
Harmonic Analysis: Problem Set II

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Abstract

This work contains solutions to the problem set II of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Solve Exercise 1.8 in Muscalu & Schlag.

Solution.

Question 2.

2. The following (non-absolutely convergent) series define functions in $H^{\frac{1}{2}}(\mathbb{T})$. (Why?)

$$f_S(x) := \sum_{n=2}^{\infty} \frac{\sin(2\pi nx)}{n \log n}, \quad f_C(x) := \sum_{n=2}^{\infty} \frac{\cos(2\pi nx)}{n \log n}$$

Show that the first series converges uniformly (hence $f_S \in C(\mathbb{T})$), but the second does not. In fact, show that $f_C(x) \geq c \log \log \frac{1}{|x|}$ as $x \rightarrow 0$ so that f_C is not even essentially bounded. (Hint: Summation by parts.)

Remark: For an example of a $C^{1/2}(\mathbb{T})$ function which is not in $A(\mathbb{T})$, see Proposition 1.14 in Muscalu & Schlag. (There is also another example, due to Hardy-Littlewood:

$$\sum_{n=1}^{\infty} \frac{e^{in \log n}}{n} e^{2\pi i n x},$$

Proof of this is given in Zygmund's "Trigonometric Series", vol. 1, p.197.)

Solution. Define $g_{n,m} = \sum_{i=n}^m \frac{\sin(2\pi ix)}{i \log(i)}$. For $x \in [0, \frac{1}{m}]$, we have

$$\begin{aligned} |g_{n,m}| &\leq \sum_{i=n}^m \left| \frac{\sin(2\pi ix)}{i \log(i)} \right| \leq \sum_{i=n}^m \frac{2\pi ix}{i \log(i)} = \sum_{i=n}^m \frac{2\pi x}{\log(i)} \\ &\leq \frac{1}{\log(n)} \leq \frac{1}{m \log(n)} \sum_{i=n}^m 2\pi \leq \frac{2\pi}{\log(n)}. \end{aligned}$$

For $x \in [\frac{1}{m}, \frac{1}{n}]$, we obtain

Therefore, we have that

$$|g_{n,m}| = O\left(\frac{1}{\log(n)}\right),$$

and the partial sums of f_S is cauchy. Thus, f_S converges uniformly and $f \in C(\mathbb{T})$.

Question 3.

3. (Problem 1.5 in Muscalu & Schlag) Suppose $f \in H^{\frac{1}{2}}(\mathbb{T}) \cap C(\mathbb{T})$. Show that $S_N f \rightarrow f$ uniformly.
(Hint: Study $S_N f - \sigma_N f$.)

Solution. By the triangle inequality of the supnorm, we have

$$\|S_N f - f\|_\infty \leq \|S_N f - \sigma_N f\|_\infty + \|\sigma_N f - f\|_\infty,$$

for all $N \in \mathbb{Z}^+$. As $f \in C(\mathbb{T})$, we have that $\|\sigma_N f - f\|_\infty \rightarrow 0$ as $N \rightarrow \infty$. Therefore, it suffices to show that $\|S_N f - \sigma_N f\|_\infty \rightarrow 0$ as $N \rightarrow \infty$. By definition of S_N and σ_N , triangle inequality, and Cauchy-Schwarz, we obtain

$$\begin{aligned} \|S_N f - \sigma_N f\|_\infty &= \left\| \sum_{n=-N}^N \frac{|n|}{N} \hat{f}(n) \right\|_\infty \\ &\leq \sum_{n=-M}^M \frac{|n| |\hat{f}(n)|}{N} + \left(\sum_{N \geq |n| > M} \frac{|n|}{N^2} \right)^{\frac{1}{2}} \left(\sum_{N \geq |n| > M} |n| |\hat{f}(n)|^2 \right)^{\frac{1}{2}}, \\ &\leq \sum_{n=-M}^M \frac{|n| |\hat{f}(n)|}{N} + 2 \left(\sum_{N \geq |n| > M} |n| |\hat{f}(n)|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

for any $N > M$. Taking lim sup with respect to N on both sides, we get

$$\limsup_{N \rightarrow \infty} \|S_N f - \sigma_N f\|_\infty \leq 2 \left(\sum_{|n| > M} |n| |\hat{f}(n)|^2 \right)^{\frac{1}{2}},$$

for any M . As $f \in H^{\frac{1}{2}}(\mathbb{T})$, taking the limit as $M \rightarrow \infty$ gives

$$\limsup_{N \rightarrow \infty} \|S_N f - \sigma_N f\|_\infty \leq 0$$

Hence, we have shown that $\|S_N f - \sigma_N f\|_\infty \rightarrow 0$ as $N \rightarrow \infty$ as desired. \square

Question 4.

4. Let $0 < \alpha < 1$. Note by a theorem we have seen in class (which one?) that $f \in C^\alpha(\mathbb{T})$ implies $\hat{f}(n) = O(|n|^{-\alpha})$. Then, note that the exponent in this decay estimate cannot be improved by showing that the function

$$F(x) = \sum_{m=1}^{\infty} \frac{1}{3^{m\alpha}} \cos(2\pi 3^m x)$$

belongs to $C^\alpha(\mathbb{T})$. Also solve Exercise 1.9 in Muscalu & Schlag.

Solution.

A theorem that gives this result of $f \in C^\alpha(\mathbb{T}) \implies \hat{f}(n) = O(n^{-\alpha})$ is recorded in section 1.4.4, pg.18 of Schleg.

Now, we show that the exponent in the decay estimate cannot be improved. We first show that $F \in C^\alpha(\mathbb{T})$. Fix $x, y \in \mathbb{T}$, such that $x \neq y$. Choose $K \in \mathbb{N}$ such that $3^{-K-1} < |x - y| \leq 3^{-K}$. In particular, observe that, with this choice of K , we have $1 < 3^{K+1}|x - y| < 3$. It follows that

$$\begin{aligned} \frac{|F(x) - F(y)|}{|x - y|^\alpha} &\leq \sum_{m=1}^K \frac{|\cos(2\pi 3^m x) - \cos(2\pi 3^m y)|}{3^{m\alpha} |x - y|^\alpha} + \sum_{m=K+1}^{\infty} \frac{|\cos(2\pi 3^m x) - \cos(2\pi 3^m y)|}{3^{m\alpha} |x - y|^\alpha} \\ &\leq \sum_{m=1}^K \frac{2\pi 3^m |x - y|}{3^{m\alpha} |x - y|^\alpha} + \sum_{m=K+1}^{\infty} \frac{2}{3^{m\alpha} |x - y|^\alpha} \\ &= 2\pi \sum_{m=1}^K (3^m |x - y|)^{1-\alpha} + \sum_{m=K+1}^{\infty} 2((3^m |x - y|)^{-\alpha}) \\ &\leq 2 \sum_{m=0}^{\infty} 3^{-\alpha m} \\ &\leq \frac{2}{1 - 3^{-\alpha}} \end{aligned}$$

□

Question 5.

5. Draw a minimal Venn diagram that shows all possible intersections of the sets below:

$$C(\mathbb{T}), A(\mathbb{T}), C^{2/3}(\mathbb{T}), H^{1/2}(\mathbb{T}), U(\mathbb{T}) := \{f : S_N f \rightarrow f \text{ uniformly}\}.$$

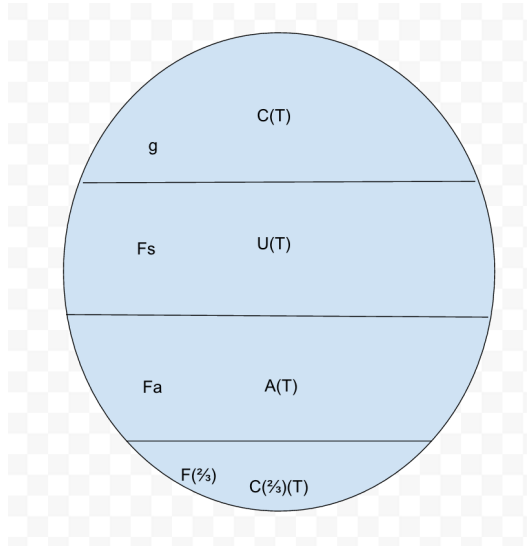
Your diagram should not have any redundancy or ambiguity, i.e., if $A \cap B = \emptyset$, $A \subset B$, or $A \neq B$, this should be visible and indicated. Give an example (or show the existence) of a function in each region of intersection.

Solution. Since $S_n f$ is continuous, and uniform limit of a continuous function is continuous, we have $U(\mathbb{T}) \subset C(\mathbb{T})$. We have previously shown that if $f \in A(\mathbb{T})$, then $\{S_n f\}$ converges uniformly. This gives $A(\mathbb{T}) \subset U(\mathbb{T})$. As $\frac{2}{3} > \frac{1}{2}$, the theorem 1.13 from Schlag gives $C^{\frac{2}{3}}(\mathbb{T}) \subset H^{\frac{1}{2}}(\mathbb{T})$.

Now, from corollary 1.10 from Schlag, gives a function $g \in C(\mathbb{T})$ such that $g \notin U(\mathbb{T})$. Hence, $C(\mathbb{T}) \setminus U(\mathbb{T}) \neq \emptyset$

In problem 2, we have shown that $f_s \notin A(\mathbb{T})$, but $f_s \in U(\mathbb{T})$. Therefore, $U(\mathbb{T}) \setminus A(\mathbb{T}) \neq \emptyset$. Now, take any $\frac{2}{3} > \alpha > \frac{1}{2}$, and consider F_α from the problem 4, parametrized by α . It follows that $F_\alpha \in A(\mathbb{T})$, and $F_\alpha \notin C^{\frac{2}{3}}(\mathbb{T})$. Therefore, we have $A(\mathbb{T}) \setminus C^{\frac{2}{3}}(\mathbb{T}) \neq \emptyset$, and $F_{\frac{2}{3}} \in C^{\frac{2}{3}}(\mathbb{T})$. Recapping the information we have gathered so far gives the following figure:

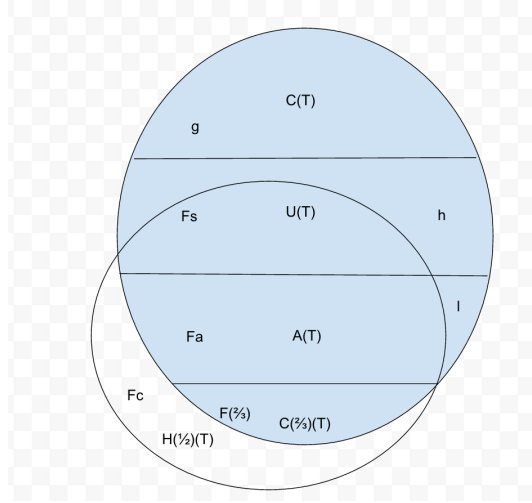
Figure 1: Function spaces on \mathbb{T}



Now, recall that f_s from the problem 2 is in $H^{\frac{1}{2}}(\mathbb{T})$, but not even essentially bounded. Hence, $H^{\frac{1}{2}}(\mathbb{T}) \setminus C^{\frac{2}{3}}(\mathbb{T}) \neq \emptyset$. Now, by the problem 3, we have that $H^{\frac{1}{2}}(\mathbb{T}) \cap C(\mathbb{T}) \subset U(\mathbb{T})$, and $f_s \in H^{\frac{1}{2}}(\mathbb{T}) \cap U(\mathbb{T})$. Recall that $F_\alpha \in H^{\frac{1}{2}}(\mathbb{T})$. Hence, by preposition 1.14 from there exists a function h on \mathbb{T} such that

$h \in U(\mathbb{T}) \setminus A(\mathbb{T})$ and $l \in A(\mathbb{T}) \setminus C^{\frac{2}{3}}(\mathbb{T})$. Incorporating the additional information gives the following figure:

Figure 2: Function spaces on \mathbb{T}



This gives the adequate description of the function spaces on \mathbb{T} for our interests.

□