Harmonic Analysis:
Problem Set V

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set IV of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. We saw that the Fourier transform is bounded from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$ for all $1 \le p \le 2$. Justify each of the following steps to show that the range of p cannot be extended beyond 2.

   (a) It suffices to show that for $p < 2$, there is no uniform constant $c_p > 0$ such that $\|\widehat{f}\|_{p'} \ge c_p \|f\|_p$ for all $f \in \mathcal{S}$, that is, the direction of the Hausdorff-Young inequality cannot be reversed. Why?

   (b) Show that there is an absolute constant C such that for all a and b

   $$\left| \int_a^b e^{-ix^2} dx \right| \le C.$$

   (c) Let $f_\lambda(x) := e^{-\pi x^2} e^{-\pi i \lambda x^2}$. Using the above result, and after an appropriate change of variables and integration by parts, show that $\|\widehat{f}_\lambda\|_\infty \le C\lambda^{-1/2}$ for some absolute constant C.

   (d) Let $p < 2$. Justify the chain of inequalities

   $$\left\|\widehat{f}_\lambda\right\|_{p'} \le \left\|\widehat{f}_\lambda\right\|_2^{2/p'} \left\|\widehat{f}_\lambda\right\|_\infty^{1-2/p'} \le C\lambda^{1/p'-1/2}$$

   and deduce the result in (a).

   (e) As a by-product, argue that for $1 \le p < 2$, the Fourier transform is not a surjective map from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$. (Hint: A classical theorem of functional analysis.)

Solution.
(a) Let $p < 2$. Assume that there is no uniform constant $c_p > 0$ such that $||\hat{f}||_{p'} \ge c_p ||f||_p$ for all $f \in S$. Since Fourier transform is a bijective mapping from S to S, and, for $f \in S$, $||\hat{\hat{f}}||_{p'} = ||f||_{p'}$ (Fourier transform composed twice on the Schwarz space is a parity operator, which does not change the integration result for $f \in S$), substituting $\hat{f}$ in the position of f implies that there is no uniform constant $c_p > 0$ such that

$$||f||_{p'} \quad \ge \quad c_p ||\hat{f}||_p,$$

for any $f \in S$. Since $S \subset L^p$, we conclude that the given statement is sufficient.

(b) With $u = \sqrt{i}x$, it follows that

$$\int e^{ix^2} = \frac{1}{\sqrt{i}} \int e^{u^2} du = (\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})\text{erf}(u) + C,$$

where the last equality holds with the well-known error function. Since $|\text{erf}(z)| \leq 1$ for any $z \in \mathbb{C}$, it follows that

$$\begin{aligned} |\int_a^b e^{ix^2} dx| &= |(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})||\text{erf}(b) - \text{erf}(a)| \\ &\leq 2|(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})| \leq C, \end{aligned}$$

for some absolute constant C.

(c) It follows that, for some absolute constant C, with any $\xi \in \mathbb{R}$,

$$|\hat{f}_\lambda(\xi)| \leq C\lambda^{-\frac{1}{2}},$$

so

$$||\hat{f}_\lambda||_\infty \leq C\lambda^{-\frac{1}{2}},$$

as required.

(d) By generalized Holder's inequality with $n = 2$ and $\theta = \frac{2}{p'}$, we obtain

$$||\hat{f}_\lambda||_{p'} \quad \leq \quad ||\hat{f}_\lambda||_2^{\frac{2}{p'}} ||\hat{f}_\lambda||_\infty^{1-\frac{2}{p'}}.$$

As $|e^{-\pi i\lambda x^2}| = 1$ for any $x \in \mathbb{R}$, and $||f||_2 = ||\hat{f}||_2$ for $f \in L^2$, we see that

$$||\hat{f}_\lambda||_2 \quad = \quad ||f_\lambda||_2 = (\int_{\mathbb{R}} |e^{-\pi x^2} e^{-\pi i\lambda x^2}|^p)^{\frac{1}{p}} = ||e^{-\pi x^2}||_p = C,$$

for some absolute constant C. Therefore, by (c) and the above equality, it follows that

$$||\hat{f}_\lambda||_{p'} \quad \leq \quad ||\hat{f}_\lambda||_2^{\frac{2}{p'}} ||\hat{f}_\lambda||_\infty^{1-\frac{2}{p'}} \leq C\lambda^{\frac{1}{p'}-\frac{1}{2}},$$

for some absolute constant C. Now, we deduce (a). Similarly, we have

$$||f_\lambda||_p \quad = \quad (\int_{\mathbb{R}} |e^{-\pi x^2} e^{-\pi i\lambda x^2}|^p)^{\frac{1}{p}} = ||e^{-\pi x^2}||_p = \delta,$$

for some constant $\delta > 0$, independent of λ. Let $c_p > 0$ be given. Then, for $\lambda > 0$ sufficiently small, it follows that

$$||\hat{f}_\lambda||_{p'} \leq C\lambda^{\frac{1}{p'}-\frac{1}{2}} \leq c_p||f_\lambda||_p,$$

where C is some absolute constant. Therefore, there does not exist a uniform constant $c_p > 0$ such that $||\hat{f}||_{p'} \geq c_p||f||_p$ for all $f \in S$.

(e) Let $1 \leq p < 2$. Suppose for sake of contradiction that Fourier transform is a surjective map from L^p to $L^{p'}$. As the Fourier transform is unique on L^p for $1 \leq p < 2$, this implies that the Fourier transform is a bijective map from L^p to $L^{p'}$. Now, by the Open Mapping theorem, we have that the Fourier transform is a bijective, open map from L^p to $L^{p'}$. Then, the inverse map of the Fourier transform is well-defined and continuous from $L^{p'}$ to L^p. This is a contradiction to (a).

Question 2.

2. For any $\Lambda < \infty$, let $\mathcal{B}^1_\Lambda(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [-\Lambda, \Lambda]\}$.

   (a) Show that $\mathcal{B}^1_\Lambda(\mathbb{R}) \subset L^p(\mathbb{R})$ for all $1 \le p \le \infty$.

   (b) Show that every function in $\mathcal{B}^1_\Lambda(\mathbb{R})$ is (Lebesgue) equivalent to a C^∞ function.

   (c) Show that for any $f \in \mathcal{B}^1_\Lambda(\mathbb{R})$, $1 \le p \le \infty$, and $m \ge 1$, we have

   $$\|f^{(m)}\|_p \le (c\Lambda)^m \|f\|_p$$

   where c is some absolute constant.
   (Hint: Pick any $\varphi \in \mathcal{S}$ that satisfies $\widehat{\varphi}(\xi) = 1$ for $|\xi| \le 1$, and scale it as $\varphi_\Lambda(x) = \Lambda\varphi(\Lambda x)$ to match the bandwidth of f. What is $f * \varphi_\Lambda$ equal to? Next, differentiate both sides and apply a familiar inequality.)

   (d) Let $f \in \mathcal{B}^1_\Lambda(\mathbb{R})$ and extend its domain to $\mathbb{C}$ via $f(z) := \int_{-\Lambda}^{\Lambda} \widehat{f}(\xi) e^{2\pi i \xi z}\, d\xi$, $z \in \mathbb{C}$. Show that f is entire.

Solution.
(a) Let $f \in B^1_\lambda$, and X be the characteristic function of $[-\Lambda, \Lambda]$. As $f \in L^1$, applying the inversion formula to $\hat{f} = \hat{f}X$ gives

$$f \quad = \quad f * \check{X}.$$

Since $X \in S$, it follows that $\check{X} \in S$, so $\check{X} \in L^p$ for any $p \ge 1$. Therefore for any $p \ge 1$, by Young's inequality, we obtain

$$||f||_p \le ||f||_1 ||\check{X}||_p < \infty,$$

as required.

(b) Let $f \in L^1$. It follows that $\hat{f}' = -2\pi i \hat{f}$. Define g by

$$g(x) \quad = \quad -2\pi i \int \hat{f}(\xi) e^{ix\xi} d\xi.$$

(d) From classical complex analysis, we have the following theorem.

Theorem. Let $F(z, \xi)$ be defined for $(z, \xi) \in \Omega \times [-M, M]$ where Ω is an open set in $\mathbb{C}$. Suppose F satisfies the following properties: (i) $F(z, \xi)$ is holomorphic in z for each ξ. (ii) F is continuous on $\Omega \times [-M, M]$. Then, $f(z) = \int_{[-M,M]} \hat{f}(\xi) e^{2\pi i \xi z} d\xi$ is holomorphic.

The proof can be found in pg. 56 of Stein's Complex Analysis. To invoke the theorem, we take $\Omega = \mathbb{C}$, $M = \Lambda$, and $F(z, \xi) = \hat{f}(\xi) e^{2\pi i \xi z}$. For any $\xi \in [-\Lambda, \Lambda]$, $\hat{f}(\xi) e^{2\pi i \xi z}$ is continuous, so (i) is satisfied. F

Question 3.

3. Let $\varphi \in L^2(\mathbb{R})$.

(a) Show that the family $\{e^{2\pi i m x}\varphi(x)\}_{m\in\mathbb{Z}}$ is an orthonormal system if and only if

$$\sum_{n\in\mathbb{Z}} |\varphi(x-n)|^2 = 1 \qquad a.e.\ x.$$

(Hint: The expression on the left hand side above defines a function in $L^1(\mathbb{T})$.)

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(b) Show that the family $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ is an orthonormal system if and only if

$$\sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\xi-m)|^2 = 1 \qquad a.e.\ \xi.$$

Solution.

(a) By definition of ONS, we have

$$\{e_n\varphi(x)\}_{m\in\mathbb{Z}} \text{ is an ONS } \iff \int e_n\varphi\overline{e_m\varphi}(x)dx = 0$$

(b) As $\varphi \in L^2$, we have $\hat{\varphi} \in L^2$. From (a), it follows that

$$\sum_{m\in\mathbb{Z}} |\hat{\varphi}(\xi-m)|^2 = 1 \text{ a.e. } \xi \iff \{e^{2\pi i m\xi}\hat{\varphi}(\xi)\}_{m\in\mathbb{Z}} \text{ is an ONS.}$$

As the Fourier transform is an isometry on L^2, we have

$$\{\varphi(\xi-m)\}_{m\in\mathbb{Z}} \text{ is an ONS } \iff \{\varphi\widehat{(\xi-m)}\}_{m\in\mathbb{Z}} \text{ is an ONS,}$$

which through the identity $\varphi\widehat{(\xi-m)} = \hat{\varphi}(\xi)e^{-2\pi i m\xi}$ and the first equivalence implies

$$\sum_{m\in\mathbb{Z}} |\hat{\varphi}(\xi-m)|^2 = 1 \text{ a.e. } \xi \iff \{\varphi(\xi-m)\}_{m\in\mathbb{Z}} \text{ is an ONS}$$

as required. $\square$

Question 4.

4. For any $\Lambda < \infty$, let $\mathcal{B}^2_\Lambda(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [-\Lambda, \Lambda]\}$.

(a) How would you restate the results of 2(a)-(d)?

(b) Show that $\mathcal{B}^2_\Lambda(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$ which is invariant under translations.

(c) Let $\operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$, $x \neq 0$, $\operatorname{sinc}(0) := 1$. Show that $\{\operatorname{sinc}(\cdot - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{B}^2_{1/2}(\mathbb{R})$. Express any $f \in \mathcal{B}^2_{1/2}(\mathbb{R})$ in this basis and show

$$f = \sum_{n\in\mathbb{Z}} f(n)\operatorname{sinc}(\cdot - n)$$

where f has been identified with its equivalent continuous version (so that $f(n)$ makes sense).

(d) Show that the 2-norm convergent series expansion in (c) is also uniformly convergent.

Solution.
(a) We restate the results for B^2_Λ instead of B^1_Λ.

(b) Let $\{f_n\}$ be a sequence in B^2_Λ such that it converges to some $f \in L^2$. As $||\hat{g}||_2 = ||g||_2$ for any $g \in L^2$, it follows that $\{\hat{f_n}\}$ converges to $\hat{f}$ in L^2. Since the compactness of the support persists through L^2 limit (this trivially can be shown using a proof by contradiction), it follows that $\text{supp}\hat{f} \subset [-\Lambda, \Lambda]$, so $f \in B^2_\Lambda$. Hence, B^2_Λ is closed.

We now show that B^2_Λ is invariant under translations. Fix $h \in \mathbb{R}$, and let η_h be defined by $(\eta_h f)(x) = f(x-h)$ on L^2. If $g \in B^2_\Lambda$,