
Complex Analysis I:

Problem Set V

Youngduck Choi
CILVR Lab
New York University
yc1104@nyu.edu

Abstract

This work contains the solutions to the problem set V of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 177.2.

Solution. Let f be continuous on a closed bounded region R , and let it be analytic and not constant throughout the interior of R . Assume that $f(z) \neq 0$ for $z \in R$. Let g be a function on R , defined by $g(z) = \frac{1}{f(z)}$ for $z \in R$. From the $g(z) = \frac{1}{f(z)}$ relation, we can deduce that g is also continuous, analytic and not constant throughout the interior of R . Then, by the given corollary of the maximum modulus principle, we have that the maximum value of $|g(z)|$ in R , which is always reached, occurs somewhere on the boundary of R and never in the interior. Observe that $|g(z)| = \left| \frac{1}{f(z)} \right| = \frac{1}{|f(z)|}$. Since a modulus is strictly positive in this case, we have that maximum value of $|g(z)|$ corresponds to the minimum value of $|f(z)|$. In other words, the z^* , which is $\operatorname{argmax}|g(z)|$ and lies on the boundary, is also the $\operatorname{argmin}|f(z)|$. Consequently, we have shown that a minimum value is reached, and it occurs in the boundary of R and never in the interior. \square

Question 2. 177.4.

Solution. From the given hint, we have that

$$|f(z)|^2 = \sin^2(x) + \sinh^2(y).$$

Observe that it reaches maximum with respect to x on $\frac{\pi}{2}$ and with respect to y on 1, simply from the known properties of \sin and \sinh functions. Also, $(\frac{\pi}{2}, 1)$ is a feasible point. Hence, we obtain that $|f(x)|^2$ reaches its maximum at $\frac{\pi}{2} + i$ on the boundary. \square

Question 3. 177.5.

Solution. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and not constant throughout the interior of R .

Question 4. 195.3.

Solution. We wish to find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4 + 4} = \frac{z}{4} \cdot \frac{1}{1 + (z^4/4)}.$$

From the geometric series, we have

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k,$$

for $|z| < 1$. Hence, by a change of variable, we have

$$\begin{aligned} \frac{1}{1 + \left(\frac{z^4}{4}\right)} &= \sum_{k=0}^{\infty} \left(-\frac{z^4}{4}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} z^{4k}, \end{aligned}$$

for $|z| < \sqrt[4]{2}$. It follows that

$$\begin{aligned} f(z) &= \frac{z}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} z^{4k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^{k+1}} z^{4k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+2}} z^{4k+1}, \end{aligned}$$

for $|z| < \sqrt[4]{2}$ as desired. \square

Question 5. 195.6.

Solution. Observe that we can write \tanh as

$$\tanh = \frac{\sinh}{\cosh}.$$

Observe that singularity happens at $\cosh = 0$, which entails $z = \left(\frac{\pi}{2} + n\pi\right)i$. Therefore, we have analyticity for $|z| < \frac{\pi}{2}$, which is the largest circle within which the Maclaurin series is defined. Taking derivatives of \tanh yields

$$\begin{aligned} \tanh'(z) &= \frac{1}{\cosh^2(x)} \\ \tanh''(z) &= -2 \frac{\sinh(x)}{\cosh^3(x)} \\ \tanh'''(z) &= -2 \frac{(1 - 2\sinh(x))}{\cosh^4(x)}. \end{aligned}$$

Substituting 0 into x , we get

$$\tanh(z) = z - \frac{1}{3}z^3 + \dots$$

as desired for the first two nonzero terms of the series.

Question 6. 195.11.

Solution. Observe that

$$\frac{1}{4z - z^2} = \frac{1}{4z} \cdot \frac{1}{1 - \frac{z}{4}}.$$

By the geometric series, we have

$$\frac{1}{1 - \frac{z}{4}} = \sum_{k=0}^{\infty} \frac{z^k}{4^k},$$

for $|z| < 4$. It follows that

$$\begin{aligned} \frac{1}{4z - z^2} &= \frac{1}{4z} \sum_{k=0}^{\infty} \frac{z^k}{4^k} \\ &= \frac{1}{4z} + \sum_{k=0}^{\infty} \frac{z^k}{4^{k+2}}, \end{aligned}$$

for $|z| < 4$ as desired. \square

Question 10. 224.1.

Solution.

Question 11. 224.3.

Solution.

Question 12. 224.5.

Solution.

Question 13. 224.8.

Solution.

Question 14. 224.9.

Solution.