
Harmonic Analysis: Problem Set II

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Abstract

This work contains solutions to the problem set II of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Solve Exercise 1.8 in Muscalu & Schlag.

Solution.

(a) Assume $f \in C^\infty$. We have that for any $f \in C^1(\mathbb{T})$, we have $\hat{f}'(n) = 2\pi i n \hat{f}(n)$ for all $n \in \mathbb{Z}$, which gives $\hat{f}(n) = O(|n|^{-1})$. Using the same argument inductively, we obtain that $\hat{f}(n) = O(|n|^{-M})$ for any $M \geq 1$. Now, conversely, assume \hat{f} decays rapidly. Since $\hat{f}(n) = O(|n|^{-2})$, we have that $f \in A(\mathbb{T})$, thus $f \in C(\mathbb{T})$. Suppose that $f \in C^1(\mathbb{T})$ for $n \geq 1$. Observe that

$$(S_N f)' = \sum_{|n| \leq N} 2\pi i n \hat{f}(n) e(n),$$

and

$$\left| \sum_{N \geq |n| > M} 2\pi i n \hat{f}(n) e(n) \right| \leq \sum_{N \geq |n| > M} 2\pi |n| |\hat{f}(n)|,$$

for any $M < N$. As $\hat{f}(n) = O(|n|^{-3})$, it follows that $\{(S_N f)'\}$ converges uniformly and we obtain that $\lim_{N \rightarrow \infty} (S_N f)' = f'$, as uniform convergence allows us to commute the differential operator and the limit. Furthermore, f' is continuous by uniform convergence. Hence, we have that $f \in C^2(\mathbb{T})$. By induction, the argument is complete, and we have that $f \in C^\infty(\mathbb{T})$.

(b) Assume that F is analytic on some neighborhood of $\{|z| = 1\}$. By analyticity, we can apply the Laurent's theorem on the annulus, and obtain that the coefficients of the series is in fact the fourier coefficients. Take $r > 1$ such that it still lies in the neighborhood of $\{|z| = 1\}$, we see that

$$F(r) = \sum_{n \in \mathbb{Z}} \hat{f}(n) r^n,$$

converges absolutely. By the absolute convergence, we have that F decays exponentially. Now, the converse is obvious, as the Fourier series will give emit the convergent power series with respect to some ϵ neighborhood around $\{|z| = 1\}$, which is the definition of analyticity. \square

Question 2.

2. The following (non-absolutely convergent) series define functions in $H^{\frac{1}{2}}(\mathbb{T})$. (Why?)

$$f_S(x) := \sum_{n=2}^{\infty} \frac{\sin(2\pi nx)}{n \log n}, \quad f_C(x) := \sum_{n=2}^{\infty} \frac{\cos(2\pi nx)}{n \log n}$$

Show that the first series converges uniformly (hence $f_S \in C(\mathbb{T})$), but the second does not. In fact, show that $f_C(x) \geq c \log \log \frac{1}{|x|}$ as $x \rightarrow 0$ so that f_C is not even essentially bounded. (Hint: Summation by parts.)

Remark: For an example of a $C^{1/2}(\mathbb{T})$ function which is not in $A(\mathbb{T})$, see Proposition 1.14 in Muscalu & Schlag. (There is also another example, due to Hardy-Littlewood:

$$\sum_{n=1}^{\infty} \frac{e^{in \log n}}{n} e^{2\pi i n x},$$

Proof of this is given in Zygmund's "Trigonometric Series", vol. 1, p.197.)

Solution. We first show that the two functions f_s and f_c are in $H^{\frac{1}{2}}(\mathbb{T})$. By definition of fourier coefficients, we have

$$\hat{f}_s(n) = \begin{cases} 0 & \text{if } |n| < 2 \\ \frac{1}{2in \log(|n|)} & \text{otherwise} \end{cases},$$

and

$$\hat{f}_c(n) = \begin{cases} 0 & \text{if } |n| < 2 \\ \frac{1}{2|n| \log(|n|)} & \text{otherwise} \end{cases}.$$

By the comparison test, we obtain

$$\begin{aligned} \|f_s\|_{H^{\frac{1}{2}}(\mathbb{T})} &= |\hat{f}(0)| + \sum_{n \in \mathbb{Z}} |n| |\hat{f}_s(n)|^2 \\ &= \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{n \log^2(n)} \leq \frac{1}{2} \int_2^{\infty} \frac{1}{x \log^2(x)} dx = \frac{1}{\log(2)} < \infty. \end{aligned}$$

Similarly, we have $\|f_c\|_{H^{\frac{1}{2}}(\mathbb{T})} < \infty$. Therefore, we have shown that $f_s, f_c \in H^{\frac{1}{2}}(\mathbb{T})$.

By the odd symmetry, it suffices to consider the interval $[0, 1/2]$. We will deduce the uniform convergence by providing estimates for three separate intervals: $[0, \frac{1}{m}]$, $[\frac{1}{m}, \frac{1}{n}]$, $[\frac{1}{n}, \frac{1}{2}]$ for fixed n, m such that $n > m$. Fix n, m such that $n > m$ and define $g_{n,m} = \sum_{i=n}^m \frac{\sin(2\pi nx)}{n \log(n)}$. For $x \in [0, \frac{1}{m}]$, we have

$$\begin{aligned} |g_{n,m}| &\leq \sum_{i=n}^m \left| \frac{\sin(2\pi ix)}{i \log(i)} \right| \leq \sum_{i=n}^m \frac{2\pi ix}{i \log(i)} = \sum_{i=n}^m \frac{2\pi x}{\log(i)} \\ &\leq \frac{1}{\log(n)} \leq \frac{1}{m \log(n)} \sum_{i=n}^m 2\pi \leq \frac{2\pi}{\log(n)}. \end{aligned}$$

For $x \in [\frac{1}{m}, \frac{1}{n}]$, we obtain

If $x \in [\frac{1}{m}, \frac{1}{n}]$, it follows that $S_{n,m} = S_{n, \lceil \frac{1}{x} \rceil} + S_{\lceil \frac{1}{x} \rceil + 1, m}$. Hence, the addition of the two bounds establishes before gives the bound on this interval as well. Therefore, we have that

$$|g_{n,m}| = O\left(\frac{1}{\log(n)}\right),$$

and the partial sums of f_S is cauchy. Thus, f_S converges uniformly and $f \in C(\mathbb{T})$.

We now show that $f_C(x) \geq c \log \log(\frac{1}{|x|})$ as $x \rightarrow 0$.

Question 3.

3. (Problem 1.5 in Muscalu & Schlag) Suppose $f \in H^{\frac{1}{2}}(\mathbb{T}) \cap C(\mathbb{T})$. Show that $S_N f \rightarrow f$ uniformly.
(Hint: Study $S_N f - \sigma_N f$.)

Solution. By the triangle inequality of the supnorm, we have

$$\|S_N f - f\|_{\infty} \leq \|S_N f - \sigma_N f\|_{\infty} + \|\sigma_N f - f\|_{\infty},$$

for all $N \in \mathbb{Z}^+$. As $f \in C(\mathbb{T})$, we have that $\|\sigma_N f - f\|_{\infty} \rightarrow 0$ as $N \rightarrow \infty$. Therefore, it suffices to show that $\|S_N f - \sigma_N f\|_{\infty} \rightarrow 0$ as $N \rightarrow \infty$. By definition of S_N and σ_N , triangle inequality, and Cauchy-Schwarz, we obtain

$$\begin{aligned} \|S_N f - \sigma_N f\|_{\infty} &= \left\| \sum_{n=-N}^N \frac{|n|}{N} \hat{f}(n) \right\|_{\infty} \\ &\leq \sum_{n=-M}^M \frac{|n| |\hat{f}(n)|}{N} + \left(\sum_{N \geq |n| > M} \frac{|n|}{N^2} \right)^{\frac{1}{2}} \left(\sum_{N \geq |n| > M} |n| |\hat{f}(n)|^2 \right)^{\frac{1}{2}}, \\ &\leq \sum_{n=-M}^M \frac{|n| |\hat{f}(n)|}{N} + 2 \left(\sum_{N \geq |n| > M} |n| |\hat{f}(n)|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

for any $N > M$. Taking lim sup with respect to N on both sides, we get

$$\limsup_{N \rightarrow \infty} \|S_N f - \sigma_N f\|_{\infty} \leq 2 \left(\sum_{|n| > M} |n| |\hat{f}(n)|^2 \right)^{\frac{1}{2}},$$

for any M . As $f \in H^{\frac{1}{2}}(\mathbb{T})$, taking the limit as $M \rightarrow \infty$ gives

$$\limsup_{N \rightarrow \infty} \|S_N f - \sigma_N f\|_{\infty} \leq 0$$

Hence, we have shown that $\|S_N f - \sigma_N f\|_{\infty} \rightarrow 0$ as $N \rightarrow \infty$ as desired. \square

Question 4.

4. Let $0 < \alpha < 1$. Note by a theorem we have seen in class (which one?) that $f \in C^\alpha(\mathbb{T})$ implies $\hat{f}(n) = O(|n|^{-\alpha})$. Then, note that the exponent in this decay estimate cannot be improved by showing that the function

$$F(x) = \sum_{m=1}^{\infty} \frac{1}{3^{m\alpha}} \cos(2\pi 3^m x)$$

belongs to $C^\alpha(\mathbb{T})$. Also solve Exercise 1.9 in Muscalu & Schlag.

Solution.

A theorem that gives this result of $f \in C^\alpha(\mathbb{T}) \implies \hat{f}(n) = O(n^{-\alpha})$ is recorded in section 1.4.4, pg.18 of Schleg.

Now, we show that the exponent in the decay estimate cannot be improved. We first show that $F \in C^\alpha(\mathbb{T})$. Fix $x, y \in \mathbb{T}$, such that $x \neq y$. Choose $K \in \mathbb{N}$ such that $3^{-K-1} < |x - y| \leq 3^{-K}$. In particular, observe that, with this choice of K , we have $1 < 3^{K+1}|x - y| < 3$. It follows that

$$\begin{aligned} \frac{|F(x) - F(y)|}{|x - y|^\alpha} &\leq \sum_{m=1}^K \frac{|\cos(2\pi 3^m x) - \cos(2\pi 3^m y)|}{3^{m\alpha} |x - y|^\alpha} + \sum_{m=K+1}^{\infty} \frac{|\cos(2\pi 3^m x) - \cos(2\pi 3^m y)|}{3^{m\alpha} |x - y|^\alpha} \\ &\leq \sum_{m=1}^K \frac{2\pi 3^m |x - y|}{3^{m\alpha} |x - y|^\alpha} + \sum_{m=K+1}^{\infty} \frac{2}{3^{m\alpha} |x - y|^\alpha} \\ &\leq 2\pi \sum_{m=1}^K (3^{m-K})^{1-\alpha} + 2 \frac{1}{1 - 3^{-\alpha}} \\ &= 2\pi (3^{1-K})^{1-\alpha} \frac{3^{K(1-\alpha)} - 1}{3^{1-\alpha} - 1} + \frac{2}{1 - 3^{-\alpha}} \\ &\leq 2\pi \frac{3^{1-\alpha}}{3^{1-\alpha} - 1} + \frac{2}{1 - 3^{-\alpha}}. \end{aligned}$$

Since the bound on LHS is a constant, we have shown that $f \in C^\alpha(\mathbb{T})$.

By definition of Fourier coefficients, it follows that

$$\hat{F}(\pm 3^m) = \frac{1}{2 \cdot 3^{m\alpha}}.$$

Therefore, for any $\beta > \alpha$, we have that $\hat{F}(\pm 3^m) 3^{m\beta} = \frac{3^{m(\beta-\alpha)}}{2} \rightarrow \infty$ as $M \rightarrow \infty$. Hence, it follows that $\hat{f}(m) \neq O(|n|^{-\beta})$ for $\beta > \alpha$, and the decay estimate cannot be improved.

Now, for the exercise 1.9, the above argument also yields that the given lacunary series is in $C^\alpha(\mathbb{T})$. Computing the Sobolev norm of the lacunary series with respect to β , we obtain

$$\begin{aligned} \|f\|_{H^\beta(\mathbb{T})}^2 &= \sum_{k \in \mathbb{N}} |2^k|^{2\beta} |2^{-\alpha k}|^2 \\ &= \sum_{k \in \mathbb{N}} 2^{2k(\beta-\alpha)} = \infty, \end{aligned}$$

as $\beta - \alpha > 0$. Hence, the given series shows that $C^\alpha(\mathbb{T})$ does not embed into $H^\beta(\mathbb{T})$ for any $\beta > \alpha$ as required. □

Question 5.

5. Draw a minimal Venn diagram that shows all possible intersections of the sets below:

$$C(\mathbb{T}), A(\mathbb{T}), C^{2/3}(\mathbb{T}), H^{1/2}(\mathbb{T}), U(\mathbb{T}) := \{f : S_N f \rightarrow f \text{ uniformly}\}.$$

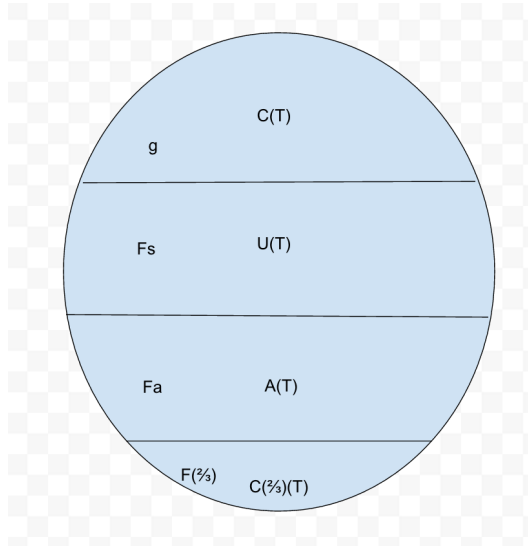
Your diagram should not have any redundancy or ambiguity, i.e., if $A \cap B = \emptyset$, $A \subset B$, or $A \neq B$, this should be visible and indicated. Give an example (or show the existence) of a function in each region of intersection.

Solution. Since $S_n f$ is continuous, and uniform limit of a continuous function is continuous, we have $U(\mathbb{T}) \subset C(\mathbb{T})$. We have previously shown that if $f \in A(\mathbb{T})$, then $\{S_n f\}$ converges uniformly. This gives $A(\mathbb{T}) \subset U(\mathbb{T})$. As $\frac{2}{3} > \frac{1}{2}$, the theorem 1.13 from Schlag gives $C^{\frac{2}{3}}(\mathbb{T}) \subset H^{\frac{1}{2}}(\mathbb{T})$.

Now, from corollary 1.10 from Schlag, gives a function $g \in C(\mathbb{T})$ such that $g \notin U(\mathbb{T})$. Hence, $C(\mathbb{T}) \setminus U(\mathbb{T}) \neq \emptyset$

In problem 2, we have shown that $f_s \notin A(\mathbb{T})$, but $f_s \in U(\mathbb{T})$. Therefore, $U(\mathbb{T}) \setminus A(\mathbb{T}) \neq \emptyset$. Now, take any $\frac{2}{3} > \alpha > \frac{1}{2}$, and consider F_α from the problem 4, parametrized by α . It follows that $F_\alpha \in A(\mathbb{T})$, and $F_\alpha \notin C^{\frac{2}{3}}(\mathbb{T})$. Therefore, we have $A(\mathbb{T}) \setminus C^{\frac{2}{3}}(\mathbb{T}) \neq \emptyset$, and $F_{\frac{2}{3}} \in C^{\frac{2}{3}}(\mathbb{T})$. Recapping the information we have gathered so far gives the following figure:

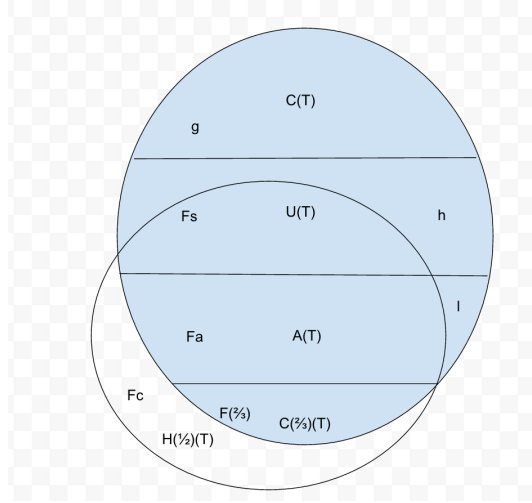
Figure 1: Function spaces on \mathbb{T}



Now, recall that f_s from the problem 2 is in $H^{\frac{1}{2}}(\mathbb{T})$, but not even essentially bounded. Hence, $H^{\frac{1}{2}}(\mathbb{T}) \setminus C^{\frac{2}{3}}(\mathbb{T}) \neq \emptyset$. Now, by the problem 3, we have that $H^{\frac{1}{2}}(\mathbb{T}) \cap C(\mathbb{T}) \subset U(\mathbb{T})$, and $f_s \in H^{\frac{1}{2}}(\mathbb{T}) \cap U(\mathbb{T})$. Recall that $F_\alpha \in H^{\frac{1}{2}}(\mathbb{T})$. Hence, by preposition 1.14 from there exists a function h on \mathbb{T} such that

$h \in U(\mathbb{T}) \setminus A(\mathbb{T})$ and $l \in A(\mathbb{T}) \setminus C^{\frac{2}{3}}(\mathbb{T})$. Incorporating the additional information gives the following figure:

Figure 2: Function spaces on \mathbb{T}



This gives the adequate description of the function spaces on \mathbb{T} for our interests.

□