
Harmonic Analysis: Problem Set I

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Abstract

This work contains solutions to the problem set I of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

EXERCISE 1.1. Verify that for each integer $N \geq 0$

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \quad (1.5)$$

and draw the graph of D_N for several different values of N , say $N = 2$ and $N = 5$. Prove the bound

$$|D_N(x)| \leq C \min\left(N, \frac{1}{|x|}\right) \quad (1.6)$$

for all $N \geq 1$ and some absolute constant C . Finally, prove the bound

$$C^{-1} \log N \leq \|D_N\|_{L^1(\mathbb{T})} \leq C \log N \quad (1.7)$$

for all $N \geq 2$ where C is another absolute constant.

Solution. We first verify that the given closed form formula for the Dirichlet Kernel D_n . Fix $x \in \mathbb{T}$ and $N \in \mathbb{N}$. From the sum formula for geometric series, and the Euler's identity $\sin(2\pi nx) = \frac{e(-nx) + e(nx)}{2i}$, it follows that

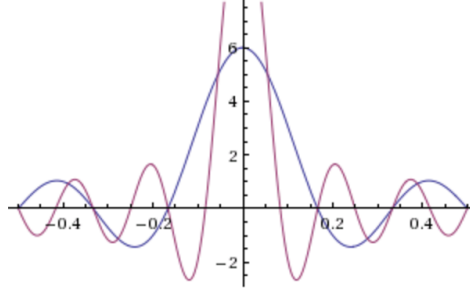
$$\begin{aligned} D_n(x) &= \sum_{n=-N}^N e(nx) = e(-Nx) \sum_{n=0}^{2N} e(nx) \\ &= e(-Nx) \frac{1 - e((2N+1)x)}{1 - e(x)} = \frac{e(-Nx) - e((N+1)x)}{1 - e(x)} \\ &= \frac{e(-(N+\frac{1}{2})x) - e((N+\frac{1}{2})x)}{e(-\frac{1}{2}x) - e(\frac{1}{2}x)} = \frac{\sin(2\pi(N+\frac{1}{2})x)}{\sin(2\pi(\frac{1}{2})x)} = \frac{\sin(2(N+1)\pi x)}{\sin(\pi x)}, \end{aligned}$$

as required.

The graphs of D_2 and D_5 are attached below. The blue graph corresponds to D_2 and the green corresponds to D_5 .

Figure 1: The graph of D_n for $n = 2, 5$

Plot:



We proceed to prove the given bound. Fix $x \in \mathbb{T}$ and $n \in \mathbb{Z}_+$. By the triangle inequality, we have

$$\begin{aligned} |D_n(x)| &= \left| \sum_{k=-N}^N e(kx) \right| \\ &\leq \sum_{k=-N}^N |e(kx)| = 2N + 1 \leq 3N. \end{aligned}$$

For $x \in (0, \frac{1}{2}]$, we have $2x \leq \sin(\pi x)$. Hence,

$$\begin{aligned} |D_n(x)| &= \left| \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \right| = \frac{|\sin((2N+1)\pi x)|}{|\sin(\pi x)|} \\ &\leq \frac{|\sin((2N+1)\pi x)|}{2|x|} \leq \frac{1}{2|x|}. \end{aligned}$$

Therefore, we obtain that

$$|D_N(x)| \leq 3 \min(N, \frac{1}{|x|}).$$

Now, using the monotonicity of Lebesgue integration and additivity over domain gives

$$\begin{aligned} \|D_N\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} |D_n(x)| dx \leq \int_{\mathbb{T}} 3 \min(N, \frac{1}{|x|}) dx \\ &= 6 \int_0^{\frac{1}{2}} \min(N, \frac{1}{|x|}) dx = 6 \left(\int_0^{\frac{1}{N}} N dx + \int_{\frac{1}{N}}^{\frac{1}{2}} \frac{1}{|x|} dx \right) \\ &= 6 + 6(\log(\frac{1}{2}) - \log(\frac{1}{N})) = 6 + 6 \log(\frac{1}{2}) + 6 \log(N) \leq C_1 \log(N), \end{aligned}$$

where a sufficiently large C_1 , that satisfies the last inequality when $N = 2$. Now, for the lower bound, using the fact that $\sin(\pi x) \leq \pi x$ for $x \in [0, \frac{1}{2}]$, we have

$$\begin{aligned} \|D_N\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} |D_n(x)| dx = 2 \int_0^{\frac{1}{2}} \left| \frac{\sin(2N+1)\pi x}{\sin(\pi x)} \right| dx \\ &\geq \frac{2}{\pi} \int_0^{\frac{1}{2}} \frac{|\sin(2N+1)\pi x|}{x} dx. \end{aligned}$$

Now, using change of variable with $x = (2N + 1)\pi t$, we can continue the computation as follows:

$$\begin{aligned}
\|D_N\|_{L^1(\mathbb{T})} &\geq C \int_0^{(N+\frac{1}{2})\pi} \frac{|\sin(t)|}{t} dt. \geq C \int_0^{N\pi} \frac{|\sin(t)|}{t} dt. \\
&= C \sum_{i=1}^N \int_{(i-1)\pi}^{i\pi} \frac{|\sin(t)|}{t} dt. \geq C \sum_{i=1}^N \frac{1}{(i+1)\pi} \int_{(i-1)\pi}^{i\pi} |\sin(t)| dt. \\
&= C \sum_{i=1}^N \frac{1}{(i+1)\pi} \cdot \frac{\pi}{2} \geq C' \sum_{i=1}^N \frac{1}{i} \geq C'' \log(N),
\end{aligned}$$

as $\sum_{i=1}^N \frac{1}{i} \geq c \log(N)$ for some c for $N \geq 2$. Choosing the maximum C from the upper and the lower bound, we have shown the desired bound on the L_1 norm of D_n . \square

Question 2.

EXERCISE 1.2. Let $\mu \in \mathcal{M}(\mathbb{T})$ have the property that

$$\sum_{n \in \mathbb{Z}} |\hat{\mu}(n)| < \infty \quad (1.11)$$

Show that $\mu(dx) = f(x) dx$ where $f \in C(\mathbb{T})$. Denote the space of all measures with this property by $\mathbb{A}(\mathbb{T})$ and identify these measure with their respective densities. Show that $\mathbb{A}(\mathbb{T})$ is an algebra under multiplication, and that

$$\widehat{fg}(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m) \hat{g}(n-m) \quad \forall n \in \mathbb{Z}$$

where the sum on the right-hand side is absolutely convergent for every $n \in \mathbb{Z}$, and itself is absolutely convergent over all n . Moreover, $\|f * g\|_{\mathbb{A}} \leq \|f\|_{\mathbb{A}} \|g\|_{\mathbb{A}}$ where $\|f\|_{\mathbb{A}} := \|\hat{f}\|_{\ell^1}$. Finally, verify that if $f, g \in L^2(\mathbb{T})$, then $f * g \in \mathbb{A}(\mathbb{T})$.

Solution.

Let $u \in \mathbb{M}(\mathbb{T})$ such that $\sum_{n \in \mathbb{Z}} |\hat{u}(n)| < \infty$. By the Lebesgue-Radon-Nikodym theorem (Rudin pg.121), there exists $f \in L_1(\mathbb{T})$ such that $u(dx) = f(x)dx$, where dx is the Lebesgue measure, restricted to Borel sets of \mathbb{T} . Let f be such function in $L_1(\mathbb{T})$. As $u(dx) = f(x)dx$, it follows that $\hat{u}(n) = \hat{f}(n)$, thus $\sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$. Recall that a uniform limit of continuous functions is continuous. As we have that $\{S_n f\}$ is a sequence of continuous functions, and that the tail ($|n| > M$) terms can be arbitrarily bounded by a sufficiently large M by the assumption, we have that $\{S_n f\}$ is Cauchy in $C(\mathbb{T})$. By completeness of $C(\mathbb{T})$ $\{S_n f\}$ converges and by problem 1.1, know have that $\{S_n f\}$ converges uniformly to f . Therefore, $f \in C(\mathbb{T})$.

Let $f, g \in \mathbb{A}(\mathbb{T})$. By linearity of integration and the triangle inequality, it follows that

$$\sum_{n \in \mathbb{Z}} |\widehat{f+g}(n)| \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)| + \sum_{n \in \mathbb{Z}} |\hat{g}(n)| < \infty.$$

Therefore, $f + g \in \mathbb{A}(\mathbb{T})$. Let $\alpha \in \mathbb{C}$ and $f \in \mathbb{A}(\mathbb{T})$. It follows that

$$\sum_{n \in \mathbb{Z}} |\alpha \hat{f}(n)| \leq |\alpha| \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty.$$

Therefore, $\alpha f \in \mathbb{A}(\mathbb{T})$. So far, we have shown that $\mathbb{A}(\mathbb{T})$ is a linear space.

Let $f, g \in A(\mathbb{T})$. It follows that

$$\begin{aligned} fg &= \left(\sum_{i \in \mathbb{Z}} \hat{f}(i) e(ix) \right) \left(\sum_{k \in \mathbb{Z}} \hat{g}(k) e(kx) \right) = \sum_{n \in \mathbb{Z}} \sum_{i+k=n} \hat{f}(i) \hat{g}(k) e(nx) \\ &= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \hat{f}(m) \hat{g}(n-m) e(nx) \end{aligned}$$

From the above equality and uniform convergence, we can further deduce that

$$\hat{f}g(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m) \hat{g}(n-m),$$

and consequently

$$\sum_{n \in \mathbb{Z}} |\hat{f}g(n)| \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)| \sum_{m \in \mathbb{Z}} |\hat{g}(m)| < \infty.$$

Therefore, $fg \in A(\mathbb{T})$. This shows that $A(\mathbb{T})$ is an algebra under multiplication. For the remaining part, it follows that

$$\|fg\|_A = \|\hat{f}g\|_{l_1} \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)| \sum_{m \in \mathbb{Z}} |\hat{g}(m)| = \|f\|_A \|g\|_A$$

If $f, g \in L^2(\mathbb{T})$, we have that $\sum |\hat{f}(n)|^2, \sum |\hat{g}(n)|^2 < \infty$. Hence, by the established inequality, it follows that $f * g \in A(\mathbb{T})$.

Question 3.

EXERCISE 1.3. Let K_N be the Fejér kernel with N a positive integer.

- Verify that \hat{K}_N looks like a triangle, i.e., for all $n \in \mathbb{Z}$

$$\hat{K}_N(n) = \left(1 - \frac{|n|}{N}\right)^+ \quad (1.16)$$

- Show that

$$K_N(x) = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2 \quad (1.17)$$

- Conclude that

$$0 \leq K_N(x) \leq C N^{-1} \min(N^2, x^{-2}) \quad (1.18)$$

Solution. Let K_N be the Fejér kernel with the positive integer n .

(1.16) From definition of n th Fourier coefficient, we obtain

$$\begin{aligned} \hat{K}_N(n) &= \int_{\mathbb{T}} K_N(x) e(-nx) dx = \int_{\mathbb{T}} \left(\frac{1}{N} \sum_{k=0}^{N-1} D_k(x) \right) e(-nx) dx \\ &= \int_{\mathbb{T}} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{|m| \leq k} e(mx) e(-nx) dx = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{|m| \leq k} \int_{\mathbb{T}} e((m-n)x) dx. \end{aligned}$$

Observe that the integral in the summation is 1 if $m = n$ and 0 otherwise. For $n \leq N$, we have $(N - |n|)$ terms in the sum where $m = n$ happens, for $n > N$, we have no such term, where the equality holds. Therefore, it follows that

$$\hat{K}_N(n) = \frac{1}{N} (N - |n|)^+ = \left(1 - \frac{|n|}{N}\right)^+,$$

which is precisely the given closed form formula for the kernel. \square

(1.17) Fix $x \in \mathbb{T}$, and $N \in \mathbb{N}$. Now, by definition of Fejer Kernel, we have

$$\begin{aligned} K_N(x) &= \frac{1}{N} \sum_{n=0}^{N-1} D_n(x) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{\sin((2n+1)\pi x)}{\sin(\pi x)} \\ &= \frac{1}{N \sin(\pi x)^2} \sum_{n=0}^{N-1} \sin((2n+1)\pi x) \sin(\pi x). \end{aligned}$$

By the use of the trig identity, $\sin(a)\sin(b) = \frac{1}{2} \cos(a-b) - \cos(a+b)$, and cancellation from a telescoping sum, it follows that

$$\begin{aligned} K_N(x) &= \frac{1}{2N \sin(\pi x)^2} \sum_{n=0}^{N-1} (\cos(2n\pi x) - \cos((2n+2)\pi x)) \\ &= \frac{1}{2N \sin(\pi x)^2} (1 - \cos(2N\pi x)) \end{aligned}$$

Lastly, from the trig identity, $2 \sin(a)^2 = 1 - \cos(2a)$, we finally obtain that

$$K_N(x) = \frac{1}{2N \sin(\pi x)^2} (2 \sin(N\pi x)^2) = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2,$$

as required.

(1.18) Fix $x \in \mathbb{T}$, and $N \in \mathbb{N}$. From (1.17), it is clear that $0 \leq K_n(x)$. By the triangle inequality, and the result from the exercise 1.1, it follows that

$$K_N(x) \leq \frac{1}{N} \sum_{n=0}^{N-1} |D_n(x)| \leq \frac{1}{N} \sum_{n=0}^{N-1} 2n+1 = \frac{1}{N} (2 \frac{N(N-1)}{2} + N) = N$$

Now, by the (1.17) result, and the fact that $|\sin(x)| \leq 2|x|$ for $x \in [0, \frac{1}{2}]$, we obtain

$$K_N(x) = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2 \leq \frac{1}{N} \left(\frac{\sin(N\pi x)}{2x} \right)^2 \leq \frac{1}{N} \cdot \frac{1}{4x^2} \leq \frac{1}{Nx^2}.$$

Hence, we have shown that

$$0 \leq K_n(x) \leq N^{-1} \min(N^2, \frac{1}{x^2}),$$

as required. \square

Question 4.

Solution.

(a) Let $0 \leq s \leq 1$, and $h = \tau_{-\theta} f$, where $\tau_{\theta} f(x) = f(x - \theta)$ is the translation operator, parametrized by θ . By the Corollary 1.6, and the linearity of integration, it follows that

$$\begin{aligned} \|h - f\|_2^2 &= \sum_{n=-\infty}^{\infty} |(\widehat{h - f})(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} |\hat{h}(n) - \hat{f}(n)|^2. \end{aligned}$$

Exercise 1.6 For any $s \in \mathbb{R}$ define the Hilbert space $H^s(\mathbb{T})$ by means of the norm

$$\|f\|_{H^s}^2 := |\hat{f}(0)|^2 + \sum_{n \in \mathbb{Z}} |n|^{2s} |\hat{f}(n)|^2. \quad (1.21)$$

Obtain the following quantitative improvements in certain qualitative convergence properties.

- (a) Show that for any $0 \leq s \leq 1$ one has $\|f(\cdot + \theta) - f\|_2 \leq 2\pi \|f\|_{H^s} |\theta|^s$.
- (b) Derive a rate of convergence for $\|S_N f - f\|_2$ in terms of N alone, assuming that $\|f\|_{H^s} \leq 1$ where $s > 0$ is fixed.

Now, we have a particular relation between the Fourier transform and translation as follows (pg. 4 in Schleg):

$$\widehat{\tau_{-\theta} f}(n) = e(n\theta) \hat{f}(n).$$

Hence, it follows that

$$\begin{aligned} \|h - f\|_2^2 &= \sum_{-\infty}^{\infty} |e(n\theta) \hat{f}(n) - \hat{f}(n)|^2 \\ &= \sum_{-\infty}^{\infty} |e(n\theta) - 1|^2 |\hat{f}(n)|^2 \end{aligned}$$

Recall that $\theta \in [0, 1)$ and $0 \leq s \leq 1$. For $|n\theta| \geq 1$, we have

$$|e(n\theta) - 1|^2 \leq 4 \leq 4\pi^2 |n|^{2s} |\theta|^{2s}.$$

Now, when $|n\theta| < 1$, by rudimentary trig identities, we obtain

$$\begin{aligned} |e(n\theta) - 1|^2 &\leq (\cos(2\pi n\theta)^2 - 1)^2 + \sin(2\pi n\theta)^2 \\ &= 2 - 2\cos(2\pi n\theta) = 4\sin^2(\pi n\theta) \\ &\leq 4\pi^2 n^2 |\theta|^2 \leq 4\pi^2 |n|^{2s} |\theta|^{2s}. \end{aligned}$$

Hence, we have shown $|e(n\theta) - 1|^2 \leq 4\pi^2 |n|^{2s} |\theta|^{2s}$. Plugging the above inequality into the two norm inequality above gives

$$\begin{aligned} \|h - f\|_2^2 &\leq 4\pi^2 \left(\sum_{-\infty}^{\infty} |n|^{2s} |\hat{f}(n)|^2 \right) |\theta|^{2s} \\ &\leq 4\pi^2 \|f\|_{H^s(\mathbb{T})}^2 |\theta|^{2s}, \end{aligned}$$

and consequently,

$$\|h - f\|_2 \leq 2\pi \|f\|_{H^s(\mathbb{T})} |\theta|^s,$$

as required.

(b) Fix $s > 0$. By definition of $S_N f$, and the given, we have

$$\|S_N f - f\|_2^2 = \sum_{|n| > N} |\hat{f}(n)|^2 \quad \text{and} \quad \|f\|_{L^2} \leq \|f\|_{H^s} \leq 1$$

With the given norm, it follows that

$$\begin{aligned} 1 &\geq \|f\|_{H^s} \\ &\geq \sum_{|n|>N} (N+1)^{2s} |\hat{f}(n)|^2 = (N+1)^{2s} \|S_N f - f\|_2^2, \end{aligned}$$

which can be simplified to

$$\|S_N f - f\|_2 \leq \frac{1}{(N+1)^{2s}},$$

which reveals the rate of convergence as required. \square

Question 5.

PROBLEM 1.1. Suppose that $f \in L^1(\mathbb{T})$ and that $\{S_n f\}_{n=1}^\infty$ (the sequence of partial sums of the Fourier series) converges in $L^p(\mathbb{T})$ to g for some $p \in [1, \infty]$ and some $g \in L^p$. Prove that $f = g$. If $p = \infty$ conclude that f is continuous.

Solution. Let $p \geq 1$. Observe that for $x \in [0, 1)$, we have $|x| \leq |x|^p$. Hence, it follows that

$$0 \leq \int_{\mathbb{T}} |S_n f - g| \leq \int_{\mathbb{T}} |S_n f - g|^p.$$

As we are given that $\int_{\mathbb{T}} |S_n f - g|^p \rightarrow 0$ as $n \rightarrow \infty$ it follows that $\int_{\mathbb{T}} |S_n f - g| \rightarrow 0$ in $L^1(\mathbb{T})$. Now, from the triangle inequality of L_1 , we have that

$$|f - g|_{L_1} \leq |f - \sigma_n f|_{L_1} + |\sigma_n f - g|_{L_1},$$

for all n . Recall that convergence of cesaro sum is more inclusive, thus implied by the convergence of the original sequence. Hence, as $|S_n f - g|_{L_1} \rightarrow 0$ as $n \rightarrow \infty$, we have $|\sigma_n f - g|_{L_1} \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, since the Fejer kernel $\{K_n\}$ is an approximate identity, we have that $|\sigma_n f - f|_{L_1} \rightarrow 0$ as $n \rightarrow \infty$. Hence, it follows that $|f - g|_{L_1} = 0$, which implies that $f = g$ almost everywhere. For the case when $p = \infty$, as we have $\{S_n f\}$ is a sequence of continuous function, by the convergence in supnorm, we have g is continuous. \square

Question 6.

Problem 1.9 Show that

$$\|f * g\|_{L^2(\mathbb{T})}^2 \leq \|f * f\|_{L^2(\mathbb{T})} \|g * g\|_{L^2(\mathbb{T})}$$

for all $f, g \in L^2(\mathbb{T})$.

Solution. As we have $f, g \in L^2(\mathbb{T})$, by Corollary 1.6, the given inequality is equivalent to

$$\sum_{n \in \mathbb{Z}} |\widehat{f * g}(n)|^2 \leq \sqrt{\sum_{n \in \mathbb{Z}} |\widehat{f * f}(n)|^2} \sqrt{\sum_{n \in \mathbb{Z}} |\widehat{g * g}(n)|^2}.$$

Since $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$, the above inequality is again equivalent to

$$\left(\sum_{n \in \mathbb{Z}} |\hat{f}(n)\hat{g}(n)|^2 \right)^2 \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^4 \sum_{n \in \mathbb{Z}} |\hat{g}(n)|^4.$$

Expanding the LHS of the desired inequality yields

$$\begin{aligned}
\left(\sum_{n \in \mathbb{Z}} |\hat{f}(n)\hat{g}(n)|^2\right)^2 &= \sum_{n \in \mathbb{Z}} |\hat{f}(n)\hat{g}(n)|^4 + \sum_{n > m} 2|\hat{f}(n)\hat{f}(m)\hat{g}(n)\hat{g}(m)|^2 \\
&\leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)\hat{g}(n)|^4 + \sum_{n > m} |\hat{f}(n)\hat{g}(n)|^4 + |\hat{f}(m)\hat{g}(m)|^4 \\
&= \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^4 \sum_{n \in \mathbb{Z}} |\hat{g}(n)|^4,
\end{aligned}$$

where the last inequality holds by the Cauchy-Schwarz inequality on the inner product space of $l^2(\mathbb{T})$.

□

Question Extra.

EXERCISE 1.4. Let $\{c_n\}_{n \in \mathbb{Z}}$ be an arbitrary sequence of complex numbers and associate to it formally a Fourier series

$$f(x) \sim \sum_n c_n e(nx)$$

Show that there exists $\mu \in \mathcal{M}(\mathbb{T})$ with the property that $\hat{\mu}(n) = c_n$ for all $n \in \mathbb{Z}$ if and only if $\{\sigma_n f\}_{n \geq 1}$ is bounded in $\mathcal{M}(\mathbb{T})$. Discuss the case of $L^p(\mathbb{T})$ with $1 \leq p < \infty$ and $C(\mathbb{T})$ as well.

Solution.