Complex Analysis I:
Problem Set IX

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Abstract

This work contains the solutions to the problem set IX of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

1. Evaluate the integral
$$\int_\gamma \frac{dz}{z^2+1}$$
for
$$\gamma(\theta) = 2\left|\cos 2\theta\right| e^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

Solution. By drawing the contour on the complex plane, we observe that γ forms 4 simple closed contours, for each direction of the axis. We denote these contours as γ_1, γ_2, γ_3, and γ_4 respectively in a counter-clockwise fashion. Observe that $f(z) = \dfrac{1}{z^2+1}$ is singular at $z = \pm i$. $z = i$ belongs to the interior of γ_2 contour, and $z = -i$ belongs to the interior of γ_4 contour. By the Cauchy-Residue formula, we obtain

$$\begin{aligned}
\int_{\gamma_1} \frac{dz}{z^2+1} &= 0 \\
\int_{\gamma_2} \frac{dz}{z^2+1} &= 2\pi i \mathrm{Res}_{z=i} \frac{1}{z^2+1} \\
\int_{\gamma_3} \frac{dz}{z^2+1} &= 0 \\
\int_{\gamma_4} \frac{dz}{z^2+1} &= 2\pi i \mathrm{Res}_{z=-i} \frac{1}{z^2+1}.
\end{aligned}$$

As it can be written that $f(z) = \dfrac{\phi(z)}{z-i}$, where $\phi(z) = \dfrac{1}{z+i}$, the residue at $z = i$ is $\phi(i) = \dfrac{1}{2i}$. On the other hand, as it can be written that $f(z) = \dfrac{\phi(z)}{z+i}$, where $\phi(z) = \dfrac{1}{z-i}$, the residue at $z = -i$

is $\phi(i) = -\dfrac{1}{2i}$. Consequently, we have

$$\begin{aligned}
\int_{\gamma_2} \frac{dz}{z^2+1} &= 2\pi i \frac{1}{2i} = \pi \\
\int_{\gamma_4} \frac{dz}{z^2+1} &= 2\pi i (-\frac{1}{2i}) = -\pi.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\int_{\gamma} \frac{dz}{z^2+1} &= \int_{\gamma_1} \frac{dz}{z^2+1} + \int_{\gamma_2} \frac{dz}{z^2+1} + \int_{\gamma_3} \frac{dz}{z^2+1} + \int_{\gamma_4} \frac{dz}{z^2+1} \\
&= 0.
\end{aligned}$$

□

Question 2.

2. Let

$$\gamma(\theta) = \begin{cases} \theta e^{i\theta}, & 0 \le \theta \le 2\pi, \\ 4\pi - \theta, & 2\pi \le \theta \le 4\pi. \end{cases}$$

Calculate

$$\int_{\gamma} \frac{dz}{z^2+\pi^2}.$$

Solution. Observe that the function has isolated singularities at $z = \pm i\pi$. By observing the contour, we see that $i\pi$ lies outside of the contour, as $\gamma(\frac{\pi}{2}) = \frac{\pi}{2} e^{i\frac{\pi}{2}} = i\frac{\pi}{2}$. On the other hand, $z = -i\pi$ lies on the interior of the contour as $\gamma(\frac{3\pi}{2}) = \frac{3\pi}{2} e^{i\frac{3\pi}{2}} = -\frac{3\pi}{2} i$. Hence, by the Cauchy Residue theorem, we have

$$\int_{\gamma} \frac{dz}{z^2+\pi^2} = 2\pi i \mathrm{Res}_{z=-i\pi} \frac{1}{z^2+\pi^2}.$$

As it can be written that $f(z) = \dfrac{\phi(z)}{z+i\pi}$, where $\phi(z) = \dfrac{1}{z-i\pi}$, the residue at $z = -i\pi$ is $\phi(-i\pi) = -\dfrac{1}{2i\pi}$. Hence, it follows that

$$\begin{aligned}
\int_{\gamma} \frac{dz}{z^2+\pi^2} &= 2\pi i (-\frac{1}{2i\pi}) \\
&= -1.
\end{aligned}$$

□

Question 3.

3. Let $\lambda > 1$ and show the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the right half plane $\{z : \mathrm{Re}\, z > 0\}$.

Solution. Firstly, the equation can be re-written as $\lambda - z = e^{z}$. Observe that it is necessary to have $|\lambda - z| = e^{-\mathrm{Re}z}$ to satisfy the above equation. As we only limit the space of possible solutions to be $\{z : \mathrm{Re}z > 0\}$, it follows that it is necessary to have $|\lambda - z| < 1$. Define $C = \{z \in \mathbb{C} \,|\, |\lambda - z| < 1\}$. So far, we have shown that the solutions to the given equation, if it exists must lie on the interior of C. Let $f(z) = e^{-z}$ and $g(z) = \lambda - z$. Then, it follows that on C, $|g(z)| = |\lambda - z| = 1$, and as $\lambda > 1$, $|f(z)| = |e^{-z}| = e^{-\mathrm{Re}z} < 1$. As $f(z)$ and $g(z)$ are entire, they are also analytic inside and on C. The conditions of Rouche's theorem are thus satisfied. Hence, $\lambda - z$ and $\lambda - z - e^{-z}$ have the same number of zeros, counting multiplicities inside C. Observe that $\lambda - z$ has a zero on $z = \lambda$. Thus, $\lambda - z - e^{-z}$ has one solution inside C. As we have shown that a solution to $\lambda - z - e^{-\lambda}$ must lie inside C, we have shown that $\lambda - z - e^{\lambda}$ has exactly one solution. $\square$

Question 4.

4. How many roots of

$$z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$$

lie in the right half plane $\{z : \mathrm{Re}\, z > 0\}$.

Solution.

Question 5.

5. Let $f \in H(B_R)$ for some $R > 1$. If $|f(z)| > 2$ for $|z| = 1$ and $f(0) = 1$. Must f have zero in B_1?

Solution. As B_1 is a circle of radius 1, centered around the origin, we have that the winding number of B_1 is simply 1. Observe that the given function is holomorphic, hence meromorphic with zero poles. interior to B_1 and is analytic on B_1. Furthermore, as $|f(z)| > 2$ for $z| = 1$, we have f is nonzero on B_1. Therefore, by the argument principle, we have that the winding number is equal to $Z - P$ where Z is the number of zeros and P is the number of poles of $f(z)$ inside B_1. Since $P = 0$ and the winding number is 1, we have that $Z = 1$. f must have zero in B_1. $\square$

Question 6.

6. Let $f \in H(B_R)$ for some $R > 1$. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. What can you say if we only have $|f(z)| \leq 1$ for $|z| = 1$ instead.

Solution.