Complex Analysis I: Problem Set IX

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Abstract

This work contains the solutions to the problem set IX of Complex Analysis I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

1. Evaluate the integral

$$\int_{\gamma} \frac{dz}{z^2+1}$$

for

$$\gamma(\theta) = 2\left|\cos 2\theta\right| e^{i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Solution. By drawing the contour on the complex plane, we observe that γ forms 4 simple closed contours, for each direction of the axis. We denote these contours as γ_1, γ_2, γ_3, and γ_4 respectively in a counter-clockwise fashion. Observe that $f(z) = \dfrac{1}{z^2+1}$ is singular at $z = \pm i$. $z = i$ belongs to the interior of γ_2 contour, and $z = -i$ belongs to the interior of γ_4 contour. By the Cauchy-Residue formula, we obtain

$$\begin{aligned}
\int_{\gamma_1} \frac{dz}{z^2+1} &= 0 \\
\int_{\gamma_2} \frac{dz}{z^2+1} &= \mathrm{Res}_{z=i} \frac{1}{z^2+1} \\
\int_{\gamma_3} \frac{dz}{z^2+1} &= 0 \\
\int_{\gamma_4} \frac{dz}{z^2+1} &= \mathrm{Res}_{z=-i} \frac{1}{z^2+1}.
\end{aligned}$$

As it can be written that $f(z) = \dfrac{\phi(z)}{z-i}$, where $\phi(z) = \dfrac{1}{z+i}$, the residue at $z = i$ is $\phi(i) = \dfrac{1}{2i}$. On the other hand, as it can be written that $f(z) = \dfrac{\phi(z)}{z+i}$, where $\phi(z) = \dfrac{1}{z-i}$, the residue at $z = -i$

is $\phi(i) = -\dfrac{1}{2i}$. Consequently, we have

$$\begin{aligned}\int_{\gamma_2} \frac{dz}{z^2+1} &= \frac{1}{2i} \\ \int_{\gamma_4} \frac{dz}{z^2+1} &= -\frac{1}{2i}.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\int_{\gamma} \frac{dz}{z^2+1} &= \int_{\gamma_1} \frac{dz}{z^2+1} + \int_{\gamma_2} \frac{dz}{z^2+1} + \int_{\gamma_3} \frac{dz}{z^2+1} + \int_{\gamma_4} \frac{dz}{z^2+1} \\ &= 0.\end{aligned}$$

$\square$

Question 2.

2. Let

$$\gamma(\theta) = \begin{cases} \theta e^{i\theta}, & 0 \leq \theta \leq 2\pi, \\ 4\pi - \theta, & 2\pi \leq \theta \leq 4\pi. \end{cases}$$

Calculate

$$\int_{\gamma} \frac{dz}{z^2+\pi^2}.$$

Solution. Observe that the function has isolated singularities at $z = \pm i\pi$. By observing the contour, we see that $i\pi$ lies outside of the contour, as $\gamma(\frac{\pi}{2}) = \frac{\pi}{2} e^{i\frac{\pi}{2}} = i\frac{\pi}{2}$. On the other hand, $z = -i\pi$ lies on the interior of the contour as $\gamma(\frac{3\pi}{2}) = \frac{3\pi}{2} e^{i\frac{3\pi}{2}} = -\frac{3\pi}{2} i$. Hence, by the Cauchy Residue theorem, we have

$$\int_{\gamma} \frac{dz}{z^2+\pi^2} = \text{Res}_{z=-i\pi} \frac{1}{z^2+\pi^2}.$$

As it can be written that $f(z) = \dfrac{\phi(z)}{z+i\pi}$, where $\phi(z) = \dfrac{1}{z-i\pi}$, the residue at $z = -i\pi$ is $\phi(-i\pi) = \dfrac{1}{-2i\pi}$.

Question 3.

Solution. As we have $\lambda - z - e^{-\lambda} = 0$, we have

$$|\lambda - z| = e^{-\text{Re} z}.$$

Since Re$z > 0$, to satisfy the inequality, we must have

$$|\lambda - z| < 1.$$

Question 4.

Solution.

Question 5.

Solution.