Harmonic Analysis:
Problem Set II

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Abstract

This work contains solutions to the problem set II of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Solve Exercise 1.8 in Muscalu & Schlag.

Solution.
(a) Assume $f \in C^\infty$. We have that for any $f \in C^1(\mathbb{T})$, we have $\hat{f}'(n) = 2\pi in\hat{f}(n)$ for all $n \in \mathbb{Z}$, which gives $\hat{f}(n) = O(|n|^{-1})$. Using the same argument inductively, we obtain that $f = O(|n|^{-M})$ for any $M \geq 1$. Now, conversely, assume $\hat{f}$ decays rapidly. Since $\hat{f}(n) = O(|n|^{-2})$, we have that $f \in A(\mathbb{T})$, thus $f \in C(\mathbb{T})$. Suppose that $f \in C^1(\mathbb{T})$ for $n \geq 1$. Observe that

$$(S_N f)^{'} = \sum_{|n| \leq N} 2\pi in\hat{f}(n)e(n),$$

and

$$\left| \sum_{N \geq |n| > M} 2\pi in\hat{f}(n)e(n) \right| \leq \sum_{N \geq |n| > M} 2\pi|n||\hat{f}(n)|,$$

for any $M < N$. As $f = O(|n|^{-3})$, it follows that $\{(S_N f)'\}$ converges uniformly and we obtain that $\lim_{N\to\infty}(S_N f)' = f'$, as uniform convergence allows us to commute the differential operator and the limit. Furthermore, f' is continuous by uniform convergence. Hence, we have that $f \in C^2(\mathbb{T})$. By induction, the argument is complete, and we have that $f \in C^\infty(\mathbb{T})$.

(b) Assume that F is analytic on some neighborhood of $\{|z| = 1\}$. By analyticity, we can apply the Laurent's theorem on the annulus, and obtain a Laurent series, whose coefficients are in fact the fourier coefficients. Take $r > 1$ such that it still lies in the neighborhood of $\{|z| = 1\}$, we see that

$$F(r) = \sum_{n \in \mathbb{Z}} \hat{f}(n)r^n,$$

converges absolutely. By the absolute convergence, we have that F decays exponentially. Now, the converse is obvious, as the Fourier series will emit the convergent power series with respect to some ϵ neighborhood around $\{|z| = 1\}$, which is the definition of analyticity. $\square$

Question 2.

2. The following (non-absolutely convergent) series define functions in $H^{\frac{1}{2}}(\mathbb{T})$. (Why?)

$$f_S(x) := \sum_{n=2}^{\infty} \frac{\sin(2\pi nx)}{n \log n}, \qquad f_C(x) := \sum_{n=2}^{\infty} \frac{\cos(2\pi nx)}{n \log n}$$

Show that the first series converges uniformly (hence $f_S \in C(\mathbb{T})$), but the second does not. In fact, show that $f_C(x) \geq c \log\log \frac{1}{|x|}$ as $x \to 0$ so that f_C is not even essentially bounded.
(Hint: Summation by parts.)

Remark: For an example of a $C^{1/2}(\mathbb{T})$ function which is not in $A(\mathbb{T})$, see Proposition 1.14 in Muscalu & Schlag. (There is also another example, due to Hardy-Littlewood:

$$\sum_{n=1}^{\infty} \frac{e^{in \log n}}{n} e^{2\pi inx},$$

Proof of this is given in Zygmund's "Trigonometric Series", vol. 1, p.197.)

Solution. We first show that the two functions f_s and f_c are in $H^{\frac{1}{2}}(\mathbb{T})$. By definition of fourier coefficients, we have

$$\hat{f}_s(n) = \begin{cases} 0 & \text{if } |n| < 2 \\ \dfrac{1}{2in\log(|n|)} & \text{otherwise} \end{cases},$$

and

$$\hat{f}_c(n) = \begin{cases} 0 & \text{if } |n| < 2 \\ \dfrac{1}{2|n|\log(|n|)} & \text{otherwise} \end{cases}.$$

By the comparison test, we obtain

$$\begin{aligned} ||f_s||_{H^{\frac{1}{2}}(\mathbb{T})} &= |\hat{f}(0)| + \sum_{n\in\mathbb{Z}} |n||\hat{f}_s(n)|^2 \\ &= \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{n\log^2(n)} \leq \frac{1}{2}\int_2^{\infty} \frac{1}{x\log^2(x)}dx = \frac{1}{\log(2)} < \infty. \end{aligned}$$

Similarly, we have $||f_c||_{H^{\frac{1}{2}}(\mathbb{T})} < \infty$. Therefore, we have shown that $f_s, f_c \in H^{\frac{1}{2}}(\mathbb{T})$.

By the odd symmetry, it suffices to consider the interval $[0, 1/2]$. We will deduce the uniform convergence by providing estimates for three separate intervals: $[0, \frac{1}{m}], [\frac{1}{m}, \frac{1}{n}], [\frac{1}{n}, \frac{1}{2}]$ for fixed n, m such that $n > m$. Fix n, m such that $n > m$ and define $g_{n,m} = \sum_{i=n}^{m} \dfrac{\sin(2\pi nx)}{n\log(n)}$. For $x \in [0, \dfrac{1}{m}]$, we have

$$\begin{aligned} |g_{n,m}| &\leq \sum_{i=n}^{m} \left|\frac{\sin(2\pi ix)}{i\log(i)}\right| \leq \sum_{i=n}^{m} \frac{2\pi ix}{i\log(i)} = \sum_{i=n}^{m} \frac{2\pi x}{\log(i)} \\ &\leq \frac{1}{\log(n)} \leq \frac{1}{m\log(n)}\sum_{i=n}^{m} 2\pi \leq \frac{2\pi}{\log(n)}. \end{aligned}$$

For $x \in [\dfrac{1}{n}, \dfrac{1}{2}]$, we first note the following teloscooing identity:

$$\sum_{k=2}^{n} \sin(2\pi kx) = \frac{\cos(\pi x)\cos(2\pi x) - \cos(n\pi x)\cos((n+1)\pi x)}{\sin(\pi x)}.$$

From this identity, it follows that $|\sum_{k=n}^{i}\sin(2\pi kx)| \leq \dfrac{4}{\sin(\pi x)} \leq \dfrac{2}{x} \leq 2n$, for all $i \geq n$. Therefore, by summation by parts, we obtain that

$$|g_{n,m}| = \left|\sum_{k=n}^{m}(\frac{1}{n\log(n)} - \frac{1}{\log(n+1)})\sum_{i=n}^{k}\sin(2\pi nx)\right| \leq \frac{2}{\log(n)}$$

If $x \in [\frac{1}{m}, \frac{1}{n}]$, it follows that $S_{n,m} = S_{n,\lceil\frac{1}{x}\rceil} + S_{\lceil\frac{1}{x}\rceil+1,m}$. Hence, the addition of the two bounds establishes before gives the bound on this interval as well. Therefore, we have that

$$|g_{n,m}| = O(\frac{1}{log(n)}),$$

and the partial sums of f_S is cauchy. Thus, f_S converges uniformly and $f \in C(\mathbb{T})$.

We now show that $f_C(x) \geq c\log\log(\frac{1}{|x|}$ as $x \to 0$. Fix $m > 2$, and consider $x \in [\dfrac{1}{8m}, \dfrac{1}{4m}]$. It follows that

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\cos(2\pi nx)}{n\log(n)} &= \sum_{n=2}^{m}\frac{\cos(2\pi nx)}{n\log(n)} + \sum_{n=m+1}^{\infty}\frac{\cos(2\pi nx)}{n\log(n)} \\ &\geq \sum_{n=2}^{m}\frac{1-2\pi^2n^2x^2}{n\log(n)} + \sum_{n=m+1}^{\infty}\frac{\cos(2\pi nx)}{n\log(n)}\end{aligned}$$

Observe that for any $p > m+1$, we have

$$\begin{aligned}\left|\sum_{n=m+1}^{p}\frac{\cos(2\pi nx)}{n\log(n)}\right| &= |\sum_{n=m+1}^{p-1}(\frac{1}{n\log(n)} - \frac{1}{(n+1)\log(n+1)})\sum_{k=m+1}^{n}\cos(2\pi kx) \\ &+ \frac{1}{p\log(p)}\sum_{k=m+1}^{p}\cos(2\pi kx)| \leq \frac{4}{\log(m)}.\end{aligned}$$

Hence, for m large enough, we have

$$\begin{aligned}f_c(x) &\leq \int_2^m \frac{1}{x\log(x)}dx - \frac{4}{\log(m)} - C \\ &\leq C''\log\log(m) \leq C'''\log(\log(\frac{1}{|x|})),\end{aligned}$$

with the constants being appropriately chosen. Therefore, we have shown that $f_C(x) \geq c\log(\log(\dfrac{1}{|x|})$ as $x \to 0$.

Question 3.

3. (Problem 1.5 in Muscalu & Schlag) Suppose $f \in H^{\frac{1}{2}}(\mathbb{T}) \cap C(\mathbb{T})$. Show that $S_N f \to f$ uniformly. (Hint: Study $S_N f - \sigma_N f$.)

Solution. By the triangle inequality of the supnorm, we have

$$||S_N f - f||_\infty \leq ||S_N f - \sigma_N f||_\infty + ||\sigma_N f - f||_\infty,$$

for all $N \in \mathbb{Z}^+$. As $f \in C(\mathbb{T})$, we have that $||\sigma_N f - f||_\infty \to 0$ as $N \to \infty$. Therefore, it suffices to show that $||S_N f - \sigma_N f||_\infty \to 0$ as $N \to \infty$. By definition of S_N and σ_N, triangle inequality, and Cauchy-Schwarz, we obtain

$$
\begin{aligned}
||S_N f - \sigma_N f||_\infty = \quad &\leq \sum_{n=-N}^{N} \frac{|n|}{N} |\hat{f}(n)| \\
&\leq \sum_{n=-M}^{M} \frac{|n||\hat{f}(n)|}{N} + (\sum_{N \geq |n| > M} \frac{|n|}{N^2})^{\frac{1}{2}} (\sum_{N \geq |n| > M} |n||\hat{f}(n)|^2)^{\frac{1}{2}}, \\
&\leq \sum_{n=-M}^{M} \frac{|n||\hat{f}(n)|}{N} + 2(\sum_{N \geq |n| > M} |n||\hat{f}(n)|^2)^{\frac{1}{2}},
\end{aligned}
$$

for any $N > M$. Taking $\limsup$ with respect to N on both sides, we get

$$
\limsup_{N\to\infty} ||S_N f - \sigma_N f||_\infty \quad \leq \quad 2(\sum_{|n|>M} |n||\hat{f}(n)|^2)^{\frac{1}{2}},
$$

for any M. As $f \in H^{\frac{1}{2}}(\mathbb{T})$, taking the limit as $M \to \infty$ gives

$$
\limsup_{N\to\infty} ||S_N f - \sigma_N f||_\infty \quad \leq \quad 0
$$

Hence, we have shown that $||S_N f - \sigma_N f||_\infty \to 0$ as $N \to \infty$ as desired. $\square$

Question 4.

4. Let $0 < \alpha < 1$. Note by a theorem we have seen in class (which one?) that $f \in C^\alpha(\mathbb{T})$ implies $\hat{f}(n) = O(|n|^{-\alpha})$. Then, note that the exponent in this decay estimate cannot be improved by showing that the function

$$
F(x) = \sum_{m=1}^{\infty} \frac{1}{3^{m\alpha}} \cos(2\pi 3^m x)
$$

belongs to $C^\alpha(\mathbb{T})$. Also solve Exercise 1.9 in Muscalu & Schlag.

Solution.
A theorem that gives this result of $f \in C^\alpha(\mathbb{T}) \implies \hat{f}(n) = O(n^{-\alpha})$ is recorded in section 1.4.4, pg.18 of Schleg.

Now, we show that the exponent in the decay estimate cannot be improved. We first show that $F \in C^\alpha(\mathbb{T})$. Fix $x, y \in \mathbb{T}$, such that $x \neq y$. Choose $K \in \mathbb{N}$ such that $3^{-K-1} < |x-y| \leq 3^{-K}$. In particular, observe that, with this choice of K, we have $1 < 3^{K+1}|x-y| < 3$. It follows that

$$
\begin{aligned}
\frac{|F(x)-F(y)|}{|x-y|^\alpha} \quad &\leq \sum_{m=1}^{K} \frac{|\cos(2\pi 3^m x) - \cos(2\pi 3^m y)|}{3^{m\alpha}|x-y|^\alpha} + \sum_{m=K+1}^{\infty} \frac{|\cos(2\pi 3^m x) - \cos(2\pi 3^m y)|}{3^{m\alpha}|x-y|^\alpha} \\
&\leq \sum_{m=1}^{K} \frac{2\pi 3^m |x-y|}{3^{m\alpha}|x-y|^\alpha} + \sum_{m=K+1}^{\infty} \frac{2}{3^{m\alpha}|x-y|^\alpha} \\
&\leq 2\pi \sum_{m=1}^{K} (3^{m-K})^{1-\alpha} + 2\frac{1}{1-3^{-\alpha}} \\
&= 2\pi (3^{1-K})^{1-\alpha} \frac{3^{K(1-\alpha)}-1}{3^{1-\alpha}-1} + \frac{2}{1-3^{-\alpha}} \\
&\leq 2\pi \frac{3^{1-\alpha}}{3^{1-\alpha}-1} + \frac{2}{1-3^{-\alpha}}.
\end{aligned}
$$

Since the bound on LHS is a constant, we have shown that $f \in C^{\alpha}(\mathbb{T})$.

By definition of Fourier coefficients, it follows that

$$\hat{F}(\pm 3^m) = \frac{1}{2 \cdot 3^{m\alpha}}.$$

Therefore, for any $\beta > \alpha$, we have that $\hat{F}(\pm 3^m)3^{m\beta} = \dfrac{3^{m(\beta-\alpha)}}{2} \to \infty$ as $M \to \infty$. Hence, it follows that $\hat{f}(m) \neq O(|n|^{-\beta})$ for $\beta > \alpha$, and the decay estimate cannot be improved.

Now, for the excercise 1.9, the above argument also yields that the given lacunary series is in $C^{\alpha}(\mathbb{T})$. Computing the Sobelev norm of the lacunary series with respect to β, we obtain

$$\begin{aligned} ||f||^2_{H^{\beta}(\mathbb{T})} &= \sum_{k \in \mathbb{N}} |2^k|^{2\beta} |2^{-\alpha k}|^2 \\ &= \sum_{k \in \mathbb{N}} 2^{2k(\beta-\alpha)} = \infty, \end{aligned}$$

as $\beta - \alpha > 0$. Hence, the given series shows that $C^{\alpha}(\mathbb{T})$ does not embed into $H^{\beta}(\mathbb{T})$ for any $\beta > \alpha$ as required.

$\square$

Question 5.

5. Draw a minimal Venn diagram that shows all possible intersections of the sets below:

$$C(\mathbb{T}), \quad A(\mathbb{T}), \quad C^{2/3}(\mathbb{T}), \quad H^{1/2}(\mathbb{T}), \quad U(\mathbb{T}) := \{f : S_N f \to f \text{ uniformly}\}.$$

Your diagram should not have any redundancy or ambiguity, i.e., if $A \cap B = \emptyset$, $A \subset B$, or $A \neq B$, this should be visible and indicated. Give an example (or show the existence) of a function in each region of intersection.

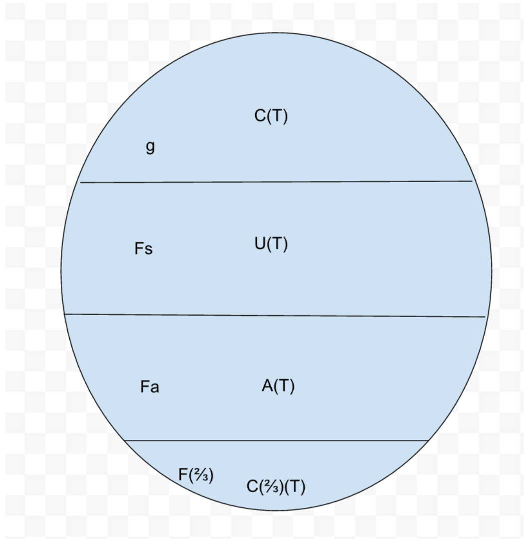
Solution. Since $S_n f$ is continuous, and uniform limit of a continuous function is continuous, we have $U(\mathbb{T}) \subset C(\mathbb{T})$. We have previously shown that if $f \in A(\mathbb{T})$, then $\{S_n f\}$ converges uniformly. This gives $A(\mathbb{T}) \subset U(\mathbb{T})$. As $\frac{2}{3} > \frac{1}{2}$, the theorem 1.13 from Schlag gives $C^{\frac{2}{3}}(\mathbb{T}) \subset H^{\frac{1}{2}}(\mathbb{T})$.

Now, from corollary 1.10 from Schlag, gives a function $g \in C(\mathbb{T})$ such that $g \notin U(\mathbb{T})$. Hence, $C(\mathbb{T}) \setminus U(\mathbb{T}) \neq \emptyset$

In problem 2, we have shown that $f_s \notin A(\mathbb{T})$, but $f_s \in U(\mathbb{T})$. Therefore, $U(\mathbb{T}) \setminus A(\mathbb{T}) \neq \emptyset$. Now, take any $\dfrac{2}{3} > \alpha > \dfrac{1}{2}$, and consider F_α from the problem 4, parametrized by α. It follows that $F_\alpha \in A(\mathbb{T})$, and $F_\alpha \notin C^{\frac{2}{3}}(\mathbb{T})$. Therefore, we have $A(\mathbb{T}) \setminus C^{\frac{2}{3}}(\mathbb{T}) \neq \emptyset$, and $F_{\frac{2}{3}} \in C^{\frac{2}{3}}(\mathbb{T})$. Recapping the information we have gathered so far gives the following figure:

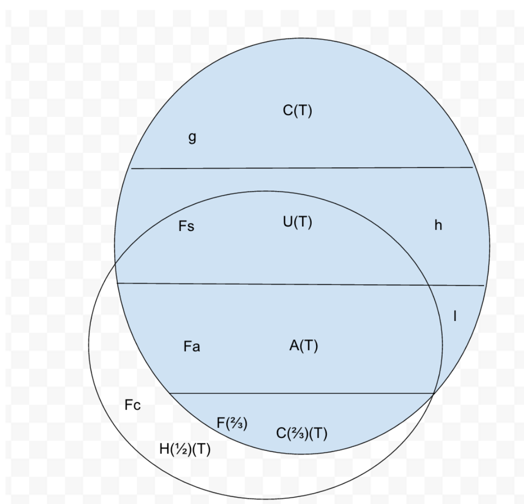
Now, recall that f_s from the problem 2 is in $H^{\frac{1}{2}}(\mathbb{T})$, but not even essentially bounded. Hence, $H^{\frac{1}{2}}(\mathbb{T}) \setminus C^{\frac{2}{3}}(\mathbb{T}) \neq \emptyset$. Now, by the problem 3, we have that $H^{\frac{1}{2}}(\mathbb{T}) \cap C(\mathbb{T}) \subset U(\mathbb{T})$, and $f_s \in H^{\frac{1}{2}}(\mathbb{T}) \cap U(\mathbb{T})$. Recall that $F_\alpha \in H^{\frac{1}{2}}(\mathbb{T})$. Hence, by preposition 1.14 from there exists a function h on $\mathbb{T}$ such that

Figure 1: Function spaces on $\mathbb{T}$



$h \in U(\mathbb{T}) \setminus A(\mathbb{T})$ and $l \in A(\mathbb{T}) \setminus C^{\frac{2}{3}}(\mathbb{T})$. Incorporating the additional information gives the following figure:

Figure 2: Function spaces on $\mathbb{T}$



This gives the adequate description of the function spaces on $\mathbb{T}$ for our interests. $\square$