Harmonic Analysis:
Final Exam

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Abstract

This work contains a solution to the Final Exam of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Watching these videos, you will see that the synthesis operation is quite intuitive, but the analysis operation (which was the very reason Michelson built this machine) requires knowing a bit more mathematics (shall we say, *harmonic analysis*). On pages 98 and 99 of their book you will find the basic mathematical explanation (but watch the videos first!). In this problem, we will make things a bit more rigorous.

(a) We defined the Discrete Fourier Transform (DFT) on our very first class (as the Fourier transform on $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$). Namely, if $x := (x_n)_{n\in\mathbb{Z}}$ is an N-periodic sequence seen as a function on $\mathbb{Z}_N$, then its DFT $\widehat{x}$ is the N-periodic sequence given by

$$\widehat{x}_k := \sum_{n\in\mathbb{Z}_N} x_n e^{-2\pi i nk/N}, \qquad k \in \mathbb{Z},$$

which we may also identify with a function on $\mathbb{Z}_N$. Then we have the inversion formula

$$x_n = \frac{1}{N}\sum_{k\in\mathbb{Z}_N} \widehat{x}_k e^{2\pi i nk/N}, \qquad n \in \mathbb{Z},$$

(which you can easily verify even if it's new to you).

Let P be a trigonometric polynomial of degree N, i.e., $P \in \operatorname{span}\{e_k : -N \le k \le N\}$. Define $p := (p_n)_{n\in\mathbb{Z}}$ to be the $(2N+1)$-periodic sequence given by $p_n := \frac{1}{2N+1}P(\frac{n}{2N+1})$, $n \in \mathbb{Z}$, and let $\widehat{p}$ be its DFT (as a $(2N+1)$-periodic sequence). Show that $\widehat{p}_k = \widehat{P}(k)$ for all $|k| \le N$. (Here $\widehat{P}(k)$ is the kth Fourier series coefficient of P.) What happens for $|k| > N$?

Solution.
(a) Let P be a trig polynomial defined on $\mathbb{T}$ of degree N, i.e.

$$P = \sum_{|k|\leq N} a_k e_k,$$

where a_ks are the complex coefficients. Suppose $|k| \leq N$. We trivially know that $\hat{P}(k) = a_k$. We compute

$$\begin{aligned}
\widehat{p_k} &= \sum_{n\in\mathbb{Z}_{2N+1}} p_n e^{\frac{-2\pi ink}{2N+1}} = \sum_{n\in\mathbb{Z}_{2N+1}} \frac{1}{2N+1} P(\frac{n}{2N+1}) e^{-\frac{2\pi ink}{2N+1}} \\
&= \frac{1}{2N+1} \sum_{n\in\mathbb{Z}_{2N+1}} \left(\sum_{|l|\leq N} a_l e_l(\frac{n}{2N+1}) \right) e^{-\frac{2\pi ink}{2N+1}} = \frac{1}{2N+1} \sum_{n\in\mathbb{Z}_{2N+1}} \left(\sum_{|l|\leq N} a_l e^{\frac{2\pi iln}{2N+1}} \right) e^{-\frac{2\pi ink}{2N+1}} \\
&= \frac{1}{2N+1} \sum_{n\in\mathbb{Z}_{2N+1}} \left(\sum_{|l|\leq N} a_l e^{\frac{2\pi i(l-k)n}{2N+1}} \right) = \frac{1}{2N+1} \sum_{|l|\leq N} \left(a_l \sum_{n\in\mathbb{Z}_{2N+1}} e^{\frac{2\pi i(l-k)n}{2N+1}} \right) \\
&= \frac{1}{2N+1} \sum_{|l|\leq N; l\neq k} \left(a_l \sum_{n\in\mathbb{Z}_{2N+1}} e^{\frac{2\pi i(l-k)n}{2N+1}} \right) + \frac{1}{2N+1} \sum_{|l|\leq N; l=k} \left(a_l \sum_{n\in\mathbb{Z}_{2N+1}} e^{\frac{2\pi i(l-k)n}{2N+1}} \right) \\
&= 0 + \frac{1}{2N+1}(2N+1)a_k = a_k,
\end{aligned}$$

as the sum of any N-th root of unity is zero, thereby forcing the first term of the second last equation to be 0. For $|k| > N$, we trivially see that $\widehat{p_k} = 0$, and $\hat{P}(k) = 0$ as well. $\square$

Question 1-2.

(b) For any $F \in C(\mathbb{T})$, define $f := (f_n)_{n\in\mathbb{Z}}$ to be the $(2N+1)$-periodic sequence given by $f_n := \frac{1}{2N+1} F(\frac{n}{2N+1})$, $n \in \mathbb{Z}$, and let $\widehat{f}$ be its DFT (as a $(2N+1)$-periodic sequence). Show that for any trigonometric polynomial P of degree N, we have

$$\max_{|k|\leq N} |\widehat{f}_k - \widehat{F}(k)| \leq 2\|F - P\|_\infty.$$

(Hint: Use (a). In particular, write $F = (F - P) + P$ and $f = (f - p) + p$ where p is as in (a).)

(c) Part (b) says that the DFT of regular samples of F at $2N + 1$ points approximate the Fourier series coefficients of F to the same extent F can be approximated by a trigonometric polynomial of degree N. Now explain in rigorous terms how Michelson uses his machine, basically a harmonic "synthesizer," as a harmonic "analyzer" instead. Here you may also wish to allude to the Fourier transform on $\mathbb{Z}$ (also known as the "Discrete Time Fourier Transform (DTFT)") which maps any (absolutely summable) sequence $(g_n)_{n\in\mathbb{Z}}$ to a function on the torus

$$\widehat{g}(\xi) := \sum_{n\in\mathbb{Z}} g_n e^{-2\pi in\xi}, \quad \xi \in \mathbb{T},$$

and the simple observation that the DFT of a periodic sequence is the same as the samples of the DTFT of its truncation to a period.

(d) Michelson wanted to build a new version of his analyzer with 1000 elements, but never did. How accurately would he be able to calculate the Fourier coefficients of a piecewise-linear continuous function using an analyzer with N elements?

Solution.
(b) Let $f \in C(\mathbb{T})$, and P be a trig polynomial of degree N. Define as before $f \triangleq (f_n)_{n\in\mathbb{Z}}$ to be the $(2N+1)$-periodic sequence given by

$$f_n \triangleq \frac{1}{2N+1}F(\frac{n}{2N+1}).$$

With $F = (F-P)+P$ and $f = (f-p)+p$, by the result of (a), for any $|k| \le N$, we see that

$$\begin{aligned} |\widehat{f}_k - \hat{F}_k| &= |\widehat{\{(f-p)+p\}}_k - \widehat{(F-P)+P}(k)| = |\widehat{(f-p)}_k - \widehat{(F-P)}(k) + \hat{p}_k - \hat{P}(k)| \\ &\le |\widehat{(f-p)}_k| + |\widehat{(F-P)}(k)| + |\hat{p}_k - \hat{P}(k)| = |\widehat{(f-p)}_k| + |\widehat{(F-P)}(k)|. \quad (1) \end{aligned}$$

Now, in view of (1), and the fact that N is finite, it suffices to show that for any $|k| \le N$,

$$|\widehat{(f-p)}_k| \le ||F-P||_\infty \text{ and } |\widehat{(F-P)}(k)| \le ||F-P||_\infty.$$

Now, for $|k| \le N$, the first inequality follows, as

$$\begin{aligned} |\widehat{(f-p)}_k| &= |\sum_{n\in\mathbb{Z}_{2N+1}} (f-p)_n e^{\frac{-2\pi ink}{2N+1}}| \\ &= |\frac{1}{2N+1}\sum_{n\in\mathbb{Z}_{2N+1}} (F(\frac{n}{2N+1}) - P(\frac{n}{2N+1}))e^{\frac{-2\pi ink}{2N+1}}| \\ &\le \frac{1}{2N+1}\sum_{n\in\mathbb{Z}_{2N+1}} |F(\frac{n}{2N+1}) - P(\frac{n}{2N+1})| = ||F-P||_\infty. \end{aligned}$$

Likewise, for $|k| \le N$, the second inequality, as $F - P \in C(\mathbb{T})$, and

$$\begin{aligned} |\widehat{F-P}(k)| &= \left|\int_{\mathbb{T}} (F-P)(t)e^{-ikt}dt\right| \le \int_{\mathbb{T}} |(F-P)(t)|dt \\ &= ||F-P||_1 \le ||F-P||_\infty. \end{aligned}$$

Therefore, we have that for any trig polynomial of degree N, P,

$$\max_{|k|\le N} |\hat{f}_k - \hat{F}(k)| \quad \le \quad 2||F-P||_\infty,$$

as required.

(c)

(d)

Question 2.

2. Let $\tau_h f = f(\cdot - h)$ be the translation operator for functions defined on $\mathbb{T}$.

(a) Show that a bounded linear operator T on $L^2(\mathbb{T})$ commutes with translations (i.e., $T(\tau_h f) = \tau_h(Tf)$ for all $h \in \mathbb{T}$ and $f \in L^2(\mathbb{T})$) if and only if

$$Tf = \sum_{n\in\mathbb{Z}} \xi_n \widehat{f}(n) e_n$$

for some $\xi \in l^\infty(\mathbb{Z})$. Show also that $\|T\|_{2\to 2} = \|\xi\|_\infty$.
(Hint: Consider $T(e_n)$.)

(b) Show that a bounded linear operator T on $L^1(\mathbb{T})$ commutes with translations if and only if $Tf = f * \mu$ for some $\mu \in M(\mathbb{T})$. Show also that $\|T\|_{1\to 1} = \|\mu\|_{M(\mathbb{T})}$.
(Hint: You may find the Fejer kernel useful.)

Solution.
(a) We first prove the equivalence for $\{e_n\}$. Fix $h \in \mathbb{T}$. Suppose that, for some $\xi \in l^\infty(\mathbb{Z})$, we have

$$Te_n = \sum_{n\in\mathbb{Z}} \xi_n \hat{f}(n) e_n,$$

for all n. As $\hat{e_n}(k) = 1$, if $k = n$ and 0 otherwise, it follows that, for all n,

$$Te_n = \xi_n e_n,$$

and, by linearity of T,

$$T(\tau_h(e_n)) = T(e^{2\pi i n(\cdot - h)}) = e^{2\pi i n h} T(e_n) = e^{-2\pi i n h} \xi_n e_n = \tau_h(T(e_n)).$$

Conversely, assume that $T(\tau_h e_n) = T($.

As $Te_n \in L^2$, by Riemann-Lebesgue lemma, it follows that

$$\xi_n = \widehat{Te_n}(n) \to 0,$$

as $n \to \infty$, so $\{\xi_n\} \in l^\infty(\mathbb{Z})$, as required.

Now, we argue that the equivalence can be extended to any $f \in L^2$. Suppose that the operator commutes with the translation operator for any e_n. Then, it follows that, for any $f \in L^2$,

$$\begin{aligned} T(\tau_h f) &= T(\tau_h \sum_{n\in\mathbb{Z}} \hat{f}(n) e_n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) T(\tau_h e_n) \\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n) \tau_h(Te_n) = \tau_h(Tf). \end{aligned}$$

The converse holds trivially. Now, suppose that the operator has the given description with respect to $\{e_n\}$. It follows that, for any $f \in L^2$,

$$Tf = T(\sum_{n\in\mathbb{Z}} \hat{f}(n) e_n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) T(e_n) = \sum_{n\in\mathbb{Z}} \xi_n \hat{f}(n) e_n.$$

The converse in this case also holds trivially. Thus, we have shown the desired equivalence.

We now argue that $||T||_{2\to 2} = ||\xi||_\infty$.

(b) Consider the Fejer kernel, denoted by $\{K_n\}$. Firstly, observe that as the Fejer kernel is an approximate identity, we have $\sup_N \int_0^1 |K_n(x)| dx < \infty$, so we can extract a subsequence $\{K_{n_l}\}$ such that $K_{n_l} \to \mu$ weakly in L^1, for some $\mu \in M(\mathbb{T})$. For notational convenience, we relabel the subsequence as $\{K_n\}$. Now, assume that T commutes with translations. It follows that, for any $f \in C(\mathbb{T})$, and $x \in \mathbb{T}$,

$$\begin{aligned} f * \mu(x) &= \lim_{n\to\infty} \int_{\mathbb{T}} f(t)T(K_n)(x-t)dt = \lim_{n\to\infty} \int_{\mathbb{T}} f(t)T(K_n(x-t))dt, \\ &= \lim_{n\to\infty} \int_{\mathbb{T}} T(f(t)(K_n)(x-t))dt, = \lim_{n\to\infty} T\left(\int_{\mathbb{T}} f(t)K_n(x-t)dt\right), \end{aligned}$$

which via boundedness of T and the fact that $f \in C(\mathbb{T})$,

$$\lim_{n\to\infty} (f * K_n)(x) = f(x),$$

implies that

$$f * \mu(x) = T\left(\lim_{n\to\infty} \int_{\mathbb{T}} f(t)K_n(x-t)dt\right) = Tf(x),$$

as required. Now, as $C(\mathbb{T})$ is dense in $L^1(\mathbb{T})$, it follows that

$$f * \mu = Tf$$

Question 3.

3. For any bounded measurable function m on $\mathbb{R}$, consider the operator T_m defined by

$$T_m f := (m\widehat{f})^\vee.$$

We know that T_m is bounded on $L^2(\mathbb{R})$.

(a) Suppose m is a monotonic step function given by

$$m(\xi) = \sum_{i=1}^{n} c_i \chi_{[\alpha_i,\alpha_{i+1})}(\xi),$$

where $-\infty = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \infty$. Show that

$$\|T_m\|_{p\to p} \lesssim_p \|m\|_\infty + |m|_{TV}, \qquad 1 < p < \infty,$$

where $|\cdot|_{TV}$ is the total variation.
(Hint: Express T_m in terms of the Hilbert transform.)

(b) Extend the conclusion of (a) to any m of bounded variation.

Solution.
(a) We fix the index in the definition of m to start at 0 and end at $n-1$. Suppose m is a monotonic step function given by

$$m = \sum_{i=0}^{n-1} c_i X_{[\alpha_i,\alpha_{i+1})}.$$

As $T_{[0,\infty)} = (iH + \frac{1}{2}I)$ (this fact was discussed in class), we have that

$$T_{[a,\infty)}f = e^{2\pi i a x} iHf(x) + \frac{1}{2}f(x),$$

so

$$\begin{aligned} T_m f &= T_{\sum_{i=0}^{n-1} c_i X_{[\alpha_i,\alpha_{i+1}]}} = c_0 I + \sum_{i=0}^{n-2}(c_{i+1}-c_i)T_{[\alpha_{i+1},\infty)} \\ &= c_0 I + \sum_{i=0}^{n-2}(c_{i+1}-c_i)(e^{2\pi i\alpha_{i+1}} iHf + \frac{1}{2}f). \end{aligned}$$

Now, by the triangle inequality, and the fact that the Hilbert Transform is bounded on L^p (recall that the bound constant is dependent on p) for $1 < p < \infty$, we obtain

$$\begin{aligned} ||T_m f||_p &\le |c_0|||f||_p + \sum_{i=0}^{n-2}|c_{i+1}-c_i|(||Hf||_p + \frac{1}{2}||f||_p) \\ &\le C_p(|c_0| + \sum_{i=0}^{n-2}|c_{i+1}-c_i|)||f||_p \le C_p(||m||_\infty + |m|_{TV})||f||_p, \end{aligned}$$

for some constant C_p, depended on p, so

$$||T_m||_{p\to p} \leqslant_p (||m||_\infty + |m|_{TV}),$$

as required.

(b) Let m be a function of bounded variation. As having a bounded variation implies being bounded, it follows that

$$||m||_\infty < \infty.$$

Now, as m is measurable, we can choose a sequence of monotonic step functions that converge pointwise to m such that $|m_n| \leq |m|$ for all n. Therefore, by DCT, for any $f \in L^p$, with $1 < p < \infty$, it follows that, for the pointwise limit,

$$\lim_{n\to\infty} T_{m_n} f \quad = \quad \lim_{n\to\infty} \int_{\mathbb{R}} m_n(\xi)\hat{f}(\xi)e^{2\pi i\xi}d\xi = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi i\xi}d\xi = T_m f.$$

Therefore, by Fatou

$$||T_m f||_p \leq \liminf_n ||T_{m_n} f||_p,$$

and by (a), and the choice of $\{m_n\}$

$$||T_m f||_p \leq \liminf_n ||T_{m_n} f||_p \lesssim_p (||m||_\infty + |m|_{TV}),$$

as required. □

Question 4.

4. (a) Suppose $f \in L^1(\mathbb{R})$ satisfies

$$f = f * f. \tag{1}$$

Show that $f = 0$ a.e.

(b) Suppose $f \in L^2(\mathbb{R})$ satisfies (1). Show that f is uniformly continuous and in $L^p(\mathbb{R})$ for all $2 \leq p \leq \infty$.

(c) Let $p \in (1, 2)$. Show that if $f \in L^p(\mathbb{R})$ satisfies (1), then $f \in L^2(\mathbb{R})$.

Solution.

(a) Let $f \in L^1$. Taking the Fourier transform on both sides gives

$$\hat{f} \quad = \quad \hat{f}\hat{f},$$

which, with the continuity of $\hat{f}$, implies that

$$\hat{f} = 0 \;\text{ a.e} \qquad \text{or} \qquad \hat{f} = 1 \text{ a.e.}$$

As $\hat{f} = 1$ a. e contradicts the Riemann-Lebesgue lemma, it follows that

$$\hat{f} \quad = \quad 0 \;\text{ a.e.},$$

so by the inversion formula for L^1

$$f \quad = \quad 0 \;\text{ a.e.},$$

as required.

(b) Let $f \in L^2$ such that $f = f * f$. By the same argument in the L^1 case, we have

$$\hat{f} = 0 \;\text{ a.e} \qquad \text{or} \qquad \hat{f} = 1 \text{ a.e.}$$

Let $E_1 = \{\hat{f} = 1\}$ and $E_0 = \{\hat{f} = 0\}$. As $\hat{f} \in L^2$, it follows that

$$||\hat{f}||_2 \quad = \quad (\int_{\mathbb{R}} |\hat{f}(\xi)|^2)^{-2} = m(E_1)^{-2} < \infty,$$

so

$$m(E_1) < \infty,$$

and, for any $p \geq 1$,

$$||\hat{f}||_p \quad = \quad (\int_{\mathbb{R}} |\hat{f}(\xi)|^p)^{-p} = m(E_1)^{-p} < \infty.$$

Therefore, we see that, for $1 \leq p \leq \infty$,

$$\hat{f} \in L^p,$$

as the infinity bound follows trivially.

Now, we first prove the uniform continuity of f. By the inversion formula for L^2, we obtain

$$f(x) \quad = \quad \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi = \int_{E_1} e^{2\pi i \xi x} d\xi.$$

Therefore, for any $\delta > 0$ and $x \in \mathbb{R}$, it follows that

$$\begin{aligned} |f(x+\delta) - f(x)| \quad &= \quad |\int_{E_1} e^{2\pi i \xi (x+\delta)} - e^{2\pi i \xi x}| d\xi \leq \int_{E_1} |e^{2\pi i \xi x}||e^{2\pi i \xi \delta} - 1| d\xi \\ &\leq \quad \int_{E_1} |e^{2\pi i \xi \delta} - 1| d\xi. \end{aligned}$$

Observe that the last integral is independent of x, and the integrand tends to 0, as $\delta \to 0$. Therefore, we have shown that f is uniformly continuous.

We now argue that $f \in L^p$ for $p \in [2, \infty]$. We employ Riesz-Thorin to the Fourier inversion operator. Since the Fourier inversion is bounded from L^1 to L^∞ and from L^2 to L^2, by Riesz-Thorin, we have that the inversion is bounded from p to q where $1 \leq p \leq 2$ and q is the conjugate of p. In particular, for $2 \leq p \leq \infty$, we see that

$$||f||_p \quad \leq \quad ||\hat{f}||_q,$$

where q is again the conjugate of p. As we have previously shown that $\hat{f} \in L^p$, for all $1 \leq p \leq \infty$, we are done.

(c) Let $f \in L^p$ such that $f = f * f$. As $p \in (1, 2)$, it follows that $\hat{f} \in L^q$, where q is the conjugate of p. As $\hat{f} \in L^q$, by the same argument from (b), we have that $\hat{f} \in L^2$. Therefore, we have shown that $\hat{f} \in L^2$, so $f \in L^2$, as the Fourier transform is an isometry on L^2. $\square$