Complex Analysis I: Problem Set V

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Abstract

This work contains the solutions to the problem set V of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 177.2.

Solution. Let f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assume that $f(z) \neq 0$ for $z \in R$. Let g be a function on R, defined by $g(z) = \dfrac{1}{f(z)}$ for $z \in R$. From the $g(z) = \dfrac{1}{f(z)}$ relation, we can deduce that g is also continuous, analytic and not constant throughout the interior of R. Then, by the given corollary of the maximum modulus principle, we have that the maximum value of $|g(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior. Observe that $|g(z)| = |\dfrac{1}{f(z)}| = \dfrac{1}{|f(z)|}$. Since a modulus is strictly positive in this case, we have that maximum value of $|g(z)|$ corresponds to the minimum value of $|f(z)|$. In other words, the z^*, which is argmax$|g(z)|$ and lies on the boundary, is also the argmin$|f(z)|$. Consequently, we have shown that a minimum value is reached, and it occurs in the boundary of R and never in the interior. $\square$

Question 2. 177.4.

Solution. From the given hint, we have that

$$|f(z)|^2 \quad = \quad sin^2(x) + sinh^2(y).$$

Observe that it reaches maximum with respect to x on $\dfrac{\pi}{2}$ and with respect to y on 1, simply from the known properties of sin and $sinh$ functions. Also, $(\dfrac{\pi}{2}, 1)$ is a feasible point. Hence, we obtain that $|f(x)|^2$ reaches its maximum at $\dfrac{\pi}{2} + i$ on the boundary. $\square$

Question 3. 177.5.

Solution. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and not constant throughout the interior of R. Consider an exponential function $g(x) = \exp(f(z))$. Observe that g is also continuous on a closed bounded region R and non constant through out the interior of R. Then, by the problem 2, we have $|g(x)|$, which equals to $|\exp(u(x, y))|$, has a minimum value in R, which occurs on the boundary of R, but never in the interior. Since $\exp$ is an increasing function in reals, we have that the minima of $|\exp(u(x, y))|$ coincides with the minima of $u(x, y)$. Therefore, we have shown that the component function $u(x, y)$ has a minimum value in R, which occurs on the boundary of R and never in the interior. $\square$

Question 4. 195.3.

Solution. We wish to find the Maclaurin series expansion of the function

$$f(z) \quad = \quad \frac{z}{z^4+4} = \frac{z}{4} \cdot \frac{1}{1+(z^4/4)}.$$

From the geometric series, we have

$$\frac{1}{1-z} \quad = \quad \sum_{k=0}^{\infty} z^k,$$

for $|z| < 1$. Hence, by a change of variable, we have

$$\begin{aligned} \frac{1}{1+(\frac{z^4}{4})} \quad &= \quad \sum_{k=0}^{\infty} (-\frac{z^4}{4})^k \\ &= \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} z^{4k}, \end{aligned}$$

for $|z| < \sqrt{2}$. It follows that

$$\begin{aligned} f(z) \quad &= \quad \frac{z}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} z^{4k} \\ &= \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{4^{k+1}} z^{4k+1} \\ &= \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+2}} z^{4k+1}, \end{aligned}$$

for $|z| < \sqrt{2}$ as desired. □

Question 5. 195.6.

Solution. Observe that we can write $tanh$ as

$$tanh \quad = \quad \frac{sinh}{cosh}.$$

Observe that singularity happens at $cosh = 0$, which entails $z = (\frac{\pi}{2} + n\pi)i$. Therefore, we have analyticity for $|z| < \frac{\pi}{2}$, which is the largest circle within which the Maclaurin series is defined. Taking derivatives of $tanh$ yields

$$\begin{aligned} tanh'(z) \quad &= \quad \frac{1}{cosh^2(x)} \\ tanh''(z) \quad &= \quad -2\frac{sinh(x)}{cosh^3(x)} \\ tanh'''(z) \quad &= \quad -2\frac{(1-2sinh(x))}{cosh^4(x)}. \end{aligned}$$

Substituting 0 into x, we get

$$tanh(z) \quad = \quad z - \frac{1}{3}z^3 + ...$$

as desired for the first two nonzero terms of the series. □

Question 6. 195.11.
Solution. Observe that

$$\frac{1}{4z - z^2} = \frac{1}{4z} \cdot \frac{1}{1 - \frac{z}{4}}.$$

By the geometric series, we have

$$\frac{1}{1 - \frac{z}{4}} = \sum_{k=0}^{\infty} \frac{z^k}{4^k},$$

for $|z| < 4$. It follows that

$$\begin{aligned} \frac{1}{4z - z^2} &= \frac{1}{4z} \sum_{k=0}^{\infty} \frac{z^k}{4^k} \\ &= \frac{1}{4z} + \sum_{k=0}^{\infty} \frac{z^k}{4^{k+2}}, \end{aligned}$$

for $|z| < 4$ as desired. $\square$

Question 7. 205.5.
Solution. For $z \in D_1$, we have $|z| < 1$. Hence, by the geometric series, we obtain

$$\begin{aligned} \frac{1}{z-1} &= -\frac{1}{1-z} = \sum_{k=0}^{\infty} z^n \\ -\frac{1}{z-2} &= \frac{1}{2} \frac{1}{1 - \frac{z}{2}} = \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}}. \end{aligned}$$

It follows that

$$\begin{aligned} f(z) &= -\sum_{k=0}^{\infty} z^k + \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} \\ &= \sum_{k=0}^{\infty} (-1 + 2^{-k-1}) z^k, \end{aligned}$$

for $z \in D_1$.

Question 8. 205.6.
Solution. By partial fraction decomposition, we have

$$\frac{z}{(z-1)(z-3)} = \frac{3}{2} \cdot \frac{1}{z-3} - \frac{1}{2} \cdot \frac{1}{z-1}.$$

The above equality can be written as

$$\frac{z}{(z-1)(z-3)} = -\frac{3}{4} \cdot \frac{1}{1 - \frac{z-1}{2}} + \frac{1}{2} \cdot \frac{1}{1-z}.$$

Since $0 < |z-1| < 2$, we have $0 < |\frac{z-1}{2}| < 1$. Therefore, by the geometric series, we have

$$\begin{aligned} \frac{z}{(z-1)(z-3)} &= -\frac{3}{4} \sum_{k=0}^{\infty} \frac{(z-1)^k}{2^k} - \frac{1}{2(z-1)} \\ &= -3 \sum_{k=0}^{\infty} \frac{(z-1)^k}{2^{k+2}} - \frac{1}{2(z-1)}, \end{aligned}$$

for $0 < |z-1| < 2$ as desired. $\square$

Question 9. 205.9.
Solution.

Question 10. 224.1.
Solution. Observe that

$$\begin{aligned} e^z &= \sum_{k=0}^{\infty} \frac{1}{k!} z^k \\ \frac{1}{z(z^2+1)} &= \frac{1}{z} \sum_{k=0}^{\infty} (-z^2)^k, \\ &= \sum_{k=0}^{\infty} (-1)^k z^{2k+1} \end{aligned}$$

$|z| < 1$. By multiplying out the first few terms manually, we obtain

$$\frac{e^z}{z^2+1} = \frac{1}{z} + 1 - \frac{1}{2} z - \frac{5}{6} z^2 + ...,$$

for $|z| < 1$. $\square$

Question 11. 224.3.
Solution. We have previously shown that

$$\begin{aligned} \sin(z) &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} z - \frac{z^3}{3!} + \frac{z^5}{5!} + ..., \end{aligned}$$

for $|z| < \pi$. By doing the division of the first several terms by hand, we obtain

$$\csc(z) = \frac{1}{z} + \frac{1}{3!} z + [\frac{1}{(3!)^2} - \frac{1}{5!}] z^3 + ...,$$

for $|z| < \pi$. $\square$

Question 12. 224.5.
Solution. From the Laurent series theorem, we have that

$$b_1 = \frac{1}{2\pi i} \int_C \frac{1}{z^2 \sinh z} dz,$$

where b_1 is the coefficient of the Laurent series for the $\frac{1}{z}$ and C is the positively oriented unit circle $|z| = 1$. Hence, by the given Laurent series, we obtain

$$-\frac{1}{6} = \frac{1}{2\pi i} \int_C \frac{1}{z^2 \sinh z} dz.$$

Re-arranging the terms yields

$$\int_C \frac{1}{z^2 \sinh z} dz = -\frac{\pi}{3},$$

where C is the positively oriented unit circle $|z| = 1$. $\square$

Question 13. 224.8.
Solution.

Question 14. 224.9.

Solution. We have $\dfrac{1}{\cosh z}$. This function is singular when $\cosh z = 0$, which occurs at $z = (\dfrac{\pi}{2} + n\pi)i$ for all $n \in \mathbb{Z}$. Therefore, the given function is analytic for the disk $|z| < \dfrac{\pi}{2}$. We have that

$$\cosh(z) \quad = \quad \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k!}.$$

By doing the division by hand for the first few terms, we have

$$\frac{1}{\cosh(z)} \quad = \quad 1 - \frac{1}{2!}z^2 + \frac{5}{4!}z^4 - \frac{61}{6!}z^6 + ...$$

Therefore, we have shown that

$$\begin{aligned} E_0 &= 1 \\ E_2 &= -1 \\ E_4 &= 5 \\ E_6 &= -61, \end{aligned}$$

as desired. $\square$