Harmonic Analysis:
Problem Set II

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Abstract

This work contains solutions to the problem set III of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Let α be any irrational number.

   (a) Show that for every trigonometric polynomial P

   $$\frac{1}{N}\sum_{n=1}^{N} P(n\alpha) \to \int_0^1 P(x)\,\mathrm{d}x.$$

   (b) Show that for every $f \in C(\mathbb{T})$

   $$\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) \to \int_0^1 f(x)\,\mathrm{d}x.$$

   (c) Show that the conclusion of (b) continues to hold for every Riemann integrable function on $\mathbb{T}$. What about Lebesgue integrable functions?

Solution.

Question 2.

2. Let $\mathcal{T}_n$ denote the linear space of trigonometric polynomials of degree up to n and

$$E_n(f) := \inf_{P \in \mathcal{T}_n} \|f - P\|_2 = \|f - S_N f\|_2 = \left(\sum_{|k|>n} |\widehat{f}(k)|^2 \right)^{1/2}.$$

Let $0 < \alpha < 1$. Show that $E_n(f) \lesssim n^{-\alpha}$ if and only if $\omega_f(\delta)_{L^2} \lesssim \delta^\alpha$ where

$$\omega_f(\delta)_{L^2} := \sup_{|h| \le \delta} \|f - f(\cdot - h)\|_2.$$

This class of functions is called $\mathrm{Lip}_{\alpha, L^2}(\mathbb{T})$.

(Hint: Use the dyadic decomposition trick we employed in class before.)

Solution.

Question 3.

3. Solve Exercises 2.2, 2.3, 2.4 in Muscalu & Schlag.

Solution. $\square$

Question 4.

4. Show that if a series $\sum a_n$ is Cesàro summable to A, then it is also Abel summable to A. In other words, show that

$$\lim_{N \to \infty} \sum_{n=-\infty}^{\infty} \left(1 - \frac{|n|}{N}\right)_+ a_n = A \text{ implies } \lim_{r \to 1^-} \sum_{n=-\infty}^{\infty} r^{|n|} a_n = A.$$

The reverse implication does not hold, however. Give a counter-example.
Can you generalize this result to series in arbitrary normed spaces?

(Hint: Note that you may first reduce the problem to one-sided series. Note also that the context of this problem is more general than summability of Fourier series.)

Solution.

$\square$