1. $\hat{a}^+ |n\rangle = \sqrt{n+1}\,|n+1\rangle$ by definition

Let's try $\left(\int_k \sqrt{k+1}\, |k+1\rangle\langle k| \right) |n\rangle = \int_k \sqrt{k+1}\, |k+1\rangle \underbrace{\langle k | n\rangle}_{0 \text{ for } k\neq n,\ 1 \text{ for } k=n} = \sqrt{n+1}\,|n+1\rangle$

2. Mistake: Not $|g\rangle\langle n|$, but $|g\rangle|n\rangle = |g\rangle \otimes |n\rangle$

$\hat{\sigma}_- \otimes \hat{a}^+ \; |g\rangle \otimes |n\rangle = (\hat{\sigma}_- |g\rangle) \otimes (\hat{a}^+ |n\rangle) =$

$= (0\, |g\rangle) \otimes (\sqrt{n+1}\, |n+1\rangle) = 0\; |g\rangle|n+1\rangle$

because $\hat{\sigma}_- = |g\rangle\langle e|$ and $\langle e|g\rangle = 0$

3. Shown in slides.

4. This is a good one. We have two ways of writing quantum Hamiltonian

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}$ and

$\hat{H} = E_0 \hat{I} + \hbar\omega\, \hat{a}^+\hat{a}$ where $\hat{I} = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a}$

so $\hat{H} = E_0\, \hat{a}\hat{a}^+ + (\hbar\omega - E_0)\, \hat{a}^+\hat{a}$

Note that we don't have $\hat{a}^2$ or $(\hat{a}^+)^2$, but their "mix" $\hat{a}\hat{a}^+$ and $\hat{a}^+\hat{a}$.

We have seen this when dealing with coupled oscillators and there we did variable change.

Let's try this time:

$$\hat{x} = A\hat{a} + B\hat{a}^+ \qquad \hat{x}^2 = A^2\hat{a}^2 + B^2(\hat{a}^+)^2 + AB(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})$$

$$\hat{p} = C\hat{a} + D\hat{a}^+ \qquad \hat{p}^2 = C^2\hat{a}^2 + D^2(\hat{a}^+)^2 + CD(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})$$

Then,

$$\hat{H} = \frac{C^2}{2m}\hat{a}^2 + \frac{D^2}{2m}(\hat{a}^+)^2 + \frac{CD}{2m}(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}) + \frac{m\omega^2}{2}A^2\hat{a}^2 + \frac{m\omega^2}{2}B^2(\hat{a}^+)^2 + \frac{m\omega^2}{2}AB(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})$$

To make it equal to $E_0\hat{a}\hat{a}^+ + (\hbar\omega - E_0)\hat{a}^+\hat{a}$ the coefficients must satisfy

$$\underbrace{\frac{C^2}{2m} + \frac{m\omega^2}{2}A^2 = 0 \quad (1) \qquad \frac{D^2}{2m} + \frac{m\omega^2}{2}B^2 = 0 \quad (2)}$$

$$\underbrace{\frac{CD}{2m} + \frac{m\omega^2}{2}AB = E_0 \qquad \frac{CD}{2m} + \frac{m\omega^2}{2}AB = \hbar\omega - E_0}$$

$\downarrow$

$$C^2 = -(m\omega A)^2 \quad (3)$$

$$D^2 = -(m\omega B)^2 \quad (4)$$

$$CD + (m\omega)^2 AB = m\hbar\omega \quad (5)$$

$\downarrow$

$$E_0 = \hbar\omega - E_0$$

$\downarrow$

$$\boxed{E_0 = \hbar\omega/2} \; !$$

very important

$$2E_0 = \hbar\omega$$

3 equations & 4 unknowns ... bad, but also means many solutions are possible.

From (3) & (4) follows

$$\left.\begin{aligned} C &= \pm J\, m\omega A \\ D &= \pm J\, m\omega B \end{aligned}\right\} \rightarrow \quad \begin{aligned} CD &= -(m\omega)^2 AB \\ &\text{or} \\ CD &= (m\omega)^2 AB \end{aligned}$$

From (5)

$$CD + (m\omega)^2 AB = m\hbar\omega$$

CD can't be $-(m\omega)^2 AB$, hence

$$CD = (m\omega)^2 AB \quad \text{and}$$

$$2(m\omega)^2 AB = m\hbar\omega \rightarrow \quad AB = \frac{\hbar}{2m\omega}$$

Simple choice $A = B = \sqrt{\dfrac{\hbar}{2m\omega}}$

Then $C = J m\omega \sqrt{\dfrac{\hbar}{2m\omega}} = J\sqrt{\dfrac{\hbar m\omega}{2}}$

$$D = -J m\omega \sqrt{\frac{\hbar}{2m}} = -J\sqrt{\frac{\hbar m\omega}{2}}$$

Thus,

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^{+}\right) \qquad \hat{p} = J\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a} - \hat{a}^{+}\right)$$

6. From the definition of $\hat{a}$ and $\hat{a}^{+}$ followed

$$[\hat{a}, \hat{a}^{+}] = \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} = \hat{I}$$

Then, from the meaning of $\hat{a}^{+}\hat{a}$ and the form of quantum Hamiltonian of oscillator, followed

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^{+}\right) \qquad \hat{p} = J\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a} - \hat{a}^{+}\right)$$

Now it is easy to find

$$\hat{x}\hat{p} = J\frac{\hbar}{2}(\hat{a}+\hat{a}^{+})(\hat{a}-\hat{a}^{+}) = J\frac{\hbar}{2}\left[(\hat{a})^2 - (\hat{a}^{+})^2 + \hat{a}^{+}\hat{a} - \hat{a}\hat{a}^{+}\right)$$

$$\hat{p}\hat{x} = J\frac{\hbar}{2}(\hat{a}-\hat{a}^{+})(\hat{a}+\hat{a}^{+}) = J\frac{\hbar}{2}\left((\hat{a})^2 - (\hat{a}^{+})^2 + \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a}\right)$$

Then $\hat{x}\hat{p} - \hat{p}\hat{x} = J\frac{\hbar}{2}\left(2\hat{a}^{+}\hat{a} - 2\hat{a}\hat{a}^{+}\right) = J\hbar\,[\hat{a}^{+},\hat{a}] = -J\hbar$

There is "−" sign mistake somewhere in the derivation of $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^{+}$. If you fix it, you will arrive at

canonical quantization:

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

It is usually postulated, but we derived it from the oscillator Hamiltonian!

COOL.