Harmonic Analysis:
Problem Set II

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Abstract

This work contains solutions to the problem set III of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Let α be any irrational number.

   (a) Show that for every trigonometric polynomial P
   $$\frac{1}{N}\sum_{n=1}^{N} P(n\alpha) \to \int_0^1 P(x)\,\mathrm{d}x.$$

   (b) Show that for every $f \in C(\mathbb{T})$
   $$\frac{1}{N}\sum_{n=1}^{N} f(n\alpha) \to \int_0^1 f(x)\,\mathrm{d}x.$$

   (c) Show that the conclusion of (b) continues to hold for every Riemann integrable function on $\mathbb{T}$. What about Lebesgue integrable functions?

Solution.
Throughout this problem, a domain of any function will be consistently defined as $\mathbb{T}$.
(a) Let α be a irrational number. First, consider $e(kx)$. For $k = 0$, we have 1 converges to 1. Now, for any $k \neq 0$, we have that $\int_0^1 e(kx) = 0$. Furthermore, as α is an irrational, we have that $e(k\alpha) \neq 1$. Hence, by the geometric series formula, it follows that

$$\frac{1}{N}\sum_{n=1}^{N} e(n\alpha) \quad = \quad \frac{e(k\alpha)}{N}\frac{1-e(kN\alpha)}{1-e(k\alpha)},$$

which converges to 0 as $N \to \infty$. Therefore, we have shown the convergence holds true for exponentials. As trig polynomials are finite linear combinations of exponentials, by the linearity of limit, the convergence is true for any trig polynomial.

(b) Fix $\epsilon > 0$. It follows that By the density of trig polynomials in $C(\mathbb{T})$, there exists a polynomial P such that $||P - f||_{max} < \epsilon$. By part (a), the triangle inequality, and rules of integration, it follows that,

$$
\begin{aligned}
\left|\frac{1}{N}\sum_{n=1}^{N} f(n\alpha)-\int_0^1 f(x)dx\right| &\leq \left|\frac{1}{N}\sum_{n=1}^{N} f(n\alpha)-\frac{1}{N}\sum_{n=1}^{N} P(n\alpha)\right| \\
&+ \left|\frac{1}{N}\sum_{n=1}^{N} P(n\alpha)-\int_0^1 P(x)dx\right| \\
&+ \left|\int_0^1 P(x)dx-\int_0^1 f(x)dx\right| \\
&\leq \frac{1}{N}\sum_{n=1}^{N} |f(n\alpha)-P(n\alpha)| \\
&+ \left|\frac{1}{N}\sum_{n=1}^{N} P(n\alpha)-\int_0^1 P(x)dx\right| \\
&+ \int_0^1 |P(x)-f(x)|dx < \epsilon+\epsilon+\epsilon = 3\epsilon,
\end{aligned}
$$

for all N large enough. Hence, we have shown that the convergence holds true for all continuous functions.

(c) Before proceeding to the main part of the proof, we prove that the asserted convergence holds true for all characteristic functions of $(a,b)\in\mathbb{T}$, which is quite natural as the definition of Riemann integration involves a partition with the associated upper sum and lower sum:

$$\frac{1}{N}\sum_{n=1}^{N}\chi_{(a,b)}(n\alpha)\to\int_0^1\chi_{(a,b)}(x)dx, \text{ as } N\to\infty. \tag{1}$$

Fix $\epsilon>0$, and $(a,b)\in\mathbb{T}$. Let f_ϵ^+ and f_ϵ^- be continuous functions on $\mathbb{T}$, defined by

$$
f_\epsilon^+(x) = \begin{cases}
0 & \text{if } x\in[0,a-\epsilon) \\
\frac{1}{\epsilon}(x-a)+1 & \text{if } x\in[a-\epsilon,a) \\
1 & \text{if } x\in[a,b) \\
-\frac{1}{\epsilon}(x-b)+1 & \text{if } x\in[b,b+\epsilon) \\
0 & \text{if } x\in[b+\epsilon,1],
\end{cases}
$$

and

$$
f_\epsilon^-(x) = \begin{cases}
0 & \text{if } x\in[0,a) \\
\frac{1}{\epsilon}(x-a) & \text{if } x\in[a,a+\epsilon) \\
1 & \text{if } x\in[a+\epsilon,b-\epsilon) \\
-\frac{1}{\epsilon}(x-b) & \text{if } x\in[b-\epsilon,b) \\
0 & \text{if } x\in[b,1].
\end{cases}
$$

In particular, we have

$$b-a-2\epsilon\leq\int_0^1 f_\epsilon^-(x)dx \text{ and } \int_0^1 f_\epsilon^+(x)dx\leq b-a+2\epsilon, \tag{2}$$

along with

$$\frac{1}{N}\sum_{n=1}^{N} f_\epsilon^-(n\alpha)\leq\frac{1}{N}\sum_{n=1}^{N}\chi_{(a,b)}(n\alpha)\leq\frac{1}{N}\sum_{n=1}^{N} f_\epsilon^+(n\alpha). \tag{3}$$

Now, letting $N\to\infty$ on both sides of (3) respectively, we obtain

$$b-a-2\epsilon\leq\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{(a,b)}(n\alpha) \text{ and } \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{(a,b)}(n\alpha)\leq b-a+2\epsilon.$$

As ϵ was arbitrary, we have shown that the asserted convergence is true for all characteristic functions in $\mathbb{T}$. As before, the result can be extended to all finite linear combinations of characteristic functions.

We proceed to the main part of the proof. Fix $\epsilon > 0$. As f is Riemann integrable, taking a fine enough partition, we have

$$\int_0^1 f(x)dx - \epsilon \leq \int_0^1 L_f(dx) \text{ and } \int_0^1 U_f(x)dx \leq \int_0^1 f(x)dx + \epsilon,$$

with

$$\frac{1}{N}\sum_{i=1}^{N} L_f(n\alpha) \leq \frac{1}{N}\sum_{i=1}^{N} f(n\alpha) \leq \frac{1}{N}\sum_{i=1}^{N} U_f(n\alpha),$$

where U_f, and L_f denote the upper, lower Riemann sum of the chosen partition respectively. In view of (1), letting $N \to \infty$, we obtain

$$\int_0^1 f(x)dx - \epsilon \leq \liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} f(n\alpha) \text{ and } \limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} f(n\alpha) \leq \int_0^1 f(x)dx + \epsilon.$$

As ϵ was arbitrary, we have shown that the asserted convergence is true for all Riemann integrable functions on $\mathbb{T}$.

Now, for the case of Lebesgue integration, we know that the Riemann integrable functions on $\mathbb{T}$

Question 2.

2. Let $\mathcal{T}_n$ denote the linear space of trigonometric polynomials of degree up to n and

$$E_n(f) := \inf_{P \in \mathcal{T}_n} \|f - P\|_2 = \|f - S_N f\|_2 = \left(\sum_{|k|>n} |\widehat{f}(k)|^2 \right)^{1/2}.$$

Let $0 < \alpha < 1$. Show that $E_n(f) \lesssim n^{-\alpha}$ if and only if $\omega_f(\delta)_{L^2} \lesssim \delta^\alpha$ where

$$\omega_f(\delta)_{L^2} := \sup_{|h| \le \delta} \|f - f(\cdot - h)\|_2.$$

This class of functions is called $\mathrm{Lip}_{\alpha, L^2}(\mathbb{T})$.

(Hint: Use the dyadic decomposition trick we employed in class before.)

Solution.

Question 3.

3. Solve Exercises 2.2, 2.3, 2.4 in Muscalu & Schlag.

Solution. □

Question 4.

4. Show that if a series $\sum a_n$ is Cesàro summable to A, then it is also Abel summable to A. In other words, show that

$$\lim_{N\to\infty} \sum_{n=-\infty}^{\infty} \left(1 - \frac{|n|}{N}\right)_+ a_n = A \text{ implies } \lim_{r \to 1^-} \sum_{n=-\infty}^{\infty} r^{|n|} a_n = A.$$

The reverse implication does not hold, however. Give a counter-example.
Can you generalize this result to series in arbitrary normed spaces?

(Hint: Note that you may first reduce the problem to one-sided series. Note also that the context of this problem is more general than summability of Fourier series.)

Solution.

□