Complex Analysis I:
Problem Set III

Youngduck Choi
CILVR Lab
New York University
yc1104@nyu.edu

Abstract

This work contains the solutions to the problem set III of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. Brown p.95-4.
Solution. We are given the following branch:

$$log(z) = ln(r) + i\theta \text{ and } r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}.$$

With the given branch, the computations yield

$$\begin{aligned} log(i^2) &= log(-1) = ln(1) + i\pi \\ 2log(i) &= 2(ln(1) + i\frac{5\pi}{2}) = i\frac{5\pi}{2}. \end{aligned}$$

Hence, we have that $log(i^2) = 2log(i)$ for this particular branch. □

Question 2. Brown p.95-11.
Solution. We wish to show that $ln(x^2 + y^2)$ is harmonic. Firstly, we can compute the partials as follow:

$$\begin{aligned} u_x &= \frac{2x}{x^2 + y^2} \\ u_{xx} &= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}, \end{aligned}$$

for $x, y \neq 0$. By symmtry, we also have

$$\begin{aligned} u_y &= \frac{2y}{x^2 + y^2} \\ u_{yy} &= \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}, \end{aligned}$$

for $x, y \neq 0$. Hence, we have that $u_{xx} + u_{yy} = 0$ for $x, y \neq 0$, and consequently $ln(x^2 + y^2)$ is harmonic. Now, we show that $ln(x^2 + y^2)$ is harmonic in a different way. Using the polar cordinats, we have $f = ln(r^2)$, excluding the origin. Computing the partials yields

$$\begin{aligned} f_r &= \frac{2}{r} \\ f_{rr} &= \frac{-2}{r^2} \\ f_{\theta\theta} &= 0. \end{aligned}$$

It follows that

$$f_{rr} = \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta theta} \quad = \quad \frac{-2}{r^2} + \frac{1}{r}\frac{2}{r} + 0 = 0.$$

Therefore, the given function is harmonic on $\mathbb{C} \setminus \{0\}$. $\square$

Question 3. Brown p.103-1.

Solution. We can re-write the expression $(1+i)^i$ as

$$\begin{aligned}
(1+i)^i &= \exp\left(i\ln(1+i)\right) \\
&= \exp\left(i(\ln(\sqrt{2}) + i(\frac{\pi}{4} + 2\pi n))\right) \\
&= \exp\left(-\frac{\pi}{4} - 2\pi n\right)\exp\left(i(\frac{\ln(2)}{2})\right) \\
&= \exp\left(-\frac{\pi}{4} + 2\pi n\right)\exp\left(i(\frac{\ln(2)}{2})\right)
\end{aligned}$$

for $n = 0, \pm 1, \pm 2, \ldots$. Now, we can also re-write the expression $\frac{1}{i^{2i}}$ as

$$\begin{aligned}
\frac{1}{i^{2i}} &= e^{-2i} \\
&= e^{-2i\log(i)} \\
&= \exp\left(-2i(\ln(1) + i(\frac{\pi}{2} + 2n\pi)\right) \\
&= \exp\left((4n+1)\pi\right),
\end{aligned}$$

for $n = 0, \pm 1, \pm 2, \ldots$ as desired. $\square$

Question 4. Brown p.133-3.

Solution. We are given a function $f(z) = \pi\exp(\pi\bar{z})$, contour C as the boundary of the square with vertices at the points $0, 1, 1+i, i$, and the orientation of C being the counter-clockwise direction. Separating the integral with 4 different legs, we have

$$\begin{aligned}
\int_C \pi\exp(\pi\bar{z})dz &= \int_{C_1} \pi\exp(\pi\bar{z})dz + \int_{C_2} \pi\exp(\pi\bar{z})dz \\
&+ \int_{C_3} \pi\exp(\pi\bar{z})dz + \int_{C_4} \pi\exp(\pi\bar{z})dz,
\end{aligned}$$

where the legs can be written as

$$\begin{aligned}
C_1 : z &= x(0 \leq x \leq 1) \\
C_2 : z &= 1 + iy(0 \leq y \leq 1) \\
C_3 : z &= 1 - x + i(0 \leq x \leq 1) \\
C_4 : z &= i(1-y)(0 \leq y \leq 1).
\end{aligned}$$

Simplifying the leg integrals with their particular values, we obtain

$$\begin{aligned}
\int_{C_1} \pi\exp(\pi\bar{z})dz &= \pi\int_0^1 \exp(\pi x)dx \\
&= e^{\pi} - 1 \\
\int_{C_2} \pi\exp(\pi\bar{z})dz &= \pi\int_0^1 \exp(\pi(1-iy))dy \\
&= 2e^{\pi} \\
\int_{C_3} \pi\exp(\pi\bar{z})dz &= \pi\int_0^1 \exp(\pi(1-x-i))dx \\
&= e^{\pi} - 1 \\
\int_{C_4} \pi\exp(\pi\bar{z})dz &= \pi\int_0^1 \exp(\pi i(y-1))dy \\
&= -2.
\end{aligned}$$

Hence, adding them up, we obtain

$$\int_C \pi e^{\pi \bar{z}} dz = 4(e^{\pi} - 1),$$

as desired. $\square$

Question 5. Brown p.133-8.

Solution. We are given that

$$\int_C f(z)dz \quad = \quad \int_\theta f[z(\theta)]z^{'}(\theta).$$

We can compute the inner term of the integral as

$$f[z(\theta)]z^{'}(\theta) = R^{a-}e^{i(a-1)\theta}Rie^{i\theta}d\theta \quad = \quad R^a ie^{ia\theta}d\theta.$$

Hence, the integral can be evaluated as

$$\begin{aligned} \int_C f(z)dz &= \int_{-\pi}^{\pi} R^a ie^{ia\theta}d\theta \\ &= \frac{R^a}{a}(e^{ia\pi} - e^{-ia\pi}) \\ &= i\frac{2R^a}{a}sin(a\pi), \end{aligned}$$

as desired. $\square$

Question 6. Brown p.138-1.

Solution. (a) We wish to show that $|\int_C \frac{z+4}{z^3-1}dz| \leq \frac{6\pi}{7}$, where C is a quarter circle from 2 to $2i$. We have the length of the contour is π. Now, we compute the upper bound of $|f(z)|$ along the contour:

$$\left|\frac{z+4}{z^3-1}\right| \quad = \quad \frac{|z+4|}{|z^3-1|}.$$

By the triangle inequality, we have

$$|z+4| \leq |z| + |4| = 6.$$

Again, by the triangle inequality, we have

$$|z^3 - 1| \geq |z^3| - |1| = |z|^3 - 1 = 8 - 1 = 7.$$

It follows that

$$\frac{1}{|z^3-1|} \leq \frac{1}{7}.$$

Consequently, combining the two inequalities yields

$$\left|\frac{z+4}{z^3-1}\right| \leq \frac{6}{7}.$$

Since f is piece-wise continous on C, we have

$$\left|\int_C \frac{z+}{z^3-1}dz\right| \leq \frac{6\pi}{7},$$

as desired.

(b) We wish to show that $\left|\int_C \frac{dz}{z^2-1}\right| \leq \frac{\pi}{3}$, where C is a quarter circle from 2 to $2i$. The length of the contour as π. We now compute the upper bound of $|f(z)|$ along the contour, which can be written as $\left|\frac{1}{z^2-1}\right|$. By the triangle inequality, we have

$$|z^2-1| \geq |z^2| - |1| = 4 - 1 = 3.$$

Hence, we have

$$\left|\frac{1}{z^2-1}\right| = \frac{1}{|z^2-1|} \leq \frac{1}{3}.$$

Since f is piece-wise continuous on C, we have

$$\left|\int_C \frac{dz}{z^2-1}\right| \leq \frac{\pi}{3},$$

as desired. $\square$

Question 7. Brown p.138-2.

Solution. We wish to show that $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$, where C is a straight line from i to 1. The contour length is $\sqrt{2}$. Observe that of all the points on the line segment, the midpoint is the closest to the origin, that distance being $d = \frac{\sqrt{2}}{2}$. Hence, we obtain

$$|z| \geq \frac{\sqrt{2}}{2}.$$

Consequently, it follows that

$$|z|^4 \geq \frac{1}{4} \text{ and } \frac{1}{|z|^4} \leq 4.$$

Since f is piece-wise continuous on C, we have

$$\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2},$$

as desired. $\square$

Question 8. Brown p.138-3.

Solution. We wish to show that $\left|\int_C (e^z - \bar{z})dz\right| \leq 60$, where C is the boundary of the triangle with vertices at the points 0, $3i$, and -4. Notice that the contour length is simply 12, as the triangle is a $3-4-5$ triangle. Now, we wish to compute an upper bound of $|f(z)|$ along the contour. Observe that

$$|e^z - \bar{z}| \leq e^x + \sqrt{x^2+y^2},$$

where $z = x + iy$. As $x \leq 0$, we have that $e^x \leq 1$. Furthermore, $\sqrt{x^2+y^2} \leq 4$, as the longer side of the triangle from the origin is 4. Therefore, we have

$$|e^z - \bar{z}| \leq 5,$$

and consequently, as f is piecewise continuous, we obtain

$$\left|\int_C (e^z - \bar{z})dz\right| \leq 60,$$

as desired. $\square$

Question 9. Brown p.138-5.

Solution. We wish to show the following inequality:

$$\left|\int_{C_R} \frac{\log(z)}{z^2} dz\right| \leq 2\pi(\frac{\pi + \ln(R)}{R}),$$

where C_R denotes the contour along the circle $|z| = R(R > 1)$. Hence, the length of the contour is simply $2\pi R$. Now, we compute an upper bound of $\left|\frac{\log(z)}{z^2}\right|$ along the contour. Basic algebraic manipulations and using triangle inequality, we have

$$\begin{aligned}\left|\frac{\log(z)}{z^2}\right| &= \frac{|\log(z)|}{|z^2|} = \frac{|\ln(R) + i\theta|}{R^2} \\ &\leq \frac{\ln(R) + |\theta|}{R^2} \leq \frac{\ln(R) + \pi}{R^2},\end{aligned}$$

as $-\pi \leq \theta \leq \pi$. Since the given function is piecewise continuous on C, we obtain

$$\left|\int_{C_R} \frac{\log(z)}{z^2} dz\right| \leq 2\pi(\frac{\pi + \ln(R)}{R}),$$

as desired. By using the l'Hospital's rule. we have

$$\lim_{R\to\infty} 2\pi(\frac{\pi + \ln(R)}{R}) \quad = \quad \lim_{R\to\infty} \frac{1}{R} = 0.$$

Therefore, as $R \to \infty$ the upper bound of the absolute value of the integral tends to 0. Consequently, we see that the integral must tend to 0 as well. $\square$

Question 10. Brown p.138-8.

Solution. (a) On the vertical side of the square, we have $x = (N + \frac{1}{2})\pi$. Therefore, $\sin(x) = -1$ or 1. Hence, as we have $|\sin(z)| \geq |\sin(x)|$, we obtain $|\sin(z)| \geq 1$. On the horizontal side of the square, we have $y = (N + \frac{1}{2})\pi$. Therefore, $\sinh(y) = \sinh(\pm\frac{1}{2}\pi)$. Hence, as we have $|\sin(z)| \geq |\sinh(y)|$, we obtain $|\sin(z)| \geq |\sinh(\frac{\pi}{2})|$. Cosnequently, there is a positive constant A, independent of N, such that $|\sin(z)| \leq A$ for all points z lying on the contour C_N.

(b) We wish to show that $\left|\int_{C_N} \frac{1}{z^2 sin(z)} dz\right| \leq \frac{16}{(2N+1)\pi A}$. The length of the C_N contour is $8(N + \frac{1}{2})\pi$. Now, we compute an upper bound of $\left|\frac{1}{z^2 sin(z)}\right|$ along the contour. We have that $|z^2| \geq ((N + \frac{1}{2})\pi)^2$ and $|sin(z)| \geq A$ on C_N. It follows that

$$\begin{aligned}\left|\frac{1}{z^2 sin(z)}\right| &= \frac{1}{|z^2||sin(z)|} \\ &\leq \frac{1}{((N + \frac{1}{2})\pi)^2 A},\end{aligned}$$

holds on C_N. Therfore, as $\frac{1}{z^2 sin(z)}$ is piece-wise continuous, we have

$$\begin{aligned}\left|\int_{C_N} \frac{1}{z^2 sin(z)} dz\right| &\leq \frac{8(N + \frac{1}{2})\pi}{((N + \frac{1}{2})\pi)^2 A} \\ &= \frac{16}{(2N+1)\pi A},\end{aligned}$$

as desired. $\square$