Harmonic Analysis:
Final Exam

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains a solution to the Final Exam of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Watching these videos, you will see that the synthesis operation is quite intuitive, but the analysis operation (which was the very reason Michelson built this machine) requires knowing a bit more mathematics (shall we say, *harmonic analysis*). On pages 98 and 99 of their book you will find the basic mathematical explanation (but watch the videos first!). In this problem, we will make things a bit more rigorous.

(a) We defined the Discrete Fourier Transform (DFT) on our very first class (as the Fourier transform on $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$). Namely, if $x := (x_n)_{n\in\mathbb{Z}}$ is an N-periodic sequence seen as a function on $\mathbb{Z}_N$, then its DFT $\widehat{x}$ is the N-periodic sequence given by

$$\widehat{x}_k := \sum_{n\in\mathbb{Z}_N} x_n e^{-2\pi i n k/N}, \quad k \in \mathbb{Z},$$

which we may also identify with a function on $\mathbb{Z}_N$. Then we have the inversion formula

$$x_n = \frac{1}{N} \sum_{k\in\mathbb{Z}_N} \widehat{x}_k e^{2\pi i n k/N}, \quad n \in \mathbb{Z},$$

(which you can easily verify even if it's new to you).

Let P be a trigonometric polynomial of degree N, i.e., $P \in \operatorname{span}\{e_k : -N \leq k \leq N\}$. Define $p := (p_n)_{n\in\mathbb{Z}}$ to be the $(2N+1)$-periodic sequence given by $p_n := \frac{1}{2N+1} P(\frac{n}{2N+1})$, $n \in \mathbb{Z}$, and let $\widehat{p}$ be its DFT (as a $(2N+1)$-periodic sequence). Show that $\widehat{p}_k = \widehat{P}(k)$ for all $|k| \leq N$. (Here $\widehat{P}(k)$ is the kth Fourier series coefficient of P.) What happens for $|k| > N$?

Solution.
(a) Let P be a trig polynomial defined on $\mathbb{T}$ of degree N, i.e.

$$P = \sum_{|k|\leq N} a_k e_k,$$

where a_ks are the complex coefficients. Suppose $|k| \leq N$. We trivially know that $\hat{P}(k) = a_k$. We compute

$$\begin{aligned}
\widehat{p_k} &= \sum_{n\in\mathbb{Z}_{2N+1}} p_n e^{\frac{-2\pi i n k}{2N+1}} = \sum_{n\in\mathbb{Z}_{2N+1}} \frac{1}{2N+1} P(\frac{n}{2N+1}) e^{-\frac{2\pi i n k}{2N+1}} \\
&= \frac{1}{2N+1} \sum_{n\in\mathbb{Z}_{2N+1}} \left(\sum_{|l|\leq N} a_l e_l(\frac{n}{2N+1}) \right) e^{-\frac{2\pi i n k}{2N+1}} = \frac{1}{2N+1} \sum_{n\in\mathbb{Z}_{2N+1}} \left(\sum_{|l|\leq N} a_l e^{\frac{2\pi i l n}{2N+1}} \right) e^{-\frac{2\pi i n k}{2N+1}} \\
&= \frac{1}{2N+1} \sum_{n\in\mathbb{Z}_{2N+1}} \left(\sum_{|l|\leq N} a_l e^{\frac{2\pi i (l-k) n}{2N+1}} \right) = \frac{1}{2N+1} \sum_{|l|\leq N} \left(a_l \sum_{n\in\mathbb{Z}_{2N+1}} e^{\frac{2\pi i (l-k) n}{2N+1}} \right) \\
&= \frac{1}{2N+1} \sum_{|l|\leq N; l\neq k} \left(a_l \sum_{n\in\mathbb{Z}_{2N+1}} e^{\frac{2\pi i (l-k) n}{2N+1}} \right) + \frac{1}{2N+1} \sum_{|l|\leq N; l=k} \left(a_l \sum_{n\in\mathbb{Z}_{2N+1}} e^{\frac{2\pi i (l-k) n}{2N+1}} \right) \\
&= 0 + \frac{1}{2N+1}(2N+1)a_k = a_k,
\end{aligned}$$

as the sum of any N-th root of unity is zero, thereby forcing the first term of the second last equation to be 0. For $|k| > N$, we trivially see that $\widehat{p_k} = 0$, and $\hat{P}(k) = 0$ as well. $\square$

Question 1-2.

(b) For any $F \in C(\mathbb{T})$, define $f := (f_n)_{n\in\mathbb{Z}}$ to be the $(2N+1)$-periodic sequence given by $f_n := \frac{1}{2N+1}F(\frac{n}{2N+1})$, $n \in \mathbb{Z}$, and let $\widehat{f}$ be its DFT (as a $(2N+1)$-periodic sequence). Show that for any trigonometric polynomial P of degree N, we have

$$\max_{|k|\leq N} |\widehat{f}_k - \widehat{F}(k)| \leq 2\|F - P\|_\infty.$$

(Hint: Use (a). In particular, write $F = (F - P) + P$ and $f = (f - p) + p$ where p is as in (a).)

(c) Part (b) says that the DFT of regular samples of F at $2N + 1$ points approximate the Fourier series coefficients of F to the same extent F can be approximated by a trigonometric polynomial of degree N. Now explain in rigorous terms how Michelson uses his machine, basically a harmonic "synthesizer," as a harmonic "analyzer" instead. Here you may also wish to allude to the Fourier transform on $\mathbb{Z}$ (also known as the "Discrete Time Fourier Transform (DTFT)") which maps any (absolutely summable) sequence $(g_n)_{n\in\mathbb{Z}}$ to a function on the torus

$$\widehat{g}(\xi) := \sum_{n\in\mathbb{Z}} g_n e^{-2\pi i n\xi}, \quad \xi \in \mathbb{T},$$

and the simple observation that the DFT of a periodic sequence is the same as the samples of the DTFT of its truncation to a period.

(d) Michelson wanted to build a new version of his analyzer with 1000 elements, but never did. How accurately would he be able to calculate the Fourier coefficients of a piecewise-linear continuous function using an analyzer with N elements?

Solution.

(b) Let $f \in C(\mathbb{T})$, and P be a trig polynomial of degree N. Define as before $f \triangleq (f_n)_{n\in\mathbb{Z}}$ to be the $(2N+1)$-periodic sequence given by

$$f_n \triangleq \frac{1}{2N+1} F(\frac{n}{2N+1}).$$

With $F = (F-P)+P$ and $f = (f-p)+p$, by the result of (a), for any $|k| \leq N$, we see that

$$\begin{aligned} |\widehat{f}_k - \hat{F}_k| &= |\widehat{\{(f-p)+p\}}_k - \widehat{(F-P)+P}(k)| = |\widehat{(f-p)}_k - \widehat{(F-P)}(k) + \hat{p}_k - \hat{P}(k)| \\ &\leq |\widehat{(f-p)}_k| + |\widehat{(F-P)}(k)| + |\hat{p}_k - \hat{P}(k)| = |\widehat{(f-p)}_k| + |\widehat{(F-P)}(k)|. \quad (1) \end{aligned}$$

Now, in view of (1), and the fact that N is finite, it suffices to show that for any $|k| \leq N$,

$$|\widehat{(f-p)}_k| \leq ||F-P||_\infty \text{ and } |\widehat{(F-P)}(k)| \leq ||F-P||_\infty.$$

Now, for $|k| \leq N$, the first inequality follows, as

$$\begin{aligned} |\widehat{(f-p)}_k| &= |\sum_{n\in\mathbb{Z}_{2N+1}} (f-p)_n e^{\frac{-2\pi i n k}{2N+1}}| \\ &= |\frac{1}{2N+1} \sum_{n\in\mathbb{Z}_{2N+1}} (F(\frac{n}{2N+1}) - P(\frac{n}{2N+1}))e^{\frac{-2\pi i n k}{2N+1}}| \\ &\leq \frac{1}{2N+1} \sum_{n\in\mathbb{Z}_{2N+1}} |F(\frac{n}{2N+1}) - P(\frac{n}{2N+1})| = ||F-P||_\infty. \end{aligned}$$

Likewise, for $|k| \leq N$, the second inequality, as $F - P \in C(\mathbb{T})$, and

$$\begin{aligned} |\widehat{F-P}(k)| &= \left| \int_{\mathbb{T}} (F-P)(t)e^{-ikt}dt \right| \leq \int_{\mathbb{T}} |(F-P)(t)|dt \\ &= ||F-P||_1 \leq ||F-P||_\infty. \end{aligned}$$

Therefore, we have that for any trig polynomial of degree N, P,

$$\max_{|k|\leq N} |\hat{f}_k - \hat{F}(k)| \quad \leq \quad 2||F-P||_\infty,$$

as required.

(c) To begin with, in the video, we can see a particular physical interpretation of the mathematical setup we have. F is the square wave and the f are the dots drawn on the square wave, which corresponds to the regular samples of F. Now, setting the rocker arms, which controls the coefficients of the cosine waves, to the values of f_n, we get the DFT by reading the output wave's discrete positions. The important distinction here is that, with the modeling assumption that the given data is an even function, Michelson managed to perform both the synthesis and analysis with the "syntehsis" (adding the cosine waves, multiplied by the appropriate coefficients) as the DFT and inversion of DFT are the same operations when restricted to cosines. This could be regarded further as a truncated DTFT as mentioned.

(d) Here, we want to see how close the discrete Fourier coefficients gets to the continuous Fourier coefficient as a function of N, which mathematically corresponds to the cardinality of the finite Abelian group under consideration. This is precisely the question of convergence of Riemann sum to an integral. Here, we are given that the function is piecewise continuous and linear, so by a well-known result which says the rate of convergence Riemann sum to Riemann integral of x^d is $\Theta(n^{-(1+d)}$, we see that with our assumption the rate of convergence is $\theta(\frac{1}{n^2})$. So, for 1000, we approximately have that the difference is bounded above by $\frac{1}{1000^2}$.

Question 2.

2. Let $\tau_h f = f(\cdot - h)$ be the translation operator for functions defined on $\mathbb{T}$.

   (a) Show that a bounded linear operator T on $L^2(\mathbb{T})$ commutes with translations (i.e., $T(\tau_h f) = \tau_h(Tf)$ for all $h \in \mathbb{T}$ and $f \in L^2(\mathbb{T})$) if and only if
$$Tf = \sum_{n\in\mathbb{Z}} \xi_n \widehat{f}(n) e_n$$
for some $\xi \in l^\infty(\mathbb{Z})$. Show also that $\|T\|_{2\to 2} = \|\xi\|_\infty$.
(Hint: Consider $T(e_n)$.)

   (b) Show that a bounded linear operator T on $L^1(\mathbb{T})$ commutes with translations if and only if $Tf = f * \mu$ for some $\mu \in M(\mathbb{T})$. Show also that $\|T\|_{1\to 1} = \|\mu\|_{M(\mathbb{T})}$.
(Hint: You may find the Fejer kernel useful.)

Solution.
(a) We first prove the equivalence for $\{e_n\}$. Fix $h \in \mathbb{T}$. Suppose that, for some $\xi \in l^\infty(\mathbb{Z})$, we have
$$Te_n = \sum_{k\in\mathbb{Z}} \xi_k \hat{e_n}(k) e_k,$$
for all n. As $\hat{e_n}(k) = 1$, if $k = n$ and 0 otherwise, it follows that, for all n,
$$Te_n = \xi_n e_n,$$
and, by linearity of T,
$$T(\tau_h(e_n)) = T(e^{2\pi i n(\cdot - h)}) = e^{2\pi i n h} T(e_n) = e^{-2\pi i n h} \xi_n e_n = \tau_h(T(e_n)).$$
Conversely, assume that $T(\tau_h e_n) = \tau_h T(e_n)$, for all n. Set
$$\{\xi_n\} = \{\widehat{Te_n}(n)\}.$$
We claim that the above construction gives the desired property. Firstly, $\{\xi_n\}$ is bounded, as $Te_n \in L^2 \subset L^1$ (we are on $\mathbb{T}$), so by Riemann-Lebesgue lemma,
$$\xi_n = \widehat{Te_n}(n) \to 0,$$
as $n \to \infty$, and $\{\xi_n\} \in l^\infty(\mathbb{Z})$. Now, it remains to be shown that, for all n,
$$Te_n = \sum_{k\in\mathbb{Z}} \widehat{Te_k}(k) \widehat{e_n}(k) e_k.$$
As T is bounded on L^2, it follows that
$$Te_n = \sum_{k\in\mathbb{Z}} \widehat{Te_n}(k) e_k,$$
and by the fact that T commutes with the translation we get
$$\sum_{k\in\mathbb{Z}} \widehat{Te_n}(k) e^{2\pi i (k-n)h} e^{2\pi i k(\cdot - h)} = \sum_{k\in\mathbb{Z}} \widehat{Te_n}(k) e^{2\pi i k(\cdot - h)},$$
so
$$\widehat{Te_n}(k) e^{2\pi i (k-n)h} = \widehat{Te_n}(k).$$
Now, the equality holds for all $h \in \mathbb{T}$, so we have that $\widehat{Te_n}(k) = 0$ for any $k \neq n$, and hence,
$$Te_n = \widehat{Te_n}(n) e_n,$$
as required.

Now, we argue that the equivalence can be extended to any $f \in L^2$. Suppose that the operator commutes with the translation operator for any e_n. Then, it follows that, for any $f \in L^2$,

$$\begin{aligned} T(\tau_h f) &= T(\tau_h \sum_{n\in\mathbb{Z}} \hat{f}(n)e_n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)T(\tau_h e_n) \\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n)\tau_h(Te_n) = \tau_h(Tf). \end{aligned}$$

The converse holds trivially. Now, suppose that the operator has the given description with respect to $\{e_n\}$. It follows that, for any $f \in L^2$,

$$Tf = T(\sum_{n\in\mathbb{Z}} \hat{f}(n)e_n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)T(e_n) = \sum_{n\in\mathbb{Z}} \xi_n \hat{f}(n)e_n.$$

The converse in this case also holds trivially. Thus, we have shown the desired equivalence.

We now argue that $||T||_{2\to 2} = ||\xi||_\infty$. By Parseval, we obtain

$$\begin{aligned} ||T||_{2\to 2} &= \sup_{||f||_2=1} ||Tf||_2 = \sup_{||f||_2=1} (\sum_{n\in\mathbb{Z}} |\xi_n \hat{f}(n)|^2)^{\frac{1}{2}} \\ &\leq \sup_{||f||_2=1} ||\xi||_\infty (\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2)^{\frac{1}{2}} = ||\xi||_\infty. \end{aligned}$$

Now, we construct $f \in L^2$, such that $||Tf|| = ||\xi||_\infty$. If $||\xi||_\infty = 0$, then, the equality immediately follows. If $||\xi||_\infty > 0$, as $\{\xi\}_\infty$ was shown to converge to 0, via Riemann-Lebesgue lemma, there exists a finite index N such that $|\xi_N| = ||\xi||_\infty$. Then, we have that

$$||Te_N|| = |\xi_N|,$$

so the equality holds.

(b) First, suppose that $Tf = f * \mu$ for some $\mu \in M(\mathbb{T})$. It follows that

$$T(\tau_h f) = (\tau_h f) * \mu = \tau_h(f * \mu) = \tau_h(Tf).$$

Now, conversely, suppose that T commutes with translations. Consider the Fejer kernel, denoted by $\{K_n\}$. Firstly, observe that as the Fejer kernel is an approximate identity, it follows that $\{TK_n\}$ are bounded on $M(\mathbb{T})$. By the weak-* compactness of $M(\mathbb{T})$, we can extract a subsequence $\{K_{n_l}\}$ such that $K_{n_l} \to \mu$ weakly, for some $\mu \in M(\mathbb{T})$. For notational convenience, we relabel the subsequence as $\{K_n\}$. Now, assume that T commutes with translations. It follows that, for any $f \in C(\mathbb{T})$, and $x \in \mathbb{T}$,

$$\begin{aligned} f * \mu(x) &= \lim_{n\to\infty} \int_{\mathbb{T}} f(t)T(K_n)(x-t)dt = \lim_{n\to\infty} \int_{\mathbb{T}} f(t)T(K_n(x-t))dt, \\ &= \lim_{n\to\infty} \int_{\mathbb{T}} T(f(t)(K_n)(x-t))dt, = \lim_{n\to\infty} T\left(\int_{\mathbb{T}} f(t)K_n(x-t)dt\right), \end{aligned}$$

which via boundedness of T and the fact that for any $f \in C(\mathbb{T})$,

$$\lim_{n\to\infty} (f * K_n)(x) = f(x),$$

implies that

$$f * \mu(x) = T\left(\lim_{n\to\infty} \int_{\mathbb{T}} f(t)K_n(x-t)dt\right) = Tf(x),$$

as required. Now, as $C(\mathbb{T})$ is dense in $L^1(\mathbb{T})$, via standard density argument, it follows that, for any $f \in L^1(\mathbb{T})$,

$$f * \mu = Tf.$$

By Young's inequality for convolution, it follows that

$$||T||_{1\to 1} = \sup_{||f||=1} ||f * \mu||_1 \leq \sup_{||f||=1} ||f||_1 ||\mu||_{M(\mathbb{T})} = ||\mu||_{M(\mathbb{T})}.$$

We now construct $f \in L^1$ such that $||Tf||_1 = ||\mu||_{M(\mathbb{T})}$. As $||u|| < \infty$, we can choose, a sequence of partition, whose sum of absolute value of the measures of each part, converges to the total variation norm. Now, consider the sequence of functions defined by by the sum of characteristic functions of the corresponding nth partition, denoted by f_n. By construction, it follows that

$$||Tf_n||_1 = ||f_n * \mu|| \to ||\mu||,$$

as $n \to \infty$, and the equality holds. $\square$

Question 3.

3. For any bounded measurable function m on $\mathbb{R}$, consider the operator T_m defined by

$$T_m f := (m\widehat{f})^{\vee}.$$

We know that T_m is bounded on $L^2(\mathbb{R})$.

(a) Suppose m is a monotonic step function given by

$$m(\xi) = \sum_{i=1}^{n} c_i \chi_{[\alpha_i, \alpha_{i+1})}(\xi),$$

where $-\infty = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \infty$. Show that

$$\|T_m\|_{p\to p} \lesssim_p \|m\|_\infty + |m|_{TV}, \qquad 1 < p < \infty,$$

where $|\cdot|_{TV}$ is the total variation.
(Hint: Express T_m in terms of the Hilbert transform.)

(b) Extend the conclusion of (a) to any m of bounded variation.

Solution.
(a) We fix the index in the definition of m to start at 0 and end at $n-1$. Suppose m is a monotonic step function given by

$$m = \sum_{i=0}^{n-1} c_i X_{[\alpha_i, \alpha_{i+1})}.$$

As $T_{[0,\infty)} = (iH + \frac{1}{2}I)$ (this fact was discussed in class), we have that

$$T_{[a,\infty)} f = e^{2\pi i a x} iHf(x) + \frac{1}{2} f(x),$$

so

$$\begin{aligned} T_m f &= T_{\sum_{i=0}^{n-1} c_i X_{[\alpha_i, \alpha_{i+1}]}} = c_0 I + \sum_{i=0}^{n-2} (c_{i+1} - c_i) T_{[\alpha_{i+1}, \infty)} \\ &= c_0 I + \sum_{i=0}^{n-2} (c_{i+1} - c_i)(e^{2\pi i \alpha_{i+1}} iHf + \frac{1}{2} f). \end{aligned}$$

Now, by the triangle inequality, and the fact that the Hilbert Transform is bounded on L^p (recall that the bound constant is dependent on p) for $1 < p < \infty$, we obtain

$$\begin{aligned} ||T_m f||_p &\le |c_0| ||f||_p + \sum_{i=0}^{n-2} |c_{i+1} - c_i| (||Hf||_p + \frac{1}{2} ||f||_p) \\ &\le C_p(|c_0| + \sum_{i=0}^{n-2} |c_{i+1} - c_i|) ||f||_p \le C_p(||m||_\infty + |m|_{TV}) ||f||_p, \end{aligned}$$

for some constant C_p, depended on p, so

$$||T_m||_{p\to p} \lesssim_p (||m||_\infty + |m|_{TV}),$$

as required.

(b) Let m be a function of bounded variation. As having a bounded variation implies being bounded, it follows that

$$||m||_\infty < \infty.$$

Now, as m is measurable, we can choose a sequence of monotonic step functions that converge pointwise to m such that $|m_n| \leq |m|$ for all n. Therefore, by DCT, for any $f \in L^p$, with $1 < p < \infty$, it follows that, for the pointwise limit,

$$\lim_{n\to\infty} T_{m_n} f = \lim_{n\to\infty} \int_{\mathbb{R}} m_n(\xi)\hat{f}(\xi)e^{2\pi i\xi} d\xi = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi i\xi} d\xi = T_m f.$$

Therefore, by Fatou

$$||T_m f||_p \leq \liminf_n ||T_{m_n} f||_p,$$

and by (a), and the choice of $\{m_n\}$

$$||T_m f||_p \leq \liminf_n ||T_{m_n} f||_p \lesssim_p (||m||_\infty + |m|_{TV}),$$

as required. $\square$

Question 4.

4. (a) Suppose $f \in L^1(\mathbb{R})$ satisfies

$$f = f * f. \tag{1}$$

Show that $f = 0$ a.e.

(b) Suppose $f \in L^2(\mathbb{R})$ satisfies (1). Show that f is uniformly continuous and in $L^p(\mathbb{R})$ for all $2 \leq p \leq \infty$.

(c) Let $p \in (1, 2)$. Show that if $f \in L^p(\mathbb{R})$ satisfies (1), then $f \in L^2(\mathbb{R})$.

Solution.

(a) Let $f \in L^1$. Taking the Fourier transform on both sides gives

$$\hat{f} = \hat{f}\hat{f},$$

which, with the continuity of $\hat{f}$, implies that

$$\hat{f} = 0 \text{ a.e} \quad \text{or} \quad \hat{f} = 1 \text{ a.e.}$$

As $\hat{f} = 1$ a. e contradicts the Riemann-Lebesgue lemma, it follows that

$$\hat{f} = 0 \text{ a.e.},$$

so by the inversion formula for L^1

$$f = 0 \text{ a.e.},$$

as required.

(b) Let $f \in L^2$ such that $f = f * f$. By the same argument in the L^1 case, we have

$$\hat{f} = 0 \text{ a.e} \quad \text{ or } \quad \hat{f} = 1 \text{ a.e.}$$

Let $E_1 = \{\hat{f} = 1\}$ and $E_0 = \{\hat{f} = 0\}$. As $\hat{f} \in L^2$, it follows that

$$||\hat{f}||_2 \quad = \quad (\int_{\mathbb{R}} |\hat{f}(\xi)|^2)^{\frac{1}{2}} = m(E_1)^{\frac{1}{2}} < \infty,$$

so

$$m(E_1) < \infty,$$

and, for any $p \geq 1$,

$$||\hat{f}||_p \quad = \quad (\int_{\mathbb{R}} |\hat{f}(\xi)|^p)^{\frac{1}{p}} = m(E_1)^{\frac{1}{p}} < \infty.$$

Therefore, we see that, for $1 \leq p \leq \infty$,

$$\hat{f} \in L^p,$$

as the infinity bound follows trivially.

Now, we first prove the uniform continuity of f. By the inversion formula for L^2, we obtain

$$f(x) \quad = \quad \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi = \int_{E_1} e^{2\pi i \xi x} d\xi.$$

Therefore, for any $\delta > 0$ and $x \in \mathbb{R}$, it follows that

$$\begin{aligned} |f(x+\delta) - f(x)| \quad &= \quad |\int_{E_1} e^{2\pi i \xi (x+\delta)} - e^{2\pi i \xi x} | d\xi \leq \int_{E_1} |e^{2\pi i \xi x}||e^{2\pi i \xi \delta} - 1| d\xi \\ &\leq \quad \int_{E_1} |e^{2\pi i \xi \delta} - 1| d\xi. \end{aligned}$$

Observe that the last integral is independent of x, and the integrand tends to 0, as $\delta \to 0$. Therefore, we have shown that f is uniformly continuous.

We now argue that $f \in L^p$ for $p \in [2, \infty]$. We employ Riesz-Thorin to the Fourier inversion operator. Since the Fourier inversion is bounded from L^1 to L^∞ and from L^2 to L^2, by Riesz-Thorin, we have that the inversion is bounded from p to q where $1 \leq p \leq 2$ and q is the conjugate of p. In particular, for $2 \leq p \leq \infty$, we see that

$$||f||_p \quad \leq \quad ||\hat{f}||_q,$$

where q is again the conjugate of p. As we have previously shown that $\hat{f} \in L^p$, for all $1 \leq p \leq \infty$, we are done.

(c) Let $f \in L^p$ such that $f = f * f$. As $p \in (1, 2)$, it follows that $\hat{f} \in L^q$, where q is the conjugate of p. As $\hat{f} \in L^q$, by the same argument from (b), we have that $\hat{f} \in L^2$. Therefore, we have shown that $\hat{f} \in L^2$, so $f \in L^2$, as the Fourier transform is an isometry on L^2. $\square$