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# Complex Analysis I:

## Problem Set IV

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Youngduck Choi  
CILVR Lab  
New York University  
yc1104@nyu.edu

### Abstract

This work contains the solutions to the problem set IV of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

#### Question 1. Brown p.147-2.

**Solution. (b)** We first have that  $\cos(z/2)$  is continuous everywhere on the complex plane. Therefore, any contour from 0 to  $\pi + 2i$  will have the same value of  $F(\pi + 2i) - F(0)$ , where  $F$  denotes the antiderivative of  $\cos(z/2)$ . We can compute the exact value as follows:

$$\begin{aligned}\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz &= \left[2\sin\left(\frac{z}{2}\right)\right]_0^{\pi+2i} \\ &= 2\sin\left(\frac{\pi}{2} + i\right) \\ &= 2\cos(i) \\ &= e + \frac{1}{e},\end{aligned}$$

as desired.  $\square$

#### Question 5. Brown p.147-5.

**Solution.** Observe that along the lower horizontal leg, we have  $z = x(-a \leq x \leq a)$ .

#### Question 1. Brown p.159-2.

**Solution. (b)** Let  $C_1$  denote the positively oriented boundary of the square whose sides lie along the line  $x = \pm 1$ ,  $y = \pm 1$  and let  $C_2$  be the positively oriented circle  $|z| = 4$ . Observe that  $C_1$  is interior to  $C_2$  and the given function  $\frac{z+2}{\sin(\frac{z}{2})}$  is analytic in the closed region consisting of the  $C_1$  and  $C_2$  contours and all points between them. Hence, by the corollary, we have

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz,$$

for  $f(z) = \frac{z+2}{\sin(\frac{z}{2})}$ .  $\square$

#### Question 2. Brown 159-4.

**Solution. (a)** Observe that along the lower horizontal leg, we have  $z = x$  ( $-a \leq x \leq a$ ). Hence, the integral along the lower horizontal leg from  $-a$  to  $a$  can be written as

$$2 \int_0^a e^{-x^2} dx.$$

For the upper horizontal leg, we have  $z = x + ib$  ( $-a \leq x \leq a$ ). Hence, the integral along the upper horizontal from  $a$  to  $-a$  can be written as

$$\int_a^{-a} e^{-(x+ib)^2} dx,$$

which can be simplified as follows:

$$\begin{aligned} \int_a^{-a} e^{-(x+ib)^2} dx &= -e^{b^2} \int_{-a}^a e^{-x^2-2ibx} dx \\ &= -2e^{b^2} \int_0^a e^{-x^2} \cos(2bx) dx. \end{aligned}$$

Hence, we have

$$2 \int_0^a e^{-x^2} dx + -2e^{b^2} \int_0^a e^{-x^2} \cos(2bx) dx,$$

for the sum of contour integrals along each horizontal leg. Now, observe that along the right vertical leg, we have  $z = a + iy$  ( $0 \leq y \leq b$ ). Hence,

#### Question 1. Brown p.170-1.

**Solution.** Let  $C$  denote the positively oriented boundary of the square whose sides lie along the lines  $x = \pm 2$  and  $y = \pm 2$ . We evaluate the following integrals.

(b) We are given the following integral:

$$\int_C \frac{\cos(z)}{z(z^2+8)} dz,$$

which can be written as

$$\int_C \frac{\frac{\cos(z)}{(z^2+8)}}{z} dz.$$

As  $\frac{\cos(z)}{(z^2+8)}$  is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the Cauchy Integral formula, we obtain

$$\frac{\cos(0)}{8} = \frac{1}{2\pi i} \int_C \frac{\frac{\cos(z)}{(z^2+8)}}{z} dz,$$

which simplifies to

$$\int_C \frac{\frac{\cos(z)}{(z^2+8)}}{z} dz = \frac{\pi i}{4}.$$

(d) We are given the following integral:

$$\int_C \frac{\cosh(z)}{z^4} dz.$$

As  $\frac{\cosh(z)}{z^4}$  is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the extended Cauchy Integral formula, we obtain

$$\cosh^{(3)}(z_0) = \frac{3!}{2\pi i} \int_C \frac{\cosh(z)}{(z-z_0)^4} dz,$$

for  $z_0$  inside and on the given contour. Observe that  $\cosh^{(3)} = \sinh$ . Hence, taking  $z_0 = 0$  yields

$$0 = \frac{3!}{2\pi i} \int_C \frac{\cosh(z)}{z^4} dz,$$

which simplifies to

$$\int_C \frac{\cosh(z)}{z^4} dz = 0.$$

(e) We are given the following integral:

$$\int_C \frac{\tan(\frac{z}{2})}{(z - x_0)^2} dz,$$

for  $-2 < x_0 < 2$ . Notice that  $x_0$  is inside the given contour. As  $\frac{\tan(\frac{z}{2})}{(z - x_0)^2}$  is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the extended Cauchy Integral formula, we obtain

$$\frac{1}{2} \sec^2\left(\frac{x_0}{2}\right) = \frac{1!}{2\pi i} \int_C \frac{\tan(\frac{z}{2})}{(z - x_0)^2} dz,$$

which simplifies to

$$\int_C \frac{\tan(\frac{z}{2})}{(z - x_0)^2} dz = i\pi \sec^2\left(\frac{x_0}{2}\right),$$

for  $-2 < x_0 < 2$ .  $\square$

### Question 2. Brown 170.3.

**Solution.** Let  $C$  be the circle  $|z| = 3$ , described in the positive sense. As  $2s^2 - s - 2$  is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the extended Cauchy Integral formula, we obtain

$$2z^2 - z - 2 = \frac{1}{2\pi i} \int_C \frac{2s^2 - s - 2}{s - z} ds,$$

for  $|z| < 3$ . As  $g(z) = \int_C \frac{2s^2 - s - 2}{s - z} ds$ , we have

$$g(z) = 2\pi i(2z^2 - z - 2),$$

for  $|z| < 3$ . Hence, it follows that  $g(2) = 8\pi i$ . For  $|z| > 3$ , we have that  $\frac{2s^2 - s - 2}{s - z}$  is analytic at all points interior to and on  $C$ . Hence, by the Cauchy-Goursat theorem, we obtain

$$\int_C \frac{2s^2 - s - 2}{s - z} dz = 0,$$

for  $|z| > 3$ . Therefore,  $g(z) = 0$  when  $|z| > 3$ , which completes the solution for the problem.  $\square$

### Question 3. Brown 170-4.

**Solution.** Let  $C$  be any simple closed contour, described in the positive sense in the  $z$  plane. As  $s^3 + 2s$  is entire, by the extended Cauchy Integral formula, we obtain

$$6z = \frac{2!}{2\pi i} \int_C \frac{s^3 + 2s}{(s - z)^3} ds,$$

for  $z$  at the interior of  $C$ . As  $g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3} ds$ , we have

$$g(z) = 6\pi iz,$$

for  $z$  inside  $C$ . Now, if  $z$  is outside of  $C$ , then  $\frac{s^3 + 2s}{s - z}$  is analytic at points interior to and on  $C$ . Hence, by the Cauchy-Goursat Theorem, we have that

$$\int_C \frac{s^3 + 2s}{(s - z)^3} ds = 0,$$

for  $z$  outside of  $C$ . Hence  $g(z) = 0$  when  $z$  is outside.  $\square$

**Question 4. Brown 170-7.**

**Solution.** Let  $C$  be the unit circle. As  $e^{az}$  is entire, by the Cauchy Integral formula, we obtain

$$e^{az_0} = \frac{1}{2\pi i} \int_C \frac{e^{az}}{z - z_0} dz,$$

for  $z_0$  inside  $C$ . By taking  $z_0 = 0$ , we get

$$1 = \frac{1}{2\pi i} \int_C \frac{e^{az}}{z} dz,$$

which simplifies to

$$\int_C \frac{e^{az}}{z} dz = 2\pi i.$$

**Question 5. Brown 170-8.**

**Solution.** The Legendre polynomials are defined by

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2 - 1)^n}{(s - z)^{n+1}} ds,$$

for any simple closed contour surrounding  $z$ . For  $z = -1$ , and by having  $C$  to be any arbitrary simple closed contour that surrounds  $z = -1$ , it follows that

$$P_n(-1) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2 - 1)^n}{(s + 1)^{n+1}} ds,$$

which, by using the suggestion, simplifies to

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s - 1)^n}{s + 1} ds.$$

Since  $(s - 1)^n$  is entire,  $(s - 1)^n$  is analytic inside and on  $C$ . Hence, by the Cauchy Integral formula, we have

$$(-2)^n 2\pi i = \int_C \frac{(s - 1)^n}{s + 1} ds.$$

Substituting the above equality into the simplified formula of Legendre polynomials yields

$$\begin{aligned} P_n(z) &= \frac{(-2)^n 2\pi i}{2^{n+1}\pi i} \\ &= (-1)^n, \end{aligned}$$

as desired.  $\square$