Harmonic Analysis: Problem Set III

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set III of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Let α be any irrational number.

   (a) Show that for every trigonometric polynomial P

$$\frac{1}{N}\sum_{n=1}^{N} P(n\alpha) \to \int_0^1 P(x)\,\mathrm{d}x.$$

   (b) Show that for every $f \in C(\mathbb{T})$

$$\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) \to \int_0^1 f(x)\,\mathrm{d}x.$$

   (c) Show that the conclusion of (b) continues to hold for every Riemann integrable function on $\mathbb{T}$. What about Lebesgue integrable functions?

Solution.
Throughout this problem, a domain of any function will be consistently defined as $\mathbb{T}$.
(a) Let α be a irrational number. First, consider $e(kx)$. For $k = 0$, we have 1 converges to 1. Now, for any $k \neq 0$, we have that $\int_0^1 e(kx) = 0$. Furthermore, as α is an irrational, we have that $e(k\alpha) \neq 1$. Hence, by the geometric series formula, it follows that

$$\frac{1}{N}\sum_{n=1}^{N} e(n\alpha) \quad = \quad \frac{e(k\alpha)}{N}\frac{1-e(kN\alpha)}{1-e(k\alpha)},$$

which converges to 0 as $N \to \infty$. Therefore, we have shown the convergence holds true for exponentials. As trig polynomials are finite linear combinations of exponentials, by the linearity of limit, the convergence is true for any trig polynomial.

(b) Fix $\epsilon > 0$. It follows that By the density of trig polynomials in $C(\mathbb{T})$, there exists a polynomial P such that $||P - f||_{max} < \epsilon$. By part (a), the triangle inequality, and rules of integration, it follows that,

$$
\begin{aligned}
\left|\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) - \int_0^1 f(x)dx\right| &\leq \left|\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) - \frac{1}{N}\sum_{n=1}^{N} P(n\alpha)\right| \\
&+ \left|\frac{1}{N}\sum_{n=1}^{N} P(n\alpha) - \int_0^1 P(x)dx\right| \\
&+ \left|\int_0^1 P(x)dx - \int_0^1 f(x)dx\right| \\
&\leq \frac{1}{N}\sum_{n=1}^{N} |f(n\alpha) - P(n\alpha)| \\
&+ \left|\frac{1}{N}\sum_{n=1}^{N} P(n\alpha) - \int_0^1 P(x)dx\right| \\
&+ \int_0^1 |P(x) - f(x)|dx < \epsilon + \epsilon + \epsilon = 3\epsilon,
\end{aligned}
$$

for all N large enough. Hence, we have shown that the convergence holds true for all continuous functions.

(c) Before proceeding to the main part of the proof, we prove that the asserted convergence holds true for all characteristic functions of $(a,b) \in \mathbb{T}$, which is quite natural as the definition of Riemann integration involves a partition with the associated upper sum and lower sum:

$$\frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\alpha) \to \int_0^1 \chi_{(a,b)}(x)dx, \text{ as } N \to \infty. \tag{1}$$

Fix $\epsilon > 0$, and $(a,b) \in \mathbb{T}$. Let f_ϵ^+ and f_ϵ^- be continuous functions on $\mathbb{T}$, defined by

$$
f_\epsilon^+(x) = \begin{cases} 0 & \text{if } x \in [0, a-\epsilon) \\ \frac{1}{\epsilon}(x-a)+1 & \text{if } x \in [a-\epsilon, a) \\ 1 & \text{if } x \in [a,b) \\ -\frac{1}{\epsilon}(x-b)+1 & \text{if } x \in [b, b+\epsilon) \\ 0 & \text{if } x \in [b+\epsilon, 1], \end{cases}
$$

and

$$
f_\epsilon^-(x) = \begin{cases} 0 & \text{if } x \in [0, a) \\ \frac{1}{\epsilon}(x-a) & \text{if } x \in [a, a+\epsilon) \\ 1 & \text{if } x \in [a+\epsilon, b-\epsilon) \\ -\frac{1}{\epsilon}(x-b) & \text{if } x \in [b-\epsilon, b) \\ 0 & \text{if } x \in [b, 1]. \end{cases}
$$

In particular, we have

$$b - a - 2\epsilon \leq \int_0^1 f_\epsilon^-(x)dx \text{ and } \int_0^1 f_\epsilon^+(x)dx \leq b - a + 2\epsilon, \tag{2}$$

along with

$$\frac{1}{N}\sum_{n=1}^{N} f_\epsilon^-(n\alpha) \leq \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\alpha) \leq \frac{1}{N}\sum_{n=1}^{N} f_\epsilon^+(n\alpha). \tag{3}$$

Now, letting $N \to \infty$ on both sides of (3) respectively, we obtain

$$b - a - 2\epsilon \leq \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\alpha) \text{ and } \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\alpha) \leq b - a + 2\epsilon.$$

As ϵ was arbitrary, we have shown that the asserted convergence is true for all characteristic functions in $\mathbb{T}$. As before, the result can be extended to all finite linear combinations of characteristic functions.

We proceed to the main part of the proof. Fix $\epsilon > 0$. As f is Riemann integrable, taking a fine enough partition, we have

$$\int_0^1 f(x)dx - \epsilon \leq \int_0^1 L_f(dx) \text{ and } \int_0^1 U_f(x)dx \leq \int_0^1 f(x)dx + \epsilon,$$

with

$$\frac{1}{N}\sum_{i=1}^{N} L_f(n\alpha) \leq \frac{1}{N}\sum_{i=1}^{N} f(n\alpha) \leq \frac{1}{N}\sum_{i=1}^{N} U_f(n\alpha),$$

where U_f, and L_f denote the upper, lower Riemann sum of the chosen partition respectively. In view of (1), letting $N \to \infty$, we obtain

$$\int_0^1 f(x)dx - \epsilon \leq \liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} f(n\alpha) \text{ and } \limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} f(n\alpha) \leq \int_0^1 f(x)dx + \epsilon.$$

As ϵ was arbitrary, we have shown that the asserted convergence is true for all Riemann integrable functions on $\mathbb{T}$.

Now, for the case of Lebesgue integration, consider χ_A, where $A = \{< n\alpha > \ | n \in \mathbb{N}\}$ ($<>$ denotes the fractional part of a number. By definition, we have $\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = 1$ for all N. As A is countable, and χ_A only 1 on measure zero set, we have that $\int_0^1 f(x)dx = 0$. Therefore, the convergence does not hold true for all Lebesgue measurable functions. $\square$

Question 2.

2. Let $\mathcal{T}_n$ denote the linear space of trigonometric polynomials of degree up to n and

$$E_n(f) := \inf_{P\in\mathcal{T}_n} \|f - P\|_2 = \|f - S_N f\|_2 = \left(\sum_{|k|>n} |\widehat{f}(k)|^2\right)^{1/2}.$$

Let $0 < \alpha < 1$. Show that $E_n(f) \lesssim n^{-\alpha}$ if and only if $\omega_f(\delta)_{L^2} \lesssim \delta^\alpha$ where

$$\omega_f(\delta)_{L^2} := \sup_{|h|\leq\delta} \|f - f(\cdot - h)\|_2.$$

This class of functions is called $\mathrm{Lip}_{\alpha,L^2}(\mathbb{T})$.

(Hint: Use the dyadic decomposition trick we employed in class before.)

Solution. Let $0 < \alpha < 1$. First, suppose that $\omega_f(\delta)_{L^2} \lesssim \delta^\alpha$. We proceed by the dyadic decomposition trick introduced in class. Let

$$\triangle_m = \pm[2^m n, 2^{m+1}n) \cap \mathbb{Z}.$$

Then, for any $k \in \triangle_m$, we obtain that

$$\frac{2\pi}{3} \leq \frac{2\pi|k|}{3\cdot 2^m \cdot n} \leq \frac{4\pi}{3}, \text{ and } |e^{2\pi i \frac{k}{3\cdot 2^m\cdot n}} - 1| \geq \sqrt{3}.$$

Now, with $h_m = \dfrac{1}{3 \cdot 2^m \cdot n}$, it follows that

$$\begin{aligned}
E_n^2(f) &= \sum_{|k|>n} |\hat{f}(k)|^2 \le \sum_{m=0}^{\infty} \sum_{k \in \triangle_m} |\hat{f}(k)|^2 \le \sum_{m=0}^{\infty} \sum_{k \in \triangle_m} |e^{2\pi i k h_m} - 1|^2 |\hat{f}(k)|^2 \\
&= \sum_{m=0}^{\infty} ||f(k) - f(\cdot - h_m)(k)||_2^2 \le \sum_{m=0}^{\infty} \omega_f(h_m)_{L_2}^2 \lesssim \sum_{m=0}^{\infty} h_m^{2\alpha} = \sum_{m=0}^{\infty} (3 \cdot 2^m)^{-2\alpha} n^{-2\alpha} \\
&= n^{-2\alpha} \sum_{m=0}^{\infty} (3 \cdot 2^m)^{-2\alpha} \lesssim n^{-2\alpha}
\end{aligned}$$

Hence, we have $E_n(f) = O(n^{-\alpha})$.

Now, suppose that $E_n(f) = O(n^{-\alpha})$. Consider h small, such that $|h| \le \delta$. It follows that

$$\begin{aligned}
||f - f(\cdot - h)||_2^2 &= \sum_{k \in \mathbb{Z}} |e^{2\pi i k h} - 1|^2 |\hat{f}(k)^2| \\
&= \sum_{|k| \le \frac{1}{2\delta}} 4 \sin^2(\pi k h) |\hat{f}(k)|^2 + \sum_{|k| > \frac{1}{2\delta}} 4 \sin^2(\pi k h) |\hat{f}(k)|^2 \\
&\le 4\pi^2 \delta^2 \sum_{|k| \le \frac{1}{2\delta}} k^2 |\hat{f}(k)|^2 + 4 \sum_{|k| > \frac{1}{2\delta}} |\hat{f}(k)|^2 \\
&= 4\pi^2 \delta^2 \sum_{|k| \le \frac{1}{2\delta}} k^2 (E_k^2(f) - E_{k-1}^2(f)) + 4E^2_{\lfloor \frac{1}{2\delta} \rfloor}(f).
\end{aligned}$$

Now, as $E_n(f) = O(n^{-\alpha})$, it follows that the RHS of the derived inequality is constant + $O(\delta^{2\alpha})$ term. Hence, we have shown that $\omega_f(\delta) = O(\delta^\alpha)$. $\square$

Question 3.

3. Solve Exercises 2.2, 2.3, 2.4 in Muscalu & Schlag.

Solution. (2.2) We first prove the closed form formula for the Poisson Kernel. Let $0 \le r < 1$. Then, the Poisson Kernel is given by

$$P_r(\theta) = \sum_{n \in \mathbb{Z}} r^{|n|} e(n\theta).$$

As the series is absolutely convergent, with $\omega = re^{i\theta}$, by the geometric series formula, it follows that

$$\begin{aligned}
P_r(\theta) &= \sum_{n=0}^{\infty} \omega^n + \sum_{n=1}^{\infty} \overline{\omega}^n \\
&= \frac{1}{1-\omega} + \frac{\overline{\omega}}{1-\overline{\omega}} = \frac{1-|\omega|^2}{|1-\omega|^2} \\
&= \frac{1-r^2}{1-2r\cos(2\pi\theta)+r^2}.
\end{aligned}$$

(2.3) Now, we show that the Poisson Kernel is an approximate identity. First, by the absolute convergence, we have

$$\int_{\mathbb{T}} P_r(\theta) = \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} r^{|n|} e(n\theta) = \sum_{n \in \mathbb{Z}} \int_{\mathbb{T}} r^{|n|} e(n\theta) = 1,$$

as the $n = 0$ term is the only term that integrates to 1, when all other terms integrates to 0. Hence, $(A1)$ from the definition 1.3 is verified. Now, observe that

$$1 - 2r\cos(2\pi\theta) + r^2 = (1-r)^2 + 2r(1-\cos(2\pi\theta)). \tag{4}$$

Hence, $Pr(\theta) \geq 0$ for $\theta \in \mathbb{T}$. It follows that

$$\sup_{0\leq r<1} \int_{\mathbb{T}} |P_r(\theta)| d\theta = 1 < \infty,$$

which shows that $(A2)$ is satisfied. Now, we verify the $(A3)$ property. Fix $\delta > 0$. For $\frac{1}{2} \leq r \leq 1$, and $\delta \leq \theta \leq 1$, by (4), it follows that

$$0 < c_\delta \leq 1 - 2r\cos(2\pi\theta) + r^2, \tag{5}$$

for some constant c_δ. In view of (5), we obtain

$$\int_{|\theta|>\delta} |P_r(\theta)| d\theta \quad \leq \quad \int_{|\theta|>\delta} \frac{|1-r^2|}{c_\delta} d\theta = \frac{1-\delta}{c_\delta}|1-r^2|.$$

Therefore, it follows that

$$\frac{1-\delta}{c_\delta}|1-r^2| \to 0 \text{ as } r \to 1^- \text{ , and } \int_{|\theta|>\delta} |P_r(\theta)| d\theta \to 0 \text{ as } r \to 1^-,$$

as required. This completes the proof that the Poisson Kernel is an approximate identity.

(2.4)-(a) From Schlag pg. 33, we have the closed form as

$$Q_r(\theta) \quad = \quad \frac{2r\sin(2\pi\theta)}{1 - 2r\cos(2\pi\theta) + r^2}.$$

As $\sin(-2\pi\theta) = -\sin(2\pi\theta)$ and $\cos(-2\pi\theta) = \cos(2\pi\theta)$, we have

$$\begin{aligned} Q_r(1-\theta) \quad &= \quad \frac{2r\sin(2\pi(1-\theta))}{1 - 2r\cos(2\pi(1-\theta)) + r^2} \\ &= \quad -\frac{2r\sin(2\pi\theta)}{1 - 2r\cos(2\pi\theta) + r^2} = -Q_r(\theta). \end{aligned}$$

Thus, Q_r has an odd symmetry, centered at $\theta = 0.5$, implying that $\int_0^1 Q_r(\theta)d\theta = 0$ for all $0 < r < 1$, which violates one of the conditions being an approximate identity.

(2.4)-(b) By the double-angle trig identity, we have

$$Q_1(\theta) \quad = \quad \frac{2\sin(2\pi\theta)}{2 - 2\cos(2\pi\theta)} = \frac{2\sin(2\pi\theta)}{2\sin^2(\pi\theta)} = \cot(\pi\theta).$$

For the graph, we have

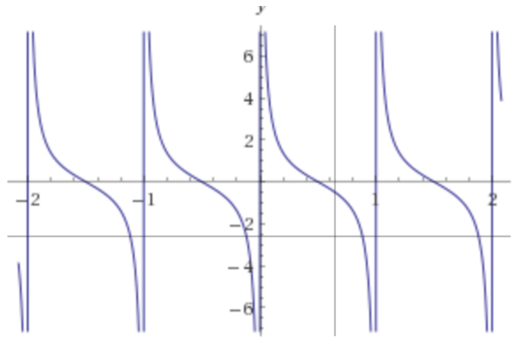
Now, as $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1$, and again using the identity, we have

$$Q_1(\theta) = \frac{\cos(\pi\theta)}{\sin(\pi\theta)} = \frac{\sin(2\pi\theta)}{2\sin^2(\pi\theta)} \sim_{\theta\to 0} \frac{1}{\theta}, \text{ and equivalently } Q_1(\theta) = \Theta(\frac{1}{\theta}),$$

as required.

$\square$

Figure 1: A graph of $\cot(\pi\theta)$



Question 4.

4. Show that if a series $\sum a_n$ is Cesàro summable to A, then it is also Abel summable to A. In other words, show that

$$\lim_{N\to\infty} \sum_{n=-\infty}^{\infty} \left(1 - \frac{|n|}{N}\right)_+ a_n = A \text{ implies } \lim_{r\to 1^-} \sum_{n=-\infty}^{\infty} r^{|n|} a_n = A.$$

The reverse implication does not hold, however. Give a counter-example.
Can you generalize this result to series in arbitrary normed spaces?

(Hint: Note that you may first reduce the problem to one-sided series. Note also that the context of this problem is more general than summability of Fourier series.)

Solution.
We first consider the case of one-sided series, with $A = 0$. Let $\sum_{n=0}^{\infty} a_n$ be Cesaro summable to 0. Let $s_n = \sum_{k=0}^{n} a_n$, $\sigma_n = \sum_{k=0}^{n} s_n$, and $r \in [0,1)$. By Cesaro summability, we have $\sigma_n = O(n)$, and the fact that $\sum_{n=0}^{\infty} \sigma_n r^n$ converges. As $s_n = \sigma_n - \sigma_{n-1}$, with $\sigma_{-1} = 0$, we have

$$\begin{aligned} \sum_{n=0}^{k} s_n r^n &= \sum_{n=0}^{k} (\sigma_n - \sigma_{n-1}) r^n \\ &= \sum_{n=0}^{k} \sigma_n r^n - r \sum_{n=0}^{k-1} \sigma_n r^n. \end{aligned}$$

Hence, in the limit, we have the following recursive relation:

$$(1-r)^2 \sum_{n=0}^{\infty} \sigma_n r^n = (1-r) \sum_{n=0}^{\infty} s_n r^n = \sum_{n=0}^{\infty} a_n r^n, \tag{6}$$

which justifies the convergence of the series $\sum_{n=0}^{\infty} s_n r^n$ and $\sum_{n=0}^{\infty} a_n r^n$ inductively. In view of (6), to prove Abel summability to 0, it suffices to show that

$$\limsup_{r\to 1-} |(1-r)^2 \sum_{n=0}^{\infty} \sigma_n r^n| \le 0. \tag{7}$$

Now, fix $\epsilon > 0$. As $\dfrac{\sigma_n}{n+1} \to 0$, we have

$$\begin{aligned}
|(1-r)^2 \sum_{n=0}^{\infty} \sigma_n r^n| &\leq |(1-r)^2 \sum_{n=0}^{N} \sigma_n r^n| + |(1-r)^2 \sum_{n=N+1}^{\infty} \sigma_n r^n| \\
&= |(1-r)^2 \sum_{n=0}^{N} \sigma_n r^n| + |(1-r)^2 \sum_{n=N+1}^{\infty} (n+1)\frac{\sigma_n}{n+1} r^n| \\
&\leq |(1-r)^2 \sum_{n=0}^{N} \sigma_n r^n| + |(1-r)^2 \sum_{n=N+1}^{\infty} (n+1) r^n|\epsilon,
\end{aligned}$$

for an N large enough. As $|(1-r)^2 \sum_{N+1}^{\infty}(n+1)r^n|$ is finite, and ϵ is arbitrary, we obtain

$$|(1-r)^2 \sum_{n=0}^{\infty} \sigma_n r^n| \quad \leq \quad |(1-r)^2 \sum_{n=0}^{N} \sigma_n r^n|$$

Now, with N fixed, and letting $r \to 1^-$, we obtain (7). Hence, we have shown that that $\sum_{n=0}^{\infty} a_n$ is Abel summable to 0.

Consider the series $\sum_{n=0}^{\infty} a_n$ where $a_n = (-1)^n n$. For every $0 \leq r < 1$, the Abel sum is formally given by

$$A(r) \quad \sim \quad \sum_{n=0}^{\infty} (-r)^n n.$$

Employing the summation by parts formula, with $a_n = n$ and $b_n = (-r)^n$, we obtain

$$\begin{aligned}
\sum_{n=0}^{N} n(-r)^n &= N \sum_{n=0}^{N} (-r)^n - \sum_{n=0}^{N-1} \sum_{k=0}^{n} (-r)^k \\
&= N \frac{1-(-r)^{N+1}}{1+r} - \sum_{n=0}^{N-1} \frac{1-(-r)^{n+1}}{1+r} \\
&= \frac{N}{1+r} - \frac{N(-r)^{N+1}}{1+r} - \frac{N-1}{1+r} + \frac{1}{1+r} \sum_{n=0}^{N-1} (-r)^{n+1} \\
&= \frac{1}{1+r} - \frac{N(-r)^{N+1}}{1+r} + \frac{1-(-r)^N}{(1+r)^2}.
\end{aligned}$$

Letting $N \to \infty$, we obtain

$$\sum_{n=0}^{\infty} (-r)^n n = \frac{2+r}{(1+r)^2},$$

and letting $r \to 1-$,

$$\lim_{r \to 1-} A(r) = \frac{3}{4},$$

which shows that the series considered is Abel summable. Now, we claim that for any Cesaro summable series $\sum_{n=0}^{\infty} a_n$, we have $\lim_{n\to\infty} \dfrac{a_n}{n} = 0$. Denote σ_n as the nth Cesaro term. Then, by basic properties of limit, it follows that

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \sigma_n - \frac{n-1}{n}\sigma_{n-1} = \lim_{n\to\infty} \sigma_n - \lim_{n\to\infty} \frac{n-1}{n} \lim_{n\to\infty} \sigma_{n-1} = 0,$$

where the last equality holds by the Cesaro summability of $\{a_n\}$. Observe that $\dfrac{a_n}{n} = (-1)^n$, for all n. Hence, the necessary condition of Cesaro summability does not hold for the above series. Thus, the series is not Cesaro summable, but Abel summable. We have shown that the reverse implication does not hold. $\square$