Quantum Physics

2024

The Theory/Framework Of *Almost* Everything *Today*

Yury Deshko

Algebras of Arrows and Operators
&
Linearity

Course Overview

Course Structure And Goals

- Part 1 : Mathematical Concepts And Tools
- Part 2 : Classical Physics
- Part 3 : Quantum Physics

Round 2

$$i\hbar\frac{\delta|\Psi\rangle}{\delta t} = \hat{H}|\Psi\rangle$$

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$$i\hbar\frac{\delta|\Psi\rangle}{\delta t} = \hat{H}|\Psi\rangle$$

$$\frac{\delta|\Psi\rangle}{\delta t} = \frac{1}{\delta t}|\Psi_{t+\delta t}\rangle - \frac{1}{\delta t}|\Psi_t\rangle$$

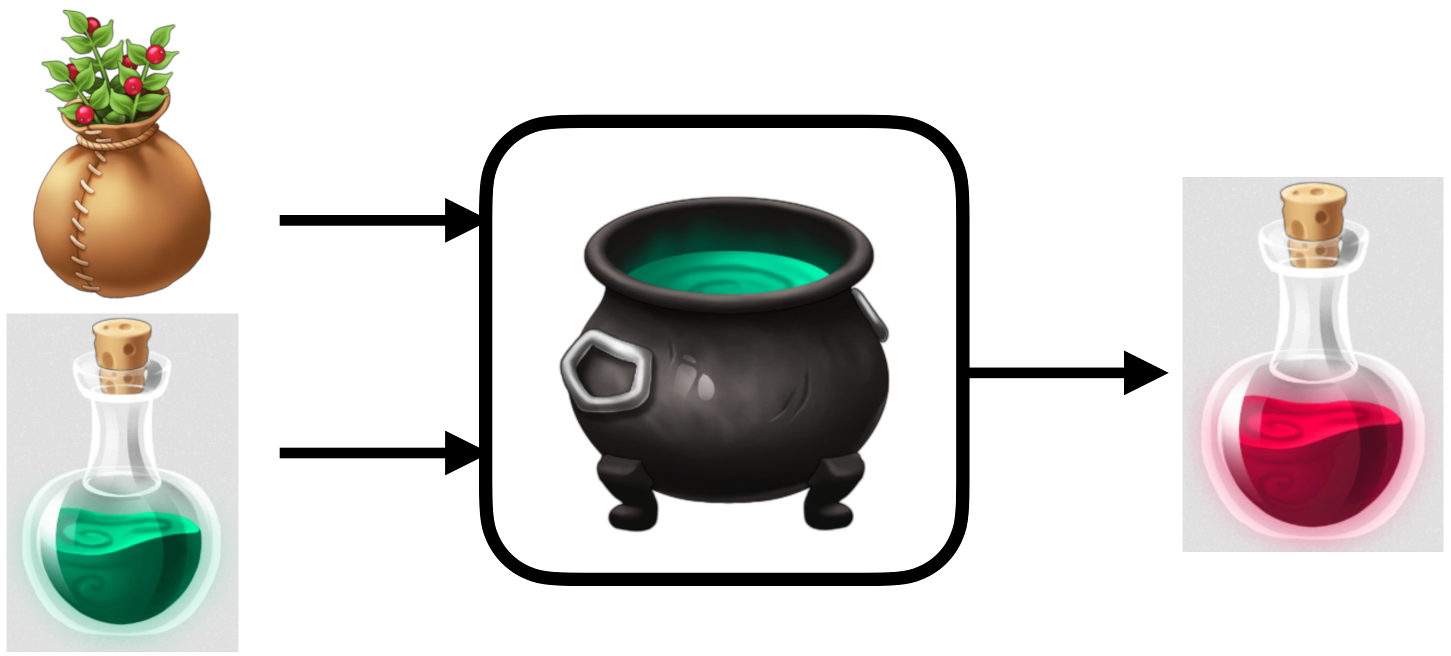
State $|\Psi\rangle$ must be multipliable and addable. Like a number.

Algebra

Is a Mathematical Version of a Game

Game

- Elements
- Interaction of elements
- Rules



Algebra

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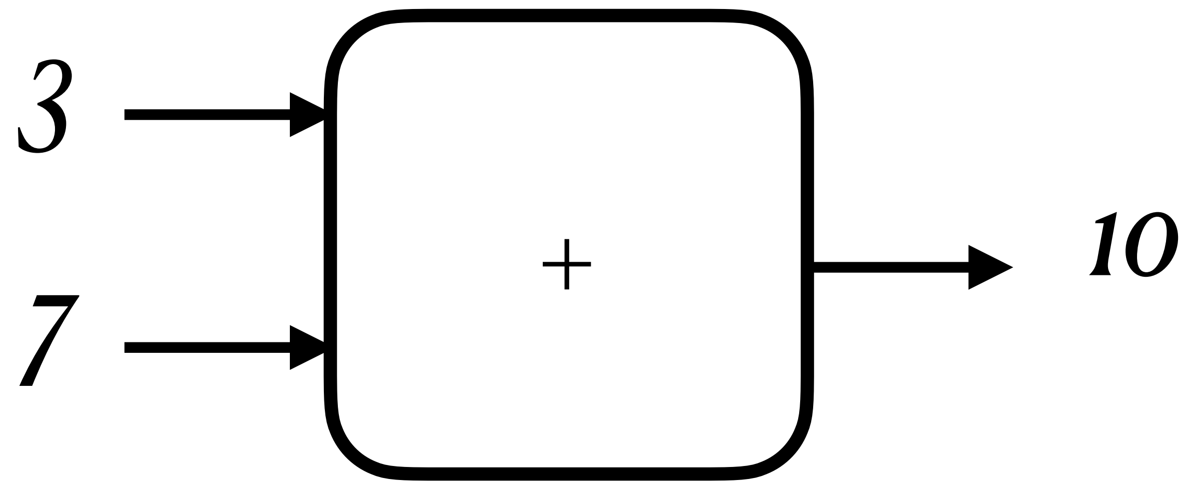
- Operations with elements
- Rules



Algebra

Formal Definitions

Algebra is the branch of mathematics that studies certain abstract systems, known as algebraic structures, and the manipulation of statements within those systems. It is a generalization of arithmetic that introduces variables and algebraic operations other than the standard arithmetic operations such as addition and multiplication.

Abstract algebra studies algebraic structures, which consist of a set of mathematical objects together with one or several operations defined on that set. It is a generalization of elementary and linear algebra, since it allows mathematical objects other than numbers and non-arithmetic operations. It distinguishes between different types of algebraic structures, such as groups, rings, and fields, based on the number of operations they use and the laws they follow. Universal algebra and category theory provide general

In mathematics, an **algebraic structure** consists of a nonempty set *A* (called the **underlying set**, **carrier set** or **domain**), a collection of operations on *A* (typically binary operations such as addition and multiplication), and a finite set of identities (known as *axioms*) that these operations must satisfy.

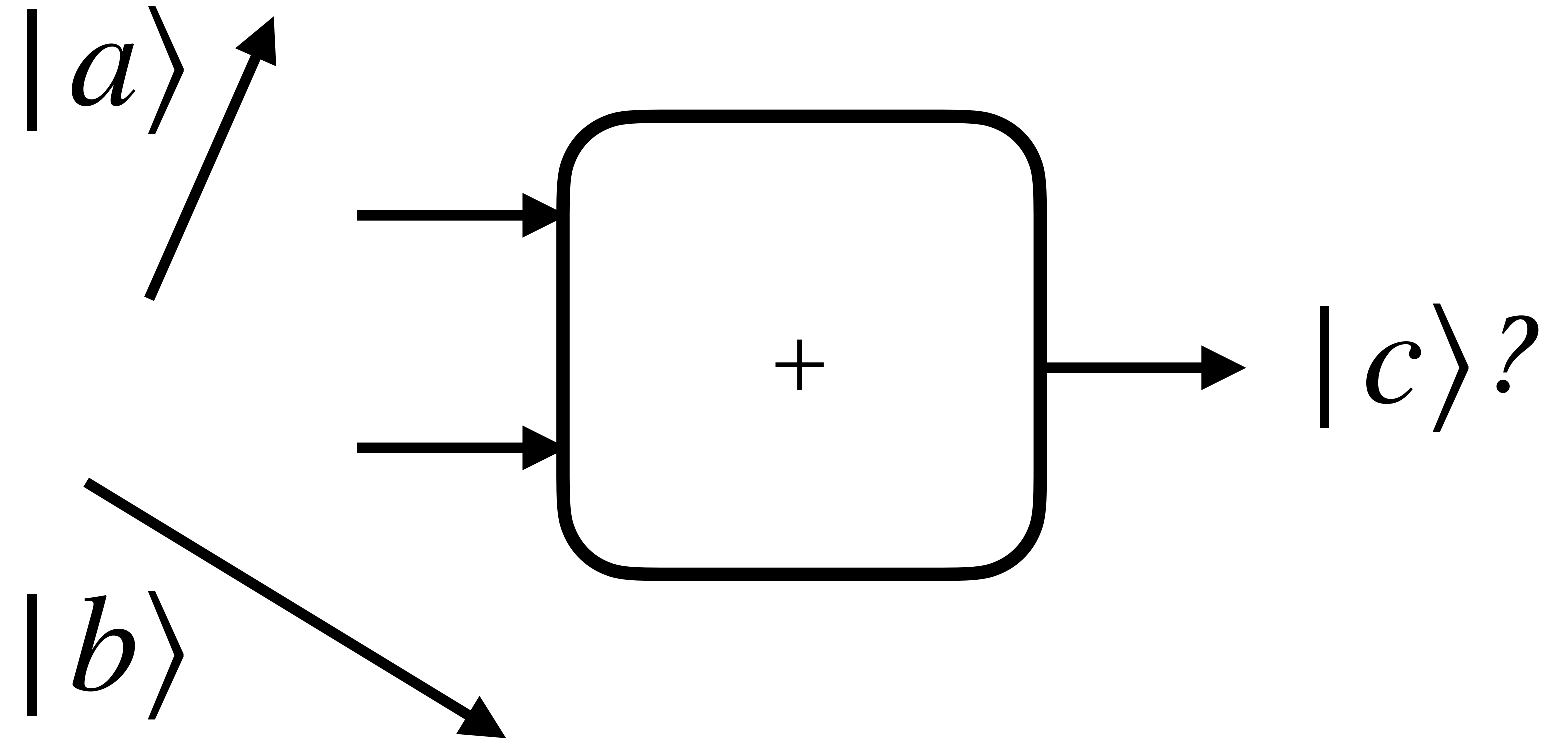
Game of Arrows and Operators

A.k.a. Vector Algebra and Operator Algebra

How to combine arrows?

How to combine operators?

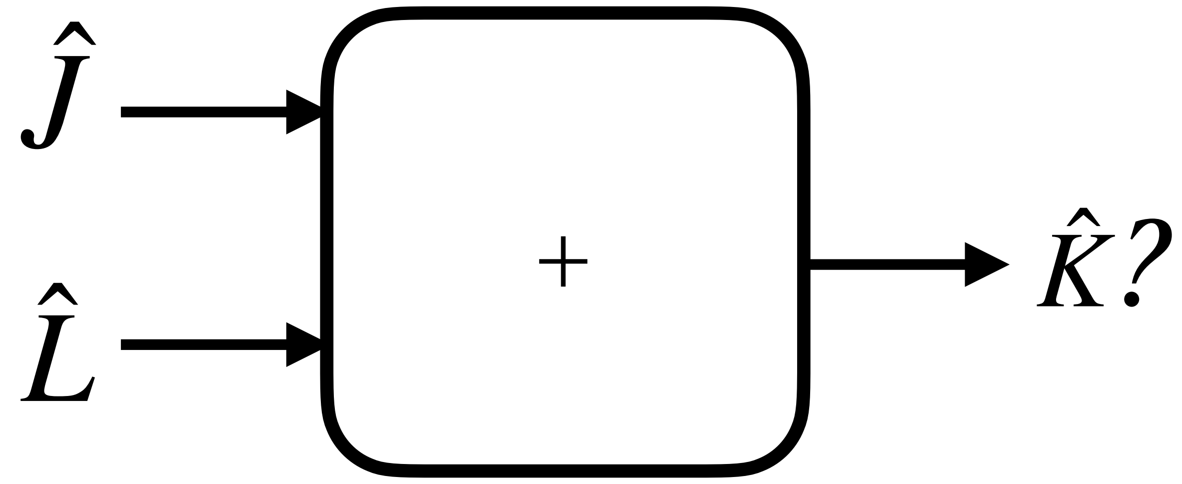
Arrows



$$|a\rangle + |b\rangle = |c\rangle$$

Operators

$$\hat{L} = \sqrt{\hat{J}}$$

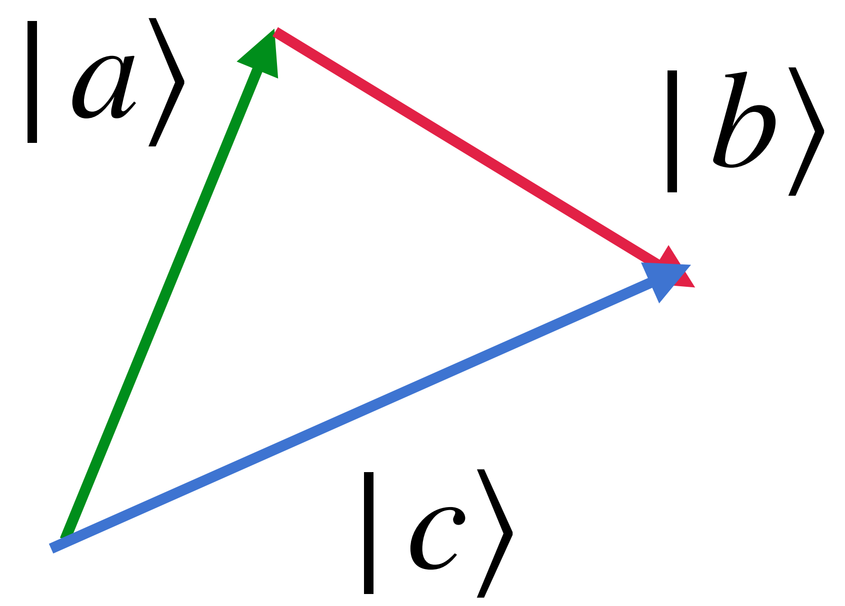


$$\hat{J} + \hat{L} = \hat{K}$$

Same symbol "+" used in three different contexts. OK if clear. But must be careful!

Game of Arrows

Algebra of Arrows and Vector Algebra



$$|a\rangle + |b\rangle = |c\rangle$$

1. View arrows as *instructions*: go this far in this direction.
2. Instructions can be *combined/sequenced/composed* to create a new instruction: First go this far in this direction, then go that far in that direction.

Game of Arrows

Algebra of Arrows and Vector Algebra

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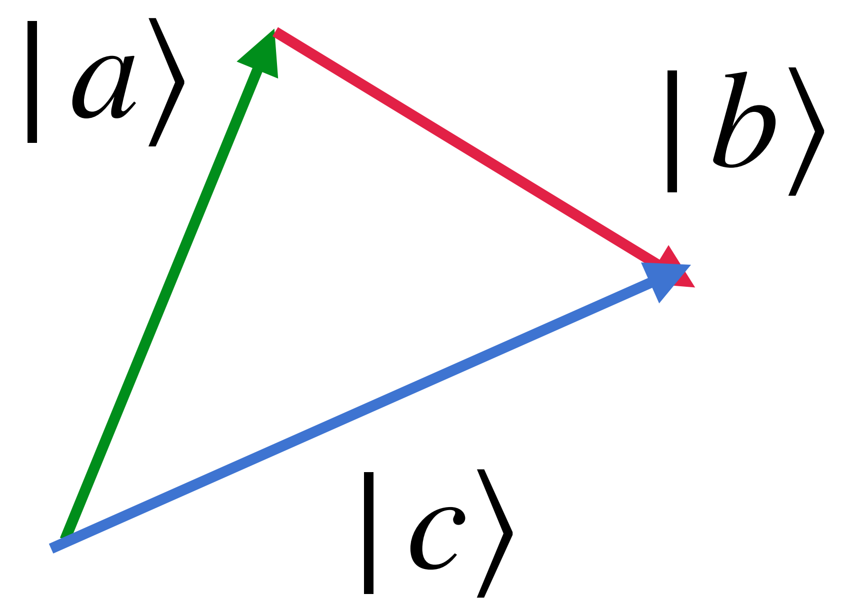
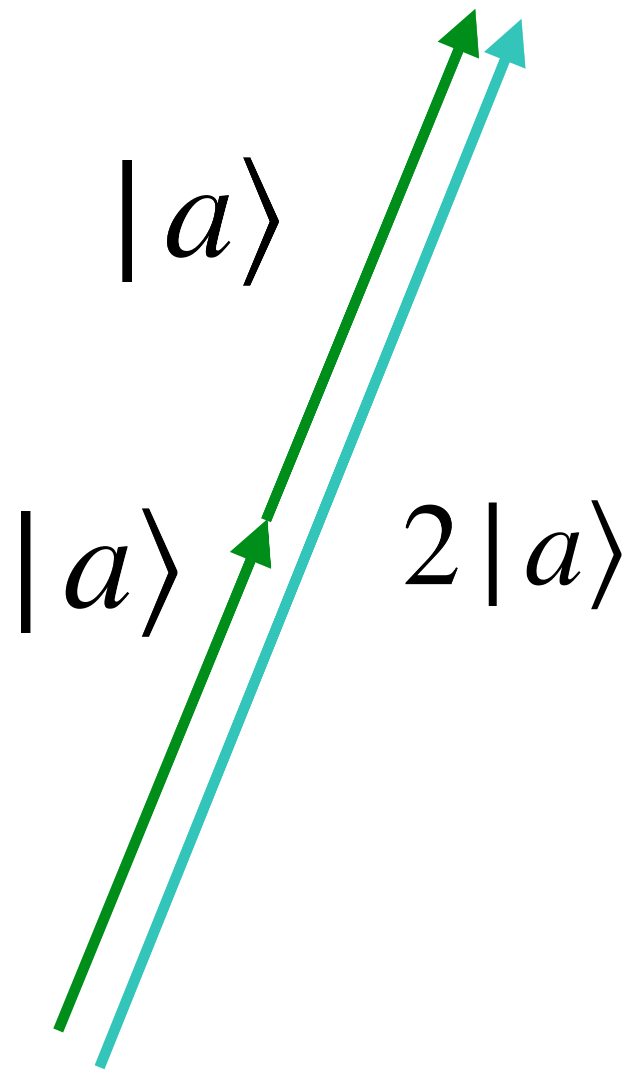
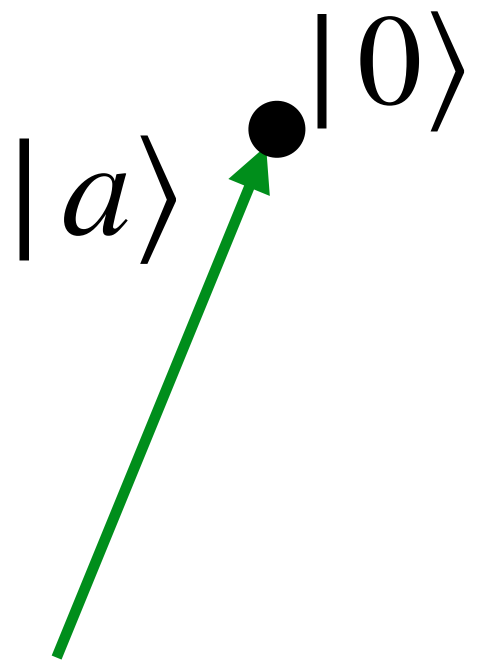


$|a\rangle + |b\rangle = |c\rangle$

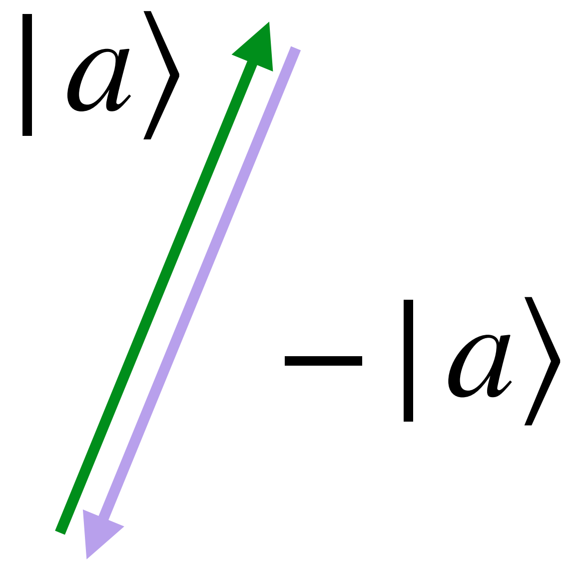


Number are “needed”/useful to express the relationship $|a\rangle + |a\rangle = 2|a\rangle$



Special arrow $|0\rangle$ is “needed”/useful to have a complete and closed arrow family:

$|a\rangle + |0\rangle = |a\rangle$

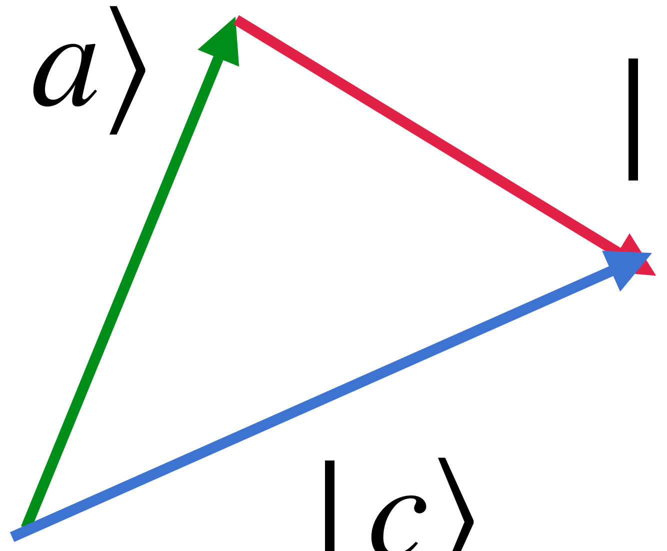


Every arrow has “anti-arrow

$|a\rangle + (-|a\rangle) = |0\rangle$

Game of Arrows

Building Blocks

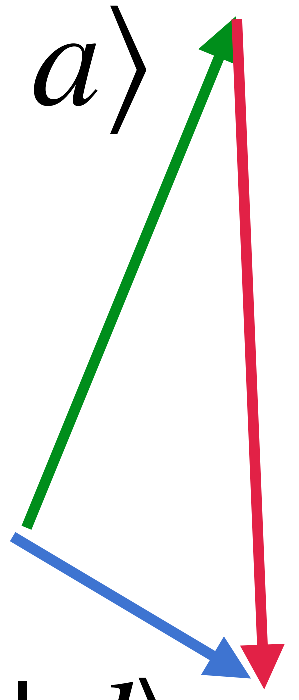


$|a\rangle + |b\rangle = |c\rangle$

$|c\rangle = |a\rangle + |b\rangle$

<u>Observation</u>: **Any** arrow in a plane can be written as a combination of two other "non-parallel" arrows:

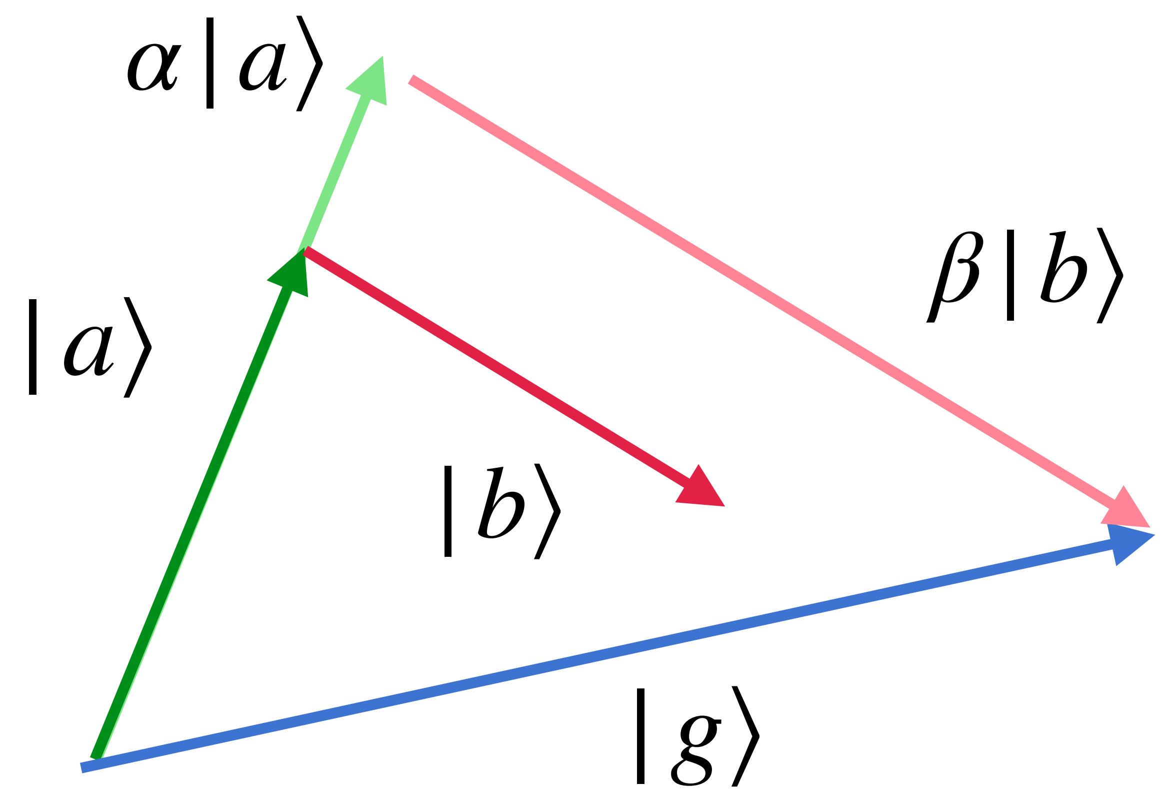
$|c\rangle = |a\rangle + |b\rangle = |d\rangle + |e\rangle$



<u>Basis:</u> Combinations like $\{|a\rangle, |b\rangle\}$ or $\{|d\rangle, |e\rangle\}$ (or others) are *building blocks* or **basis** for all other arrows in a plane.

Game of Arrows

Component Representation



$$|g\rangle = \alpha|a\rangle + \beta|b\rangle$$

<u>Components</u>

$$|g\rangle \sim (\alpha, \beta)$$

Arrow can *represented* (~) as a pair of components.

<u>Components</u>: Basis is a nice tool for converting geometrical problem (drawing arrows) into numerical/algebraic problem

$$|g\rangle = \alpha|a\rangle + \beta|b\rangle$$

$$|g\rangle = \alpha|a\rangle + \beta|b\rangle \qquad |g\rangle \sim (\alpha, \beta)$$

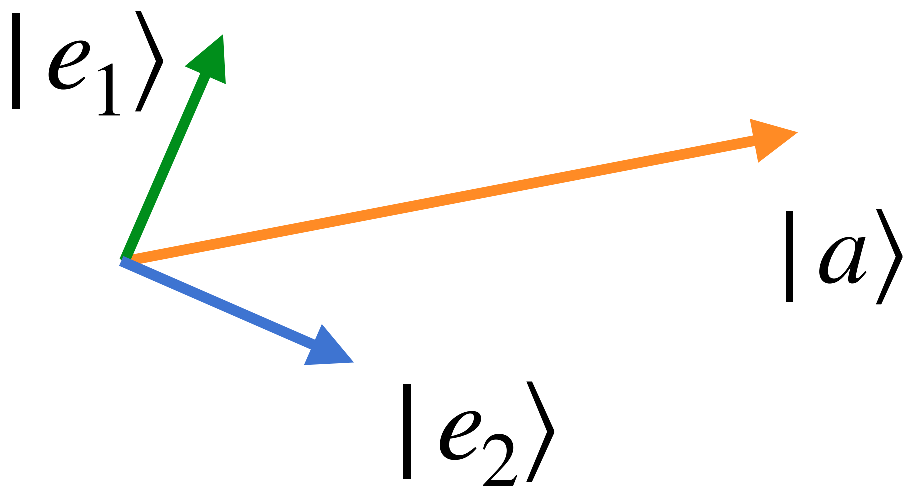
$$|h\rangle = \mu|a\rangle + \nu|b\rangle \qquad |h\rangle \sim (\mu, \nu)$$

$$|g\rangle + |h\rangle = (\alpha + \mu)|a\rangle + (\beta + \nu)|b\rangle \qquad |g\rangle + |h\rangle \sim (\alpha + \mu, \beta + \nu)$$

Operations on arrows become operation with pairs of numbers — often easier. Especially with computers.

Game of Arrows

Arrows Are Independent of Bases Components



$$|a\rangle = a_1 |e_1\rangle + a_2 |e_2\rangle$$

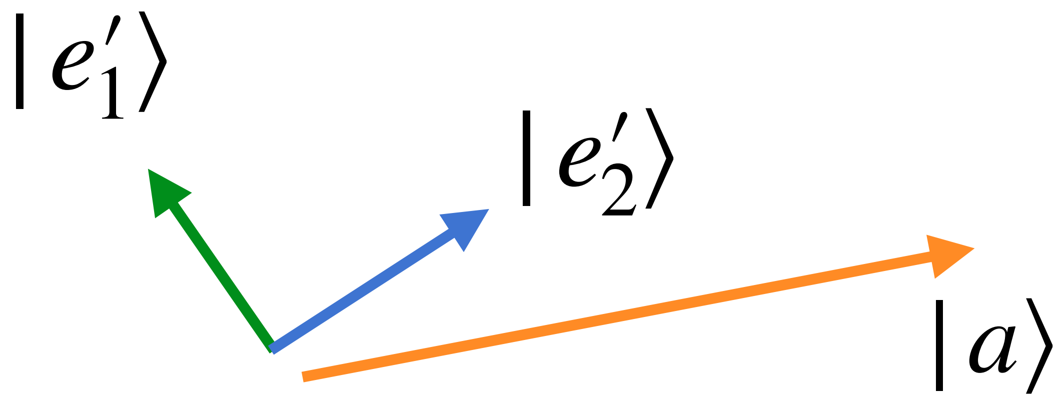
Components

$$|a\rangle \sim (a_1, a_2)$$

$$\boxed{\begin{gathered} |e_1\rangle \perp |e_2\rangle \\ len\,|e_1\rangle = 1 = len\,|e_2\rangle \end{gathered}}$$

Ortho-normal

Orthonormal basis: Useful bases have unit length and are perpendicular to each other.



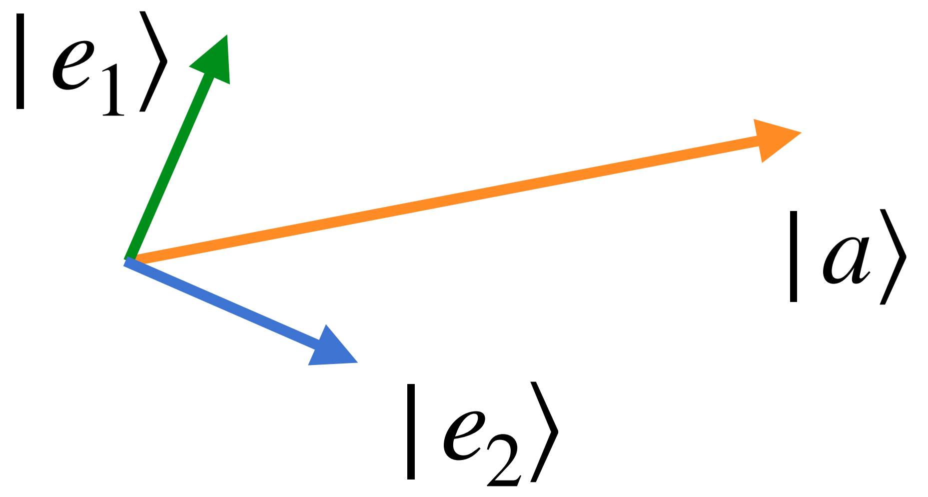
$$|a\rangle = a_1' |e_1'\rangle + a_2' |e_2'\rangle$$

Same arrow is **represented** by different components in different bases.

$$|a\rangle = a_1 |e_1\rangle + a_2 |e_2\rangle \qquad |a\rangle = a_1' |e_1'\rangle + a_2' |e_2'\rangle$$

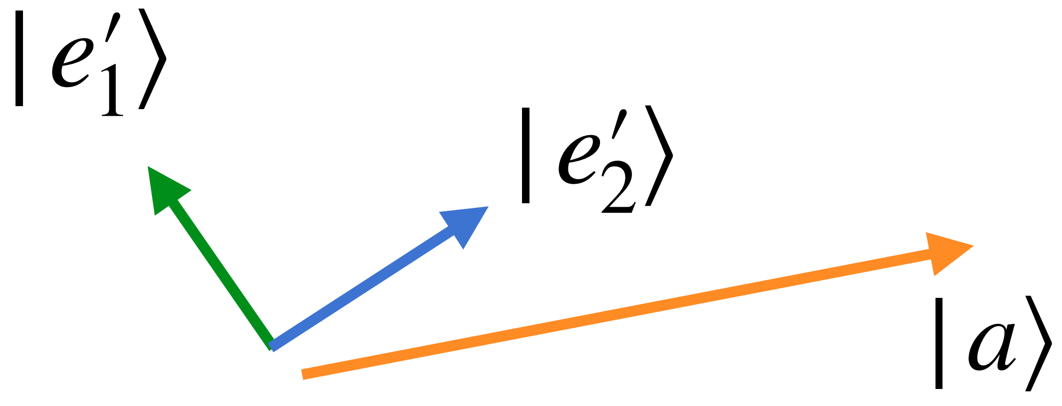
Game of Arrows

Arrows Are Vectors (One Type)



$$|a\rangle = a_1|e_1\rangle + a_2|e_2\rangle$$

Components



$$|a\rangle = a_1'|e_1'\rangle + a_2'|e_2'\rangle$$

Vectors: Objects that are addable like numbers, have "building blocks" (**bases**), can be **represented** by **components**, but are truly **independent** of them. When bases change, components change, **object remains the same**. Components are related by a certain rule — transformation rule.

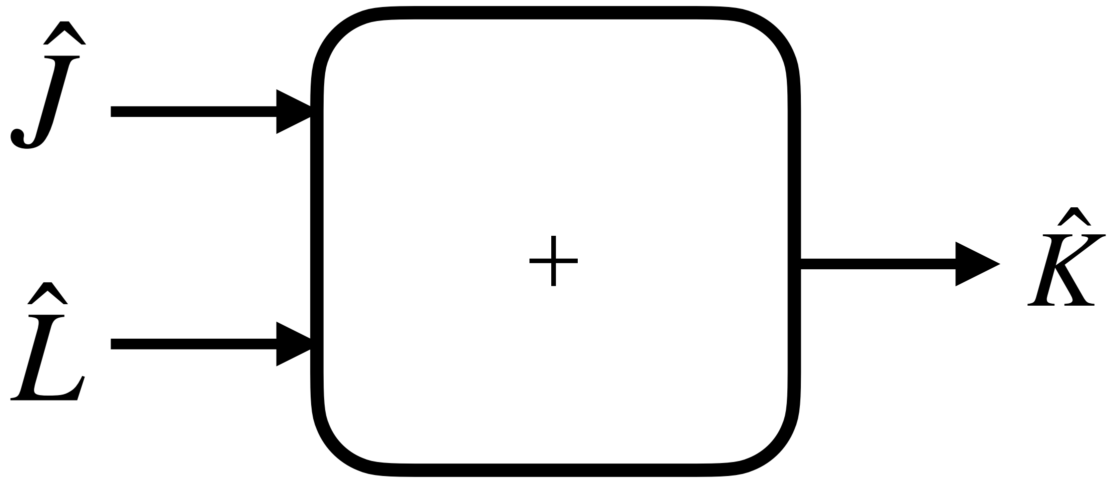
Arrows are NOT the only objects that satisfy vector properties. Orientable areas and volumes do. Functions do. Operators do.

Note how important numbers are for vectors!

Game of Operators

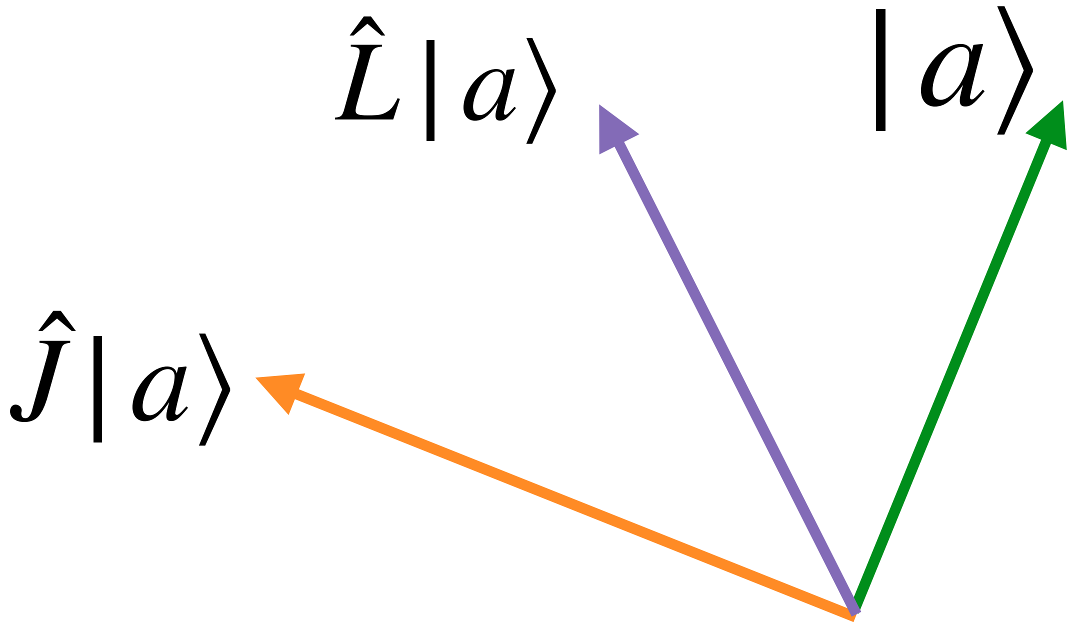
Algebra of Operators

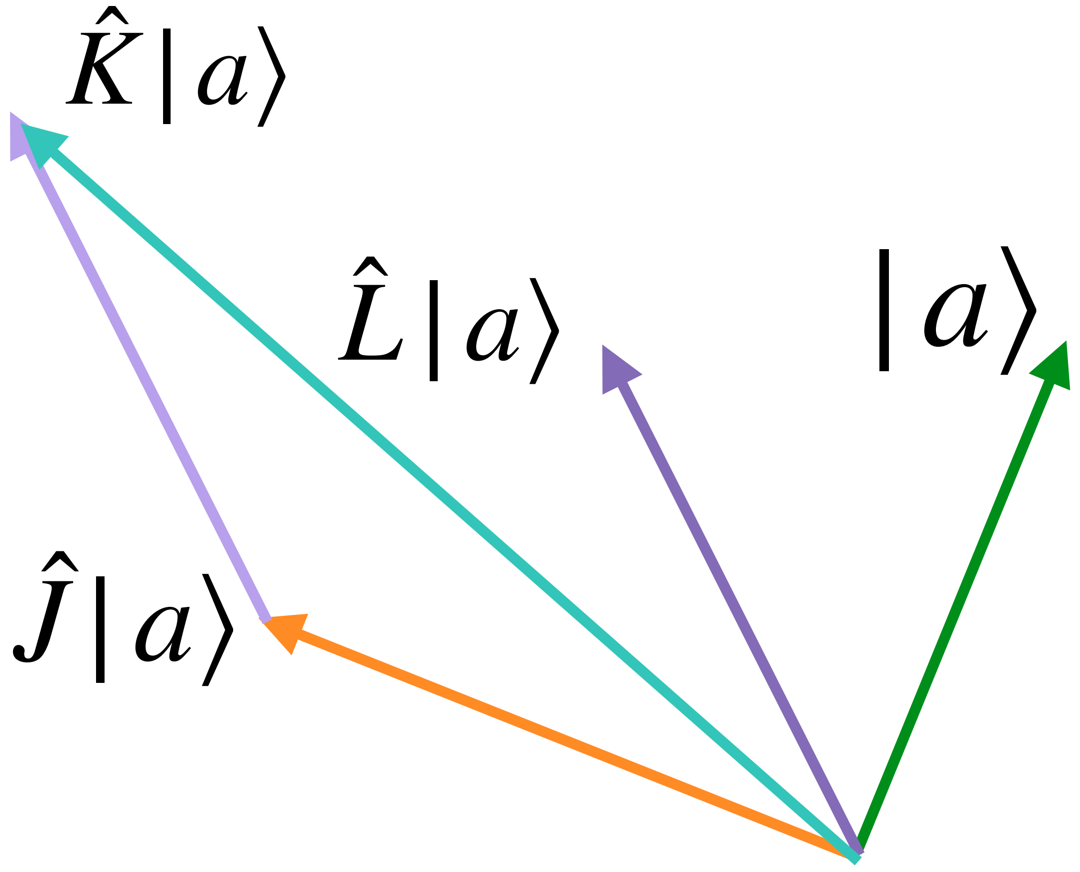
$$\hat{L} = \sqrt{\hat{J}}$$



$$\hat{J} + \hat{L} = \hat{K}$$

1. View operators as *instructions*: Do this to this vector.
2. Instructions can be *combined/sequenced/composed* to create a new instruction: First do this to this vector, then do that for the resulting vectors.
3. **Operators can be "added" like numbers.**
4. Operators can be multiplied by numbers.





If we know how to add vectors, we can make sense of adding operators.

$$\hat{J} + \hat{L} = \hat{K}$$

$$\hat{L} = \frac{\hat{I}}{\sqrt{2}} + \frac{\hat{J}}{\sqrt{2}}$$

Game of Operators

Algebra of Operators

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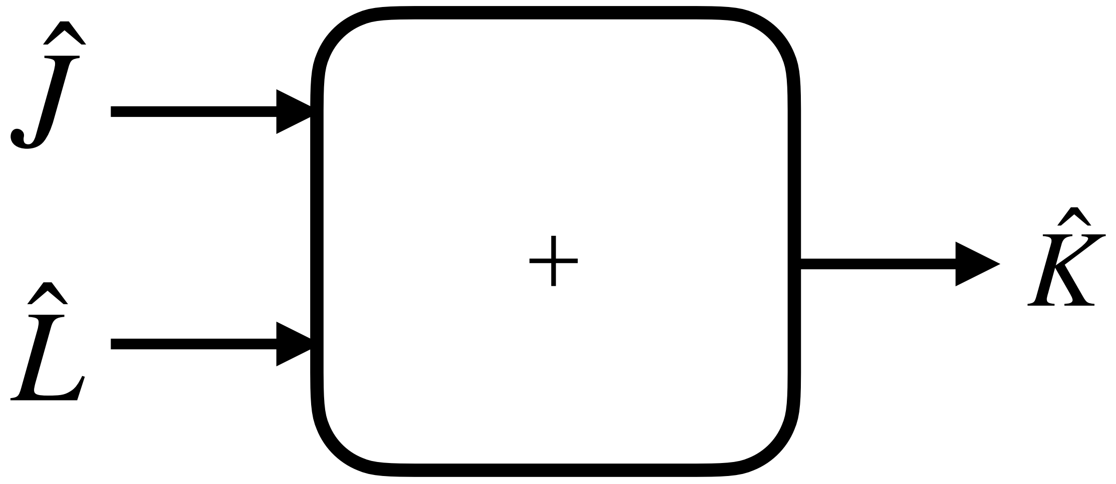

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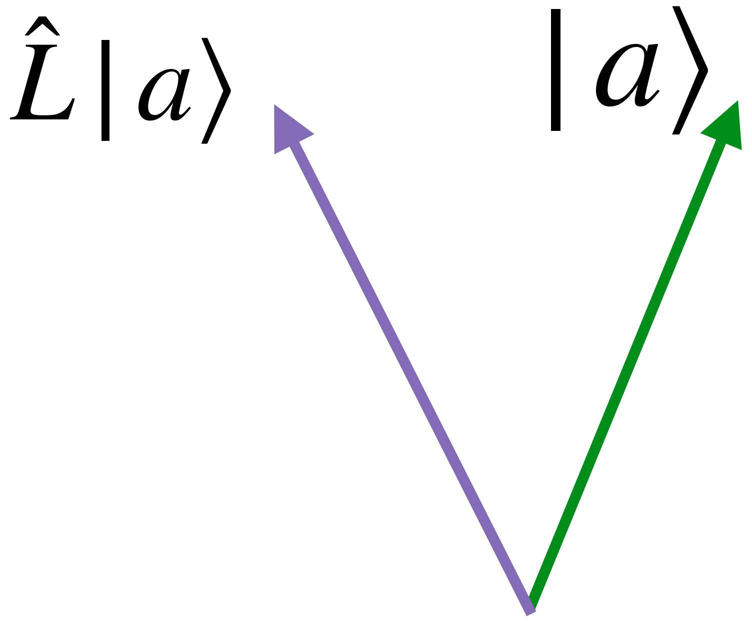

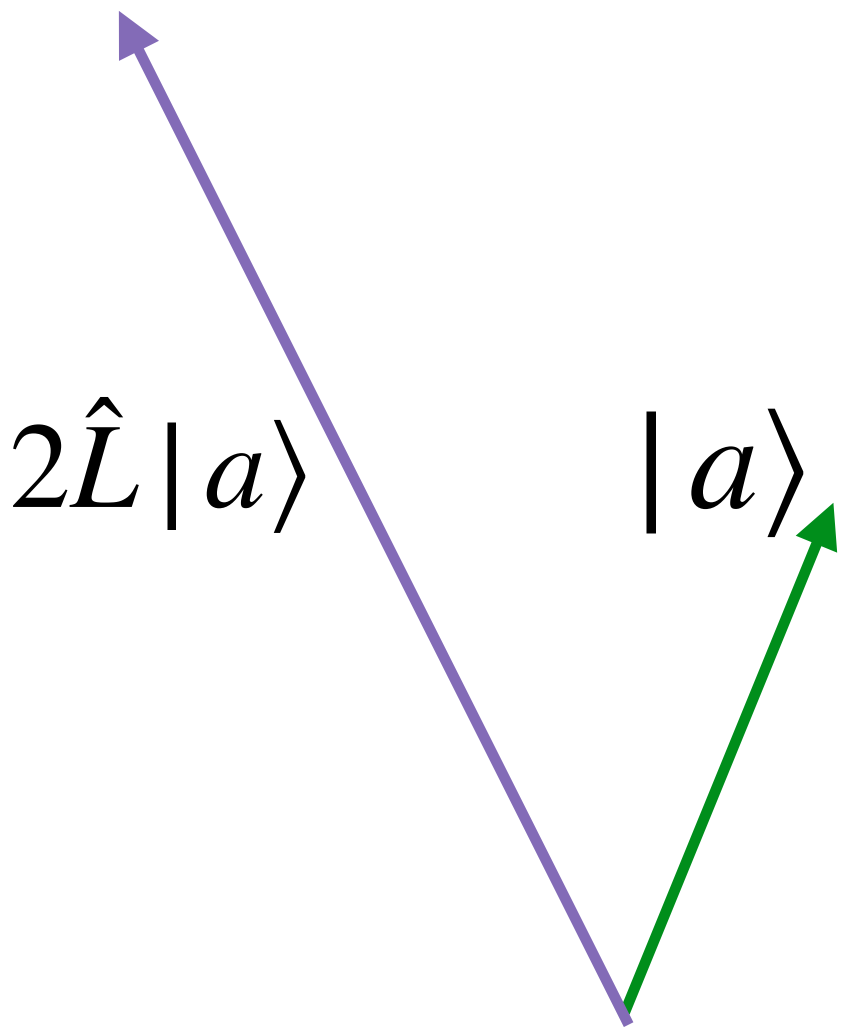

Multiplying operator by a number is just scaling the resulting vector

$$2\hat{L}$$

Game of Arrows and Operators

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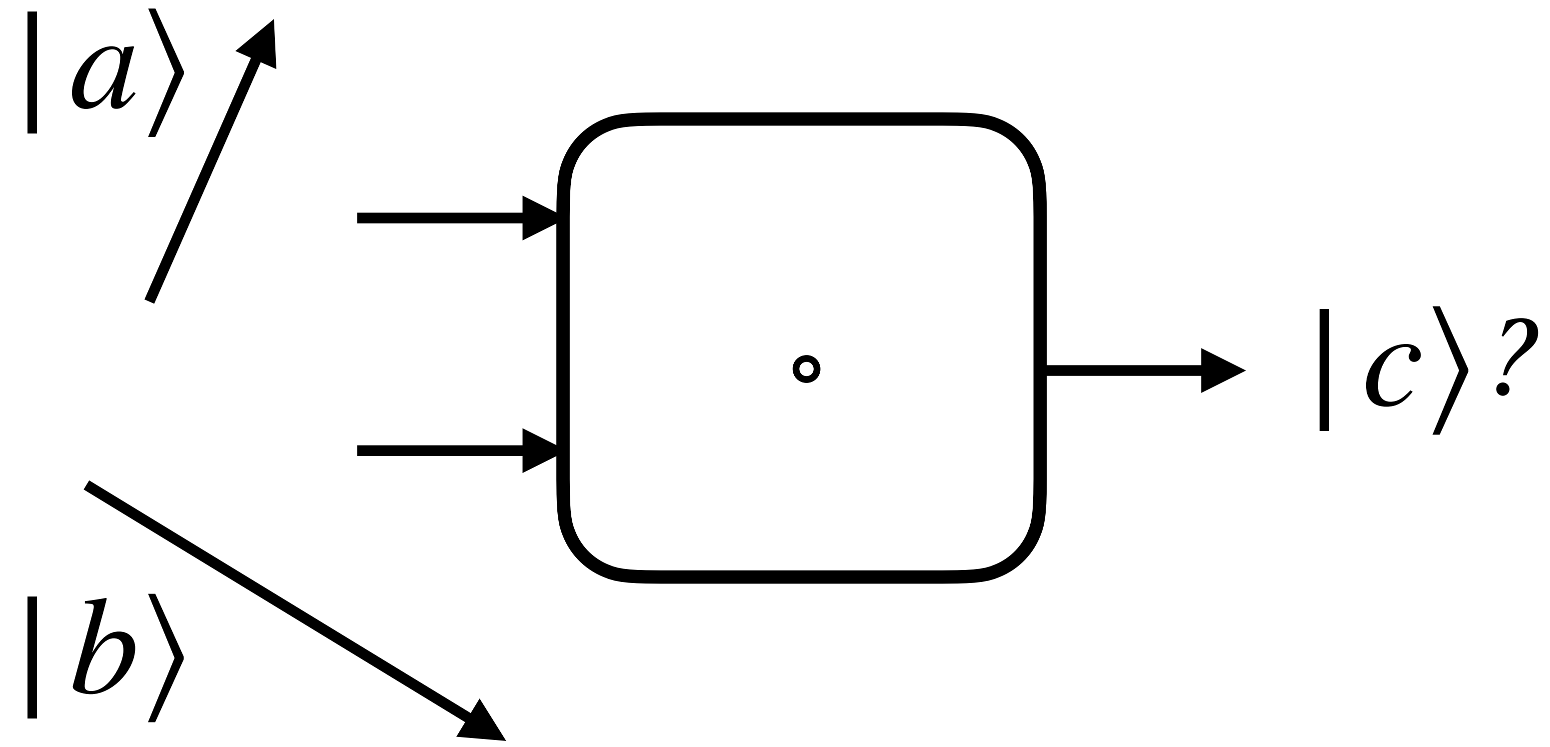
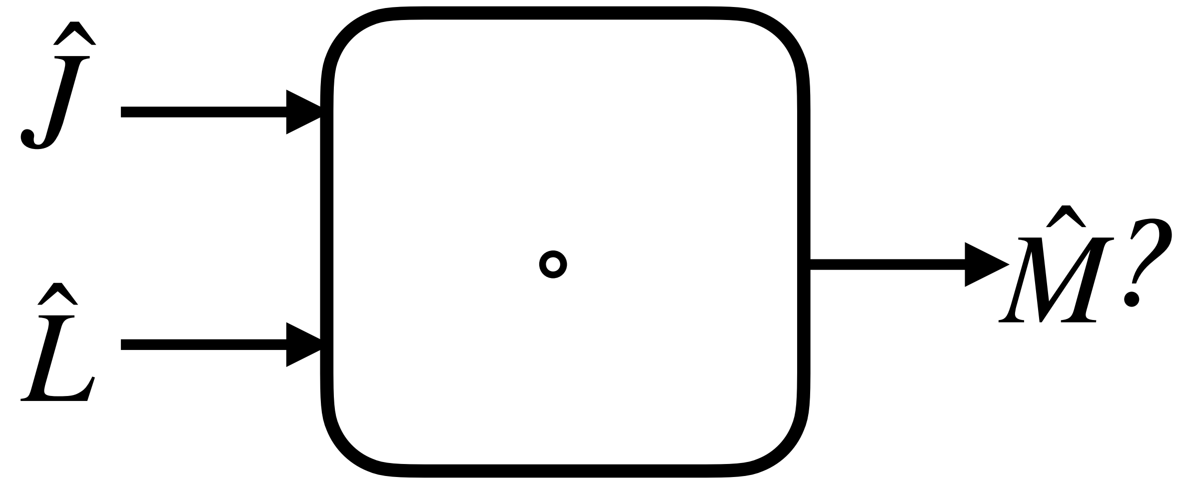
Arrows



$$|a\rangle \circ |b\rangle \stackrel{?}{=} |c\rangle$$

Operators

$$\hat{L} = \sqrt{\hat{J}}$$



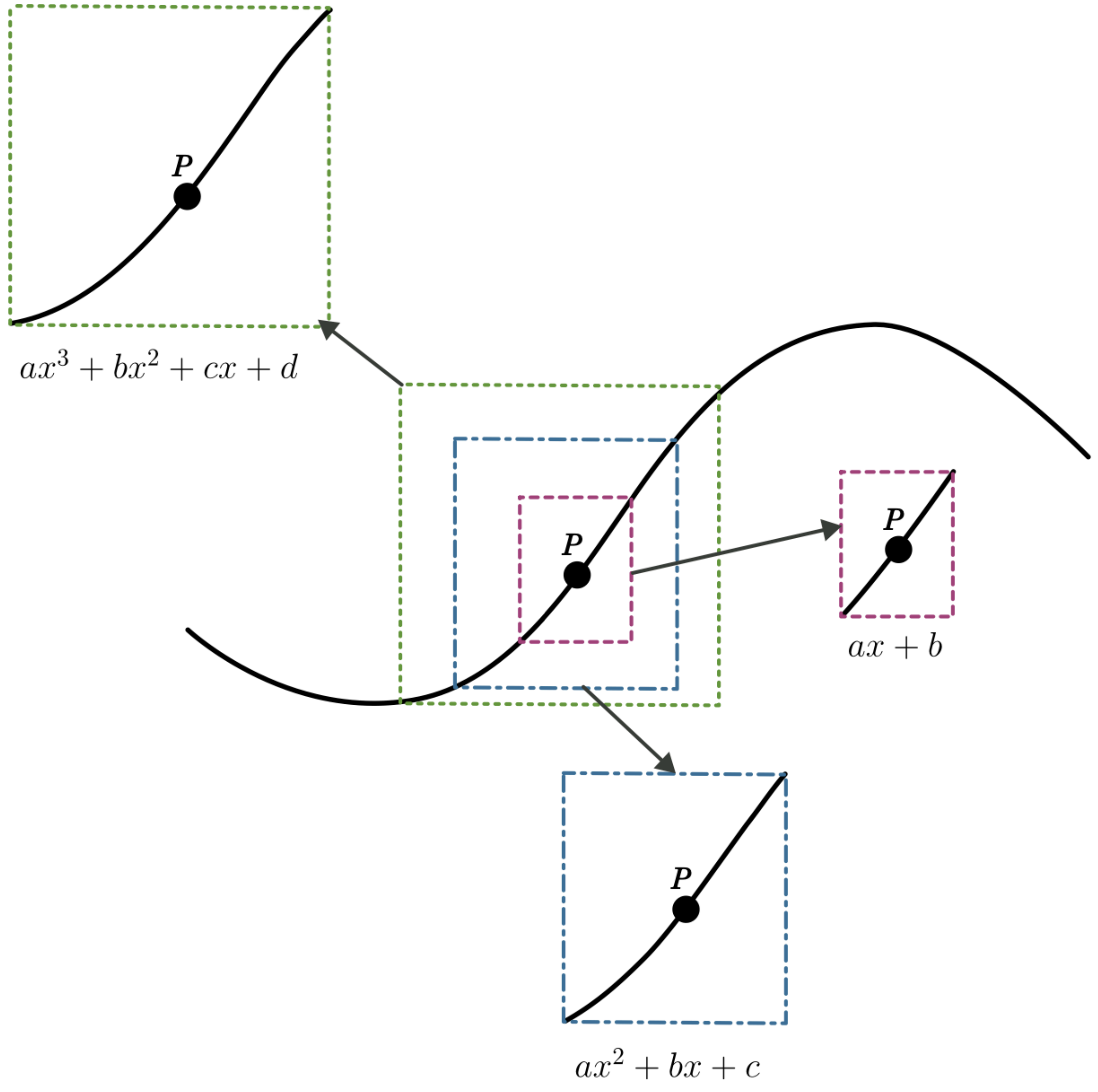
$$\hat{J} \circ \hat{L} = \hat{M}$$

Operators, like functions, can also be composed. Can arrows be composed?

Game of Operators

Algebra of Operators

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$



With addition and composition of operators we can write any polynomial operator, e.g. $\hat{J}^3 + \hat{J}^2 + \hat{J} + \hat{I}$

For any operator $\hat{X}$, we can write a polynomial with coefficients a_k:

$$a_0 + a_1\hat{X} + a_2\hat{X}^2 + \cdots + a_n\hat{X}^n + \cdots$$

Since ***almost all*** *"good physical functions"* (e^x, $\cos x$, $1/(1+x^2)$) etc.) can be approximated with polynomials, we can use operators as their arguments:

$e^{\hat{J}}$ or $\cos\hat{J}$ or $1/(1-\hat{J}/2)$.

Of course, we have to be careful, but we can do **A LOT**.

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \ldots$$

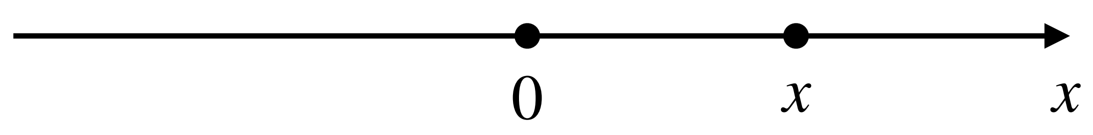
$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \ldots$$

$$\frac{1}{1+x^2} \approx 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$$

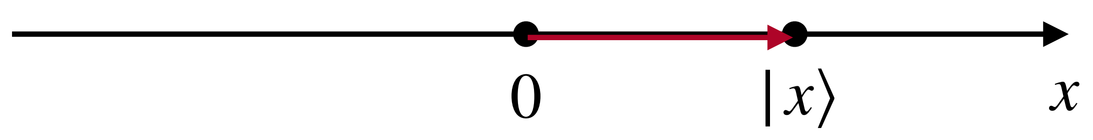
$$\frac{1}{1+e^{x^2}} \approx \frac{1}{2} - \frac{x^2}{4} + \frac{x^6}{48} - \frac{x^{10}}{480} + \ldots$$

Operators

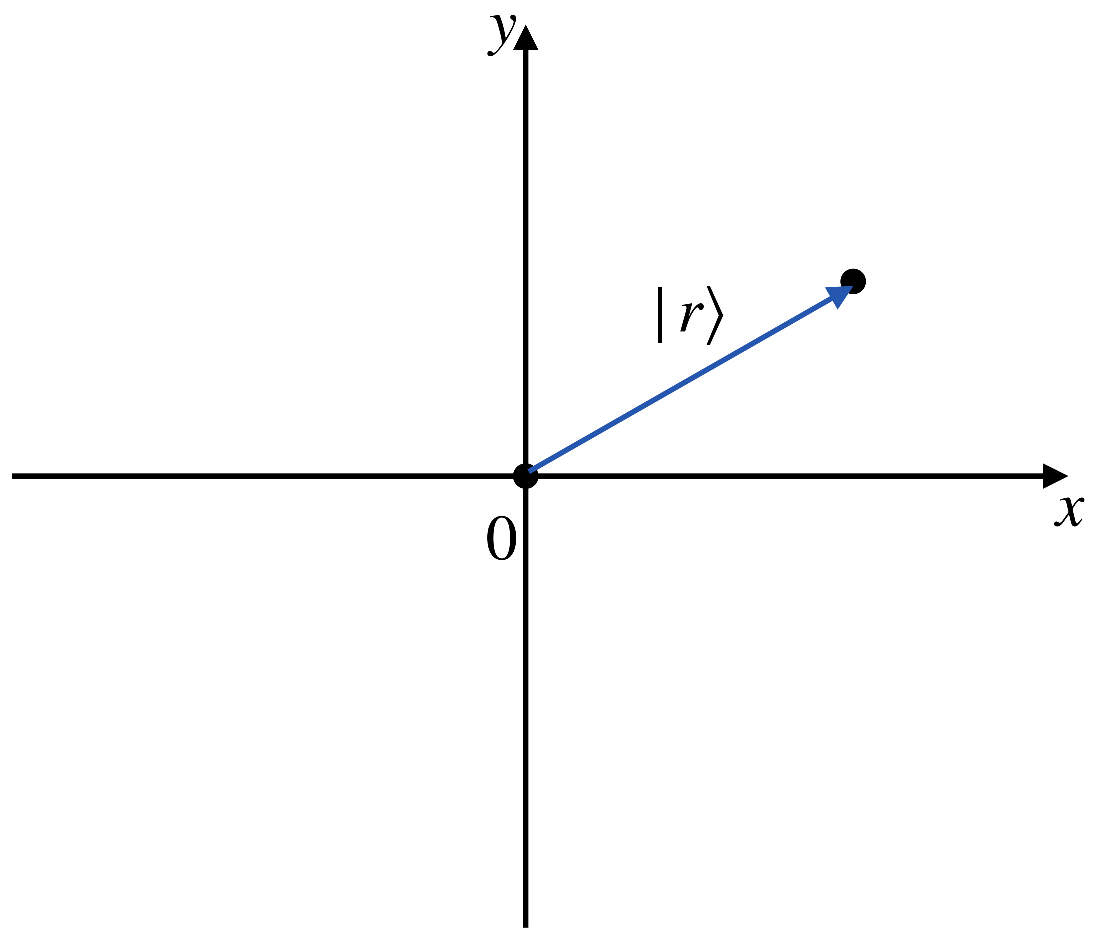
Learn them. Love them. Use them.



Numbers, 1D



Arrow-like vectors



Arrow-like vectors, 2D numbers

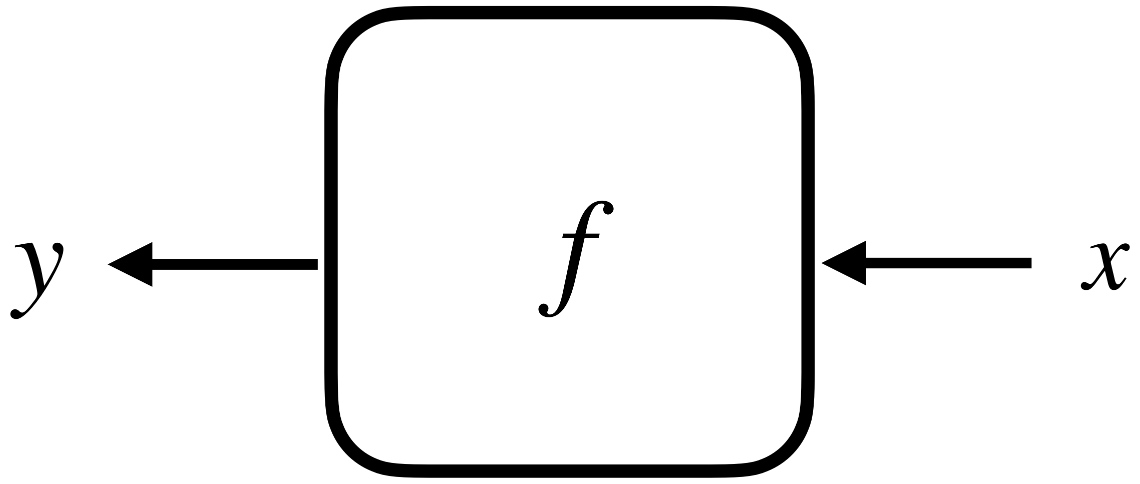
$$|r\rangle \xrightarrow{\text{Scalar product}} \hat{P}_r = |r\rangle\langle r|$$

Operators (projectors) for every arrow-like vectors.

- Equation $x^3 + x^2 + x + 1 = 0$ can be interpreted in terms of numbers (not natural!) and even better — in terms of operators: $\hat{F} = -\hat{I}, \hat{J}$, and $\hat{G} = -\hat{J}$.
- Operators include special simple ones: Projectors, which correspond 1-1 for every arrow-vector.
- Some arrow-vectors (1D and 2D) correspond 1-1 to “ordinary” numbers.
- Operators give you more power that “ordinary” numbers.

Linearity

Surprisingly Effective Simplicity



You can review the idea of functional equations .

Функціональний підхід Коші

Вам часто доводиться розв'язувати рівняння, тобто шукати такі значення змінної, при підстановці яких у рівняння отримуємо правильну рівність. Такі рівняння можна було б назвати числовими, оскільки їхніми розв'язками є числа. У математиці вивчають й інші рівняння, розв'язками яких є не числа, а функції. Природно, що їх називають **функціональними рівняннями**.

З функціональними рівняннями ви стикалися раніше. Наприклад, рівність

$$f(x) = f(-x),\ x \in D(f),$$

яка задає парні функції, можна розглядати як функціональне рівняння. Розв'язком цього рівняння є будь-яка парна функція.

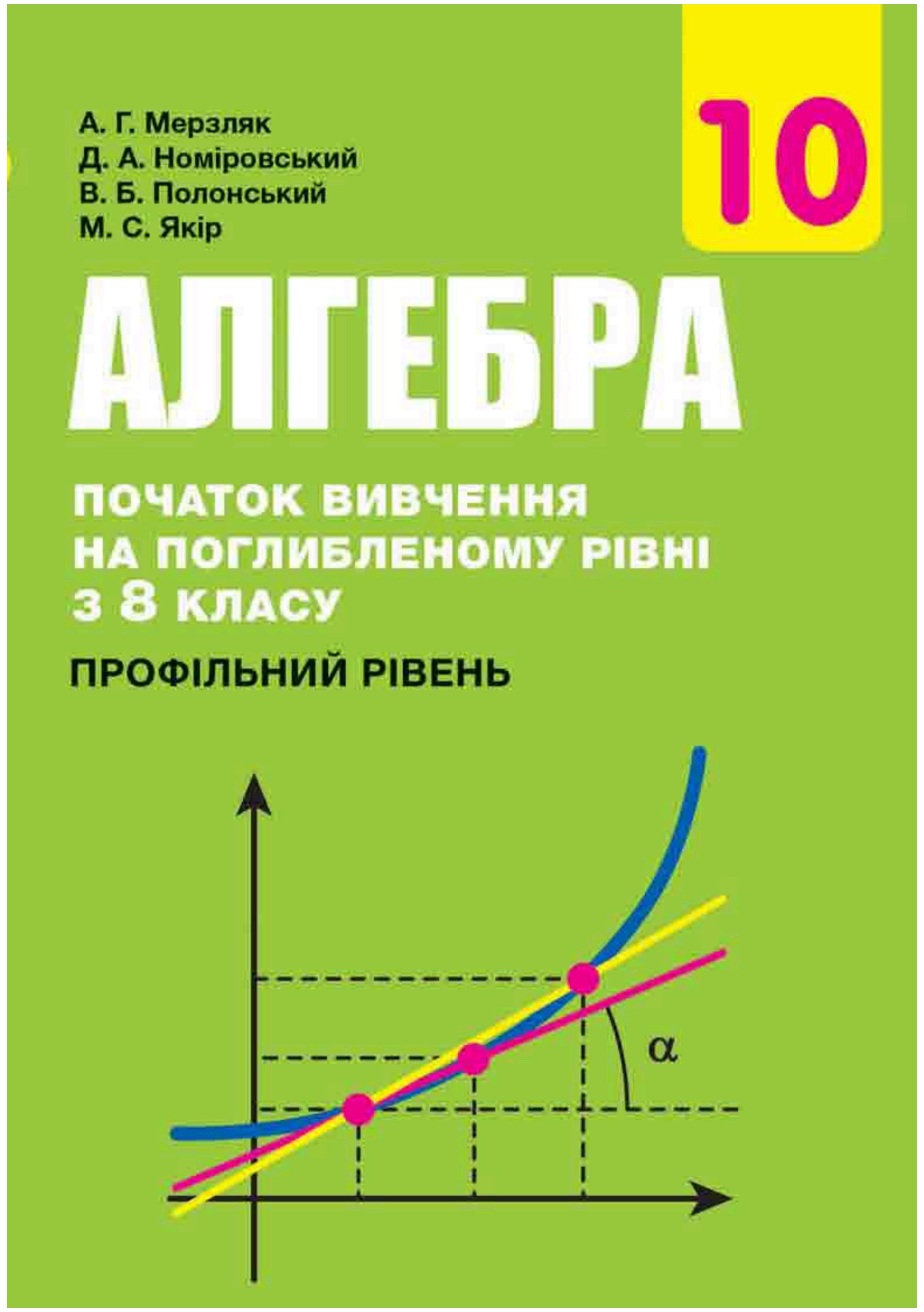
функцій, то за допомогою функціональних рівнянь можна визначати конкретні класи функцій. Такий спосіб визначення функцій через опис їхніх характерних властивостей у вигляді функціональних рівнянь запровадив відомий французький математик О. Коші. Його ім'я носять такі функціональні рівняння:

$$f(x+y) = f(x) + f(y),$$

$$f(xy) = f(x) + f(y),$$

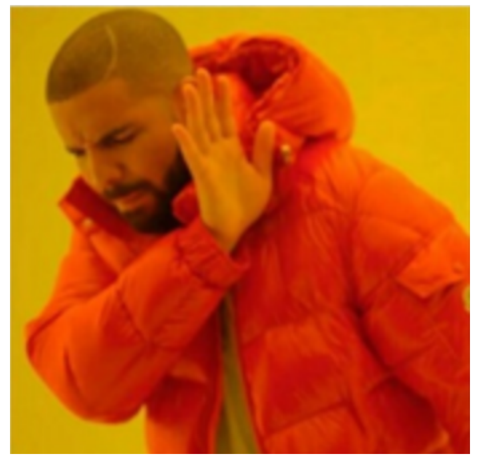
$$f(x+y) = f(x)f(y),$$

$$f(xy) = f(x)f(y).$$



Linearity

Surprisingly Effective Simplicity



$$y \longleftarrow \boxed{f} \longleftarrow x$$

Linear numeric functions are boring. They are just multiplication by a certain number.

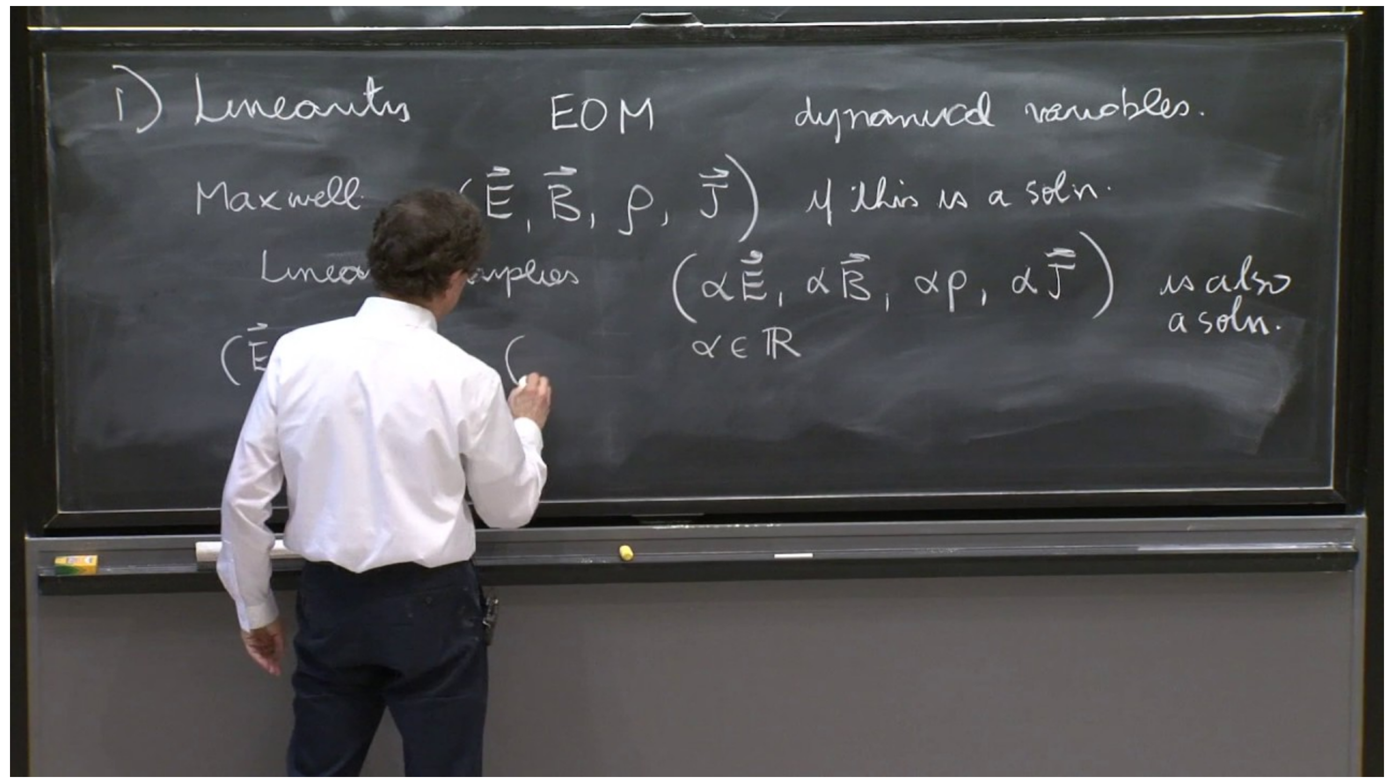
$$|b\rangle \longleftarrow \boxed{\hat{L}} \longleftarrow |a\rangle$$

Linear operators are *amazingly powerful*, yet still simple.

Linearity:

$$\hat{L}(|a\rangle + |b\rangle) = \hat{L}|a\rangle + \hat{L}|b\rangle$$

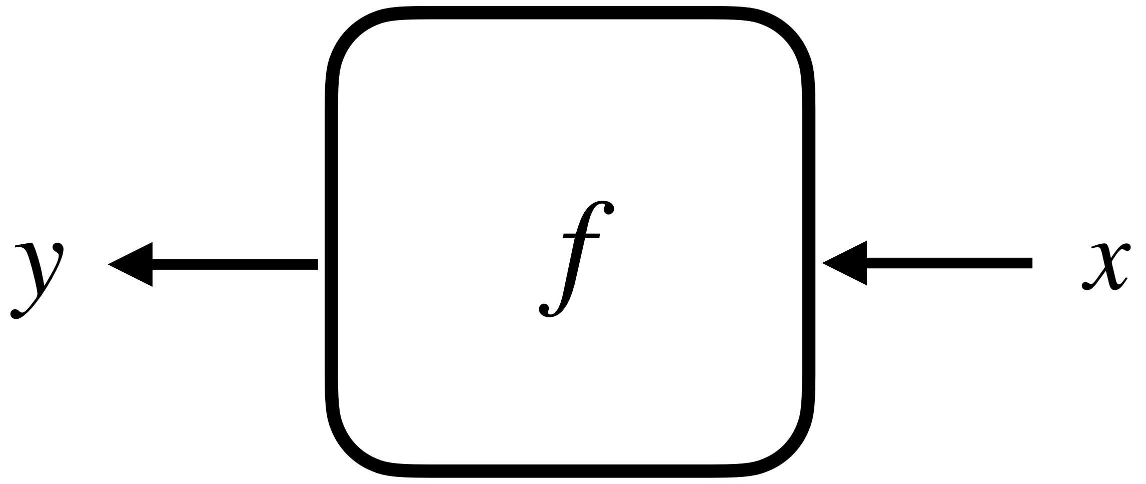
$$\hat{L}(k|a\rangle) = k(\hat{L}|a\rangle)$$



Quantum mechanics as a framework. Defining linearity

Linearity

Surprisingly Effective Simplicity



Some functions are clearly simpler than others.

Simplicity can be expressed in terms of how many properties a function has, how many conditions/"constraints" it satisfies.

The simplest kind.

Only trivial functions are simpler.

Symmetric	$f(-x) = f(x)$	x^2
Anti-symmetric	$f(-x) = -f(x)$	$1/x^3$
Periodic	$f(x+a) = f(x)$	$\cos(2\pi x/a)$
Affine	$f(x+y) = f(x) + f(y)$	$4x + 7$
Homogeneous	$f(kx) = k^n f(x)$	$3x^n$
Linear	$f(x+y) = f(x) + f(y)$ $f(kx) = k\,f(x)$	$\frac{1}{2}x$

Functions with special properties

Self-Test

Answer These Questions 1hr After Class

1. What makes a mathematical object “number-like”?
2. What do games and algebras have in common?
3. How can we “add” two arrows? How can we “add” two operators?
4. What role do “normal” numbers play in the additions of arrows and operators?
5. What is the essential difference between an arrow and its components?
6. What useful properties do “good” functions have?
7. What allows us to write polynomials of operators?
8. What is linearity and why it is important?

Homework Problems

Mathematical Concepts and Notation Day 3

- How many linear numerical functions $f\,x = ax$ are there?
- For a vector $|a\rangle = a_1|e_1\rangle + a_2|e_2\rangle = a_1'|e_1'\rangle + a_2'|e_2'\rangle$, find the relationship between the components (a_1, a_2) and (a_1', a_2').
- Is the function $len\,|a\rangle = a$ linear?
- Consider the "normalizing" operator $\hat{N}|a\rangle = |a\rangle/a$. Is it a linear operator?
- Use mathematical induction method to show that $(1+x)^n \approx 1 + nx$ for small values of x? Start with $n = 1,2,3$ and then generalize.
- Using the previous result, show that $\partial_x x^n = nx^{n-1}$.
- Using the previous result, show that $\partial_x^n\, x^n = n!$
- Using Schrödinger's equation, show that $|\Psi_{t+\delta t}\rangle = \left(\hat{I} + \frac{-i\delta t\hat{H}}{\hbar}\right)|\Psi_t\rangle$. Then show that $|\Psi_{t+\delta t}\rangle = \left(\hat{I} + \frac{-it\hat{H}}{N\hbar}\right)^N |\Psi_0\rangle$.

Quantum Theory

In a Nutshell

II. POSTULATES FOR QUANTUM MECHANICS

In this paper, all state vectors are supposed to be normalized, and mixed states are represented by density operators, i.e., positive operators with unit trace. Let A be an observable with a nondegenerate purely discrete spectrum. Let $\phi_1, \phi_2, \ldots$ be a complete orthonormal sequence of eigenvectors of A and $a_1, a_2, \ldots$ the corresponding eigenvalues; by assumption, all different from each other.

According to the standard formulation of quantum mechanics, on the result of a measurement of the observable A the following postulates are posed:

(A1) *If the system is in the state ψ at the time of measurement, the eigenvalue a_n is obtained as the outcome of measurement with the probability $|\langle\phi_n|\psi\rangle|^2$.*

(A2) *If the outcome of measurement is the eigenvalue a_n, the system is left in the corresponding eigenstate ϕ_n at the time just after measurement.*

The postulate (A1) is called the *statistical formula*, and (A2) the *measurement axiom*. The state change $\psi \mapsto \phi_n$ described by the measurement axiom is called the *state reduction*.

You will understand this paragraph in the end of the course.