Complex Analysis I:
Problem Set V

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Abstract

This work contains the solutions to the problem set V of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 177.2.

Solution. Let f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assume that $f(z) \neq 0$ for $z \in R$. Let g be a function on R, defined by $g(z) = \dfrac{1}{f(z)}$ for $z \in R$. From the $g(z) = \dfrac{1}{f(z)}$ relation, we can deduce that g is also continuous, analytic and not constant throughout the interior of R. Then, by the given corollary of the maximum modulus principle, we have that the maximum value of $|g(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior. Observe that $|g(z)| = |\dfrac{1}{f(z)}| = \dfrac{1}{|f(z)|}$. Since a modulus is strictly positive in this case, we have that maximum value of $|g(z)|$ corresponds to the minimum value of $|f(z)|$. In other words, the z^*, which is argmax$|g(z)|$ and lies on the boundary, is also the argmin$|f(z)|$. Consequently, we have shown that a minimum value is reached, and it occurs in the boundary of R and never in the interior. $\square$

Question 2. 177.4.

Solution. From the given hint, we have that

$$|f(z)|^2 \quad = \quad sin^2(x) + sinh^2(y).$$

Observe that it reaches maximum with respect to x on $\dfrac{\pi}{2}$ and with respect to y on 1, simply from the known properties of sin and $sinh$ functions. Also, $(\dfrac{\pi}{2}, 1)$ is a feasible point. Hence, we obtain that $|f(x)|^2$ reaches its maximum at $\dfrac{\pi}{2} + i$ on the boundary. $\square$

Question 3. 177.5.

Solution. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and nont constant throughout the interior of R.

Question 4. 195.3.

Solution. We wish to find the Maclaurin series expansion of the function

$$f(z) \quad = \quad \frac{z}{z^4 + 4} = \frac{z}{4} \cdot \frac{1}{1 + (z^4/4)}.$$

From the geometric series, we have

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k,$$

for $|z| < 1$. Hence, by a change of variable, we have

$$\begin{aligned}\frac{1}{1+(\frac{z^4}{4})} &= \sum_{k=0}^{\infty}(-\frac{z^4}{4})^k \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{4^k}z^{4k},\end{aligned}$$

for $|z| < \sqrt{2}$. It follows that

$$\begin{aligned}f(z) &= \frac{z}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k}z^{4k} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{4^{k+1}}z^{4k+1} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k+2}}z^{4k+1},\end{aligned}$$

for $|z| < \sqrt{2}$ as desired. $\square$

Question 5. 195.6.

Solution. Observe that we can write $tanh$ as

$$tanh = \frac{sinh}{cosh}.$$

Observe that singularity happens at $cosh = 0$, which entails $z = (\frac{\pi}{2} + n\pi)i$. Therefore, we have analyticity for $|z| < \frac{\pi}{2}$, which is the largest circle within which the Maclaurin series is defined. Taking derivatives of $tanh$ yields

$$\begin{aligned}tanh'(z) &= \frac{1}{cosh^2(x)} \\ tanh''(z) &= -2\frac{sinh(x)}{cosh^3(x)} \\ tanh'''(z) &= -2\frac{(1-2sinh(x))}{cosh^4(x)}.\end{aligned}$$

Substituting 0 into x, we get

$$tanh(z) = z - \frac{1}{3}z^3 + ...$$

as desired for the first two nonzero terms of the series. $\square$

Question 6. 195.11.

Solution. Observe that

$$\frac{1}{4z-z^2} = \frac{1}{4z}\cdot\frac{1}{1-\frac{z}{4}}.$$

By the geometric series, we have

$$\frac{1}{1-\frac{z}{4}} = \sum_{k=0}^{\infty} \frac{z^k}{4^k},$$

for $|z| < 4$. It follows that

$$\begin{aligned}
\frac{1}{4z - z^2} &= \frac{1}{4z} \sum_{k=0}^{\infty} \frac{z^k}{4^k} \\
&= \frac{1}{4z} + \sum_{k=0}^{\infty} \frac{z^k}{4^{k+2}},
\end{aligned}$$

for $|z| < 4$ as desired. $\square$

Question 8. 205.6.

Solution. By partial fraction decomposition, we have

$$\frac{z}{(z-1)(z-3)} = \frac{3}{2} \cdot \frac{1}{z-3} - \frac{1}{2} \cdot \frac{1}{z-1}.$$

The above equality can be written as

$$\frac{z}{(z-1)(z-3)} = -\frac{3}{4} \cdot \frac{1}{1-\frac{z-1}{2}} + \frac{1}{2} \cdot \frac{1}{1-z}.$$

Since $0 < |z-1| < 2$, we have $0 < |\frac{z-1}{2}| < 1$. Therefore, by the geometric series, we have

$$\begin{aligned}
\frac{z}{(z-1)(z-3)} &= -\frac{3}{4} \sum_{k=0}^{\infty} \frac{(z-1)^k}{2^k} - \frac{1}{2(z-1)} \\
&= -3 \sum_{k=0}^{\infty} \frac{(z-1)^k}{2^{k+2}} - \frac{1}{2(z-1)},
\end{aligned}$$

for $0 < |z-1| < 2$ as desired. $\square$

Question 10. 224.1.

Solution. Observe that

$$\begin{aligned}
e^z &= \sum_{k=0}^{\infty} \frac{1}{k!} z^k \\
\frac{1}{z(z^2+1)} &= \frac{1}{z} \sum_{k=0}^{\infty} (-z^2)^k, \\
&= \sum_{k=0}^{\infty} (-1)^k z^{2k+1}
\end{aligned}$$

$|z| < 1$. By multiplying out the first few terms manually, we obtain

$$\frac{e^z}{z^2+1} = \frac{1}{z} + 1 - \frac{1}{2} z - \frac{5}{6} z^2 + \ldots,$$

for $|z| < 1$. $\square$

Question 11. 224.3.

Solution.

Question 12. 224.5.

Solution. From the Laurent series theorem, we have that

$$b_1 = \frac{1}{2\pi i}\int_C \frac{1}{z^2 \sinh z}dz,$$

where b_1 is the coefficient of the Laurent series for the $\frac{1}{z}$ and C is the positively oriented unit circle $|z| = 1$. Hence, by the given Laurent series, we obtain

$$-\frac{1}{6} = \frac{1}{2\pi i}\int_C \frac{1}{z^2 \sinh z}.$$

Re-arranging the terms yields

$$\int_C \frac{1}{z^2 \sinh z}dz = -\frac{\pi}{3},$$

where C is the positively oriented unit circle $|z| = 1$. $\square$

Question 13. 224.8.

Solution.

Question 14. 224.9.

Solution.