Complex Analysis I:
Problem Set VIII

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Abstract

This work contains the solutions to the problem set VIII of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 273-12.
Solution.

Question 278-1.

Solution. Let $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$. On the indented contour in figure 108, by Cauchy-Goursat theorem, we obtain

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = -\int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Observe that on L_1 and L_2, we have the following parametric representations:

$$L_1 : z = re^{i0} = r(\rho \leq r \leq R) \text{ and } -L_2 : z = re^{i\pi} = -r(\rho \leq r \leq R).$$

It follows that

$$\begin{aligned}
\int_{L_1} f(z)dz + \int_{L_2} f(z)dz &= \int_{L_1} f(z)dz - \int_{-L_2} f(z)dz \\
&= \int_\rho^R \frac{e^{iar} - e^{ibr}}{r^2} dr + \int_\rho^R \frac{e^{-iar} - e^{-ibr}}{r^2} dr \\
&= \int_\rho^R \frac{(e^{iar} + e^{-iar}) - (e^{ibr} + e^{-ibr})}{r^2} dr \\
&= 2\int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2} dr.
\end{aligned}$$

Therefore, we obtain

Question 287-1.

Solution. Observe that the following equality holds, by the linearity of integration:

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{1}{5}\int_0^{2\pi} \frac{d\theta}{1 + \frac{4}{5}\sin\theta}$$

From the example 1 from pg.285 in the section 92, it follows that

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} &= \frac{1}{5}\int_0^{2\pi} \frac{d\theta}{1+\frac{4}{5}\sin\theta} \\
&= \frac{1}{5}\frac{2\pi}{\sqrt{1-\frac{4}{5}^2}} \\
&= \frac{1}{5}\frac{2\pi}{\frac{3}{5}} \\
&= \frac{2}{3}\pi.
\end{aligned}$$

□

Question 287-6.

Solution. As $|\sin(-\pi+\theta)| = |\sin(\theta)|$, we have $\sin^{2n}(\theta) = \sin^{2n}(-\pi+\theta)$, and $\int_0^\pi \sin^{2n}(\theta)d\theta = \int_{-\pi}^0 \sin^{2n}(\theta)d\theta$. Hence, it follows that

$$\int_0^\pi \sin^{2n}(\theta)d\theta = \frac{1}{2}\int_C \sin^{2n}(\theta),$$

where C is the positively oriented unit circle $|z| = 1$. Substituting $\dfrac{z-z^{-1}}{2i}$ for $\sin(\theta)$, using the binomial formula, and linearity of integration, we obtain

$$\begin{aligned}
\frac{1}{2}\int_C \sin^{2n}(\theta) &= \frac{1}{2}\int_C \left(\frac{z-z^{-1}}{2i}\right)^{2n}\frac{dz}{iz} \\
&= \frac{1}{2^{2n+1}(-1)^n i}\int_C \frac{(z-z^{-1})^{2n}}{z}dz \\
&= \frac{1}{2^{2n+1}(-1)^n i}\int_C \sum_{k=0}^{n}\binom{2n}{k} z^k z^{2n-k}\left(-z^{-1}\right)^k z^{-1}dz \\
&= \frac{1}{2^{2n+1}(-1)^n i}\sum_{k=0}^{n}\binom{2n}{k}(-1)^k\int_C z^{2n-2k-1}dz.
\end{aligned}$$

Observe that we only get non-zero integral value for $k = n$ case, and $\int_C z^{-1}dz = 2\pi i$. Therefore, it follows that

$$\begin{aligned}
\int_0^\pi \sin^{2n}(\theta)d\theta &= \frac{1}{2^{2n+1}(-1)^n i}\frac{(2n)!}{n!n!}(-1)^n 2\pi i \\
&= \frac{2n!}{2^{2n}(n!)^2}\pi,
\end{aligned}$$

as desired. □

Question 293-6.

Solution. (a) Inside the circle $|z| = 1$, write

$$f(z) = -5z^4 \ \text{ and } \ g(z) = z^6 + z^3 - 2z.$$

Then, observe that when $|z| = 1$,

$$|f(z)| = 5|z|^4 = 5 \ \text{ and } \ |g(z)| \le |z|^6 + |z|^3 + 2|z| = 4.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 4 zeroes, counting multiplicities, inside the circle $|z| = 1$, $f(z) + g(z)$ has 4 zeroes, inside the circle $|z| = 1$. Therefore, the polynomial $z^6 - 5z^4 + z^3 - 2z$ has 4 zeroes, inside the circle $|z| = 1$. □

(b) Inside the circle $|z| = 1$, write

$$f(z) = 9 \text{ and } g(z) = 2z^4 - 2z^3 + 2z^2 - 2z.$$

Then, observe that when $|z| = 1$,

$$|f(z)| = 9 \text{ and } |g(z)| = 2|z|^4 + 2|z|^3 + 2|z|^2 + 2|z| = 8.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 0 zero, counting multiplicities, inside the circle $|z| = 1$, $f(z) + g(z)$ has 0 zero, inside the circle $|z| = 1$. Therefore, the polynomial $2z^4 - 2z^3 + 2z^2 - 2z + 9$ has 0 zero, inside the circle $|z| = 1$. □

(c) Inside the circle $|z| = 1$, write

$$f(z) = -4z^3 \text{ and } g(z) = z^7 + z - 1.$$

Then, observe that when $|z| = 1$,

$$|f(z)| = 4|z|^3 = 4 \text{ and } |g(z)| \leq |z|^7 + |z| - 1 = 1.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 3 zeroes, counting multiplicities, inside the circle $|z| = 1$, $f(z)+g(z)$ has 3 zeroes. Therefore, the polynomial $z^7 - 4z^3 + z - 1$ has 3 zeroes inside the circle $|z| = 1$. □

Question 293-8.

Solution. Inside the circle $|z| = 2$, write

$$f(z) = 2z^5 \text{ and } g(z) = 6z^2 + z + 1.$$

Then, observe that when $|z| = 2$,

$$|f(z)| = 2|z|^5 = 64 \text{ and } |g(z)| \leq 6|z|^2 + |z| + |1| = 8.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 5 zeroes, counting multiplicities, inside the circle $|z| = 2$, $f(z) + g(z)$ has 5 zeroes. On the other hand, inside the circle $|z| = 1$, write

$$f(z) = -6z^2 \text{ and } g(z) = 2z^5 + z + 1.$$

Then, observe that when $|z| = 1$,

$$|f(z)| = 6|z|^2 = 6 \text{ and } |g(z)| \leq 2|z|^5 + |z| + 1 = 4.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 2 zeroes, counting multiplicities, inside the circle $|z| = 1$, $f(z) + g(z)$ has 2 zeroes. Therefore, we have shown that in the annulus $1 \leq |z| \leq 2$, we have $5 - 2 = 3$ zeroes. □