

Atiyah Commutative Algebra: Problems

Youngduck Choi *

Abstract

This work contains solutions to some exercises from Atiyah's Commutative Algebra text.

1 Chapter 2: Modules

Question 2.1.

1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

Solution.

As m, n are coprime, by Bezout, there exists $a, b \in \mathbb{Z}$ such that

$$am + bn = 1.$$

Therefore,

$$x \otimes y = am(x \otimes y) + bn(x \otimes y) = a(mx \otimes y) + b(x \otimes ny) = a(0 \otimes y) + b(x \otimes 0) = 0$$

for any $x \otimes y \in \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$. □

Question 2.4.

4. Let M_i ($i \in I$) be any family of A -modules, and let M be their direct sum. Prove that M is flat \Leftrightarrow each M_i is flat.

Solution.

Let $M = \bigoplus_{i \in I} M_i$. Then, M is exact iff for any exact sequence of A -modules

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

*Department of Mathematics, Courant Institute of Mathematical Sciences, yc1104@nyu.edu; If you find an error and want to share with me, you can reach me via email.

the tensored sequence

$$0 \rightarrow N' \otimes M \rightarrow N \otimes M \rightarrow N'' \otimes M \rightarrow 0$$

is exact. Observe that the above tensored sequence can be written as

$$0 \rightarrow \bigoplus_{i \in I} N' \otimes M_i \rightarrow \bigoplus_{i \in I} N \otimes M_i \rightarrow \bigoplus_{i \in I} N'' \otimes M_i \rightarrow 0.$$

Hence M is exact iff for any $i \in I$ and for any exact sequence of A -modules

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

the tensored sequence

$$0 \rightarrow N' \otimes M_i \rightarrow N \otimes M_i \rightarrow N'' \otimes M_i \rightarrow 0,$$

but the later is equivalent to M_i being flat for all $i \in I$, so we are done. \square

Question 2.5.

5. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A . Prove that $A[x]$ is a flat A -algebra. [Use Exercise 4.]

Solution.

2 Chapter 1: Rings and Ideals

Question 1.1.

1. Let x be a nilpotent element of a ring A . Show that $1 + x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit.

Solution.

As x is nilpotent, we can choose $n \in \mathbb{N}$ such that $x^n = 0$. Then,

$$(1 + x)(1 + (-1)x + \dots + (-1)^{n-1}x^{n-1}) = 1 - x^n = 1.$$

Therefore, $1 + x$ is a unit. Now, let y be a unit. Then,

$$(y + x)(z - z^2x + \dots + (-1)^{n-1}z^n x^{n-1}) = yz - z^n x^n = 1.$$

Therefore, $x + y$ is a unit. □

Question 1.2.

2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x , with coefficients in A . Let $f = a_0 + a_1x + \dots + a_nx^n \in A[x]$. Prove that



EXERCISES 11

- i) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and a_1, \dots, a_n are nilpotent. [If $b_0 + b_1x + \dots + b_mx^m$ is the inverse of f , prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex. 1.]
- ii) f is nilpotent $\Leftrightarrow a_0, a_1, \dots, a_n$ are nilpotent.
- iii) f is a zero-divisor \Leftrightarrow there exists $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1x + \dots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ ($0 \leq r \leq n$).]
- iv) f is said to be *primitive* if $(a_0, a_1, \dots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

Solution.

Question 1.3.

3. Generalize the results of Exercise 2 to a polynomial ring $A[x_1, \dots, x_r]$ in several indeterminates.

4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution.

Question 1.4.

Solution.

Let \mathfrak{N} , and \mathfrak{R} be nilradical, and Jacobson radical respectively. As any primal ideal is a maximal ideal,

$$\begin{aligned}\mathfrak{N} &= \bigcap_{\{\mathfrak{p} : \text{a primal ideal of } A\}} \mathfrak{p} \\ &\subset \bigcap_{\{\mathfrak{p} : \text{a maximal ideal of } A\}} \mathfrak{p} = \mathfrak{R}.\end{aligned}$$

Hence, it now suffices to show that $\mathfrak{R} \subset \mathfrak{N}$. Let $f = a_0 + a_1x + \dots + a_nx^n \in \mathfrak{R}$. Then, by proposition 1.9,

$$1 + xf = 1 + a_0x + \dots + a_nx^{n+1} \text{ is a unit in } A[x].$$

Now, by exercise 1.2-i, a_0, \dots, a_n are nilpotent, and by 1-2-ii, $f \in \mathfrak{N}$. Therefore, $\mathfrak{R} = \mathfrak{N}$. □

Question 1.5.

7. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Solution.

Let \mathfrak{p} be a prime ideal of A .

Question 1.15.

The prime spectrum of a ring

15. Let A be a ring and let X be the set of all prime ideals of A . For each subset E of A , let $V(E)$ denote the set of all prime ideals of A which contain E . Prove that

- i) if \mathfrak{a} is the ideal generated by E , then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.
- ii) $V(0) = X$, $V(1) = \emptyset$.
- iii) if $(E_i)_{i \in I}$ is any family of subsets of A , then

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

- iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A .

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology*. The topological space X is called the *prime spectrum* of A , and is written $\text{Spec}(A)$.

Solution.

(i) Any prime ideal containing $r(\mathfrak{a})$ contains \mathfrak{a} , so $V(r(\mathfrak{a})) \subset V(\mathfrak{a})$. Now, if $\mathfrak{p} \in V(\mathfrak{a})$, then by closure of multiplication of \mathfrak{p} , $r(\mathfrak{a}) \subset \mathfrak{p}$. Hence, $V(\mathfrak{a}) \subset V(r(\mathfrak{a}))$, and $V(\mathfrak{a}) = V(r(\mathfrak{a}))$. Now, any prime ideal containing E , as \mathfrak{a} is generated by E , contains E , therefore $V(\mathfrak{a}) \subset V(E)$. Now, let $\mathfrak{p} \in V(E)$. Then, as by definition, \mathfrak{a} is the smallest ideal that contains E , so $\mathfrak{a} \subset \mathfrak{p}$, and therefore $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$. This shows that

$$E \subset \mathfrak{p} \iff \mathfrak{a} \subset \mathfrak{p} \iff r(\mathfrak{a}) \subset \mathfrak{p}$$

for any prime ideal \mathfrak{p} of A .

(ii) It is clear that $V(0) \subset X$. Any prime ideal \mathfrak{p} of A contains 0, so $V(0) = X$. And as there is no prime ideal that contains 1, $V(1) = \emptyset$.

(iii) Let $\mathfrak{p} \in V(\bigcup_{i \in I} E_i)$. Then, $E_i \subset \mathfrak{p}$ for all $i \in I$. Therefore, $\mathfrak{p} \in V(E_i)$ for all $i \in I$, so $\mathfrak{p} \in \bigcap_{i \in I} V(E_i)$. This shows $V(\bigcup_{i \in I} E_i) \subset \bigcap_{i \in I} V(E_i)$. Now, suppose $\mathfrak{p} \in \bigcap_{i \in I} V(E_i)$. Then, $E_i \subset \mathfrak{p}$ for all $i \in I$, so $\bigcup_{i \in I} E_i \subset \mathfrak{p}$. Therefore, $\mathfrak{p} \in V(\bigcup_{i \in I} E_i)$, so we are done.

We can also see

$$\begin{aligned} \mathfrak{p} \in \bigcap_{i \in I} V(E_i) &\iff \mathfrak{p} \in V(E_i) \quad \forall i \in I \iff E_i \subset \mathfrak{p} \quad \forall i \in I \\ &\iff \bigcup_{i \in I} E_i \subset \mathfrak{p} \iff \mathfrak{p} \in V\left(\bigcup_{i \in I} E_i\right) \end{aligned}$$

for any prime ideal \mathfrak{p} in A .

(iv) For any prime ideal \mathfrak{p} of A ,

$$\begin{aligned}
 \mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b}) &\implies \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p} \implies \mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p} \\
 &\implies \mathfrak{p} \in V(\mathfrak{a}\mathfrak{b}) \implies \mathfrak{a} \subset \mathfrak{p} \text{ or } \mathfrak{b} \subset \mathfrak{p} \\
 &\implies \mathfrak{p} \in V(\mathfrak{a}) \text{ and } \mathfrak{p} \in V(\mathfrak{b}) \implies \mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b}) \\
 &\implies \mathfrak{a} \subset \mathfrak{p} \text{ or } \mathfrak{b} \subset \mathfrak{p} \implies \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p} \implies \mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})
 \end{aligned} \tag{1}$$

where (1) holds by primality of \mathfrak{p} . Therefore,

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a})V(\mathfrak{b}),$$

as required. □

Question 1.16.

16. Draw pictures of $\text{Spec}(\mathbb{Z})$, $\text{Spec}(\mathbb{R})$, $\text{Spec}(\mathbb{C}[x])$, $\text{Spec}(\mathbb{R}[x])$, $\text{Spec}(\mathbb{Z}[x])$.

Solution.

Question 1.17.

17. For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \text{Spec}(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that
- i) $X_f \cap X_g = X_{fg}$;
 - ii) $X_f = \emptyset \Leftrightarrow f$ is nilpotent;
 - iii) $X_f = X \Leftrightarrow f$ is a unit;
 - iv) $X_f = X_g \Leftrightarrow r((f)) = r((g))$;
 - v) X is quasi-compact (that is, every open covering of X has a finite sub-covering).
 - vi) More generally, each X_f is quasi-compact.
 - vii) An open subset of X is quasi-compact if and only if it is a finite union of sets X_f .
- The sets X_f are called *basic open sets* of $X = \text{Spec}(A)$.
 [To prove (v), remark that it is enough to consider a covering of X by basic open sets X_{f_i} ($i \in I$). Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i \in J} g_i f_i \quad (g_i \in A)$$

where J is some finite subset of I . Then the X_{f_i} ($i \in J$) cover X .]

Solution.

We claim that $\mathcal{B} = \{X_f = X \setminus V(f)\}_{f \in A}$ is a basis of the Zariski topology on X . Let O be an open subset in the Zariski topology, so $O = X \setminus V(E)$ for some $E \in 2^A$. Then,

$$\begin{aligned} O &= X \setminus V(E) = X \setminus V\left(\bigcup_{f \in E} \{f\}\right) \\ &= X \setminus \bigcap_{f \in E} V(\{f\}) = X \setminus \bigcap_{f \in E} X_f \end{aligned} \tag{2}$$

$$= \bigcup_{f \in E} X \setminus X_f \tag{3}$$

where (2) holds by 1-15-(iii) and (3) holds by DeMorgan's law. Therefore, any open O in the Zariski topology is a union of elements in \mathcal{B} , so \mathcal{B} is a basis for the Zariski topology.

(i) For any prime ideal \mathfrak{p} of A ,

$$\begin{aligned} \mathfrak{p} \in X_f \cap X_g &\iff \mathfrak{p} \in X_f \text{ and } \mathfrak{p} \in X_g \iff \mathfrak{p} \notin V(f) \text{ and } \mathfrak{p} \notin V(g) \\ &\iff f \notin \mathfrak{p} \text{ and } g \notin \mathfrak{p} \iff fg \notin \mathfrak{p} \\ &\iff \mathfrak{p} \notin V(fg) \iff \mathfrak{p} \in X_{fg} \end{aligned} \tag{4}$$

where (4) holds by primality of \mathfrak{p} . Therefore, $X_f \cap X_g = X_{fg}$.

(ii) Observe that

$$\begin{aligned}
 X_f = \emptyset &\iff X \subset V(f) \iff \forall \text{ prime ideal } \mathfrak{p} \text{ of } A \quad f \in \mathfrak{p} \\
 &\iff f \in \bigcap_{\mathfrak{p}: \text{ prime ideal of } A} \mathfrak{p} \iff f \text{ belongs to the nilradical of } A \\
 &\iff f \text{ is nilpotent.}
 \end{aligned} \tag{5}$$

where (5) holds by proposition 1-8.

(iii)

Question 1.18.

- 18.** For psychological reasons it is sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X = \text{Spec}(A)$. When thinking of x as a prime ideal of A , we denote it by \mathfrak{p}_x (logically, of course, it is the same thing). Show that
- i) the set $\{x\}$ is closed (we say that x is a “closed point”) in $\text{Spec}(A) \iff \mathfrak{p}_x$ is maximal;
 - ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$;
 - iii) $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subseteq \mathfrak{p}_y$;
 - iv) X is a T_0 -space (this means that if x, y are distinct points of X , then either there is a neighborhood of x which does not contain y , or else there is a neighborhood of y which does not contain x).

Solution.

Question 1.19.

- 19.** A topological space X is said to be *irreducible* if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect, or equivalently if every non-empty open set is dense in X . Show that $\text{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal.

Solution.

Question 1.21.

Solution.

21. Let $\phi: A \rightarrow B$ be a ring homomorphism. Let $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. If $\mathfrak{q} \in Y$, then $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A , i.e., a point of X . Hence ϕ induces a mapping $\phi^*: Y \rightarrow X$. Show that
- i) If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous.
 - ii) If \mathfrak{a} is an ideal of A , then $\phi^{*-1}(\overline{V(\mathfrak{a})}) = \overline{V(\mathfrak{a}^e)}$.
 - iii) If \mathfrak{b} is an ideal of B , then $\phi^*(\overline{V(\mathfrak{b})}) = \overline{V(\mathfrak{b}^c)}$.
 - iv) If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\text{Ker}(\phi))$ of X . (In particular, $\text{Spec}(A)$ and $\text{Spec}(A/\mathfrak{N})$ (where \mathfrak{N} is the nilradical of A) are naturally homeomorphic.)
 - v) If ϕ is injective, then $\phi^*(Y)$ is dense in X . More precisely, $\phi^*(Y)$ is dense in $X \Leftrightarrow \text{Ker}(\phi) \subseteq \mathfrak{N}$.
 - vi) Let $\psi: B \rightarrow C$ be another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
 - vii) Let A be an integral domain with just one non-zero prime ideal \mathfrak{p} , and let K be the field of fractions of A . Let $B = (A/\mathfrak{p}) \times K$. Define $\phi: A \rightarrow B$ by $\phi(x) = (\bar{x}, x)$, where \bar{x} is the image of x in A/\mathfrak{p} . Show that ϕ^* is bijective but not a homeomorphism.

2. Let M be an A -module. If every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.

3 Chapter 6: Chain Conditions

Question 6.2.

Solution.

Suppose M is not Noetherian. Then, there exists a non finitely-generated submodule N of M . Choose x_1 from N . Then, since N is non finitely-generated, $N - \{x_1\} \neq \emptyset$, so we can choose $x_2 \neq x_1$ from N . Repeat this process inductively, then we have a nonempty set of finitely generated submodules of M

$$\{(x_1), (x_1, x_2), (x_1, x_2, x_3) \dots\}$$

such that it does not have a maximal element. Therefore, by contradiction, we have that M must be Noetherian. \square

4 Chapter 3: Rings and Modules of Fractions

Question 3.1.

1. Let S be a multiplicatively closed subset of a ring A , and let M be a finitely generated A -module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Solution.

Suppose there exists $s_0 \in S$ such that $s_0 M = 0$. Then, for any $(a, s), (b, t) \in A \times S$,

$$(at - bs)s_0 = 0$$

so $(a, s) \sim (b, t)$, and hence $S = 0$. Conversely, suppose $S^{-1}M = 0$. As M is finitely generated, there exists $\{x_1, \dots, x_m\}$ for some m that generates M . As $S^{-1}M = 0$, we can choose $\{s_1, \dots, s_m\}$ such that

$$s_i x_i = 0$$

for all $1 \leq i \leq m$. Consider $s^* = s_1 \dots s_m$. Then,

$$\begin{aligned} s^* x &= s_1 \dots s_m x = s_1 \dots s_m \left(\sum_{i=1}^m a_i x_i \right) \\ &= \sum_{i=1}^m s_1 \dots s_m a_i x_i = 0 \end{aligned}$$

for some $a_1, \dots, a_m \in A$, for all $x \in M$. Therefore, $s^* M = 0$. □

Question 3.2.

2. Let \mathfrak{a} be an ideal of a ring A , and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.

Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants. [If $M = \mathfrak{a}M$, then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$, hence by Nakayama we have $S^{-1}M = 0$. Now use Exercise 1.]

Solution.

If $1 + a, 1 + a' \in 1 + \mathfrak{a}$, then

$$(1 + a)(1 + a') = 1a + a' + aa' \in 1 + \mathfrak{a}.$$

Therefore $1 + \mathfrak{a}$ is a multiplicatively closed set in A . Now, for any $\in 1 + \mathfrak{a}$,

$$1 - 1$$

Question 3.5.

- 5. Let A be a ring. Suppose that, for each prime ideal \mathfrak{p} , the local ring $A_{\mathfrak{p}}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_{\mathfrak{p}}$ is an integral domain, is A necessarily an integral domain?**

Solution.

Let \mathfrak{N} be a nilradical of A . By Corollary 3.12, $(\mathfrak{N})_{\mathfrak{p}}$ is a nilradical of $A_{\mathfrak{p}}$, which is 0 by assumption, for any prime ideal \mathfrak{p} . Therefore, by proposition 3.8, $\mathfrak{N} = 0$.

Question 3.12.

12. Let A be an integral domain and M an A -module. An element $x \in M$ is a *torsion element* of M if $\text{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A . Show that the torsion elements of M form a submodule of M . This submodule is called the *torsion submodule* of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that
- i) If M is any A -module, then $M/T(M)$ is torsion-free.
 - ii) If $f: M \rightarrow N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.
 - iii) If $0 \rightarrow M' \rightarrow M \rightarrow M''$ is an exact sequence, then the sequence $0 \rightarrow T(M') \rightarrow T(M) \rightarrow T(M'')$ is exact.
 - iv) If M is any A -module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A .
- [For iv), show that K may be regarded as the direct limit of its submodules $A\xi$ ($\xi \in K$); using Chapter 1, Exercise 15 and Exercise 20, show that if $1 \otimes x = 0$ in $K \otimes M$ then $1 \otimes x = 0$ in $A\xi \otimes M$ for some $\xi \neq 0$. Deduce that $\xi^{-1}x = 0$.]

Solution.

We first show that $T(M)$ is a A -sub-module of M . Let $x, y \in T(M)$, and $r \in A$. Then, there exists $a, b \in A$, such that $a, b \neq 0$, $ax = 0$ and $by = 0$. Then, as A is an integral domain, $ab \neq 0$ and $ab(x + y) = abx + aby = bax + aby = 0$, so $x + y \in T(M)$. Also, $arx = rax = 0$, so $rx \in T(M)$ and $T(M)$ is a A -sub-module of M .

(i) To show that $M/T(M)$ is torsion-free, it suffices to show that $T(M/T(M)) = 0$.

(ii) Let $y \in f(T(M))$. Then, there exists $x \in T(M)$ such that $f(x) = y$, and $0 \neq a \in A$ with $ax = 0$. Therefore, $ay = af(x) = f(ax) = 0$, as f is a module homomorphism. Therefore, $f(T(M)) \subseteq T(N)$.

(iii)

Question 3.13.

13. Let S be a multiplicatively closed subset of an integral domain A . In the notation of Exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$. Deduce that the following are equivalent:
- i) M is torsion-free.
 - ii) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals \mathfrak{p} .
 - iii) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals \mathfrak{m} .

Solution.