Atiyah Commutative Algebra: Problems

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Abstract

This work contains solutions to some exercises from Atiyah's Commutative Algebra text.

1 Chapter 2: Modules

Question 2.1.

1. Show that $(\mathbf{Z}/m\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z}) = 0$ if m, n are coprime.

Solution.
As m, n are coprime, by Bezout, there exists $a, b \in \mathbb{Z}$ such that

$$am + bn = 1.$$

Therefore,

$$x \otimes y \quad = \quad am(x \otimes y) + bn(x \otimes y) = a(mx \otimes y) + b(x \otimes ny) = a(0 \otimes y) + b(x \otimes 0) = 0$$

for any $x \otimes y \in \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$. □

Question 2.4.

4. Let M_i $(i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Solution.
Let $M = \bigoplus_{i \in I} M_i$. Then, M is exact iff for any exact sequence of $A-$modules

$$0 \to N' \to N \to N'' \to 0$$

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the tensored sequence

$$0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$$

is exact. Observe that the above tensored sequence can be written as

$$0 \to \bigoplus_{i \in I} N' \otimes M_i \to \bigoplus_{i \in I} N \otimes M_i \to \bigoplus_{i \in I} N'' \otimes M_i \to 0.$$

Hence M is exact iff for any $i \in I$ and for any exact sequence of $A-$modules

$$0 \to N' \to N \to N'' \to 0$$

the tensored sequence

$$0 \to N' \otimes M_i \to N \otimes M_i \to N'' \otimes M_i \to 0,$$

but the later is equivalent to M_i being flat for all $i \in I$, so we are done. □

Question 2.5.

5. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra. [Use Exercise 4.]

Solution.

2 Chapter 1: Rings and Ideals

Question 1.1.

1. Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution.

Question 1.2.

2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

i) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent. [If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex. 1.]

ii) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.

iii) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ $(0 \leqslant r \leqslant n)$.]

iv) f is said to be *primitive* if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

Solution.

Question 1.3.

3. Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \ldots, x_r]$ in several indeterminates.

Solution.

Question 1.4.

4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution.

Question 1.5.

7. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution.

Question 1.15.

The prime spectrum of a ring

15. Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

i) if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

ii) $V(0) = X$, $V(1) = \varnothing$.

iii) if $(E_i)_{i \in I}$ is any family of subsets of A, then

$$V\Big(\bigcup_{i \in I} E_i\Big) = \bigcap_{i \in I} V(E_i).$$

iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology*. The topological space X is called the *prime spectrum* of A, and is written Spec (A).

Solution.
(i) Any prime ideal containing $r(\mathfrak{a})$ contains $\mathfrak{a}$, so $V(r(\mathfrak{a})) \subset V(\mathfrak{a})$. Now, if $\mathfrak{p} \in V(\mathfrak{a})$, then by closure of multiplication of $\mathfrak{p}$, $r(\mathfrak{a}) \subset \mathfrak{p}$. Hence, $V(\mathfrak{a}) \subset V(r(\mathfrak{a}))$, and $V(\mathfrak{a}) = V(r(\mathfrak{a}))$. Now, any prime ideal containing E, as $\mathfrak{a}$ is generated by E, contains E, therefore $V(\mathfrak{a}) \subset V(E)$. Now, let $\mathfrak{p} \in V(E)$. Then, as by definition, $\mathfrak{a}$ is the smallest ideal that contains E, so $\mathfrak{a} \subset \mathfrak{p}$, and therefore $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$. This shows that

$$E \subset \mathfrak{p} \iff \mathfrak{a} \subset \mathfrak{p} \iff r(\mathfrak{a}) \subset \mathfrak{p}$$

for any prime ideal $\mathfrak{p}$ of A.

(ii) It is clear that $V(0) \subset X$. Any prime ideal $\mathfrak{p}$ of A contains 0, so $V(0) = X$. And as there is no prime ideal that contains 1, $V(1) = \emptyset$.

(iii) Let $\mathfrak{p} \in V(\bigcup_{i \in I} E_i)$. Then, $E_i \subset \mathfrak{p}$ for all $i \in I$. Therefore, $\mathfrak{p} \in V(E_i)$ for all $i \in I$, so $\mathfrak{p} \in \bigcap_{i \in I} V(E_i)$. This shows $V(\bigcup_{i \in I} E_i) \subset \bigcap_{i \in I} V(E_i)$. Now, suppose $\mathfrak{p} \in \bigcap_{i \in I} V(E_i)$. Then, $E_i \subset \mathfrak{p}$ for all $i \in I$, so $\bigcup_{i \in I} V(E_i) \subset \mathfrak{p}$. Therefore, $p \in V(\bigcup_{i \in I} E_i)$, so we are done.

We can also see

$$\begin{aligned} \mathfrak{p} \in \bigcap_{i \in I} V(E_i) \iff p \in V(E_i) \ \ \forall i \in I \iff E_i \subset \mathfrak{p} \ \ \forall i \in I \\ \iff \bigcup_{i \in I} E_i \subset \mathfrak{p} \iff \mathfrak{p} \in V(\bigcup_{i \in I} E_i) \end{aligned}$$

for any prime ideal $\mathfrak{p}$ in A.

(iv) For any prime ideal $\mathfrak{p}$ of A,

$$\begin{aligned} \mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b}) \quad &\Longrightarrow \quad \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p} \Longrightarrow \mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p} \\ &\Longrightarrow \quad \mathfrak{p} \in V(\mathfrak{a}\mathfrak{b}) \Longrightarrow \mathfrak{a} \subset \mathfrak{p} \text{ or } \mathfrak{b} \subset \mathfrak{p} \qquad (1) \\ &\Longrightarrow \quad \mathfrak{p} \in V(\mathfrak{a}) \text{ and } \mathfrak{p} \in V(\mathfrak{b}) \Longrightarrow \mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b}) \\ &\Longrightarrow \quad \mathfrak{a} \subset \mathfrak{p} \text{ or} \mathfrak{b} \subset \mathfrak{p} \Longrightarrow \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p} \Longrightarrow \mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b}) \end{aligned}$$

where (**??**) holds by primality of $\mathfrak{p}$. Therefore,

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a})V(\mathfrak{b}),$$

as required. □

Question 1.16.

16. Draw pictures of Spec (Z), Spec (R), Spec (C[x]), Spec (R[x]), Spec (Z[x]).

Solution.

Question 1.17.

17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that

i) $X_f \cap X_g = X_{fg}$;

ii) $X_f = \varnothing \Leftrightarrow f$ is nilpotent;

iii) $X_f = X \Leftrightarrow f$ is a unit;

iv) $X_f = X_g \Leftrightarrow r((f)) = r((g))$;

v) X is quasi-compact (that is, every open covering of X has a finite sub-covering).

vi) More generally, each X_f is quasi-compact.

vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.

The sets X_f are called *basic open sets* of $X = \operatorname{Spec}(A)$.

[To prove (v), remark that it is enough to consider a covering of X by basic open sets X_{f_i} $(i \in I)$. Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i \in J} g_i f_i \qquad (g_i \in A)$$

where J is some *finite* subset of I. Then the X_{f_i} $(i \in J)$ cover X.]

Solution.

We claim that $\mathscr{B} = \{X_f = X \setminus V(f)\}_{f \in A}$ is a basis of the Zariski topology on X. Let O be an open subset in the Zariski topology, so $O = X \setminus V(E)$ for some $E \in 2^A$. Then,

$$\begin{aligned} O &= X \setminus V(E) = X \setminus V(\bigcup_{f \in E} \{f\}) \\ &= X \setminus \bigcap_{f \in E} V(\{f\}) = X \setminus \bigcap_{f \in E} X_f \qquad (2) \\ &= \bigcup_{f \in E} X \setminus X_f \qquad (3) \end{aligned}$$

where (**??**) holds by 1-15-(iii) and (**??**) holds by DeMorgan's law. Therefore, any open O in the Zariski topology is a union of elements in $\mathscr{B}$, so $\mathscr{B}$ is a basis for the Zariski topology.

(i) For any prime ideal $\mathfrak{p}$ of A,

$$\begin{aligned} \mathfrak{p} \in X_f \cap X_g &\iff \mathfrak{p} \in X_f \text{ and } \mathfrak{p} \in X_g \iff \mathfrak{p} \notin V(f) \text{ and } \mathfrak{p} \notin V(g) \\ &\iff f \notin \mathfrak{p} \text{ and } g \notin \mathfrak{p} \iff fg \notin \mathfrak{p} \qquad (4) \\ &\iff \mathfrak{p} \notin V(fg) \iff \mathfrak{p} \in X_{fg} \end{aligned}$$

where (**??**) holds by primality of $\mathfrak{p}$. Therefore, $X_f \cap X_g = X_{fg}$.

(ii) Observe that

$$\begin{aligned} X_f = \emptyset \quad &\iff \quad X \subset V(f) \iff \forall \text{ prime ideal } \mathfrak{p} \text{ of } A \quad f \in \mathfrak{p} \\ &\iff \quad f \in \bigcap_{\mathfrak{p}\text{: prime ideal of } A} \mathfrak{p} \iff f \text{ belongs to the nilradical of } f \\ &\iff \quad f \text{ is nilpotent.} \end{aligned} \tag{5}$$

where (**??**) holds by preposition 1-8.

(iii)

Question 1.18.

18. For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \text{Spec}(A)$. When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that

i) the set $\{x\}$ is closed (we say that x is a "closed point") in Spec $(A) \Leftrightarrow \mathfrak{p}_x$ is maximal;

ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;

iii) $y \in \overline{\{x\}} \Leftrightarrow \mathfrak{p}_x \subseteq \mathfrak{p}_y$;

iv) X is a T_0-space (this means that if x, y are distinct points of X, then either there is a neighborhood of x which does not contain y, or else there is a neighborhood of y which does not contain x).

Solution.

Question 1.19.

19. A topological space X is said to be *irreducible* if $X \neq \varnothing$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X. Show that Spec (A) is irreducible if and only if the nilradical of A is a prime ideal.

Solution.

Question 1.21.

Solution.

21. Let $\phi: A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A, i.e., a point of X. Hence ϕ induces a mapping $\phi^*: Y \to X$. Show that

i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous.

ii) If $\mathfrak{a}$ is an ideal of A, then $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$.

iii) If $\mathfrak{b}$ is an ideal of B, then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\operatorname{Ker}(\phi))$ of X. (In particular, $\operatorname{Spec}(A)$ and $\operatorname{Spec}(A/\mathfrak{N})$ (where $\mathfrak{N}$ is the nilradical of A) are naturally homeomorphic.)

v) If ϕ is injective, then $\phi^*(Y)$ is dense in X. More precisely, $\phi^*(Y)$ is dense in $X \Leftrightarrow \operatorname{Ker}(\phi) \subseteq \mathfrak{N}$.

vi) Let $\psi: B \to C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

vii) Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi: A \to B$ by $\phi(x) = (\bar{x}, x)$, where $\bar{x}$ is the image of x in $A/\mathfrak{p}$. Show that ϕ^* is bijective but not a homeomorphism.

2. Let M be an A-module. If every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.

3 Chapter 6: Chain Conditions

Question 6.2.

Solution.
Suppose M is not Noetherian. Then, there exists a non finitely-generated submodule N of M. Choose x_1 from N. Then, since N is non finitely-generated, $N - \{x_1\} \neq \emptyset$, so we can choose $x_2 \neq x_1$ from N. Repeat this process inductively, then we have a nonempty set of finitely generated submodules of M

$$\{(x_1), (x_1, x_2), (x_1, x_2, x_3)...\}$$

such that it does not have a maximal element. Therefore, by contradiction, we have that M must be Noetherian. $\square$

4 Chapter 3: Rings and Modules of Fractions

Question 3.1.

1. Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Solution.
Suppose there exists $s_0 \in S$ such that $s_0 M = 0$. Then, for any $(a, s), (b, t) \in A \times S$,

$$(at - bs)s_0 = 0$$

so $(a, s) \sim (b, t)$, and hence $S = 0$. Conversely, suppose $S^{-1}M = 0$. As M is finitely generated, there exists $\{x_1, ..., x_m\}$ for some m that generates M. As $S^{-1}M = 0$, we can choose $\{s_1, ..., s_m\}$ such that

$$s_i x_i = 0$$

for all $1 \leq i \leq m$. Consider $s^* = s_1 ... s_m$. Then,

$$\begin{aligned} s^* x &= s_1 ... s_m x = s_1 ... s_m (\sum_{i=1}^{m} a_i x_i) \\ &= \sum_{i=1}^{m} s_1 ... s_m a_i x_i = 0 \end{aligned}$$

for some $a_1, ... a_m \in A$, for all $x \in M$. Therefore, $s^* M = 0$. $\square$

Question 3.2.

2. Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants. [If $M = \mathfrak{a}M$, then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$, hence by Nakayama we have $S^{-1}M = 0$. Now use Exercise 1.]

Solution.
If $1 + a, 1 + a' \in 1 + \mathbf{a}$, then

$$(1 + a)(1 + a') \quad = \quad 1a + a' + aa' \in 1 + \mathbf{a}.$$

Therfore $1 + \mathbf{a}$ is a multiplicatively closed set in A. Now, for any $\in 1 + \mathbf{a}$,

$$1 - 1$$

Question 3.5.

5. Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?

Solution.
Let $\mathfrak{N}$ be a nilradical of A. By Corollary 3.12, $(\mathfrak{N})_\mathfrak{p}$ is a nilradical of $A_\mathfrak{p}$, which is 0 by assumption, for any prime ideal $\mathfrak{p}$. Therefore, by proposition 3.8, $\mathfrak{N} = 0$.

Question 3.12.

12. Let A be an integral domain and M an A-module. An element $x \in M$ is a *torsion element* of M if $\mathrm{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule of M. This submodule is called the *torsion submodule* of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

i) If M is any A-module, then $M/T(M)$ is torsion-free.

ii) If $f: M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

iii) If $0 \to M' \to M \to M''$ is an exact sequence, then the sequence $0 \to T(M') \to T(M) \to T(M'')$ is exact.

iv) If M is any A-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A.

[For iv), show that K may be regarded as the direct limit of its submodules $A\xi$ ($\xi \in K$); using Chapter 1, Exercise 15 and Exercise 20, show that if $1 \otimes x = 0$ in $K \otimes M$ then $1 \otimes x = 0$ in $A\xi \otimes M$ for some $\xi \neq 0$. Deduce that $\xi^{-1}x = 0$.]

Solution.
We first show that $T(M)$ is a A-sub-module of M. Let $x, y \in T(M)$, and $r \in A$. Then, there exists $a, b \in A$, such that $a, b \neq 0$, $ax = 0$ and $by = 0$. Then, as A is an integral domain, $ab \neq 0$ and $ab(x+y) = abx + aby = bax + aby = 0$, so $x + y \in T(M)$. Also, $arx = rax = 0$, so $rx \in T(M)$ and $T(M)$ is a $A-$sub-module of M.

(i) To show that $M/T(M)$ is torsion-free, it suffices to show that $T(M/T(M)) = 0$.

(ii) Let $y \in f(T(M))$. Then, there exists $x \in T(M)$ such that $f(x) = y$, and $0 \neq a \in A$ with $ax = 0$. Therefore, $ay = af(x) = f(ax) = 0$, as f is a module homomorphism. Therefore, $f(T(M)) \subset T(N)$.

(iii)

Question 3.13.

13. Let S be a multiplicatively closed subset of an integral domain A. In the notation of Exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$. Deduce that the following are equivalent:

i) M is torsion-free.

ii) $M_\mathfrak{p}$ is torsion-free for all prime ideals $\mathfrak{p}$.

iii) $M_\mathfrak{m}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

Solution.