Complex Analysis I: Problem Set VII

Youngduck Choi
CILVR Lab
New York University
yc1104@nyu.edu

Abstract

This work contains the solutions to the problem set VII of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 247-7.

Solution. (a) Observe that

$$\frac{1}{z^2} f(\frac{1}{z}) \quad = \quad \frac{(3+2z)^2}{z(1-z)(2+5z)}.$$

As $\frac{1}{z^2} f(\frac{1}{z})$ has a simple pole at $z = 0$, we have

$$\begin{aligned} \int_C \frac{(3+2z)^2}{z(z-1)(2z+5)} dz \quad &= \quad 2\pi i \cdot \text{Res}_{z=0}[\frac{1}{z^2} f(\frac{1}{z})] \\ &= \quad 2\pi i \cdot \frac{9}{2} = 9\pi i. \end{aligned}$$

(b) Observe that

$$\frac{1}{z^2} f(\frac{1}{z}) = \frac{e^z}{z^2(1+z^3)}.$$

As $\frac{1}{z^2} f(\frac{1}{z})$ has a pole of order 2 at $z = 0$, we have

$$\int_C \frac{z^3 e^{\frac{1}{z}}}{1+z^3} dz \quad = \quad 2\pi i \cdot \text{Res}_{z=0}[\frac{1}{z^2} f(\frac{1}{z})],$$

where $\phi(z) = \frac{e^z}{1+z^3}$. We have

$$\phi^{'}(z) \quad = \quad \frac{(1+z^3)e^z - e^z 3z^2}{(1+z^3)^2}.$$

By substituting $z = 0$, we see that $\phi^{'}(0) = 1$, which is the residue at $z = 0$. It follows that

$$\int_C \frac{z^3 e^{\frac{1}{z}}}{1+z^3} dz \quad = \quad 2\pi i.$$

□

Question 2. 254-5.

Solution. (a) The given integral can be written as

$$\begin{aligned}\int_C \tan(z)dz &= \int_C \frac{p(z)}{q(z)}dz \\ &= \int_C \frac{\sin(z)}{\cos(z)}dz.\end{aligned}$$

As the zeros of $\cos(z)$ are $z = \frac{\pi}{2} + n\pi$ and C is the positively oriented circle $|z| = 2$, there are two isolated singularities of $\tan(z)$ interior to C, namely $z = \pm\frac{\pi}{2}$. It follows that

$$\begin{aligned}\mathrm{Res}_{z=\frac{\pi}{2}} \tan(z) &= \frac{p(\frac{\pi}{2})}{q'(\frac{\pi}{2})} = \frac{\sin(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = -1, \\ \mathrm{Res}_{z=\frac{-\pi}{2}} \tan(z) &= \frac{p(\frac{-\pi}{2})}{q'(\frac{-\pi}{2})} = \frac{\sin(\frac{-\pi}{2})}{-\sin(\frac{-\pi}{2})} = -1.\end{aligned}$$

Consequently, by the residue theorem, we have

$$\int_C \tan(z)dz = -4\pi i.$$

(b) We wish to evaluate the integral $\int_C \frac{dz}{\sinh(2z)}$. As $\sinh(z) = 0$ for $\frac{n\pi i}{2}$, we can conclude that the isolated singularities of the integrand happens at $z = 0$ and $z = \pm\frac{\pi i}{2}$. It follows that

$$\begin{aligned}\mathrm{Res}_{z=0}\frac{1}{\sinh(2z)} &= \frac{1}{2\cosh(0)} = \frac{1}{2} \\ \mathrm{Res}_{z=\frac{\pi i}{2}}\frac{1}{\sinh(2z)} &= \frac{1}{2\cosh(\pi i)} = -\frac{1}{2} \\ \mathrm{Res}_{z=-\frac{\pi i}{2}}\frac{1}{\sinh(2z)} &= \frac{1}{2\cosh(-\pi i)} = -\frac{1}{2}\end{aligned}$$

Consequently, by the residue theorem, we have

$$\int_C \frac{1}{\sinh(2z)}dz = -\pi i.$$

□

Question 3. 254-6.

Solution.

Question 1. 237-2.

Solution.