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# Harmonic Analysis: Problem Set V

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## Abstract

This work contains solutions to the problem set IV of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

### Question 1.

1. We saw that the Fourier transform is bounded from  $L^p(\mathbb{R})$  to  $L^{p'}(\mathbb{R})$  for all  $1 \leq p \leq 2$ . Justify each of the following steps to show that the range of  $p$  cannot be extended beyond 2.

- (a) It suffices to show that for  $p < 2$ , there is no uniform constant  $c_p > 0$  such that  $\|\hat{f}\|_{p'} \geq c_p \|f\|_p$  for all  $f \in S$ , that is, the direction of the Hausdorff-Young inequality cannot be reversed. Why?
- (b) Show that there is an absolute constant  $C$  such that for all  $a$  and  $b$

$$\left| \int_a^b e^{-ix^2} dx \right| \leq C.$$

- (c) Let  $f_\lambda(x) := e^{-\pi x^2} e^{-\pi i \lambda x^2}$ . Using the above result, and after an appropriate change of variables and integration by parts, show that  $\|\hat{f}_\lambda\|_\infty \leq C \lambda^{-1/2}$  for some absolute constant  $C$ .
- (d) Let  $p < 2$ . Justify the chain of inequalities

$$\|\hat{f}_\lambda\|_{p'} \leq \|\hat{f}_\lambda\|_2^{2/p'} \|\hat{f}_\lambda\|_\infty^{1-2/p'} \leq C \lambda^{1/p'-1/2}$$

and deduce the result in (a).

- (e) As a by-product, argue that for  $1 \leq p < 2$ , the Fourier transform is not a surjective map from  $L^p(\mathbb{R})$  to  $L^{p'}(\mathbb{R})$ . (Hint: A classical theorem of functional analysis.)

### Solution.

(a) Let  $p < 2$ . Assume that there is no uniform constant  $c_p > 0$  such that  $\|\hat{f}\|_{p'} \geq c_p \|f\|_p$  for all  $f \in S$ . Since Fourier transform is a bijective mapping from  $S$  to  $S$ , and, for  $f \in S$ ,  $\|\hat{\hat{f}}\|_{p'} = \|f\|_{p'}$  (Fourier transform composed twice on the Schwarz space is a parity operator, which does not change any  $L^p$  norm for  $f \in S$ ), substituting  $\hat{f}$  in the position of  $f$  implies that there is no uniform constant  $c_p > 0$  such that

$$\|f\|_{p'} \geq c_p \|\hat{f}\|_p,$$

for any  $f \in S$ . Since  $S \subset L^p$ , we conclude that the given statement is sufficient.

(b) With  $u = \sqrt{i}x$ , it follows that

$$\int e^{ix^2} = \frac{1}{\sqrt{i}} \int e^{u^2} du = \left( \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) \text{erf}(u) + C,$$

where the last equality holds with the well-known error function. Since  $|\operatorname{erf}(z)| \leq 1$  for any  $z \in \mathbb{C}$ , it follows that

$$\begin{aligned} \left| \int_a^b e^{ix^2} dx \right| &= \left| \left( \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) (\operatorname{erf}(b) - \operatorname{erf}(a)) \right| \\ &\leq 2 \left| \left( \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) \right| \leq C, \end{aligned}$$

for some absolute constant  $C$ .

(c) By definition of Fourier transform, we have

$$\hat{f}_\lambda(\xi) = \int_{\mathbb{R}} f_\lambda(x) e^{-2\pi i x \xi} dx = \int_{\mathbb{R}} e^{-\pi x^2} e^{-\pi i \lambda x^2} e^{-2\pi i x \xi} dx,$$

By integration by parts, it follows that

$$\int_{-N}^N e^{-\pi x^2} e^{-\pi i \lambda x^2} e^{-2\pi i x \xi} dx =$$

It follows that, for some absolute constant  $C$ , with any  $\xi \in \mathbb{R}$ ,

$$|\hat{f}_\lambda(\xi)| \leq C \lambda^{-\frac{1}{2}},$$

so

$$\|\hat{f}_\lambda\|_\infty \leq C \lambda^{-\frac{1}{2}},$$

as required.

(d) By generalized Holder's inequality with  $n = 2$  and  $\theta = \frac{2}{p'}$ , we obtain

$$\|\hat{f}_\lambda\|_{p'} \leq \|\hat{f}_\lambda\|_2^{\frac{2}{p'}} \|\hat{f}_\lambda\|_\infty^{1 - \frac{2}{p'}}.$$

As  $|e^{-\pi i \lambda x^2}| = 1$  for any  $x \in \mathbb{R}$ , and  $\|f\|_2 = \|\hat{f}\|_2$  for  $f \in L^2$ , we see that

$$\|\hat{f}_\lambda\|_2 = \|f_\lambda\|_2 = \left( \int_{\mathbb{R}} |e^{-\pi x^2} e^{-\pi i \lambda x^2}|^p dx \right)^{\frac{1}{p}} = \|e^{-\pi x^2}\|_p = C,$$

for some absolute constant  $C$ . Therefore, by (c) and the above equality, it follows that

$$\|\hat{f}_\lambda\|_{p'} \leq \|\hat{f}_\lambda\|_2^{\frac{2}{p'}} \|\hat{f}_\lambda\|_\infty^{1 - \frac{2}{p'}} \leq C \lambda^{\frac{1}{p'} - \frac{1}{2}},$$

for some absolute constant  $C$ . Now, we deduce (a). Similarly, we have

$$\|f_\lambda\|_p = \left( \int_{\mathbb{R}} |e^{-\pi x^2} e^{-\pi i \lambda x^2}|^p dx \right)^{\frac{1}{p}} = \|e^{-\pi x^2}\|_p = \delta,$$

for some constant  $\delta > 0$ , independent of  $\lambda$ . Let  $c_p > 0$  be given. Then, for  $\lambda > 0$  sufficiently small, it follows that

$$\|\hat{f}_\lambda\|_{p'} \leq C \lambda^{\frac{1}{p'} - \frac{1}{2}} \leq c_p \|f_\lambda\|_p,$$

where  $C$  is some absolute constant. Therefore, there does not exist a uniform constant  $c_p > 0$  such that  $\|\hat{f}\|_{p'} \geq c_p \|f\|_p$  for all  $f \in S$ .

(e) Let  $1 \leq p < 2$ . Suppose for sake of contradiction that Fourier transform is a surjective map from  $L^p$  to  $L^{p'}$ . As the Fourier transform is unique on  $L^p$  for  $1 \leq p < 2$ , this implies that the Fourier transform is a bijective map from  $L^p$  to  $L^{p'}$ . Now, by the Open Mapping theorem, we have that the Fourier transform is a bijective, open map from  $L^p$  to  $L^{p'}$ . Then, the inverse map of the Fourier transform is well-defined and continuous from  $L^{p'}$  to  $L^p$ . This is a contradiction to (a).

## Question 2.

2. For any  $\Lambda < \infty$ , let  $\mathcal{B}_\Lambda^1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \text{supp } \hat{f} \subset [-\Lambda, \Lambda]\}$ .

- (a) Show that  $\mathcal{B}_\Lambda^1(\mathbb{R}) \subset L^p(\mathbb{R})$  for all  $1 \leq p \leq \infty$ .
- (b) Show that every function in  $\mathcal{B}_\Lambda^1(\mathbb{R})$  is (Lebesgue) equivalent to a  $C^\infty$  function.
- (c) Show that for any  $f \in \mathcal{B}_\Lambda^1(\mathbb{R})$ ,  $1 \leq p \leq \infty$ , and  $m \geq 1$ , we have

$$\|f^{(m)}\|_p \leq (c\Lambda)^m \|f\|_p$$

where  $c$  is some absolute constant.

(Hint: Pick any  $\varphi \in \mathcal{S}$  that satisfies  $\hat{\varphi}(\xi) = 1$  for  $|\xi| \leq 1$ , and scale it as  $\varphi_\Lambda(x) = \Lambda\varphi(\Lambda x)$  to match the bandwidth of  $f$ . What is  $f * \varphi_\Lambda$  equal to? Next, differentiate both sides and apply a familiar inequality.)

- (d) Let  $f \in \mathcal{B}_\Lambda^1(\mathbb{R})$  and extend its domain to  $\mathbb{C}$  via  $f(z) := \int_{-\Lambda}^{\Lambda} \hat{f}(\xi) e^{2\pi i \xi z} d\xi$ ,  $z \in \mathbb{C}$ . Show that  $f$  is entire.

## Solution.

(a) Let  $f \in \mathcal{B}_\Lambda^1$ , and  $X$  be the characteristic function of  $[-\Lambda, \Lambda]$ . As  $f \in L^1$ , applying the inversion to  $\hat{f} = \hat{f}X$  gives

$$f = f * \check{X}.$$

Arising from the fact that the Fourier transform is a bijective mapping on  $\mathcal{S}$  to  $\mathcal{S}$ ,  $\check{X}$  is in  $L^p$  for any  $1 \leq p \leq \infty$ , and  $X \in \mathcal{S}$ , it follows that  $\check{X} \in \mathcal{S}$ , so  $\check{X} \in L^p$  for any  $p \geq 1$ . Therefore for any  $1 \leq p \leq \infty$ , by Young's inequality for convolution (to be exact, it is the form recorded in Schlag pg.74), we obtain

$$\|f\|_p = \|f * \check{X}\|_p \leq \|f\|_1 \|\check{X}\|_p < \infty.$$

Hence  $f \in L^p$  for any  $1 \leq p \leq \infty$ , so we have shown that  $\mathcal{B}_\Lambda^1 \subset L^p$  for  $1 \leq p \leq \infty$ .

(b) Let  $f \in L^1$ . It follows that  $\hat{f}' = -2\pi i \hat{f}$ . Define  $g$  by

$$g(x) = -2\pi i \int \hat{f}(\xi) e^{ix\xi} d\xi.$$

(c) Set  $\varphi_\Lambda(\xi) = \Lambda\varphi(\Lambda\xi)$ . Then, by a change of variable  $y = \Lambda\xi$ , we obtain

$$\varphi_\Lambda(\xi) = \int \varphi(\Lambda x) e^{i\frac{\xi}{\Lambda}\Lambda x} d\Lambda x = \int \varphi(y) e^{i\frac{\xi}{\Lambda}y} dy = \hat{\varphi}\left(\frac{\xi}{\Lambda}\right),$$

so

$$\hat{\varphi}_\Lambda(\xi) = 1 \quad \text{for } |\xi| \leq \Lambda.$$

Therefore, by the noted observation and the inversion, it follows that

$$\widehat{f * \varphi_\Lambda} = \hat{f} \hat{\varphi}_\Lambda = \hat{f} \quad \text{and} \quad f * \varphi_\Lambda = f.$$

Differentiating the last identity  $m$  times, which is justified by (b), and choosing to differentiate  $\varphi_\Lambda$  term in the convolution yields

$$f^{(m)} = f * \varphi_\Lambda^{(m)}.$$

By the inequality employed in part (a), it follows that

$$\|f^{(m)}\|_p = \|f * \varphi_\Lambda^{(m)}\|_p \leq \|\varphi_\Lambda^{(m)}\|_1 \|f\|_p \leq (c\Lambda)^m \|f\|_p,$$

for some absolute constant  $c$ , as the Schwarz class is closed under differentiation and the last inequality follows from the definition of Schwarz class.

(d) From classical complex analysis, we have the following theorem.

**Theorem.** Let  $F(z, \xi)$  be defined for  $(z, \xi) \in \Omega \times [-M, M]$  where  $\Omega$  is an open set in  $\mathbb{C}$ . Suppose  $F$  satisfies the following properties: (i)  $F(z, \xi)$  is holomorphic in  $z$  for each  $\xi$ . (ii)  $F$  is continuous on  $\Omega \times [-M, M]$ . Then,  $f(z) = \int_{[-M, M]} \hat{f}(\xi) e^{2\pi i \xi z} d\xi$  is holomorphic.

The proof can be found in pg. 56 of Stein's Complex Analysis. To invoke the theorem, we take  $\Omega = \mathbb{C}$ ,  $M = \Lambda$ , and  $F(z, \xi) = \hat{f}(\xi) e^{2\pi i \xi z}$ . For any  $\xi \in [-\Lambda, \Lambda]$ ,  $\hat{f}(\xi) e^{2\pi i \xi z}$  is continuous, so (i) is satisfied. As  $\hat{f}$  is uniformly continuous, (ii) is also satisfied, so we conclude that  $f$  is entire.  $\square$

**Question 3.**

3. Let  $\varphi \in L^2(\mathbb{R})$ .

(a) Show that the family  $\{e^{2\pi imx}\varphi(x)\}_{m \in \mathbb{Z}}$  is an orthonormal system if and only if

$$\sum_{n \in \mathbb{Z}} |\varphi(x - n)|^2 = 1 \quad \text{a.e. } x.$$

(Hint: The expression on the left hand side above defines a function in  $L^1(\mathbb{T})$ .)

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(b) Show that the family  $\{\varphi(x - n)\}_{n \in \mathbb{Z}}$  is an orthonormal system if and only if

$$\sum_{m \in \mathbb{Z}} |\hat{\varphi}(\xi - m)|^2 = 1 \quad \text{a.e. } \xi.$$

**Solution.**

(a) By definition of ONS, we have

$$\{e_n \varphi(x)\}_{m \in \mathbb{Z}} \text{ is an ONS} \iff \int e_n \overline{e_m} \varphi(x) dx = 0$$

(b) As  $\varphi \in L^2$ , we have  $\hat{\varphi} \in L^2$ . From (a), it follows that

$$\sum_{m \in \mathbb{Z}} |\hat{\varphi}(\xi - m)|^2 = 1 \quad \text{a.e. } \xi \iff \{e^{2\pi im\xi} \hat{\varphi}(\xi)\}_{m \in \mathbb{Z}} \text{ is an ONS.}$$

As the Fourier transform is an isometry on  $L^2$ , we have

$$\{\varphi(\xi - m)\}_{m \in \mathbb{Z}} \text{ is an ONS} \iff \{\widehat{\varphi(\xi - m)}\}_{m \in \mathbb{Z}} \text{ is an ONS,}$$

which through the identity  $\widehat{\varphi(\xi - m)} = \hat{\varphi}(\xi) e^{-2\pi im\xi}$  and the first equivalence implies

$$\sum_{m \in \mathbb{Z}} |\hat{\varphi}(\xi - m)|^2 = 1 \quad \text{a.e. } \xi \iff \{\varphi(\xi - m)\}_{m \in \mathbb{Z}} \text{ is an ONS}$$

as required. □

**Question 4.**

4. For any  $\Lambda < \infty$ , let  $\mathcal{B}_\Lambda^2(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \text{supp } \hat{f} \subset [-\Lambda, \Lambda]\}$ .
- (a) How would you restate the results of 2(a)-(d)?
  - (b) Show that  $\mathcal{B}_\Lambda^2(\mathbb{R})$  is a closed subspace of  $L^2(\mathbb{R})$  which is invariant under translations.
  - (c) Let  $\text{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$ ,  $x \neq 0$ ,  $\text{sinc}(0) := 1$ . Show that  $\{\text{sinc}(\cdot - n)\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $\mathcal{B}_{1/2}^2(\mathbb{R})$ . Express any  $f \in \mathcal{B}_{1/2}^2(\mathbb{R})$  in this basis and show

$$f = \sum_{n \in \mathbb{Z}} f(n) \text{sinc}(\cdot - n)$$

where  $f$  has been identified with its equivalent continuous version (so that  $f(n)$  makes sense).

- (d) Show that the 2-norm convergent series expansion in (c) is also uniformly convergent.

**Solution.**

(a) We restate the results for  $B_\Lambda^2$  instead of  $B_\Lambda^1$ .

(b) Let  $\{f_n\}$  be a sequence in  $B_\Lambda^2$  such that it converges to some  $f \in L^2$ . As  $\|\hat{g}\|_2 = \|g\|_2$  for any  $g \in L^2$ , it follows that  $\{\hat{f}_n\}$  converges to  $\hat{f}$  in  $L^2$ . Since the compactness of the support persists through  $L^2$  limit (this trivially can be shown using a proof by contradiction), it follows that  $\text{supp } \hat{f} \subset [-\Lambda, \Lambda]$ , so  $f \in B_\Lambda^2$ . Hence,  $B_\Lambda^2$  is closed.

We now show that  $B_\Lambda^2$  is invariant under translations. Fix  $h \in \mathbb{R}$ , and let  $\eta_h$  be defined by  $(\eta_h f)(x) = f(x - h)$  on  $L^2$ . If  $g \in B_\Lambda^2$ ,