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# Complex Analysis I: Problem Set VIII

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## Abstract

This work contains the solutions to the problem set VIII of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

### Question 273-12.

**Solution.** (a) As  $\exp(iz^2)$  is entire, by Cauchy-Goursat theorem, we have

$$\int_{C_1} \exp(iz^2) dz + \int_{C_R} \exp(iz^2) dz + \int_{C_2} \exp(iz^2) dz = 0,$$

where  $C_1$  refers to the segment from 0 to  $R$ , and  $C_2$  refers to the segment from  $Re^{i\frac{\pi}{4}}$  to 0. Observe that we can parametrize points on  $C_1$  as  $z = x(0 \leq x \leq R)$ , and points on  $C_2$  as  $z = re^{i\frac{\pi}{4}}(0 \leq r \leq R)$ . Therefore, the above equality can be written as

$$\int_0^R \exp(ix^2) dx = e^{i\frac{\pi}{4}} \int_0^R e^{-r^2} dr - \int_{C_R} e^{iz^2} dz.$$

Equating the real and imaginary parts separately from the above equation yields

$$\begin{aligned} \int_0^R \cos(x^2) dx &= \frac{1}{\sqrt{2}} \int_0^R e^{-r^2} dr - \operatorname{Re} \int_{C_R} e^{iz^2} dz, \\ \int_0^R \sin(x^2) dx &= \frac{1}{\sqrt{2}} \int_0^R e^{-r^2} dr - \operatorname{Im} \int_{C_R} e^{iz^2} dz. \end{aligned}$$

□

(b) We can parametrize the points on  $C_R$  as  $z = Re^{i\theta}(0 \leq \theta \leq \frac{\pi}{4})$ . It follows that

$$\begin{aligned} \int_{C_R} e^{iz^2} &= \int_0^{\frac{\pi}{4}} e^{iR^2 e^{i2\theta}} Rie^{i\theta} d\theta \\ &= iR \int_0^{\frac{\pi}{4}} e^{-R^2 \sin(2\theta)} e^{iR^2 \cos(2\theta)} e^{i\theta} d\theta. \end{aligned}$$

Consequently, we have

$$\begin{aligned}
\left| \int_{C_R} e^{iz^2} \right| &= \left| iR \int_0^{\frac{\pi}{4}} e^{-R^2 \sin(2\theta)} e^{iR^2 \cos(2\theta)} e^{i\theta} d\theta \right| \\
&\leq R \int_0^{\frac{\pi}{4}} \left| e^{-R^2 \sin(2\theta)} \right| \left| e^{iR^2 \cos(2\theta)} \right| \left| e^{i\theta} \right| d\theta \\
&= R \int_0^{\frac{\pi}{4}} \left| e^{-R^2 \sin(2\theta)} \right| d\theta \\
&= R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin(2\theta)} d\theta \\
&= \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-R^2 \sin(\phi)} d\theta,
\end{aligned}$$

by change of variable  $\phi = 2\theta$ . From Jordan's lemma in section 88, pg. 271, continues to

$$\begin{aligned}
\left| \int_{C_R} e^{iz^2} \right| &\leq \frac{R}{2} \frac{\pi}{2R^2} \\
&= \frac{\pi}{4R},
\end{aligned}$$

which limits 0 as  $R \rightarrow \infty$ . Hence, the integral limits 0 as  $R \rightarrow \infty$ . □

(c) Combining the result from (a),(b) and the given formula, we have

$$\begin{aligned}
\int_0^\infty \cos(x^2) dx &= \frac{1}{\sqrt{2}} \int_0^\infty e^{-x^2} dx \\
&= \frac{1}{2} \sqrt{\frac{\pi}{2}}, \\
\int_0^\infty \sin(x^2) dx &= \frac{1}{\sqrt{2}} \int_0^\infty e^{-x^2} dx \\
&= \frac{1}{2} \sqrt{\frac{\pi}{2}},
\end{aligned}$$

as desired. □

### Question 278-1.

**Solution.** Let  $f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}$ . On the indented contour in figure 108, by Cauchy-Goursat theorem, we obtain

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

Observe that on  $L_1$  and  $L_2$ , we have the following parametric representations:

$$L_1 : z = re^{i0} = r(\rho \leq r \leq R) \text{ and } -L_2 : z = re^{i\pi} = -r(\rho \leq r \leq R).$$

It follows that

$$\begin{aligned}
\int_{L_1} f(z) dz + \int_{L_2} f(z) dz &= \int_{L_1} f(z) dz - \int_{-L_2} f(z) dz \\
&= \int_\rho^R \frac{e^{iar} - e^{ibr}}{r^2} dr + \int_\rho^R \frac{e^{-iar} - e^{-ibr}}{r^2} dr \\
&= \int_\rho^R \frac{(e^{iar} + e^{-iar}) - (e^{ibr} + e^{-ibr})}{r^2} dr \\
&= 2 \int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2} dr.
\end{aligned}$$

Now, writing the Laurent series of  $f(z)$  gives

$$\begin{aligned} f(z) &= \frac{e^{iaz} - e^{ibz}}{z^2} \\ &= \frac{1}{z^2} \left[ \left( 1 + \frac{iaz}{1!} + \frac{(iaz)^2}{2!} \dots \right) + \left( 1 + \frac{ibz}{1!} + \frac{(ibz)^2}{2!} \dots \right) \right] \\ &= \frac{i(a-b)}{z} + \dots \end{aligned}$$

Therefore, we see that  $z = 0$  is a simple pole of  $f(z)$  and the residue  $B_0$  equals  $i(a-b)$ . Consequently, we obtain that

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z) dz = -i(a-b)\pi i = \pi(a-b).$$

Observe that

$$|f(z)| \leq \frac{|e^{iaz}| + |e^{ibz}|}{|z|^2}.$$

For  $z$  on  $C_R$ , we have

$$|f(z)| \leq \frac{2}{R^2}.$$

Therefore, by  $ML$  inequality, it follows that

$$\begin{aligned} \left| \int_{C_R} f(z) dz \right| &\leq \int_{C_R} |f(z)| dz \\ &\leq \frac{2}{R^2} \pi R \\ &= \frac{2\pi}{R}. \end{aligned}$$

Therefore, as  $R \rightarrow \infty$   $\frac{2\pi}{R} \rightarrow 0$ , the integral  $\int_{C_R} f(z) dz$  tends to 0 as  $R \rightarrow \infty$ . Consequently,

$$\int_0^\infty \frac{\cos(ar) - \cos(br)}{r^2} dr = \frac{\pi}{2}(b-a).$$

Setting  $b = 2$  and  $a = 0$  and using the identity  $1 - \cos(2x) = 2\sin^2(x)$ , we finally obtain

$$\int_0^\infty \frac{\sin^2(x)}{x^2} dx = \frac{\pi}{2},$$

as desired.  $\square$

### Question 283-2.

**Solution.** We use the following branch as suggested:

$$\begin{aligned} f(z) &= \frac{z^{-\frac{1}{2}}}{z^2 + 1} \\ &= \frac{e^{-\frac{1}{2} \log(z)}}{z^2 + 1} (|z| > 0, -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}). \end{aligned}$$

We integrate the branch over the indented path in figure 109. Starting with Cauchy's residue theorem, we write

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = 2\pi i \operatorname{Res}_{z=i} f(z) - \int_{C_\rho} f(z) dz - \int_{C_R} f(z) dz.$$

By using the parametrizations of

$$\begin{aligned} L_1 : z &= re^{i0} = r(\rho \leq r \leq R), \\ -L_2 : z &= re^{i\pi} = -r(\rho \leq r \leq R), \end{aligned}$$

we obtain

$$\begin{aligned} \int_{L_1} f(z)dz + \int_{L_2} f(z)dz &= \int_{\rho}^R \frac{1}{\sqrt{r}(r^2+1)}dr - i \int_{\rho}^R \frac{1}{\sqrt{r}(r^2+1)}dr \\ &= (1-i) \int_{\rho}^R \frac{1}{\sqrt{r}(r^2+1)}dr. \end{aligned}$$

Observe that  $z = i$  is a simple pole of  $f(z)$ , as it can be written that  $f(z) = \frac{z^{-\frac{1}{2}}}{(z+i)(z-i)}$ , and  $\frac{z^{-\frac{1}{2}}}{(z+i)}$  is analytic and nonzero at  $z = i$ . Therefore, the residue at  $z = i$  can be computed by

$$\begin{aligned} \text{Res}_{z=i} f(z) &= \left[ \frac{z^{-\frac{1}{2}}}{z+i} \right]_{z=i} \\ &= \frac{\exp(-\frac{1}{2}(\ln(1) + i\frac{\pi}{2}))}{2i} \\ &= \frac{e^{-i\frac{\pi}{4}}}{2i} \\ &= \frac{1}{2i} \left( \frac{1-i}{\sqrt{2}} \right). \end{aligned}$$

For the integral terms on  $C_{\rho}$  and  $C_R$ , by the ML inequality, we have

$$\begin{aligned} \left| \int_{C_{\rho}} f(z)dz \right| &\leq \int_{C_{\rho}} |f(z)|dz \\ &\leq \frac{\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{\pi\sqrt{\rho}}{1-\rho^2}, \\ \left| \int_{C_R} f(z)dz \right| &\leq \int_{C_R} |f(z)|dz \\ &\leq \frac{\pi\sqrt{R}}{(R^2-1)}. \end{aligned}$$

As  $R \rightarrow \infty$  and  $\rho \rightarrow 0$ , we have  $\frac{\pi\sqrt{R}}{(R^2-1)} \rightarrow 0$ , and  $\frac{\pi\sqrt{\rho}}{1-\rho^2} \rightarrow 0$  respectively. Hence, it follows that

$$(1-i) \int_0^{\infty} \frac{1}{\sqrt{r}(r^2+1)}dr = \frac{\pi(1-i)}{\sqrt{2}},$$

which can be re-written as

$$\int_0^{\infty} \frac{1}{\sqrt{r}(r^2+1)}dr = \frac{\pi}{\sqrt{2}},$$

as desired □

#### Question 283-4.

**Solution.** We use the branch  $f(z) = \frac{\exp(\frac{1}{3}\log(z))}{(z+a)(z+b)}$ , for  $|z| > 0, 0 < \arg(z) < 2\pi$  over the contour on section 91. By the residue theorem, we have

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz + \int_{C_{\rho}} f(z)dz + \int_{C_R} f(z)dz = 2\pi i(B_1 + B_2),$$

where  $B_1$  and  $B_2$  refer to the residues from  $-a$  and  $-b$  respectively. Firstly, we show that The integrals over the circular paths goes to 0 as follows:

$$\begin{aligned} \left| \int_{C_\rho} f(z) dz \right| &\leq \frac{\rho^{\frac{1}{3}}}{(a-\rho)(b-\rho)} 2\pi\rho \\ &= \frac{2\pi\rho^{\frac{4}{3}}}{(a-\rho)(b-\rho)}. \end{aligned}$$

As  $\rho \rightarrow 0$ , we see that the upper bound limits to 0. Therefore, the integral goes to 0. By the same computation, we also see that the integral over  $C_R$ . It follows that

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = 2\pi i (B_1 + B_2).$$

The residues can be computed as follows:

$$\begin{aligned} B_1 &= \text{Res}_{z=-a} f(z) \\ &= \frac{\exp(\frac{1}{3} \log(-b))}{-a+b} \\ &= -\frac{\exp(\frac{1}{3}(\ln(a) + i\pi))}{a-b} \\ &= -\frac{e^{i\frac{\pi}{3}} a^{\frac{1}{3}}}{a-b} \\ B_2 &= \frac{e^{i\frac{\pi}{3}} b^{\frac{1}{3}}}{a-b}, \end{aligned}$$

by symmetry.

#### Question 283-5.

**Solution.** First, as before, we have

$$\int_0^\pi \frac{d\theta}{(a + \cos(\theta))^2} = \frac{1}{2} \int_{-\pi}^\pi \frac{d\theta}{(a + \cos(\theta))^2}.$$

We substitute  $\cos(\theta) = \frac{z + z^{-1}}{2}$ , and  $d\theta = \frac{1}{iz} dz$  to the last integral. Then, it follows that

$$\begin{aligned} \int_{-\pi}^\pi \frac{d\theta}{(a + \cos(\theta))^2} &= \int_C \frac{1}{\left(a + \frac{z+z^{-1}}{2}\right)^2} \frac{1}{iz} dz \\ &= \frac{4}{i} \int_C \frac{z}{(2az + z^2 + 1)^2} dz, \end{aligned}$$

where  $C$  denotes the contour of the unit circle. Using the quadratic formulat, we obtain that the integrand has a pole of order 2 at  $z = -a \pm \sqrt{a^2 - 1}$ . As  $a > 1$ , we have that  $z = -a - \sqrt{a^2 - 1}$  is the only singular point in  $C$ , which results in the residue of  $\frac{a}{4(\sqrt{a^2 - 1})^3}$ . Therefore, by the residue theorem, we have

$$\begin{aligned} \int_0^\pi \frac{d\theta}{(a + \cos(\theta))^2} &= \frac{1}{2} \frac{4}{i} \frac{2\pi i a}{4\sqrt{a^2 - 1}^3} \\ &= \frac{\pi a}{(\sqrt{a^2 - 1})^3}, \end{aligned}$$

as desired. □

#### Question 287-6.

**Solution.** Observe that the following equality holds, by the linearity of integration:

$$\int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} = \frac{1}{5} \int_0^{2\pi} \frac{d\theta}{1 + \frac{4}{5} \sin \theta}$$

From the example 1 from pg.285 in the section 92, it follows that

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} &= \frac{1}{5} \int_0^{2\pi} \frac{d\theta}{1 + \frac{4}{5} \sin \theta} \\ &= \frac{1}{5} \frac{2\pi}{\sqrt{1 - \frac{4^2}{5^2}}} \\ &= \frac{1}{5} \frac{2\pi}{\frac{3}{5}} \\ &= \frac{2}{3} \pi. \end{aligned}$$

□

**Question 287-6.**

**Solution.** As  $|\sin(-\pi + \theta)| = |\sin(\theta)|$ , we have  $\sin^{2n}(\theta) = \sin^{2n}(-\pi + \theta)$ , and  $\int_0^\pi \sin^{2n}(\theta) d\theta = \int_{-\pi}^0 \sin^{2n}(\theta) d\theta$ . Hence, it follows that

$$\int_0^\pi \sin^{2n}(\theta) d\theta = \frac{1}{2} \int_C \sin^{2n}(\theta),$$

where  $C$  is the positively oriented unit circle  $|z| = 1$ . Substituting  $\frac{z - z^{-1}}{2i}$  for  $\sin(\theta)$ , using the binomial formula, and linearity of integration, we obtain

$$\begin{aligned} \frac{1}{2} \int_C \sin^{2n}(\theta) &= \frac{1}{2} \int_C \left( \frac{z - z^{-1}}{2i} \right)^{2n} \frac{dz}{iz} \\ &= \frac{1}{2^{2n+1}(-1)^n i} \int_C \frac{(z - z^{-1})^{2n}}{z} dz \\ &= \frac{1}{2^{2n+1}(-1)^n i} \int_C \sum_{k=0}^n \binom{2n}{k} z^k z^{2n-k} (-z^{-1})^k z^{-1} dz \\ &= \frac{1}{2^{2n+1}(-1)^n i} \sum_{k=0}^n \binom{2n}{k} (-1)^k \int_C z^{2n-2k-1} dz. \end{aligned}$$

Observe that we only get non-zero integral value for  $k = n$  case, and  $\int_C z^{-1} dz = 2\pi i$ . Therefore, it follows that

$$\begin{aligned} \int_0^\pi \sin^{2n}(\theta) d\theta &= \frac{1}{2^{2n+1}(-1)^n i} \frac{(2n)!}{n!n!} (-1)^n 2\pi i \\ &= \frac{2n!}{2^{2n}(n!)^2} \pi, \end{aligned}$$

as desired. □

**Question 293-6.**

**Solution. (a)** Inside the circle  $|z| = 1$ , write

$$f(z) = -5z^4 \text{ and } g(z) = z^6 + z^3 - 2z.$$

Then, observe that when  $|z| = 1$ ,

$$|f(z)| = 5|z|^4 = 5 \text{ and } |g(z)| \leq |z|^6 + |z|^3 + 2|z| = 4.$$

The conditions of Rouché's theorem are thus satisfied. Consequently, since  $f(z)$  has 4 zeroes, counting multiplicities, inside the circle  $|z| = 1$ ,  $f(z) + g(z)$  has 4 zeroes, inside the circle  $|z| = 1$ . Therefore, the polynomial  $z^6 - 5z^4 + z^3 - 2z$  has 4 zeroes, inside the circle  $|z| = 1$ .  $\square$

(b) Inside the circle  $|z| = 1$ , write

$$f(z) = 9 \text{ and } g(z) = 2z^4 - 2z^3 + 2z^2 - 2z.$$

Then, observe that when  $|z| = 1$ ,

$$|f(z)| = 9 \text{ and } |g(z)| = 2|z|^4 + 2|z|^3 + 2|z|^2 + 2|z| = 8.$$

The conditions of Rouché's theorem are thus satisfied. Consequently, since  $f(z)$  has 0 zero, counting multiplicities, inside the circle  $|z| = 1$ ,  $f(z) + g(z)$  has 0 zero, inside the circle  $|z| = 1$ . Therefore, the polynomial  $2z^4 - 2z^3 + 2z^2 - 2z + 9$  has 0 zero, inside the circle  $|z| = 1$ .  $\square$

(c) Inside the circle  $|z| = 1$ , write

$$f(z) = -4z^3 \text{ and } g(z) = z^7 + z - 1.$$

Then, observe that when  $|z| = 1$ ,

$$|f(z)| = 4|z|^3 = 4 \text{ and } |g(z)| \leq |z|^7 + |z| - 1 = 1.$$

The conditions of Rouché's theorem are thus satisfied. Consequently, since  $f(z)$  has 3 zeroes, counting multiplicities, inside the circle  $|z| = 1$ ,  $f(z) + g(z)$  has 3 zeroes. Therefore, the polynomial  $z^7 - 4z^3 + z - 1$  has 3 zeroes inside the circle  $|z| = 1$ .  $\square$

### Question 293-8.

**Solution.** Inside the circle  $|z| = 2$ , write

$$f(z) = 2z^5 \text{ and } g(z) = 6z^2 + z + 1.$$

Then, observe that when  $|z| = 2$ ,

$$|f(z)| = 2|z|^5 = 64 \text{ and } |g(z)| \leq 6|z|^2 + |z| + |1| = 8.$$

The conditions of Rouché's theorem are thus satisfied. Consequently, since  $f(z)$  has 5 zeroes, counting multiplicities, inside the circle  $|z| = 2$ ,  $f(z) + g(z)$  has 5 zeroes. On the other hand, inside the circle  $|z| = 1$ , write

$$f(z) = -6z^2 \text{ and } g(z) = 2z^5 + z + 1.$$

Then, observe that when  $|z| = 1$ ,

$$|f(z)| = 6|z|^2 = 6 \text{ and } |g(z)| \leq 2|z|^5 + |z| + 1 = 4.$$

The conditions of Rouché's theorem are thus satisfied. Consequently, since  $f(z)$  has 2 zeroes, counting multiplicities, inside the circle  $|z| = 1$ ,  $f(z) + g(z)$  has 2 zeroes. Therefore, we have shown that in the annulus  $1 \leq |z| \leq 2$ , we have  $5 - 2 = 3$  zeroes.  $\square$