Complex Analysis I: Problem Set VII

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Abstract

This work contains the solutions to the problem set VII of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 247-7.

Solution. (a) Observe that

$$\frac{1}{z^2}f(\frac{1}{z}) \quad = \quad \frac{(3+2z)^2}{z(1-z)(2+5z)}.$$

As $\frac{1}{z^2}f(\frac{1}{z})$ has a simple pole at $z = 0$, we have

$$\begin{aligned} \int_C \frac{(3+2z)^2}{z(z-1)(2z+5)}dz &= 2\pi i \cdot \text{Res}_{z=0}[\frac{1}{z^2}f(\frac{1}{z})] \\ &= 2\pi i \cdot \frac{9}{2} = 9\pi i. \end{aligned}$$

(b) Observe that

$$\frac{1}{z^2}f(\frac{1}{z}) = \frac{e^z}{z^2(1+z^3)}.$$

As $\frac{1}{z^2}f(\frac{1}{z})$ has a pole of order 2 at $z = 0$, we have

$$\int_C \frac{z^3 e^{\frac{1}{z}}}{1+z^3}dz \quad = \quad 2\pi i \cdot \text{Res}_{z=0}[\frac{1}{z^2}f(\frac{1}{z})],$$

where $\phi(z) = \frac{e^z}{1+z^3}$. We have

$$\phi^{'}(z) \quad = \quad \frac{(1+z^3)e^z - e^z 3z^2}{(1+z^3)^2}.$$

By substituting $z = 0$, we see that $\phi^{'}(0) = 1$, which is the residue at $z = 0$. It follows that

$$\int_C \frac{z^3 e^{\frac{1}{z}}}{1+z^3}dz \quad = \quad 2\pi i.$$

□

Question 2. 254-5.

Solution. **(a)** The given integral can be written as

$$\begin{aligned}\int_C \tan(z)dz &= \int_C \frac{p(z)}{q(z)}dz \\ &= \int_C \frac{\sin(z)}{\cos(z)}dz.\end{aligned}$$

As the zeros of $\cos(z)$ are $z = \frac{\pi}{2} + n\pi$ and C is the positively oriented circle $|z| = 2$, there are two isolated singularities of $\tan(z)$ interior to C, namely $z = \pm\frac{\pi}{2}$. It follows that

$$\begin{aligned}\mathrm{Res}_{z=\frac{\pi}{2}} \tan(z) &= \frac{p(\frac{\pi}{2})}{q'(\frac{\pi}{2})} = \frac{\sin(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = -1, \\ \mathrm{Res}_{z=\frac{-\pi}{2}} \tan(z) &= \frac{p(\frac{-\pi}{2})}{q'(\frac{-\pi}{2})} = \frac{\sin(\frac{-\pi}{2})}{-\sin(\frac{-\pi}{2})} = -1.\end{aligned}$$

Consequently, by the residue theorem, we have

$$\int_C \tan(z)dz = -4\pi i.$$

(b) We wish to evaluate the integral $\int_C \frac{dz}{\sinh(2z)}$. As $\sinh(z) = 0$ for $\frac{n\pi i}{2}$, we can conclude that the isolated singularities of the integrand happens at $z = 0$ and $z = \pm\frac{\pi i}{2}$. It follows that

$$\begin{aligned}\mathrm{Res}_{z=0}\frac{1}{\sinh(2z)} &= \frac{1}{2\cosh(0)} = \frac{1}{2} \\ \mathrm{Res}_{z=\frac{\pi i}{2}}\frac{1}{\sinh(2z)} &= \frac{1}{2\cosh(\pi i)} = -\frac{1}{2} \\ \mathrm{Res}_{z=-\frac{\pi i}{2}}\frac{1}{\sinh(2z)} &= \frac{1}{2\cosh(-\pi i)} = -\frac{1}{2}\end{aligned}$$

Consequently, by the residue theorem, we have

$$\int_C \frac{1}{\sinh(2z)}dz = -\pi i.$$

□

Question 3. 254-6.

Solution. Observe that interior to C_N, the function $\frac{1}{z^2\sin(z)}$ has singularities at $z = 0$ and $z = \pm n\pi$ for $n = 1, 2, ..., N$. The residue at $z = 0$ can be found by finding the $\frac{1}{z}$ coefficient of $\frac{1}{z^2\sin(z)}$, which was found to be $\frac{1}{6}$ in a previous homework problem. For the residues at $z = \pm n\pi$ for $n = 1, 2, ..., N$, we have

$$\frac{1}{z^2\sin(z)} = \frac{p(z)}{q(z)} \text{ where } p(z) = 1, q(z) = z^2\sin(z).$$

As $q'(z) = z^2 cos(z)$, we have for $z = \pm n\pi$, we have $q'(\pm n\pi) = (-1)^n n^2\pi^2$. By the residue theorem, it follows that

$$\int_{C_N} \frac{1}{z^2\sin(z)}dz = 2\pi i[\frac{1}{6} + 2\sum_{n=1}^{N}\frac{(-1)^n}{n^2\pi^2}],$$

which can be re-written as

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} - \frac{\pi}{4i} \int_{C_N} \frac{1}{z^2 \sin(z)} dz.$$

We have that as $N \to \infty$, the integral goes to 0. Therefore, it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

$\square$

Question 254-8.

Solution.

Question 264.2.

Solution. Consider the function $f(z) = \dfrac{1}{(z^2+1)^2}$ and the simple, closed, and positively oriented contour of a half-circle above the real-axis, centered around the origin with the radius $R > 1$. It follows that

$$\int_{-R}^{R} \frac{1}{(x^2+1)^2} dx + \int_{C_R} \frac{1}{(z^2+1)^2} dz \quad = \quad 2\pi B,$$

where C_R denotes the contour of the curve part of the half-circle, and B is the residue of the complex integrand at $z = i$. For computing the residue, we have

$$\frac{1}{(z^2+1)} \quad = \quad \frac{\phi(z)}{(z-i)^2},$$

where $\phi(z) = \dfrac{1}{(z+i)^2}$. It follows that $B = \phi^{'}(i) = \dfrac{1}{4i}$. Therefore, we obtain

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2} - \int_{C_R} \frac{1}{(z^2+1)^2} dz.$$

For z on C_R, we have $|z^2+1| \geq R^2 - 1$. Therefore, we have

$$\left| \int_{C_R} \frac{1}{(z^2+1)^2} dz \right| \leq \frac{\pi R}{(R^2-1)^2}.$$

Consequently, we have as $R \to \infty$, $\dfrac{\pi R}{(R^2-1)^2} \to 0$. It follows that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx \quad = \quad \frac{\pi}{2}.$$

As the integrand is an even function, we obtain that

$$\int_{0}^{\infty} \frac{1}{(x^2+1)^2} dx = \frac{\pi}{4}.$$

$\square$

Question 264.6.

Solution. Consider the function $f(z) = \dfrac{z^2}{(z^2+9)(z^+4)^2}$ and the simple, closed, and positively oriented contour of a half-circle above the real-axis, centered around the origin with the radius $R > 3$. It follows that

$$\int_{-R}^{R} \frac{x^2}{(x^2+9)(z^2+4)^2} dx + \int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2} dz \quad = \quad 2\pi(B_1 + B_2).$$

where C_R denotes the contour of the curve part of the half-circle, and B_1 and B_2 are the residues of the complex integrand at $z = 3i$ and $z = zi$ respectively. First, observe that

$$\begin{aligned} B_1 &= \text{Res}_{z=3i} \frac{z^2}{(z^2+9)(z^2+4)^2} \\ &= \frac{z^2}{(z+3i)(z^2+4)^2} \|_{z=3i} = -\frac{3}{50i}. \end{aligned}$$

For computing the B_2 term, observe that

$$\frac{z^2}{(z^2+9)(z^2+4)^2} = \frac{\phi(z)}{(z-2i)^2},$$

where

$$\phi(z) = \frac{z^2}{(z^2+9)(z+2i)^2}.$$

It follows that

$$B_2 = \phi^{'}(2i) = \frac{13}{200i}.$$

Hence, we have

$$\int_{-R}^{R} \frac{x^2}{(x^2+9)(z^2+4)^2} dx = \frac{\pi}{100} - \int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2} dz.$$

Observe that

$$\left| \int_{C_R} \frac{z^2}{(z^2+9)(z^2+4)^2} \right| \leq \frac{R^2}{(R^2-9)(R^2-4)^2} \pi R$$

The RHS of the inequality goes to 0 as $R \to \infty$. With the fact that the integrand is even, we finally obtain

$$\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx = \frac{\pi}{200}.$$

□

Question 264.8.

Solution. Consider the function $f(z) = \dfrac{z}{(z^2+1)(x^2+2x+2)}$, and the simple, closed, and positively oriented contour of a semi-circle with the radius strictly larger than $\sqrt(2)$ and the center of the circle at the origin. It follows that the isolated singularities are at $z = i$ and $z = -1 + i$. By the residue theorem, it follows that

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i(B_0 + B_1),$$

where B_0 and B_1 are the residues at the singularities respectively. It follows that

$$\begin{aligned} B_0 &= \text{Res}_{z=i} f(z) \\ &= [\frac{z}{(z+i)(z^2+2z+2)}]_{z=i} = \frac{1}{10} - \frac{1}{5}i \\ B_1 &= \text{Res}_{z=-1+i} f(z) \\ &= [\frac{z}{(z^2+1)(z+1+i})]_{z=-1+i} = -\frac{1}{10} - \frac{3}{10}i. \end{aligned}$$

By the residue theorem, we obtain

$$\int_{-R}^{R} \frac{x}{(x^2+1)(x^2+2x+2)} dx = -\frac{\pi}{5} - \int_{C_N} \frac{z}{(z^2+1)(z^2+2z+2)}.$$

Observe that by the RL-inequality,

$$\left|\int_{C_R} \frac{z}{(z^2+1)(z^2+2z+2)}\right| \quad \leq \quad \frac{\pi R^2}{(R^2-1)(R-\sqrt{2})^2}.$$

Hence, the integral over the C_N contour tends to 0, has $R \to \infty$. Therefore, we have

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x}{(x^2+1)(x^2+2x+2)} \quad = \quad -\frac{\pi}{5}.$$

□

Question 272.2.

Solution. We wish to evaluate $\int_0^\infty \frac{\cos(ax)}{x^2+1}dx$. Consider the function $f(z) = \frac{1}{z^2+1}$ with a semi-circle contour above the real-axis of the radius $R > 1$. Observe that the singularity interior to the contour is at $z = i$. It follows that the residue at $z = i$ can be computed as

$$\begin{aligned} B &= \text{Res}_{z=i}[f(z)e^{iaz}] = \frac{e^{iaz}}{z+i}]_{z=i} \\ &= \frac{e^{-a}}{2i}. \end{aligned}$$

By the residue theorem, it follows that

$$\int_{-R}^{R} \frac{\cos(ax)}{x^2+1}dx \quad = \quad \pi e^{-a} - \text{Re}\int_{C_R} f(z)e^{iaz}dz.$$

Observe that

$$\begin{aligned} \left|\text{Re}\int_{C_R} f(z)e^{iaz}dz\right| &\leq \left|\int_{C_R} f(z)e^{iaz}dz\right| \\ &\leq \frac{\pi R}{R^2-1}. \end{aligned}$$

As $\frac{\pi R}{R^2-1}$ tends to 0 as $R \to \infty$, by the Jordan's Lemma, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}dx \quad = \quad \pi e^{-a}.$$

□

Question 272.7.

Solution. We wish to evaluate $\int_0^\infty \frac{x^3\sin(x)}{(x^2+1)(x^2+9)}$. Consider the function $f(z) = \frac{z^3}{(z^2+1)(z^2+9)}$ with a semi-circle contour around the origin above the real-axis of the radius $R > 3$. Observe that singularities occur at $z = i$ and $z = 3i$. It follows that

$$\begin{aligned} \text{Res}_{z=i}[f(z)e^{iz}] &= \frac{z^3e^{iz}}{(z+i)(z^2+9)}]_{z=i} \\ &= -\frac{1}{16e}, \\ \text{Res}_{z=3i}[f(z)e^{iz}] &= \frac{z^3e^{iz}}{(z^2+1)(z+3i)}_{z=3i} \\ &= \frac{9}{16e^3}. \end{aligned}$$

By the residue theorem and the Jordan's Lemma, applied as above, we have

$$\int_{-\infty}^{\infty} \frac{x^3\sin(x)}{(x^2+1)(x^2+9)}dx \quad = \quad \frac{\pi}{8e}(\frac{9}{e^2}-1).$$

□