Harmonic Analysis: Problem Set IV

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set IV of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Let H denote the Hilbert transform defined for $f \in L^2(\mathbb{T})$ by

$$Hf = \sum_{n \neq 0} -i \operatorname{sign}(n) \widehat{f}(n) e_n. \tag{1}$$

Let $1 < p < \infty$. Explain why knowing L^p-boundedness of H on the space of trigonometric polynomials allows us to define it uniquely as a bounded operator on the whole of $L^p(\mathbb{T})$ and why the resulting operator satisfies, for all $f \in L^p(\mathbb{T})$, the relation

$$\widehat{Hf}(n) = -i \operatorname{sign}(n) \widehat{f}(n), \quad n \in \mathbb{Z}, \tag{2}$$

with the convention $\operatorname{sign}(0) = 0$.

Show also that for any $f \in L^p(\mathbb{T})$, we have

$$H^2 f = -f + \widehat{f}(0). \tag{3}$$

Solution.
(a) Firstly, note that H is a bounded linear operator on the space of trig polynomials for any p by assumption. For any $f \in L^p$, by density of trig polynomials, there exists $\{f_n\}$ such that $f_n \stackrel{L^p}{\to} f$. Observe that by linearity and boundedness of H on the space of trig polynomials, we have

$$||Hf_n - Hf_m||_p \quad = \quad ||H(f_n - f_m)||_p \leq C_p ||f_n - f_m||_p,$$

where C_p is the operator bound on the space of trig polynomials. By the above estimate, $\{Hf_n\}$ is Cauchy in L^p, thus convergent by completeness of L^p. Therefore, we can define Hf as

$$Hf \quad = \quad \lim_{n\to\infty} Hf_n.$$

We first show that the extension is well-defined. Let $f \in L^p(\mathbb{T})$, and consider $\{g_n\}$ and $\{h_n\}$ trig polynomials such that they converge to f in L^p. Analogous the previous estimate, we have

$$||Hg_n - Hh_n||_p \quad = \quad ||H(g_n - h_n)||_p \leq C_p ||g_n - h_n|| \to 0,$$

as $n \to \infty$. Therefore, $\lim_{n\to\infty} Hg_n = \lim_{n\to\infty} Hh_n$ and the extension is well-defined. Similarly, it follows that the operator is linear and bounded on the whole L^p. We now argue the uniqueness of the operator. Let H_1, H_2 be bounded linear operators on $L^p(\mathbb{T})$ that agree on the space of trig polynomials. As the trig polynomials are dense in $L^p(\mathbb{T})$, we have that $H_1 = H_2$ on the entire $L^p(\mathbb{T})$, by the continuity of H_1 and H_2. Therefore, the defined operator is unique.

Let $f \in L^p(\mathbb{T})$, and $\{p_n\}$ be trig polynomials such that $p_n \xrightarrow{L^p} f$, as $n \to \infty$. Since

$$\int_{\mathbb{T}} |f_n - f||e^{2\pi i n x}|dx \leq ||f_n - f||_p,$$

we have

$$\hat{f}_k(n) \to \hat{f}(n) \text{ as } k \to \infty.$$

It follows that

$$\begin{aligned} \widehat{Hf}(n) &= \int Hf e^{-2\pi n\theta} d\theta = \lim_{k\to\infty} \int Hf_k e^{-2\pi n\theta} d\theta \\ &= \lim_{k\to\infty} \widehat{Hf_k}(n) = -i\text{sgn}(n)\hat{f}(n). \end{aligned}$$

(b) Applying the above result twice to $H^2 f$ yields

$$\widehat{H^2 f}(n) = \begin{cases} -\hat{f}(n) & \text{if } n \neq 0 \\ 0 & \text{if } n = 0, \end{cases}$$

Since,

$$\widehat{-f + \hat{f}(0)}(n) = \begin{cases} -\hat{f}(n) & \text{if } n \neq 0 \\ 0 & \text{if } n = 0, \end{cases}$$

it follows that, for any $n \in \mathbb{Z}$,

$$\widehat{H^2 f}(n) = \widehat{-f + \hat{f}(0)}(n),$$

which by uniqueness of fourier coefficients implies

$$H^2 f = -f + \hat{f}(0),$$

as required.

$\square$

Question 2.

2. Let again H denote the Hilbert transform on the torus.

   (a) Let f be any real-valued trigonometric polynomial. Using the identity $(u+iv)^2 = u^2 - v^2 + i(2uv)$, argue that

   $$H(f^2 - (Hf)^2) = 2fHf. \tag{4}$$

   Deduce that

   $$(Hf)^2 = f^2 - \hat{f}(0)^2 + 2H(fHf). \tag{5}$$

   (b) Show that (5) continues to hold for any complex-valued trigonometric polynomial. (Hint: Expand out $[H(f+ig)]^2$. In order to handle the cross-terms, utilize the algebraic identity $[H(f+g)]^2 - [H(f-g)]^2 = 4(Hf)(Hg)$.)

Solution.
(a) By the given identity, we have

$$(f + iHf)^2 = f^2 - (Hf)^2 + i(2fHf),$$

which by linearity of H implies that

$$H(f^2 - (Hf)^2) = H((f + iHf)^2) - H(i2fHf).$$

Therefore, to prove (4), it suffices to show that

$$\widehat{H((f + iHf)^2)}(n) = \widehat{2(fHf + iH(fHf)}(n),$$

for any $n \in \mathbb{Z}$. We compute

$$f + iHf = \hat{f}(0) + 2\sum_{n>0} \hat{f}(n)e^{2\pi in\theta},$$

and

$$\widehat{f+iH}f(n) = \begin{cases} 2\hat{f}(n) & \text{if } n > 0 \\ \hat{f}(0) & \text{if } n = 0, \\ 0 & \text{if } n < 0 \end{cases}$$

so

$$\begin{aligned} \widehat{(f+iHf)^2}(n) &= \sum_{m\in\mathbb{Z}} \widehat{f+iHf}(n-m)\widehat{f+iHf}(m) \\ &= \begin{cases} \sum_{m=1}^{n} 4\hat{f}(n-m)\hat{f}(m) & \text{if } n > 0 \\ \hat{f}(0)^2 & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases} \end{aligned}$$

Thus

$$\widehat{H(f+iHf)}(n) = \begin{cases} -4i\sum_{m=1}^{n} \hat{f}(n-m)\hat{f}(m) & \text{if } n > 0 \\ 0 & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

We again compute

$$\begin{aligned} \widehat{fHf}(n) &= \sum_{m\in\mathbb{Z}} \hat{f}(n-m)(-i)\mathrm{sgn}(m)\hat{f}(m) \\ &= -i\sum_{m\in\mathbb{Z}} \hat{f}(n-m)\hat{f}(m), \end{aligned}$$

so

$$\widehat{2(fHf+iH(fHf))}(n) = \begin{cases} 4\widehat{fHf}(n) = -4i\sum_{m=1}^{n} \hat{f}(n-m)\hat{f}(m) & \text{if } n > 0 \\ 0 & \text{if } n = 0, \\ 0 & \text{if } n < 0. \end{cases}$$

Therefore, we have

$$\widehat{H(f+iHf)^2}(n) = 2\widehat{fHf+iH(fHf)}(n),$$

for any $n \in \mathbb{Z}$ as required. Now, from (4) we deduce (5). Applying H to both sides of (4) yields

$$2H(fHf) = H^2(f^2 - (Hf)^2) = -f^2 + (Hf)^2 + \widehat{f^2 - (Hf)^2}(0).$$

Since

$$\begin{aligned} \widehat{f^2 - (Hf)^2}(0) &= \sum_{m\in\mathbb{Z}} \hat{f}(-m)\hat{f}(m) + \mathrm{sgn}(-m)\mathrm{sgn}(m)\hat{f}(-m)\hat{f}(m) \\ &= \hat{f}(0)^2, \end{aligned}$$

we obtain (5) as desired. □

(b) Let $Z = f + ig$ be any complex-valued trig polynomial, where f and g are the real-valued trig polynomials. Observe that

$$(HF)^2 = (Hf)^2 + 2iHfHg - (Hg)^2$$

and

$$H(FHF) = H((f+ig)(Hf+iHg)) = H(fHf - gHg - i(fHg + gHf)).$$

Therefore, it suffices to show that

$$HfHg = fg - \hat{f}(0)\hat{g}(0) + H(fHg + gHf).$$

Expanding the RHS of the given identity gives

$$\begin{aligned}
4HfHg &= (H(f+g)^2) - (H(f-g))^2 \\
&= (f+g)^2 - (f-g)^2 + 2H((f-g)(Hf+Hg)) - 2H((f-g)(Hf-Hg)) \\
&- (\hat{f}(0)+\hat{g}(0))^2 + (\hat{f}(0)^2 - \hat{g}(0))^2 \\
&= 4fg - 4\hat{f}(0)\hat{g}(0) + 4H(fHg + gHf).
\end{aligned}$$

Dividing the above identity by 4 gives the desired identity as desired. □

Question 3.

3. For any $1 < p < \infty$, let C_p denote the supremum of $\|Hf\|_p$ over all trigonometric polynomials f of p-norm at most 1. Using the relation (5), show that

$$\|Hf\|_{2p}^2 \leq 2\|f\|_{2p}^2 + 2C_p\|f\|_{2p}\|Hf\|_{2p}.$$

Deduce that

$$C_{2p} \leq C_p + \sqrt{C_p^2 + 2}.$$

Hence, knowing $C_2 = 1$, it follows that C_4, C_8, etc. are all finite. With interpolation and duality, this provides another route to showing that $C_p < \infty$ for all $1 < p < \infty$.

Solution.
Substituting the second identity from the problem 2 with, via Minkowski's inequality, we obtain

$$\begin{aligned}
||Hf||_{2p}^2 &= ||Hf^2||_p \leq ||f^2||_p + ||\hat{f}(0)^2||_p + ||2H(fHf)||_p \\
&\leq ||f||_p^2 + |\hat{f}(0)|^2 + ||2H(fHf)||_p,
\end{aligned}$$

where we have simply used the fact that a constant function in $L^p(\mathbb{T})$ attains the modulus of the value for the norm of any $p \in (1, \infty)$. Now, by monotonicity of $L^p(\mathbb{T})$ norms, it follows that

$$|\hat{f}(0)|^2 = |\int_{\mathbb{T}} f(x)dx|^2 \leq (\int_{\mathbb{T}} |f(x)|dx)^2 = ||f||_1^2 \leq ||f||_{2p}^2.$$

Furthermore, by generalized Holder's inequality, we have

$$||2H(fHf)||_p \leq 2C_p||fHf||_p \leq 2C_p||f||_{2p}||Hf||_{2p}.$$

Therefore, combining the two estimates gives

$$||Hf||_{2p}^2 \leq 2||f||_{2p}^2 + 2C_p||f||_{2p}||Hf||_{2p},$$

as required. Since

$$||Hf||_{2p}^2 - 2||f||_{2p}^2 - 2C_p||f||_{2p}||Hf||_{2p} \leq 0,$$

utilizing the quadratic formula, we have

$$||Hf||_{2p}^2 \leq \frac{2C_p||f||_{2p} + \sqrt{4C_p^2||f||_{2p}^2 + 2||Hf||_{2p}^2}}{2},$$

which can be simplified to

$$||Hf||_{2p} \leq C_p||f||_{2p} + \sqrt{C_p^2 + 2}||f||_{2p},$$

which then implies

$$C_{2p} \leq C_p + \sqrt{C_p^2 + 2},$$

as required. As $C_2 = 1$, by the above inequality, we obtain that $C_{2^n} < \infty$ for $n \geq 1$. Therefore, by Riesz-Thorin Interpolation theorem and duality, we have $C_p < \infty$ for any $p \in (1, \infty)$. □

4. Let $1 \le p \le \infty$ and $\phi \in L^p(\mathbb{T})$. Prove the following generalization of Young's inequality: For any $1 \le r \le p'$ and $f \in L^r(\mathbb{T})$, the convolution $\phi * f$ belongs to $L^s(\mathbb{T})$ and $\|\phi * f\|_s \le \|\phi\|_p \|f\|_r$, where $p^{-1} + r^{-1} = s^{-1} + 1$.
(Hint: Riesz-Thorin.)

Question 4.

Solution.
In this solution, all integral are taken over the torus. We view $\phi * f$ as an operator, which we denote as T. We first show that the statement holds for the case $r = 1$, and $r = q$. Suppose $r = 1$, so $s = p$. By general Minkowski inequality, we obtain

$$\begin{aligned} ||Tf||_p &= \left(\int |\int \phi(x-y)f(y)dy|^p dx\right)^{\frac{1}{p}} \\ &\le \int \left(\int |\phi(x-y)|^p |f(y)|^p dx\right)^{\frac{1}{p}} dy = ||\phi||_p ||f||_1 < \infty, \end{aligned}$$

which implies that for $T : L^1 \to L^p$,

$$||T||_{1\to p} \le ||\phi||_p,$$

thus bounded by assumption. Now, suppose $r = q$, so $s = \infty$. By Holder inequality, we obtain

$$||Tf||_\infty \le |\sup_{x\in\mathbb{T}} f(x)\phi(x-y)dy| \le ||\phi f||_1 \le ||\phi||_p ||f||_q,$$

which implies that for $T : L^q \to L^\infty$,

$$|||T||_{q\to\infty} \le ||\phi||_p,$$

thus bounded by assumption.

Hence, by Riesz-Thorin, we conclude that, for $1 \le r \le q$ and T maps L^r into L^s with

$$||T||_{r\to s} \le ||T||_{1\to p}^{\theta} ||T||_{q\to\infty}^{1-\theta} \le ||\phi||_p,$$

where $r^{-1} = \dfrac{\theta}{r_1} + \dfrac{1-\theta}{r_2}$, $s^{-1} = \dfrac{\theta}{s_1} + \dfrac{1-\theta}{s_2}$ with $(1, q)$ and (p, ∞) being interpolation points respectively. $\square$

Question 5.

5. Let (X, μ) be a measure space, $1 \le p < \infty$, and consider the space weak-$L^p(X, \mu)$, which we defined to be the set of measurable functions on X such that

$$[f]_{w,p} := \sup_{\alpha > 0} \alpha[\lambda_f(\alpha)]^{1/p} < \infty,$$

where $\lambda_f(\alpha) := \mu\{x : |f(x)| > \alpha\}$. As we saw, if $f \in L^p$, then $[f]_{w,p} \le \|f\|_p$. Hence $L^p \subset$ weak-L^p.

(a) Show that weak-L^p is a linear space.
(b) Show that for all $\beta \in \mathbb{C}$, we have $[\beta f]_{w,p} = |\beta| [f]_{w,p}$.
(c) Show that $[f]_{w,p} = 0$ if and only if $f = 0$ μ-a.e. in X.
(d) Show that the triangle inequality can fail for $[\cdot]_{w,p}$ for all p. Hence $[\cdot]_{w,p}$ is not a norm in general. (Hint: It happens even when X consists of just 2 points.)
(e) A *quasi*-norm $[\cdot]$ satisfies all the properties of a norm except the triangle inequality is replaced by $[f + g] \le C([f] + [g])$. Show that $[\cdot]_{w,p}$ is a quasi-norm with $C \le 2$.

Solution.
(a) If $f = 0$ μ-a.e in X, then by (c), $[f] = 0$, so $f \in$ weak-L^p. Now, let $\gamma \in \mathbb{C}$, and $f \in$ weak-L^p. By (b), we see that $\gamma f \in$ weak-L^p. Furthermore, let $f, g \in$ weak-L^p. By (e), it follows that

$$[f + g] \le 2([f] + [g]) < \infty,$$

which implies that $f+g \in$ weak-L^p. Therefore, the space is a linear space. All the constituents will be proven in the later parts.

(b) Let $\beta \in \mathbb{C}$. If $\beta = 0$, then $\big[\beta f\big] = 0$, thus the equality holds trivially. Suppose $\beta \neq 0$. By the definition of weak-L^p, we have

$$\begin{aligned}\big[\beta f\big] &= \sup_{\alpha>0} \alpha \big[\mu\{|\beta f| > \alpha\}\big]^{\frac{1}{p}} = |\beta| \sup_{\alpha>0} \frac{\alpha}{|\beta|} \big[\mu\{|f| > \frac{\alpha}{|\beta|}\}\big]^{\frac{1}{p}} \\ &= |\beta| \sup_{\alpha>0} \alpha \big[\mu\{|f| > \alpha\}\big]^{\frac{1}{p}} = |\beta| \big[f\big]\end{aligned}$$

(c) Let $f = 0$ μ-a.e. in X. By monotonicity of measure, we have

$$\mu\{|f| > \alpha\} = 0,$$

for any $\alpha > 0$, which implies that $\big[f\big]_{w,p} = 0$. Conversely, let $\big[f\big]_{w,p} = 0$. As $\mathbb{C}$ is an integral domain, this implies that for any $n \in \mathbb{N}$,

$$\mu\{|f| > \frac{1}{n}\} = 0,$$

so

$$\mu\{|f| > 0\} = \mu \bigcup_{n=1}^{\infty} \{|f| > \frac{1}{n}\} \leq \sum_{n=1}^{\infty} \mu\{|f| > \frac{1}{n}\} = 0.$$

Therefore,

$$\mu\{|f| = 0\} = 1,$$

which completes the proof.

(d) Let $X = \{x_0, x_1\}$, equipped with the uniform measure with $\mu(X) = 1$. Let $0 < a$, and $p \in [1, \infty)$. Define $f, g : X \to \mathbb{C}$ by

$$f(x) = \begin{cases} a & \text{if } x = x_0 \\ 2a & \text{if } x = x_1, \end{cases}$$

and

$$g(x) = \begin{cases} k-a & \text{if } x = x_0 \\ k-2a & \text{if } x = x_1, \end{cases}$$

so

$$f+g(x) = \begin{cases} k & \text{if } x = x_0 \\ k & \text{if } x = x_1, \end{cases}$$

with some $k > 2a$ to be determined. It follows that

$$\big[f\big] = 2a\frac{1}{2}^{\frac{1}{p}}, \ \big[g\big] = k-2a \ , \big[f+g\big] = k,$$

for k chosen large enough such that $(\frac{1}{2})^{\frac{1}{p}}(k-2a) \leq k-a$ holds. With the above equality, granted with the appropriate choice for k, dependent on p, in order to violate the triangle inequality, we must have

$$2a(\frac{1}{2})^{\frac{1}{p}} + (k-2a) < k.$$

However, the above inequality is equivalent to

$$2a((\frac{1}{2})^{\frac{1}{p}} - 1) < 0,$$

showing that the construction is valid for any $p \in [1, \infty)$. We have shown that $\big[\,\cdot\,\big]$ is not a norm in general.

(e) Let $p \in [1, \infty)$. If $\big[f\big] = \infty$ or $\big[g\big] = \infty$, then the inequality holds trivially. Suppose $f, g \in$ weak-L^p. Observe that, for any $\alpha > 0$,

$$\mu\{|f+g| > \alpha\} \leq \mu\{|f| + |g| > \alpha\} \leq \mu\{|f| > \frac{\alpha}{2}\} + \mu\{|g| > \frac{\alpha}{2}\},$$

which implies that

$$\begin{aligned}\big[f+g\big] &= \sup_{\alpha>0} \alpha\big[\lambda_{f+g}(\alpha)\big]^{\frac{1}{p}} \leq \sup_{\alpha>0} \alpha\big(\big[\lambda_f(\frac{\alpha}{2})\big]^{\frac{1}{p}} + \big[\lambda_g(\frac{\alpha}{2})\big]^{\frac{1}{p}} \\ &\leq 2\big(\sup_{\alpha>0} \alpha\big[\lambda_f(\alpha)\big]^{\frac{1}{p}} + \sup_{\alpha>0} \alpha\big[\lambda_g(\alpha)\big]^{\frac{1}{p}}\big) = 2(\big[f\big] + \big[g\big]),\end{aligned}$$

as required. Hence, we have shown that $\big[\,\cdot\,\big]$ is a quasi-norm with $C \leq 2$. □

Question 6.

6. Assume the notation in problem 5.

(a) Show that $\lambda_f(\alpha)$ is a decreasing, right continuous function of α.

(b) Show that $f \in L^p$ if and only if $\sum_{n=-\infty}^{\infty} 2^{np}\lambda_f(2^n) < \infty$.

(c) Conclude that if $f \in L^p$, then $\lambda_f(\alpha) = o(\alpha^{-p})$, both as $\alpha \to 0$ and $\alpha \to \infty$. This improves the Markov-Chebyshev inequality.

Solution.
(a) Let $0 < \alpha < \beta$. By monotonicity of measure, we have

$$\mu\{|f| > \alpha\} \quad \geq \quad \mu\{|f| > \beta\}.$$

Hence, λ_f is decreasing. Let $\{\alpha_n\}$ be a sequence from (α, ∞) such that and $\alpha_n \to \alpha$ as $n \to \infty$. Choose a subsequence from the sequence, which is strictly decreasing, denoted as $\{\alpha_{n_k}\}$. Now, by continuity of measure, it follows that

$$\begin{aligned}\lim_{k\to\infty} \lambda_f(\alpha_{n_k}) &= \lim_{k\to\infty} \mu\{|f| > \alpha_{n_k}\} = \mu \bigcup_{k=1}^{\infty} \{|f| > \alpha_{n_k}\} \\ &= \mu\{|f| > \alpha\} = \lambda_f(\alpha).\end{aligned}$$

Hence, $\lambda_f(\alpha_n) \to \lambda_f(\alpha)$ as $n \to \infty$ and λ_f is right continuous.

(b) Assume $f \in L^p$. We have the following result from class:

$$f \in L^p \implies ||f||_p^p = \int_0^\infty p\alpha^{p-1}\lambda_f(\alpha)d\alpha.$$

Therefore, by DCT, it follows that

$$\begin{aligned}||f||_p^p &= \int_0^\infty p\alpha^{p-1}\lambda_f(\alpha)d\alpha \\ &= \int_0^\infty \sum_{n=-\infty}^{\infty} p\alpha^{p-1}\lambda_f(\alpha)\mathbf{1}_{(2^{n-1},2^n]}(\alpha)d\alpha \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_0^\infty p\alpha^{p-1}\lambda_f(\alpha)\mathbf{1}_{(2^{n-1},2^n]}(\alpha)d\alpha \\ &\geq \lim_{N\to\infty} \sum_{n=-N}^{N} p(2^{n-1})^{p-1}\lambda_f(2^n)2^{n-1} = \frac{p}{2^p} \sum_{n=-\infty}^{\infty} 2^{np}\lambda_f(2^n).\end{aligned}$$

Hence, $\sum_{n=-\infty}^{\infty} 2^{np}\lambda_f(2^n) < \infty$. Conversely, suppose that $\sum_{n=-\infty}^{\infty} 2^{np}\lambda_f(2^n) < \infty$. Set

$$A_n \quad = \quad = \{2^n \leq |f| \leq 2^{n+1}\}.$$

By MCT, it follows that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} 2^{np}\lambda_f(2^n) &\leq \sum_{n=-\infty}^{\infty} 2^{np}\mu(A_n) = 2^{-p}\lim_{N\to\infty}\int_X \sum_{n=-N}^{N} 2^{(n+1)p}1_{A_n}d\mu \\
&= 2^{-p}\int_X \sum_{n=-\infty}^{\infty} 2^{(n+1)p}1_{A_n}d\mu \\
&\leq 2^{-p}\int_X |f|^p d\mu,
\end{aligned}$$

which implies that $f \in L^p$.

(c) Let $p \in [1, \infty)$. Let f be a simple function in L^p such that $f = \sum_{i=1}^{k} a_i 1_{E_i}$. As $L^p \subset$ weak-L^p, for any $\alpha \in (0, \infty)$, we have $\lambda_f(\alpha) < \infty$. Furthermore, it follows that

$$\lambda_f(\alpha) = 0, \text{ if } \alpha \geq \max\{|a_i|\},$$

and, by right continuity of λ_f,

$$\lambda_f(\alpha) = \lim_{\alpha'\to 0}\lambda_f(\alpha') \text{ if } \alpha < \min\{|a_i|\}.$$

Without loss of generality, assume that $a_i \neq 0$ for all i. By the above observation, it follows that

$$\lim_{\alpha\to\infty}\alpha^p\lambda_f(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha\to 0}\alpha^p\lambda_f(\alpha) = 0,$$

so

$$\lambda_f(\alpha) = o(\alpha^{-p})$$

for both $a \to \infty$ and $a \to 0$ limits. Now, we use a standard density argument to show that the claim holds true for any $f \in L^p$. Let $f \in L^p$. Fix $\epsilon > 0$. As simple functions are dense in L^p, there exists a simple function $s \in L^p$ such that

$$||f - s||_p < \epsilon$$

Then

$$\begin{aligned}
0 \leq \alpha^p\lambda_f(\alpha) &= \alpha^p\lambda_{f-s}(\frac{\alpha}{2}) + \alpha^p\lambda_s(\frac{\alpha}{2}) \\
&\leq 2^p||f - s||_p^p + 2^p(\frac{\alpha}{2})^p\lambda_s(\frac{\alpha}{2}) < C\epsilon,
\end{aligned}$$

for some constant C provided that α is small or large enough. Therefore,

$$\lambda_f(\alpha) = o(a^{-p}),$$

for both $a \to \infty$ and $a \to 0$ limits as required. $\square$