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# Harmonic Analysis: Final Exam

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## Abstract

This work contains a solution to the Final Exam of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

### Question 1.

Watching these videos, you will see that the synthesis operation is quite intuitive, but the analysis operation (which was the very reason Michelson built this machine) requires knowing a bit more mathematics (shall we say, *harmonic analysis*). On pages 98 and 99 of their book you will find the basic mathematical explanation (but watch the videos first!). In this problem, we will make things a bit more rigorous.

- (a) We defined the Discrete Fourier Transform (DFT) on our very first class (as the Fourier transform on  $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ ). Namely, if  $x := (x_n)_{n \in \mathbb{Z}}$  is an  $N$ -periodic sequence seen as a function on  $\mathbb{Z}_N$ , then its DFT  $\hat{x}$  is the  $N$ -periodic sequence given by

$$\hat{x}_k := \sum_{n \in \mathbb{Z}_N} x_n e^{-2\pi i n k / N}, \quad k \in \mathbb{Z},$$

which we may also identify with a function on  $\mathbb{Z}_N$ . Then we have the inversion formula

$$x_n = \frac{1}{N} \sum_{k \in \mathbb{Z}_N} \hat{x}_k e^{2\pi i n k / N}, \quad n \in \mathbb{Z},$$

(which you can easily verify even if it's new to you).

Let  $P$  be a trigonometric polynomial of degree  $N$ , i.e.,  $P \in \text{span}\{e_k : -N \leq k \leq N\}$ . Define  $p := (p_n)_{n \in \mathbb{Z}}$  to be the  $(2N+1)$ -periodic sequence given by  $p_n := \frac{1}{2N+1} P(\frac{n}{2N+1})$ ,  $n \in \mathbb{Z}$ , and let  $\hat{p}$  be its DFT (as a  $(2N+1)$ -periodic sequence). Show that  $\hat{p}_k = \hat{P}(k)$  for all  $|k| \leq N$ . (Here  $\hat{P}(k)$  is the  $k$ th Fourier series coefficient of  $P$ .) What happens for  $|k| > N$ ?

**Solution.**

(a) Let  $P$  be a trig polynomial defined on  $\mathbb{T}$  of degree  $N$ , i.e.

$$P = \sum_{|k| \leq N} a_k e_k,$$

where  $a_k$ s are the complex coefficients. Suppose  $|k| \leq N$ . We trivially know that  $\hat{P}(k) = a_k$ . We compute the LHS as

$$\begin{aligned} \hat{p}_k &= \sum_{n \in \mathbb{Z}_{2N+1}} p_n e^{-\frac{2\pi i n k}{2N+1}} = \sum_{n \in \mathbb{Z}_{2N+1}} \frac{1}{2N+1} P\left(\frac{n}{2N+1}\right) e^{-\frac{2\pi i n k}{2N+1}} \\ &= \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_{2N+1}} \left( \sum_{|l| \leq N} a_l e_l\left(\frac{n}{2N+1}\right) \right) e^{-\frac{2\pi i n k}{2N+1}} \\ &= \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_{2N+1}} \left( \sum_{|l| \leq N} a_l e^{\frac{2\pi i l n}{2N+1}} \right) e^{-\frac{2\pi i n k}{2N+1}} \\ &= \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_{2N+1}} \left( \sum_{|l| \leq N} a_l e^{\frac{2\pi i (l-k)n}{2N+1}} \right) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_{2N+1}} a_n = a_n \end{aligned}$$

as required. Now, for  $|k| > N$ , we see that  $\hat{p}_k = 0$ , and  $\hat{P}(k) = 0$  as well.

**Question 1-2.**

- (b) For any  $F \in C(\mathbb{T})$ , define  $f := (f_n)_{n \in \mathbb{Z}}$  to be the  $(2N+1)$ -periodic sequence given by  $f_n := \frac{1}{2N+1} F(\frac{n}{2N+1})$ ,  $n \in \mathbb{Z}$ , and let  $\hat{f}$  be its DFT (as a  $(2N+1)$ -periodic sequence). Show that for any trigonometric polynomial  $P$  of degree  $N$ , we have

$$\max_{|k| \leq N} |\hat{f}_k - \hat{P}(k)| \leq 2\|F - P\|_\infty.$$

(Hint: Use (a). In particular, write  $F = (F - P) + P$  and  $f = (f - p) + p$  where  $p$  is as in (a).)

- (c) Part (b) says that the DFT of regular samples of  $F$  at  $2N + 1$  points approximate the Fourier series coefficients of  $F$  to the same extent  $F$  can be approximated by a trigonometric polynomial of degree  $N$ . Now explain in rigorous terms how Michelson uses his machine, basically a harmonic “synthesizer,” as a harmonic “analyzer” instead. Here you may also wish to allude to the Fourier transform on  $\mathbb{Z}$  (also known as the “Discrete Time Fourier Transform (DTFT)”) which maps any (absolutely summable) sequence  $(g_n)_{n \in \mathbb{Z}}$  to a function on the torus

$$\hat{g}(\xi) := \sum_{n \in \mathbb{Z}} g_n e^{-2\pi i n \xi}, \quad \xi \in \mathbb{T},$$

and the simple observation that the DFT of a periodic sequence is the same as the samples of the DTFT of its truncation to a period.

- (d) Michelson wanted to build a new version of his analyzer with 1000 elements, but never did. How accurately would he be able to calculate the Fourier coefficients of a piecewise-linear continuous function using an analyzer with  $N$  elements?

**Solution.**

(b) Let  $f \in C(\mathbb{T})$ , and  $P$  be a trig polynomial of degree  $N$ . Define as before  $f \triangleq (f_n)_{n \in \mathbb{Z}}$  to be the  $(2N + 1)$ -periodic sequence given by

$$f_n \triangleq \frac{1}{2N+1} F\left(\frac{n}{2N+1}\right).$$

With  $F = (F - P) + P$  and  $f = (f - p) + p$ , by the result of (a), for any  $|k| \leq N$ , we see that

$$\begin{aligned} |\hat{f}_k - \hat{P}_k| &= |\widehat{\{(f - p) + p\}}_k - \widehat{(F - P) + P}(k)| \\ &= |\widehat{(f - p)}_k - \widehat{(F - P)}(k) + \hat{p}_k - \hat{P}(k)| \\ &\leq |\widehat{(f - p)}_k| + |\widehat{(F - P)}(k)| + |\hat{p}_k - \hat{P}(k)|. \\ &= |\widehat{(f - p)}_k| + |\widehat{(F - P)}(k)|. \end{aligned}$$

Hence, as  $N$  is finite, it suffices to show that for any  $|k| \leq N$ ,

$$|\widehat{(f - p)}_k| \leq \|F - P\|_\infty \text{ and } |\widehat{(F - P)}(k)| \leq \|F - P\|_\infty.$$

Now, for  $|k| \leq N$ , the first inequality follows, as

$$\begin{aligned} |\widehat{(f - p)}_k| &= \left| \sum_{n \in \mathbb{Z}_{2N+1}} (f - p)_n e^{\frac{-2\pi i n k}{2N+1}} \right| \\ &= \left| \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_{2N+1}} \left( F\left(\frac{n}{2N+1}\right) - P\left(\frac{n}{2N+1}\right) \right) e^{\frac{-2\pi i n k}{2N+1}} \right| \\ &\leq \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_{2N+1}} \left| F\left(\frac{n}{2N+1}\right) - P\left(\frac{n}{2N+1}\right) \right| = \|F - P\|_\infty. \end{aligned}$$

Likewise, for  $|k| \leq N$ , the second inequality, as  $F - P \in C(\mathbb{T})$ , and

$$\begin{aligned} |\widehat{F - P}(k)| &= \left| \int_{\mathbb{T}} (F - P)(t) e^{-ikt} dt \right| \leq \int_{\mathbb{T}} |(F - P)(t)| dt \\ &= \|F - P\|_1 \leq \|F - P\|_{\infty}. \end{aligned}$$

Therefore, we have that for any trig polynomial of degree  $N$ ,  $P$ ,

$$\max_{|k| \leq N} |\hat{f}_k - \hat{F}(k)| \leq 2\|F - P\|_{\infty},$$

as required.

**Question 2.**

2. Let  $\tau_h f = f(\cdot - h)$  be the translation operator for functions defined on  $\mathbb{T}$ .

- (a) Show that a bounded linear operator  $T$  on  $L^2(\mathbb{T})$  commutes with translations (i.e.,  $T(\tau_h f) = \tau_h(Tf)$  for all  $h \in \mathbb{T}$  and  $f \in L^2(\mathbb{T})$ ) if and only if

$$Tf = \sum_{n \in \mathbb{Z}} \xi_n \hat{f}(n) e_n$$

for some  $\xi \in l^\infty(\mathbb{Z})$ . Show also that  $\|T\|_{2 \rightarrow 2} = \|\xi\|_\infty$ .  
(Hint: Consider  $T(e_n)$ .)

- (b) Show that a bounded linear operator  $T$  on  $L^1(\mathbb{T})$  commutes with translations if and only if  $Tf = f * \mu$  for some  $\mu \in M(\mathbb{T})$ . Show also that  $\|T\|_{1 \rightarrow 1} = \|\mu\|_{M(\mathbb{T})}$ .  
(Hint: You may find the Fejer kernel useful.)

**Solution.**

(a) Fix  $h \in \mathbb{T}$ . Suppose that, for some  $\xi \in l^\infty(\mathbb{Z})$ , we have

$$Tf = \sum_{n \in \mathbb{Z}} \xi_n \hat{f}(n) e_n,$$

for some  $\xi \in l^\infty(\mathbb{Z})$ . It follows that, for any  $n \in \mathbb{Z}$ , by linearity of  $T$ ,

$$T(\tau_h(e_n)) = e^{-2\pi i n h} T(e_n) = e^{-2\pi i n h} \xi_n e_n = \tau_h(T(e_n)).$$

With the fact that the trig polynomials are dense in  $L^2$ , through a standard density argument, we obtain

$$T(\tau_h(f)) = \tau_h(T(f)),$$

as required. Conversely,

Now, let  $f = \frac{\|\xi\|_\infty}{|\xi_n|} e_n$  for some fixed  $n$ . By the established equivalence, it follows that

$$\|Tf\|_2 =$$

(b) Consider the Fejer kernel, denoted by  $\{K_n\}$ . Firstly, observe that as the Fejer kernel is an approximate identity, we have  $\sup_N \int_0^1 |K_n(x)| dx < \infty$ , so we can extract a subsequence  $\{K_{n_i}\}$  such that  $K_{n_i} \rightarrow \mu$  weakly in  $L^1$ , for some  $\mu \in M(\mathbb{T})$ . For notational convenience, we relabel the subsequence as  $\{K_n\}$ . Now, assume that  $T$  commutes with translations. It follows that, for any  $f \in C(\mathbb{T})$ , and  $x \in \mathbb{T}$ ,

$$\begin{aligned} f * \mu(x) &= \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f(t) T(K_n)(x - t) dt = \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f(t) T(K_n(x - t)) dt, \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{T}} T(f(t)(K_n)(x - t)) dt = \lim_{n \rightarrow \infty} T \left( \int_{\mathbb{T}} f(t) K_n(x - t) dt \right), \end{aligned}$$

which via boundedness of  $T$  and the fact that  $f \in C(\mathbb{T})$ ,

$$\lim_{n \rightarrow \infty} (f * K_n)(x) = f(x),$$

implies that

$$f * \mu(x) = T \left( \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f(t) K_n(x - t) dt \right) = Tf(x),$$

as required. Now, as  $C(\mathbb{T})$  is dense in  $L^1(\mathbb{T})$ , it follows that

$$f * \mu = Tf$$

**Question 3.**

3. For any bounded measurable function  $m$  on  $\mathbb{R}$ , consider the operator  $T_m$  defined by

$$T_m f := (m\hat{f})^\vee.$$

We know that  $T_m$  is bounded on  $L^2(\mathbb{R})$ .

(a) Suppose  $m$  is a monotonic step function given by

$$m(\xi) = \sum_{i=1}^n c_i \chi_{[\alpha_i, \alpha_{i+1})}(\xi),$$

where  $-\infty = \alpha_0 < \alpha_1 < \dots < \alpha_n = \infty$ . Show that

$$\|T_m\|_{p \rightarrow p} \lesssim_p \|m\|_\infty + |m|_{TV}, \quad 1 < p < \infty,$$

where  $|\cdot|_{TV}$  is the total variation.

(Hint: Express  $T_m$  in terms of the Hilbert transform.)

(b) Extend the conclusion of (a) to any  $m$  of bounded variation.

**Solution.**

(a) Suppose  $m$  is a monotonic step function given by

$$T_m f = (m\hat{f})^\wedge = \left( \sum_{i=1}^n c_i X_{[\alpha_i, \alpha_{i+1})} \hat{f} \right)^\wedge.$$

Under this setting, we simply have that

$$\|m\|_\infty = \max_i |c_i| \quad \text{and} \quad |m|_{TV} = \sum_{i=1}^{n-1} |c_{i+1} - c_i|$$

We have that

$$\|T_m\|_{p \rightarrow p} \leq C(p)$$

**(b)**

**Question 4.**

4. (a) Suppose  $f \in L^1(\mathbb{R})$  satisfies

$$f = f * f. \quad (1)$$

Show that  $f = 0$  a.e.

- (b) Suppose  $f \in L^2(\mathbb{R})$  satisfies (1). Show that  $f$  is uniformly continuous and in  $L^p(\mathbb{R})$  for all  $2 \leq p \leq \infty$ .

- (c) Let  $p \in (1, 2)$ . Show that if  $f \in L^p(\mathbb{R})$  satisfies (1), then  $f \in L^2(\mathbb{R})$ .

**Solution.**

- (a) Let  $f \in L^1$ . Taking the Fourier transform on both sides gives

$$\hat{f} = \hat{f}\hat{f},$$

which, with the continuity of  $\hat{f}$ , implies that

$$\hat{f} = 0 \text{ a.e.} \quad \text{or} \quad \hat{f} = 1 \text{ a.e.}$$

As  $\hat{f} = 1$  a. e contradicts the Riemann-Lebesgue lemma, it follows that

$$\hat{f} = 0 \text{ a.e.,}$$

so by the inversion formula for  $L^1$

$$f = 0 \text{ a.e.,}$$

as required.

- (b) Let  $f \in L^2$  such that  $f = f * f$ . By the same argument in the  $L^1$  case, we have

$$\hat{f} = 0 \text{ a.e.} \quad \text{or} \quad \hat{f} = 1 \text{ a.e.}$$

Let  $E_1 = \{\hat{f} = 1\}$  and  $E_0 = \{\hat{f} = 0\}$ . Note that as  $\hat{f} \in L^2$ , it follows that  $m(E_1) < \infty$ . Now, by the inversion formula for  $L^2$ , we obtain

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi = \int_{E_1} e^{2\pi i \xi x} d\xi.$$

Now, for any  $\delta > 0$  and  $x \in \mathbb{R}$ , it follows that

$$\begin{aligned} |f(x + \delta) - f(x)| &= \left| \int_{E_1} e^{2\pi i \xi (x + \delta)} - e^{2\pi i \xi x} d\xi \right| \leq \int_{E_1} |e^{2\pi i \xi x}| |e^{2\pi i \xi \delta} - 1| d\xi \\ &\leq \int_{E_1} |e^{2\pi i \xi \delta} - 1| d\xi. \end{aligned}$$

Observe that the last integral is independent of  $x$ , and the integrand tends to 0, as  $\delta \rightarrow 0$ . Therefore, we have shown that  $f$  is uniformly continuous. We now argue that  $f \in L^p$  for  $p \in [2, \infty]$ . Fix  $p \in [2, \infty]$ .

- (c)