Harmonic Analysis:
Problem Set II

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Abstract

This work contains solutions to the problem set II of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Solve Exercise 1.8 in Muscalu & Schlag.

Solution.

Question 2.

2. The following (non-absolutely convergent) series define functions in $H^{\frac{1}{2}}(\mathbb{T})$. (Why?)

$$f_S(x) := \sum_{n=2}^{\infty} \frac{\sin(2\pi n x)}{n \log n}, \qquad f_C(x) := \sum_{n=2}^{\infty} \frac{\cos(2\pi n x)}{n \log n}$$

Show that the first series converges uniformly (hence $f_S \in C(\mathbb{T})$), but the second does not. In fact, show that $f_C(x) \geq c \log \log \frac{1}{|x|}$ as $x \to 0$ so that f_C is not even essentially bounded.
(Hint: Summation by parts.)

Remark: For an example of a $C^{1/2}(\mathbb{T})$ function which is not in $A(\mathbb{T})$, see Proposition 1.14 in Muscalu & Schlag. (There is also another example, due to Hardy-Littlewood:

$$\sum_{n=1}^{\infty} \frac{e^{in \log n}}{n} e^{2\pi i n x},$$

Proof of this is given in Zygmund's "Trigonometric Series", vol. 1, p.197.)

Solution.

Question 3.

3. (Problem 1.5 in Muscalu & Schlag) Suppose $f \in H^{\frac{1}{2}}(\mathbb{T}) \cap C(\mathbb{T})$. Show that $S_N f \to f$ uniformly. (Hint: Study $S_N f - \sigma_N f$.)

Solution.

Question 4.

4. Let $0 < \alpha < 1$. Note by a theorem we have seen in class (which one?) that $f \in C^\alpha(\mathbb{T})$ implies $\hat{f}(n) = O(|n|^{-\alpha})$. Then, note that the exponent in this decay estimate cannot be improved by showing that the function

$$F(x) = \sum_{m=1}^{\infty} \frac{1}{3^{m\alpha}} \cos(2\pi 3^m x)$$

belongs to $C^\alpha(\mathbb{T})$. Also solve Exercise 1.9 in Muscalu & Schlag.

Solution.

$\square$

Question 5.

5. Draw a minimal Venn diagram that shows all possible intersections of the sets below:

$$C(\mathbb{T}),\ \ A(\mathbb{T}),\ \ C^{2/3}(\mathbb{T}),\ \ H^{1/2}(\mathbb{T}),\ \ U(\mathbb{T}) := \{f : S_N f \to f \text{ uniformly}\}.$$

Your diagram should not have any redundancy or ambiguity, i.e., if $A \cap B = \emptyset$, $A \subset B$, or $A \neq B$, this should be visible and indicated. Give an example (or show the existence) of a function in each region of intersection.

Solution.

$\square$