Complex Analysis I: Problem Set V

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Abstract

This work contains the solutions to the problem set V of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. 177.2.

Solution. Let f be continuous on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assume that $f(z) \neq 0$ for $z \in R$. Let g be a function on R, defined by $g(z) = \frac{1}{f(z)}$ for $z \in R$. From the $g(z) = \frac{1}{f(z)}$ relation, we can deduce that g is also continuous, analytic and not constant throughout the interior of R. Then, by the given corollary of the maximum modulus principle, we have that the maximum value of $|g(z)|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior. Observe that $|g(z)| = |\frac{1}{f(z)}| = \frac{1}{|f(z)|}$. Since a modulus is strictly positive in this case, we have that maximum value of $|g(z)|$ corresponds to the minimum value of $|f(z)|$. In other words, the z^*, which is argmax$|g(z)|$ and lies on the boundary, is also the argmin$|f(z)|$. Consequently, we have shown that a minimum value is reached, and it occurs in the boundary of R and never in the interior. $\square$

Question 2. 177.4.

Solution.

Question 1.177.5.

Solution. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and nont constant throughout the interior of R.

Question 4. 195.3.

Solution. We wish to find the Maclaurin series expansion of the function

$$f(z) \quad = \quad \frac{z}{z^4 + 4} = \frac{z}{4} \cdot \frac{1}{1 + (z^4/4)}.$$

As the given function f is analytic throughout a disk $D = \{z \mid |z - z_0| < R_0\}$, where $z_0 = 0$ and $R_0 = \sqrt{2}$, by the Taylor's theorem, we have a Maclaurin series expansion for f as

$$f(z) \quad = \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n,$$

for $|z| < \sqrt{2}$. For f, the nth derivative can be computed, using the chain rule and product rule of differentiation, as follows:

$$\begin{aligned} f^{(1)}(z) &= \frac{1}{4} \cdot \frac{1}{1 + (z^4/4))} + \frac{z}{4} \cdot \frac{-z^3}{1 + (z^4/4)} \\ &= \frac{1}{4} \left(\frac{1 - z^4}{1 + (z^4/4)} \right) \\ f^{(2)}(z) &= \frac{1}{4} \end{aligned}$$

Question 5. 195.6.

Solution. From pg.193, we are given a Maclaurin series expansion of $\sinh$ as

$$\sinh(z) = \sum_{i=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$

for all $z \in \mathbb{C}$. From pg.196, we are given that $\sinh(z + \pi i) = -\sinh(z)$ and $\sinh$ is $2\pi i$ periodic. Consequently, we have that $\sinh(z - \pi i) = -\sinh(z)$. Substituting the equality into the above Taylor series expansion yields

$$\begin{aligned} \sinh(z - \pi i) &= \sum_{i=0}^{\infty} \frac{(z - \pi i)^{2n+1}}{(2n+1)!} \\ \sinh(z) &= -\sum_{i=0}^{\infty} \frac{(z - \pi i)^{2n+1}}{(2n+1)!}, \end{aligned}$$

for any $z \in \mathbb{C}$. as desired. $\square$

Question 6. 195.11.

Solution.