Complex Analysis I: Problem Set VIII

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Abstract

This work contains the solutions to the problem set VIII of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 273-12.

Solution. (a) As $\exp(iz^2)$ is entire, by Cauchy-Goursat theorem, we have

$$\int_{C_1} \exp(iz^2)dz + \int_{C_R} \exp(iz^2)dz + \int_{C_2} \exp(iz^2)dz \quad = \quad 0,$$

where C_1 refers to the segment from 0 to R, and C_2 refers to the segment from $Re^{i\frac{\pi}{4}}$ to 0. Observe that we can parametrize points on C_1 as $z = x(0 \leq x \leq R)$, and points on C_2 as $z = re^{i\frac{\pi}{4}}(0 \leq r \leq R)$. Therefore, the above equality can be written as

$$\int_0^R \exp(ix^2)dx \quad = \quad e^{i\frac{\pi}{4}} \int_0^R e^{-r^2}dr - \int_{C_R} e^{iz^2}dz.$$

Equating the real and imaginary parts separately from the above equation yields

$$\begin{aligned}
\int_0^R \cos(x^2)dx \quad &= \quad \frac{1}{\sqrt{2}} \int_0^R e^{-r^2}dr - \text{Re} \int_{C_R} e^{iz^2}dz, \\
\int_0^R \sin(x^2)dx \quad &= \quad \frac{1}{\sqrt{2}} \int_0^R e^{-r^2}dr - \text{Im} \int_{C_R} e^{iz^z}dz.
\end{aligned}$$

□

(b) We can parametrize the points on C_R as $z = Re^{i\theta}(0 \leq \theta \leq \frac{\pi}{4})$. It follows that

$$\begin{aligned}
\int_{C_R} e^{iz^2} \quad &= \quad \int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}} Rie^{i\theta}d\theta \\
&= \quad iR \int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)} e^{iR^2\cos(2\theta)} e^{i\theta}d\theta.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\left|\int_{C_R} e^{iz^2}\right| &= \left|iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}e^{iR^2\cos(2\theta)}e^{i\theta}d\theta\right| \\
&\leq R\int_0^{\frac{\pi}{4}} \left|e^{-R^2\sin(2\theta)}\right|\left|e^{iR^2\cos(2\theta)}\right|\left|e^{i\theta}\right|d\theta \\
&= R\int_0^{\frac{\pi}{4}} \left|e^{-R^2\sin(2\theta)}\right|d\theta \\
&= R\int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}d\theta \\
&= \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\sin(\phi)}d\theta,
\end{aligned}$$

by change of variable $\phi = 2\theta$. From Jordan's lemma in section 88, pg. 271, continues to

$$\begin{aligned}
\left|\int_{C_R} e^{iz^2}\right| &\leq \frac{R}{2}\frac{\pi}{2R^2} \\
&= \frac{\pi}{4R},
\end{aligned}$$

which limits 0 as $R \to \infty$. Hence, the integral limits 0 as $R \to \infty$. □

(c) Combining the result from (a),(b) and the given formula, we have

$$\begin{aligned}
\int_0^\infty \cos(x^2)dx &= \frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}dx \\
&= \frac{1}{2}\sqrt{\frac{\pi}{2}}, \\
\int_0^\infty \sin(x^2)dx &= \frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}dx \\
&= \frac{1}{2}\sqrt{\frac{\pi}{2}},
\end{aligned}$$

as desired. □

Question 278-1.

Solution. Let $f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2}$. On the indented contour in figure 108, by Cauchy-Goursat theorem, we obtain

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = -\int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Observe that on L_1 and L_2, we have the following parametric representations:

$$L_1 : z = re^{i0} = r(\rho \leq r \leq R) \text{ and } -L_2 : z = re^{i\pi} = -r(\rho \leq r \leq R).$$

It follows that

$$\begin{aligned}
\int_{L_1} f(z)dz + \int_{L_2} f(z)dz &= \int_{L_1} f(z)dz - \int_{-L_2} f(z)dz \\
&= \int_\rho^R \frac{e^{iar} - e^{ibr}}{r^2}dr + \int_\rho^R \frac{e^{-iar} - e^{-ibr}}{r^2}dr \\
&= \int_\rho^R \frac{(e^{iar} + e^{-iar}) - (e^{ibr} + e^{-ibr})}{r^2}dr \\
&= 2\int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2}dr.
\end{aligned}$$

Now, writing the Laurent series of $f(z)$ gives

$$\begin{aligned} f(z) &= \frac{e^{iaz}-e^{ibz}}{z^2} \\ &= \frac{1}{z^2}\left[\left(1+\frac{iaz}{1!}+\frac{(iaz)^2}{2!}...\right)+\left(1+\frac{ibz}{1!}+\frac{(ibz)^2}{2!}...\right)\right] \\ &= \frac{i(a-b)}{z}+... \end{aligned}$$

Therefore, we see that $z=0$ is a simple pole of $f(z)$ and the residue B_0 equals $i(a-b)$. Consequently, we obtain that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz = -i(a-b)\pi i = \pi(a-b).$$

Observe that

$$|f(z)| \le \frac{|e^{iaz}|+e^{ibz}}{|z|^2}.$$

For z on C_R, we have

$$|f(z)| \le \frac{2}{R^2}.$$

Therefore, by ML inequality, it follows that

$$\begin{aligned} \left|\int_{C_R} f(z)dz\right| &\le \int_{C_R}|f(z)|\,dz \\ &\le \frac{2}{R^2}\pi R \\ &= \frac{2\pi}{R}. \end{aligned}$$

Therefore, as $R\to\infty$ $\dfrac{2\pi}{R}\to 0$, the integral $\int_{C_R} f(z)dz$ tends to 0 as $R\to\infty$. Consequently,

$$\int_0^\infty \frac{\cos(ar)-\cos(br)}{r^2}dr = \frac{\pi}{2}(b-a).$$

Setting $b=2$ and $a=0$ and using the identity $1-\cos(2x)=2\sin^2(x)$, we finally obtain

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2},$$

as desired. $\square$

Question 283-2.

Solution. We use the following branch as suggested:

$$\begin{aligned} f(z) &= \frac{z^{\frac{-1}{2}}}{z^2+1} \\ &= \frac{e^{-\frac{1}{2}\log(z)}}{z^2+1}(|z|>0, -\frac{\pi}{2}<\arg(z)<\frac{3\pi}{2}). \end{aligned}$$

We integrate the branch over the indented path in figure 109. Starting with Cauchy's residue theorem, we write

$$\int_{L_1} f(z)dz+\int_{L_2} f(z)dz = 2\pi i\mathrm{Res}_{z=i}f(z)-\int_{C_\rho} f(z)dz-\int_{C_R} f(z)dz.$$

By using the parametrizations of

$$\begin{aligned} L_1 : z &= re^{i0} = r(\rho \leq r \leq R), \\ -L_2 : z &= re^{i\pi} = -r(\rho \leq r \leq R), \end{aligned}$$

we obtain

$$\begin{aligned} \int_{L_1} f(z)dz + \int_{L_2} f(z)dz &= \int_{\rho}^{R} \frac{1}{\sqrt{r}(r^2+1)}dr - i\int_{\rho}^{R} \frac{1}{\sqrt{r}(r^2+1)}dr \\ &= (1-i)\int_{\rho}^{R} \frac{1}{\sqrt{r}(r^2+1)}dr. \end{aligned}$$

Observe that $z = i$ is a simple pole of $f(z)$, as it can be written that $f(z) = \dfrac{z^{-\frac{1}{2}}}{(z+i)(z-i)}$, and $\dfrac{z^{-\frac{1}{2}}}{(z+i)}$ is analytic and nonzero at $z = i$. Therefore, the residue at $z = i$ can be computed by

$$\begin{aligned} \text{Res}_{z=i} f(z) &= \frac{z^{-\frac{1}{2}}}{z+i}]_{z=i} \\ &= \frac{exp(-\frac{1}{2}(\ln(1) + i\frac{\pi}{2}))}{2i} \\ &= \frac{e^{-i\frac{\pi}{4}}}{2i} \\ &= \frac{1}{2i}(\frac{1-i}{\sqrt{2}}). \end{aligned}$$

For the integral terms on C_ρ and C_R, by the ML inequality, we have

$$\begin{aligned} \left|\int_{C_\rho} f(z)dz\right| &\leq \int_{C_\rho} |f(z)|dz \\ &\leq \frac{\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{\pi\sqrt{\rho}}{1-\rho^2}, \\ \left|\int_{C_R} f(z)dz\right| &\leq \int_{C_R} |f(z)|dz \\ &\leq \frac{\pi\sqrt{R}}{(R^2-1)}. \end{aligned}$$

As $R \to \infty$ and $\rho \to 0$, we have $\dfrac{\pi\sqrt{R}}{(R^2-1)} \to 0$, and $\dfrac{\pi\sqrt{\rho}}{1-\rho^2} \to 0$ respectively. Hence, it follows that

$$(1-i)\int_0^\infty \frac{1}{\sqrt{r}(r^2+1)}dr = \frac{\pi(1-i)}{\sqrt{2}},$$

which can be re-written as

$$\int_0^\infty \frac{1}{\sqrt{r}(r^2+1)}dr = \frac{\pi}{\sqrt{2}},$$

as desired □

Question 283-4.

Solution. We use the branch $f(z) = \dfrac{\exp(\frac{1}{3}\log(z))}{(z+a)(z+b)}$, for $|z| > 0, 0 < \arg(z) < 2\pi$ over the contour on section 91. By the reisude theorem, we have

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz + \int_{C_\rho} f(z)dz + \int_{C_R} f(z)dz = 2\pi i(B_1 + B_2),$$

where B_1 and B_2 refer to the residues from $-a$ and $-b$ respectively. Firstly, we show that The integrals over the circular paths goes to 0 as follows:

$$\begin{aligned}\left|\int_{C_\rho} f(z)dz\right| &\leq \frac{\rho^{\frac{1}{3}}}{(a-\rho)(b-\rho)}2\pi\rho \\ &= \frac{2\pi\rho^{\frac{1}{3}}\rho}{(a-\rho)(b-\rho)}.\end{aligned}$$

As $\rho \to 0$, we see that the upper bound limits to 0. Therefore, the integral goes to 0. By the same computation, we also see that the integral over C_R. It follows that

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 2\pi i(B_1 + B_2).$$

The residues can be computed as follows:

$$\begin{aligned}B_1 &= \text{Res}_{z=-a} f(z) \\ &= \frac{\exp(\frac{1}{3}\log(-b))}{-a+b} \\ &= -\frac{\exp(\frac{1}{3}(\ln(a)+i\pi))}{a-b} \\ &= -\frac{e^{i\frac{\pi}{3}}a^{\frac{1}{3}}}{a-b} \\ B_2 &= \frac{e^{i\frac{\pi}{3}}b^{\frac{1}{3}}}{a-b},\end{aligned}$$

by symmetry. Using the $z = r$ parametrization on L_1 and L_2, we obtain

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = \int_\rho^R (1 - e^{i2\frac{\pi}{3}})\frac{r^{\frac{1}{3}}}{(r+a)(r+b)}dr.$$

Combining all the results, we obtain

$$\int_0^\infty \frac{x^{\frac{1}{3}}}{(x+a)(x+b)}dx = \frac{2\pi}{\sqrt{3}}\frac{a^{\frac{1}{3}} - b^{\frac{1}{3}}}{a-b}.$$

Question 283-5.

Solution. First, as before, we have

$$\int_0^\pi \frac{d\theta}{(a+\cos(\theta))^2} = \frac{1}{2}\int_{-\pi}^\pi \frac{d\theta}{(a+\cos(\theta))^2}.$$

We substitute $\cos(\theta) = \dfrac{z+z^{-1}}{2}$, and $d\theta = \dfrac{1}{iz}dz$ to the last integral. Then, it follows that

$$\begin{aligned}\int_{-\pi}^\pi \frac{d\theta}{(a+\cos(\theta))^2} &= \int_C \frac{1}{\left(a + \frac{z+z^{-1}}{2}\right)^2}\frac{1}{iz}dz \\ &= \frac{4}{i}\int_C \frac{z}{(2az+z^2+1)^2}dz,\end{aligned}$$

where C denotes the contour of the unit circle. Using the quadratic formulat, we obtain that the integrand has a pole of order 2 at $z = -a \pm \sqrt{a^2-1}$. As $a > 1$, we have that $z = -a - \sqrt{a^2-1}$ is the only singular point in C, which results in the residue of $\dfrac{a}{4(\sqrt{a^2-1})^3}$. Therefore, by the residue theorem, we have

$$\begin{aligned}\int_0^\pi \frac{d\theta}{(a+\cos(\theta))^2} &= \frac{1}{2}\frac{4}{i}\frac{2\pi ia}{4\sqrt{a^2-1}^3} \\ &= \frac{\pi a}{(\sqrt{a^2-1})^3},\end{aligned}$$

as desired. $\square$

Question 287-6.

Solution. Observe that the following equality holds, by the linearity of integration:

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{1}{5}\int_0^{2\pi} \frac{d\theta}{1+\frac{4}{5}\sin\theta}$$

From the example 1 from pg.285 in the section 92, it follows that

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} &= \frac{1}{5}\int_0^{2\pi} \frac{d\theta}{1+\frac{4}{5}\sin\theta} \\
&= \frac{1}{5}\frac{2\pi}{\sqrt{1-\frac{4}{5}^2}} \\
&= \frac{1}{5}\frac{2\pi}{\frac{3}{5}} \\
&= \frac{2}{3}\pi.
\end{aligned}$$

□

Question 287-6.

Solution. As $|\sin(-\pi+\theta)| = |\sin(\theta)|$, we have $\sin^{2n}(\theta) = \sin^{2n}(-\pi+\theta)$, and $\int_0^{\pi}\sin^{2n}(\theta)d\theta = \int_{-\pi}^{0}\sin^{2n}(\theta)d\theta$. Hence, it follows that

$$\int_0^{\pi}\sin^{2n}(\theta)d\theta = \frac{1}{2}\int_C \sin^{2n}(\theta),$$

where C is the positively oriented unit circle $|z| = 1$. Substituting $\dfrac{z-z^{-1}}{2i}$ for $\sin(\theta)$, using the binomial formula, and linearity of integration, we obtain

$$\begin{aligned}
\frac{1}{2}\int_C \sin^{2n}(\theta) &= \frac{1}{2}\int_C \left(\frac{z-z^{-1}}{2i}\right)^{2n}\frac{dz}{iz} \\
&= \frac{1}{2^{2n+1}(-1)^n i}\int_C \frac{(z-z^{-1})^{2n}}{z}dz \\
&= \frac{1}{2^{2n+1}(-1)^n i}\int_C \sum_{k=0}^{n}\binom{2n}{k} z^k z^{2n-k}\left(-z^{-1}\right)^k z^{-1}dz \\
&= \frac{1}{2^{2n+1}(-1)^n i}\sum_{k=0}^{n}\binom{2n}{k}(-1)^k\int_C z^{2n-2k-1}dz.
\end{aligned}$$

Observe that we only get non-zero integral value for $k = n$ case, and $\int_C z^{-1}dz = 2\pi i$. Therefore, it follows that

$$\begin{aligned}
\int_0^{\pi}\sin^{2n}(\theta)d\theta &= \frac{1}{2^{2n+1}(-1)^n i}\frac{(2n)!}{n!n!}(-1)^n 2\pi i \\
&= \frac{2n!}{2^{2n}(n!)^2}\pi,
\end{aligned}$$

as desired. □

Question 293-6.

Solution. (a) Inside the circle $|z| = 1$, write

$$f(z) = -5z^4 \ \text{ and } \ g(z) = z^6 + z^3 - 2z.$$

Then, observe that when $|z| = 1$,

$$|f(z)| = 5|z|^4 = 5 \text{ and } |g(z)| \leq |z|^6 + |z|^3 + 2|z| = 4.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 4 zeroes, counting multiplicities, inside the circle $|z| = 1$, $f(z) + g(z)$ has 4 zeroes, inside the circle $|z| = 1$. Therefore, the polynomial $z^6 - 5z^4 + z^3 - 2z$ has 4 zeroes, inside the circle $|z| = 1$. $\square$

(b) Inside the circle $|z| = 1$, write

$$f(z) = 9 \text{ and } g(z) = 2z^4 - 2z^3 + 2z^2 - 2z.$$

Then, observe that when $|z| = 1$,

$$|f(z)| = 9 \text{ and } |g(z)| = 2|z|^4 + 2|z|^3 + 2|z|^2 + 2|z| = 8.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 0 zero, counting multiplicities, inside the circle $|z| = 1$, $f(z) + g(z)$ has 0 zero, inside the circle $|z| = 1$. Therefore, the polynomial $2z^4 - 2z^3 + 2z^2 - 2z + 9$ has 0 zero, inside the circle $|z| = 1$. $\square$

(c) Inside the circle $|z| = 1$, write

$$f(z) = -4z^3 \text{ and } g(z) = z^7 + z - 1.$$

Then, observe that when $|z| = 1$,

$$|f(z)| = 4|z|^3 = 4 \text{ and } |g(z)| \leq |z|^7 + |z| - 1 = 1.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 3 zeroes, counting multiplicities, inside the circle $|z| = 1$, $f(z)+g(z)$ has 3 zeroes. Therefore, the polynomial $z^7 - 4z^3 + z - 1$ has 3 zeroes inside the circle $|z| = 1$. $\square$

Question 293-8.

Solution. Inside the circle $|z| = 2$, write

$$f(z) = 2z^5 \text{ and } g(z) = 6z^2 + z + 1.$$

Then, observe that when $|z| = 2$,

$$|f(z)| = 2|z|^5 = 64 \text{ and } |g(z)| \leq 6|z|^2 + |z| + |1| = 8.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 5 zeroes, counting multiplicities, inside the circle $|z| = 2$, $f(z) + g(z)$ has 5 zeroes. On the other hand, inside the circle $|z| = 1$, write

$$f(z) = -6z^2 \text{ and } g(z) = 2z^5 + z + 1.$$

Then, observe that when $|z| = 1$,

$$|f(z)| = 6|z|^2 = 6 \text{ and } |g(z)| \leq 2|z|^5 + |z| + 1 = 4.$$

The conditions of Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has 2 zeroes, counting multiplicities, inside the circle $|z| = 1$, $f(z) + g(z)$ has 2 zeroes. Therefore, we have shown that in the annulus $1 \leq |z| \leq 2$, we have $5 - 2 = 3$ zeroes. $\square$