Quantum Physics

2025

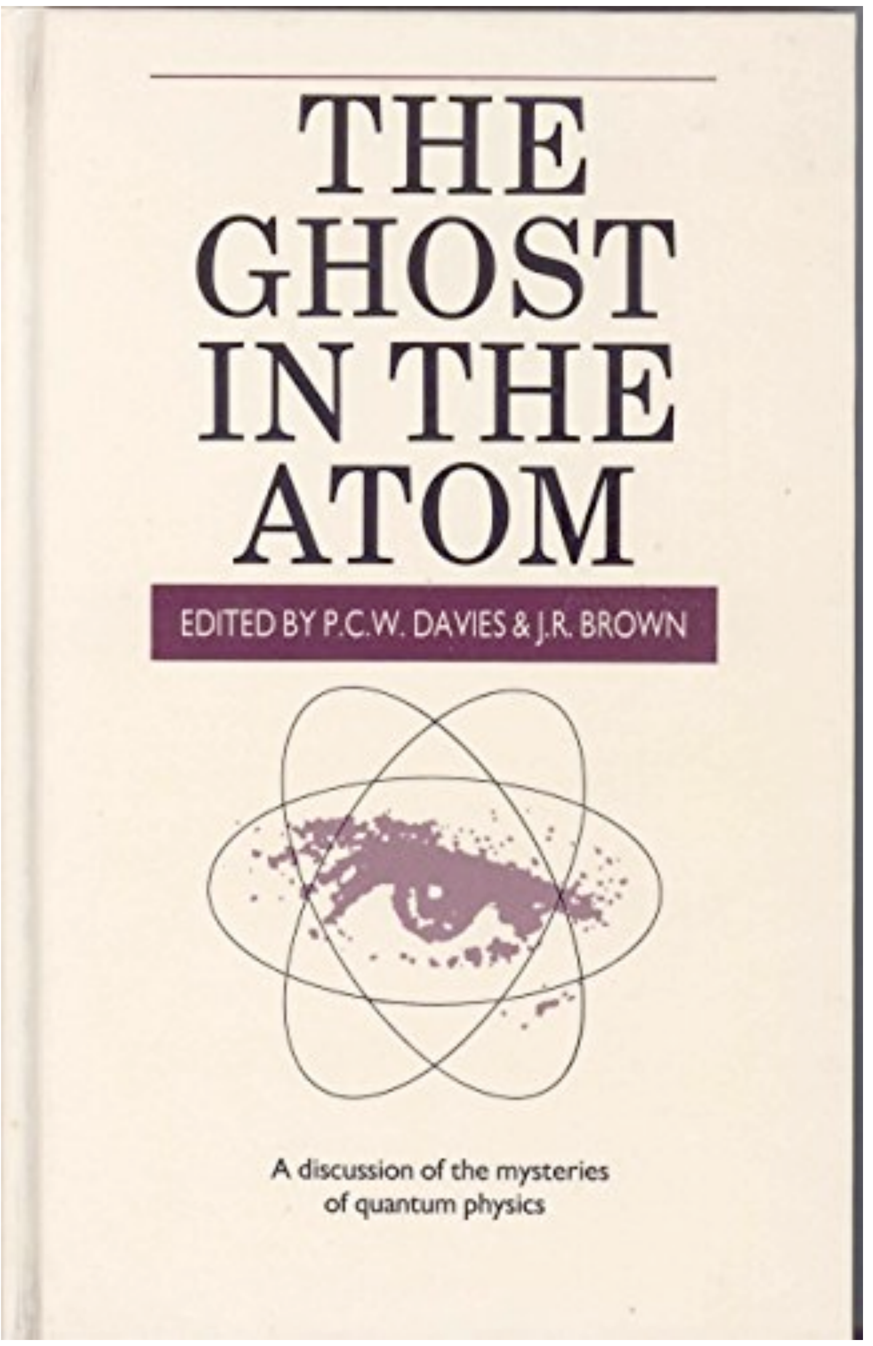
The Theory/Framework Of ***Almost*** Everything ***Today***

But Most Likely NOT of Tomorrow

Yury Deshko

“I’m quite convinced of that: quantum theory is only a temporary expedient.”

John Bell in “*The Ghost In The Atom*”.



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Course Overview

Course Structure And Goals

- **Part 1** : Mathematical Concepts And Tools.
- **Part 2** : Classical Physics.
- **Part 3** : Quantum Physics.

- Learn the language of quantum physics.
- Enhance the knowledge of classical physics.
- Develop modern quantum thinking.

Projectors & Matrices

As Tools in Quantum Theory

I like operators. I recommend you using them as much as possible.

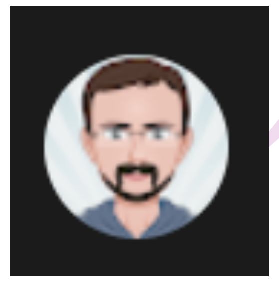
Projectors & Matrices

As Tools in Quantum Theory

I like operators. I recommend you use them as much as possible.

I don’t like matrices. I recommend you avoid them for as long as possible.

They are not bad. But operator methods are more general.
Matrices quickly become monstrous.



Projectors & Matrices

Are Related to Linear Operators

All **linear** operators: $\hat{I}, \hat{J}, \hat{H}, \ldots, \hat{L}, \hat{P}, \ldots$

Special **linear** operators — projectors. $\hat{P}_1, \hat{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \ldots$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

"Components" of the operators — numeric representation **relative to a specific basis!**

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Projectors

In Quantum Theory

- *Every* state vector $|\Psi\rangle$ has an *operator* associated with it: $\hat{P}_\psi = |\Psi\rangle \otimes \langle\Psi| = |\Psi\rangle\langle\Psi|$.
- This operator simply "extracts" from *any* vector the part which is *parallel to* $|\Psi\rangle$*:* $|\Psi\rangle \otimes \langle\Psi|\,|\Phi\rangle = |\Psi\rangle\langle\Psi|\Phi\rangle = s|\Psi\rangle$, where $s = \langle\Psi|\Phi\rangle$ is the scalar product (bracket).
- The operation is called *projection* of $|\Phi\rangle$ *onto* $|\Psi\rangle$.
- The operator $|\Psi\rangle\langle\Psi|$ is called **projector**. It is a linear operator. (**Exercise**: Check)
- <u>Not every linear operator is projector.</u>
- Projectors are very useful in quantum theory. E.g., they describe state just as well as their vector counterparts. $|\Psi\rangle \leftrightarrow |\Psi\rangle\langle\Psi|$.

Projectors

Examples

- $\hat{P}_1 = |e_1\rangle\langle e_1|$ for basis vector $|e_1\rangle$. Generally, $\hat{P}_i = |e_i\rangle\langle e_i|$.
- *NOTE*: **Not** a projector: $\hat{F} = |e_1\rangle\langle e_2|$ even though the result $\hat{F}|\Psi\rangle = s|e_1\rangle$ looks similar to how projectors work. Why not? See Problem 1.

Matrices

As Numeric Representation of <u>Linear</u> Operators

- A **linear** operator is defined by its action on *any* vector.
- Consider $\hat{L}\,|a\rangle$. Choose your favorite basis. Expand $|a\rangle = a_1|e_1\rangle + a_2|e_2\rangle$.
- Use linearity of $\hat{L}$ and write $\hat{L}\ \left(a_1|e_1\rangle + a_2|e_2\rangle\right) = a_1\left(\hat{L}\,|e_1\rangle\right) + a_2\left(\hat{L}\,|e_2\rangle\right)$.
- Thus, we only need to know how $\hat{L}$ transforms basis vectors.
- $\hat{L}\,|e_1\rangle$ is also a vector. It can also be expanded in the chosen basis. We then have

$$\hat{L}\,|e_1\rangle = A|e_1\rangle + B|e_2\rangle$$

$$\hat{L}\,|e_2\rangle = C|e_1\rangle + D|e_2\rangle$$

$$\boxed{\hat{L} \longleftrightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix}}$$

Matrix of the operator $\hat{L}$

"Components" of the operators — numeric representation **relative to a specific basis!**

Matrices

Example

- Take familiar linear operator $\hat{J}$.

$$\hat{J}\,|e_1\rangle = +\,0\,|e_1\rangle + 1\,|e_2\rangle$$

$$\hat{J}\,|e_2\rangle = -\,1\,|e_1\rangle + 0\,|e_2\rangle$$

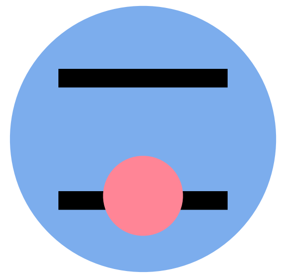
$$\boxed{\hat{J} \longleftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$

Matrix of the operator $\hat{L}$

That is enough for now. We need to be familiar with the idea, but do not need to be proficient.

Projectors

In Quantum Theory



- Consider qubit with energy levels E_0 and E_1 for states $|0\rangle$ and $|1\rangle$.
- Hamiltonian is simply $\hat{H} = E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1|$.
- It is easy to check (see HW problems) that $\hat{H}|n\rangle = E_n|n\rangle$.
- Since there is a one-to-one correspondence between state vectors and projectors: $|\Psi\rangle \leftrightarrow |\Psi\rangle\langle\Psi|$, either one can be used in quantum theory for calculations.
- For $|\Psi\rangle\langle\Psi|$ special notation exists: $\hat{\rho} = |\Psi\rangle\langle\Psi|$ and the operator $\hat{\rho}$ is called **density operator**. Matrix representation of $\hat{\rho}$ is called **density matrix**.

Density Operator

And Its Equation of Motion

- For state vector $|\Psi\rangle$ we have Schrödinger equation: $i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle$.
- Do we have something equivalent for density operator $\hat{\rho} = |\Psi\rangle\langle\Psi|$? Can we write something like $i\hbar\partial_t\,\hat{\rho} = \tilde{L}\,\hat{\rho}$? Yes, see below. It is not very important for now, just for education.
- The first step would be to have Schrödinger equation for $\langle\Psi|$: $-i\hbar\partial_t\langle\Psi| = \langle\Psi|\hat{H}$
- Second, recall a rule $\partial_t\left(fg\right) = (\partial_t f)g + f(\partial_t g)$ and apply it to $|\Psi\rangle\langle\Psi|$: $\partial_t\left(|\Psi\rangle\langle\Psi|\right) = (\partial_t|\Psi\rangle)\langle\Psi| + |\Psi\rangle(\partial_t\langle\Psi|)$.
- Third, use Schrödinger equations: $i\hbar\partial_t\left(|\Psi\rangle\langle\Psi|\right) = \hat{H}|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|\hat{H}$.
- Finally, replace $|\Psi\rangle\langle\Psi|$ with $\hat{\rho}$: $\boxed{i\hbar\partial_t\hat{\rho} = \hat{H}\hat{\rho} - \hat{\rho}\hat{H}}$

This is the operator version of Schrödinger Equation. It is called Liouville-von Neumann equation.

Density Operator

Is As Powerful as State Vector

- Density operator $\hat{\rho} = |\Psi\rangle\langle\Psi|$ contains the same information about measurements and all predictions as the state vector $|\Psi\rangle$.
- It is an alternative mathematical tool.
- It is more versatile and more powerful that just $|\Psi\rangle$.
- To use it effectively, one needs to master various operator techniques.
- We will not need use it much in the course.

$$|\Psi\rangle \leftrightarrow \hat{\rho}_\Psi$$

Self-Test

Answer These Questions 1hr After Class

1. What is the difference between a vector and its components?
2. What is the similarity between vector components and matrices?
3. Is the sum of linear operators also a linear operator?
4. What is a projector?
5. Is the sum of projectors also a projector?
6. How many numbers are needed to fully specify a linear operator that acts on vectors in three dimensions?
7. What is the operator analogue of Schrödinger equation?

Homework Problems

Tensor Product

1. Prove that projectors have simple property $\hat{P}_\psi \circ \hat{P}_\psi = \hat{P}_\psi$. (Mathematicians say that such objects are *idempotent,* meaning "same power"). Generalize to $\hat{P}^n_\psi = \hat{P}_\psi$. What does this mean in terms of projection operation?

2. Write the tensor product form of the operator $\hat{S}$ that "swaps" basis vectors $|e_1\rangle \leftrightarrow |e_2\rangle$. Then its matrix representation.

3. Convince yourself that the qubit states $|0\rangle$ and $|1\rangle$ are eigen-states of the Hamiltonian $\hat{H} = E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1|$. What are the eigen-values?

4. Solve the eigen-problem for a projector $\hat{P} = |\Psi\rangle\langle\Psi|$. That is, find its eigen-vectors and corresponding eigen-values.

5. Read the last slide and the "Postulates For Quantum Mechanics" and try to see how much of it you can already comprehend.

Quantum Theory

In a Nutshell

II. POSTULATES FOR QUANTUM MECHANICS

In this paper, all state vectors are supposed to be normalized, and mixed states are represented by density operators, i.e., positive operators with unit trace. Let A be an observable with a nondegenerate purely discrete spectrum. Let $\phi_1, \phi_2, \ldots$ be a complete orthonormal sequence of eigenvectors of A and $a_1, a_2, \ldots$ the corresponding eigenvalues; by assumption, all different from each other.

According to the standard formulation of quantum mechanics, on the result of a measurement of the observable A the following postulates are posed:

(A1) *If the system is in the state ψ at the time of measurement, the eigenvalue a_n is obtained as the outcome of measurement with the probability $|\langle\phi_n|\psi\rangle|^2$.*

(A2) *If the outcome of measurement is the eigenvalue a_n, the system is left in the corresponding eigenstate ϕ_n at the time just after measurement.*

The postulate (A1) is called the *statistical formula*, and (A2) the *measurement axiom*. The state change $\psi \mapsto \phi_n$ described by the measurement axiom is called the *state reduction*.

You will understand this paragraph in the end of the course.