Complex Analysis I:
Problem Set VI

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Abstract

This work contains the solutions to the problem set VI of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 237-2.

Solution. (a) We have

$$\begin{aligned}\frac{1}{z+z^2} &= \frac{1}{z}\frac{1}{1+z} \\ &= \frac{1}{z}(1-z+z^2\ldots) \\ &= \frac{1}{z}-1+z\ldots\end{aligned}$$

for $0<|z|<1$. The coefficient of $\frac{1}{z}$ term is 1. Hence, the residue at $z=0$ is 1.

(b) We have

$$\begin{aligned}z\cos(\frac{1}{z}) &= z(1-\frac{1}{2!}\frac{1}{z^2}+\frac{1}{4!}\frac{1}{z^4}\ldots) \\ &= z-\frac{1}{2!}\frac{1}{z}+\frac{1}{4!}\frac{1}{z^3}\ldots\end{aligned}$$

for $|z|<\infty$. The coefficient of $\frac{1}{z}$ term is 0. Hence, the residue at $z=0$ is 0.

(c) We have

$$\begin{aligned}\frac{z-\sin(z)}{z} &= \frac{1}{z}\cdot\frac{1}{\sin(z)} \\ &= \frac{1}{z}(z-(z-\frac{z^3}{3!}+\frac{z^5}{5!}\ldots))\end{aligned}$$

for $0<|z|<\infty$. The coefficient of $\frac{1}{z}$ term is 0. Hence, the residue at $z=0$ is 0.

(d) We have

$$\frac{\cot(z)}{z^4} = \frac{1}{z^4}\cdot\frac{\cos(z)}{\sin(z)}$$

By dividing the Maclaurin series representation of $\cos$ by $\sin$, we obtain

$$\frac{\cos(z)}{\sin(z)} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45}\dots$$

for $0 < |z| < \pi$. It follows that

$$\frac{\cot(z)}{z^4} = \frac{1}{z^5} - \frac{1}{3}\cdot\frac{1}{z^3} - \frac{1}{45}\cdot\frac{1}{z}\dots$$

for $0 < |z| < \pi$. The coefficient of $\frac{1}{z}$ is $-\frac{1}{45}$. Hence, the residue at $z = 0$ is $-\frac{1}{45}$.

(e) We have

$$\frac{\sinh(z)}{z^4(1-z^2)} = \sinh(z)\cdot\frac{1}{z^4}\cdot\frac{1}{1-z^2}$$

By substituting the Maclaurin series, we obtain

$$\begin{aligned}\frac{\sinh(z)}{z^4(1-z^2)} &= \frac{1}{z^4}(z + \frac{1}{6}z^3 + \frac{1}{120}z^5\dots)(1 + z^2 + z^4\dots)\\ &= \frac{1}{z^3} + \frac{7}{6}\frac{1}{z}\dots\end{aligned}$$

The coefficient of $\frac{1}{z}$ is $\frac{7}{6}$. Hence, the residue at $z = 0$ is $\frac{7}{6}$. □

Question 2. Brown p.237-2.

Solution. By the Cauchy's residue theorem, we can evaluate the integral by computing the residues.

(a) We compute the residue of the integrand at $z = 0$. Using the Laurent series of $\frac{\exp(-z)}{z^2}$, we obtain

$$\begin{aligned}\frac{\exp(-z^2)}{z^2} &= \frac{1}{z^2}(1 - \frac{1}{1!}z + \frac{1}{2!}z^2\dots)\\ &= \frac{1}{z^2} - \frac{1}{1!}\frac{1}{z}\dots\end{aligned}$$

Hence, the residue at $z = 0$ is -1. Therefore, by the Cauchy's residue theorem, we obtain

$$\int_C \frac{\exp(-z)}{z^2}dz = 2\pi i(-1) = -2\pi i.$$

(b) Observe that

$$\begin{aligned}e^{-z} &= e^{-1}e^{-(z-1)}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{e(z-1)^n}.\end{aligned}$$

Therefore, the coefficient of $\frac{1}{z-1}$ is $-\frac{1}{e}$. Hence, by the residue theorem, we have

$$\int_C \frac{\exp(-z)}{(z-1)^2}dz = -\frac{2\pi i}{e},$$

for C being the positively oriented circle $|z| = 3$.

(c) Observe that

$$z^2\exp(\frac{1}{z}) = z^2(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3}...)$$

for $0 < |z| < \infty$. The coefficient of $\frac{1}{z}$ term is $\frac{1}{6}$. Hence, by the residue theorem, we have

$$\int_C \frac{\exp(-z)}{(z-1)^2} dz = \frac{\pi i}{3}.$$

(d) Observe that the singularities happen at $z = 0$ and $z = 2$. Using poles, we have

$$\begin{aligned} \text{Res}_{z=0} \frac{z+1}{z(z-2)} &= \frac{-1}{2} \\ \text{Res}_{z=2} \frac{z+1}{z(z-2)} &= \frac{3}{2}. \end{aligned}$$

Hence, by the residue theorem, we have

$$\int_C \frac{z+1}{z^2 - 2z} dz = 2\pi i,$$

for the given contour. □

Question 3. Brown p.242-1.

Solution. Simple computations show the following results: **(a)** $z = 0$ is essential singular point, and the principal part is $\frac{1}{2!}\frac{1}{z} + \frac{1}{3!}\frac{1}{z^2} \cdots$.

(b) $z = -1$ is the singular point with the principal part being $\frac{1}{z+1}$. $z = -1$ is a simple pole.

(c) $z = 0$ is the singular point with the 0 principal part. $z = 0$ is removable.

(d) $z = 0$ is the singular point with the principal part $\frac{1}{z}$. $z = 0$ is a simple pole.

(e) $z = 2$ is the singular point and the principal part is the given function itself. It is a pole of order 3. □

Question 4. Brown p.246-2.

Solution. (a) Observe that

$$f(z) = \frac{\phi(z)}{z+1},$$

where $\phi(z) = z^{\frac{1}{4}}$. It follows that

$$\text{Res}_{z=-1} f(z) = \phi(-1) = \frac{1+i}{\sqrt{2}}.$$

(b) Observe that

$$\frac{\text{Log}(z)}{(z^2+1)^2} = \frac{\phi(z)}{(z-i)^2},$$

where $\phi(z) = \frac{\text{Log}(z)}{(z+i)^2}$. It follows that

$$\begin{aligned} \text{Res}_{z=i} \frac{\text{Log}(z)}{(z^2+1)^2} &= \phi'(i) \\ &= \frac{\pi + 2i}{8}. \end{aligned}$$

Question 5. Brown p.246-4.

Solution. We wish to evaluate

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz,$$

for C being the circle $|z-2|=2$, taken in the counterclockwise direction. Clearly, the singularities of the integrand are 1 and $\pm 3i$. The residues at the singularities can be computed as follows:

$$\begin{aligned}
\mathrm{Res}_{z=1}\frac{3z^3+2}{(z-1)(z^2+9)} &= \frac{3z^3+2}{z^2+9}]_{z=1} = \frac{1}{2} \\
\mathrm{Res}_{z=3i}\frac{3z^3+2}{(z-1)(z^2+9)} &= \frac{3z^3+2}{(z-1)(z+3i)}]_{z=3i} = \frac{15+49i}{12} \\
\mathrm{Res}_{z=-3i}\frac{3z^3+2}{(z-1)(z^2+9)} &= \frac{3z^3+2}{(z-1)(z-3i)}]_{z=-3i} = \frac{15-49i}{12}.
\end{aligned}$$

For the part (a), we have $z=1$ is the only singularity inside C. Hence, by Cauchy's residue theorem, we have

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz = \pi i.$$

For the part (b), we have all singularities inside C. It follows that

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz = 2\pi i(\frac{1}{2}+\frac{15+49i}{12}+\frac{15-49i}{12}) = 6\pi i.$$

This completes the computations. $\square$

Question 1. Brown p.246-5.

Solution. We wish to evaluate

$$\int_C \frac{1}{z^3(z+4)}dz,$$

for C being the positively oriented circle $|z|=2$. Clearly, the singularities are at $z=0$ and $z=-4$. The residues of the singularities can be computed as follows:

$$\begin{aligned}
\mathrm{Res}_{z=0}\frac{1}{z^3(z+4)} &= \frac{1}{64} \\
\mathrm{Res}_{z=-4}\frac{1}{z^3(z+4)} &= -\frac{1}{64}.
\end{aligned}$$

For the part (a), we have $z=0$ is the only singularity inside C. By the residue theorem, we have

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz = \frac{\pi i}{32}.$$

For the part (b), we have both singularities inside C. By the residue theorem, we have

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)}dz = 0.$$

This completes the computation. $\square$

Question 1. Brown p.246-6.

Solution. We wish to evaluate

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz$$

where C is the circle $|z| = 2$, described in the positive sense. The singularities of the integrand, that are interior to C, are $0, \pm i$. The residues are respectively

$$\begin{aligned} \frac{\cosh(\pi z)}{z^2+1}|_{z=0} &= 1 \\ \frac{\cosh(\pi z)}{z(z+i)}|_{z=i} &= \frac{1}{2} \\ \frac{\cosh(\pi z)}{z(z-i)}|_{z=-i} &= \frac{1}{2}. \end{aligned}$$

Hence, by the Cauchy residue theorem, we have

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)} dz = 4\pi i,$$

as desired. $\square$