Atiyah Commutative Algebra: Problems

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Abstract

This work contains solutions to some exercises from Atiyah's Commutative Algebra text.

1 Chapter 2: Modules

Question 2.1.

1. Show that $(\mathbf{Z}/m\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z}) = 0$ if m, n are coprime.

Solution.
As m, n are coprime, by Bezout, there exists $a, b \in \mathbb{Z}$ such that

$$am + bn = 1.$$

Therefore,

$$x \otimes y = am(x \otimes y) + bn(x \otimes y) = a(mx \otimes y) + b(x \otimes ny) = a(0 \otimes y) + b(x \otimes 0) = 0$$

for any $x \otimes y \in \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$. $\square$

Question 2.4.

4. Let M_i ($i \in I$) be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Solution.
Let $M = \bigoplus_{i \in I} M_i$. Then, M is exact iff for any exact sequence of $A-$modules

$$0 \to N' \to N \to N'' \to 0$$

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the tensored sequence

$$0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$$

is exact. Observe that the above tensored sequence can be written as

$$0 \to \bigoplus_{i \in I} N' \otimes M_i \to \bigoplus_{i \in I} N \otimes M_i \to \bigoplus_{i \in I} N'' \otimes M_i \to 0.$$

Hence M is exact iff for any $i \in I$ and for any exact sequence of $A-$modules

$$0 \to N' \to N \to N'' \to 0$$

the tensored sequence

$$0 \to N' \otimes M_i \to N \otimes M_i \to N'' \otimes M_i \to 0,$$

but the later is equivalent to M_i being flat for all $i \in I$, so we are done. □

Question 2.5.

5. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra. [Use Exercise 4.]

Solution.

2 Chapter 6: Chain Conditions

Question 6.2.

2. Let M be an A-module. If every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.

Solution.
Suppose M is not Noetherian. Then, there exists a non finitely-generated submodule N of M. Choose x_1 from N. Then, since N is non finitely-generated, $N - \{x_1\} \neq \emptyset$, so we can choose $x_2 \neq x_1$ from N. Repeat this process inductively, then we have a nonempty set of finitely generated submodules of M

$$\{(x_1), (x_1, x_2), (x_1, x_2, x_3) ...\}$$

such that it does not have a maximal element. Therefore, by contradiction, we have that M must be Noetherian. □

3 Chapter 3: Rings and Modules of Fractions

Question 3.1.

> **1.** Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Solution.
Suppose there exists $s_0 \in S$ such that $s_0 M = 0$. Then, for any $(a, s), (b, t) \in A \times S$,

$$(at - bs)s_0 = 0$$

so $(a, s) \sim (b, t)$, and hence $S = 0$. Conversely, suppose $S^{-1}M = 0$. As M is finitely generated, there exists $\{x_1, ..., x_m\}$ for some m that generates M. As $S^{-1}M = 0$, we can choose $\{s_1, ..., s_m\}$ such that

$$s_i x_i = 0$$

for all $1 \leq i \leq m$. Consider $s^* = s_1 ... s_m$. Then,

$$\begin{aligned} s^* x &= s_1 ... s_m x = s_1 ... s_m (\sum_{i=1}^{m} a_i x_i) \\ &= \sum_{i=1}^{m} s_1 ... s_m a_i x_i = 0 \end{aligned}$$

for some $a_1, ... a_m \in A$, for all $x \in M$. Therefore, $s^* M = 0$. □

Question 3.2.

> **2.** Let $\mathfrak{a}$ be an ideal of a ring A, and let $S = 1 + \mathfrak{a}$. Show that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$.
>
> Use this result and Nakayama's lemma to give a proof of (2.5) which does not depend on determinants. [If $M = \mathfrak{a}M$, then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$, hence by Nakayama we have $S^{-1}M = 0$. Now use Exercise 1.]

Solution.
If $1 + a, 1 + a' \in 1 + \mathbf{a}$, then

$$(1 + a)(1 + a') = 1a + a' + aa' \in 1 + \mathbf{a}.$$

Therfore $1 + \mathbf{a}$ is a multiplicatively closed set in A. Now, for any $\in 1 + \mathbf{a}$,

$$1 - 1$$

12. Let A be an integral domain and M an A-module. An element $x \in M$ is a *torsion element* of M if $\mathrm{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule of M. This submodule is called the *torsion submodule* of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

i) If M is any A-module, then $M/T(M)$ is torsion-free.

ii) If $f: M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

iii) If $0 \to M' \to M \to M''$ is an exact sequence, then the sequence $0 \to T(M') \to T(M) \to T(M'')$ is exact.

iv) If M is any A-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A.

[For iv), show that K may be regarded as the direct limit of its submodules $A\xi$ ($\xi \in K$); using Chapter 1, Exercise 15 and Exercise 20, show that if $1 \otimes x = 0$ in $K \otimes M$ then $1 \otimes x = 0$ in $A\xi \otimes M$ for some $\xi \neq 0$. Deduce that $\xi^{-1}x = 0$.]

Question 3.12.

Solution.
If $1+a, 1+a' \in 1+\mathbf{a}$, then

$$(1+a)(1+a') \quad = \quad 1a + a' + aa' \in 1+\mathbf{a}.$$

Therfore $1+\mathbf{a}$ is a multiplicatively closed set in A. Now, for any $\in 1+\mathbf{a}$,

$$1-1$$

ddd

Question 3.13.

13. Let S be a multiplicatively closed subset of an integral domain A. In the notation of Exercise 12, show that $T(S^{-1}M) = S^{-1}(TM)$. Deduce that the following are equivalent:

i) M is torsion-free.

ii) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals $\mathfrak{p}$.

iii) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

Solution.