Complex Analysis I: Problem Set VI

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Abstract

This work contains the solutions to the problem set VI of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 237-2.

Solution. (a) We have

$$\begin{aligned} \frac{1}{z+z^2} &= \frac{1}{z}\frac{1}{1+z} \\ &= \frac{1}{z}(1 - z + z^2 \dots) \\ &= \frac{1}{z} - 1 + z \dots \end{aligned}$$

for $0 < |z| < 1$. The coefficient of $\frac{1}{z}$ term is 1. Hence, the residue at $z = 0$ is 1.

(b) We have

$$\begin{aligned} z\cos(\frac{1}{z}) &= z(1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} \dots) \\ &= z - \frac{1}{2!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} \dots \end{aligned}$$

for $|z| < \infty$. The coefficient of $\frac{1}{z}$ term is 0. Hence, the residue at $z = 0$ is 0.

(c) We have

$$\begin{aligned} \frac{z - \sin(z)}{z} &= \frac{1}{z} \cdot \frac{1}{\sin(z)} \\ &= \frac{1}{z}(z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots)) \end{aligned}$$

for $0 < |z| < \infty$. The coefficient of $\frac{1}{z}$ term is 0. Hence, the residue at $z = 0$ is 0.

(d) We have

$$\frac{\cot(z)}{z^4} = \frac{1}{z^4} \cdot \frac{\cos(z)}{\sin(z)}$$

By dividing the Maclaurin series representation of $\cos$ by $\sin$, we obtain

$$\frac{\cos(z)}{\sin(z)} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} \dots$$

for $0 < |z| < \pi$. It follows that

$$\frac{\cot(z)}{z^4} = \frac{1}{z^5} - \frac{1}{3} \cdot \frac{1}{z^3} - \frac{1}{45} \cdot \frac{1}{z} \dots$$

for $0 < |z| < \pi$. The coefficient of $\frac{1}{z}$ is $-\frac{1}{45}$. Hence, the residue at $z = 0$ is $-\frac{1}{45}$.

(e) We have

$$\frac{\sinh(z)}{z^4(1-z^2)} = \sinh(z) \cdot \frac{1}{z^4} \cdot \frac{1}{1-z^2}$$

By substituting the Maclaurin series, we obtain

$$\begin{aligned}\frac{\sinh(z)}{z^4(1-z^2)} &= \frac{1}{z^4}(z + \frac{1}{6}z^3 + \frac{1}{120}z^5 \dots)(1 + z^2 + z^4 \dots) \\ &= \frac{1}{z^3} + \frac{7}{6}\frac{1}{z} \dots\end{aligned}$$

The coefficient of $\frac{1}{z}$ is $\frac{7}{6}$. Hence, the residue at $z = 0$ is $\frac{7}{6}$. □

Question 2. Brown p.237-2.
Solution. By the Cauchy's residue theorem, we can evaluate the integral by computing the residues.
(a) We compute the residue of the integrand at $z = 0$. Using the Laurent series of $\frac{\exp(-z)}{z^2}$, we obtain

$$\begin{aligned}\frac{\exp(-z^2)}{z^2} &= \frac{1}{z^2}(1 - \frac{1}{1!}z + \frac{1}{2!}z^2 \dots) \\ &= \frac{1}{z^2} - \frac{1}{1!}\frac{1}{z} \dots\end{aligned}$$

Hence, the residue at $z = 0$ is -1. Therefore, by the Cauchy's residue theorem, we obtain

$$\int_C \frac{\exp(-z)}{z^2} dz = 2\pi i(-1) = -2\pi i.$$

(b)

(c)

(d)

Question 1. Brown p.147-2.
Solution.

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Question 1. Brown p.246-6.

Solution. We wish to evaluate

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz$$

where C is the circle $|z| = 2$, described in the positive sense. The singularities of the integrand, that are interior to C, are $0, \pm i$. The residues are respectively

$$\begin{aligned} \frac{\cosh(\pi z)}{z^2+1}|_{z=0} &= 1 \\ \frac{\cosh(\pi z)}{z(z+i)}|_{z=i} &= \frac{1}{2} \\ \frac{\cosh(\pi z)}{z(z-i)}|_{z=-i} &= \frac{1}{2}. \end{aligned}$$

Hence, by the Cauchy residue theorem, we have

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz = 4\pi i,$$

as desired. $\square$