Complex Analysis I:
Problem Set I

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Abstract

This work contains the solutions to the problem set I of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. Brown p61. 1.

Solution. We wish to give a direct proof that

$$\frac{dw}{dz} = 2z \text{ when } \; w = z^2,$$

using the definition (3) in section 19, which is

$$\frac{dw}{dz} = \lim_{\triangle z \to 0} \frac{\triangle w}{\triangle z}.$$

We proceed to compute $\lim_{\triangle z \to 0} \frac{\triangle w}{\triangle z}$, given that $w = z^2$.

$$\begin{aligned}
\lim_{\triangle z \to 0} \frac{\triangle w}{\triangle z} &= \lim_{\triangle z \to 0} ((\triangle z + z)^2 - z^2) \frac{1}{\triangle z} \\
&= \lim_{\triangle z \to 0} (\triangle z^2 + 2z\triangle z) \frac{1}{\triangle z} \\
&= \lim_{\triangle z \to 0} \triangle z + 2z \\
&= 2z.
\end{aligned}$$

Therefore, we have shown that $\frac{dw}{dz} = 2z$. $\square$

Question 2. Brown p61. 2.

Solution. We differentiate four given functions of z.

(a) We wish to differentiate $f(z) = 3z^2 - 2z + 4$. Simply applying the power rule, we obtain $f^{'}(z) = 6z - 2$.

(b) We wish to differentiate $f(z) = (2z^2 + i)^5$. Applying the chain rule, we obtain

$$\begin{aligned}
f^{'}(z) &= 5(2z^2 + i)^4 \cdot 4z \\
&= 20z(2z^2 + i)^4.
\end{aligned}$$

(c) We wish to differentiate $f(z) = \dfrac{z-1}{2z+1}$ $(z \neq \dfrac{1}{2})$. Applying the quotient rule, we obtain

$$\begin{aligned} f^{'}(z) &= \frac{(2z+1)(1)-(z-1)(2)}{(2z+1)^2} \\ &= \frac{3}{(2z+1)^2}, \end{aligned}$$

for $z \neq \dfrac{1}{2}$.

(d) We wish to differentiate $f(z) = \dfrac{(1+z^2)^4}{z^2}$ $(z \neq 0)$. Applying the quotient rule, we obtain

$$f^{'}(z) = \frac{(z^2)(\frac{d}{dz}(1+z^2)^4) - (1+z^2)^4(2z)}{z^4},$$

for $z \neq 0$. Using the chain rule to resolve $\dfrac{d}{dz}(1+z^2)^4$ term, we finally get

$$\begin{aligned} f^{'}(z) &= \frac{(z^2)(4)(1+z^2)^3(2z) - (1+z^2)^4(2z)}{z^4} \\ &= \frac{8z^3(1+z^2)^3 - 2z(1+z^2)^4}{z^4} \\ &= \frac{8z^2(1+z^2)^3 - 2(1+z^2)^4}{z^3} \\ &= \frac{2(1+z^2)^3(3z^2-1)}{z^3}, \end{aligned}$$

for $z \neq 0$.

Question 3. Brown p76. 4.

Solution. We determine the singular points of the three given functions of z.

(a) We wish to determine the singular points of $f(z) = \dfrac{2z+1}{z(z^2+1)}$. Notice that $2z+1$ and $z(z^2+1)$ are differentiable everywhere. Hence, by the quotient rule, we know that $f(z)$ is differentiable, thus analytic everywhere, other than where $z(z^2+1) = 0$. Therefore, The singular points are $z = 0, \pm i$ and $f(z)$ is analytic everywhere else.

(b) We wish to determine the singular points of $f(z) = \dfrac{z^3+i}{z^2-3z+2}$. Notice that the function definition can be factorized as

$$f(z) = \frac{z^3+i}{(z-2)(z-1)}.$$

Again, with the same reasoning through the quotient rule, we have that the singular points are $z = 1, 2$, and $f(z)$ is analytic everywhere else.

(c) We wish to determine the singular points of $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$. Again, with the same reasoning through the quotient rule, we have that the singular points are $z = -2, 1 \pm i$, and $f(z)$ is analytic everywhere else.

Question 4. Brown p90. 5.
Solution. The terms $|\exp(2z+i)|$ and $|\exp(iz^2)|$ can be written as

$$|\exp(2z+i)| = |\exp(2x+i(2y+1))| = e^{2x} \quad (1)$$
$$|\exp(iz^2) = |\exp(-2xy+i(x^2-y^2))| = e^{-2xy}. \quad (2)$$

By the triangle-inequality, we have that

$$|\exp(2z+i)+\exp(iz^2)| \leq |\exp(2z+i)| + |\exp(iz^2)|.$$

Using the 1 substitution, we can conclude that

$$|\exp(2z+i)+\exp(iz^2)| \leq \mathrm{e}^{2x} + \mathrm{e}^{-2xy}.$$

$\square$

Question 5. Brown p185. 4.
Solution. We have the following summation formulation:

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}, \quad (3)$$

whenever $|z| < 1$. Substituting $r\mathrm{e}^{i\theta}$ for z and separating the real and imaginary parts, we can re-write the LHS as

$$\begin{aligned}
\sum_{n=1}^{\infty} z^n &= \sum_{n=0}^{\infty} (r\mathrm{e}^{i\theta})^n \\
&= \sum_{n=1}^{\infty} r^n \mathrm{e}^{in\theta} \\
&= \sum_{n=1}^{\infty} r^n \mathrm{cos}(n\theta) + i \sum_{n=0}^{\infty} r^n \mathrm{sin}(n\theta). \quad (4)
\end{aligned}$$

Now, multiplying both denominator and numerator by the conjugate of $1-z$, $1-\overline{z}$, we can re-write the RHS as

$$\begin{aligned}
\frac{z}{1-z} &= \frac{z(1-\overline{z})}{(1-z)(1-\overline{z})} \\
&= \frac{z - z\overline{z}}{1 + z\overline{z} - (z+\overline{z})}.
\end{aligned}$$

Substituting $z = r\mathrm{cos}(\theta) + i\mathrm{sin}(\theta)$, $z\overline{z} = r^2$, and $z + \overline{z} = 2r\mathrm{cos}(\theta)$ to the last expression, and separating the real and imaginary parts, we obtain

$$\begin{aligned}
\frac{z}{1-z} &= \frac{r\mathrm{cos}(\theta) - r^2 + ir\mathrm{sin}(\theta)}{1 + r^2 - 2r\mathrm{cos}(\theta)} \\
&= \frac{r\mathrm{cos}(\theta) - r^2}{1 + r^2 - 2r\mathrm{cos}(\theta)} + \frac{r\mathrm{sin}(\theta)}{1 + r^2 - 2r\mathrm{cos}(\theta)} i. \quad (5)
\end{aligned}$$

Substituting 5 and 4 to 3, we obtain

$$\sum_{n=1}^{\infty} r^n \mathrm{cos}(n\theta) + i \sum_{n=0}^{\infty} r^n \mathrm{sin}(n\theta) = \frac{r\mathrm{cos}(\theta) - r^2}{1 + r^2 - 2r\mathrm{cos}(\theta)} + \frac{r\mathrm{sin}(\theta)}{1 + r^2 - 2r\mathrm{cos}(\theta)} i. \quad (6)$$

By the Theorem from section 61, we know that the real and imaginary part of the series must equal the real and imaginary part of the convergent value respectively. Hence, we have that

$$\sum_{n=1}^{\infty} r^n \mathrm{cos}(n\theta) = \frac{r\mathrm{cos}(\theta) - r^2}{1 - 2r\mathrm{cos}(\theta) + r^2}$$
$$\sum_{n=1}^{\infty} r^n \mathrm{sin}(n\theta) = \frac{r\mathrm{sin}(\theta)}{1 - 2r\mathrm{cos}(\theta) + r^2},$$

when $0 < r < 1$. $\square$

Question 6. Radius of Convergence I.

Solution. We wish to find the radius of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} z^{k(k+1)}.$$

Let s_k denote the kth term of the above power series. From the ratio test, we know that a series $\sum_{k=1} s_k$ converges if $\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| < 1$ and diverges if $|\frac{s_{k+1}}{s_k}| \geq 1$ for all $k \geq k_0$, where k_0 is some fixed integer. We proceed to compute $\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}|$.

$$\begin{aligned}
\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| &= \limsup_{k\to\infty} |\frac{\frac{(-1)^{k+1}}{k+1} z^{(k+1)(k+2)}}{\frac{(-1)^k}{k} z^{k(k+1)}}| \\
&= \limsup_{k\to\infty} |\frac{-k}{k+1} z^{2(k+1)}| \\
&= \limsup_{k\to\infty} |\frac{k}{k+1}||z^{2(k+1)}|.
\end{aligned}$$

As $\frac{k}{k+1} > 0$ for all k, and $|z^{2(k+1)}| \leq 0$, we have

$$\begin{aligned}
\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| &= \limsup_{k\to\infty} |\frac{k}{k+1}||z^{2(k+1)}| \\
&\leq \limsup_{k\to\infty} |\frac{k}{k+1}| \limsup_{k\to\infty} |z^{2(k+1)}| \\
&= \limsup_{k\to\infty} |z^{2(k+1)}|.
\end{aligned}$$

For $|z| < 1$, we have that $\limsup_{k\to\infty} |z^{2(k+1)}| = 0$. As $0 \leq \limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| \leq \limsup_{k\to\infty} |z^{2(k+1)}| = 0$, we obtain that $\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| = 0$, and $\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| < 1$. Hence, the series converges when $|z| < 1$. In the above computation, we have shown that $|\frac{s_{k+1}}{s_k}| = |\frac{k}{k+1}||z^{2(k+1)}|$. Notice that for $|z| > 1$, $\lim_{k\to\infty} |z^{2(k+1)}| = \infty$. Therefore, there exists k_0, such that for all $k \geq k_0$, $|\frac{s_{k+1}}{s_k}| \geq 1$. Hence, the series diverges when $|z| > 1$.

Now, we discuss the convergence for $z = 1$ and -1, and i. For $z = 1, -1$, as $z^{k(k+1)} = 1$ for all k, the series simplifies to

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}.$$

Notice that $-\frac{1}{k} + \frac{1}{k+1} = -\frac{1}{k(k+1)}$ for $k \geq 1$. Hence, the above series can be re-written in the following way:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k)}.$$

Observe that

$$-\frac{1}{(2k-1)(2k-1)} \leq -\frac{1}{(2k-1)(2k)} \leq 0, \tag{7}$$

for $k \leq 1$. As $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges absolutely and (7) holds, by the direct comparison test, $\sum_{k=0}^{\infty} -\frac{1}{(2k-1)(2k-1)}$ converges absolutely. Thus, the given series is convergent for $z = 1$ and -1.

For $z = i$, notice that the given series can be simplified as

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} i^{k(k+1)} \quad = \quad \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{2k} - \frac{1}{2k-1}\right).$$

Hence, the series is an alternating harmonic series, which converges. Thus, the given series is convergent for $z = i$. Note that this technique could be applied to $z = 1, -1$ cases as well. $\square$

Question 7. Radius of Convergence II.

Solution. We wish to find the radius of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{k^2}{4^k + 5k} z^k.$$

Let s_k denote the kth term of the above power series. From the root test, we know that a series $\sum_{k=1}^{\infty} s_k$ converges if $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} < 1$ and diverges if $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} > 1$. We proceed to compute $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}}$.

$$\begin{aligned}
\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} &= \limsup_{k\to\infty} \left(\frac{k^2}{4^k + 5k} z^k\right)^{\frac{1}{k}} \\
&= \limsup_{k\to\infty} \left|\frac{k^2}{4^k + 5k}\right|^{\frac{1}{k}} |z^k|^{\frac{1}{k}} \\
&= |z| \limsup_{k\to\infty} \left(\frac{k^2}{4^k + 5k}\right)^{\frac{1}{k}} \\
&= \frac{1}{4}|z|. \qquad (8)
\end{aligned}$$

For the sake of completeness, we should note that, the equation, $\limsup_{k\to\infty} \left(\frac{k^2}{4^k + 5k}\right)^{\frac{1}{k}} = \frac{1}{4}$, can be rigorously proven by taking the log of the sequence, and using L'hopital's rule. For $|z| < 4$, we have that $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} = \frac{1}{4}|z| < 1$. Hence, the series converges when $|z| < 4$. For $|z| > 4$, we have that $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} = \frac{1}{4}|z| > 1$. Hence, the series diverges when $|z| > 4$. Therefore, we have that the radius of convergence of the given sequence is 4, with the center being the origin. $\square$