Complex Analysis I: Problem Set IV

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Abstract

This work contains the solutions to the problem set IV of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. Brown p.147-2.

Solution. (b) We first have that $\cos(z/2)$ is continous everywhere on the complex plane. Therefore, any contour from 0 to $\pi + 2i$ will have the same value of $F(\pi + 2i) - F(0)$, where F denotes the antiderivative of $\cos(z/2)$. We can compute the exact value as follows:

$$\begin{aligned} \int_0^{\pi+2i} \cos(\frac{z}{2}) dz &= \Big[2\sin(\frac{z}{2}) \Big]_0^{\pi+2i} \\ &= 2\sin(\frac{\pi}{2} + i) \\ &= 2\cos(i) \\ &= e + \frac{1}{e}, \end{aligned}$$

as desired. □

Question 5. Brown p.147-5.

Solution. Observe that along the lower horizontal leg, we have $z = x(-a \leq x \leq a)$.

Question 1. Brown p.159-2.

Solution. (b) Let C_1 denote the positively oriented boundary of the square whose sides lie along the line $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$. Observe that C_1 is interior to C_2 and the given function $\frac{z+2}{\sin(\frac{z}{2})}$ is analytic in the closed region consisting of the C_1 and C_2 contours and all points between them. Hence, by the corollary, we have

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz,$$

for $f(z) = \frac{z+2}{\sin(\frac{z}{2})}$. □

Question 2. Brown 159-4.

Solution. (a) Observe that along the lower horizontal leg, we have $z = x \ (-a \leq x \leq a)$. Hence, the integral along the lower horizontal leg from $-a$ to a can be written as

$$2\int_0^a e^{-x^2}dx.$$

For the upper horizontal leg, we have $z = x + ib \ (-a \leq x \leq a)$. Hence, the integra along the upper horizontal from a to $-a$ can be written as

$$\int_a^{-a} e^{-(x+ib)^2}dx,$$

wich can be simplified as follows:

$$\begin{aligned}\int_a^{-a} e^{-(x+ib)^2}dx &= -e^{b^2}\int_{-a}^{a} e^{-x^2-2ibx}dx \\ &= -2e^{b^2}\int_0^a e^{-x^2}\cos(2bx)dx.\end{aligned}$$

Hence, we have

$$2\int_0^a e^{-x^2}dx + -2e^{b^2}\int_0^a e^{-x^2}\cos(2bx)dx,$$

for the sum of contour integrals along each horizontal leg. Now, observe that along the right verical leg, we have $z = a + iy \ (0 \leq y \leq b)$. Hence,

Question 1. Brown p.170-1.

Solution. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. We evaluate the following integrals.

(b) We are given the following integral:

$$\int_C \frac{\cos(z)}{z(z^2+8)}dz,$$

which can be written as

$$\int_C \frac{\frac{\cos(z)}{(z^2+8)}}{z}dz.$$

As $\dfrac{\cos(z)}{(z^2+8)}$ is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the Cauchy Integral formula, we obtain

$$\frac{\cos(0)}{8} = \frac{1}{2\pi i}\int_C \frac{\frac{\cos(z)}{(z^2+8)}}{z}dz,$$

which simplifies to

$$\int_C \frac{\frac{\cos(z)}{(z^2+8)}}{z}dz = \frac{\pi i}{4}.$$

(d) We are given the following integral:

$$\int_C \frac{\cosh(z)}{z^4}dz.$$

As $\dfrac{\cosh(z)}{z^4}$ is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the extended Cauchy Integral formula, we obtain

$$\cosh^{(3)}(z_0) = \frac{3!}{2\pi i}\int_C \frac{\cosh(z)}{(z-z_0)^4}dz,$$

for z_0 inside and on the given contour. Observe that $\cosh^{(3)} = \sinh$. Hence, taking $z_0 = 0$ yields

$$0 = \frac{3!}{2\pi i}\int_C \frac{\cosh(z)}{z^4}dz,$$

which simplifies to

$$\int_C \frac{\cosh(z)}{z^4}dz = 0.$$

(e) We are given the following integral:

$$\int_C \frac{\tan(\frac{z}{2})}{(z-x_0)^2}dz,$$

for $-2 < x_0 < 2$. Notice that x_0 is inside the given contour. As $\dfrac{\tan(\frac{z}{2})}{(z-x_0)^2}$ is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the extended Cauchy Integral formula, we obtain

$$\frac{1}{2}\sec^2(\frac{x_0}{2}) = \frac{1!}{2\pi i}\int_C \frac{\tan(\frac{z}{2})}{(z-x_0)^2}dz,$$

which simplifies to

$$\int_C \frac{\tan(\frac{z}{2})}{(z-x_0)^2}dz = i\pi\sec^2(\frac{x_0}{2}),$$

for $-2 < x_0 < 2$. $\square$

Question 2. Brown 170.3.

Solution. Let C be the circle $|z| = 3$, described in the positive sense. As $2s^2 - s - 2$ is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the extended Cauchy Integral formula, we obtain

$$2z^2 - z - 2 = \frac{1}{2\pi i}\int_C \frac{2s^2 - s - 2}{s - z}ds,$$

for $|z| < 3$. As $g(z) = \int_C \dfrac{2s^2 - s - 2}{s - z}ds$, we have

$$g(z) = 2\pi i(2z^2 - z - 2),$$

for $|z| < 3$. Hence, it follows that $g(2) = 8\pi i$. For $|z| > 3$, we have that $\dfrac{2s^2 - s - 2}{s - z}$ is analytic at all points interior to and on C. Hence, by the Cauchy-Goursat theorem, we obtain

$$\int_C \frac{2s^2 - s - 2}{s - z}dz = 0,$$

for $|z| > 3$. Therefore, $g(z) = 0$ when $|z| > 3$, which completes the solution for the problem. $\square$

Question 3. Brown 170-4.

Solution. Let C be any simple closed contour, described in the positive sense in the z plane. As $s^3 + 2s$ is entire, by the extended Cauchy Integral formula, we obtain

$$6z = \frac{2!}{2\pi i}\int_C \frac{s^3 + 2s}{(s - z)^3 i}ds,$$

for z at the interior of C. As $g(z) = \int_C \dfrac{s^3 + 2s}{(s - z)^3}ds$, we have

$$g(z) = 6\pi i z,$$

for z inside C. Now, if z is outside of C, then $\dfrac{s^3+2s}{s-z}$ is analytic at points interior to and on C. Hence, by the Cauchy-Goursat Theorem, we have that

$$\int_C \frac{s^3+2s}{(s-z)^3}ds \quad = \quad 0,$$

for z outside of C. Hence $g(z) = 0$ when z is outside. $\square$

Question 4. Brown 170-7.

Solution. Let C be the unit circle. As e^{az} is entire, by the Cauchy Integral formula, we obtain

$$e^{az_0} \quad = \quad \frac{1}{2\pi i}\int_C \frac{e^{az}}{z-z_0}dz,$$

for z_0 inside C. By taking $z_0 = 0$, we get

$$1 \quad = \quad \frac{1}{2\pi i}\int_C \frac{e^{az}}{z}dz,$$

which simplifies to

$$\int_C \frac{e^{az}}{z}dz \quad = \quad 2\pi i.$$

Question 5. Brown 170-8.

Solution. The Legendre polynomials are defined by

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}ds,$$

for any simple closed contour surrounding z. For $z = -1$, and by having C to be any arbitrary simple closed contour that surrounds $z = -1$, it follows that

$$P_n(-1) \quad = \quad \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s+1)^{n+1}}ds,$$

which, by using the suggestion, simplifies to

$$P_n(z) \quad = \quad \frac{1}{2^{n+1}\pi i}\int_C \frac{(s-1)^n}{s+1}ds.$$

Since $(s-1)^n$ is entire, $(s-1)^n$ is analytic inside and on C. Hence, by the Cauchy Integral formula, we have

$$(-2)^n 2\pi i \quad = \quad \int_C \frac{(s-1)^n}{s+1}ds.$$

Substituting the above equality into the simplified formula of Legendre polynomials yields

$$\begin{aligned} P_n(z) &= \frac{(-2)^n 2\pi i}{2^{n+1}\pi i} \\ &= (-1)^n, \end{aligned}$$

as desired. $\square$