Complex Analysis I: Problem Set IX

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Abstract

This work contains the solutions to the problem set IX of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

1. Evaluate the integral

$$\int_\gamma \frac{dz}{z^2+1}$$

for

$$\gamma(\theta) = 2\left|\cos 2\theta\right| e^{i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Solution. By drawing the contour on the complex plane, we observe that γ forms 4 simple closed contours, for each direction of the axis. We denote these contours as γ_1, γ_2, γ_3, and γ_4 respectively in a counter-clockwise fashion. Observe that $f(z) = \frac{1}{z^2+1}$ is singular at $z = \pm i$. $z = i$ belongs to the interior of γ_2 contour, and $z = -i$ belongs to the interior of γ_4 contour. By the Cauchy-Residue formula, we obtain

$$\begin{aligned}
\int_{\gamma_1} \frac{dz}{z^2+1} &= 0 \\
\int_{\gamma_2} \frac{dz}{z^2+1} &= 2\pi i \mathrm{Res}_{z=i} \frac{1}{z^2+1} \\
\int_{\gamma_3} \frac{dz}{z^2+1} &= 0 \\
\int_{\gamma_4} \frac{dz}{z^2+1} &= 2\pi i \mathrm{Res}_{z=-i} \frac{1}{z^2+1}.
\end{aligned}$$

As it can be written that $f(z) = \frac{\phi(z)}{z-i}$, where $\phi(z) = \frac{1}{z+i}$, the residue at $z = i$ is $\phi(i) = \frac{1}{2i}$. On the other hand, as it can be written that $f(z) = \frac{\phi(z)}{z+i}$, where $\phi(z) = \frac{1}{z-i}$, the residue at $z = -i$

is $\phi(i) = -\frac{1}{2i}$. Consequently, we have

$$\begin{aligned}
\int_{\gamma_2} \frac{dz}{z^2+1} &= 2\pi i \frac{1}{2i} = \pi \\
\int_{\gamma_4} \frac{dz}{z^2+1} &= 2\pi i (-\frac{1}{2i}) = -\pi.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\int_{\gamma} \frac{dz}{z^2+1} &= \int_{\gamma_1} \frac{dz}{z^2+1} + \int_{\gamma_2} \frac{dz}{z^2+1} + \int_{\gamma_3} \frac{dz}{z^2+1} + \int_{\gamma_4} \frac{dz}{z^2+1} \\
&= 0.
\end{aligned}$$

□

Question 2.

2. Let

$$\gamma(\theta) = \begin{cases} \theta e^{i\theta}, & 0 \leq \theta \leq 2\pi, \\ 4\pi - \theta, & 2\pi \leq \theta \leq 4\pi. \end{cases}$$

Calculate

$$\int_{\gamma} \frac{dz}{z^2+\pi^2}.$$

Solution. Observe that the function has isolated singularities at $z = \pm i\pi$. By observing the contour, we see that $i\pi$ lies outside of the contour, as $\gamma(\frac{\pi}{2}) = \frac{\pi}{2} e^{i\frac{\pi}{2}} = i\frac{\pi}{2}$. On the other hand, $z = -i\pi$ lies on the interior of the contour as $\gamma(\frac{3\pi}{2}) = \frac{3\pi}{2} e^{i\frac{3\pi}{2}} = -\frac{3\pi}{2} i$. Hence, by the Cauchy Residue theorem, we have

$$\int_{\gamma} \frac{dz}{z^2+\pi^2} = 2\pi i \mathrm{Res}_{z=-i\pi} \frac{1}{z^2+\pi^2}.$$

As it can be written that $f(z) = \frac{\phi(z)}{z+i\pi}$, where $\phi(z) = \frac{1}{z-i\pi}$, the residue at $z = -i\pi$ is $\phi(-i\pi) = -\frac{1}{2i\pi}$. Hence, it follows that

$$\begin{aligned}
\int_{\gamma} \frac{dz}{z^2+\pi^2} &= 2\pi i (-\frac{1}{2i\pi}) \\
&= -1.
\end{aligned}$$

□

Question 3.

3. Let $\lambda > 1$ and show the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the right half plane $\{z : \text{Re}\, z > 0\}$.

Solution. Firstly, the equation can be re-written as $\lambda - z = e^{z}$. Observe that it is necessary to have $|\lambda - z| = e^{-\text{Re}z}$ to satisfy the above equation. As we only limit the space of possible solutions to be $\{z : \text{Re}z > 0\}$, it follows that it is necessary to have $|\lambda - z| < 1$. Define $C = \{z \in \mathbb{C} \,|\, |\lambda - z| < 1\}$. So far, we have shown that the solutions to the given equation, if it exists must lie on the interior of C. Let $f(z) = e^{-z}$ and $g(z) = \lambda - z$. Then, it follows that on C, $|g(z)| = |\lambda - z| = 1$, and as $\lambda > 1$, $|f(z)| = |e^{-z}| = e^{-\text{Re}z} < 1$. As $f(z)$ and $g(z)$ are entire, they are also analytic inside and on C. The conditions of Rouche's theorem are thus satisfied. Hence, $\lambda - z$ and $\lambda - z - e^{-z}$ have the same number of zeros, counting multiplicities inside C. Observe that $\lambda - z$ has a zero on $z = \lambda$. Thus, $\lambda - z - e^{-z}$ has one solution inside C. As we have shown that a solution to $\lambda - z - e^{-\lambda}$ must lie inside C, we have shown that $\lambda - z - e^{\lambda}$ has exactly one solution. $\square$

Question 4.

4. How many roots of

$$z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$$

lie in the right half plane $\{z : \text{Re}\, z > 0\}$.

Solution. Firstly, observe that the number of poles P is 0, as the given function $z^4 + 8z^3 + 3z^2 + 8z + 3$, is entire. Let γ be a contour, which moves from $-iR$ to iR as a semi-circle of a radius R. Since $P = 0$, as $R \to \infty$, by the argument principle $Z = \frac{1}{2\pi} \triangle_{\gamma} \text{arg} f(z)$, where Z is the number of zeros in the right plane. On γ, we can parametrize z has $z = Re^{i\theta}$. Then, the given equation can be re-written as

$$R^4 e^{i4\theta}\left(1 + \frac{8}{Re^{i\theta}} + \frac{3}{R^2 e^{i2\theta}} + \frac{8}{R^3 e^{i3\theta}} + \frac{3}{R^4 e^{i4\theta}}\right) \quad = \quad 0.$$

Observe that as $R \to \infty$, LHS $\to R^4 e^{i4\theta}$. Hence, the argument changes by 4π from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. It follows that $Z = \frac{1}{2\pi} 4\pi = 2$. There are two zeros in the right plane. $\square$

Question 5.

5. Let $f \in H(B_R)$ for some $R > 1$. If $|f(z)| > 2$ for $|z| = 1$ and $f(0) = 1$. Must f have zero in B_1?

Solution. As B_1 is a circle of radius 1, centered around the origin, we have that the winding number of B_1 is simply 1. Observe that the given function is holomorphic, hence meromorphic with zero poles, interior to B_1 and is analytic on B_1. Furthermore, as $|f(z)| > 2$ for $|z| = 1$, we have f is nonzero on B_1. Therefore, by the argument principle, we have that the winding number is equal to $Z - P$ where Z is the number of zeros and P is the number of poles of $f(z)$ inside B_1. Since $P = 0$ and the winding number is 1, we have that $Z = 1$. f must have zero in B_1. □

Question 6.

6. Let $f \in H(B_R)$ for some $R > 1$. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. What can you say if we only have $|f(z)| \leq 1$ for $|z| = 1$ instead.

Solution. As B_1 is a circle of radius 1, centered around the origin, we have that the winding number of B_1 is simply 1. As f is holomorphic on B_R for some $R > 1$, we have f is analytic on B_1. On B_1, as we have $|f(z)| < 1$, it follows that $|f(z) - z| \leq ||f(z)| - |z|| = 1 - |f(z)| > 0$. Therefore, $f(z) - z$ is nonzero on B_1. Therefore, by the argument principle, as above, with $P = 0$, we have $Z = 1$. Therefore, there exists a unique solution to the equation $f(z) - z = 0$ inside B_1, which is also a solution to $f(z) = z$ as well. Hence, there exists a unique solution to $f(z) = z$ inside B_1. When we only have $|f(z) \leq 1$ for $|z|| = 1$, we lose the nonzero property of $f(z) - z$ on B_1. Therefore, the argument will not work in that case. □

Question 7.

7. Let $f, g \in C(\overline{B}_1) \cap H(B_1)$. If for $|z| = 1$, we have

$$|f(z) - g(z)| < |f(z)| + |g(z)|,$$

then show f and g have the same number of zeroes (counting the multiplicities) in B_1.

Solution. We are given that $f, g \in H(B_1)$ and f, g are continuous on B_1. The conditions of symmetric Rouche's theorem are satisfied. Then, by the Symmetric Rouche's theorem. we have the same number of roots for f and g, counting the multiplicies in B_1. □