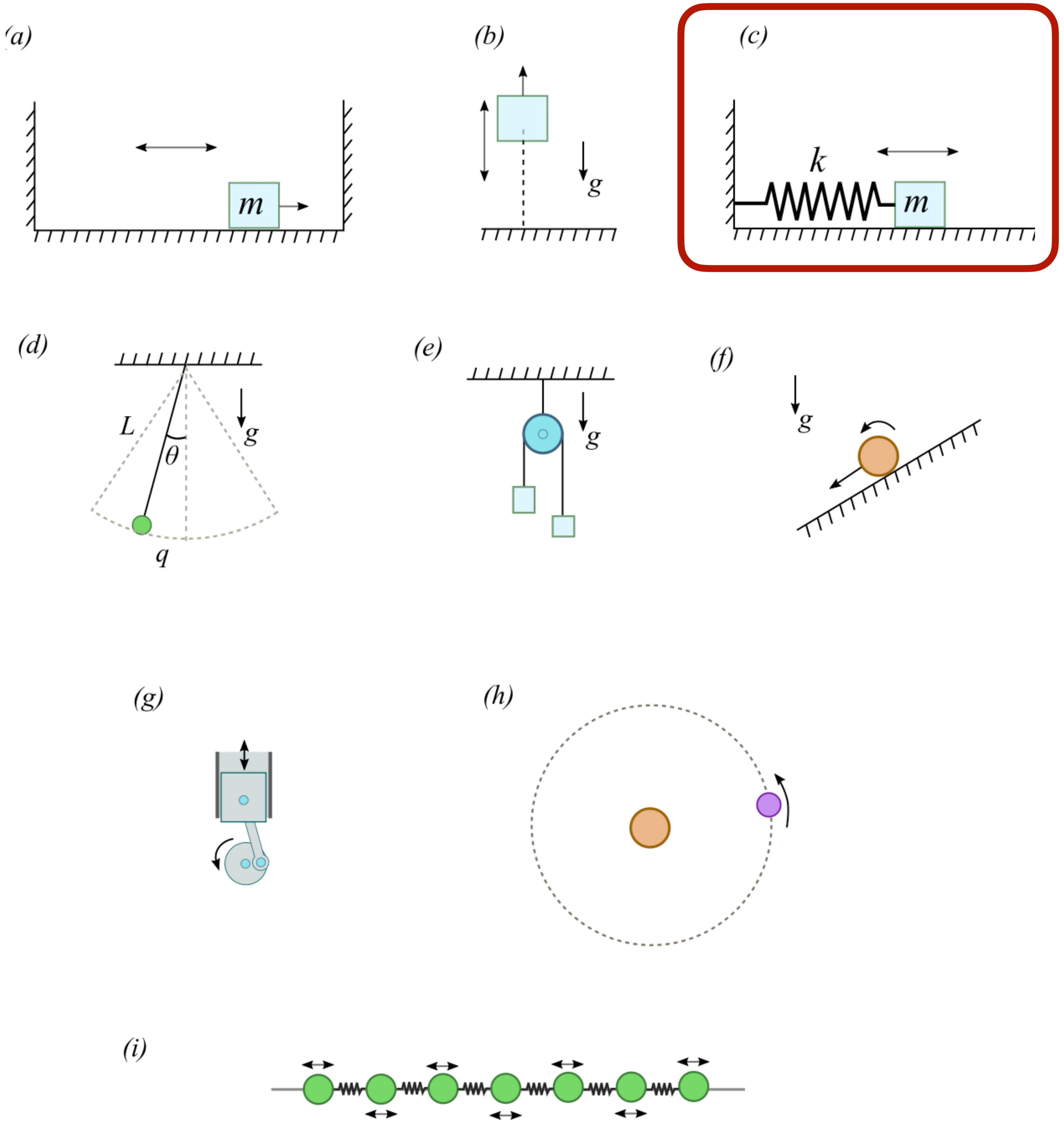
"Mega"-function, one of the most powerful functions in applied math

HW: Review properties of a^x.

State

Basic Notion of Physics

Physical Systems



Mathematical/symbolic expression of **state**

$$i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle$$

Physics studies the world. To simplify, parts of the world are isolated and studied separately.

System — part of the world that can be isolated and studied.

Goals of physics are

- ***Describe*** behavior of a system (D)
- ***Explain*** behavior of a system (E)
- ***Predict*** behavior of a system (P)

State — all *information*/knowledge we need to (D+P). A *complete* description of the system. All there is to know about a system at a given moment.

Describe: x, v

$E_k = mv^2/2$

$F = f(x, v)$

$a = F/m$

$L = mvx$

State

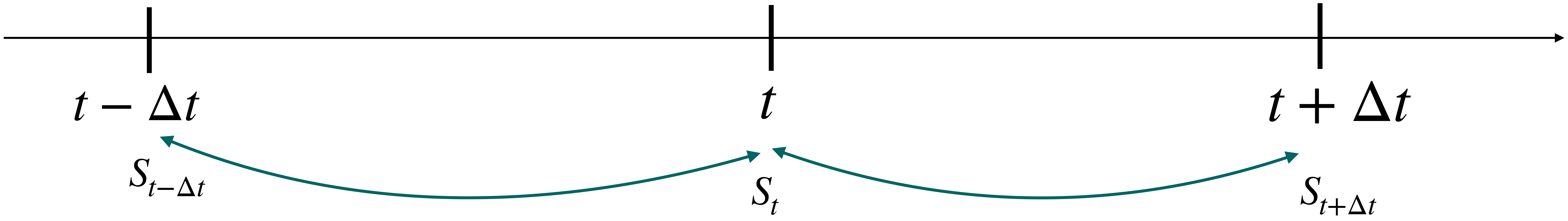
State Evolution

Explanation needs rules that connect “forces” to acceleration — dynamical laws: $v = \partial_t x, a = F/m$

Describe:
$x, v, \cancel{a, F, E_k, E_p, L}$

Predict: $\partial_t S$

State



Predict: If we know everything about the system now (time t) then we can find everything about the system later (time $t + \Delta t$) or earlier ($t + \Delta t$). *Know state now — know state at any time.*

$\partial_t S = \hat{D}\, S$ Equation of state *evolution* or state *dynamics* (how state changes in time).

$i\hbar\partial_t |\Psi\rangle = \hat{H}|\Psi\rangle$ How state changes in time depends on “forces”(in $\hat{D}$) and state.

State Evolution in Newtonian Dynamics

Basic Idea

- Use (x, v) to calculate everything important **now** (time t):

  $E_k = mv^2/2, F = f(x, v), a = F/m, L = mvx$

- Use (x, v) at t to calculate(x, v) at $t + \delta t$ and, consequently, at $t + \Delta t$

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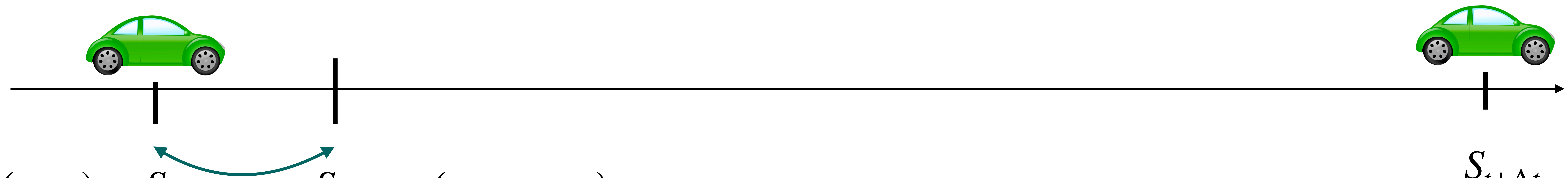
t $\quad$ $t + \delta t$ $\quad$ $t + \Delta t$



$(x_t, v_t) = S_t$ $\quad$ $S_{t+\delta t} = (x_{t+\delta t}, v_{t+\delta t})$ $\quad$ $S_{t+\Delta t}$

$$x_{t+\delta t} = x_t + v_t \delta t$$

$$v_{t+\delta t} = v_t + a_t \delta t$$

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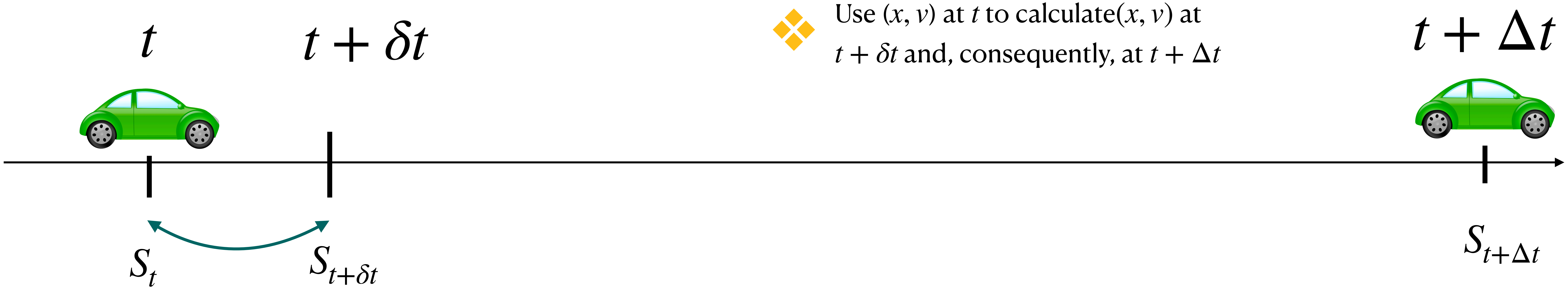

$x_{t+\delta t} = x_t + v_t\delta t$

$v_{t+\delta t} = v_t + a_t\delta t$

$\boxed{\cancel{a_{t+\delta t} = a_t + b_t\delta t}}$ $ma_t = F_t$ and $F = f(x, v)$

$F = kx$ — Hooke's law

$F = \dfrac{A}{x^2}$ — Newton's gravitation or Coulomb's law

$F = Bv$ — Lorentz force

State Evolution in Newtonian Dynamics

Basic Idea

- Use (x, v) to calculate everything important **now** (time t):

  $E_k = mv^2/2, F = f(x, v), a = F/m, L = mvx$

- Use (x, v) at t to calculate(x, v) at $t + \delta t$ and, consequently, at $t + \Delta t$

$x_{t+\delta t} = x_t + v_t \delta t$

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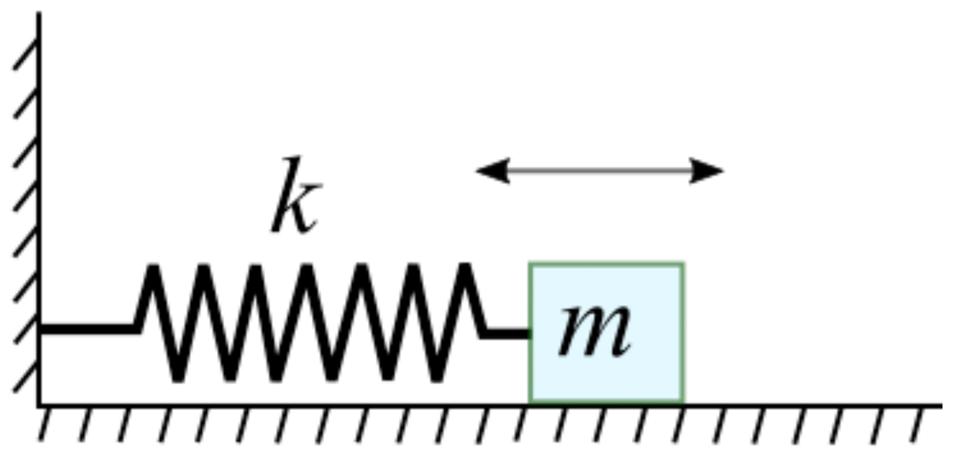
State in Newtonian mechanics: $S_t = (x, v)$

State Evolution in Newtonian Dynamics

Harmonic Oscillator Example

State in Newtonian mechanics: $S_t = (x, v)$



$x_0 \rightarrow x_t$

$v_0 \rightarrow v_t$

$S_0 \rightarrow S_t$

$$x_{t+\delta t} = x_t + v_t \delta t$$

$$v_{t+\delta t} = v_t + a_t \delta t$$

$$\partial_t x = v$$

$$\partial_t v = F/m$$

$$\partial_t S = ?$$

How to combine two equations into one for state S?

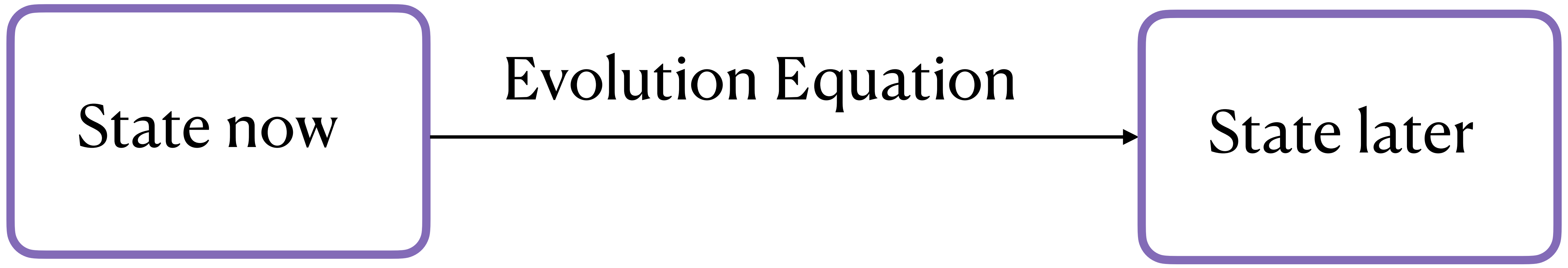
One solution: Make x and v parts/components of some vector $|\xi\rangle = (x, v)$

Study the script with numerical calculations.

Determinism

General Concept

Given initial state S_0 all later states S_t are uniquely determined. Nothing random.



The equation $\partial_t S = \hat{D}\, S_t$ is a symbolic expression of a *deterministic* evolution of state.

$$S_{t+\delta t} = S_t + \delta t \cdot (\partial_t\, S) = S_t + \delta t \hat{D}\, S_t$$

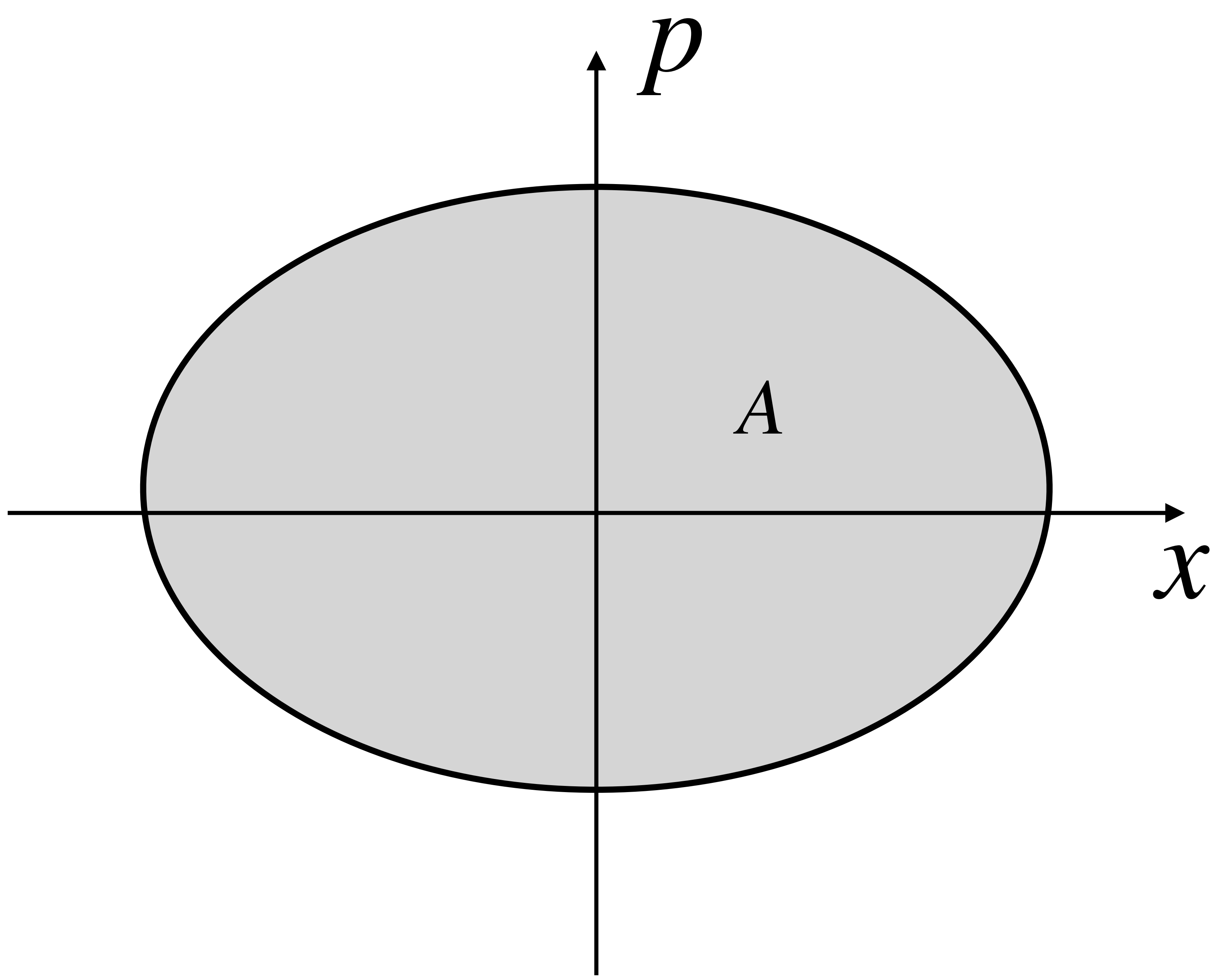
$$S_{t+\delta t} = (\hat{I} + \delta t \hat{D})\, S_t$$

$$S_{t+2\delta t} = (\hat{I} + \delta t \hat{D})\, S_{t+\delta t} = (\hat{I} + \delta t \hat{D})^2\, S_t$$

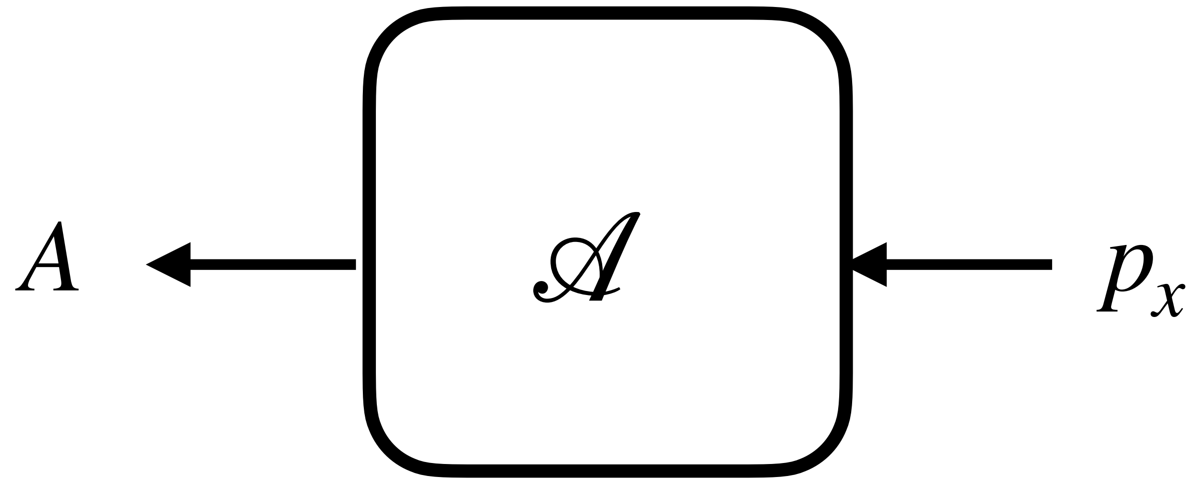
$$S_{t+\Delta t} = (\hat{I} + \delta t \hat{D})^N\, S_t \qquad N = \Delta t / \delta t$$

Functionals

Very Important Functional: Action



Calculate total area of the figure in phase space.



$\mathscr{A}$: **function** —> **number**

Works for confined motion only.

Functionals

Very Important Functional: Action

$x_t \quad \rightarrow \quad v_t = \partial_t x$

$$E_k = \frac{m v_t^2}{2}$$

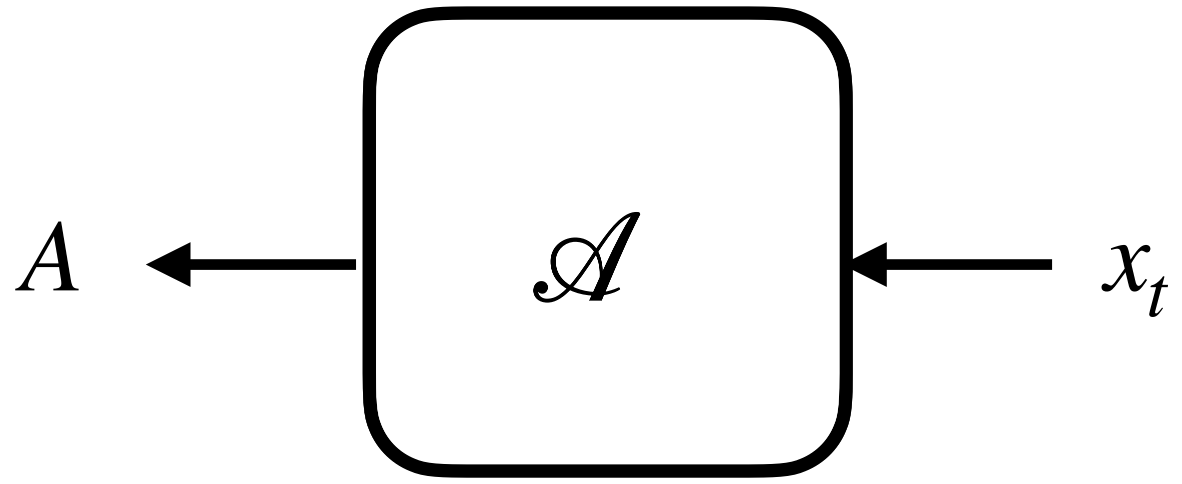
$$E_p = \frac{k x_t^2}{2}$$

$$L_t = E_k - E_p$$

$$\delta A = L \delta t$$

$$A = \int \delta A$$

Calculate total imbalance of kinetic energy over potential energy accumulated during the motion



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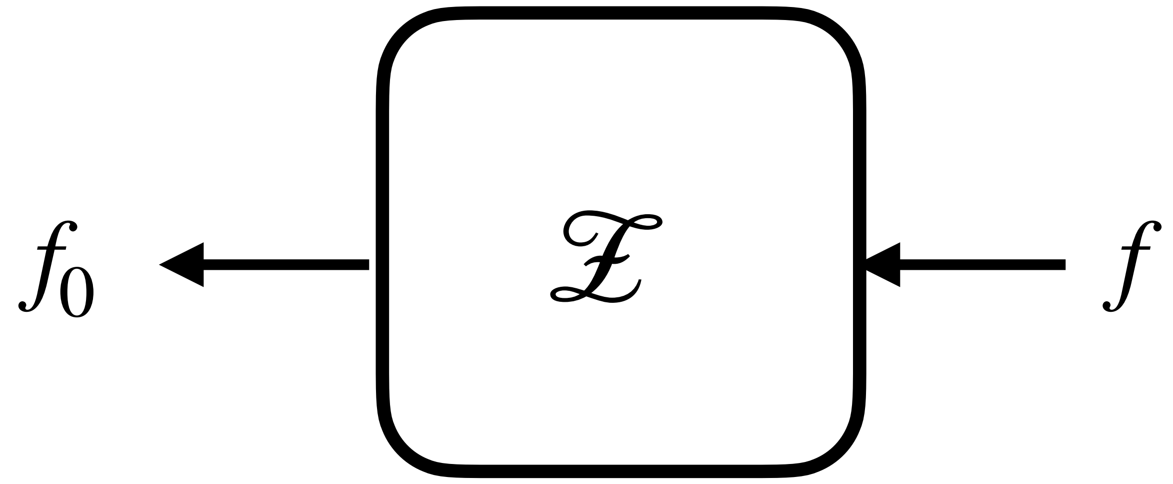
Functionals

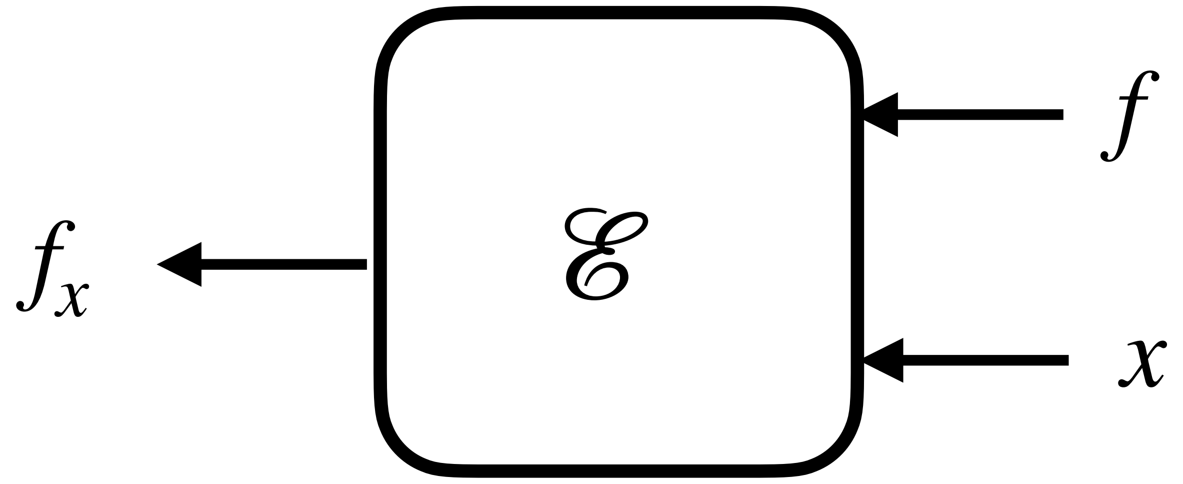
Function Application

Evaluate function at zero



$\mathcal{Z}$: **function** —> **number**

Evaluate function at a given point



$\mathcal{E}$: **function and number** —> **number**

$$\mathcal{E} f x = f \mathcal{E} x = f(x) = f x$$

$$\partial t f = \partial_t f$$

Sin of dropping infix operator: $3xy = 3 \times x \times y$

Self-Test

Answer These Questions 1hr After Class

1. What is an operator and why ∂_t is an operator?
2. What is Dirac notation?
3. What do circular motion and Schrödinger equation have in common?
4. What is a system?
5. What is a state?
6. What are three main goals of physics?
7. What is determinism?
8. How is the equation for rate of change of state in time called?

Homework Problems

Mathematical Concepts and Notation Day 3

- Review the properties of the function a^x.
- Solve the equation $x^3 + x^2 + x + 1 = 0$ in terms of real numbers.
- Evaluate $\hat{J}^3 + \hat{J}^2 + \hat{J} + \hat{I}$.
- Consider the "flipping" operator $\hat{F} = -\hat{I}$. Evaluate $\hat{F}^3 + \hat{F}^2 + \hat{F} + \hat{I}$.
- Consider the operator of 90-degree rotation clock-wise: $\hat{G}$. How is it related to $\hat{J}$ and $\hat{F}$?
- Evaluate $\hat{G}^3 + \hat{G}^2 + \hat{G} + \hat{I}$
- **Challenge and fun**: Express $\dfrac{1}{1 - \hat{J}/2}$ in terms of $\hat{I}$ and $\hat{J}$.
- Play with the script: Add friction, change computation method to use energy conservation, use non-linear formulas for δx.

Quantum Theory

In a Nutshell

II. POSTULATES FOR QUANTUM MECHANICS

In this paper, all state vectors are supposed to be normalized, and mixed states are represented by density operators, i.e., positive operators with unit trace. Let A be an observable with a nondegenerate purely discrete spectrum. Let $\phi_1, \phi_2, \ldots$ be a complete orthonormal sequence of eigenvectors of A and $a_1, a_2, \ldots$ the corresponding eigenvalues; by assumption, all different from each other.

According to the standard formulation of quantum mechanics, on the result of a measurement of the observable A the following postulates are posed:

(A1) *If the system is in the state* ψ *at the time of measurement, the eigenvalue* a_n *is obtained as the outcome of measurement with the probability* $|\langle\phi_n|\psi\rangle|^2$

(A2) *If the outcome of measurement is the eigenvalue* a_n*, the system is left in the corresponding eigenstate* ϕ_n *at the time just after measurement.*

The postulate (A1) is called the *statistical formula*, and (A2) the *measurement axiom*. The state change $\psi \mapsto \phi_n$ described by the measurement axiom is called the *state reduction.*

You will understand this paragraph in the end of the course.

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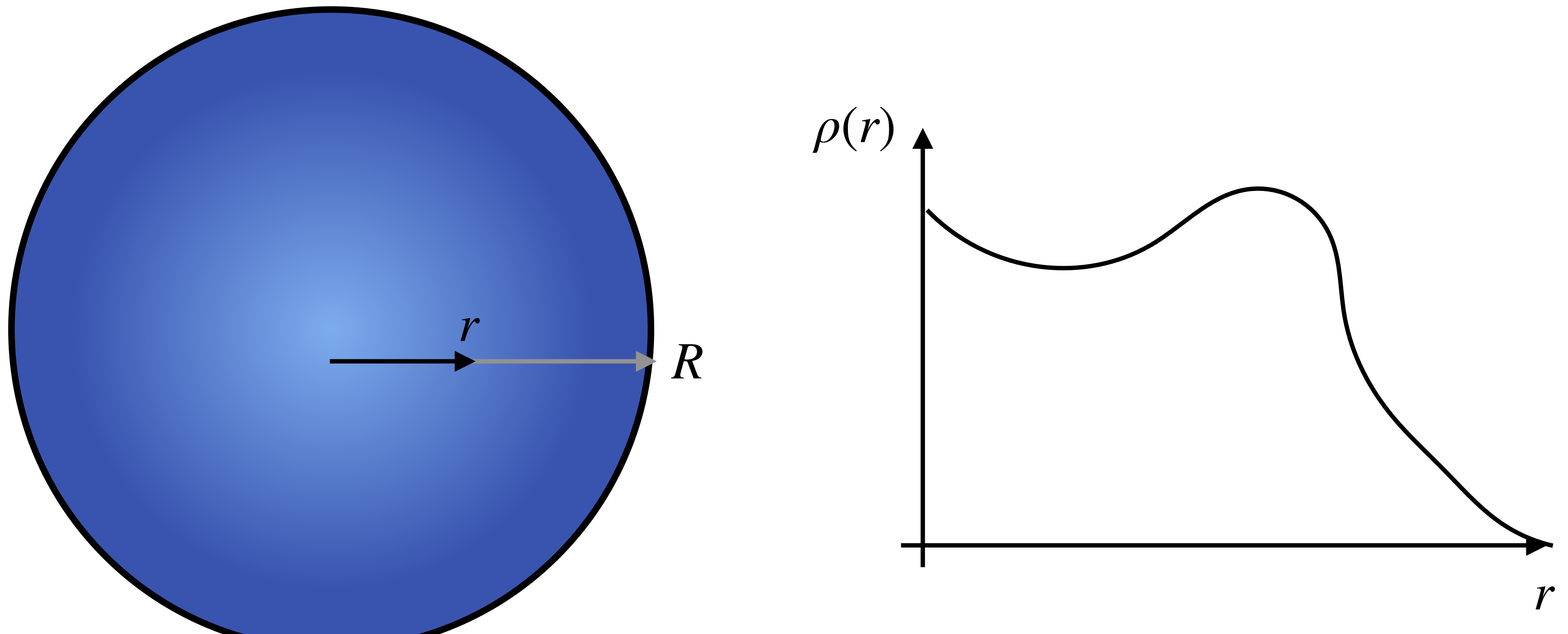

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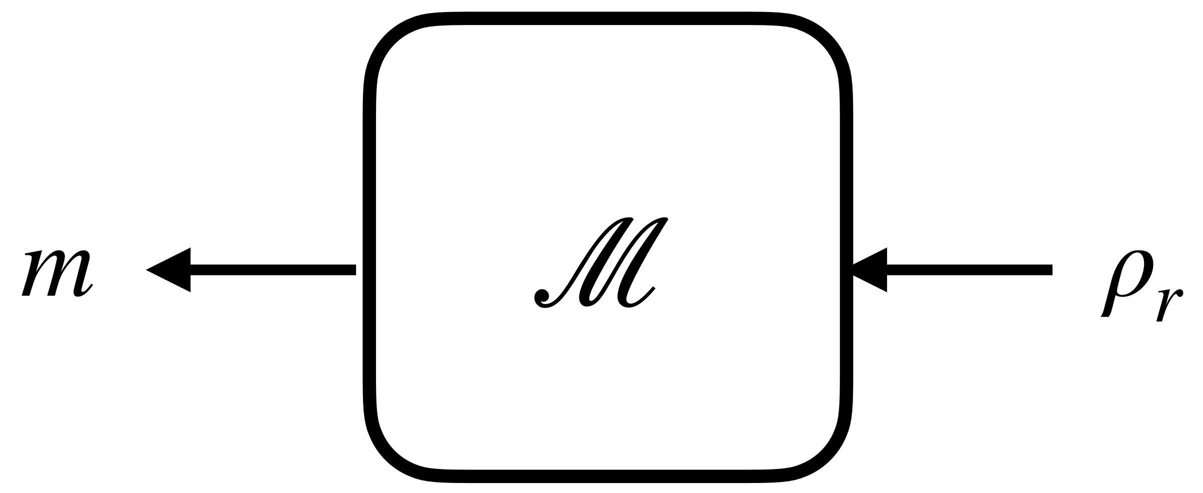

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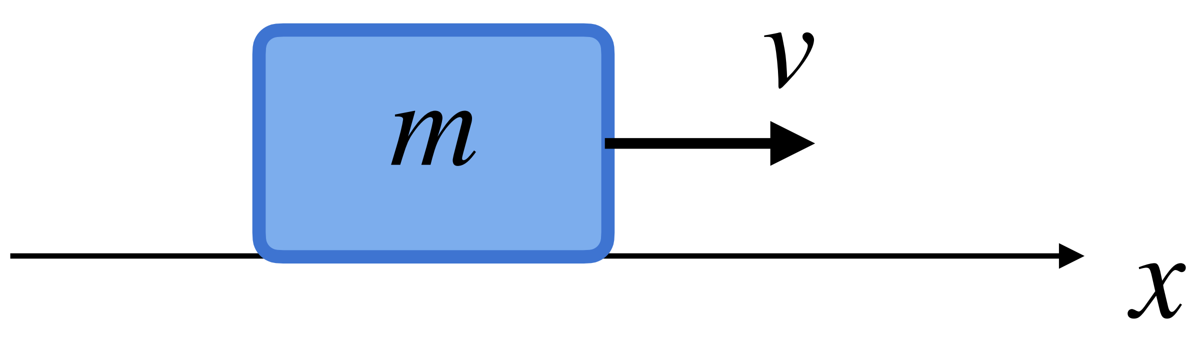

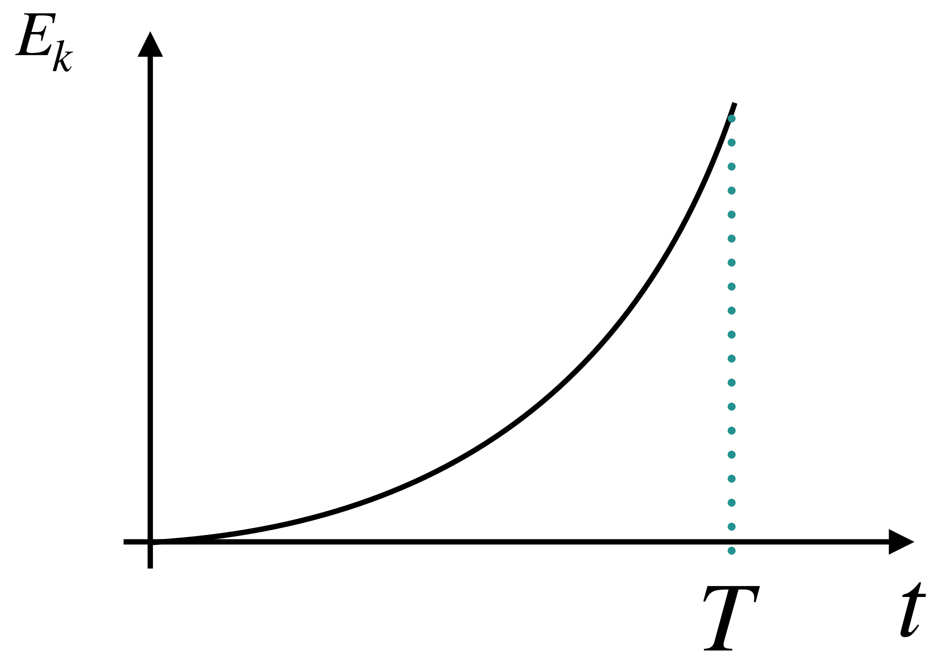

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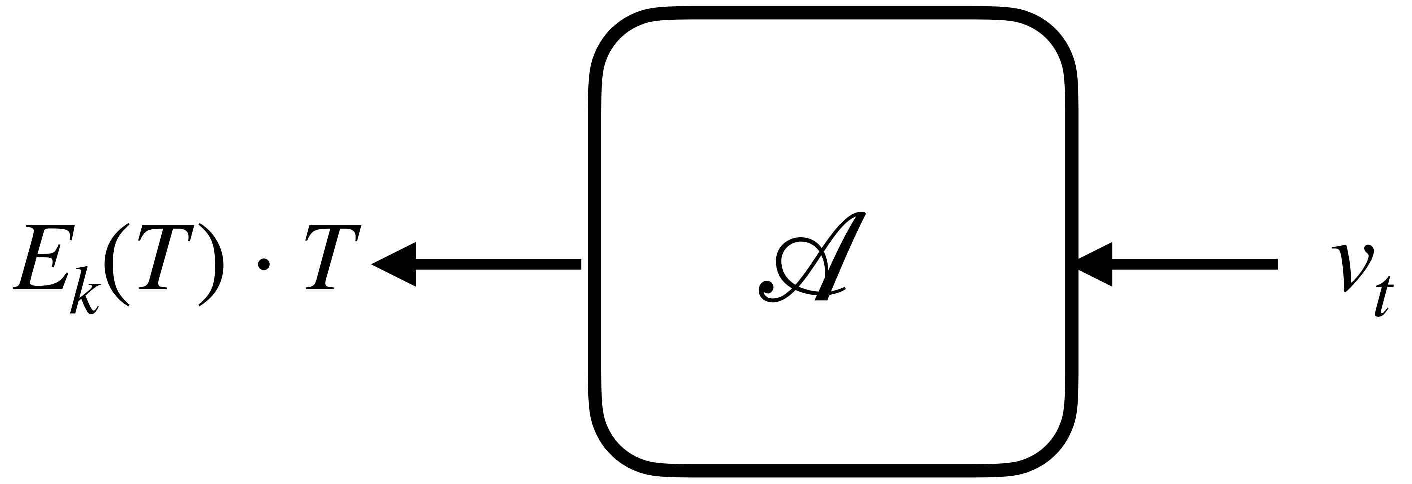

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