Complex Analysis I: Problem Set VII

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Abstract

This work contains the solutions to the problem set VII of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 247-7.

Solution. (a) Observe that

$$\frac{1}{z^2}f(\frac{1}{z}) \quad = \quad \frac{(3+2z)^2}{z(1-z)(2+5z)}.$$

As $\frac{1}{z^2}f(\frac{1}{z})$ has a simple pole at $z=0$, we have

$$\begin{aligned}\int_C \frac{(3+2z)^2}{z(z-1)(2z+5)}dz &= 2\pi i \cdot \text{Res}_{z=0}[\frac{1}{z^2}f(\frac{1}{z})] \\ &= 2\pi i \cdot \frac{9}{2} = 9\pi i.\end{aligned}$$

(b) Observe that

$$\frac{1}{z^2}f(\frac{1}{z}) = \frac{e^z}{z^2(1+z^3)}.$$

As $\frac{1}{z^2}f(\frac{1}{z})$ has a pole of order 2 at $z=0$, we have

$$\int_C \frac{z^3 e^{\frac{1}{z}}}{1+z^3}dz \quad = \quad 2\pi i \cdot \text{Res}_{z=0}[\frac{1}{z^2}f(\frac{1}{z})],$$

where $\phi(z) = \frac{e^z}{1+z^3}$. We have

$$\phi^{'}(z) \quad = \quad \frac{(1+z^3)e^z - e^z 3z^2}{(1+z^3)^2}.$$

By substituting $z=0$, we see that $\phi^{'}(0) = 1$, which is the residue at $z=0$. It follows that

$$\int_C \frac{z^3 e^{\frac{1}{z}}}{1+z^3}dz \quad = \quad 2\pi i.$$

□

Question 2. 254-5.

Solution. (a) The given integral can be written as

$$\begin{aligned}\int_C \tan(z)dz &= \int_C \frac{p(z)}{q(z)}dz \\ &= \int_C \frac{\sin(z)}{\cos(z)}dz.\end{aligned}$$

As the zeros of $\cos(z)$ are $z = \frac{\pi}{2} + n\pi$ and C is the positively oriented circle $|z| = 2$, there are two isolated singularities of $\tan(z)$ interior to C, namely $z = \pm\frac{\pi}{2}$. It follows that

$$\begin{aligned}\text{Res}_{z=\frac{\pi}{2}} \tan(z) &= \frac{p(\frac{\pi}{2})}{q'(\frac{\pi}{2})} = \frac{\sin(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = -1, \\ \text{Res}_{z=\frac{-\pi}{2}} \tan(z) &= \frac{p(\frac{-\pi}{2})}{q'(\frac{-\pi}{2})} = \frac{\sin(\frac{-\pi}{2})}{-\sin(\frac{-\pi}{2})} = -1.\end{aligned}$$

Consequently, by the residue theorem, we have

$$\int_C \tan(z)dz = -4\pi i.$$

(b) We wish to evaluate the integral $\int_C \frac{dz}{\sinh(2z)}$. As $\sinh(z) = 0$ for $\frac{n\pi i}{2}$, we can conclude that the isolated singularities of the integrand happens at $z = 0$ and $z = \pm\frac{\pi i}{2}$. It follows that

$$\begin{aligned}\text{Res}_{z=0}\frac{1}{\sinh(2z)} &= \frac{1}{2\cosh(0)} = \frac{1}{2} \\ \text{Res}_{z=\frac{\pi i}{2}}\frac{1}{\sinh(2z)} &= \frac{1}{2\cosh(\pi i)} = -\frac{1}{2} \\ \text{Res}_{z=-\frac{\pi i}{2}}\frac{1}{\sinh(2z)} &= \frac{1}{2\cosh(-\pi i)} = -\frac{1}{2}\end{aligned}$$

Consequently, by the residue theorem, we have

$$\int_C \frac{1}{\sinh(2z)}dz = -\pi i.$$

□

Question 3. 254-6.

Solution.

Question 1. 264-2.

Solution. Consider the function $f(z) = \frac{1}{(z^2+1)^2}$ and the simple, closed, and positively oriented contour of a half-circle above the real-axis, centered around the origin with the radius $R > 1$. It follows that

$$\int_{-R}^{R} \frac{1}{(x^2+1)^2}dx + \int_{C_R} \frac{1}{(z^2+1)^2}dz = 2\pi B,$$

where C_R denotes the contour of the curve part of the half-circle, and B is the residue of the complex integrand at $z = i$. For computing the residue, we have

$$\frac{1}{(z^2+1)} = \frac{\phi(z)}{(z-i)^2},$$

where $\phi(z) = \dfrac{1}{(z+i)^2}$. It follows that $B = \phi^{'}(i) = \dfrac{1}{4i}$. Therefore, we obtain

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} - \int_{C_R} \frac{1}{(z^2+1)^2} dz.$$

For z on C_R, we have $|z^2+1| \leq R^2 - 1$.