
Harmonic Analysis: Problem Set IV

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set IV of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Let H denote the Hilbert transform defined for $f \in L^2(\mathbb{T})$ by

$$Hf = \sum_{n \neq 0} -i \operatorname{sign}(n) \hat{f}(n) e_n. \quad (1)$$

Let $1 < p < \infty$. Explain why knowing L^p -boundedness of H on the space of trigonometric polynomials allows us to define it uniquely as a bounded operator on the whole of $L^p(\mathbb{T})$ and why the resulting operator satisfies, for all $f \in L^p(\mathbb{T})$, the relation

$$\widehat{Hf}(n) = -i \operatorname{sign}(n) \hat{f}(n), \quad n \in \mathbb{Z}, \quad (2)$$

with the convention $\operatorname{sign}(0) = 0$.

Show also that for any $f \in L^p(\mathbb{T})$, we have

$$H^2 f = -f + \hat{f}(0). \quad (3)$$

Solution.

(a) Firstly, note that H is a bounded linear operator on the space of trig polynomials for any p by assumption. For any $f \in L^p$, by density of trig polynomials, there exists $\{f_n\}$ such that $f_n \xrightarrow{L^p} f$. Observe that by linearity and boundedness of H on the space of trig polynomials, we have

$$\|Hf_n - Hf_m\|_p = \|H(f_n - f_m)\|_p \leq C_p \|f_n - f_m\|_p,$$

where C_p is the operator bound on the space of trig polynomials. By the above estimate, $\{Hf_n\}$ is Cauchy in L^p , thus convergent by completeness of L^p . Therefore, we can define Hf as

$$Hf = \lim_{n \rightarrow \infty} Hf_n.$$

We first show that the extension is well-defined. Let $f \in L^p(\mathbb{T})$, and consider $\{g_n\}$ and $\{h_n\}$ trig polynomials such that they converge to f in L^p . Analogous the previous estimate, we have

$$\|Hg_n - Hh_n\|_p = \|H(g_n - h_n)\|_p \leq C_p \|g_n - h_n\|_p \rightarrow 0,$$

as $n \rightarrow \infty$. Therefore, $\lim_{n \rightarrow \infty} Hg_n = \lim_{n \rightarrow \infty} Hh_n$ and the extension is well-defined. Similarly, it follows that the operator is linear and bounded on the whole L^p . We now argue the uniqueness of the operator. Let H_1, H_2 be bounded linear operators on $L^p(\mathbb{T})$ that agree on the space of trig polynomials. As the trig polynomials are dense in $L^p(\mathbb{T})$, we have that $H_1 = H_2$ on the entire $L^p(\mathbb{T})$, by the continuity of H_1 and H_2 . Therefore, the defined operator is unique.

Let $f \in L^p(\mathbb{T})$, and $\{p_n\}$ be trig polynomials such that $p_n \xrightarrow{L^p} f$, as $n \rightarrow \infty$. Since

$$\int_{\mathbb{T}} |f_n - f| |e^{2\pi i n x}| dx \leq \|f_n - f\|_p,$$

we have

$$\widehat{f_k}(n) \rightarrow \widehat{f}(n) \text{ as } k \rightarrow \infty.$$

It follows that

$$\begin{aligned} \widehat{Hf}(n) &= \int Hf e^{-2\pi i n \theta} d\theta = \lim_{k \rightarrow \infty} \int Hf_k e^{-2\pi i n \theta} d\theta \\ &= \lim_{k \rightarrow \infty} \widehat{Hf_k}(n) = -i \operatorname{sgn}(n) \widehat{f}(n). \end{aligned}$$

(b) Applying the above result twice to $H^2 f$ yields

$$\widehat{H^2 f}(n) = \begin{cases} -\widehat{f}(n) & \text{if } n \neq 0 \\ 0 & \text{if } n = 0, \end{cases}$$

Since,

$$-\widehat{f + \widehat{f}(0)}(n) = \begin{cases} -\widehat{f}(n) & \text{if } n \neq 0 \\ 0 & \text{if } n = 0, \end{cases}$$

it follows that, for any $n \in \mathbb{Z}$,

$$\widehat{H^2 f}(n) = -\widehat{f + \widehat{f}(0)}(n),$$

which by uniqueness of fourier coefficients implies

$$H^2 f = -f + \widehat{f}(0),$$

as required. □

Question 2.

2. Let again H denote the Hilbert transform on the torus.

- (a) Let f be any real-valued trigonometric polynomial. Using the identity $(u+iv)^2 = u^2 - v^2 + i(2uv)$, argue that

$$H(f^2 - (Hf)^2) = 2fHf. \quad (4)$$

Deduce that

$$(Hf)^2 = f^2 - \widehat{f}(0)^2 + 2H(fHf). \quad (5)$$

- (b) Show that (5) continues to hold for any complex-valued trigonometric polynomial.
(Hint: Expand out $[H(f+ig)]^2$. In order to handle the cross-terms, utilize the algebraic identity $[H(f+g)]^2 - [H(f-g)]^2 = 4(Hf)(Hg)$.)

Solution.

(a) By the given identity, we have

$$(f + iHf)^2 = f^2 - (Hf)^2 + i(2fHf),$$

which by linearity of H implies that

$$H(f^2 - (Hf)^2) = H((f + iHf)^2) - H(i2fHf).$$

Therefore, to prove (4), it suffices to show that

$$\widehat{H((f + iHf)^2)}(n) = \widehat{2(fHf + iH(fHf))}(n),$$

for any $n \in \mathbb{Z}$. We compute

$$f + iHf = \hat{f}(0) + 2 \sum_{n>0} \hat{f}(n) e^{2\pi i n \theta},$$

and

$$\widehat{f + iHf}(n) = \begin{cases} 2\hat{f}(n) & \text{if } n > 0 \\ \hat{f}(0) & \text{if } n = 0, \\ 0 & \text{if } n < 0 \end{cases}$$

so

$$\begin{aligned} \widehat{(f + iHf)^2}(n) &= \sum_{m \in \mathbb{Z}} \widehat{f + iHf}(n-m) \widehat{f + iHf}(m) \\ &= \begin{cases} \sum_{m=1}^n 4\hat{f}(n-m)\hat{f}(m) & \text{if } n > 0 \\ \hat{f}(0)^2 & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases} \end{aligned}$$

Thus

$$\widehat{H(f + iHf)}(n) = \begin{cases} -4i \sum_{m=1}^n \hat{f}(n-m)\hat{f}(m) & \text{if } n > 0 \\ 0 & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

We again compute

$$\begin{aligned} \widehat{fHf}(n) &= \sum_{m \in \mathbb{Z}} \hat{f}(n-m)(-i)\text{sgn}(m)\hat{f}(m) \\ &= -i \sum_{m \in \mathbb{Z}} \hat{f}(n-m)\hat{f}(m), \end{aligned}$$

so

$$\widehat{2(fHf + iH(fHf))}(n) = \begin{cases} 4\widehat{fHf}(n) = -4i \sum_{m=1}^n \hat{f}(n-m)\hat{f}(m) & \text{if } n > 0 \\ 0 & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Therefore, we have

$$\widehat{H(f + iHf)^2}(n) = \widehat{2fHf + iH(fHf)}(n),$$

for any $n \in \mathbb{Z}$ as required. Now, from (4) we deduce (5). Applying H to both sides of (4) yields

$$2H(fHf) = H^2(f^2 - (Hf)^2) = -f^2 + (Hf)^2 + \widehat{f^2 - (Hf)^2}(0).$$

Since

$$\begin{aligned} \widehat{f^2 - (Hf)^2}(0) &= \sum_{m \in \mathbb{Z}} \hat{f}(-m)\hat{f}(m) + \text{sgn}(-m)\text{sgn}(m)\hat{f}(-m)\hat{f}(m) \\ &= \hat{f}(0)^2, \end{aligned}$$

we obtain (5) as desired. □

(b) Let $Z = f + ig$ be any complex-valued trig polynomial, where f and g are the real-valued trig polynomials. Observe that

$$(HF)^2 = (Hf)^2 + 2iHfHg - (Hg)^2$$

and

$$H(FHF) = H((f + ig)(Hf + iHg)) = H(fHf - gHg - i(fHg + gHf)).$$

Therefore, it suffices to show that

$$HfHg = fg - \hat{f}(0)\hat{g}(0) + H(fHg + gHf).$$

Expanding the RHS of the given identity gives

$$\begin{aligned}
4HfHg &= (H(f+g))^2 - (H(f-g))^2 \\
&= (f+g)^2 - (f-g)^2 + 2H((f-g)(Hf+Hg)) - 2H((f-g)(Hf-Hg)) \\
&= (\hat{f}(0) + \hat{g}(0))^2 + (\hat{f}(0)^2 - \hat{g}(0)^2) \\
&= 4fg - 4\hat{f}(0)\hat{g}(0) + 4H(fHg + gHf).
\end{aligned}$$

Dividing the above identity by 4 gives the desired identity as desired. \square

Question 3.

3. For any $1 < p < \infty$, let C_p denote the supremum of $\|Hf\|_p$ over all trigonometric polynomials f of p -norm at most 1. Using the relation (5), show that

$$\|Hf\|_{2p}^2 \leq 2\|f\|_{2p}^2 + 2C_p\|f\|_{2p}\|Hf\|_{2p}.$$

Deduce that

$$C_{2p} \leq C_p + \sqrt{C_p^2 + 2}.$$

Hence, knowing $C_2 = 1$, it follows that C_4, C_8, \dots are all finite. With interpolation and duality, this provides another route to showing that $C_p < \infty$ for all $1 < p < \infty$.

Solution.

Substituting the second identity from the problem 2 with, via Minkowski's inequality, we obtain

$$\begin{aligned}
\|Hf\|_{2p}^2 &= \|Hf^2\|_p \leq \|f^2\|_p + \|\hat{f}(0)^2\|_p + \|2H(fHf)\|_p \\
&\leq \|f\|_p^2 + |\hat{f}(0)|^2 + \|2H(fHf)\|_p,
\end{aligned}$$

where we have simply used the fact that a constant function in $L^p(\mathbb{T})$ attains the modulus of the value for the norm of any $p \in (1, \infty)$. Now, by monotonicity of $L^p(\mathbb{T})$ norms, it follows that

$$|\hat{f}(0)|^2 = \left| \int_{\mathbb{T}} f(x) dx \right|^2 \leq \left(\int_{\mathbb{T}} |f(x)| dx \right)^2 = \|f\|_1^2 \leq \|f\|_{2p}^2.$$

Furthermore, by generalized Holder's inequality, we have

$$\|2H(fHf)\|_p \leq 2C_p\|fHf\|_p \leq 2C_p\|f\|_{2p}\|Hf\|_{2p}.$$

Therefore, combining the two estimates gives

$$\|Hf\|_{2p}^2 \leq 2\|f\|_{2p}^2 + 2C_p\|f\|_{2p}\|Hf\|_{2p},$$

as required. Since

$$\|Hf\|_{2p}^2 - 2\|f\|_{2p}^2 - 2C_p\|f\|_{2p}\|Hf\|_{2p} \leq 0,$$

utilizing the quadratic formula, we have

$$\|Hf\|_{2p}^2 \leq \frac{2C_p\|f\|_{2p} + \sqrt{4C_p^2\|f\|_{2p}^2 + 2\|Hf\|_{2p}^2}}{2},$$

which can be simplified to

$$\|Hf\|_{2p} \leq C_p\|f\|_{2p} + \sqrt{C_p^2 + 2}\|f\|_{2p},$$

which then implies

$$C_{2p} \leq C_p + \sqrt{C_p^2 + 2},$$

as required. As $C_2 = 1$, by the above inequality, we obtain that $C_{2^n} < \infty$ for $n \geq 1$. Therefore, by Riesz-Thorin Interpolation theorem and duality, we have $C_p < \infty$ for any $p \in (1, \infty)$. \square

4. Let $1 \leq p \leq \infty$ and $\phi \in L^p(\mathbb{T})$. Prove the following generalization of Young's inequality: For any $1 \leq r \leq p'$ and $f \in L^r(\mathbb{T})$, the convolution $\phi * f$ belongs to $L^s(\mathbb{T})$ and $\|\phi * f\|_s \leq \|\phi\|_p \|f\|_r$, where $p^{-1} + r^{-1} = s^{-1} + 1$.
(Hint: Riesz-Thorin.)

Question 4.

Solution.

In this solution, all integral are taken over the torus. We view $\phi * f$ as an operator, which we denote as T . We first show that the statement holds for the case $r = 1$, and $r = q$. Suppose $r = 1$, so $s = p$. By general Minkowski inequality, we obtain

$$\begin{aligned} \|Tf\|_p &= \left(\int \left| \int \phi(x-y)f(y)dy \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \int \left(\int |\phi(x-y)|^p |f(y)|^p dx \right)^{\frac{1}{p}} dy = \|\phi\|_p \|f\|_1 < \infty, \end{aligned}$$

which implies that for $T : L^1 \rightarrow L^p$,

$$\|T\|_{1 \rightarrow p} \leq \|\phi\|_p,$$

thus bounded by assumption. Now, suppose $r = q$, so $s = \infty$. By Holder inequality, we obtain

$$\|Tf\|_\infty \leq \sup_{x \in \mathbb{T}} |f(x)\phi(x-y)| \leq \|\phi f\|_1 \leq \|\phi\|_p \|f\|_q,$$

which implies that for $T : L^q \rightarrow L^\infty$,

$$\|T\|_{q \rightarrow \infty} \leq \|\phi\|_p,$$

thus bounded by assumption.

Hence, by Riesz-Thorin, we conclude that, for $1 \leq r \leq q$ and T maps L^r into L^s with

$$\|T\|_{r \rightarrow s} \leq \|T\|_{1 \rightarrow p}^\theta \|T\|_{q \rightarrow \infty}^{1-\theta} \leq \|\phi\|_p,$$

where $r^{-1} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}$, $s^{-1} = \frac{\theta}{s_1} + \frac{1-\theta}{s_2}$ with $(1, q)$ and (p, ∞) being interpolation points respectively. \square

Question 5.

5. Let (X, μ) be a measure space, $1 \leq p < \infty$, and consider the space $\text{weak-}L^p(X, \mu)$, which we defined to be the set of measurable functions on X such that

$$[f]_{w,p} := \sup_{\alpha > 0} \alpha [\lambda_f(\alpha)]^{1/p} < \infty,$$

where $\lambda_f(\alpha) := \mu\{x : |f(x)| > \alpha\}$. As we saw, if $f \in L^p$, then $[f]_{w,p} \leq \|f\|_p$. Hence $L^p \subset \text{weak-}L^p$.

- Show that $\text{weak-}L^p$ is a linear space.
- Show that for all $\beta \in \mathbb{C}$, we have $[\beta f]_{w,p} = |\beta| [f]_{w,p}$.
- Show that $[f]_{w,p} = 0$ if and only if $f = 0$ μ -a.e. in X .
- Show that the triangle inequality can fail for $[\cdot]_{w,p}$ for all p . Hence $[\cdot]_{w,p}$ is not a norm in general.
(Hint: It happens even when X consists of just 2 points.)
- A *quasi-norm* $[\cdot]$ satisfies all the properties of a norm except the triangle inequality is replaced by $[f + g] \leq C([f] + [g])$. Show that $[\cdot]_{w,p}$ is a quasi-norm with $C \leq 2$.

Solution.

(a) If $f = 0$ μ -a.e in X , then by (c), $[f] = 0$, so $f \in \text{weak-}L^p$. Now, let $\gamma \in \mathbb{C}$, and $f \in \text{weak-}L^p$. By (b), we see that $\gamma f \in \text{weak-}L^p$. Furthermore, let $f, g \in \text{weak-}L^p$. By (e), it follows that

$$[f + g] \leq 2([f] + [g]) < \infty,$$

which implies that $f + g \in \text{weak-}L^p$. Therefore, the space is a linear space. All the constituents will be proven in the later parts.

(b) Let $\beta \in \mathbb{C}$. If $\beta = 0$, then $[\beta f] = 0$, thus the equality holds trivially. Suppose $\beta \neq 0$. By the definition of weak- L^p , we have

$$\begin{aligned} [\beta f] &= \sup_{\alpha > 0} \alpha [\mu\{|\beta f| > \alpha\}]^{\frac{1}{p}} = |\beta| \sup_{\alpha > 0} \frac{\alpha}{|\beta|} [\mu\{|f| > \frac{\alpha}{|\beta|}\}]^{\frac{1}{p}} \\ &= |\beta| \sup_{\alpha > 0} \alpha [\mu\{|f| > \alpha\}]^{\frac{1}{p}} = |\beta| [f] \end{aligned}$$

(c) Let $f = 0$ μ -a.e. in X . By monotonicity of measure, we have

$$\mu\{|f| > \alpha\} = 0,$$

for any $\alpha > 0$, which implies that $[f]_{w,p} = 0$. Conversely, let $[f]_{w,p} = 0$. As \mathbb{C} is an integral domain, this implies that for any $n \in \mathbb{N}$,

$$\mu\{|f| > \frac{1}{n}\} = 0,$$

so

$$\mu\{|f| > 0\} = \mu \bigcup_{n=1}^{\infty} \{|f| > \frac{1}{n}\} \leq \sum_{n=1}^{\infty} \mu\{|f| > \frac{1}{n}\} = 0.$$

Therefore,

$$\mu\{|f| = 0\} = 1,$$

which completes the proof.

(d) Let $X = \{x_0, x_1\}$, equipped with the uniform measure with $\mu(X) = 1$. Let $0 < a$, and $p \in [1, \infty)$. Define $f, g : X \rightarrow \mathbb{C}$ by

$$f(x) = \begin{cases} a & \text{if } x = x_0 \\ 2a & \text{if } x = x_1, \end{cases}$$

and

$$g(x) = \begin{cases} k - a & \text{if } x = x_0 \\ k - 2a & \text{if } x = x_1, \end{cases}$$

so

$$f + g(x) = \begin{cases} k & \text{if } x = x_0 \\ k & \text{if } x = x_1, \end{cases}$$

with some $k > 2a$ to be determined. It follows that

$$[f] = 2a^{\frac{1}{p}}, \quad [g] = k - 2a, \quad [f + g] = k,$$

for k chosen large enough such that $(\frac{1}{2})^{\frac{1}{p}}(k - 2a) \leq k - a$ holds. With the above equality, granted with the appropriate choice for k , dependent on p , in order to violate the triangle inequality, we must have

$$2a(\frac{1}{2})^{\frac{1}{p}} + (k - 2a) < k.$$

However, the above inequality is equivalent to

$$2a((\frac{1}{2})^{\frac{1}{p}} - 1) < 0,$$

showing that the construction is valid for any $p \in [1, \infty)$. We have shown that $[\cdot]$ is not a norm in general.

(e) Let $p \in [1, \infty)$. If $[f] = \infty$ or $[g] = \infty$, then the inequality holds trivially. Suppose $f, g \in \text{weak-}L^p$. Observe that, for any $\alpha > 0$,

$$\mu\{|f+g| > \alpha\} \leq \mu\{|f| + |g| > \alpha\} \leq \mu\{|f| > \frac{\alpha}{2}\} + \mu\{|g| > \frac{\alpha}{2}\},$$

which implies that

$$\begin{aligned} [f+g] &= \sup_{\alpha>0} \alpha [\lambda_{f+g}(\alpha)]^{\frac{1}{p}} \leq \sup_{\alpha>0} \alpha \left([\lambda_f(\frac{\alpha}{2})]^{\frac{1}{p}} + [\lambda_g(\frac{\alpha}{2})]^{\frac{1}{p}} \right) \\ &\leq 2 \left(\sup_{\alpha>0} \alpha [\lambda_f(\alpha)]^{\frac{1}{p}} + \sup_{\alpha>0} \alpha [\lambda_g(\alpha)]^{\frac{1}{p}} \right) = 2([f] + [g]), \end{aligned}$$

as required. Hence, we have shown that $[\cdot]$ is a quasi-norm with $C \leq 2$. \square

Question 6.

6. Assume the notation in problem 5.

- (a) Show that $\lambda_f(\alpha)$ is a decreasing, right continuous function of α .
- (b) Show that $f \in L^p$ if and only if $\sum_{n=-\infty}^{\infty} 2^{np} \lambda_f(2^n) < \infty$.
- (c) Conclude that if $f \in L^p$, then $\lambda_f(\alpha) = o(\alpha^{-p})$, both as $\alpha \rightarrow 0$ and $\alpha \rightarrow \infty$. This improves the Markov-Chebyshev inequality.

Solution.

(a) Let $0 < \alpha < \beta$. By monotonicity of measure, we have

$$\mu\{|f| > \alpha\} \geq \mu\{|f| > \beta\}.$$

Hence, λ_f is decreasing. Let $\{\alpha_n\}$ be a sequence from (α, ∞) such that $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$. Choose a subsequence from the sequence, which is strictly decreasing, denoted as $\{\alpha_{n_k}\}$. Now, by continuity of measure, it follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} \lambda_f(\alpha_{n_k}) &= \lim_{k \rightarrow \infty} \mu\{|f| > \alpha_{n_k}\} = \mu \bigcup_{k=1}^{\infty} \{|f| > \alpha_{n_k}\} \\ &= \mu\{|f| > \alpha\} = \lambda_f(\alpha). \end{aligned}$$

Hence, $\lambda_f(\alpha_n) \rightarrow \lambda_f(\alpha)$ as $n \rightarrow \infty$ and λ_f is right continuous.

(b) Assume $f \in L^p$. We have the following result from class:

$$f \in L^p \implies \|f\|_p^p = \int_0^{\infty} p \alpha^{p-1} \lambda_f(\alpha) d\alpha.$$

Therefore, by DCT, it follows that

$$\begin{aligned} \|f\|_p^p &= \int_0^{\infty} p \alpha^{p-1} \lambda_f(\alpha) d\alpha \\ &= \int_0^{\infty} \sum_{n=-\infty}^{\infty} p \alpha^{p-1} \lambda_f(\alpha) \mathbf{1}_{(2^{n-1}, 2^n]}(\alpha) d\alpha \\ &= \lim_{N \rightarrow \infty} \sum_{n=-N}^N \int_0^{\infty} p \alpha^{p-1} \lambda_f(\alpha) \mathbf{1}_{(2^{n-1}, 2^n]}(\alpha) d\alpha \\ &\geq \lim_{N \rightarrow \infty} \sum_{n=-N}^N p (2^{n-1})^{p-1} \lambda_f(2^n) 2^{n-1} = \frac{p}{2^p} \sum_{n=-\infty}^{\infty} 2^{np} \lambda_f(2^n). \end{aligned}$$

Hence, $\sum_{n=-\infty}^{\infty} 2^{np} \lambda_f(2^n) < \infty$. Conversely, suppose that $\sum_{n=-\infty}^{\infty} 2^{np} \lambda_f(2^n) < \infty$. Set

$$A_n = \{2^n \leq |f| \leq 2^{n+1}\}.$$

By MCT, it follows that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} 2^{np} \lambda_f(2^n) &\leq \sum_{n=-\infty}^{\infty} 2^{np} \mu(A_n) = 2^{-p} \lim_{N \rightarrow \infty} \int_X \sum_{n=-N}^N 2^{(n+1)p} 1_{A_n} d\mu \\ &= 2^{-p} \int_X \sum_{n=-\infty}^{\infty} 2^{(n+1)p} 1_{A_n} d\mu \\ &\leq 2^{-p} \int_X |f|^p d\mu, \end{aligned}$$

which implies that $f \in L^p$.

(c) Let $p \in [1, \infty)$. Let f be a simple function in L^p such that $f = \sum_{i=1}^k a_i 1_{E_i}$. As $L^p \subset \text{weak-}L^p$, for any $\alpha \in (0, \infty)$, we have $\lambda_f(\alpha) < \infty$. Furthermore, it follows that

$$\lambda_f(\alpha) = 0, \text{ if } \alpha \geq \max\{|a_i|\},$$

and, by right continuity of λ_f ,

$$\lambda_f(\alpha) = \lim_{\alpha' \rightarrow 0} \lambda_f(\alpha') \text{ if } \alpha < \min\{|a_i|\}.$$

Without loss of generality, assume that $a_i \neq 0$ for all i . By the above observation, it follows that

$$\lim_{\alpha \rightarrow \infty} \alpha^p \lambda_f(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \rightarrow 0} \alpha^p \lambda_f(\alpha) = 0,$$

so

$$\lambda_f(\alpha) = o(\alpha^{-p})$$

for both $\alpha \rightarrow \infty$ and $\alpha \rightarrow 0$ limits. Now, we use a standard density argument to show that the claim holds true for any $f \in L^p$. Let $f \in L^p$. Fix $\epsilon > 0$. As simple functions are dense in L^p , there exists a simple function $s \in L^p$ such that

$$\|f - s\|_p < \epsilon$$

Then

$$\begin{aligned} 0 \leq \alpha^p \lambda_f(\alpha) &= \alpha^p \lambda_{f-s}\left(\frac{\alpha}{2}\right) + \alpha^p \lambda_s\left(\frac{\alpha}{2}\right) \\ &\leq 2^p \|f - s\|_p^p + 2^p \left(\frac{\alpha}{2}\right)^p \lambda_s\left(\frac{\alpha}{2}\right) < C\epsilon, \end{aligned}$$

for some constant C provided that α is small or large enough. Therefore,

$$\lambda_f(\alpha) = o(\alpha^{-p}),$$

for both $\alpha \rightarrow \infty$ and $\alpha \rightarrow 0$ limits as required. \square