Quantum Physics

2024

The Theory/Framework Of _Almost_ Everything _Today_

Yury Deshko

Harmonic Oscillator
&
Rotation Operator

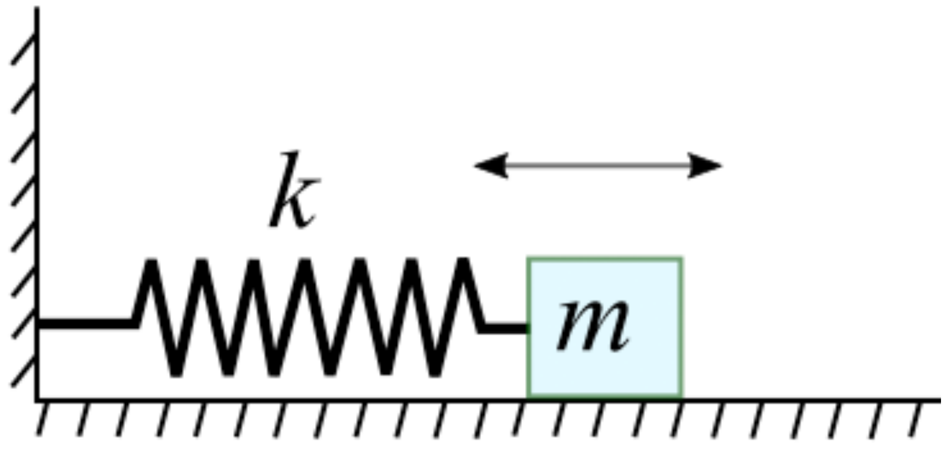
Course Overview

Course Structure And Goals

- Part 1 : Mathematical Concepts And Tools
- Part 2 : Classical Physics
- Part 3 : Quantum Physics

Harmonic Oscillator

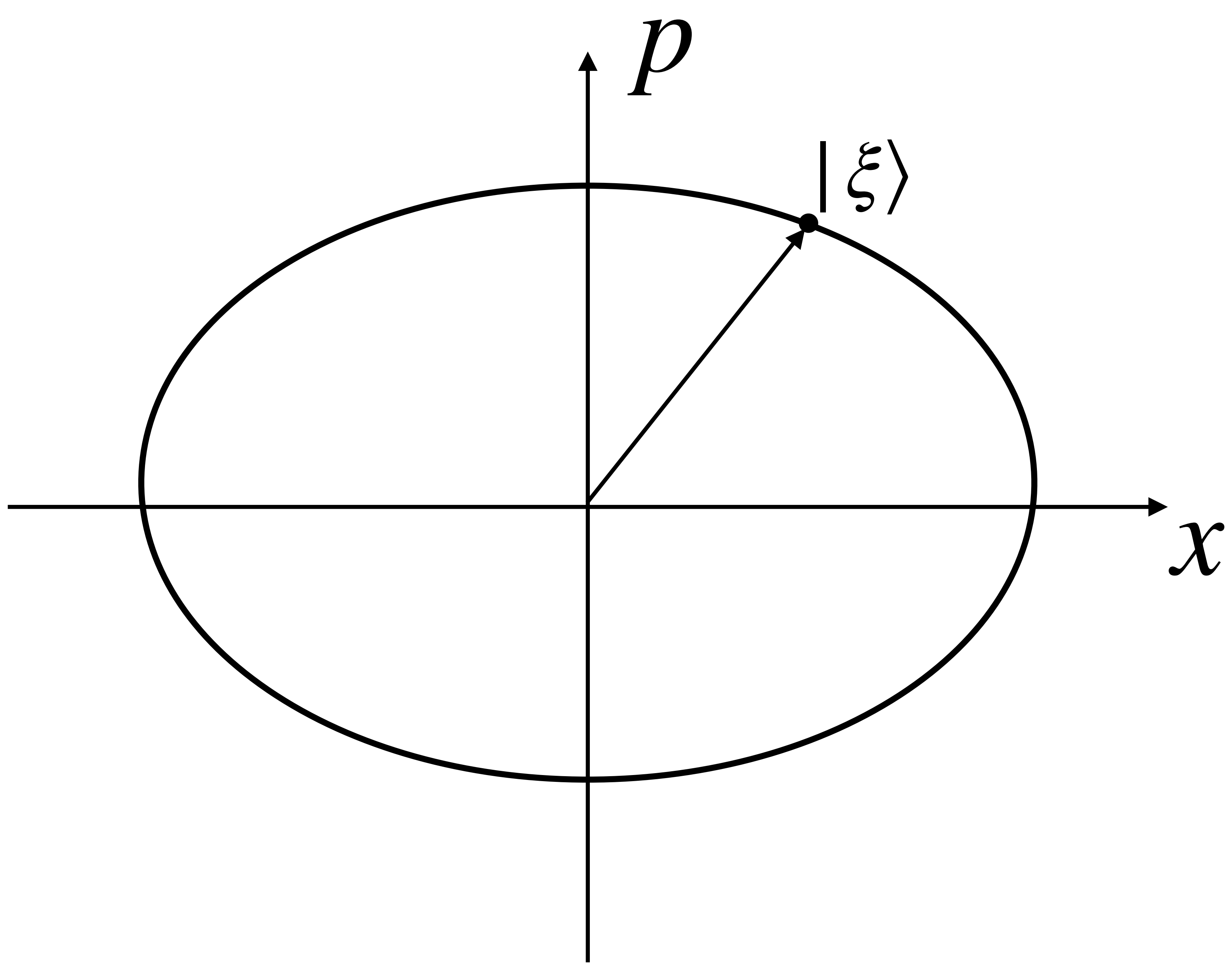
Solving Equations of Motion

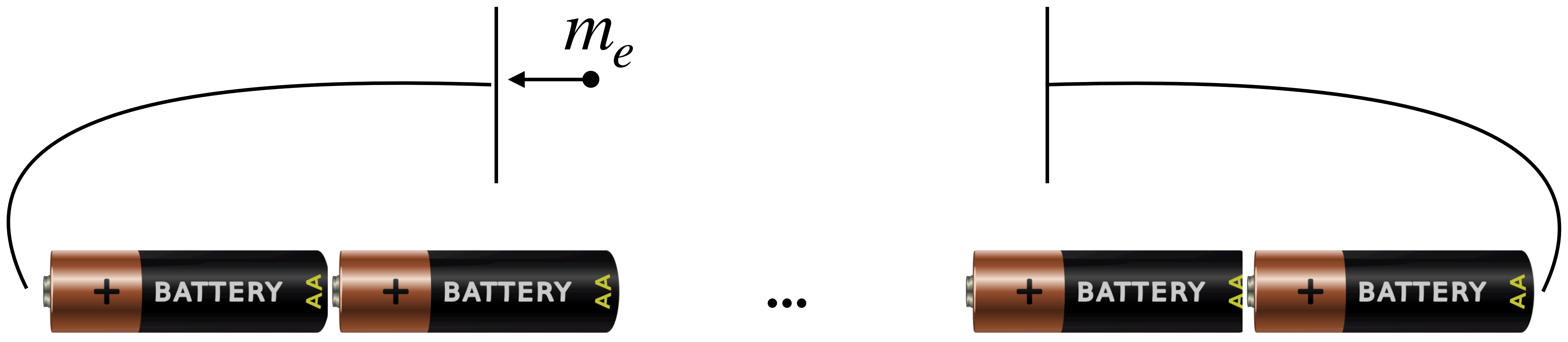


$$H = \frac{p^2}{2m} + \frac{kx^2}{2}$$

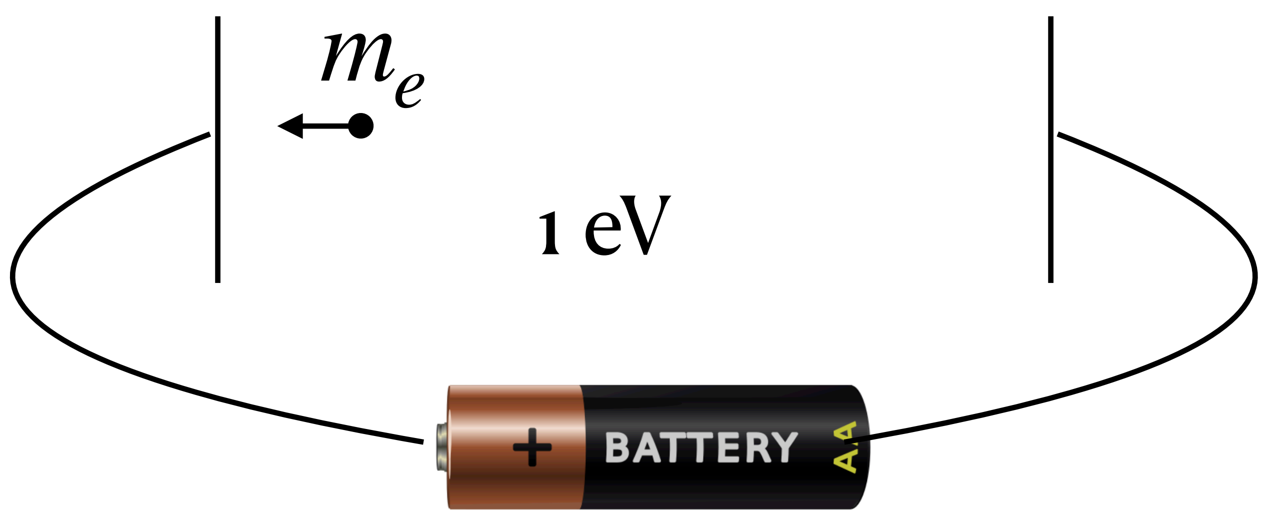
m_e $\qquad E_e = m_e c^2$ "Natural" units of energy.

$E_e = 0.511\,(MeV) = 8.187 \times 10^{-14}\,(J)$ Small value on human scale





511 000 batteries 1V each

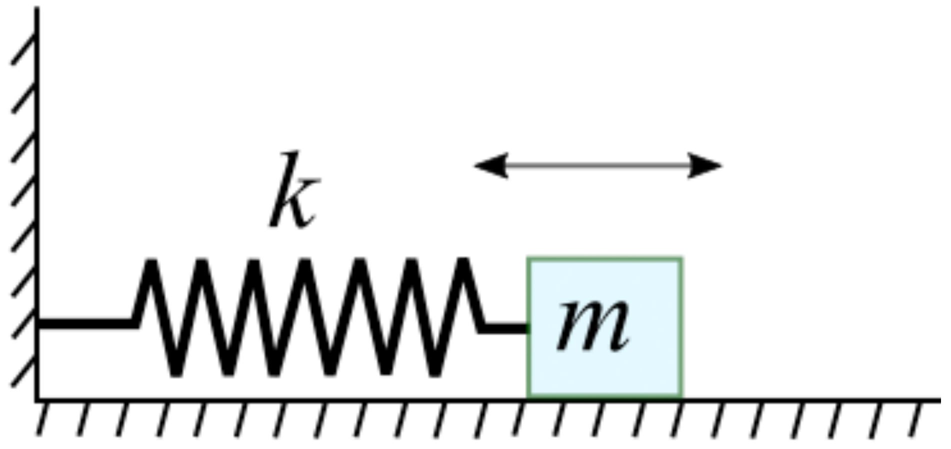


1V

$|\xi\rangle = (x, p)$ State. Problem: (apples, oranges)

Harmonic Oscillator

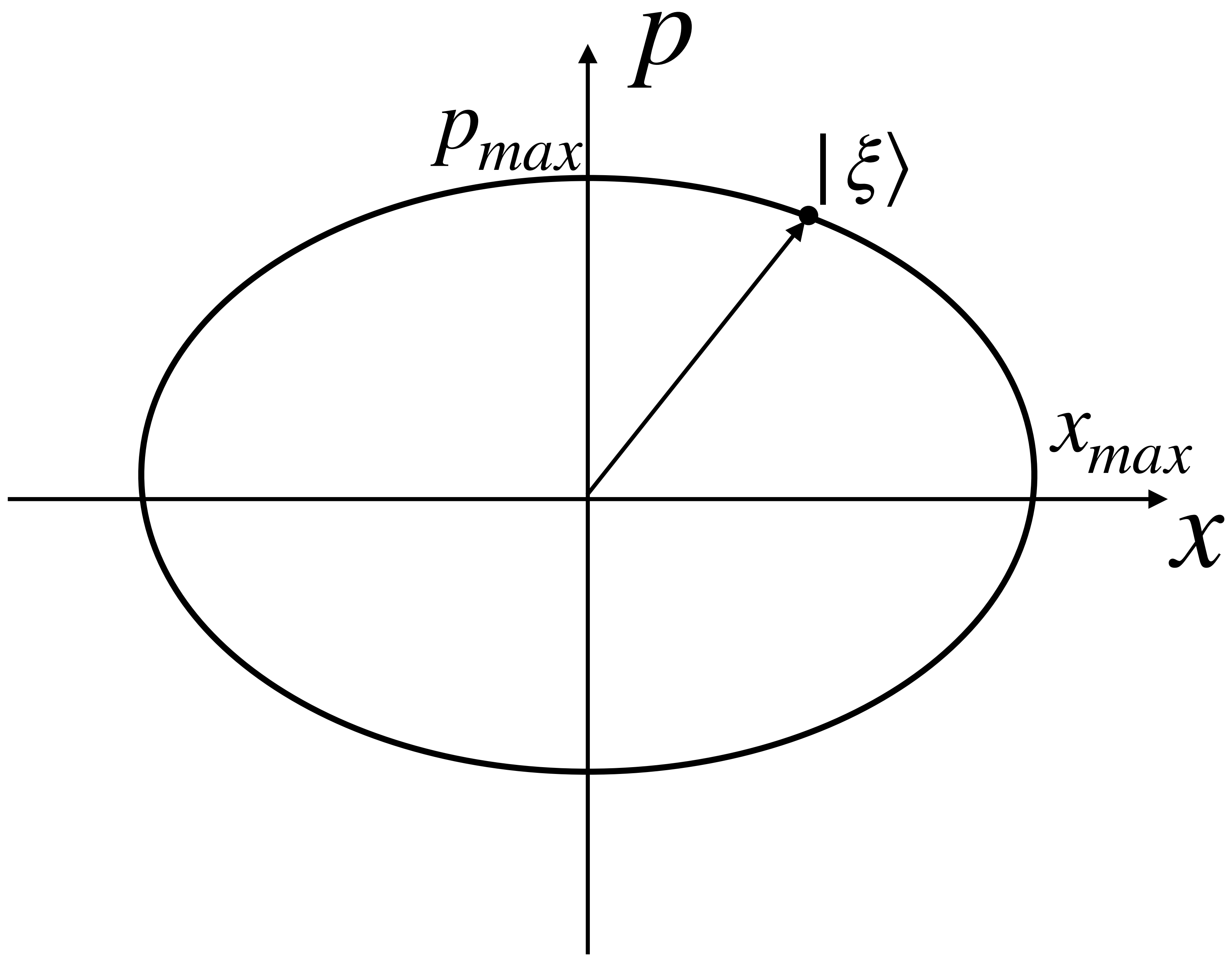
Solving Equations of Motion



$$H = \frac{p^2}{2m} + \frac{kx^2}{2}$$

m_e $\quad E_e = m_e c^2$ "Natural" units of energy.

$$H = \bar{H} E_e \qquad \bar{H} = \frac{H(J)}{E_e(J)}$$ Dimensionless number



$|\xi\rangle = (x, p)$ State. Problem: (apples, oranges)

Suppose the oscillator has energy $H = E_e$

Maximum displacement: $\frac{kx_e^2}{2} = E_e$

Maximum momentum: $\frac{p_e^2}{2m} = E_e$

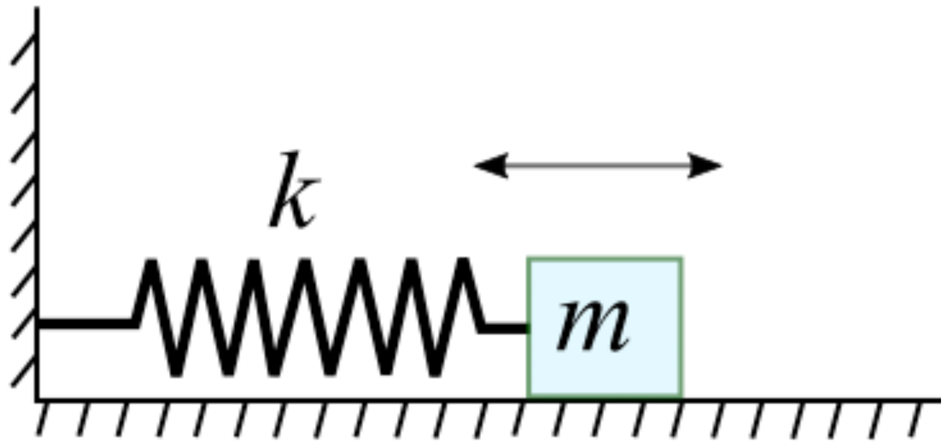
Not fully natural, since depend on k and m

Dimensionless state: $x = \bar{x} x_e \qquad p = \bar{p} p_e$

$|\bar{\xi}\rangle = (\bar{x}, \bar{p})$ Now both axes are unitless.

Harmonic Oscillator

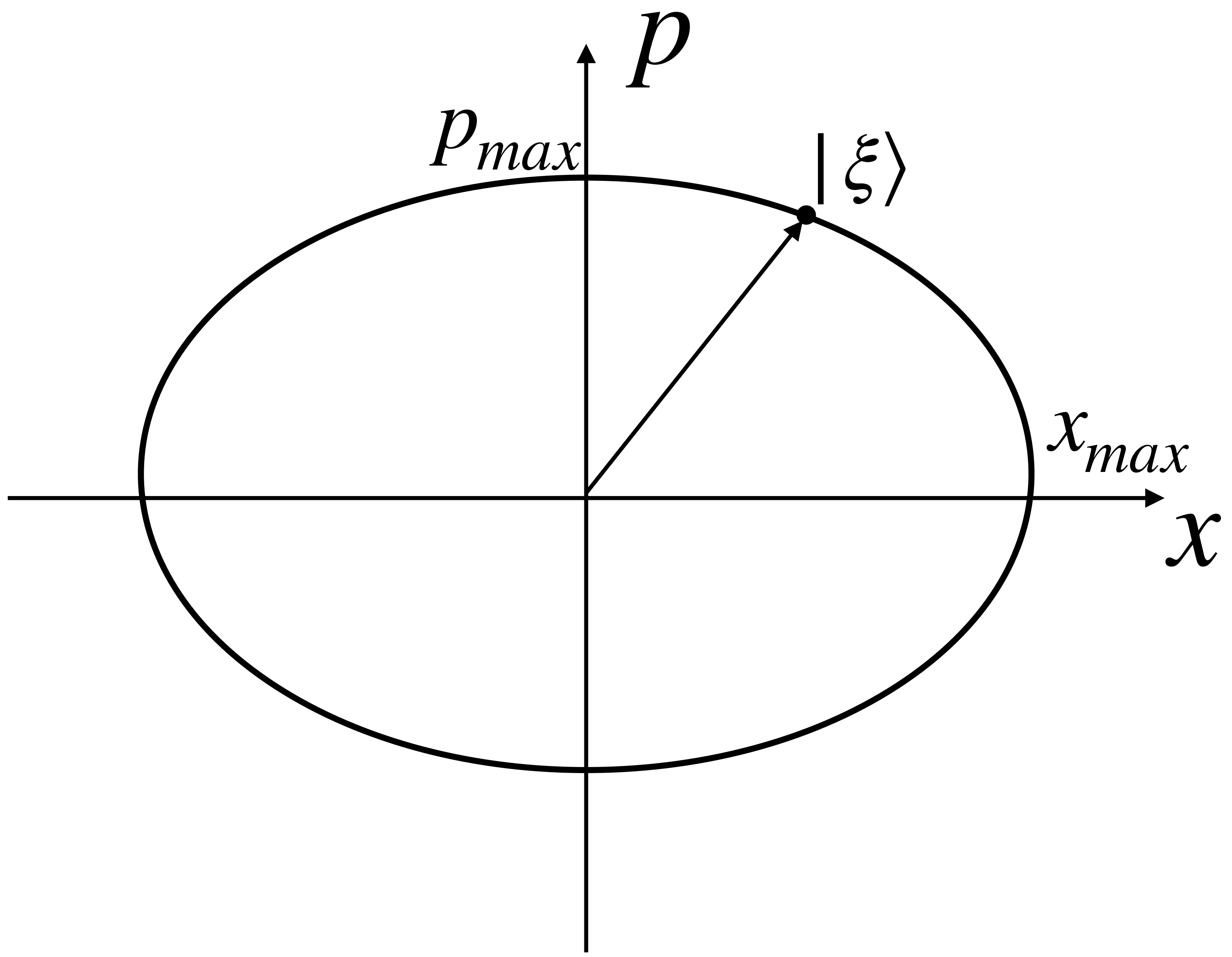
Solving Equations of Motion



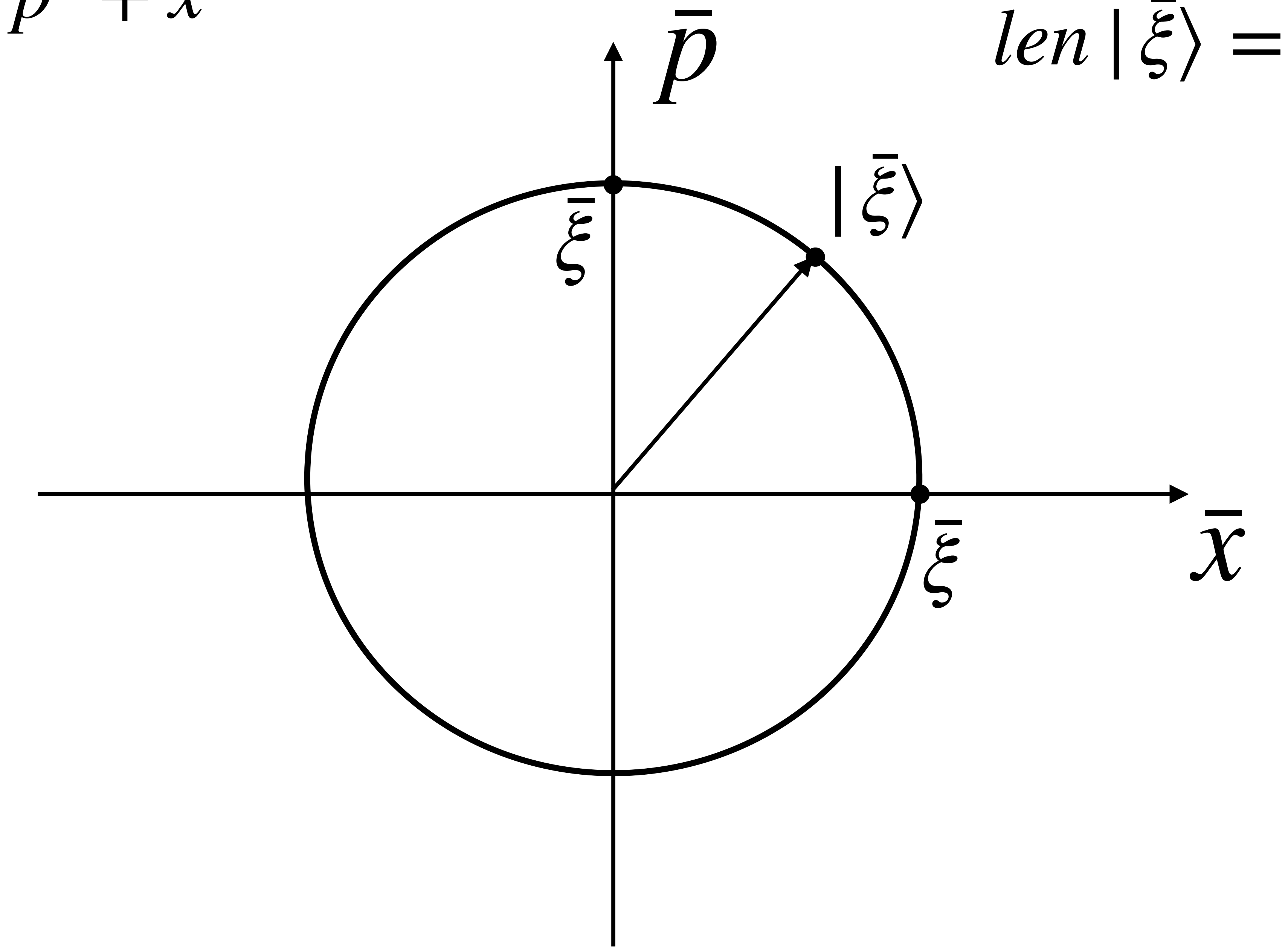
$$H = \frac{p^2}{2m} + \frac{kx^2}{2}$$

$$\bar{H}E_e = \frac{p_e^2}{2m}\bar{p}^2 + \frac{kx_e^2}{2}\bar{x}^2 = E_e\bar{p}^2 + E_e\bar{x}^2$$

$$\bar{H} = \bar{p}^2 + \bar{x}^2$$



$$len\,|\bar{\xi}\rangle = \bar{\xi} = \sqrt{\bar{H}}$$



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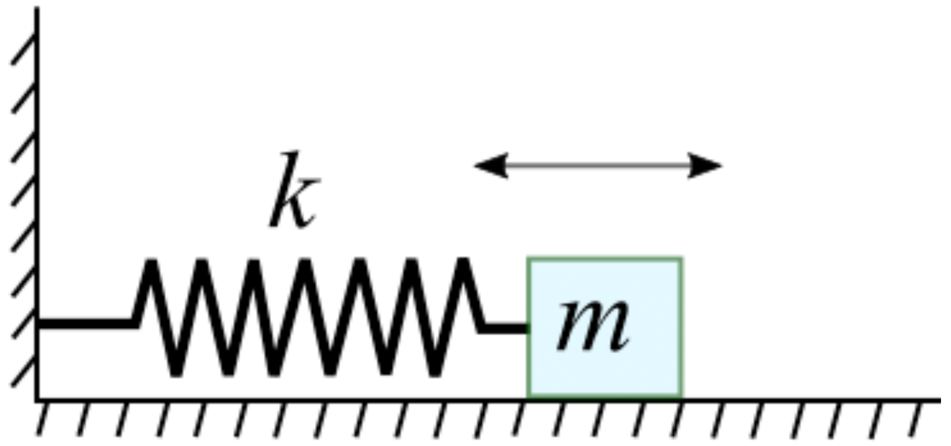
Harmonic Oscillator

Solving Equations of Motion



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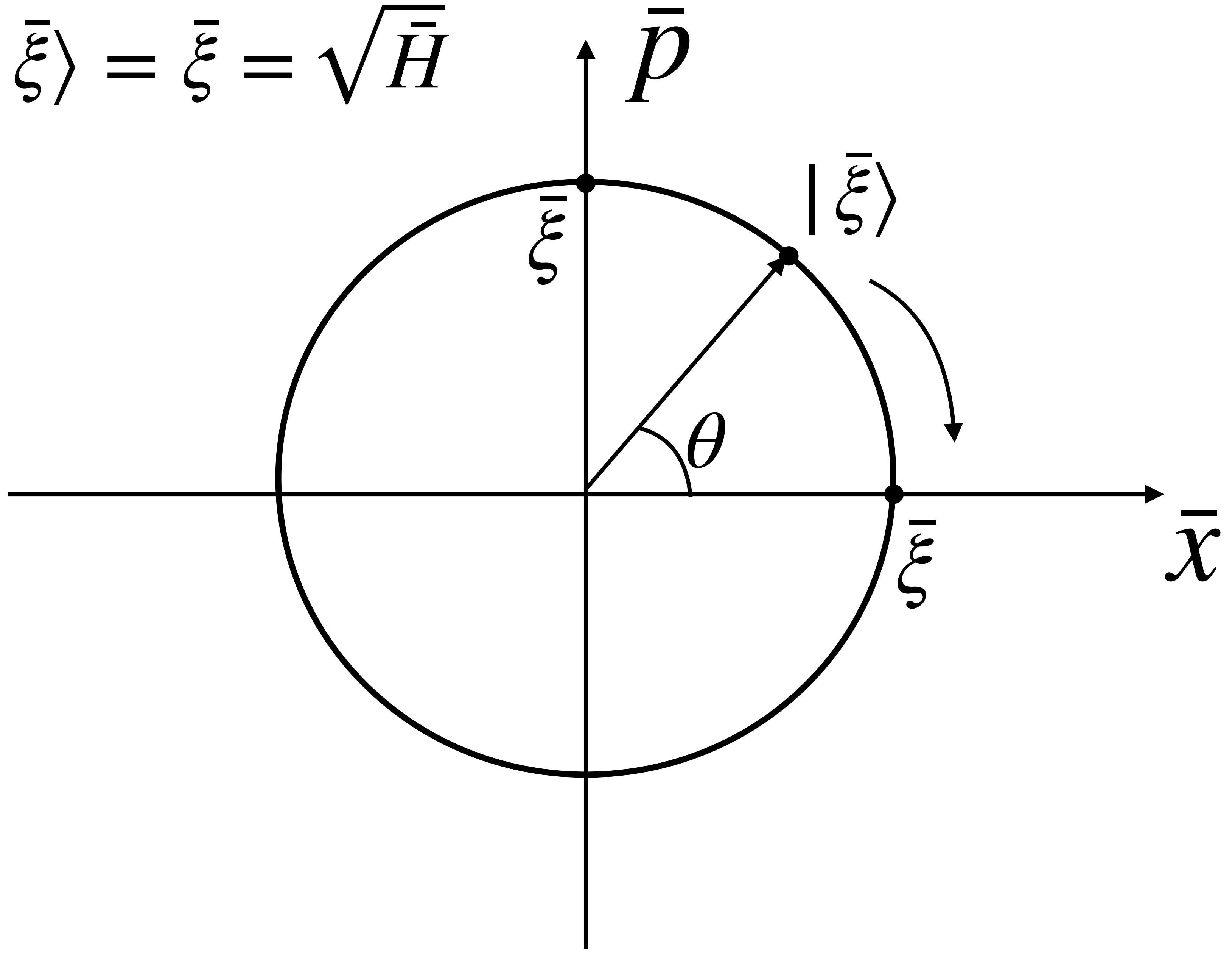


$$|\bar{\xi}\rangle = (\bar{x}, \bar{p})$$

Now both axes are unitless.

Now we have circular motion of the $|\bar{\xi}\rangle$ arrow in “normalized” phase space $(\bar{x}, \bar{p})$.

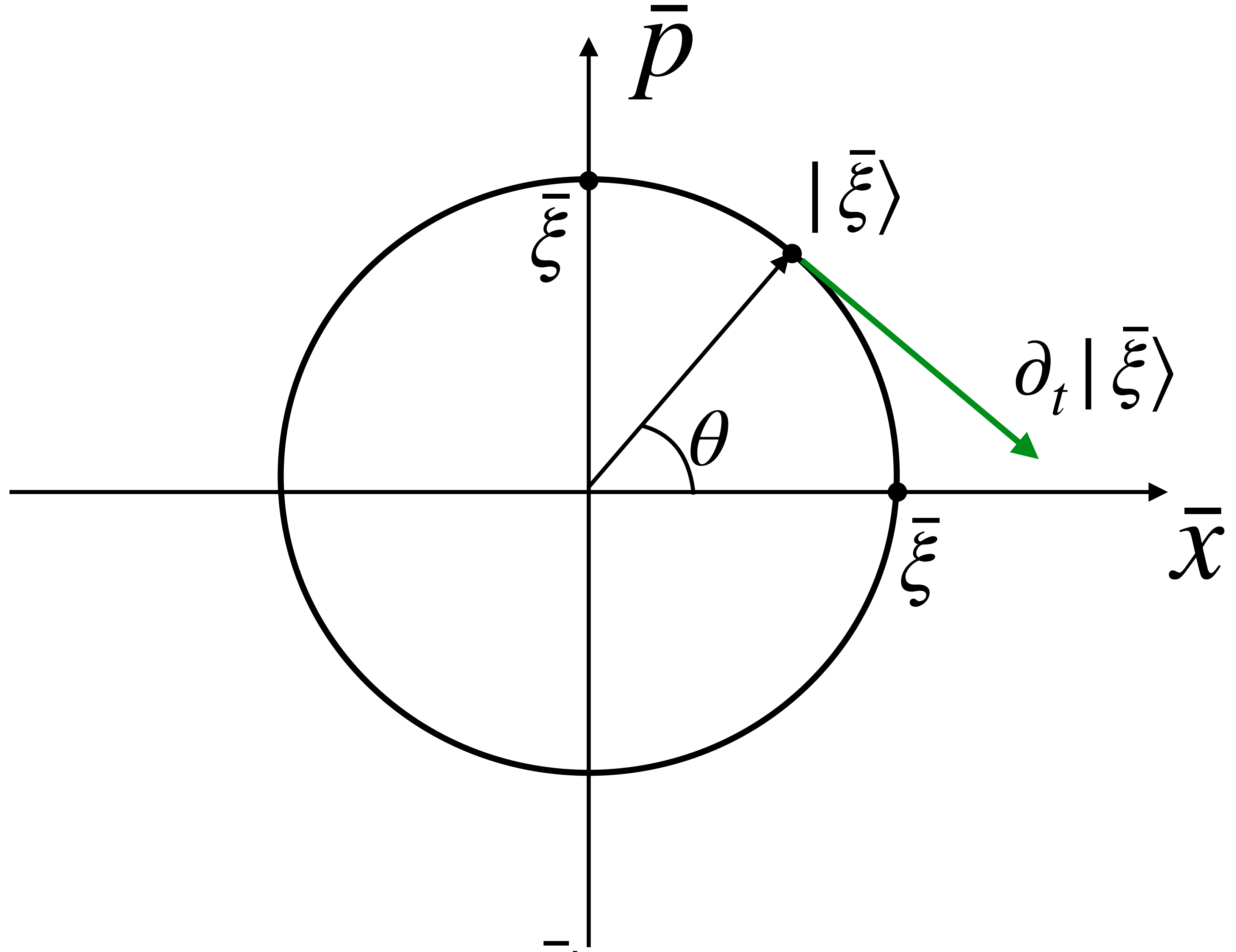
The motion is clock-wise.

It’s angular speed $\omega = \partial_t \theta$ might be constant, might be not.

We will prove that it is constant.

Harmonic Oscillator

Solving Equations of Motion



$|\bar{\xi}\rangle = (\bar{x}, \bar{p})$ $\qquad$ $\partial_t|\bar{\xi}\rangle = (\partial_t\bar{x}, \partial_t\bar{p})$

Now we have circular motion of the $|\bar{\xi}\rangle$ arrow in “normalized” phase space $(\bar{x}, \bar{p})$.

The motion is clock-wise.

It’s angular speed $\omega = \partial_t\theta$ might be constant, might be not.

We will prove that it is constant.

$$\partial_t|\bar{\xi}\rangle \perp |\bar{\xi}\rangle$$

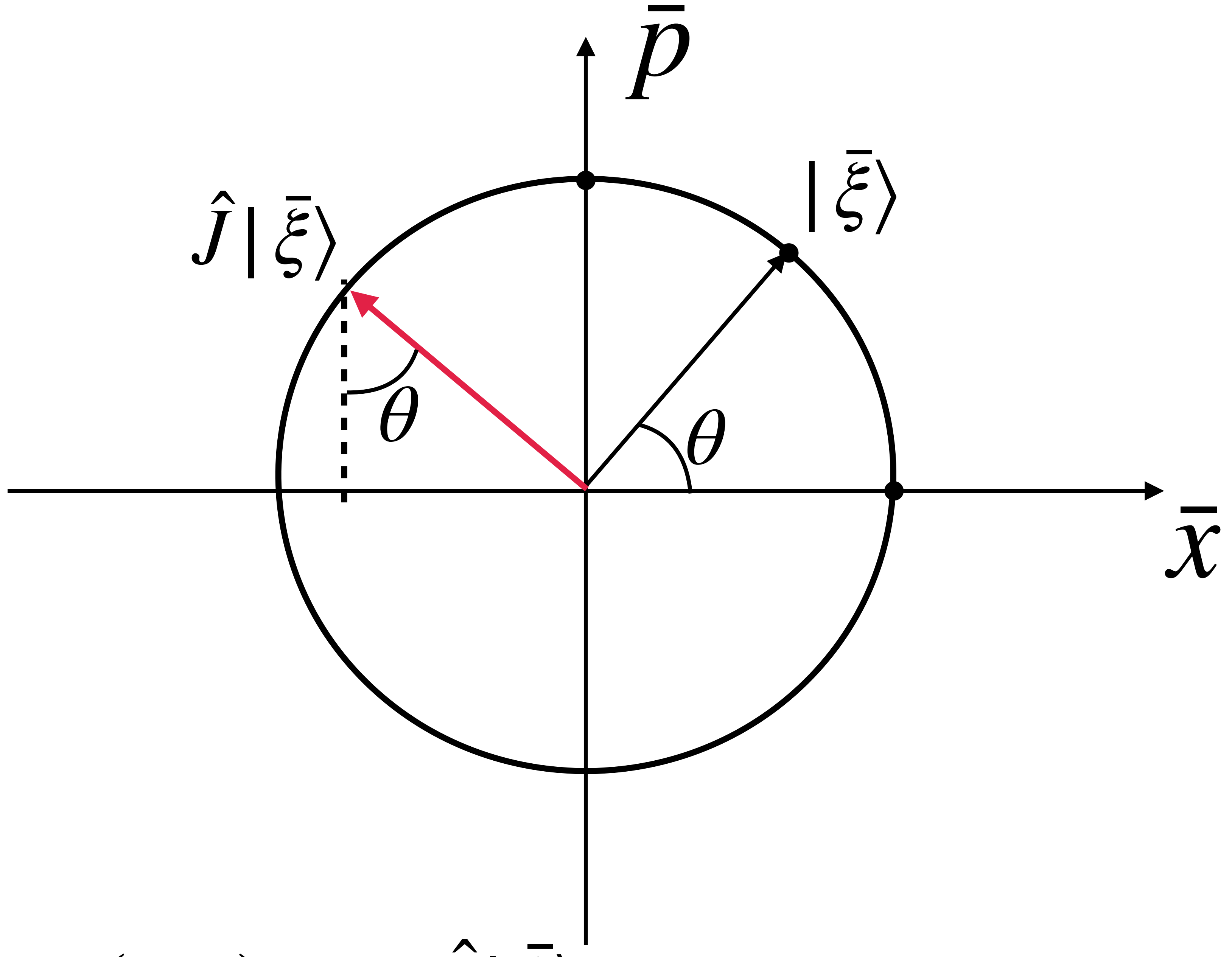
$$\partial_t|\bar{\xi}\rangle = -\omega\hat{J}|\bar{\xi}\rangle$$

See “Circular motion” lecture and slide at the end.

Right side is easy to evaluate

Harmonic Oscillator

Solving Equations of Motion



$|\bar{\xi}\rangle = (\bar{x}, \bar{p})$ $\hat{J}|\bar{\xi}\rangle = (-\bar{p}, \bar{x})$

Now we have circular motion of the $|\bar{\xi}\rangle$ arrow in "normalized" phase space $(\bar{x}, \bar{p})$.

The motion is clock-wise.

It's angular speed $\omega = \partial_t \theta$ might be constant, might be not.

We will prove that it is constant.

$$\partial_t |\bar{\xi}\rangle \perp |\bar{\xi}\rangle$$

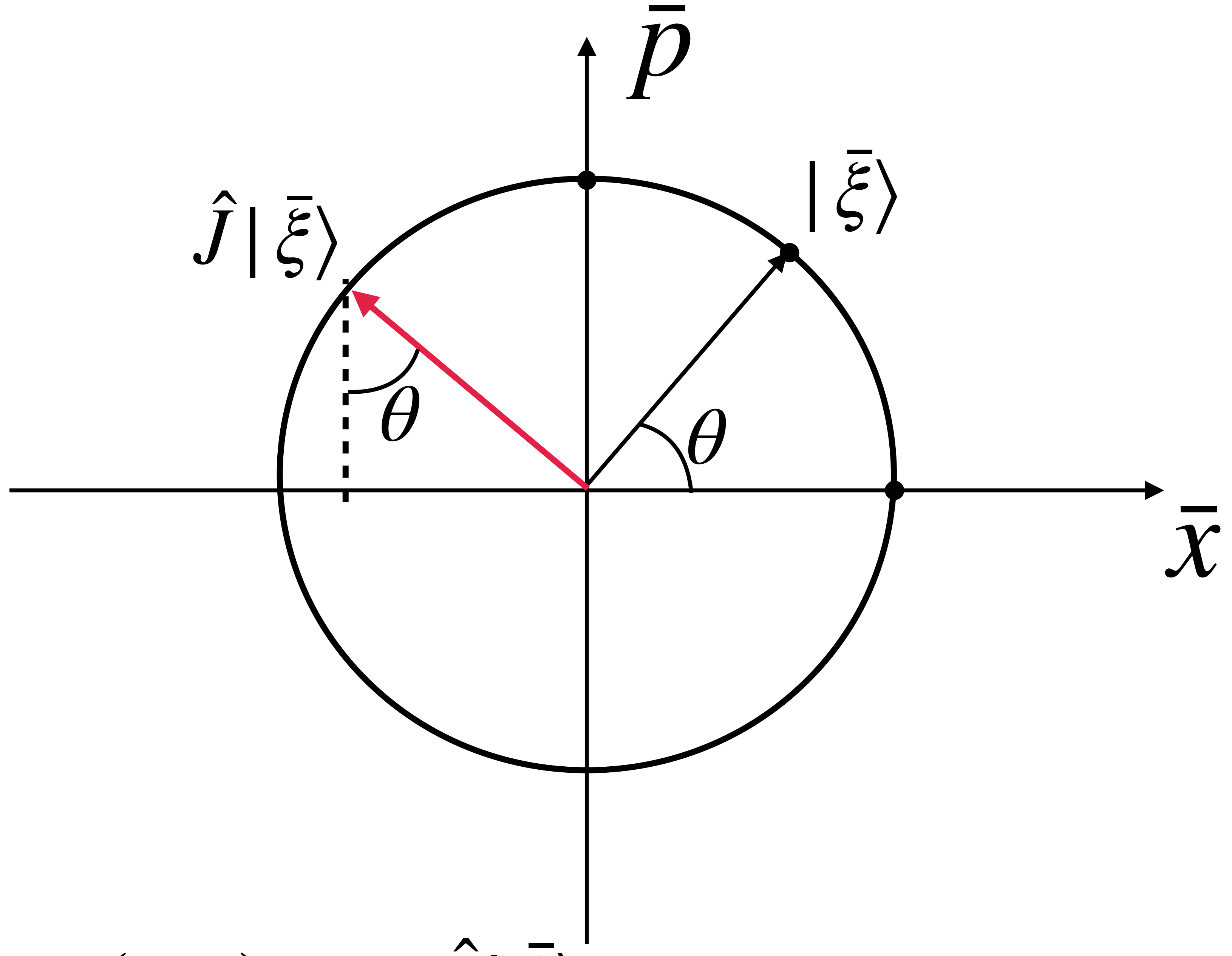
$$\partial_t |\bar{\xi}\rangle = -\omega \hat{J} |\bar{\xi}\rangle$$ See "Circular motion" lecture and slide at the end.

Right side is easy to evaluate $\hat{J}|\bar{\xi}\rangle = (-\bar{p}, \bar{x})$ — **Exercise**: Show it!

$$-\omega \hat{J} |\bar{\xi}\rangle = (\omega \bar{p}, -\omega \bar{x})$$

Harmonic Oscillator

Solving Equations of Motion



$|\bar{\xi}\rangle = (\bar{x}, \bar{p})$ $\qquad$ $\hat{J}|\bar{\xi}\rangle = (-\bar{p}, \bar{x})$

$$\partial_t|\bar{\xi}\rangle = -\omega\hat{J}|\bar{\xi}\rangle$$

$$\partial_t|\bar{\xi}\rangle = (\boxed{\partial_t\bar{x}}, \partial_t\bar{p}) \qquad -\omega\hat{J}|\bar{\xi}\rangle = (\boxed{\omega\bar{p}}, -\omega\bar{x})$$

$$\partial_t\bar{x} = \omega\bar{p}$$

Left-hand side: $\partial_t\bar{x} = \delta(x/x_e)/\delta t = v/x_e$

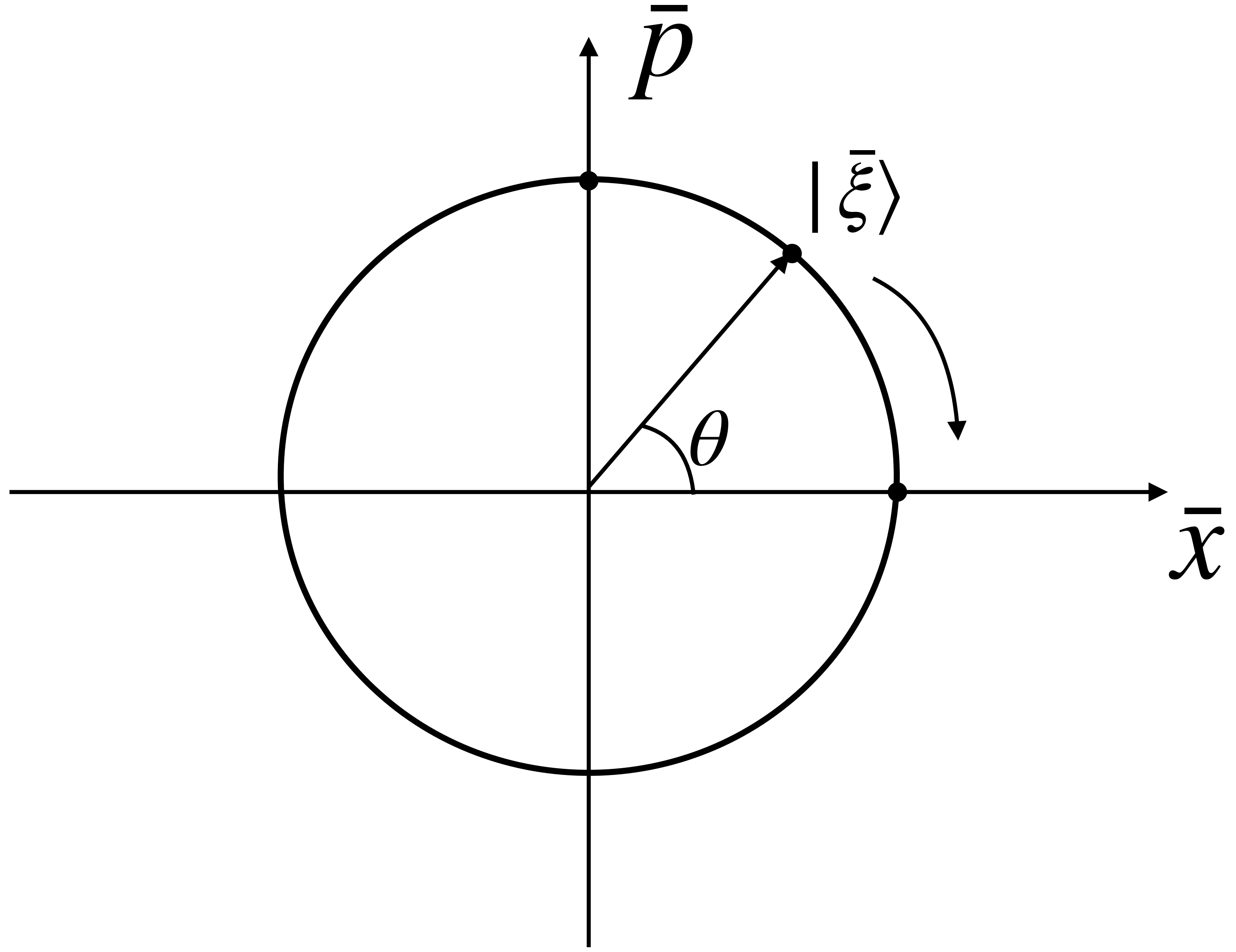
Right-hand side: $\omega\bar{p} = \omega p/p_e = \omega m v/p_e$

$$\frac{v}{x_e} = \omega\frac{mv}{p_e} \longrightarrow \omega = \frac{p_e}{m x_e} = \text{const}$$

Plug in $p_e = \sqrt{2mE_e}$ and $x_e = \sqrt{2E_e/k}$ to find $\boxed{\omega = \sqrt{\frac{k}{m}}}$

Harmonic Oscillator

Solving Equations of Motion



$|\bar{\xi}\rangle = (\bar{x}, \bar{p})$

Oscillator motion is a circular motion in "normalized" phase space with constant angular speed $\boxed{\omega = \sqrt{\frac{k}{m}}}$

$$|\bar{\xi}\rangle = (\bar{x}, \bar{p})$$

$$\bar{x} = \sqrt{\bar{H}} \cos \omega t$$

$$\bar{p} = -\sqrt{\bar{H}} \sin \omega t$$

Minus because angular *velocity* is clock-wise and negative.

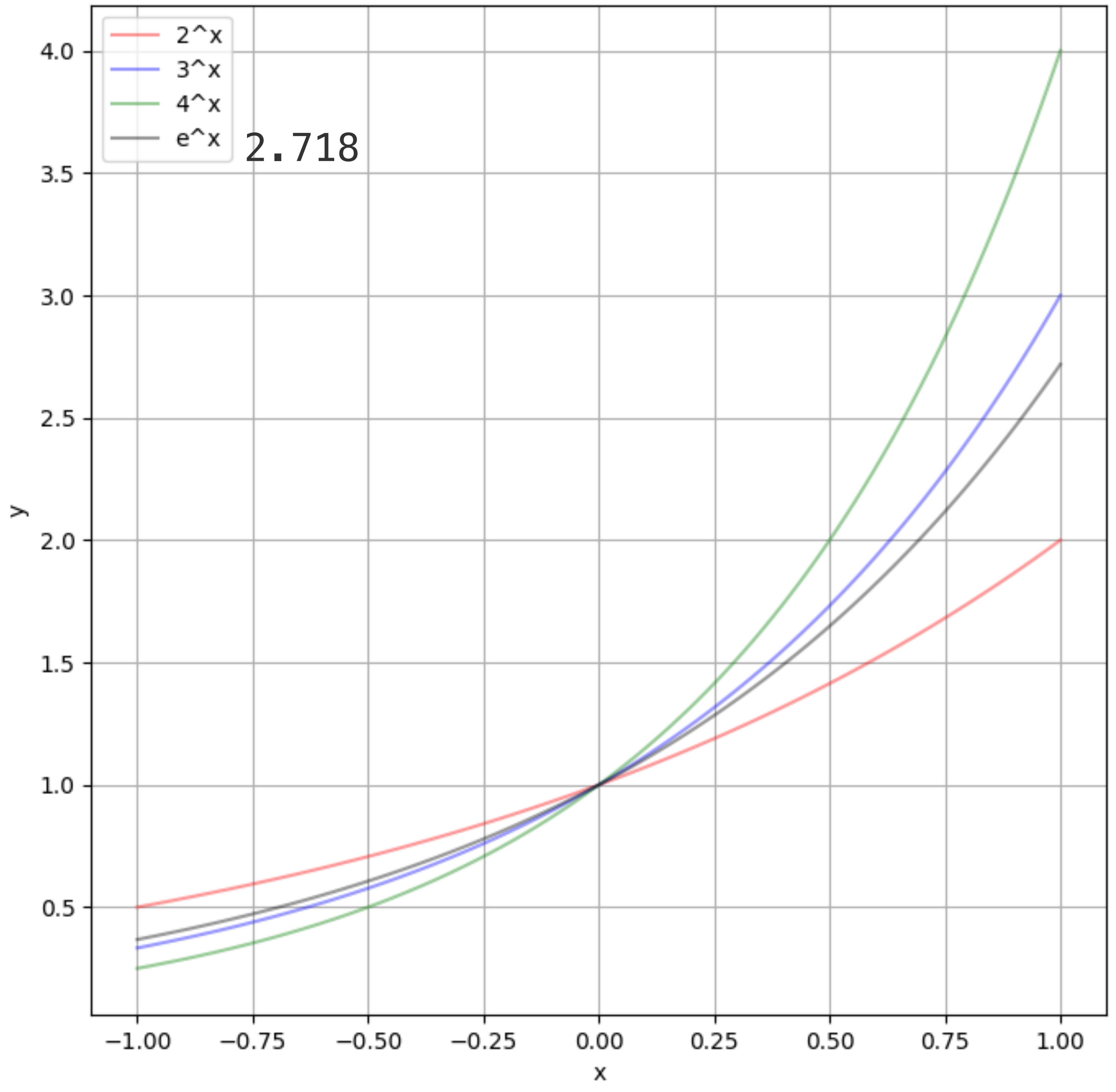
Now plugging back "normal" values, we arrive at

$$\boxed{x = \sqrt{\frac{2H}{k}} \cos \omega t \qquad p = -\sqrt{2mH} \sin \omega t \qquad \omega = \sqrt{\frac{k}{m}}}$$

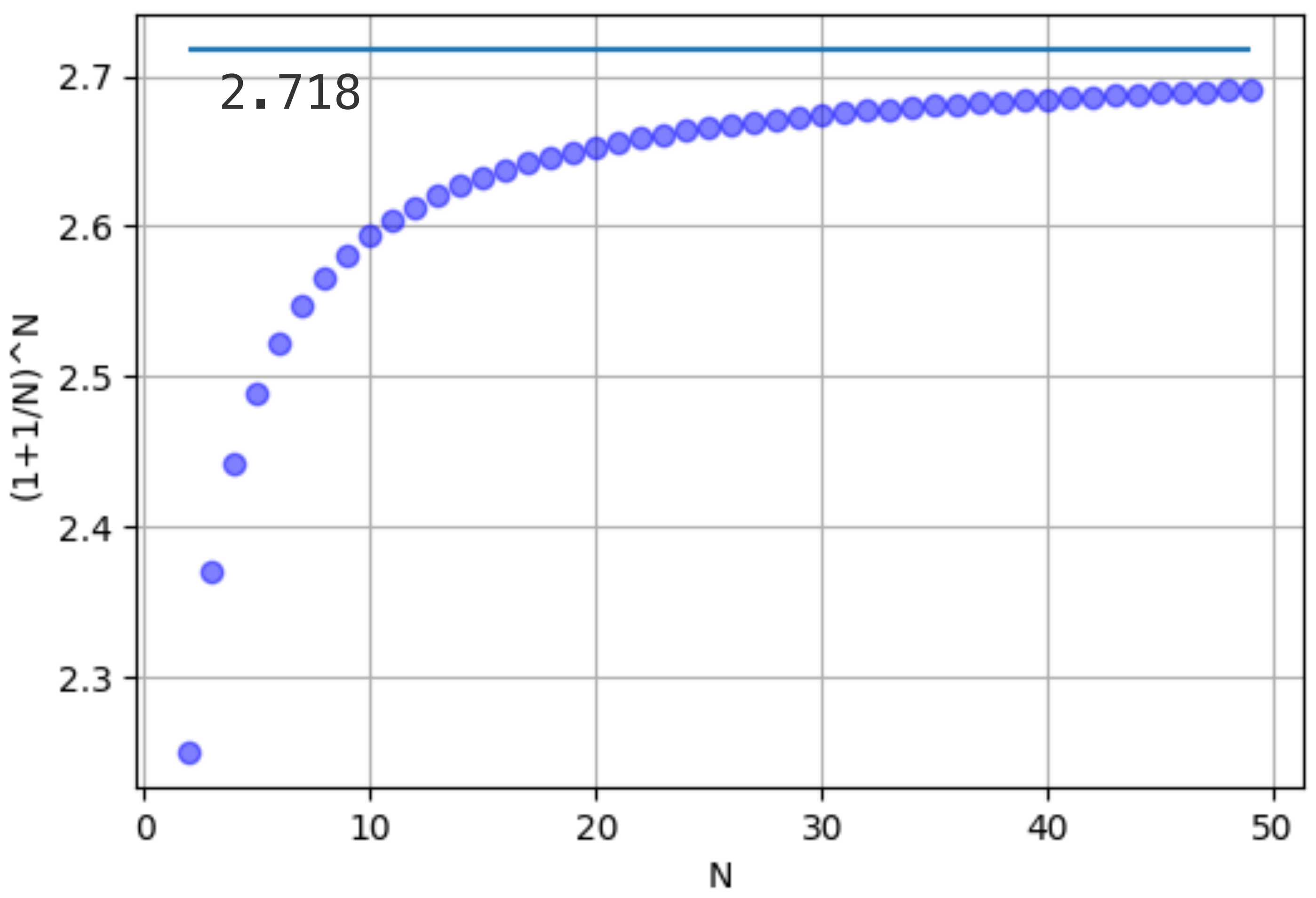
We thus solved fully harmonic oscillator.

Rotation Operator

Exponential Function



$$\left(1+\frac{1}{N}\right)^N \approx e$$



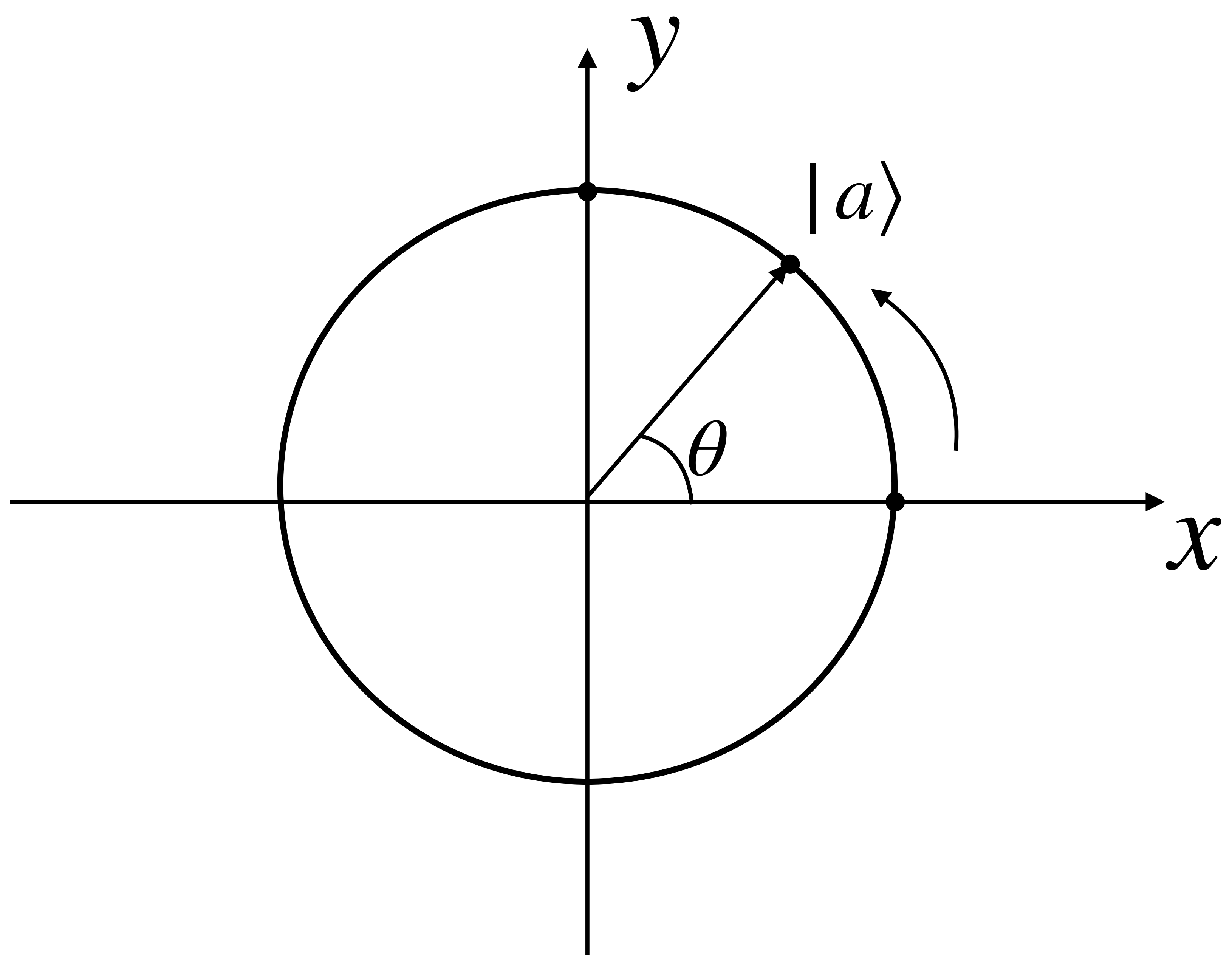
$$e^x = \left(1+\frac{1}{N}\right)^{xN}$$

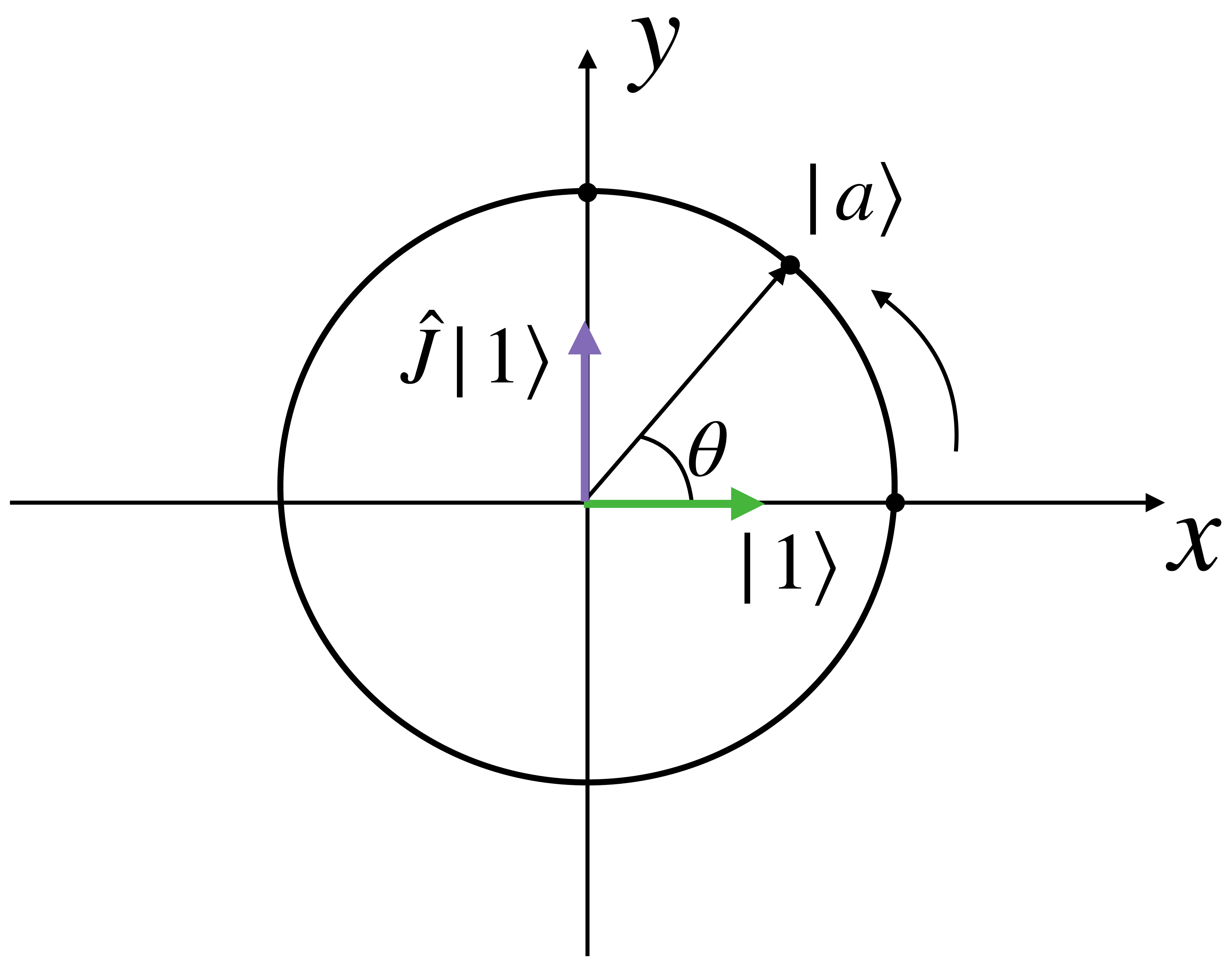
$$xN = M \rightarrow N = M/x$$

$$\boxed{e^x = \left(1+\frac{x}{M}\right)^M}$$

Rotation Operator

And Its Two Main Representations





$$|1\rangle = (1,0)$$

$$|a\rangle = (x, y)$$

$$x = a\cos\theta$$

$$y = a\sin\theta$$

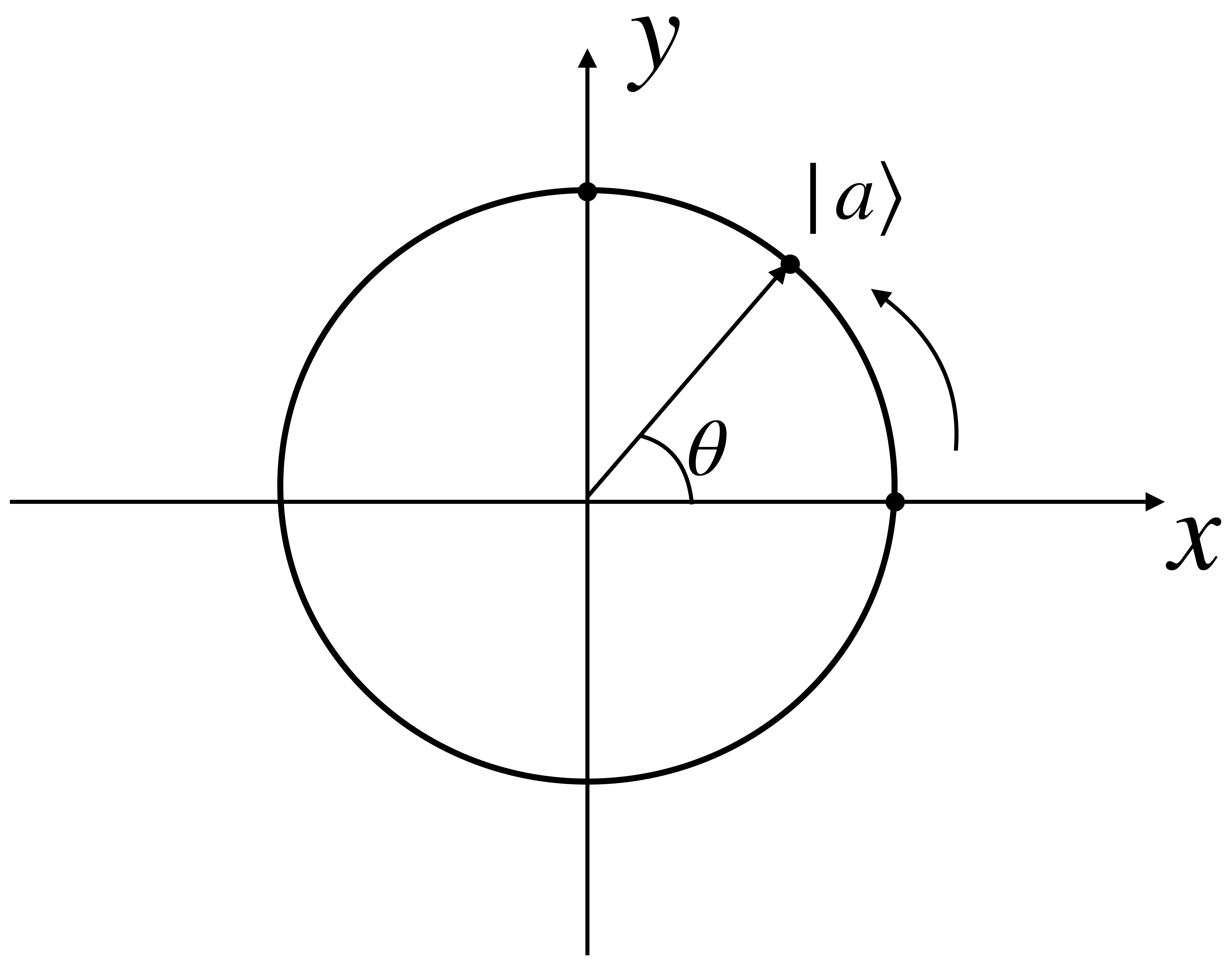
$$|a\rangle = x|1\rangle + y\hat{J}|1\rangle = a(\hat{I}\cos\theta + \hat{J}\sin\theta)|1\rangle$$

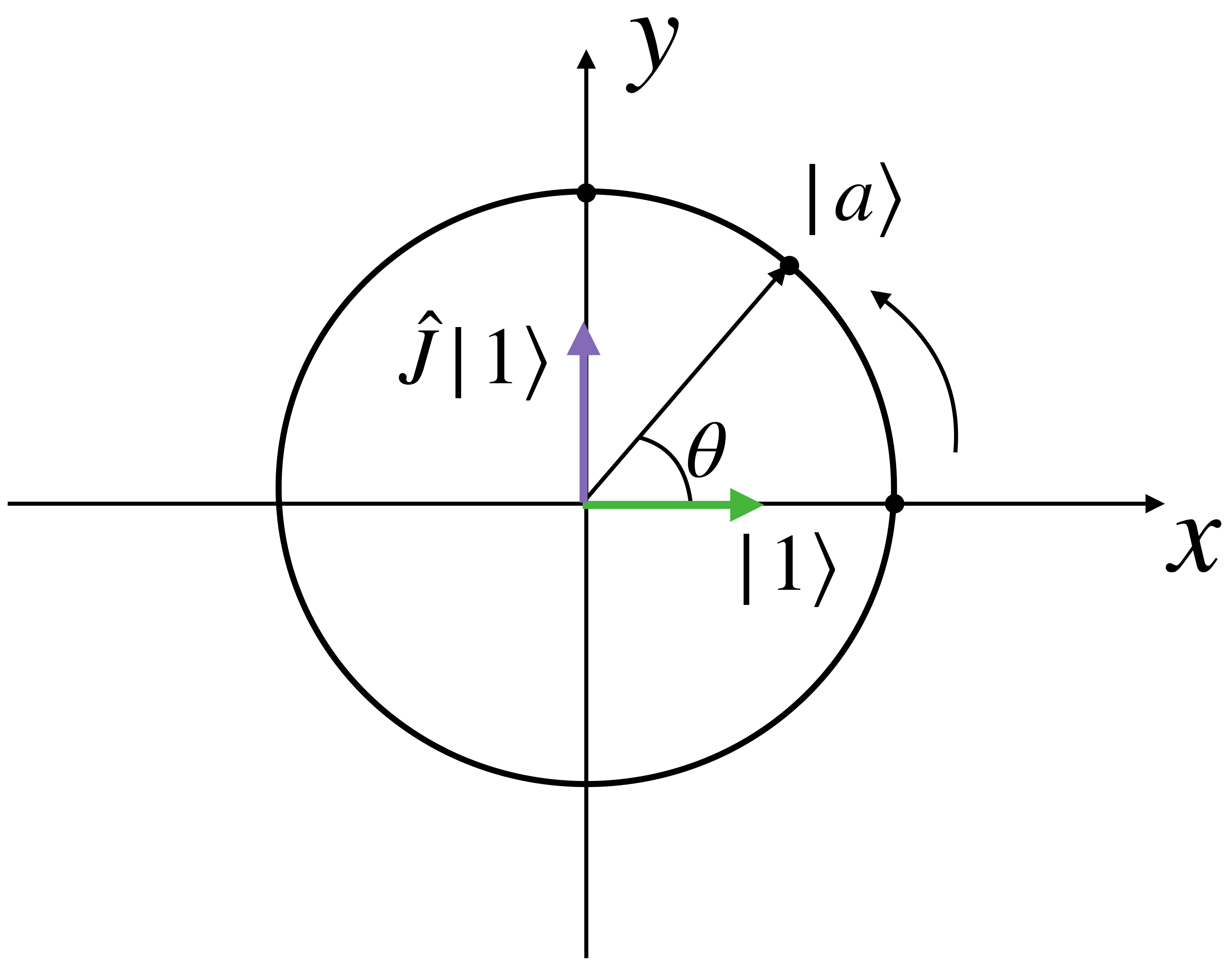
$$|a\rangle = \hat{R}_\theta\left(a|1\rangle\right)$$

$$\hat{R}_\theta = \hat{I}\cos\theta + \hat{J}\sin\theta$$

Rotation Operator

And Its Two Main Representations





$$|1\rangle = (1,0)$$

$$|a\rangle = (x, y)$$

$$x = a\cos\theta$$

$$y = a\sin\theta$$

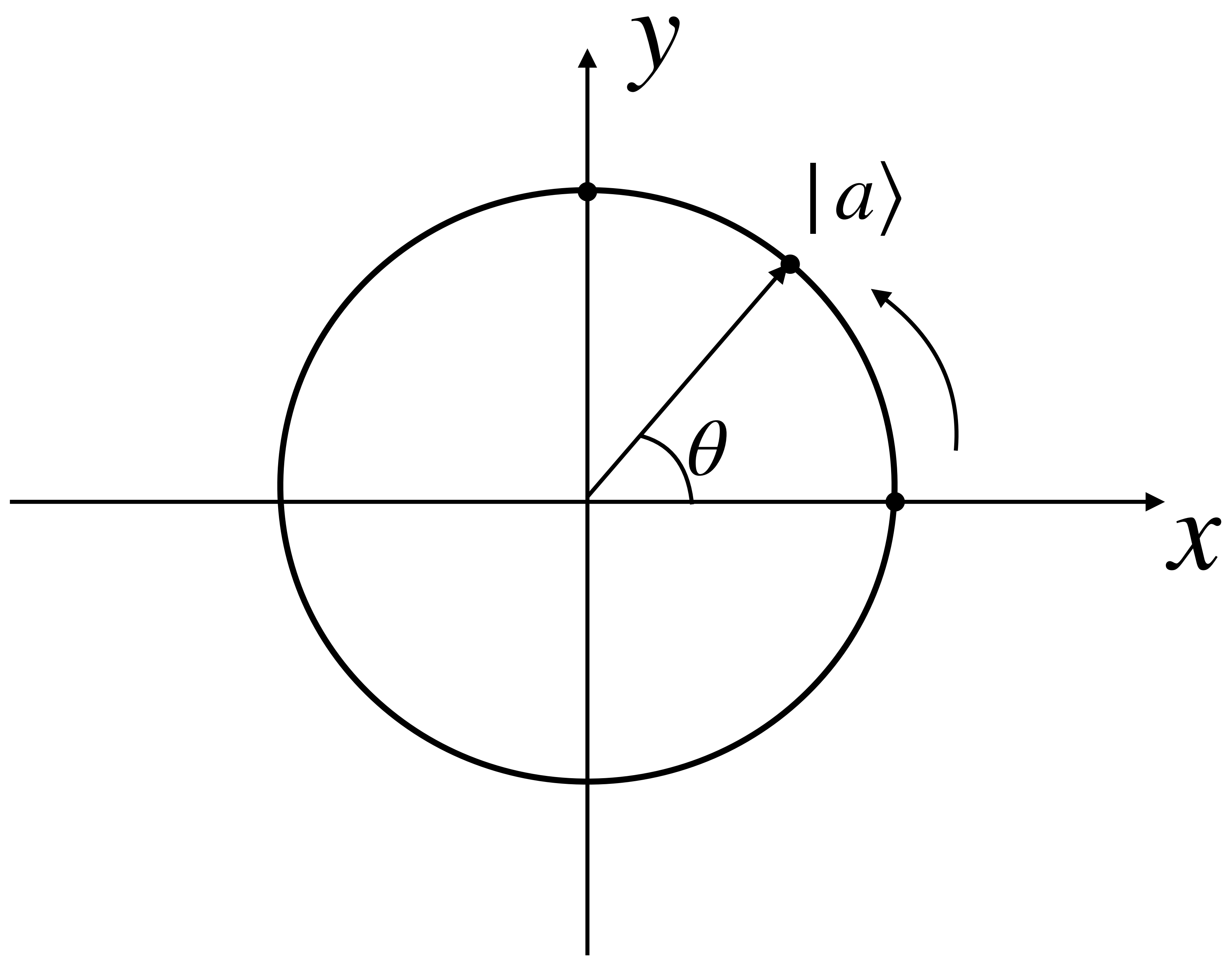
$$\hat{R}_\theta = \hat{I}\cos\theta + \hat{J}\sin\theta$$

Exercise: Show that $\hat{R}_\alpha \hat{R}_\beta = \hat{R}_{\alpha+\beta}$

We will discuss the algebra of arrows and operators more rigorously in the next lecture.

Rotation Operator

Euler's Formula



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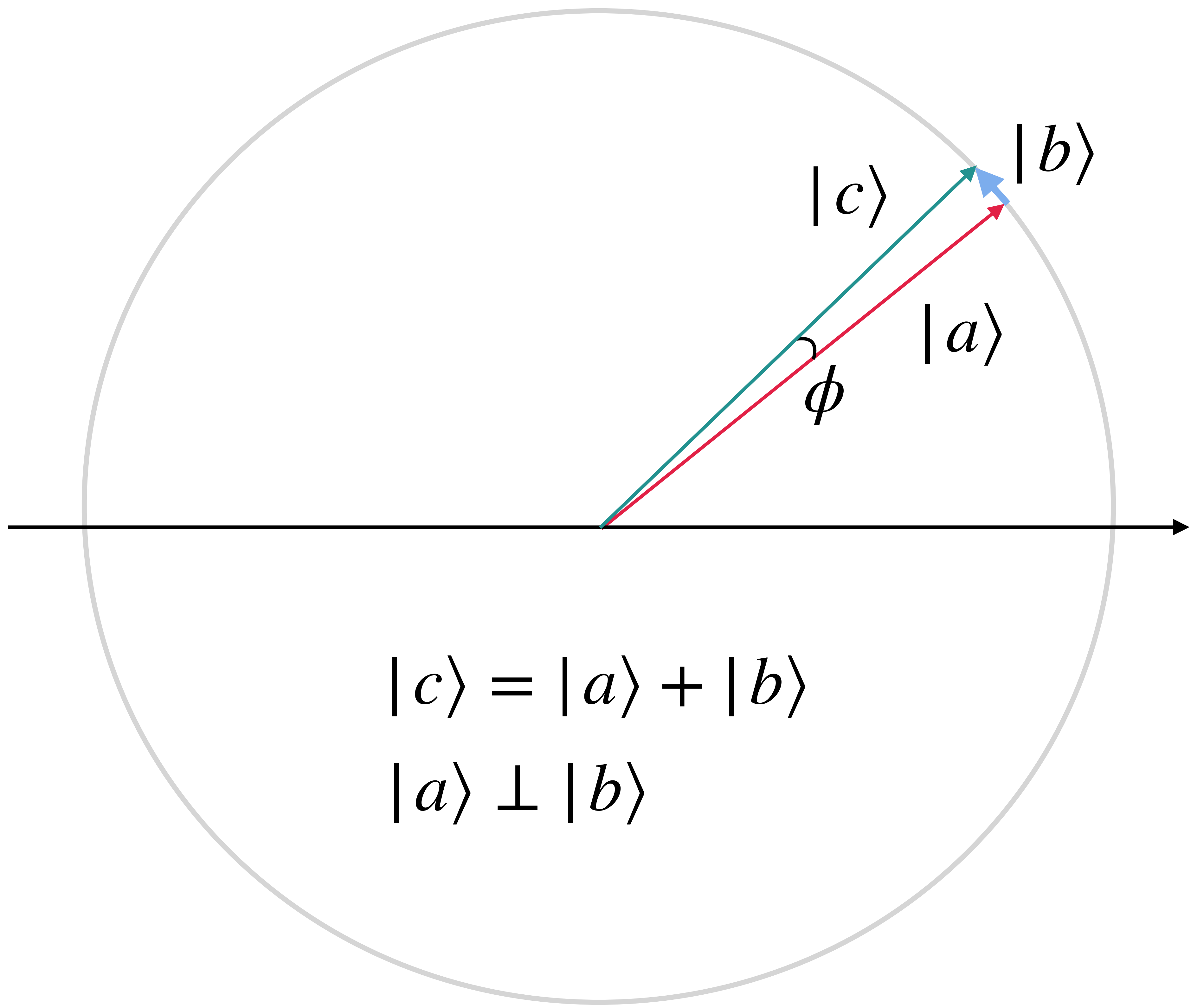


$$|c\rangle = |a\rangle + |b\rangle$$

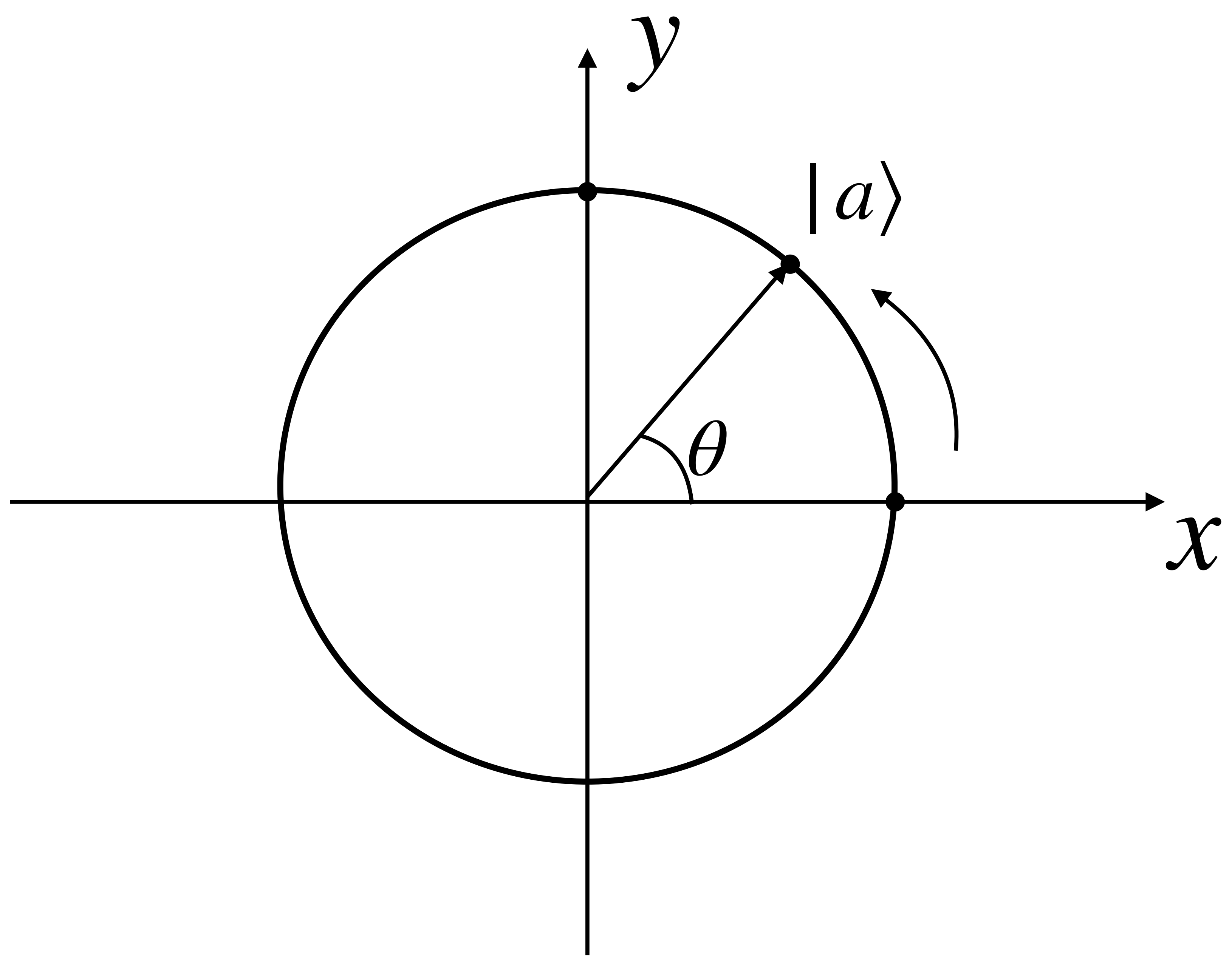
$$|a\rangle \perp |b\rangle$$

$$len\,|a\rangle = a$$

$$len\,|b\rangle = b = a\phi$$

Rotation Operator

Euler’s Formula



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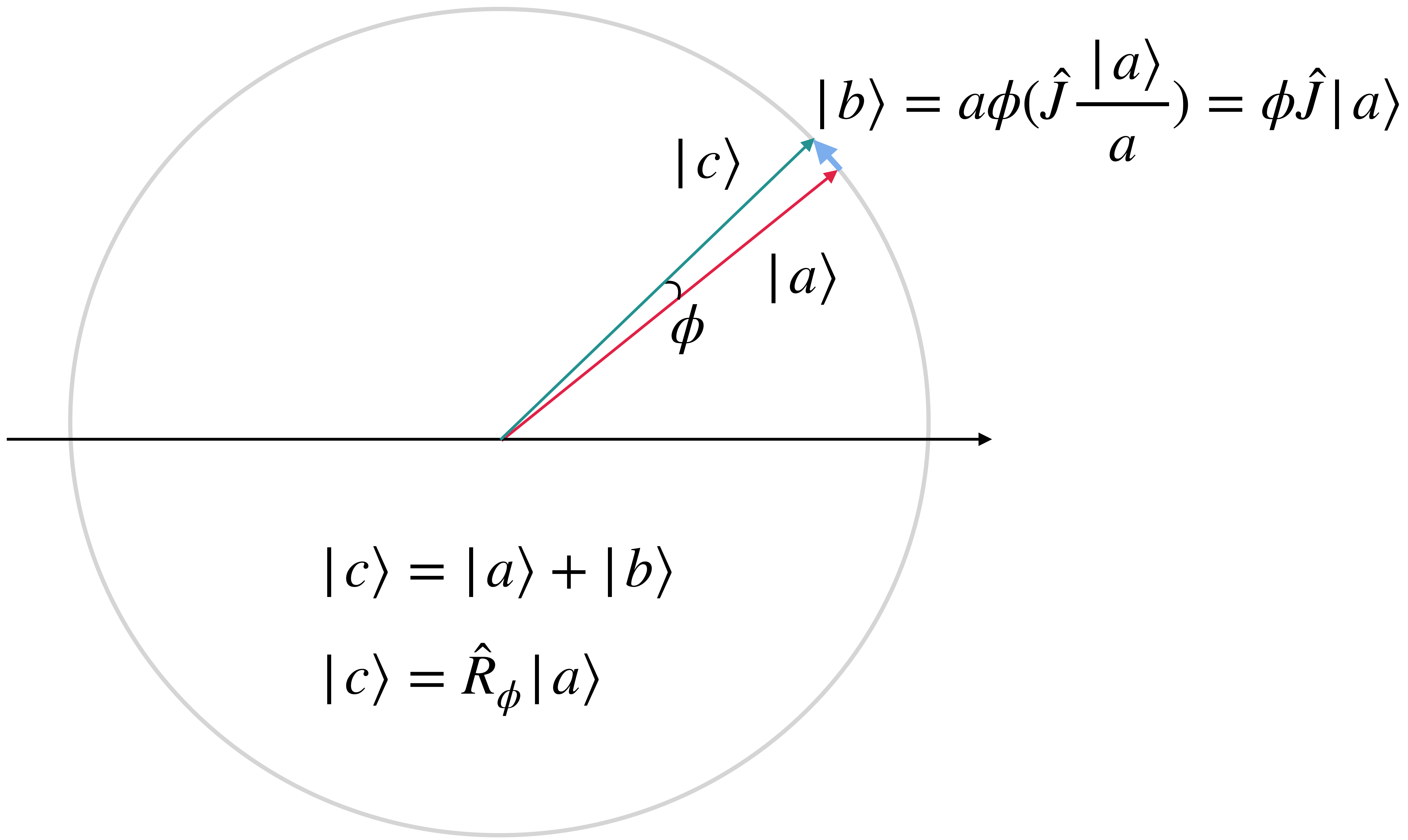
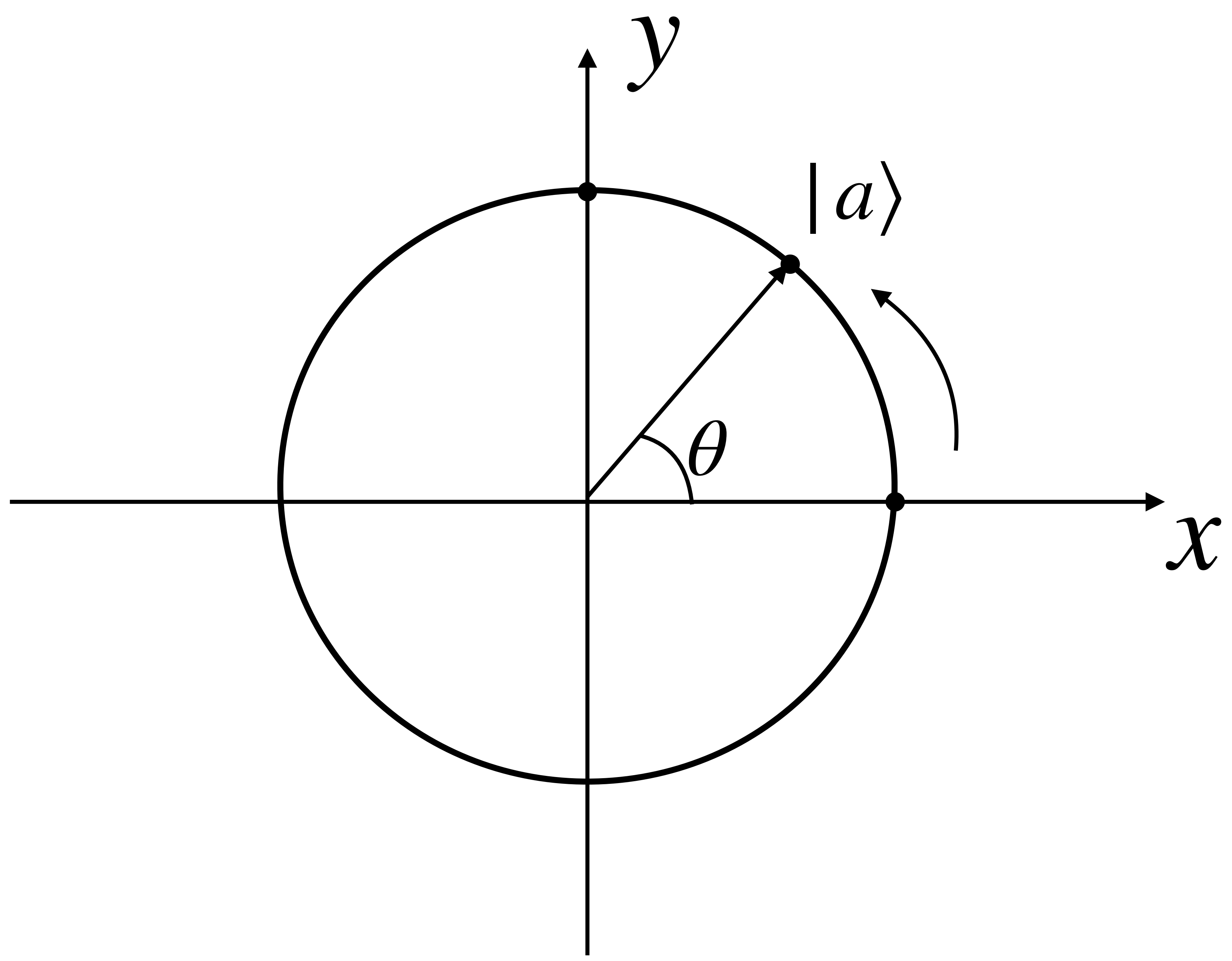


$$\hat{R}_\phi|a\rangle = |a\rangle + \phi\hat{J}|a\rangle = (\hat{I} + \hat{J}\phi)|a\rangle$$

Rotation Operator

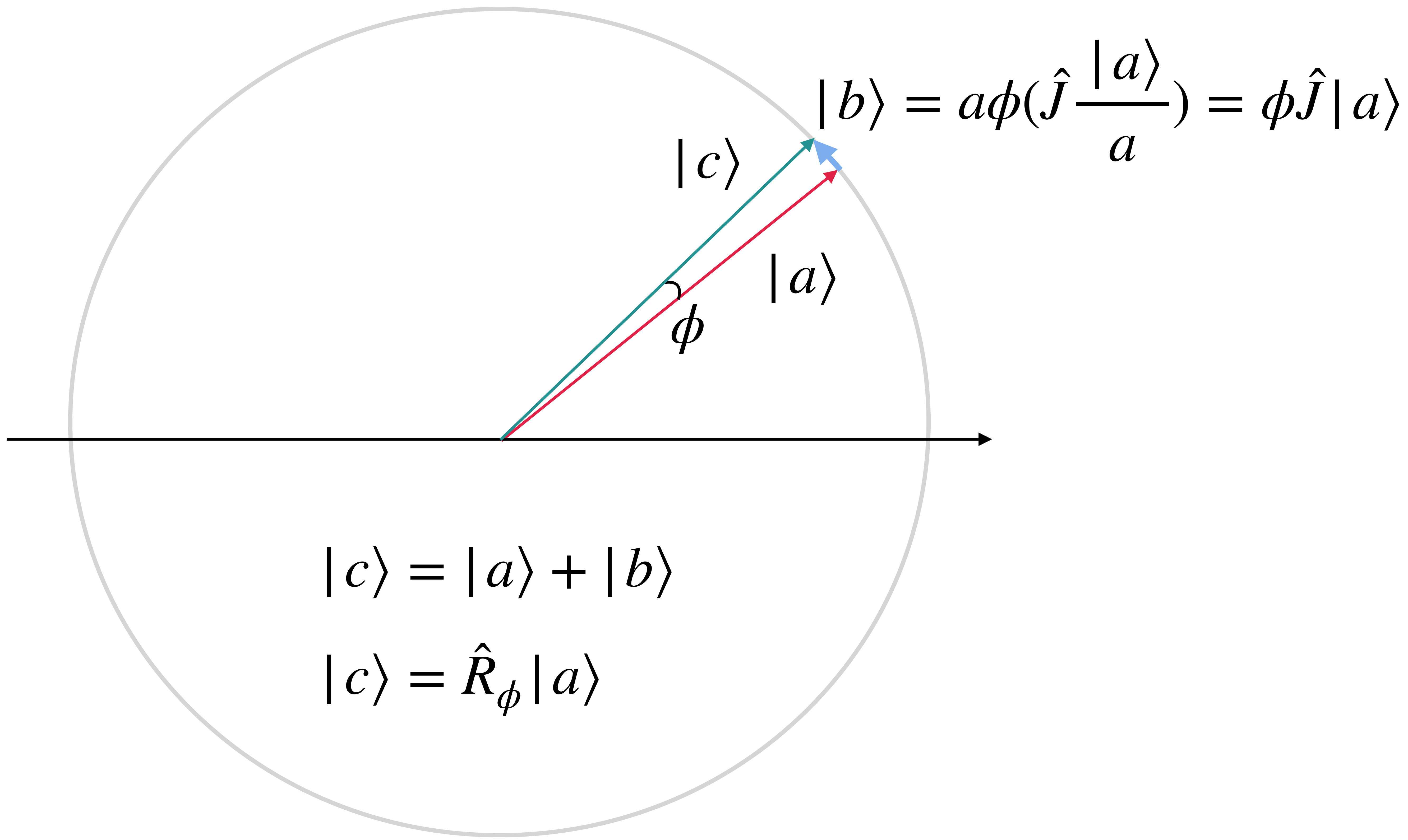
Euler's Formula



Rotation in N small steps.

$$\hat{R}_\theta = \hat{R}_\phi \hat{R}_\phi \hat{R}_\phi \cdots \hat{R}_\phi \qquad \phi = \theta / N$$

$$\hat{R}_\theta = \left(\hat{R}_\phi\right)^N = \left(\hat{I} + \frac{\hat{J}\theta}{N}\right)^N$$



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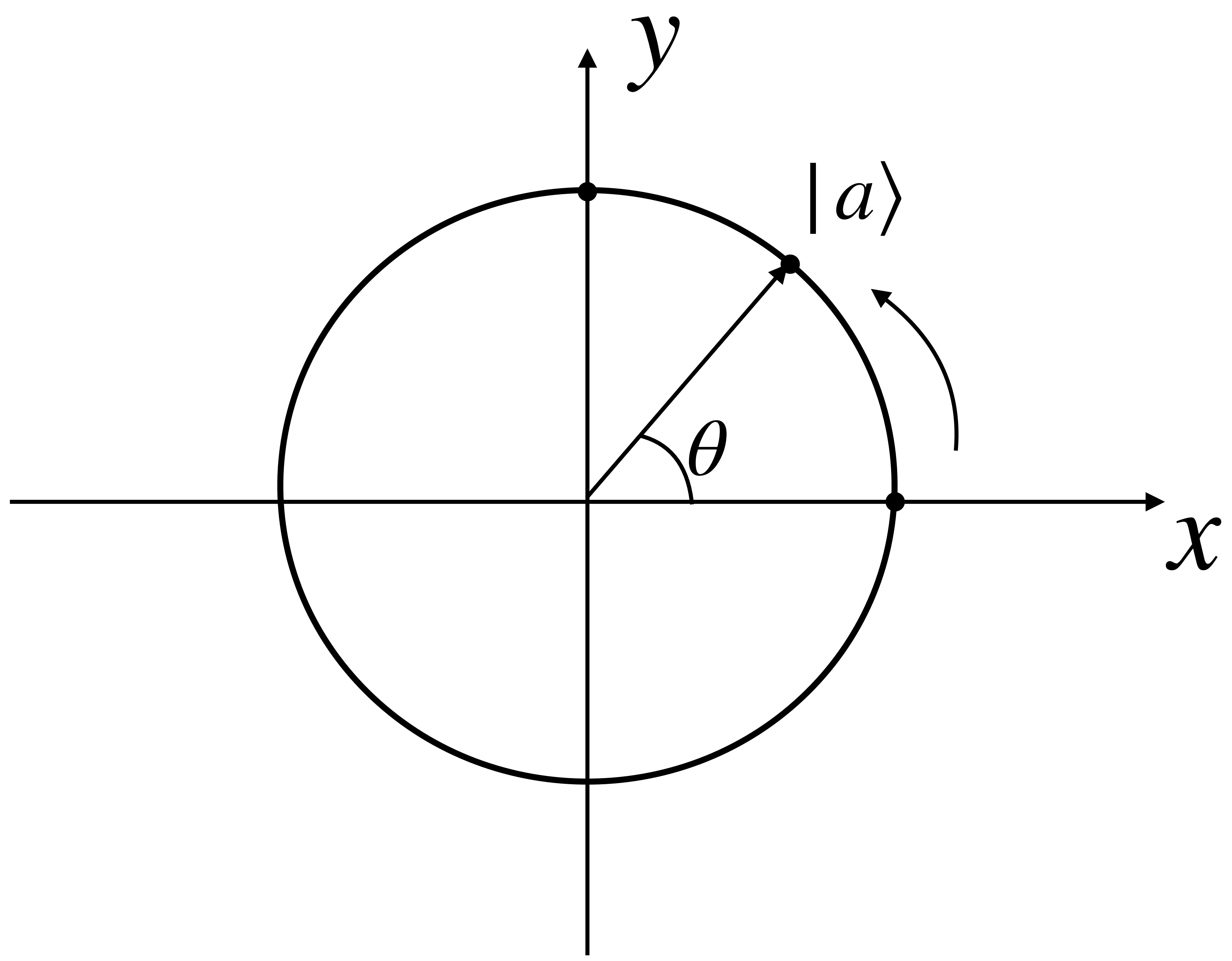
Rotation Operator

Euler's Formula



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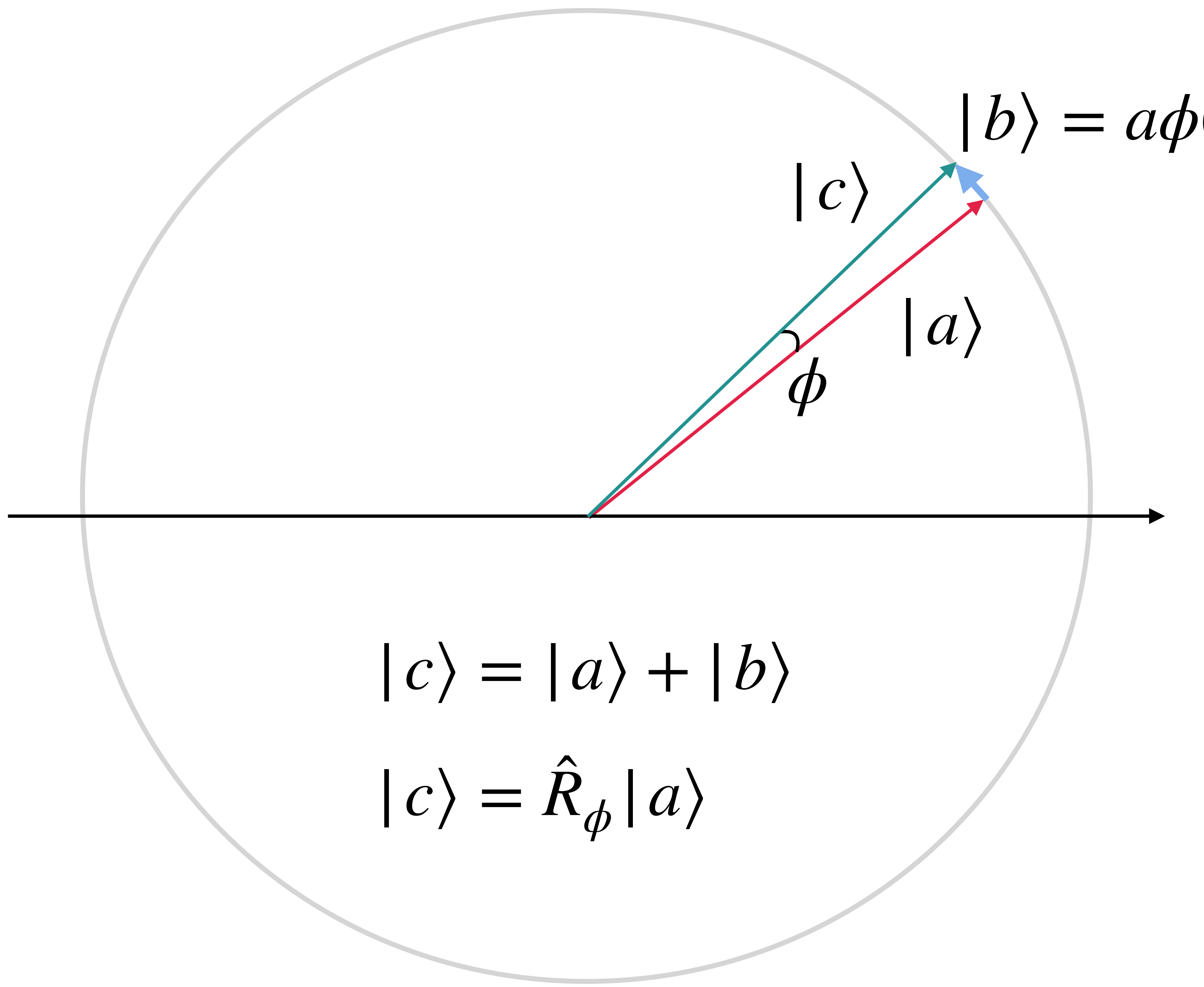
$$\hat{R}_\theta = \left(\hat{R}_\phi\right)^N = \left(\hat{I} + \frac{\hat{J}\theta}{N}\right)^N = e^{\hat{J}\theta}$$



$$|b\rangle = a\phi(\hat{J}\frac{|a\rangle}{a}) = \phi\hat{J}|a\rangle$$

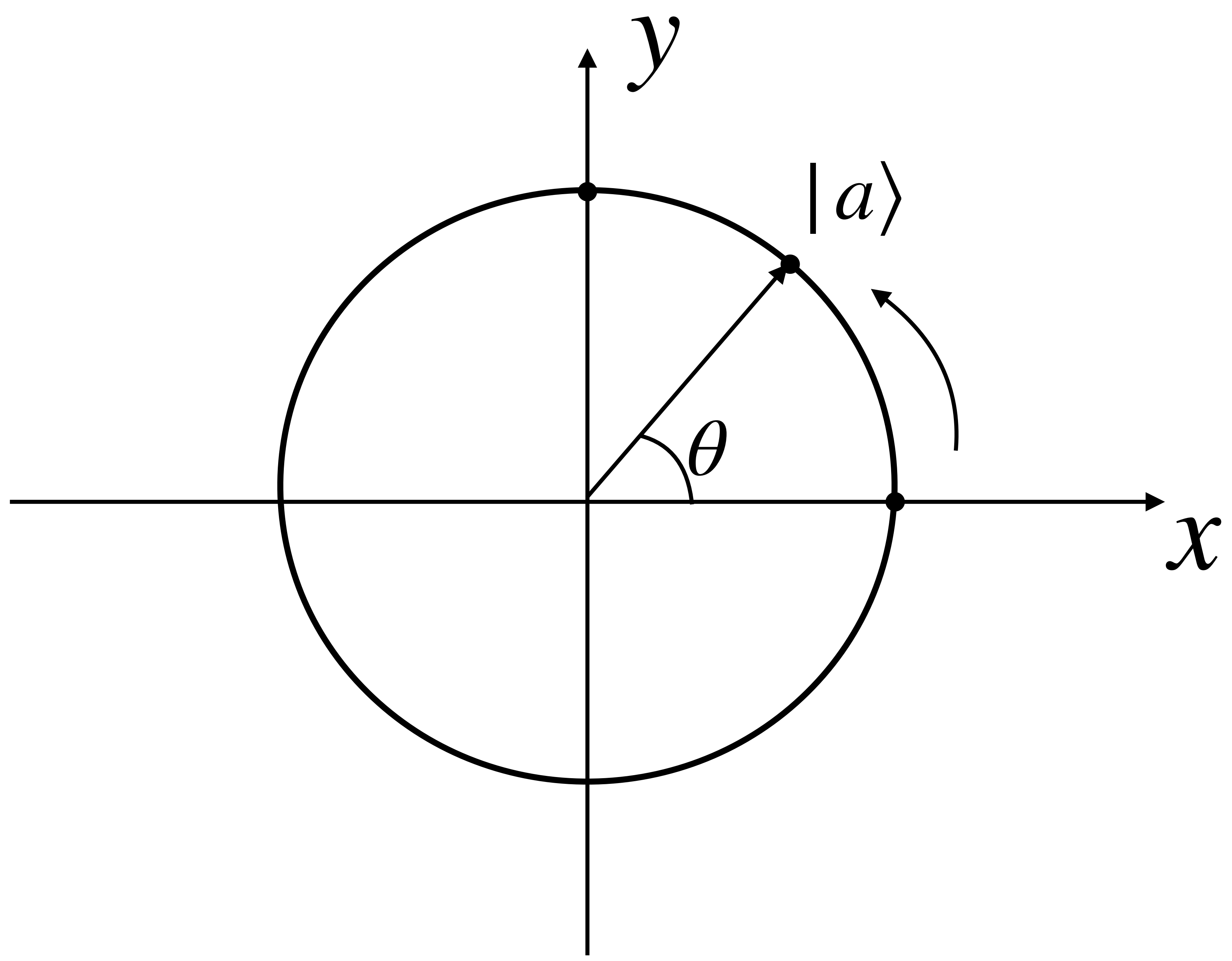
$$|c\rangle = |a\rangle + |b\rangle$$

$$|c\rangle = \hat{R}_\phi |a\rangle$$

$$\hat{R}_\phi |a\rangle = |a\rangle + \phi\hat{J}|a\rangle = (\hat{I} + \hat{J}\phi)|a\rangle$$

Rotation Operator

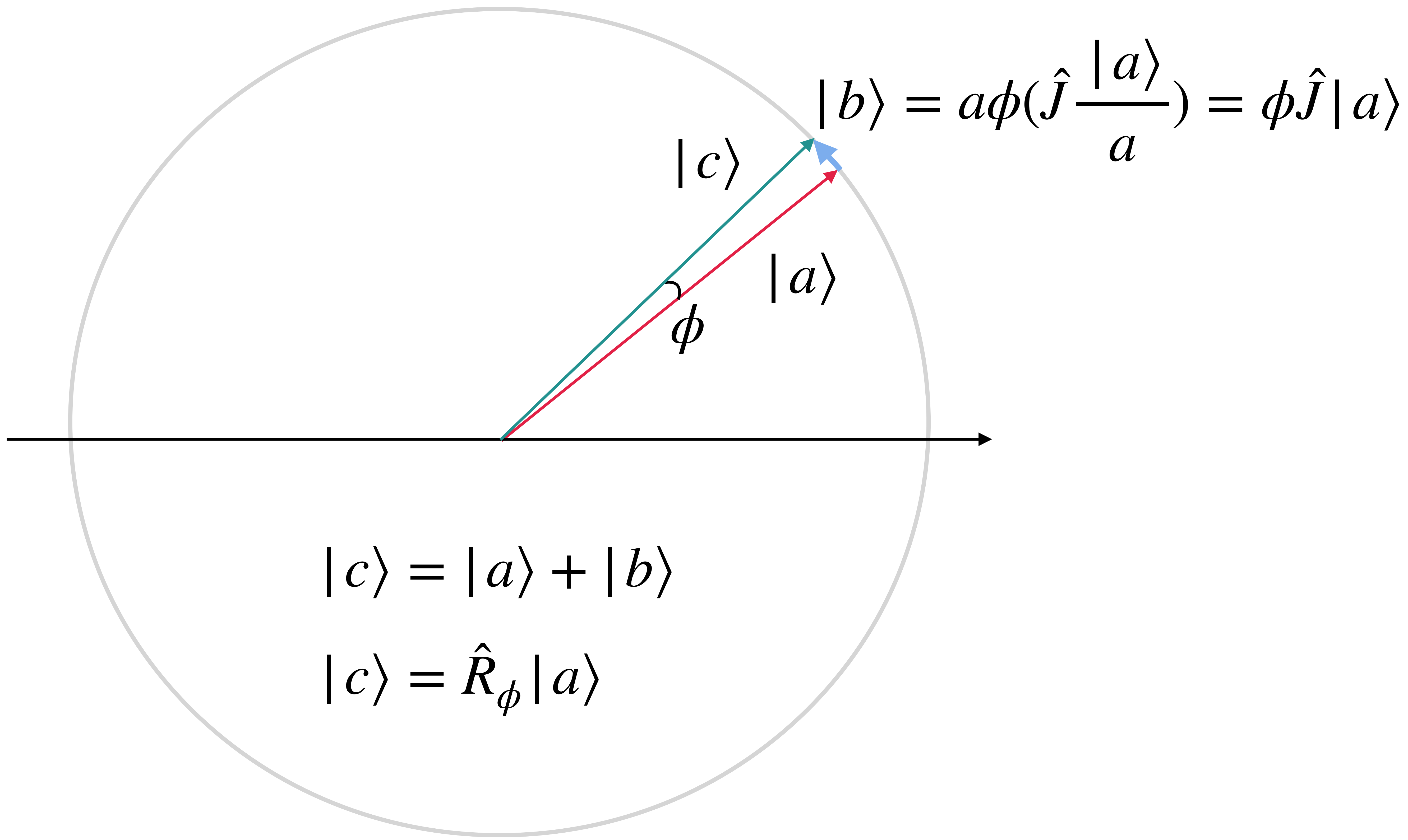
Euler's Formula



$$\hat{R}_\theta = e^{\hat{J}\theta}$$

$$\hat{R}_\theta = \hat{I}\cos\theta + \hat{J}\sin\theta$$

$$\boxed{e^{\hat{J}\theta} = \hat{I}\cos\theta + \hat{J}\sin\theta}$$



$$|c\rangle = |a\rangle + |b\rangle$$

$$|c\rangle = \hat{R}_\phi|a\rangle$$

$$\hat{R}_\phi|a\rangle = |a\rangle + \phi\hat{J}|a\rangle = (\hat{I} + \hat{J}\phi)|a\rangle$$

Self-Test

Answer These Questions 1hr After Class

1. What is a small mathematical issue with the “physical” phase space?
2. What is the meaning of 1 eV energy?
3. How does the state arrow of an oscillator moves in phase space?
4. How does the “normalization” of energy, position, and momentum help to study oscillator dynamics?
5. What does the oscillator’s frequency depend on?
6. What is special about the number e and exponential function e^x?
7. What does Euler’s Formula say?

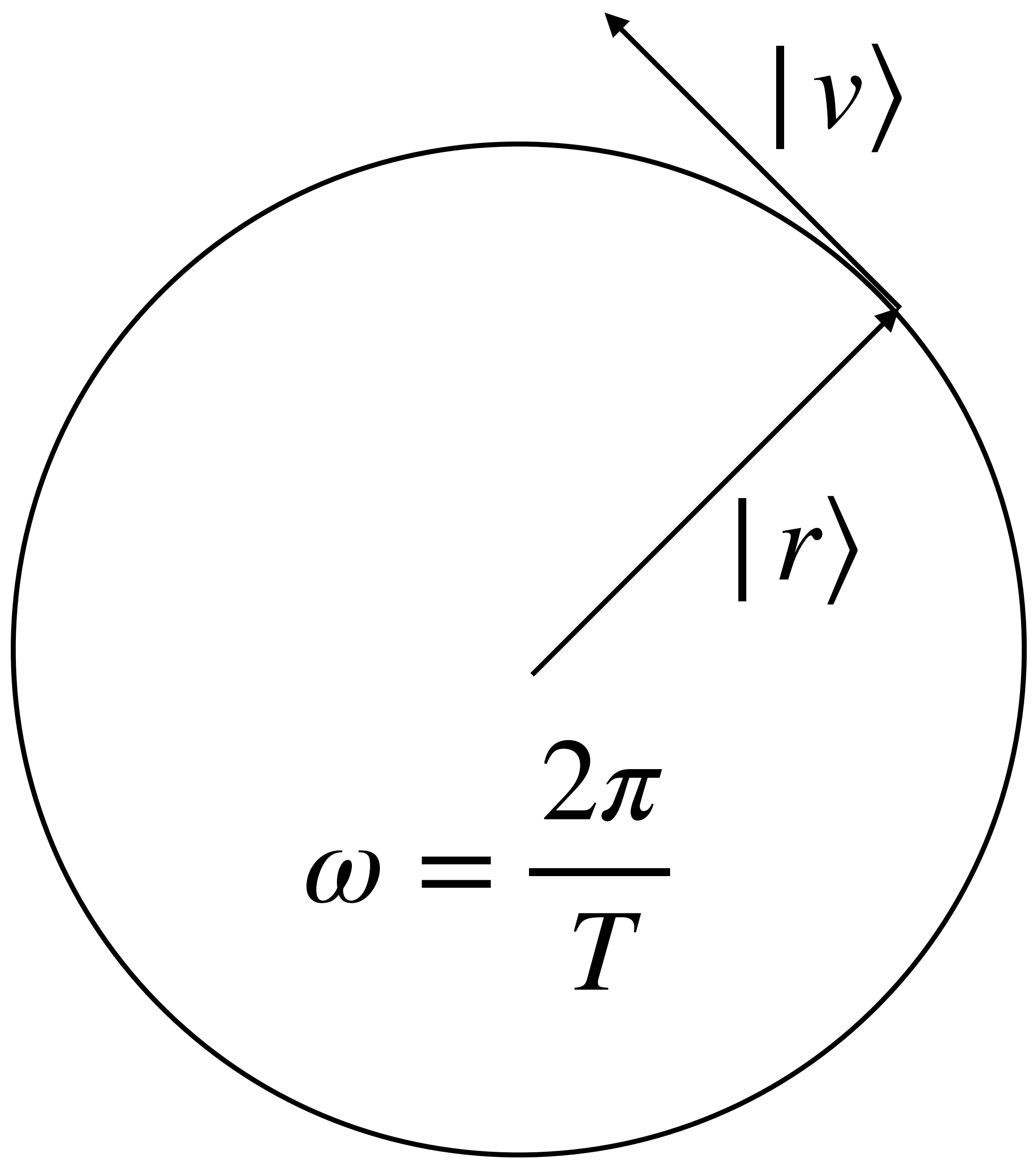
Homework Problems

Homework 6

- Consider an electron with kinetic energy 1 eV. Find its speed in meters per second. How does it compare to the energy of electrons in “normal” thermal motion.
- An average human spends 1800 kilocalories per day. How many it is in electron-volts?
- Show that $\hat{J}(x, y) = (-y, x)$ for any arrow with components (x, y).
- Assume that the Coulomb force $F = ke^2/r^2$ can be written as if electron is attached by a “spring”: $F = Kr$ where $K = ke^2/r^3$. What would be the frequency ω for an oscillator with the mass of the electron and the spring of stiffness K?
- Using Euler’s Formula, show that $(\cos\theta + \hat{J}\sin\theta)^n = \cos(n\theta) + \hat{J}\sin(n\theta)$.

Circular Motion

And $\hat{J}$ Operator



$$\omega = \frac{2\pi}{T}$$

Circular motion with constant angular speed ω

$$v = \frac{2\pi r}{T} = \omega r$$

$$t \quad \rightarrow \quad t + \delta t$$

$$|r\rangle \quad \rightarrow \quad \boxed{|r\rangle + \delta |r\rangle} \qquad\qquad \delta |r\rangle \perp |r\rangle$$

$$|v\rangle = \boxed{\partial_t |r\rangle} \qquad\qquad |v\rangle \perp |r\rangle$$

Take $|r\rangle$, scale it down to unit length (dividing by its length r), then rotate with $\hat{J}$ to make it perpendicular to $|r\rangle$. Finally, scale it up to the length of $|v\rangle$ given by $v = \omega r$

$$|v\rangle = v\hat{J}\left(\frac{1}{r}|r\rangle\right) = \omega \hat{J} |r\rangle$$

$$\partial_t |r\rangle = \omega \hat{J} |r\rangle$$

Quantum Theory

In a Nutshell

II. POSTULATES FOR QUANTUM MECHANICS

In this paper, all state vectors are supposed to be normalized, and mixed states are represented by density operators, i.e., positive operators with unit trace. Let A be an observable with a nondegenerate purely discrete spectrum. Let $\phi_1, \phi_2, \ldots$ be a complete orthonormal sequence of eigenvectors of A and $a_1, a_2, \ldots$ the corresponding eigenvalues; by assumption, all different from each other.

According to the standard formulation of quantum mechanics, on the result of a measurement of the observable A the following postulates are posed:

(A1) *If the system is in the state ψ at the time of measurement, the eigenvalue a_n is obtained as the outcome of measurement with the probability $|\langle\phi_n|\psi\rangle|^2$.*

(A2) *If the outcome of measurement is the eigenvalue a_n, the system is left in the corresponding eigenstate ϕ_n at the time just after measurement.*

The postulate (A1) is called the *statistical formula*, and (A2) the *measurement axiom*. The state change $\psi \mapsto \phi_n$ described by the measurement axiom is called the *state reduction*.

You will understand this paragraph in the end of the course.