Complex Analysis I: Problem Set VI

Youngduck Choi
CILVR Lab
New York University
yc1104@nyu.edu

Abstract

This work contains the solutions to the problem set VI of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 237-2.

Solution. (a) We have

$$\begin{aligned}\frac{1}{z+z^2} &= \frac{1}{z}\frac{1}{1+z} \\ &= \frac{1}{z}(1-z+z^2\ldots) \\ &= \frac{1}{z}-1+z\ldots\end{aligned}$$

for $0<|z|<1$. The coefficient of $\frac{1}{z}$ term is 1. Hence, the residue at $z=0$ is 1.

(b) We have

$$\begin{aligned}z\cos(\frac{1}{z}) &= z(1-\frac{1}{2!}\frac{1}{z^2}+\frac{1}{4!}\frac{1}{z^4}\ldots) \\ &= z-\frac{1}{2!}\frac{1}{z}+\frac{1}{4!}\frac{1}{z^3}\ldots\end{aligned}$$

for $|z|<\infty$. The coefficient of $\frac{1}{z}$ term is 0. Hence, the residue at $z=0$ is 0.

(c) We have

$$\begin{aligned}\frac{z-\sin(z)}{z} &= \frac{1}{z}\cdot\frac{1}{\sin(z)} \\ &= \frac{1}{z}(z-(z-\frac{z^3}{3!}+\frac{z^5}{5!}\ldots))\end{aligned}$$

for $0<|z|<\infty$. The coefficient of $\frac{1}{z}$ term is 0. Hence, the residue at $z=0$ is 0.

(d) We have

$$\frac{\cot(z)}{z^4} = \frac{1}{z^4}\cdot\frac{\cos(z)}{\sin(z)}$$

By dividing the Maclaurin series representation of $\cos$ by $\sin$, we obtain

$$\frac{\cos(z)}{\sin(z)} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} \ldots$$

for $0 < |z| < \pi$. It follows that

$$\frac{\cot(z)}{z^4} = \frac{1}{z^5} - \frac{1}{3} \cdot \frac{1}{z^3} - \frac{1}{45} \cdot \frac{1}{z} \ldots$$

for $0 < |z| < \pi$. The coefficient of $\frac{1}{z}$ is $-\frac{1}{45}$. Hence, the residue at $z = 0$ is $-\frac{1}{45}$.

(e) We have

$$\frac{\sinh(z)}{z^4(1-z^2)} = \sinh(z) \cdot \frac{1}{z^4} \cdot \frac{1}{1-z^2}$$

By substituting the Maclaurin series, we obtain

$$\begin{aligned} \frac{\sinh(z)}{z^4(1-z^2)} &= \frac{1}{z^4}(z + \frac{1}{6}z^3 + \frac{1}{120}z^5 \ldots)(1 + z^2 + z^4 \ldots) \\ &= \frac{1}{z^3} + \frac{7}{6}\frac{1}{z} \ldots \end{aligned}$$

The coefficient of $\frac{1}{z}$ is $\frac{7}{6}$. Hence, the residue at $z = 0$ is $\frac{7}{6}$. □

Question 2. Brown p.237-2.

Solution. By the Cauchy's residue theorem, we can evaluate the integral by computing the residues.

(a) We compute the residue of the integrand at $z = 0$. Using the Laurent series of $\frac{\exp(-z)}{z^2}$, we obtain

$$\begin{aligned} \frac{\exp(-z^2)}{z^2} &= \frac{1}{z^2}(1 - \frac{1}{1!}z + \frac{1}{2!}z^2 \ldots) \\ &= \frac{1}{z^2} - \frac{1}{1!}\frac{1}{z} \ldots \end{aligned}$$

Hence, the residue at $z = 0$ is -1. Therefore, by the Cauchy's residue theorem, we obtain

$$\int_C \frac{\exp(-z)}{z^2} dz = 2\pi i(-1) = -2\pi i.$$

(b)

(c)

(d)

Question 1. Brown p.246-2.

Solution.

Question 1. Brown p.246-4.

Solution. We wish to evaluate

$$\int_C \frac{3z^2 + 2}{(z-1)(z^2+9)} dz,$$

for C being the circle $|z-2|=2$, taken in the counterclockwise direction. Clearly, the singularities of the integrand are 1 and $\pm 3i$. The residues at the singularities can be computed as follows:

$$\begin{aligned}
\mathrm{Res}_{z=1}\frac{3z^3+2}{(z-1)(z^2+9)} &= \frac{3z^3+2}{z^2+9}]_{z=1} = \frac{1}{2} \\
\mathrm{Res}_{z=3i}\frac{3z^3+2}{(z-1)(z^2+9)} &= \frac{3z^3+2}{(z-1)(z+3i)}]_{z=3i} = \frac{15+49i}{12} \\
\mathrm{Res}_{z=-3i}\frac{3z^3+2}{(z-1)(z^2+9)} &= \frac{3z^3+2}{(z-1)(z-3i)}]_{z=-3i} = \frac{15-49i}{12}.
\end{aligned}$$

For the part (a), we have $z=1$ is the only singularity inside C. Hence, by Cauchy's residue theorem, we have

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz = \pi i.$$

For the part (b), we have all singularities inside C. It follows that

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz = 2\pi i(\frac{1}{2}+\frac{15+49i}{12}+\frac{15-49i}{12}) = 6\pi i.$$

This completes the computations. $\square$

Question 1. Brown p.246-5.

Solution.

Question 1. Brown p.246-6.

Solution. We wish to evaluate

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)}dz$$

where C is the circle $|z|=2$, described in the positive sense. The singularities of the integrand, that are interior to C, are $0, \pm i$. The residues are respectively

$$\begin{aligned}
\frac{\cosh(\pi z)}{z^2+1}|_{z=0} &= 1 \\
\frac{\cosh(\pi z)}{z(z+i)}|_{z=i} &= \frac{1}{2} \\
\frac{\cosh(\pi z)}{z(z-i)}|_{z=-i} &= \frac{1}{2}.
\end{aligned}$$

Hence, by the Cauchy residue theorem, we have

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)}dz = 4\pi i,$$

as desired. $\square$