Complex Analysis:
Problem Set I

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Abstract

This work contains the solutions to the problem set I of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. Brown p61. 1.

Solution. We wish to give a direct proof that

$$\frac{dw}{dz} = 2z \text{ when } \; w = z^2,$$

using the definition (3) in section 19, which is

$$\frac{dw}{dz} = \lim_{\triangle z \to 0} \frac{\triangle w}{\triangle z}.$$

We proceed to compute $\lim_{\triangle z \to 0} \frac{\triangle w}{\triangle z}$, given that $w = z^2$.

$$\begin{aligned}
\lim_{\triangle z \to 0} \frac{\triangle w}{\triangle z} &= \lim_{\triangle z \to 0} ((\triangle z + z)^2 - z^2) \frac{1}{\triangle z} \\
&= \lim_{\triangle z \to 0} (\triangle z^2 + 2z \triangle z) \frac{1}{\triangle z} \\
&= \lim_{\triangle z \to 0} \triangle z + 2z \\
&= 2z.
\end{aligned}$$

Therefore, we have shown that $\frac{dw}{dz} = 2z$. $\square$

Question 2. Brown p61. 2.

Solution. We differentiate four given functions of z.

(a) We wish to differentiate $f(z) = 3z^2 - 2z + 4$. Simply applying the power rule, we obtain $f^{'}(z) = 6z - 2$.

(b) We wish to differentiate $f(z) = (2z^2 + i)^5$. Applying the chain rule, we obtain

$$\begin{aligned}
f^{'}(z) &= 5(2z^2 + i)^4 \cdot 4z \\
&= 20z(2z^2 + i)^4.
\end{aligned}$$

(c) We wish to differentiate $f(z) = \dfrac{z-1}{2z+1}$ $(z \neq \dfrac{1}{2})$. Applying the quotient rule, we obtain

$$\begin{aligned} f^{'}(z) &= \frac{(2z+1)(1)-(z-1)(2)}{(2z+1)^2} \\ &= \frac{3}{(2z+1)^2}, \end{aligned}$$

for $z \neq \dfrac{1}{2}$.

(d) We wish to differentiate $f(z) = \dfrac{(1+z^2)^4}{z^2}$ $(z \neq 0)$. Applying the quotient rule, we obtain

$$f^{'}(z) = \frac{(z^2)(\frac{d}{dz}(1+z^2)^4) - (1+z^2)^4(2z)}{z^4},$$

for $z \neq 0$. Using the chain rule to resolve $\dfrac{d}{dz}(1+z^2)^4$ term, we finally get

$$\begin{aligned} f^{'}(z) &= \frac{(z^2)(4)(1+z^2)^3(2z) - (1+z^2)^4(2z)}{z^4} \\ &= \frac{8z^3(1+z^2)^3 - 2z(1+z^2)^4}{z^4} \\ &= \frac{8z^2(1+z^2)^3 - 2(1+z^2)^4}{z^3} \\ &= \frac{2(1+z^2)^3(3z^2-1)}{z^3}, \end{aligned}$$

for $z \neq 0$.

Question 3. Brown p76. 4.

Solution. We determine the singular points of the three given functions of z.

(a) We wish to determine the singular points of $f(z) = \dfrac{2z+1}{z(z^2+1)}$. The singular points are $z = 0, \pm i$.

(b) We wish to determine the singular points of $f(z) = \dfrac{z^3+i}{z^2-3z+2}$. Notice that the function definition can be factorized as

$$f(z) = \frac{z^3+i}{(z-2)(z-1)}.$$

The singular points are $z = 1, 2$.

(c) We wish to determine the singular points of $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$. The singular points are $z = -2, 1 \pm i$.

Question 4. Brown p90. 5.

Solution. The terms $|\exp(2z+i)|$ and $|\exp(iz^2)|$ can be written as

$$|\exp(2z+i)| = |\exp(2x+i(2y+1))| = e^{2x} \tag{1}$$
$$|\exp(iz^2) = |\exp(-2xy+i(x^2-y^2))| = e^{2xy}. \tag{2}$$

By the triangle-inequality, we have that

$$|\exp(2z+i)+\exp(iz^2)| \leq |\exp(2z+i)| + |\exp(iz^2)|.$$

Using the 1 substitution, we can conclude that

$$|\exp(2z+i)+\exp(iz^2)| \leq \mathrm{e}^{2x} + \mathrm{e}^{-2xy}.$$

□

Question 5. Brown p185. 4.

Solution. We have the following summation formulation:

$$\sum_{n=1}^{\infty} z^n \quad = \quad \frac{z}{1-z}, \tag{3}$$

whenever $|z| < 1$. Substituting $r\mathrm{e}^{i\theta}$ for z and separating the real and imaginary parts, we can re-write the LHS as

$$\begin{aligned}\sum_{n=1}^{\infty} z^n &= \sum_{n=0}^{\infty} (r\mathrm{e}^{i\theta})^n \\ &= \sum_{n=1}^{\infty} r^n \mathrm{e}^{in\theta} \\ &= \sum_{n=1}^{\infty} r^n \cos(n\theta) + i\sum_{n=0}^{\infty} r^n \sin(n\theta). \end{aligned} \tag{4}$$

Now, multiplying both denominator and numerator by the conjugate of $1-z$, $1-\overline{z}$, we can re-write the RHS as

$$\begin{aligned}\frac{z}{1-z} &= \frac{z(1-\overline{z})}{(1-z)(1-\overline{z})} \\ &= \frac{z - z\overline{z}}{1+z\overline{z}-(z+\overline{z})}.\end{aligned}$$

Substituting $z = r\cos(\theta) + i\sin(\theta)$, $z\overline{z} = r^2$, and $z + \overline{z} = 2r\cos(\theta)$ to the last expression, and separating the real and imaginary parts, we obtain

$$\begin{aligned}\frac{z}{1-z} &= \frac{r\cos(\theta) - r^2 + ir\sin(\theta)}{1+r^2-2r\cos(\theta)} \\ &= \frac{r\cos(\theta) - r^2}{1+r^2-2r\cos(\theta)} + \frac{r\sin(\theta)}{1+r^2-2r\cos(\theta)} i.\end{aligned} \tag{5}$$

Substituting 5 and 4 to 3, we obtain

$$\sum_{n=1}^{\infty} r^n \cos(n\theta) + i\sum_{n=0}^{\infty} r^n \sin(n\theta) \quad = \quad \frac{r\cos(\theta) - r^2}{1+r^2-2r\cos(\theta)} + \frac{r\sin(\theta)}{1+r^2-2r\cos(\theta)} i. \tag{6}$$

By the Theorem from section 61, we know that the real and imaginary part of the series must equal the real and imaginary part of the convergent value respectively. Hence, we have that

$$\sum_{n=1}^{\infty} r^n \cos(n\theta) = \frac{r\cos(\theta) - r^2}{1 - 2r\cos(\theta) + r^2}$$
$$\sum_{n=1}^{\infty} r^n \sin(n\theta) = \frac{r\sin(\theta)}{1 - 2r\cos(\theta) + r^2},$$

when $0 < r < 1$. □

Question 6. Radius Convergence I.

Solution. We wish to find the radius of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} z^{k(k+1)}.$$

Let s_k denote the kth term of the above power series. From the ratio test, we know that a series $\sum_{k=1} s_k$ converges if $\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| < 1$ and diverges if $|\frac{s_{k+1}}{s_k}| \geq 1$ for all $k \geq k_0$, where n_0 is some fixed integer. We proceed to compute $\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}|$.

$$\begin{aligned}
\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| &= \limsup_{k\to\infty} |\frac{\frac{(-1)^{k+1}}{k+1} z^{(k+1)(k+2)}}{\frac{(-1)^k}{k} z^{k(k+1)}}| \\
&= \limsup_{k\to\infty} |\frac{-k}{k+1} z^2| \\
&= \limsup_{k\to\infty} |\frac{k}{k+1}||z^2| \\
&= |z^2| \limsup_{k\to\infty} |\frac{k}{k+1}| \\
&= |z|^2.
\end{aligned}$$

For $|z| < 1$, we have that $\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| = |z|^2 < 1$. Hence, the series converges when $|z| < 1$. For $|z| > 1$, we have that $\limsup_{k\to\infty} |\frac{s_{k+1}}{s_k}| = |z|^2 > 1$. Hence, the series diverges when $|z| > 1$. Therefore, we have that the radius of convergence of the given sequence is 1, with the center being the origin.

Now, we discuss the convergence for $z = 1$ and -1, and i. For $z = 1, -1$, as $z^{k(k+1)} = 1$ for all k, the series simplifies to

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}.$$

Notice that $-\frac{1}{k} + \frac{1}{k+1} = -\frac{1}{k(k+1)}$ for $k \geq 1$. Hence, the above series can be re-written in the following way:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k)}.$$

Observe that

$$-\frac{1}{(2k-1)(2k-1)} \leq -\frac{1}{(2k-1)(2k)} \leq 0,$$

for $k \leq 1$. As $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges to 0, $\sum_{k=0}^{\infty} -\frac{1}{(2k-1)(2k-1)}$ converges to 0, and by the squeeze theorem, we have that $\sum_{k=1} \frac{1}{(2k-1)(2k)}$ converges to 0. Thus, the given series is convergent for $z = 1$ and -1.

For $z = i$,

Question 7. Radius of Convergence II.

Solution. We wish to find the radius of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{k^2}{4^k + 5k} z^k.$$

Let s_k denote the kth term of the above power series. From the root test, we know that a series $\sum_{k=1}^{\infty}$ converges if $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} < 1$ and diverges if $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} > 1$. We proceed to compute $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}}$.

$$\begin{aligned}
\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} &= \limsup_{k\to\infty} \left|\frac{k^2}{4^k + 5k} z^k\right|^{\frac{1}{k}} \\
&= \limsup_{k\to\infty} \left|\frac{k^2}{4^k + 5k}\right|^{\frac{1}{k}} |z^k|^{\frac{1}{k}} \\
&= |z| \limsup_{k\to\infty} \left|\left(\frac{k^2}{4^k + 5k}\right)^{\frac{1}{k}}\right| \\
&= \frac{1}{4}|z|.
\end{aligned}$$

For $|z| < 4$, we have that $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} = \frac{1}{4}|z| < 1$. Hence, the series converges when $|z| < 4$. For $|z| > 4$, we have that $\limsup_{k\to\infty} |s_k|^{\frac{1}{k}} = \frac{1}{4}|z| > 1$. Hence, the series diverges when $|z| > 4$. Therefore, we have that the radius of convergence of the given sequence is 4, with the center being the origin. $\square$