PDE-Evans

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Abstract

This work contains solutions to some problems in Evan's PDE.

Question 2-1.

1. Write down an explicit formula for a function u solving the initial-value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Solution.

Question 2-2.

2. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(Ox) \quad (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

Solution.

Question 2-3.

3. Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g\,dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f\,dx,$$

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provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

Solution.

Question 2-4.

4. We say $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Prove for subharmonic v that

$$v(x) \leq ⨍_{B(x,r)} v\, dy \quad \text{for all } B(x,r) \subset U.$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Solution.
(a) Fix $x \in U$. Define $\phi : (0, \text{dist}(x, \partial U)) \to \mathbb{R}$ by

$$\phi(r) \quad := \quad ⨍_{\partial B(x,r)} v(y) dS(y) = ⨍_{\partial B(0,1)} u(x + rz) dS(z).$$

Then

$$\phi'(r) \quad = \quad ⨍_{\partial B(0,1)} Dv(x + rz) \cdot z dS(z),$$

and by Green's formula

$$\begin{aligned} \phi'(r) \quad &= \quad ⨍_{\partial B(x,r)} Dv(y) \cdot \frac{y - x}{r} dS(y) \\ &= \quad ⨍_{\partial B(x,r)} \frac{\partial v}{\partial \nu} dS(y) \\ &= \quad \frac{r}{n} ⨍_{B(x,r)} \Delta v(y) dy. \end{aligned}$$

Since $\Delta v \geq 0$ in U, the above equality implies

$$\phi'(r) \geq 0 \text{ in } (0, \text{dist}(x, \partial U)).$$

so for a fixed $B(x, r_0) \subset U$,

$$v(x) = \lim_{r \to 0; \in (0, \text{dist}(x, \partial U))} \phi(r) \leq ⨍_{\partial B(x, r_0)} v(y) dS(y).$$

2. Now employing polar coordinates yields, for any $B(x, r) \subset U$,

$$\begin{aligned} \int_{B(x,r)} v dy \quad &= \quad \int_0^r \left(\int_{\partial B(x,r)} v dS \right) ds \\ &\geq \quad v(x) \int_0^r n\alpha(n) s^{n-1} ds = \alpha(n) r^n v(x), \end{aligned}$$

so

$$⨍_{B(x,r)} v dy \quad \geq \quad v(x),$$

as required.

(c) 1. We assert that the converse of the sub-harmonic mean-value property holds. Let $v \in C^2(\overline{U})$ such that

$$v(x) \le \fint_{B(x,r)} v dy \quad \text{for all } \; B(x,r) \subset U.$$

If $\Delta v < 0$, by the C^2 smoothness of v, there exists some ball $B(x, r_0) \subset U$ such that $\Delta v < 0$ within $B(x, r_0)$. But then for ϕ as above, and $r \in (0, r_0)$,

$$\phi'(r) \quad = \quad \frac{r}{n} \fint_{B(x,r)} \Delta u(y) dy < 0.$$

This implies that

$$v(x) = \lim_{r \to 0; r \in (0,r_0)} \phi(r) > \fint_{B(x,r_0)} v dy,$$

a contradiction.

2. Fix $x \in U$ and $B(x,r) \subset U$. Observe that $B^0(x,r)$ is open, bounded, and u is harmonic, thus summable. Thus by Jensen's inequality, and the mean value property of u, we obtain

$$\begin{aligned} \fint_{B(x,r)} v dx \quad &= \quad \fint_{B^0(x,r)} v dx = \fint_{B^0(x,r)} \phi(u) dx \\ &\ge \quad \phi\left(\fint_{B^0(x,r)} u dx\right) = \phi\left(\fint_{B(x,r)} u dx\right) = \phi(u(x)) = v(x). \end{aligned}$$

So, by the converse of the sub-harmonic mean-value property, we have shown that v is subharmonic.

(d) In view of (c), it suffices to show that $|Du|$ is harmonic. As u_{x_i} is harmonic for $i = 1, ..., n$, we have that Du is harmonic. We now claim that if $u \in C^2$ is harmonic, then $|u|$ is harmonic.

Question 2-5.

5. Prove that there exists a constant C, depending only on n, such that

$$\max_{B(0,1)} |u| \leq C(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,1) \\ u = g & \text{on } \partial B(0,1). \end{cases}$$

Solution.

Question 2-6.

6. Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

Solution. Fix $x \in B^0(0,r)$. If $y \in \partial B(0,r)$, then $|x-y| \le x + r$, so

$$\begin{aligned}
u(x) &= \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}dS(y) \ge \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{(|x|+r)^n}dS(y) \\
&= r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}\frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(0,r)} g(y)dS(y) = r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}⨍_{\partial B(0,r)} g(y)dS(y) \\
&= r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0)
\end{aligned}$$

Question 2-7.

7. Prove Theorem 15 in §2.2.4. (Hint: Since $u \equiv 1$ solves (44) for $g \equiv 1$, the theory automatically implies

$$\int_{\partial B(0,1)} K(x, y)\, dS(y) = 1$$

for each $x \in B^0(0, 1)$.)

Solution.

Question 2-8.

8. Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \le 1$. Show Du is *not* bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda e_n) - u(0)}{\lambda}$.)

Solution.

Question 2-9.

9. Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n \mid |x| < 1,\ x_n > 0\}$. Assume $u \in C(\bar{U}^+)$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for $x \in U = B^0(0,1)$. Prove v is harmonic in U.

Solution.