Complex Analysis I: Problem Set I

Youngduck Choi
CILVR Lab
New York University
yc1104@nyu.edu

Abstract

This work contains the solutions to the problem set I of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. Brown p.79-1.

Solution. The Cauchy-Riemann equation in polar coordinates can be given by

$$u_\theta = -rv_r \text{ and } ru_r = v_\theta.$$

Differentiating the first equation with respect to the θ parameter, we obtain

$$u_{\theta\theta} = -rv_{r\theta}. \tag{1}$$

Differentiating the second equation with respect to the r parameter, we have

$$v_{\theta r} = u_r + ru_{rr}. \tag{2}$$

Multiplying (2) by r and subtracting with (1), we obtain

$$-u_{\theta\theta} + rv_{\theta r} = rv_{r\theta} + ru_{rr} + r^2 u_{rr}.$$

As $v_{r\theta} = v_{\theta r}$ by Fubini's theorem, we obtain

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0.$$

Therefore, we have derived the polar form of Laplace's equation. Now, we wish to verify that the same equation holds for v. Consider the expression $r^2 v_{rr} + rv_r + v_{\theta\theta}$. Differentiating $v_r = -\frac{1}{r}u_\theta$ with respect to r, we have

$$v_{rr} = \frac{1}{r^2}u_\theta - \frac{1}{r}u_{\theta r}.$$

Differentiating $v_\theta = ru_r$ with respect to θ, we have

$$v_{\theta\theta} = ru_{r\theta}.$$

Substituting the last two equations into the considered expression, we have

$$u_\theta - ru_{\theta r} + u_\theta + ru_{r\theta},$$

which again by Fubini equals 0. $\square$

Question 1-2. Brown p.79-2.

Solution. Let the function $f(z) = u(x,y) + iv(x,y)$ be analytic on a domain D, and consider a point $z_0 = (x_0, y_0) = (c_1, c_2)$. Assume that $f'(z_0) \neq 0$. For the level curve of $u(x,y) = c_1$, by implicit differentiation with respect to x, we obtain

$$u_x + u_y \frac{dy}{dx} = 0.$$

As $f'(z_0) \neq 0$, the above equation can be simplified to

$$\frac{dy}{dx} = -\frac{u_x}{u_y}.$$

Analogously, for the corresponding level curve of $v(x,y) = c_2$, by implicit differentiation with respect to x, we obtain

$$v_x + v_y \frac{dy}{dx} = 0,$$

As $f'(z_0) \neq 0$, the above equation can be simplified to

$$\frac{dy}{dx} = -\frac{v_x}{v_y}.$$

Now, as the function is analytic in a domain D, the Cauchy-Riemann equation must hold at z_0. Hence, the last equation can be extened as

$$\frac{dy}{dx} = -\frac{v_x}{v_y} = \frac{u_y}{u_x}.$$

Therefore, we have shown that the level curves of the component functions at the analytic domain D are orthogonal. $\square$

Question 1-3. Brown p.79-4.

Solution.

Question 2. Conjugate Harmonic Function.

Solution. We are given a polynomial of the following form:

$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3.$$

Differentiating the given polynomial $u(x,y)$ with respect to x and y variables, we obtain

$$\frac{\partial u}{\partial x} = 3ax^2 + 2byx + cy^2,$$
$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by,$$
$$\frac{\partial u}{\partial y} = 3dy^2 + 2cxy + bx^2,$$
$$\frac{\partial^2 u}{\partial y^2} = 6dy + 2cx.$$

As the given polynomial is harmonic, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ must hold for all x and y. Hence, we obtain

$$6ax + 2by + 6dy + 2cx = 0,$$

which simplifies to

$$(3a + c)x + (3d + b)y = 0,$$

for all x and y. As the above equation must hold for all x and y, we have that

$$c = -3a \text{ and } b = -3d.$$

Substituting the above equations into the original polynomial, we obtain the most general harmonic polynomial of the given form as

$$u(x,y) = ax^3 - 3dx^2y - 3axy^2 + dy^3.$$

Now, we wish to compute the conjugate harmonic function of $u(x,y)$, denoted by $v(x,y)$. From the Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3ax^2 + 2byx + cy^2,$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 3dy^2 + 2cxy + bx^2.$$

Simplifying for the $v(x,y)$ term, we have

$$\frac{\partial v}{\partial y} = 3ax^2 + 2byx + cy^2,$$
$$\frac{\partial v}{\partial x} = -(3dy^2 + 2cxy + bx^2).$$

By taking the integral, we can solve for $v(x,v)$, and obtain

$$v(x,y) \quad = \quad \frac{1}{3}cy^3 - \frac{1}{3}bx^3 + bxy^2 - cx^2y$$

Question 3. The Complex Chain Rule.

Solution.

Question 4. $|f(z)| = 1$ **implies constant** f**.**

Solution. Let D be a unit disk, and $f : D \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for $z \in D$. Notice that $|f|$ is also a holomorhpic function, which can be re-written as $u(x,y) + iv(x,y)$. As $|f(z)| = 1$ for all $z \in D$, we obtain that $u^2 + v^2 = 1$ for all $(x,y) \in D$. Taking the partials, we obtain

$$2uu_x + 2vv_x = 0 \quad \text{and} \quad 2uu_y + 2vv_y = 0.$$

As f is holomorphic, from the Cauchy-Riemann equation, we have that

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Substituting the above equations to the partial equations and simplifying, we obtain

$$uv_y + vv_x = 0 \quad \text{and} \quad -uv_x + vv_y = 0,$$
$$u^2{v_y}^2 + v^2{v_x}^2 + 2uvv_xv_y = 0 \quad \text{and} \quad u^2{v_x}^2 + v^2{v_y}^2 - 2uvv_xv_y.$$

Hence, by adding the last two equations together, and factoring, we obtain

$$(u^2 + v^2)({v_x}^2 + {v_y}^2) = 0.$$

Since $|f| = 1$, $(u^2 + v^2)$ term cannot be 0, and we obtain that $v_x = 0$ and $v_y = 0$. By Cauchy-Riemann equation, we also have $u_x = 0$ and $u_y = 0$. Thus, using the Cauchy-Riemann theorem, $f'(z)$ for $z \in D$ can be written as

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= 0. \end{aligned}$$

Hence, $f'(z) = 0$ for all $z \in D$, and f is a constant function. $\square$

Question 5. Harmonic Functions.

Solution. Let u and v be real harmonic function on the unit disk, and f be a holomorphic function on the unit disk. As u and v are harmonic, we have

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0,$$

for all points on the unit disk.

(i) Assume that u^2 is harmonic. By using the chain rule, we can write u^2_{xx} and u^2_{yy} in terms of u_x and u_y respectively as

$$u^2_{xx} = 2(u_x u_x + u u_{xx}) \quad \text{and} \quad u^2_{yy} = 2(u_y u_y + u u_{yy}).$$

As u^2 is harmonic, we obtain

$$2(u_x u_x + u u_{xx}) + 2(u_y u_y + u u_{yy}) \quad = \quad 0,$$

which can be simplified and re-arranged to

$$u(u_{xx} + u_{yy}) + (u_x u_x + u_y u_y) \quad = \quad 0.$$

As $u_{xx} + u_{yy} = 0$, we finally obtain

$$u_x u_x + u_y u_y \quad = \quad 0.$$

Hence, $u_x = 0$, $u_y = 0$, and u is a constant function. $\square$

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