Harmonic Analysis: Problem Set V

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Abstract

This work contains solutions to the problem set IV of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. We saw that the Fourier transform is bounded from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$ for all $1 \leq p \leq 2$. Justify each of the following steps to show that the range of p cannot be extended beyond 2.

   (a) It suffices to show that for $p < 2$, there is no uniform constant $c_p > 0$ such that $\|\widehat{f}\|_{p'} \geq c_p \|f\|_p$ for all $f \in \mathcal{S}$, that is, the direction of the Hausdorff-Young inequality cannot be reversed. Why?

   (b) Show that there is an absolute constant C such that for all a and b

   $$\left| \int_a^b e^{-ix^2} dx \right| \leq C.$$

   (c) Let $f_\lambda(x) := e^{-\pi x^2} e^{-\pi i \lambda x^2}$. Using the above result, and after an appropriate change of variables and integration by parts, show that $\|\widehat{f}_\lambda\|_\infty \leq C\lambda^{-1/2}$ for some absolute constant C.

   (d) Let $p < 2$. Justify the chain of inequalities

   $$\left\|\widehat{f}_\lambda\right\|_{p'} \leq \left\|\widehat{f}_\lambda\right\|_2^{2/p'} \left\|\widehat{f}_\lambda\right\|_\infty^{1-2/p'} \leq C\lambda^{1/p'-1/2}$$

   and deduce the result in (a).

   (e) As a by-product, argue that for $1 \leq p < 2$, the Fourier transform is not a surjective map from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$. (Hint: A classical theorem of functional analysis.)

Solution.
(a) Let $p < 2$. Assume that there is no uniform constant $c_p > 0$ such that $||\hat{f}||_{p'} \geq c_p ||f||_p$ for all $f \in S$. Since Fourier transform is a bijective mapping from S to S, and, for $f \in S$, $||\hat{\hat{f}}||_{p'} = ||f||_{p'}$ (Fourier transform composed twice on the Schwarz space is a parity operator, which does not change any L^p norm for $f \in S$), substituting $\hat{f}$ in the position of f implies that there is no uniform constant $c_p > 0$ such that

$$||f||_{p'} \quad \geq \quad c_p ||\hat{f}||_p,$$

for any $f \in S$. Since $S \subset L^p$, we conclude that the given statement is sufficient.

(b) With $u = \sqrt{i}x$, it follows that

$$\int e^{ix^2} = \frac{1}{\sqrt{i}} \int e^{u^2} du = (\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})\text{erf}(u) + C,$$

where the last equality holds with the well-known error function. Since $|\text{erf}(z)| \leq 1$ for any $z \in \mathbb{C}$, it follows that

$$\begin{aligned} |\int_a^b e^{ix^2} dx| &= |(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})||\text{erf}(b) - \text{erf}(a)| \\ &\leq 2|(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})| \leq C, \end{aligned}$$

for some absolute constant C.

(c) With $y = \sqrt{\pi\lambda}x$, we compute with integration by parts

$$\begin{aligned} |\hat{f}_\lambda(\xi)| &= |\int_{\mathbb{R}} e^{-(\frac{y-\xi\sqrt{\frac{\pi}{\lambda}}}{2})^2} e^{-iy^2} dy| \\ &\leq |\int_{\xi\sqrt{\frac{\pi}{\lambda}}}^{\infty} e^{-(\frac{y-\xi\sqrt{\frac{\pi}{\lambda}}}{2})^2} e^{-iy^2} dy| \\ &+ |\int_{-\infty}^{\xi\sqrt{\frac{\pi}{\lambda}}} e^{-(\frac{y-\xi\sqrt{\frac{\pi}{\lambda}}}{2})^2} e^{-iy^2} dy| \\ &= |\int_{\xi\sqrt{\frac{\pi}{\lambda}}}^{\infty} 2(\frac{y-\xi\sqrt{\frac{\pi}{\lambda}}}{\lambda})\frac{1}{\lambda} e^{-(\frac{y-\xi\sqrt{\frac{\pi}{\lambda}}}{2})^2} F(y) dy| \\ &+ |\int_{-\infty}^{\xi\sqrt{\frac{\pi}{\lambda}}} 2(\frac{y-\xi\sqrt{\frac{\pi}{\lambda}}}{\lambda})\frac{1}{\lambda} e^{-(\frac{y-\xi\sqrt{\frac{\pi}{\lambda}}}{2})^2} F(y) dy| \leq C, \end{aligned}$$

for some absolute constant C, so

$$||\hat{f}_\lambda||_\infty \leq C\lambda^{-\frac{1}{2}},$$

as required.

(d) By generalized Holder's inequality with $n = 2$ and $\theta = \frac{2}{p'}$, we obtain

$$||\hat{f}_\lambda||_{p'} \leq ||\hat{f}_\lambda||_2^{\frac{2}{p'}} ||\hat{f}_\lambda||_\infty^{1-\frac{2}{p'}}.$$

As $|e^{-\pi i\lambda x^2}| = 1$ for any $x \in \mathbb{R}$, and $||f||_2 = ||\hat{f}||_2$ for $f \in L^2$, we see that

$$||\hat{f}_\lambda||_2 = ||f_\lambda||_2 = (\int_{\mathbb{R}} |e^{-\pi x^2} e^{-\pi i\lambda x^2}|^p)^{\frac{1}{p}} = ||e^{-\pi x^2}||_p = C,$$

for some absolute constant C. Therefore, by (c) and the above equality, it follows that

$$||\hat{f}_\lambda||_{p'} \leq ||\hat{f}_\lambda||_2^{\frac{2}{p'}} ||\hat{f}_\lambda||_\infty^{1-\frac{2}{p'}} \leq C\lambda^{\frac{1}{p'}-\frac{1}{2}},$$

for some absolute constant C. Now, we deduce (a). Similarly, we have

$$||f_\lambda||_p = (\int_{\mathbb{R}} |e^{-\pi x^2} e^{-\pi i\lambda x^2}|^p)^{\frac{1}{p}} = ||e^{-\pi x^2}||_p = \delta,$$

for some constant $\delta > 0$, independent of λ. Let $c_p > 0$ be given. Then, for $\lambda > 0$ sufficiently small, it follows that

$$||\hat{f}_\lambda||_{p'} \leq C\lambda^{\frac{1}{p'}-\frac{1}{2}} \leq c_p||f_\lambda||_p,$$

where C is some absolute constant. Therefore, there does not exist a uniform constant $c_p > 0$ such that $||\hat{f}||_{p'} \geq c_p||f||_p$ for all $f \in S$.

(e) Let $1 \leq p < 2$. Suppose for sake of contradiction that Fourier transform is a surjective map from L^p to $L^{p'}$. As the Fourier transform is unique on L^p for $1 \leq p < 2$, this implies that the Fourier transform is a bijective map from L^p to $L^{p'}$. Now, by the Open Mapping theorem, we have that the Fourier transform is a bijective, open map from L^p to $L^{p'}$. Then, the inverse map of the Fourier transform is well-defined and continuous from $L^{p'}$ to L^p. This is a contradiction to (a).

Question 2.

2. For any $\Lambda < \infty$, let $\mathcal{B}^1_\Lambda(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [-\Lambda, \Lambda]\}$.

(a) Show that $\mathcal{B}^1_\Lambda(\mathbb{R}) \subset L^p(\mathbb{R})$ for all $1 \le p \le \infty$.

(b) Show that every function in $\mathcal{B}^1_\Lambda(\mathbb{R})$ is (Lebesgue) equivalent to a C^∞ function.

(c) Show that for any $f \in \mathcal{B}^1_\Lambda(\mathbb{R})$, $1 \le p \le \infty$, and $m \ge 1$, we have

$$\|f^{(m)}\|_p \le (c\Lambda)^m \|f\|_p$$

where c is some absolute constant.
(Hint: Pick any $\varphi \in \mathcal{S}$ that satisfies $\widehat{\varphi}(\xi) = 1$ for $|\xi| \le 1$, and scale it as $\varphi_\Lambda(x) = \Lambda\varphi(\Lambda x)$ to match the bandwidth of f. What is $f * \varphi_\Lambda$ equal to? Next, differentiate both sides and apply a familiar inequality.)

(d) Let $f \in \mathcal{B}^1_\Lambda(\mathbb{R})$ and extend its domain to $\mathbb{C}$ via $f(z) := \int_{-\Lambda}^{\Lambda} \widehat{f}(\xi) e^{2\pi i \xi z}\, d\xi$, $z \in \mathbb{C}$. Show that f is entire.

Solution.
(a) Let $f \in B^1_\Lambda$, and X be the characteristic function of $[-\Lambda, \Lambda]$. As $f \in L^1$, applying the inversion to $\hat{f} = \hat{f}X$ gives

$$f \quad = \quad f * \check{X}.$$

Arising from the fact that the Fourier transform is a bijective mapping on S to S, S is in L^p for any $1 \le p \le \infty$, and $X \in S$, it follows that $\check{X} \in S$, so $\check{X} \in L^p$ for any $p \ge 1$. Therefore for any $1 \le p \le \infty$, by Young's inequality for convolution (to be exact, it is the form recorded in Schlag pg.74) , we obtain

$$||f||_p = ||f * \check{X}||_p \le ||f||_1 ||\check{X}||_p < \infty.$$

Hence $f \in L^p$ for any $1 \le p \le \infty$, so we have shown that $B^1_\Lambda \subset L^p$ for $1 \le p \le \infty$.

(b) As $f \in L^1$, by the inversion formula, we have, for any $x \in \mathbb{R}$(Lebesgue identifiable point),

$$f(x) = \int_{-\Lambda}^{\Lambda} \hat{f}(\xi) e^{2\pi i x\xi} dx.$$

Since $\hat{f}(\xi)$ is uniformly continuous, we have that the integrand, when viewed as a function of two variables is continuous, as a product of continuous function is continuous. Hence, as the integral is over compact domain, we have that the differentiation under integral sign is justified, and

$$\frac{d^n}{dx} f(x) = \int_{-\Lambda}^{\Lambda} \frac{d^n}{dx} \hat{f}(\xi) e^{2\pi i x\xi} dx,$$

for any $n \in \mathbb{N}$. Since $e^{2\pi i x\xi}$ is in C^∞ with respect to the x variable, we have shown that f is Lebesgue equivalent to C^∞.

(c) Set $\varphi_\Lambda(\xi) = \Lambda\varphi(\Lambda x)$. Then, by a change of variable $y = \Lambda x$, we obtain

$$\hat{\varphi_\Lambda}(\xi) \quad = \quad \int \varphi(\Lambda x) e^{i \frac{\xi}{\Lambda} \Lambda x} d\Lambda x = \int \varphi(y) e^{i \frac{\xi}{\Lambda y}} dy = \hat{\varphi}(\frac{\xi}{\Lambda}),$$

so

$$\hat{\varphi_\Lambda}(\xi) = 1 \quad \text{for } |\xi| \le \Lambda.$$

Therefore, by the noted observation and the inversion, it follows that

$$\widehat{f * \varphi_\Lambda} = \hat{f}\hat{\varphi_\Lambda} = \hat{f} \;\text{ and }\; f * \varphi_\Lambda = f.$$

Differentiating the last identity m times, which is justified by (b), and choosing to differentiate φ_Λ term in the convolution yields

$$f^{(m)} = f * \varphi_\Lambda^{(m)}.$$

By the inequality employed in part (a), it follows that

$$||f^{(m)}||_p \quad = \quad ||f * \varphi_\Lambda^{(m)}||_p \leq ||\varphi_\Lambda^{(m)}||_1 ||f||_p \leq (c\Lambda)^m ||f||_p,$$

for some absolute constant c, as the Schwartz class is closed under differentiation and the last inequality follows from the definition of Schwartz class.

(d) From classical complex analysis, we have the following theorem.

Theorem. Let $F(z, \xi)$ be defined for $(z, \xi) \in \Omega \times [-M, M]$ where Ω is an open set in $\mathbb{C}$. Suppose F satisfies the following properties: (i) $F(z, \xi)$ is holomorphic in z for each ξ. (ii) F is continuous on $\Omega \times [-M, M]$. Then, $f(z) = \int_{[-M,M]} \hat{f}(\xi) e^{2\pi i \xi z} d\xi$ is holomorphic.

The proof can be found in pg. 56 of Stein's Complex Analysis. To invoke the theorem, we take $\Omega = \mathbb{C}$, $M = \Lambda$, and $F(z, \xi) = \hat{f}(\xi) e^{2\pi i \xi z}$. For any $\xi \in [-\Lambda, \Lambda]$, $\hat{f}(\xi) e^{2\pi i \xi z}$ is continuous, so (i) is satisfied. As $\hat{f}$ is uniformly continuous, (ii) is also satisfied (argued in (b) more explicitly), so we conclude that f is entire. $\square$

Question 3.

3. Let $\varphi \in L^2(\mathbb{R})$.

(a) Show that the family $\{e^{2\pi i m x} \varphi(x)\}_{m \in \mathbb{Z}}$ is an orthonormal system if and only if

$$\sum_{n \in \mathbb{Z}} |\varphi(x - n)|^2 = 1 \qquad a.e.\ x.$$

(Hint: The expression on the left hand side above defines a function in $L^1(\mathbb{T})$.)

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(b) Show that the family $\{\varphi(x - n)\}_{n \in \mathbb{Z}}$ is an orthonormal system if and only if

$$\sum_{m \in \mathbb{Z}} |\widehat{\varphi}(\xi - m)|^2 = 1 \qquad a.e.\ \xi.$$

Solution.

(a) We use the idea of periodization to prove both directions. For the converse, we compute, for any $n, m \in \mathbb{Z}$,

$$< e_n \phi, \overline{e_m \phi} > \quad = \quad \sum_{k \in \mathbb{Z}} \int_{[0,1]-k} e^{2\pi i (n-m) x} |\phi(x)|^2 dx.$$

With a change of variable $y = x + k$, the RHS of the above equality becomes

$$\sum_{k \in \mathbb{Z}} \int_0^1 e^{2\pi i (n-m)(y-k)} |\phi(y - k)|^2 dy,$$

which can be re-written as

$$\int_0^1 e^{2\pi i (n-m) y} \sum_{k \in \mathbb{Z}} |\phi(y - k)|^2 dy,$$

and again be simplified to

$$\int_0^1 e^{2\pi i(n-m)y}dy,$$

as the summand in the integral equals 1 (absolutely convergent) by assumption and e_n is periodic to justify the interchange of the summand and the integral. With the last expression, we see that $< e_n\phi, \overline{e_m\phi} >$ equals 1 if $n = m$ and 0 if $n \neq m$, which shows that the given system is orthonormal. Now, for the forward direction, as $\phi \in L^2$, $|\phi|^2 \in L^1$, we can set

$$|\phi|^2_{\text{per}} = \sum_{k\in\mathbb{Z}} |\phi(x-k)|^2.$$

Taking a Fourier coefficient gives

$$\begin{aligned}
\widehat{|\phi|^2_{\text{per}}}(n) &= \int_{\mathbb{T}} |\phi|^2_{\text{per}} e^{-2\pi inx}dx \\
&= \int_{\mathbb{T}} \sum_{k\in\mathbb{Z}} |\phi(x-k)|^2 e^{-2\pi inx}dx \\
&= \int_{\mathbb{R}} |\phi(x)|^2 e^{-2\pi inx}dx.
\end{aligned}$$

By the orthornomral assumption, the above equality reveals that $\widehat{|\phi|^2_{\text{per}}}(n)$ is 1 for the 0th term and 0 for the rest. Using the Cesaro convergence in L^1, we see that

$$||1 - |\phi|^2_{\text{per}}|| = 0,$$

which implie that $|\phi|_{\text{per}}|^2 = 1$ a.e., so by the established identity

$$\sum_{k\in\mathbb{Z}} |\phi(x-k)|^2 = 1 \quad \text{a.e.},$$

as required.

(b) As $\varphi \in L^2$, we have $\hat{\varphi} \in L^2$. From (a), it follows that

$$\sum_{m\in\mathbb{Z}} |\hat{\varphi}(\xi - m)|^2 = 1 \ \text{ a.e. } \xi \iff \{e^{2\pi im\xi}\hat{\varphi}(\xi)\}_{m\in\mathbb{Z}} \ \text{ is an ONS.}$$

As the Fourier transform is an isometry on L^2, we have

$$\{\varphi(\xi - m)\}_{m\in\mathbb{Z}} \ \text{ is an ONS } \iff \{\widehat{\varphi(\xi - m)}\}_{m\in\mathbb{Z}} \ \text{ is an ONS,}$$

which through the identity $\widehat{\varphi(\xi - m)} = \hat{\varphi}(\xi)e^{-2\pi im\xi}$ and the first equivalence implies

$$\sum_{m\in\mathbb{Z}} |\hat{\varphi}(\xi - m)|^2 = 1 \ \text{ a.e. } \xi \iff \{\varphi(\xi - m)\}_{m\in\mathbb{Z}} \ \text{ is an ONS}$$

as required. $\square$

Question 4.

4. For any $\Lambda < \infty$, let $\mathcal{B}^2_\Lambda(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [-\Lambda, \Lambda]\}$.

(a) How would you restate the results of 2(a)-(d)?

(b) Show that $\mathcal{B}^2_\Lambda(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$ which is invariant under translations.

(c) Let $\operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$, $x \neq 0$, $\operatorname{sinc}(0) := 1$. Show that $\{\operatorname{sinc}(\cdot - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{B}^2_{1/2}(\mathbb{R})$. Express any $f \in \mathcal{B}^2_{1/2}(\mathbb{R})$ in this basis and show

$$f = \sum_{n\in\mathbb{Z}} f(n)\operatorname{sinc}(\cdot - n)$$

where f has been identified with its equivalent continuous version (so that $f(n)$ makes sense).

(d) Show that the 2-norm convergent series expansion in (c) is also uniformly convergent.

Solution.
(a) We restate the results for B^2_Λ instead of B^1_Λ, but without changing the lower bound on p from 1 to 2.

(b) Let $\{f_n\}$ be a sequence in B^2_Λ such that it converges to some $f \in L^2$. As $||\hat{g}||_2 = ||g||_2$ for any $g \in L^2$, it follows that $\{\hat{f}_n\}$ converges to $\hat{f}$ in L^2. Since the property of support being contained in compact set persists through L^2 limit (this trivially can be shown using a proof by contradiction), it follows that supp$\hat{f} \subset [-\Lambda, \Lambda]$, so $f \in B^2_\Lambda$. Hence, B^2_Λ is closed.

We now show that B^2_Λ is invariant under translations. Fix $h \in \mathbb{R}$, and let η_h be defined by $(\eta_h f)(x) = f(x-h)$ on L^2. Now, consider the modulation operator m_h to be defined by $(m_h f)(x) = e^{2\pi i h x} f(x)$. Then, it follows that, for $f \in L^2$, and $\xi \in \mathbb{R}$,

$$\widehat{\eta_h f}(\xi) = \int f(x-h)e^{-2\pi i \xi x}dx = (m_{-h}\hat{f})(\xi),$$

so

$$\text{supp } \widehat{\eta_h f} = \text{supp } m_{-h}\hat{f} = \text{supp } \hat{f},$$

as one can trivially see that a support of a function is an invariant property under the modulation operator. Therefore, we have shown that the translation operator is invariant on B^2_Λ.

(c) Consider the space of functions defined by

$$B^{2'}_\Lambda = \{f \in L^2 \mid f(\xi) = 0 \text{ if } |\xi| > \frac{1}{2}\}.$$

By Plancherel's theorem, the Fourier transform is a linear isometry from B^2_Λ to $B^{2'}_\Lambda$. We see that $\{e^{-2\pi i n\xi}1_{-(\frac{1}{2},\frac{1}{2}]}\}$ is an orthonormal basis in $B^{2'}_\Lambda$. By the isometry relation, to show that the given sinc set is an orthornal basis, it suffices to show that the the identified orthonormal basis is mapped to sinc functions by inverse of the Fourier transform, which can be shown as

$$\begin{aligned} F^{-1}(e^{-2\pi i n\xi})(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n\xi}e^{2\pi i \xi x}d\xi \\ &= \text{sinc}(x-n), \end{aligned}$$

as required.

Now, let $f \in B^1_\Lambda$ and $g = F(f)$. Using the basis identified, it follows that

$$g(\xi) = \sum_{n\in\mathbb{Z}} < g, e^{-2\pi i n\xi} > e^{-2\pi i n\xi}.$$

Taking the inverse of the Fourier transform gives

$$f(\xi) = \sum_{n\in\mathbb{Z}} < g, e^{-2\pi i n\xi} > \text{sinc}(\cdot - n),$$

which can be re-written as

$$f(\xi) \quad = \quad \sum_{n\in\mathbb{Z}} f(n)\text{sinc}(\cdot - n),$$

as $< g, e^{-2\pi i n\xi} >= f(n)$. Therefore, we are done.

(d) By Cauchy-Scwhwarz, it follows that

$$\sum_{n\in\mathbb{Z}} | < f, \text{sinc}(\cdot - n) > ||\text{sinc}(t-n)| \quad \leq \quad (\sum_{n\in\mathbb{Z}} | < f, \text{sinc}(\cdot - n) > |^2)^{\frac{1}{2}} (\sum_{n\in\mathbb{Z}} \text{sinc}^2(t-n))^{\frac{1}{2}}.$$

In view of Parseval's theorem, we see that the first sum on the RHS converges, so it suffices to show that the sinc sum is uniformly bounded, but this can be shown with the comparison test with $\sum_{n\in\mathbb{Z}} \dfrac{1}{n^2}$. Therefore, the sum is uniformly bounded for all $t \in \mathbb{R}$, so the convergence is uniform by Weistrauss M-test. $\square$