Complex Analysis I: Problem Set IV

Youngduck Choi
CILVR Lab
New York University
yc1104@nyu.edu

Abstract

This work contains the solutions to the problem set IV of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1. Brown p.147-2.

Solution. (b) We first have that $\cos(z/2)$ is continous everywhere on the complex plane. Therefore, any contour from 0 to $\pi + 2i$ will have the same value of $F(\pi + 2i) - F(0)$, where F denotes the antiderivative of $\cos(z/2)$. We can compute the exact value as follows:

$$\begin{aligned}\int_0^{\pi+2i} \cos(\frac{z}{2})dz &= \Big[2\sin(\frac{z}{2})\Big]_0^{\pi+2i} \\ &= 2\sin(\frac{\pi}{2} + i) \\ &= 2\cos(i) \\ &= e + \frac{1}{e},\end{aligned}$$

as desired. $\square$

Question 2. Brown p.147-5.

Solution. Let C be a contour from -1 to 1 that lies above the x-axis. We wish to compute the following integral:

$$\int_C z^i dz,$$

where z^i denotes the principal branch $exp(iLog(z))$ for $|z| > 0, -\pi < Arg(z) < \pi$. Notice that under the principal branch, $z = -1$ is not defined. The following branch, however, agrees with the integrand along C and is has anti-derivative along C:

$$z^i = exp(ilog(z)) \text{ for } (|z| > 0, -\frac{\pi}{2} < arg(z) < \frac{3\pi}{2}).$$

We then can compute the integral as follows:

$$\begin{aligned}
\int_C z^i dz &= \left[\frac{z^{i+1}}{i+1}\right]_{-1}^{1} \\
&= \frac{1}{i+1}(e^{(i+1)log1} - e^{(i+1)log(-1)}) \\
&= \frac{1}{i+1}(e^{(i+1)(ln1)} - e^{(i+1)ln(1+i\pi)}) \\
&= \frac{1}{i+1}(1+e^{i\pi}) \\
&= \frac{1+e^{-\pi}}{2}(1-i),
\end{aligned}$$

as desired. $\square$

Question 3. Brown p.159-2.

Solution. (b) Let C_1 denote the positively oriented boundary of the square whose sides lie along the line $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$. Observe that C_1 is interior to C_2 and the given function $\frac{z+2}{\sin(\frac{z}{2})}$ is analytic in the closed region consisting of the C_1 and C_2 contours and all points between them. Hence, by the corollary, we have

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz,$$

for $f(z) = \frac{z+2}{\sin(\frac{z}{2})}$. $\square$

Question 4. Brown 159-4.

Solution. (a) Observe that along the lower horizontal leg, we have $z = x \ (-a \leq x \leq a)$. Hence, the integral along the lower horizontal leg from $-a$ to a can be written as

$$2\int_0^a e^{-x^2}dx.$$

For the upper horizontal leg, we have $z = x+ib \ (-a \leq x \leq a)$. Hence, the integral along the upper horizontal from a to $-a$ can be written as

$$\int_a^{-a} e^{-(x+ib)^2}dx,$$

wich can be simplified as follows:

$$\begin{aligned}
\int_a^{-a} e^{-(x+ib)^2}dx &= -e^{b^2}\int_{-a}^{a} e^{-x^2-2ibx}dx \\
&= -2e^{b^2}\int_0^a e^{-x^2}\cos(2bx)dx.
\end{aligned}$$

Hence, we have

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos(2bx)dx,$$

for the sum of contour integrals along each horizontal leg.

Now, observe that along the vertical legs, we have $z = \pm a + iy \ (0 \leq y \leq b)$. Hence, the integral along the vertical legs can be written as

$$I = \int_0^b e^{-(a+iy)^2}idy + \int_b^0 e^{-(-a+iy)^2}idy,$$

which can be simplified as follows:

$$\begin{aligned} I &= \int_0^b e^{-a^2-2iay+y^2} idy - \int_0^b e^{-a^2+2iay+y^2} idy \\ &= ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} - ie^{-a^2} \int_0^b e^{y^2} e^{2iay} dy. \end{aligned}$$

Hence, by the Cauchy-Goursat Theorem, we have

$$\begin{aligned} 2\int_0^a e^{-x^2} dx - 2e^{b^2} \int_0^a e^{-x^2} \cos(2bx) dx & \\ +ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} - ie^{-a^2} \int_0^b e^{y^2} e^{2iay} dy &= 0, \end{aligned}$$

which by re-arranging and simplifying becomes

$$\int_0^a e^{-x^2} cos(2bx) dx = e^{-b^2} \int_0^a e^{-x^2} dx + e^{-a^2-b^2} \int_0^b e^{y^2} sin(2ay) dy,$$

as desired.

(b) As $a \to \infty$, we have that

$$e^{-a^2-b^2} \int_0^b e^{y^2} dy \to 0.$$

Observe that

$$\left| e^{-a^2-b^2} \int_0^b e^{y^2} sin(2ay) dy \right| \leq e^{-a^2-b^2} \int_0^b e^{y^2} dy.$$

Hence, by the squeeze theorem, we have as $a \to \infty$

$$e^{-a^2-b^2} \int_0^b e^{y^2} sin(2ay) dy \to 0.$$

Therefore, with the given result in the problem, we obtain

$$\int_0^\infty e^{-x^2} cos(2bx) dx = \frac{\sqrt{\pi}}{2} e^{-b^2},$$

as desired. $\square$

Question 5. Brown p.170-1.

Solution. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. We evaluate the following integrals.

(b) We are given the following integral:

$$\int_C \frac{\cos(z)}{z(z^2+8)} dz,$$

which can be written as

$$\int_C \frac{\frac{\cos(z)}{(z^2+8)}}{z} dz.$$

As $\dfrac{\cos(z)}{(z^2+8)}$ is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the Cauchy Integral formula, we obtain

$$\frac{\cos(0)}{8} = \frac{1}{2\pi i} \int_C \frac{\frac{\cos(z)}{(z^2+8)}}{z} dz,$$

which simplifies to

$$\int_C \frac{\frac{\cos(z)}{(z^2+8)}}{z} dz \quad = \quad \frac{\pi i}{4}.$$

(d) We are given the following integral:

$$\int_C \frac{\cosh(z)}{z^4} dz.$$

As $\dfrac{\cosh(z)}{z^4}$ is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the extended Cauchy Integral formula, we obtain

$$\cosh^{(3)}(z_0) \quad = \quad \frac{3!}{2\pi i} \int_C \frac{\cosh(z)}{(z - z_0)^4} dz,$$

for z_0 inside and on the given contour. Observe that $\cosh^{(3)} = \sinh$. Hence, taking $z_0 = 0$ yields

$$0 \quad = \quad \frac{3!}{2\pi i} \int_C \frac{\cosh(z)}{z^4} dz,$$

which simplifies to

$$\int_C \frac{\cosh(z)}{z^4} dz \quad = \quad 0.$$

(e) We are given the following integral:

$$\int_C \frac{\tan(\frac{z}{2})}{(z - x_0)^2} dz,$$

for $-2 < x_0 < 2$. Notice that x_0 is inside the given contour. As $\dfrac{\tan(\frac{z}{2})}{(z - x_0)^2}$ is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the extended Cauchy Integral formula, we obtain

$$\frac{1}{2}\sec^2(\frac{x_0}{2}) \quad = \quad \frac{1!}{2\pi i} \int_C \frac{\tan(\frac{z}{2})}{(z - x_0)^2} dz,$$

which simplifies to

$$\int_C \frac{\tan(\frac{z}{2})}{(z - x_0)^2} dz \quad = \quad i\pi \sec^2(\frac{x_0}{2}),$$

for $-2 < x_0 < 2$. $\square$

Question 6. Brown 170.3.

Solution. Let C be the circle $|z| = 3$, described in the positive sense. As $2s^2 - s - 2$ is analytic everywhere inside and on the given contour, which is simple and closed, taken in the positive sense, by the extended Cauchy Integral formula, we obtain

$$2z^2 - z - 2 \quad = \quad \frac{1}{2\pi i} \int_C \frac{2s^2 - s - 2}{s - z} ds,$$

for $|z| < 3$. As $g(z) = \int_C \dfrac{2s^2 - s - 2}{s - z} ds$, we have

$$g(z) = 2\pi i(2z^2 - z - 2),$$

for $|z| < 3$. Hence, it follows that $g(2) = 8\pi i$. For $|z| > 3$, we have that $\dfrac{2s^2 - s - 2}{s - z}$ is analytic at all points interior to and on C. Hence, by the Cauchy-Goursat theorem, we obtain

$$\int_C \frac{2s^2 - s - 2}{s - z} dz = 0,$$

for $|z| > 3$. Therefore, $g(z) = 0$ when $|z| > 3$, which completes the solution for the problem. $\square$

Question 7. Brown 170-4.

Solution. Let C be any simple closed contour, described in the positive sense in the z plane. As $s^3 + 2s$ is entire, by the extended Cauchy Integral formula, we obtain

$$6z = \frac{2!}{2\pi i} \int_C \frac{s^3 + 2s}{(s-z)^3 i} ds,$$

for z at the interior of C. As $g(z) = \int_C \frac{s^3+2s}{(s-z)^3} ds$, we have

$$g(z) = 6\pi i z,$$

for z inside C. Now, if z is outside of C, then $\frac{s^3+2s}{s-z}$ is analytic at points interior to and on C. Hence, by the Cauchy-Goursat Theorem, we have that

$$\int_C \frac{s^3+2s}{(s-z)^3} ds = 0,$$

for z outside of C. Hence $g(z) = 0$ when z is outside. □

Question 8. Brown 170-7.

Solution. Let C be the unit circle. As e^{az} is entire, by the Cauchy Integral formula, we obtain

$$e^{az_0} = \frac{1}{2\pi i} \int_C \frac{e^{az}}{z - z_0} dz,$$

for z_0 inside C. By taking $z_0 = 0$, we get

$$1 = \frac{1}{2\pi i} \int_C \frac{e^{az}}{z} dz,$$

which simplifies to

$$\int_C \frac{e^{az}}{z} dz = 2\pi i.$$

As C is the unit circle, we have a polar parametrization as $z = e^{i\theta}$. Hence, by substitution, we have

$$\begin{aligned}
\int_C \frac{e^{az}}{z} dz &= \int_{-\pi}^{\pi} \frac{exp(ae^{i\theta})}{e^{i\theta}} ie^{i\theta} d\theta \\
&= i\int_{-\pi}^{\pi} exp(ae^{i\theta}) d\theta \\
&= i\int_{-\pi}^{\pi} e^{acos(\theta)} e^{iasin(\theta)} d\theta \\
&= i\int_{-\pi}^{\pi} e^{acos(\theta)} (cos(asin(\theta)) + isin(asin(\theta)) d\theta \\
&= -\int_{-\pi}^{\pi} sin(asin(\theta) d\theta + i\int_{-\pi}^{\pi} e^{acos(\theta)} cos(asin(\theta)) d\theta
\end{aligned}$$

As the above integral equals $2\pi i$, we have

$$\int_{-\pi}^{\pi} e^{acos(\theta)} cos(asin(\theta)) d\theta = 2\pi,$$

which, as the integrand is even, simplifies to

$$\int_0^{\pi} e^{acos(\theta)} cos(asin(\theta)) d\theta = \pi,$$

as desired. □

Question 9. Brown 170-8.

Solution. The Legendre polynomials are defined by

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}ds,$$

for any simple closed contour surrounding z. For $z = -1$, and by having C to be any arbitrary simple closed contour that surrounds $z = -1$, it follows that

$$P_n(-1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s+1)^{n+1}}ds,$$

which, by using the suggestion, simplifies to

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s-1)^n}{s+1}ds.$$

Since $(s-1)^n$ is entire, $(s-1)^n$ is analytic inside and on C. Hence, by the Cauchy Integral formula, we have

$$(-2)^n 2\pi i = \int_C \frac{(s-1)^n}{s+1}ds.$$

Substituting the above equality into the simplified formula of Legendre polynomials yields

$$\begin{aligned} P_n(z) &= \frac{(-2)^n 2\pi i}{2^{n+1}\pi i} \\ &= (-1)^n, \end{aligned}$$

as desired. □

Question 10. Brown 177-1.

Solution. Assume that $f(z)$ is entire, and that the harmonic function $u(x,y) = Re[f(z)]$ has an upper bound u_0. Observe that $g(z) = e^{f(z)}$ is entire, and

$$|e^{f(z)}| = |e^{u(x,y)+iv(x,y)}| = |e^{u(x,y)}| \leq e^{u_0},$$

as u_0 is an upper bound for $u(x,y)$. Therefore, by the Liouville's theorem, we have that g is constant. Then, $g^{'}(z) = 0$ for all z. By the complex chain rule, we obtain

$$g'(z) = f'(z)e^{f(z)}.$$

Since $e^{f(z)} \neq 0$, we have $f'(z) = 0$, and $f(z)$ is constant. Hence, the real part $u(x,y)$ is constant as well. □