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# Harmonic Analysis: Problem Set I

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## Abstract

This work contains solutions to the problem set I of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

### Question 1.

EXERCISE 1.1. Verify that for each integer  $N \geq 0$

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \quad (1.5)$$

and draw the graph of  $D_N$  for several different values of  $N$ , say  $N = 2$  and  $N = 5$ . Prove the bound

$$|D_N(x)| \leq C \min\left(N, \frac{1}{|x|}\right) \quad (1.6)$$

for all  $N \geq 1$  and some absolute constant  $C$ . Finally, prove the bound

$$C^{-1} \log N \leq \|D_N\|_{L^1(\mathbb{T})} \leq C \log N \quad (1.7)$$

for all  $N \geq 2$  where  $C$  is another absolute constant.

**Solution.** We first verify that the given closed form formula for the Dirichlet Kernel  $D_n$ . Fix  $x \in \mathbb{T}$  and  $N \in \mathbb{N}$ . From the sum formula for geometric series, and the Euler's identity  $\sin(2\pi nx) = \frac{e(-nx) + e(nx)}{2i}$ , it follows that

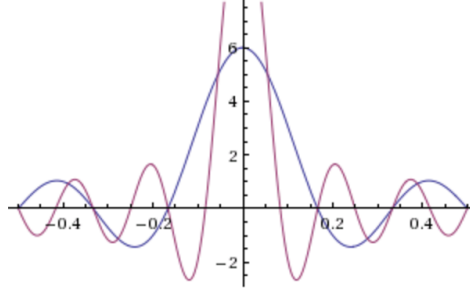
$$\begin{aligned} D_n(x) &= \sum_{n=-N}^N e(nx) = e(-Nx) \sum_{n=0}^{2N} e(nx) \\ &= e(-Nx) \frac{1 - e((2N+1)x)}{1 - e(x)} = \frac{e(-Nx) - e((N+1)x)}{1 - e(x)} \\ &= \frac{e(-(N+\frac{1}{2})x) - e((N+\frac{1}{2})x)}{e(-\frac{1}{2}x) - e(\frac{1}{2}x)} = \frac{\sin(2\pi(N+\frac{1}{2})x)}{\sin(2\pi(\frac{1}{2})x)} = \frac{\sin(2(N+1)\pi x)}{\sin(\pi x)}, \end{aligned}$$

as required.

The graphs of  $D_2$  and  $D_5$  are attached below. The blue graph corresponds to  $D_2$  and the green corresponds to  $D_5$ .

Figure 1: The graph of  $D_n$  for  $n = 2, 5$

Plot:



We proceed to prove the given bound. Fix  $x \in \mathbb{T}$  and  $n \in \mathbb{Z}_+$ . By the triangle inequality, we have

$$\begin{aligned} |D_n(x)| &= \left| \sum_{k=-N}^N e(kx) \right| \\ &\leq \sum_{k=-N}^N |e(kx)| = 2N + 1 \leq 3N. \end{aligned}$$

For  $x \in (0, \frac{1}{2}]$ , we have  $2x \leq \sin(\pi x)$ . Hence,

$$\begin{aligned} |D_n(x)| &= \left| \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \right| = \frac{|\sin((2N+1)\pi x)|}{|\sin(\pi x)|} \\ &\leq \frac{|\sin((2N+1)\pi x)|}{2|x|} \leq \frac{1}{2|x|}. \end{aligned}$$

Therefore, we obtain that

$$|D_N(x)| \leq 3 \min(N, \frac{1}{|x|}).$$

Now, using the monotonicity of Lebesgue integration and additivity over domain gives

$$\begin{aligned} \|D_N\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} |D_n(x)| dx \leq \int_{\mathbb{T}} 3 \min(N, \frac{1}{|x|}) dx \\ &= 6 \int_0^{\frac{1}{2}} \min(N, \frac{1}{|x|}) dx = 6 \left( \int_0^{\frac{1}{N}} N dx + \int_{\frac{1}{N}}^{\frac{1}{2}} \frac{1}{|x|} dx \right) \\ &= 6 + 6(\log(\frac{1}{2}) - \log(\frac{1}{N})) = 6 + 6 \log(\frac{1}{2}) + 6 \log(N) \leq C_1 \log(N), \end{aligned}$$

where a sufficiently large  $C_1$ , that satisfies the last inequality when  $N = 2$ . Now, for the lower bound, using the fact that  $\sin(\pi x) \leq \pi x$  for  $x \in [0, \frac{1}{2}]$ , we have

$$\begin{aligned} \|D_N\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} |D_n(x)| dx = 2 \int_0^{\frac{1}{2}} \left| \frac{\sin(2N+1)\pi x}{\sin(\pi x)} \right| dx \\ &\geq \frac{2}{\pi} \int_0^{\frac{1}{2}} \frac{|\sin(2N+1)\pi x|}{x} dx. \end{aligned}$$

Now, using change of variable with  $x = (2N + 1)\pi t$ , we can continue the computation as follows:

$$\begin{aligned}
\|D_N\|_{L^1(\mathbb{T})} &\geq C \int_0^{(N+\frac{1}{2})\pi} \frac{|\sin(t)|}{t} dt. \geq C \int_0^{N\pi} \frac{|\sin(t)|}{t} dt. \\
&= C \sum_{i=1}^N \int_{(i-1)\pi}^{i\pi} \frac{|\sin(t)|}{t} dt. \geq C \sum_{i=1}^N \frac{1}{(i+1)\pi} \int_{(i-1)\pi}^{i\pi} |\sin(t)| dt. \\
&= C \sum_{i=1}^N \frac{1}{(i+1)\pi} \cdot \frac{\pi}{2} \geq C' \sum_{i=1}^N \frac{1}{i} \geq C'' \log(N),
\end{aligned}$$

as  $\sum_{i=1}^N \frac{1}{i} \geq c \log(N)$  for some  $c$  for  $N \geq 2$ . Choosing the maximum  $C$  from the upper and the lower bound, we have shown the desired bound on the  $L_1$  norm of  $D_n$ .  $\square$

## Question 2.

**EXERCISE 1.2.** Let  $\mu \in \mathcal{M}(\mathbb{T})$  have the property that

$$\sum_{n \in \mathbb{Z}} |\hat{\mu}(n)| < \infty \quad (1.11)$$

Show that  $\mu(dx) = f(x) dx$  where  $f \in C(\mathbb{T})$ . Denote the space of all measures with this property by  $\mathbb{A}(\mathbb{T})$  and identify these measure with their respective densities. Show that  $\mathbb{A}(\mathbb{T})$  is an algebra under multiplication, and that

$$\widehat{fg}(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m) \hat{g}(n-m) \quad \forall n \in \mathbb{Z}$$

where the sum on the right-hand side is absolutely convergent for every  $n \in \mathbb{Z}$ , and itself is absolutely convergent over all  $n$ . Moreover,  $\|f * g\|_{\mathbb{A}} \leq \|f\|_{\mathbb{A}} \|g\|_{\mathbb{A}}$  where  $\|f\|_{\mathbb{A}} := \|\hat{f}\|_{\ell^1}$ . Finally, verify that if  $f, g \in L^2(\mathbb{T})$ , then  $f * g \in \mathbb{A}(\mathbb{T})$ .

## Solution.

Let  $u \in \mathbb{M}(\mathbb{T})$  such that  $\sum_{n \in \mathbb{Z}} |\hat{u}(n)| < \infty$ . By the Lebesgue-Radon-Nikodym theorem (Rudin pg.121), there exists  $f \in L_1(\mathbb{T})$  such that  $u(dx) = f(x)dx$ , where  $dx$  is the Lebesgue measure, restricted to Borel sets of  $\mathbb{T}$ . Let  $f$  be such function in  $L_1(\mathbb{T})$ . As  $u(dx) = f(x)dx$ , it follows that  $\hat{u}(n) = \hat{f}(n)$ , thus  $\sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty$ . Recall that a uniform limit of continuous functions is continuous. As we have that  $\{S_n f\}$  is a sequence of continuous functions, and that the tail ( $|n| > M$ ) terms can be arbitrarily bounded by a sufficiently large  $M$  by the assumption, we have that  $\{S_n f\}$  is Cauchy in  $C(\mathbb{T})$ . By completeness of  $C(\mathbb{T})$   $\{S_n f\}$  converges and by problem 1.1, know have that  $\{S_n f\}$  converges uniformly to  $f$ . Therefore,  $f \in C(\mathbb{T})$ .

Let  $f, g \in \mathbb{A}(\mathbb{T})$ . By linearity of integration and the triangle inequality, it follows that

$$\sum_{n \in \mathbb{Z}} |\widehat{f+g}(n)| \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)| + \sum_{n \in \mathbb{Z}} |\hat{g}(n)| < \infty.$$

Therefore,  $f + g \in \mathbb{A}(\mathbb{T})$ . Let  $\alpha \in \mathbb{C}$  and  $f \in \mathbb{A}(\mathbb{T})$ . It follows that

$$\sum_{n \in \mathbb{Z}} |\alpha \hat{f}(n)| \leq |\alpha| \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty.$$

Therefore,  $\alpha f \in \mathbb{A}(\mathbb{T})$ . So far, we have shown that  $\mathbb{A}(\mathbb{T})$  is a linear space.

Let  $f, g \in A(\mathbb{T})$ . It follows that

$$\begin{aligned} fg &= \left( \sum_{i \in \mathbb{Z}} \hat{f}(i) e(ix) \right) \left( \sum_{k \in \mathbb{Z}} \hat{g}(k) e(kx) \right) = \sum_{n \in \mathbb{Z}} \sum_{i+k=n} \hat{f}(i) \hat{g}(k) e(nx) \\ &= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \hat{f}(m) \hat{g}(n-m) e(nx) \end{aligned}$$

From the above equality and uniform convergence, we can further deduce that

$$\hat{f}g(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m) \hat{g}(n-m),$$

and consequently

$$\sum_{n \in \mathbb{Z}} |\hat{f}g(n)| \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)| \sum_{m \in \mathbb{Z}} |\hat{g}(m)| < \infty.$$

Therefore,  $fg \in A(\mathbb{T})$ . This shows that  $\mathbb{A}(\mathbb{T})$  is an algebra under multiplication. For the remaining part, it follows that

$$\|fg\|_A = \|\hat{f}g\|_{l_1} \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)| \sum_{m \in \mathbb{Z}} |\hat{g}(m)| = \|f\|_A \|g\|_A$$

If  $f, g \in L^2(\mathbb{T})$ , we have that  $\sum |\hat{f}(n)|^2, \sum |\hat{g}(n)|^2 < \infty$ . Hence, by the established inequality, it follows that  $f * g \in A(\mathbb{T})$ .

### Question 3.

**EXERCISE 1.3.** Let  $K_N$  be the Fejér kernel with  $N$  a positive integer.

- Verify that  $\hat{K}_N$  looks like a triangle, i.e., for all  $n \in \mathbb{Z}$

$$\hat{K}_N(n) = \left(1 - \frac{|n|}{N}\right)^+ \quad (1.16)$$

- Show that

$$K_N(x) = \frac{1}{N} \left( \frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2 \quad (1.17)$$

- Conclude that

$$0 \leq K_N(x) \leq C N^{-1} \min(N^2, x^{-2}) \quad (1.18)$$

**Solution.** Let  $K_N$  be the Fejér kernel with the positive integer  $n$ .

**(1.16)** From definition of  $n$ th Fourier coefficient, we obtain

$$\begin{aligned} \hat{K}_N(n) &= \int_{\mathbb{T}} K_N(x) e(-nx) dx = \int_{\mathbb{T}} \left( \frac{1}{N} \sum_{k=0}^{N-1} D_k(x) \right) e(-nx) dx \\ &= \int_{\mathbb{T}} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{|m| \leq k} e(mx) e(-nx) dx = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{|m| \leq k} \int_{\mathbb{T}} e((m-n)x) dx. \end{aligned}$$

Observe that the integral in the summation is 1 if  $m = n$  and 0 otherwise. For  $n \leq N$ , we have  $(N - |n|)$  terms in the sum where  $m = n$  happens, for  $n > N$ , we have no such term, where the equality holds. Therefore, it follows that

$$\hat{K}_N(n) = \frac{1}{N} (N - |n|)^+ = \left(1 - \frac{|n|}{N}\right)^+,$$

which is precisely the given closed form formula for the kernel.  $\square$

**(1.17)** Fix  $x \in \mathbb{T}$ , and  $N \in \mathbb{N}$ . Now, by definition of Fejer Kernel, we have

$$\begin{aligned} K_N(x) &= \frac{1}{N} \sum_{n=0}^{N-1} D_n(x) = \frac{1}{N} \sum_{n=0}^{N-1} \frac{\sin((2n+1)\pi x)}{\sin(\pi x)} \\ &= \frac{1}{N \sin(\pi x)^2} \sum_{n=0}^{N-1} \sin((2n+1)\pi x) \sin(\pi x). \end{aligned}$$

By the use of the trig identity,  $\sin(a) \sin(b) = \frac{1}{2} \cos(a-b) - \cos(a+b)$ , and cancellation from a telescoping sum, it follows that

$$\begin{aligned} K_N(x) &= \frac{1}{2N \sin(\pi x)^2} \sum_{n=0}^{N-1} (\cos(2n\pi x) - \cos((2n+2)\pi x)) \\ &= \frac{1}{2N \sin(\pi x)^2} (1 - \cos(2N\pi x)) \end{aligned}$$

Lastly, from the trig identity,  $2 \sin(a)^2 = 1 - \cos(2a)$ , we finally obtain that

$$K_N(x) = \frac{1}{2N \sin(\pi x)^2} (2 \sin(N\pi x)^2) = \frac{1}{N} \left( \frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2,$$

as required.

**(1.18)** Fix  $x \in \mathbb{T}$ , and  $N \in \mathbb{N}$ . From (1.17), it is clear that  $0 \leq K_n(x)$ . By the triangle inequality, and the result from the exercise 1.1, it follows that

$$K_N(x) \leq \frac{1}{N} \sum_{n=0}^{N-1} |D_n(x)| \leq \frac{1}{N} \sum_{n=0}^{N-1} 2n+1 = \frac{1}{N} (2 \frac{N(N-1)}{2} + N) = N$$

Now, by the (1.17) result, and the fact that  $|\sin(x)| \leq 2|x|$  for  $x \in [0, \frac{1}{2}]$ , we obtain

$$K_N(x) = \frac{1}{N} \left( \frac{\sin(N\pi x)}{\sin(\pi x)} \right)^2 \leq \frac{1}{N} \left( \frac{\sin(N\pi x)}{2x} \right)^2 \leq \frac{1}{N} \cdot \frac{1}{4x^2} \leq \frac{1}{Nx^2}.$$

Hence, we have shown that

$$0 \leq K_n(x) \leq N^{-1} \min(N^2, \frac{1}{x^2}),$$

as required.  $\square$

**Question 4.**

**Exercise 1.6** For any  $s \in \mathbb{R}$  define the Hilbert space  $H^s(\mathbb{T})$  by means of the norm

$$\|f\|_{H^s}^2 := |\hat{f}(0)|^2 + \sum_{n \in \mathbb{Z}} |n|^{2s} |\hat{f}(n)|^2. \quad (1.21)$$

Obtain the following quantitative improvements in certain qualitative convergence properties.

- (a) Show that for any  $0 \leq s \leq 1$  one has  $\|f(\cdot + \theta) - f\|_2 \leq 2\pi \|f\|_{H^s} |\theta|^s$ .
- (b) Derive a rate of convergence for  $\|S_N f - f\|_2$  in terms of  $N$  alone, assuming that  $\|f\|_{H^s} \leq 1$  where  $s > 0$  is fixed.

**Solution.**

(a) Let  $0 \leq s \leq 1$ , and  $h = \tau_{-\theta} f$ , where  $\tau_{\theta} f(x) = f(x - \theta)$  is the translation operator, parametrized by  $\theta$ . By the Corollary 1.6, and the linearity of integration, it follows that

$$\begin{aligned} \|h - f\|_2^2 &= \sum_{-\infty}^{\infty} |(\widehat{h - f})(n)|^2 \\ &= \sum_{-\infty}^{\infty} |\hat{h}(n) - \hat{f}(n)|^2. \end{aligned}$$

Now, we have a particular relation between the Fourier transform and translation as follows (pg. 4 in Schleg):

$$\widehat{\tau_{-\theta} f}(n) = e(n\theta) \hat{f}(n).$$

Hence, it follows that

$$\begin{aligned} \|h - f\|_2^2 &= \sum_{-\infty}^{\infty} |e(n\theta) \hat{f}(n) - \hat{f}(n)|^2 \\ &= \sum_{-\infty}^{\infty} |e(n\theta) - 1|^2 |\hat{f}(n)|^2 \end{aligned}$$

Recall that  $\theta \in [0, 1)$  and  $0 \leq s \leq 1$ . For  $|n\theta| \geq 1$ , we have

$$|e(n\theta) - 1|^2 \leq 4 \leq 4\pi^2 |n|^{2s} |\theta|^{2s}.$$

Now, when  $|n\theta| < 1$ , by rudimentary trig identities, we obtain

$$\begin{aligned} |e(n\theta) - 1|^2 &\leq (\cos(2\pi n\theta) - 1)^2 + \sin^2(2\pi n\theta) \\ &= 2 - 2\cos(2\pi n\theta) = 4\sin^2(\pi n\theta) \\ &\leq 4\pi^2 n^2 |\theta|^2 \leq 4\pi^2 |n|^{2s} |\theta|^{2s}. \end{aligned}$$

Hence, we have shown  $|e(n\theta) - 1|^2 \leq 4\pi^2 |n|^{2s} |\theta|^{2s}$ . Plugging the above inequality into the two norm inequality above gives

$$\begin{aligned} \|h - f\|_2^2 &\leq 4\pi^2 \left( \sum_{-\infty}^{\infty} |n|^{2s} |\hat{f}(n)|^2 \right) |\theta|^{2s} \\ &\leq 4\pi^2 \|f\|_{H^s(\mathbb{T})}^2 |\theta|^{2s}, \end{aligned}$$

and consequently,

$$\|h - f\|_2 \leq 2\pi \|f\|_{H^s(\mathbb{T})} |\theta|^s,$$

as required.

(b) Fix  $s > 0$ . By definition of  $S_N f$ , and the given, we have

$$\|S_N f - f\|_2^2 = \sum_{|n| > N} |\hat{f}(n)|^2 \quad \text{and} \quad \|f\|_{L^2} \leq \|f\|_{H^s} \leq 1$$

With the given norm, it follows that

$$\begin{aligned} 1 &\geq \|f\|_{H^s} \\ &\geq \sum_{|n| > N} (N+1)^{2s} |\hat{f}(n)|^2 = (N+1)^{2s} \|S_N f - f\|_2^2, \end{aligned}$$

which can be simplified to

$$\|S_N f - f\|_2 \leq \frac{1}{(N+1)^{2s}},$$

which reveals the rate of convergence as required.  $\square$

### Question 5.

**PROBLEM 1.1.** Suppose that  $f \in L^1(\mathbb{T})$  and that  $\{S_n f\}_{n=1}^\infty$  (the sequence of partial sums of the Fourier series) converges in  $L^p(\mathbb{T})$  to  $g$  for some  $p \in [1, \infty]$  and some  $g \in L^p$ . Prove that  $f = g$ . If  $p = \infty$  conclude that  $f$  is continuous.

**Solution.** Let  $p \geq 1$ . Observe that for  $x \in [0, 1)$ , we have  $|x| \leq |x|^p$ . Hence, it follows that

$$0 \leq \int_{\mathbb{T}} |S_n f - g| \leq \int_{\mathbb{T}} |S_n f - g|^p.$$

As we are given that  $\int_{\mathbb{T}} |S_n f - g|^p \rightarrow 0$  as  $n \rightarrow \infty$  it follows that  $\int_{\mathbb{T}} |S_n f - g| \rightarrow 0$  in  $L^1(\mathbb{T})$ . Now, from the triangle inequality of  $L_1$ , we have that

$$|f - g|_{L_1} \leq |f - \sigma_n f|_{L_1} + |\sigma_n f - g|_{L_1},$$

for all  $n$ . Recall that convergence of cesaro sum is more inclusive, thus implied by the convergence of the original sequence. Hence, as  $|S_n f - g|_{L_1} \rightarrow 0$  as  $n \rightarrow \infty$ , we have  $|\sigma_n f - g|_{L_1} \rightarrow 0$  as  $n \rightarrow \infty$ . Furthermore, since the Fejer kernel  $\{K_n\}$  is an approximate identity, we have that  $|\sigma_n f - f|_{L_1} \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, it follows that  $|f - g|_{L_1} = 0$ , which implies that  $f = g$  almost everywhere. For the case when  $p = \infty$ , as we have  $\{S_n f\}$  is a sequence of continuous function, by the convergence in supnorm, we have  $g$  is continuous.  $\square$

### Question 6.

**Problem 1.9** Show that

$$\|f * g\|_{L^2(\mathbb{T})}^2 \leq \|f * f\|_{L^2(\mathbb{T})} \|g * g\|_{L^2(\mathbb{T})}$$

for all  $f, g \in L^2(\mathbb{T})$ .

**Solution.** As we have  $f, g \in L^2(\mathbb{T})$ , by Corollary 1.6, the given inequality is equivalent to

$$\sum_{n \in \mathbb{Z}} |\widehat{f * g}(n)|^2 \leq \sqrt{\sum_{n \in \mathbb{Z}} |\widehat{f * f}(n)|^2} \sqrt{\sum_{n \in \mathbb{Z}} |\widehat{g * g}(n)|^2}.$$

Since  $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$ , the above inequality is again equivalent to

$$\left( \sum_{n \in \mathbb{Z}} |\hat{f}(n)\hat{g}(n)|^2 \right)^2 \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^4 \sum_{n \in \mathbb{Z}} |\hat{g}(n)|^4.$$

Expanding the LHS of the desired inequality yields

$$\begin{aligned} \left( \sum_{n \in \mathbb{Z}} |\hat{f}(n)\hat{g}(n)|^2 \right)^2 &= \sum_{n \in \mathbb{Z}} |\hat{f}(n)\hat{g}(n)|^4 + \sum_{n > m} 2|\hat{f}(n)\hat{f}(m)\hat{g}(n)\hat{g}(m)|^2 \\ &\leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)\hat{g}(n)|^4 + \sum_{n > m} |\hat{f}(n)\hat{g}(n)|^4 + |\hat{f}(m)\hat{g}(m)|^4 \\ &= \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^4 \sum_{n \in \mathbb{Z}} |\hat{g}(n)|^4, \end{aligned}$$

where the last inequality holds by the Cauchy-Schwarz inequality on the inner product space of  $l^2(\mathbb{T})$ . □

#### Question Extra.

**EXERCISE 1.4.** Let  $\{c_n\}_{n \in \mathbb{Z}}$  be an arbitrary sequence of complex numbers and associate to it formally a Fourier series

$$f(x) \sim \sum_n c_n e(nx)$$

Show that there exists  $\mu \in \mathcal{M}(\mathbb{T})$  with the property that  $\hat{\mu}(n) = c_n$  for all  $n \in \mathbb{Z}$  if and only if  $\{\sigma_n f\}_{n \geq 1}$  is bounded in  $\mathcal{M}(\mathbb{T})$ . Discuss the case of  $L^p(\mathbb{T})$  with  $1 \leq p < \infty$  and  $C(\mathbb{T})$  as well.

**Solution.**