# Complex Analysis: Problem Set I

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## Abstract

This work contains the solutions to the problem set I of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

**Question 1. Brown p61. 1.**

**Solution.**

**Question 2. Brown p61. 2.**

**Solution.** We differentiate four given functions of $z$.

**(a)** We wish to differentiate $f(z) = 3z^2 - 2z + 4$. Simply applying the power rule, we obtain $f^{'}(z) = 6z - 2$.

**(b)** We wish to differentiate $f(z) = (2z^2 + i)^5$. Applying the chain rule, we obtain

$$\begin{aligned} f^{'}(z) &= 5(2z^2 + i)^4 \cdot 4z \\ &= 20z(2z^2 + i)^4. \end{aligned}$$

**(c)** We wish to differentiate $f(z) = \dfrac{z-1}{2z+1}$ $(z \neq \dfrac{1}{2})$. Applying the quotient rule, we obtain

$$\begin{aligned} f^{'}(z) &= \frac{(2z+1)(1) - (z-1)(2)}{(2z+1)^2} \\ &= \frac{3}{(2z+1)^2}, \end{aligned}$$

for $z \neq \dfrac{1}{2}$.

**(d)** We wish to differentiate $f(z) = \dfrac{(1+z^2)^4}{z^2}$ $(z \neq 0)$. Applying the quotient rule, we obtain

$$f^{'}(z) = \frac{(z^2)(\frac{d}{dz}(1+z^2)^4) - (1+z^2)^4(2z)}{z^4},$$

for $z \neq 0$. Using the chain rule to resolve $\dfrac{d}{dz}(1+z^2)^4$ term, we finally get

$$\begin{aligned}
f^{'}(z) &= \frac{(z^2)(4)(1+z^2)^3(2z) - (1+z^2)^4(2z)}{z^4} \\
&= \frac{8z^3(1+z^2)^3 - 2z(1+z^2)^4}{z^4} \\
&= \frac{8z^2(1+z^2)^3 - 2(1+z^2)^4}{z^3} \\
&= \frac{2(1+z^2)^3(3z^2-1)}{z^3},
\end{aligned}$$

for $z \neq 0$.

**Question 3. Brown p76. 4.**

**Solution.** We determine the singular points of the three given functions of $z$.

**(a)** We wish to determine the singular points of $f(z) = \dfrac{2z+1}{z(z^2+1)}$. The singular points are $z = 0, \pm i$.

**(b)** We wish to determine the singular points of $f(z) = \dfrac{z^3+i}{z^2-3z+2}$. Notice that the function definition can be factorized as

$$f(z) = \frac{z^3+i}{(z-2)(z-1)}.$$

The singular points are $z = 1, 2$.

**(c)** We wish to determine the singular points of $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$.

**Question 4. Brown p90. 5.**

**Solution.** Hence,

$$|\exp(2z+i)|$$

**Question 5. Brown p185. 4.**

**Solution.** We have the following summation formulation:

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}, \tag{1}$$

whenever $|z| < 1$. Substituting $r\mathrm{e}^{i\theta}$ for $z$ and separating the real and imaginary parts, we can re-write the LHS as

$$\begin{aligned}\sum_{n=1}^{\infty} z^n &= \sum_{n=0}^{\infty} (r\mathrm{e}^{i\theta})^n \\ &= \sum_{n=1}^{\infty} r^n \mathrm{e}^{in\theta} \\ &= \sum_{n=1}^{\infty} r^n \cos(n\theta) + i \sum_{n=0}^{\infty} r^n \sin(n\theta). \end{aligned} \tag{2}$$

Now, multiplying both denominator and numerator by the conjugate of $1-z$, $1-\overline{z}$, we can re-write the RHS as

$$\begin{aligned}\frac{z}{1-z} &= \frac{z(1-\overline{z})}{(1-z)(1-\overline{z})} \\ &= \frac{z - z\overline{z}}{1 + z\overline{z} - (z+\overline{z})}. \end{aligned}$$

Substituting $z = r\cos(\theta) + i\sin(\theta)$, $z\overline{z} = r^2$, and $z + \overline{z} = 2r\cos(\theta)$ to the last expression, and separating the real and imaginary parts, we obtain

$$\begin{aligned}\frac{z}{1-z} &= \frac{r\cos(\theta) - r^2 + ir\sin(\theta)}{1 + r^2 - 2r\cos(\theta)} \\ &= \frac{r\cos(\theta) - r^2}{1 + r^2 - 2r\cos(\theta)} + \frac{r\sin(\theta)}{1 + r^2 - 2r\cos(\theta)} i. \end{aligned} \tag{3}$$

Substituting 3 and 2 to 1, we obtain

$$\sum_{n=1}^{\infty} r^n \cos(n\theta) + i \sum_{n=0}^{\infty} r^n \sin(n\theta) = \frac{r\cos(\theta) - r^2}{1 + r^2 - 2r\cos(\theta)} + \frac{r\sin(\theta)}{1 + r^2 - 2r\cos(\theta)} i. \tag{4}$$

By the Theorem from section 61, we know that the real and imaginary part of the series must equal the real and imaginary part of the convergent value respectively. Hence, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} r^n \cos(n\theta) &= \frac{r\cos(\theta) - r^2}{1 - 2r\cos(\theta) + r^2} \\ \sum_{n=1}^{\infty} r^n \sin(n\theta) &= \frac{r\sin(\theta)}{1 - 2r\cos(\theta) + r^2}, \end{aligned}$$

when $0 < r < 1$. $\square$