Harmonic Analysis:
Problem Set I

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Abstract

This work contains solutions to the problem set I of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

EXERCISE 1.1. Verify that for each integer $N \geqslant 0$

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \tag{1.5}$$

and draw the graph of D_N for several different values of N, say $N = 2$ and $N = 5$. Prove the bound

$$|D_N(x)| \leqslant C \min\left(N, \frac{1}{|x|}\right) \tag{1.6}$$

for all $N \geqslant 1$ and some absolute constant C. Finally, prove the bound

$$C^{-1} \log N \leqslant \|D_N\|_{L^1(\mathbb{T})} \leqslant C \log N \tag{1.7}$$

for all $N \geqslant 2$ where C is another absolute constant.

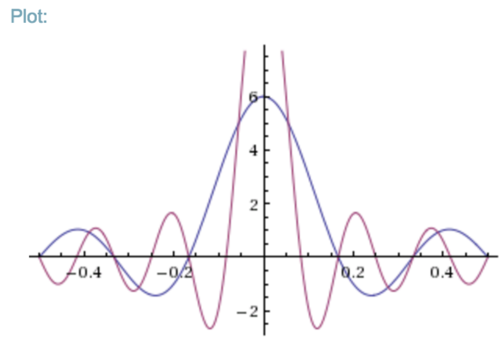
Solution. We first verify that the given closed form formula for the Dirichlet Kernel D_n. Fix $x \in \mathbb{T}$ and $N \in \mathbb{N}$. From the sum formula for geometric series, and the Euler's identity $\sin(2\pi nx) = \dfrac{e(-nx) + e(nx)}{2i}$, it follows that

$$\begin{aligned}
D_n(x) &= \sum_{n=-N}^{N} e(nx) = e(-Nx) \sum_{n=0}^{2N} e(nx) \\
&= e(-Nx) \frac{1 - e((2N+1)x)}{1 - e(x)} = \frac{e(-Nx) - e((N+1)x)}{1 - e(x)} \\
&= \frac{e(-(N+\frac{1}{2})x - e((N+\frac{1}{2})x)}{e(-\frac{1}{2}x) - e(\frac{1}{2}x)} = \frac{\sin(2\pi(N+\frac{1}{2})x)}{\sin(2\pi(\frac{1}{2})x)} = \frac{\sin(2(N+1)\pi x)}{\sin(\pi x)},
\end{aligned}$$

as required.

The graphs of D_2 and D_5 are attached below. The blue graph corresponds to D_2 and the green corresponds to D_5.

Figure 1: The graph of D_n for $n = 2, 5$



We proceed to prove the given bound. Fix $x \in \mathbb{T}$ and $n \in \mathbb{Z}_+$. By the triangle inequality, we have

$$\begin{aligned} |D_n(x)| &= \left| \sum_{k=-N}^{N} e(nx) \right| \\ &\leq \sum_{k=-N}^{N} |e(nx)| = 2N + 1 \leq 3N. \end{aligned}$$

For $x \in (0, \frac{1}{2}]$, we have $2x \leq \sin(\pi x)$. Hence,

$$\begin{aligned} |D_n(x)| &= \left| \frac{\sin((2N+1)\pi x}{\sin(\pi x)} \right| = \frac{|\sin((2N+1)\pi x|}{|\sin(\pi x)|} \\ &\leq \frac{|\sin((2N+1)\pi x)|}{2|x|} \leq \frac{1}{2|x|}. \end{aligned}$$

Therefore, we obtain that

$$|D_N(x)| \quad \leq \quad 3 \min(N, \frac{1}{|x|}).$$

Now, using the monotonicity of Lebesgue integration and additivity over domain gives

$$\begin{aligned} ||D_N||_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} |D_n(x)| dx \leq \int_{\mathbb{T}} 3 \min(N, \frac{1}{|x|}) dx \\ &= 6 \int_0^{\frac{1}{2}} \min(N, \frac{1}{|x|}) dx = 6 (\int_0^{\frac{1}{N}} N dx + \int_{\frac{1}{N}}^{\frac{1}{2}} \frac{1}{|x|} dx) \\ &= 6 + 6(\log(\frac{1}{2}) - \log(\frac{1}{N}) = 6 + 6\log(\frac{1}{2}) + 6\log(N) \leq C_1 log(N), \end{aligned}$$

where a sufficiently large C_1, that satisfies the last inequality when $N = 2$. Now, for the lower bound, using the fact that $\sin(\pi x) \leq \pi x$ for $x \in [0, \frac{1}{2}]$, we have

$$\begin{aligned} ||D_N||_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} |D_n(x)| dx = 2 \int_0^{\frac{1}{2}} |\frac{\sin(2N+1)\pi x}{\sin(\pi x)}| dx \\ &\geq \frac{2}{\pi} \int_0^{\frac{1}{2}} \frac{|\sin(2N+1)\pi x}{x} dx. \end{aligned}$$

Now, using change of variable with $x = (2N+1)\pi t$, we can continue the computation as follows:

$$\begin{aligned} ||D_N||_{L^1(\mathbb{T})} &\geq C\int_0^{(N+\frac{1}{2})\pi} \frac{|\sin(t)|}{t}dt. \geq C\int_0^{N\pi} \frac{|\sin(t)|}{t}dt. \\ &= C\sum_{i=1}^{N}\int_{(i-1)\pi}^{i\pi} \frac{|\sin(t)|}{t}dt. \geq C\sum_{i=1}^{N}\frac{1}{(i+1)\pi}\int_{(i-1)\pi}^{i\pi}|\sin(t)|dt. \\ &= C\sum_{i=1}^{N}\frac{1}{(i+1)\pi}\cdot\frac{\pi}{2} \geq C'\sum_{i=1}^{N}\frac{1}{i} \geq C''\log(N), \end{aligned}$$

as $\sum_{i=1}^{N}\frac{1}{i} \geq c\log(N)$ for some c for $N \geq 2$. Choosing the maximum C from the upper and the lower bound, we have shown the desired bound on the L_1 norm of D_n. ☐

Question 2.

EXERCISE 1.2. Let $\mu \in \mathcal{M}(\mathbb{T})$ have the property that

$$\sum_{n\in\mathbb{Z}} |\hat{\mu}(n)| < \infty \tag{1.11}$$

Show that $\mu(dx) = f(x)\,dx$ where $f \in C(\mathbb{T})$. Denote the space of all measures with this property by $\mathbb{A}(\mathbb{T})$ and identify these measure with their respective densities. Show that $\mathbb{A}(\mathbb{T})$ is an algebra under multiplication, and that

$$\widehat{fg}(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\hat{g}(n-m) \qquad \forall\, n \in \mathbb{Z}$$

where the sum on the right-hand side is absolutely convergent for every $n \in \mathbb{Z}$, and itself is absolutely convergent over all n. Moreover, $\|f * g\|_{\mathbb{A}} \leqslant \|f\|_{\mathbb{A}}\|g\|_{\mathbb{A}}$ where $\|f\|_{\mathbb{A}} := \|\hat{f}\|_{\ell^1}$. Finally, verify that if $f, g \in L^2(\mathbb{T})$, then $f * g \in \mathbb{A}(\mathbb{T})$.

Solution.

Let $u \in \mathbb{M}(\mathbb{T})$ such that $\sum_{n\in\mathbb{Z}}|\hat{u}(n)| < \infty$. By the Lebesgue-Radon-Nikodym theorem (Rudin pg.121), there exists $f \in L_1(\mathbb{T})$ such that $u(dx) = f(x)dx$, where dx is the Lebesgue measure, restricted to Borel sets of $\mathbb{T}$. Let f be such function in $L_1(\mathbb{T})$. As $u(dx) = f(x)dx$, it follows that $\hat{u}(n) = \hat{f}(n)$, thus $\sum_{n\mathbb{Z}}|\hat{f}(n)| < \infty$. Recall that an uniform limit of continuous functions is continuous. As we have that $\{S_n f\}$ is a sequence of continuous functions, and that the tail ($|n| > M$) terms can be arbitrarily bounded by a sufficiently large M by the assumption, we have that $\{S_n f\}$ is Cauchy in $C(\mathbb{T})$. By completeness of $C(\mathbb{T})$ $\{S_n f\}$ converges and by problem 1.1, know have that $\{S_n f\}$ converges uniformly to f. Therefore, $f \in C(\mathbb{T})$.

Let $f, g \in A(\mathbb{T})$. By linearity of integration and the triangle inequality, it follows that

$$\sum_{n\in\mathbb{Z}}|\widehat{f+g}(n)| \leq \sum_{n\in\mathbb{Z}}|\hat{f}(n)| + \sum_{n\in\mathbb{Z}}|\hat{g}(n)| < \infty.$$

Therefore, $f + g \in A(\mathbb{T})$. Let $\alpha \in \mathbb{C}$ and $f \in A(\mathbb{T})$. It follows that

$$\sum_{n\in\mathbb{Z}}|\alpha\hat{f}(n)| \leq |\alpha|\sum_{n\in\mathbb{Z}}|\hat{f}(n)| < \infty.$$

Therefore, $\alpha f \in A(\mathbb{T})$. So far, we have shown that $A(\mathbb{T})$ is a linear space.

Let $f, g \in A(\mathbb{T})$. It follows that

$$\begin{aligned} fg &= \big(\sum_{i\in\mathbb{Z}} \hat{f}(i)e(ix)\big)\big(\sum_{k\in\mathbb{Z}} \hat{g}(k)e(kx)\big) = \sum_{n\in\mathbb{Z}}\sum_{i+k=n} \hat{f}(i)\hat{g}(k)e(nx) \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \hat{f}(m)\hat{g}(n-m)e(nx) \end{aligned}$$

From the above equality and uniform convergence, we can further deduce that

$$\hat{fg}(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\hat{g}(m-n),$$

and consequently

$$\sum_{n\in\mathbb{Z}} |\hat{fg}(n)| \le \sum_{n\in\mathbb{Z}} |\hat{f}(n)| \sum_{m\in\mathbb{Z}} |\hat{g}(m)| < \infty.$$

Therefore, $fg \in A(\mathbb{T})$. This shows that $\mathbb{A}(\mathbb{T})$ is an algebra under multiplication. For the remaining part, it follows that

$$||fg||_A = ||\hat{fg}||_{l_1} \le \sum_{n\in\mathbb{Z}} |\hat{f}(n)| \sum_{m\in\mathbb{Z}} |\hat{g}(m)| = ||f||_A ||g||_A$$

If $f, g \in L^2(\mathbb{T})$, we have that $\sum |\hat{f}(n)|^2, \sum |\hat{f}(n)|^2 < \infty$. Hence, by the established inequality, it follows that $f * g \in A(\mathbb{T})$.

Question 3.

EXERCISE 1.3. Let K_N be the Fejér kernel with N a positive integer.

- Verify that $\widehat{K}_N$ looks like a triangle, i.e., for all $n \in \mathbb{Z}$

$$\widehat{K}_N(n) = \left(1 - \frac{|n|}{N}\right)^+ \tag{1.16}$$

- Show that

$$K_N(x) = \frac{1}{N}\left(\frac{\sin(N\pi x)}{\sin(\pi x)}\right)^2 \tag{1.17}$$

- Conclude that

$$0 \leqslant K_N(x) \leqslant C\,N^{-1}\min\left(N^2, x^{-2}\right) \tag{1.18}$$

Solution. Let K_N be the Fejer kernel with the positive integer n.

(1.16) From definition of nth Fourier coefficient, we obtain

$$\begin{aligned} \hat{K}_N(n) &= \int_{\mathbb{T}} K_n(x)e(-nx)dx = \int_{\mathbb{T}} (\frac{1}{N}\sum_{k=0}^{N-1} D_k(x))e(-nx)dx \\ &= \int_{\mathbb{T}} \frac{1}{N}\sum_{k=0}^{N-1}\sum_{|m|\le k} e(mx)e(-nx)dx = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{|m|\le k}\int_{\mathbb{T}} e((m-n)x)dx. \end{aligned}$$

Observe that the integral in the summation is 1 if $m = n$ and 0 otherwise. For $n \le N$, we have $(N - |n|)$ terms in the sum where $m = n$ happens, for $n > N$, we have no such term, where the equality holds. Therefore, it follows that

$$\hat{K}_N(n) = \frac{1}{N}(N - |n|)^+ = (1 - \frac{|n|}{N})^+,$$

which is precisely the given closed form formula for the kernel. $\square$

(1.17) Fix $x \in \mathbb{T}$, and $N \in \mathbb{N}$. Now, by definition of Fejer Kernel, we have

$$\begin{aligned} K_N(x) &= \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{1}{N}\sum_{n=0}^{N-1}\frac{\sin((2n+1)\pi x)}{\sin(\pi x)} \\ &= \frac{1}{N\sin(\pi x)^2}\sum_{n=0}^{N-1}\sin((2n+1)\pi x)\sin(\pi x). \end{aligned}$$

By the use of the trig identity, $\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \cos(a+b)$, and cancellation from a telescoping sum, it follows that

$$\begin{aligned} K_N(x) &= \frac{1}{2N\sin(\pi x)^2}\sum_{n=0}^{N-1}(\cos(2n\pi x) - \cos((2n+2)\pi x)) \\ &= \frac{1}{2N\sin(\pi x)^2}(1-\cos(2N\pi x)) \end{aligned}$$

Lastly, from the trig identity, $2\sin(a)^2 = 1-\cos(2a)$, we finally obtain that

$$K_N(x) = \frac{1}{2N\sin(\pi x)^2}(2\sin(N\pi x)^2) = \frac{1}{N}(\frac{\sin(N\pi x)}{\sin(\pi x)})^2,$$

as required.

(1.18) Fix $x \in \mathbb{T}$, and $N \in \mathbb{N}$. From (1.17), it is clear that $0 \leq K_n(x)$. By the triangle inequality, and the result from the excercise 1.1, it follows that

$$K_N(x) \leq \frac{1}{N}\sum_{n=0}^{N-1}|D_n(x)| \leq \frac{1}{N}\sum_{n=0}^{N-1} 2n+1 = \frac{1}{N}(2\frac{N(N-1)}{2}+N) = N$$

Now, by the (1.17) result, and the fact that $|\sin(x)| \leq 2|x|$ for $x \in [0, \frac{1}{2}]$, we obtain

$$K_N(x) = \frac{1}{N}(\frac{\sin(N\pi x)}{\sin(\pi x)})^2 \leq \frac{1}{N}(\frac{\sin(N\pi x)}{2x})^2 \leq \frac{1}{N}\cdot\frac{1}{4x^2} \leq \frac{1}{Nx^2}.$$

Hence, we have shown that

$$0 \leq K_n(x) \leq N^{-1}\min(N^2, \frac{1}{x^2}),$$

as required.

$\square$

Question 4.

Solution.

(a) Let $0 \leq s \leq 1$, and $h = \tau_{-\theta} f$, where $\tau_\theta f(x) = f(x-\theta)$ is the translation operator, parametrized by θ. By the Corollary 1.6, and the linearity of integration, it follows that

$$\begin{aligned} ||h-f||_2^2 &= \sum_{-\infty}^{\infty}|\widehat{(h-f)}(n)|^2 \\ &= \sum_{-\infty}^{\infty}|\hat{h}(n) - \hat{f}(n)|^2. \end{aligned}$$

Exercise 1.6 For any $s \in \mathbb{R}$ define the Hilbert space $H^s(\mathbb{T})$ by means of the norm

$$\|f\|_{H^s}^2 := |\hat{f}(0)|^2 + \sum_{n\in\mathbb{Z}} |n|^{2s}|\hat{f}(n)|^2. \qquad (1.21)$$

Obtain the following quantitative improvements in certain qualitative convergence properties.

(a) Show that for any $0 \le s \le 1$ one has $\|f(\cdot+\theta) - f\|_2 \le 2\pi\|f\|_{H^s}|\theta|^s$.

(b) Derive a rate of convergence for $\|S_N f - f\|_2$ in terms of N alone, assuming that $\|f\|_{H^s} \le 1$ where $s > 0$ is fixed.

Now, we have a particular relation between the Fourier transform and translation as follows (pg. 4 in Schleg):

$$\widehat{\tau_{-\theta} f}(n) = e(n\theta)\hat{f}(n).$$

Hence, it follows that

$$\begin{aligned} ||h - f||_2^2 &= \sum_{-\infty}^{\infty} |e(n\theta)\hat{f}(n) - \hat{f}(n)|^2 \\ &= \sum_{-\infty}^{\infty} |e(n\theta) - 1|^2|\hat{f}(n)|^2 \end{aligned}$$

Recall that $\theta \in [0,1)$ and $0 \le s \le 1$. For $|n\theta| \ge 1$, we have

$$|e(n\theta) - 1|^2 \le 4 \le 4\pi^2|n|^{2s}|\theta|^{2s}.$$

Now, when $|n\theta| < 1$, by rudimentary trig identities, we obtain

$$\begin{aligned} |e(n\theta) - 1|^2 &\le (\cos(2\pi n\theta)^2 - 1)^2 + \sin(2\pi n\theta)^2 \\ &= 2 - 2\cos(2\pi n\theta) = 4\sin^2(\pi n\theta) \\ &\le 4\pi^2 n^2|\theta|^2 \le 4\pi^2|n|^{2s}|\theta|^{2s}. \end{aligned}$$

Hence, we have shown $|e(n\theta) - 1|^2 \le 4\pi^2|n|^{2s}|\theta|^{2s}$. Plugging the above inequality into the two norm inequality above gives

$$\begin{aligned} ||h - f||_2^2 &\le 4\pi^2\big(\sum_{-\infty}^{\infty} n^2\hat{f}(n)|^2\big)|\theta|^{2s} \\ &\le 4\pi^2||f||_{H^s(\mathbb{T})}^2|\theta|^{2s}, \end{aligned}$$

and consequently,

$$||h - f||_2 \le 2\pi||f||_{H^s(\mathbb{T})}|\theta|^s,$$

as required.

(b) Fix $s > 0$. By definition of $S_N f$, and the given, we have

$$||S_N f - f||_2^2 = \sum_{|n|>N} |\hat{f}(n)|^2 \quad \text{and} \quad ||f||_{L^2} \le ||f||_{H^s} \le 1$$

With the given norm, it follows that

$$\begin{aligned} 1 &\geq ||f||_{H^s} \\ &\geq \sum_{|n|>N} (N+1)^{2s} |\hat{f}(n)|^2 = (N+1)^{2s} ||S_N f - f||_2^2, \end{aligned}$$

which can be simplified to

$$||S_N f - f||_2 \leq \frac{1}{(N+1)^{2s}},$$

which reveals the rate of convergence as required. □

Question 5.

Problem 1.1. Suppose that $f \in L^1(\mathbb{T})$ and that $\{S_n f\}_{n=1}^{\infty}$ (the sequence of partial sums of the Fourier series) converges in $L^p(\mathbb{T})$ to g for some $p \in [1, \infty]$ and some $g \in L^p$. Prove that $f = g$. If $p = \infty$ conclude that f is continuous.

Solution. Let $p \geq 1$. Observe that for $x \in [0, 1)$, we have $|x| \leq |x|^p$. Hence, it follows that

$$0 \leq \int_{\mathbb{T}} |S_n f - g| \leq \int_{\mathbb{T}} |S_n f - g|^p.$$

As we are given that $\int_{\mathbb{T}} |S_n f - g|^p \to 0$ as $n \to \infty$ it follows that $\int_{\mathbb{T}} |S_n f - g| \to 0$ in $L^1(\mathbb{T})$. Now, from the triangle inequality of L_1, we have that

$$|f - g|_{L_1} \leq |f - \sigma_n f|_{L_1} + |\sigma_n f - g|_{L_1},$$

for all n. Recall that convergence of cesaro sum is more inclusive, thus implied by the convergence of the original sequence. Hence, as $|S_n f - g|$ goes to 0, we have $|\sigma_f - g|$ going to 0 as $n \to \infty$. Furthermore, since $\{\sigma_n\}$ is an approximate identity, we have that $|\sigma f - f|$ term going to 0. Hence, it follows that $|f - g|_{L_1} = 0$, which implies that $f = g$ almost everywhere. For the case when $p = \infty$, as we have $\{S_n f\}$ is a sequence of continuous function, by the convergence in supnorm, we have g is continuous. □

Question 6.

Problem 1.9 Show that

$$\|f * g\|_{L^2(\mathbb{T})}^2 \leq \|f * f\|_{L^2(\mathbb{T})} \|g * g\|_{L^2(\mathbb{T})}$$

for all $f, g \in L^2(\mathbb{T})$.

Solution. As we have $f, g \in L^2(\mathbb{T})$, by Corollary 1.6, the given inequality is equivalent to

$$\sum_{n\in\mathbb{Z}} |\widehat{f * g}(n)|^2 \leq \sqrt{\sum_{n\in\mathbb{Z}} |\widehat{f * f}(n)|^2} \sqrt{\sum_{n\in\mathbb{Z}} |\widehat{g * g}(n)|^2}.$$

Since $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$, the above inequality is again equivalent to

$$\Big(\sum_{n\in\mathbb{Z}} |\hat{f}(n)\hat{g}(n)|^2\Big)^2 \leq \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^4 \sum_{n\in\mathbb{Z}} |\hat{g}(n)|^4.$$

Expanding the LHS of the desired inequality yields

$$\begin{aligned}
\Big(\sum_{n\in\mathbb{Z}}|\hat{f}(n)\hat{g}(n)|^2\Big)^2 &= \sum_{n\in\mathbb{Z}}|\hat{f}(n)\hat{g}(n)|^4+\sum_{n>m}2|\hat{f}(n)\hat{f}(m)\hat{g}(n)\hat{g}(n)|^2\\
&\le \sum_{n\in\mathbb{Z}}|\hat{f}(n)\hat{g}(n)|^4+\sum_{n>m}|\hat{f}(n)\hat{g}(n)|^4+|\hat{f}(m)\hat{g}(m)|^4\\
&= \sum_{n\in\mathbb{Z}}|\hat{f}(n)|^4\sum_{n\in\mathbb{Z}}|\hat{g}(n)|^4,
\end{aligned}$$

where the last inequality holds by the Cauchy-Schwarz inequality on the inner product space of $l^2(\mathbb{T})$.

$\square$

Question Extra.

Exercise 1.4. Let $\{c_n\}_{n\in\mathbb{Z}}$ be an arbitrary sequence of complex numbers and associate to it formally a Fourier series

$$f(x)\sim\sum_n c_n e(nx)$$

Show that there exists $\mu\in\mathcal{M}(\mathbb{T})$ with the property that $\hat{\mu}(n)=c_n$ for all $n\in\mathbb{Z}$ if and only if $\{\sigma_n f\}_{n\geqslant 1}$ is bounded in $\mathcal{M}(\mathbb{T})$. Discuss the case of $L^p(\mathbb{T})$ with $1\leqslant p<\infty$ and $C(\mathbb{T})$ as well.

Solution.