
Complex Analysis I:

Problem Set VI

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Abstract

This work contains the solutions to the problem set VI of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 237-2.

Solution. (a) We have

$$\begin{aligned}\frac{1}{z+z^2} &= \frac{1}{z} \frac{1}{1+z} \\ &= \frac{1}{z} (1 - z + z^2 \dots) \\ &= \frac{1}{z} - 1 + z \dots\end{aligned}$$

for $0 < |z| < 1$. The coefficient of $\frac{1}{z}$ term is 1. Hence, the residue at $z = 0$ is 1.

(b) We have

$$\begin{aligned}z \cos\left(\frac{1}{z}\right) &= z \left(1 - \frac{1}{2!} \frac{1}{z^2} + \frac{1}{4!} \frac{1}{z^4} \dots\right) \\ &= z - \frac{1}{2!} \frac{1}{z} + \frac{1}{4!} \frac{1}{z^3} \dots\end{aligned}$$

for $|z| < \infty$. The coefficient of $\frac{1}{z}$ term is 0. Hence, the residue at $z = 0$ is 0.

(c) We have

$$\begin{aligned}\frac{z - \sin(z)}{z} &= \frac{1}{z} \cdot \frac{1}{\sin(z)} \\ &= \frac{1}{z} \left(z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots \right) \right)\end{aligned}$$

for $0 < |z| < \infty$. The coefficient of $\frac{1}{z}$ term is 0. Hence, the residue at $z = 0$ is 0.

(d) We have

$$\frac{\cot(z)}{z^4} = \frac{1}{z^4} \cdot \frac{\cos(z)}{\sin(z)}$$

By dividing the Maclaurin series representation of \cos by \sin , we obtain

$$\frac{\cos(z)}{\sin(z)} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} \dots$$

for $0 < |z| < \pi$. It follows that

$$\frac{\cot(z)}{z^4} = \frac{1}{z^5} - \frac{1}{3} \cdot \frac{1}{z^3} - \frac{1}{45} \cdot \frac{1}{z} \dots$$

for $0 < |z| < \pi$. The coefficient of $\frac{1}{z}$ is $-\frac{1}{45}$. Hence, the residue at $z = 0$ is $-\frac{1}{45}$.

(e) We have

$$\frac{\sinh(z)}{z^4(1-z^2)} = \sinh(z) \cdot \frac{1}{z^4} \cdot \frac{1}{1-z^2}$$

By substituting the Maclaurin series, we obtain

$$\begin{aligned} \frac{\sinh(z)}{z^4(1-z^2)} &= \frac{1}{z^4} \left(z + \frac{1}{6}z^3 + \frac{1}{120}z^5 \dots \right) (1 + z^2 + z^4 \dots) \\ &= \frac{1}{z^3} + \frac{7}{6} \frac{1}{z} \dots \end{aligned}$$

The coefficient of $\frac{1}{z}$ is $\frac{7}{6}$. Hence, the residue at $z = 0$ is $\frac{7}{6}$. □

Question 2. Brown p.237-2.

Solution. By the Cauchy's residue theorem, we can evaluate the integral by computing the residues.

(a) We compute the residue of the integrand at $z = 0$. Using the Laurent series of $\frac{\exp(-z)}{z^2}$, we obtain

$$\begin{aligned} \frac{\exp(-z^2)}{z^2} &= \frac{1}{z^2} \left(1 - \frac{1}{1!}z + \frac{1}{2!}z^2 \dots \right) \\ &= \frac{1}{z^2} - \frac{1}{1!} \frac{1}{z} \dots \end{aligned}$$

Hence, the residue at $z = 0$ is -1 . Therefore, by the Cauchy's residue theorem, we obtain

$$\int_C \frac{\exp(-z)}{z^2} dz = 2\pi i(-1) = -2\pi i.$$

(b) Observe that

$$\begin{aligned} e^{-z} &= e^{-1} e^{-(z-1)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{e(z-1)^n}. \end{aligned}$$

Therefore, the coefficient of $\frac{1}{z-1}$ is $-\frac{1}{e}$. Hence, by the residue theorem, we have

$$\int_C \frac{\exp(-z)}{(z-1)^2} dz = -\frac{2\pi i}{e},$$

for C being the positively oriented circle $|z| = 3$.

(c) Observe that

$$z^2 \exp\left(\frac{1}{z}\right) = z^2 \left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} \dots \right)$$

for $0 < |z| < \infty$. The coefficient of $\frac{1}{z}$ term is $\frac{1}{6}$. Hence, by the residue theorem, we have

$$\int_C \frac{\exp(-z)}{(z-1)^2} dz = \frac{\pi i}{3}.$$

(d) Observe that the singularities happen at $z = 0$ and $z = 2$. Using poles, we have

$$\text{Res}_{z=0} \frac{z+1}{z(z-2)} = \frac{-1}{2}$$

$$\text{Res}_{z=2} \frac{z+1}{z(z-2)} = \frac{3}{2}.$$

Hence, by the residue theorem, we have

$$\int_C \frac{z+1}{z^2-2z} dz = 2\pi i,$$

for the given contour.

Question 3. Brown p.242-1.

Solution.

Question 1. Brown p.246-2.

Solution.

Question 1. Brown p.246-4.

Solution. We wish to evaluate

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz,$$

for C being the circle $|z-2|=2$, taken in the counterclockwise direction. Clearly, the singularities of the integrand are 1 and $\pm 3i$. The residues at the singularities can be computed as follows:

$$\text{Res}_{z=1} \frac{3z^2+2}{(z-1)(z^2+9)} = \left. \frac{3z^2+2}{z^2+9} \right|_{z=1} = \frac{1}{2}$$

$$\text{Res}_{z=3i} \frac{3z^2+2}{(z-1)(z^2+9)} = \left. \frac{3z^2+2}{(z-1)(z+3i)} \right|_{z=3i} = \frac{15+49i}{12}$$

$$\text{Res}_{z=-3i} \frac{3z^2+2}{(z-1)(z^2+9)} = \left. \frac{3z^2+2}{(z-1)(z-3i)} \right|_{z=-3i} = \frac{15-49i}{12}.$$

For the part (a), we have $z = 1$ is the only singularity inside C . Hence, by Cauchy's residue theorem, we have

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz = \pi i.$$

For the part (b), we have all singularities inside C . It follows that

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz = 2\pi i \left(\frac{1}{2} + \frac{15+49i}{12} + \frac{15-49i}{12} \right) = 6\pi i.$$

This completes the computations. \square

Question 1. Brown p.246-5.

Solution. We wish to evaluate

$$\int_C \frac{1}{z^3(z+4)} dz,$$

for C being the positively oriented circle $|z| = 2$. Clearly, the singularities are at $z = 0$ and $z = -4$. The residues of the singularities can be computed as follows:

$$\begin{aligned} \operatorname{Res}_{z=0} \frac{1}{z^3(z+4)} &= \frac{1}{64} \\ \operatorname{Res}_{z=-4} \frac{1}{z^3(z+4)} &= -\frac{1}{64}. \end{aligned}$$

For the part (a), we have $z = 0$ is the only singularity inside C . By the residue theorem, we have

$$\int_C \frac{3z^2 + 2}{(z-1)(z^2+9)} dz = \frac{\pi i}{32}.$$

For the part (b), we have both singularities inside C . By the residue theorem, we have

$$\int_C \frac{3z^2 + 2}{(z-1)(z^2+9)} dz = 0.$$

This completes the computation. \square

Question 1. Brown p.246-6.

Solution. We wish to evaluate

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz$$

where C is the circle $|z| = 2$, described in the positive sense. The singularities of the integrand, that are interior to C , are $0, \pm i$. The residues are respectively

$$\begin{aligned} \frac{\cosh(\pi z)}{z^2+1} \Big|_{z=0} &= 1 \\ \frac{\cosh(\pi z)}{z(z+i)} \Big|_{z=i} &= \frac{1}{2} \\ \frac{\cosh(\pi z)}{z(z-i)} \Big|_{z=-i} &= \frac{1}{2}. \end{aligned}$$

Hence, by the Cauchy residue theorem, we have

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz = 4\pi i,$$

as desired. \square