Complex Analysis I:
Problem Set I

Youngduck Choi
CILVR Lab
New York University
yc1104@nyu.edu

Abstract

This work contains the solutions to the problem set I of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 2. Conjugate Harmonic Function.

Solution. We are given a polynomial of the following form:

$$u(x,y) \quad = \quad ax^3 + bx^2y + cxy^2 + dy^3.$$

Differentiating the given polynomial $u(x,y)$ with respect to x and y variables, we obtain

$$\frac{\partial u}{\partial x} = 3ax^2 + 2byx + cy^2,$$
$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by,$$
$$\frac{\partial u}{\partial y} = 3dy^2 + 2cxy + bx^2,$$
$$\frac{\partial^2 u}{\partial y^2} = 6dy + 2cx.$$

As the given polynomial is harmonic, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ must hold for all x and y. Hence, we obtain

$$6ax + 2by + 6dy + 2cx = 0,$$

which simplifies to

$$(3a + c)x + (3d + b)y = 0,$$

for all x and y. As the above equation must hold for all x and y, we have that

$$c = -3a \text{ and } b = -3d.$$

Substituting the above equations into the original polynomial, we obtain the most general harmonic polynomial of the given form as

$$u(x,y) = ax^3 - 3dx^2y - 3axy^2 + dy^3.$$

Now, we wish to compute the conjugate harmonic function of $u(x,y)$, denoted by $v(x,y)$. From the Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3ax^2 + 2byx + cy^2,$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 3dy^2 + 2cxy + bx^2.$$

Simplifying for the $v(x, y)$ term, we have

$$\frac{\partial v}{\partial y} = 3ax^2 + 2byx + cy^2,$$
$$\frac{\partial v}{\partial x} = -(3dy^2 + 2cxy + bx^2).$$

By taking the integral, we can solve for $v(x, v)$, and obtain

$$v(x, y) = \frac{1}{3}cy^3 - \frac{1}{3}bx^3 + bxy^2 - cx^2y$$

Question 3. The Complex Chain Rule.

Solution.

Question 4. $|f(z)| = 1$ implies constant f.

Solution. Let D be a unit disk, and $f : D \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for $z \in D$. Notice that $|f|$ is also a holomorhpic function, which can be re-written as $u(x, y) + iv(x, y)$. As $|f(z)| = 1$ for all $z \in D$, we obtain that $u^2 + v^2 = 1$ for all $(x, y) \in D$. Taking the partials, we obtain

$$2uu_x + 2vv_x = 0 \quad \text{and} \quad 2uu_y + 2vv_y = 0.$$

As f is holomorphic, from the Cauchy-Riemann equation, we have that

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Substituting the above equations to the partial equations and simplifying, we obtain

$$uv_y + vv_x = 0 \quad \text{and} \quad -uv_x + vv_y = 0,$$
$$u^2{v_y}^2 + v^2{v_x}^2 + 2uvv_xv_y = 0 \quad \text{and} \quad u^2{v_x}^2 + v^2{v_y}^2 - 2uvv_xv_y.$$

Hence, by adding the last two equations together, and factoring, we obtain

$$(u^2 + v^2)({v_x}^2 + {v_y}^2) = 0.$$

Since $|f| = 1$, $(u^2 + v^2)$ term cannot be 0, and we obtain that $v_x = 0$ and $v_y = 0$. By Cauchy-Riemann equation, we also have $u_x = 0$ and $u_y = 0$. Thus, using the Cauchy-Riemann theorem, $f'(z)$ for $z \in D$ can be written as

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= 0. \end{aligned}$$

Hence, $f'(z) = 0$ for all $z \in D$, and f is a constant function. $\square$

Question 4. .

Solution.