Harmonic Analysis:
Problem Set II

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set II of Harmonic Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Solve Exercise 1.8 in Muscalu & Schlag.

Solution.

Question 2.

2. The following (non-absolutely convergent) series define functions in $H^{\frac{1}{2}}(\mathbb{T})$. (Why?)

$$f_S(x) := \sum_{n=2}^{\infty} \frac{\sin(2\pi nx)}{n\log n}, \qquad f_C(x) := \sum_{n=2}^{\infty} \frac{\cos(2\pi nx)}{n\log n}$$

Show that the first series converges uniformly (hence $f_S \in C(\mathbb{T})$), but the second does not. In fact, show that $f_C(x) \geq c\log\log\frac{1}{|x|}$ as $x \to 0$ so that f_C is not even essentially bounded.
(Hint: Summation by parts.)

Remark: For an example of a $C^{1/2}(\mathbb{T})$ function which is not in $A(\mathbb{T})$, see Proposition 1.14 in Muscalu & Schlag. (There is also another example, due to Hardy-Littlewood:

$$\sum_{n=1}^{\infty} \frac{e^{in\log n}}{n} e^{2\pi inx},$$

Proof of this is given in Zygmund's "Trigonometric Series", vol. 1, p.197.)

Solution. Define $g_{n,m} = \sum_{i=n}^{m} \dfrac{\sin(2\pi nx)}{n\log(n)}$. For $x \in [0, \dfrac{1}{m}]$, we have

$$\begin{aligned} |g_{n,m}| &\leq \sum_{i=n}^{m} \left|\frac{\sin(2\pi ix)}{i\log(i)}\right| \leq \sum_{i=n}^{m} \frac{2\pi ix}{i\log(i)} = \sum_{i=n}^{m} \frac{2\pi x}{\log(i)} \\ &\leq \frac{1}{\log(n)} \leq \frac{1}{m\log(n)} \sum_{i=n}^{m} 2\pi \leq \frac{2\pi}{\log(n)}. \end{aligned}$$

For $x \in [\dfrac{1}{m}, \dfrac{1}{n}]$, we obtain

Therefore, we have that

$$|g_{n,m}| = O(\frac{1}{log(n)}),$$

and the partial sums of f_S is cauchy. Thus, f_S converges uniformly and $f \in C(\mathbb{T})$.

Question 3.

3. (Problem 1.5 in Muscalu & Schlag) Suppose $f \in H^{\frac{1}{2}}(\mathbb{T}) \cap C(\mathbb{T})$. Show that $S_N f \to f$ uniformly. (Hint: Study $S_N f - \sigma_N f$.)

Solution. By the triangle inequality of the supnorm, we have

$$||S_N f - f||_\infty \leq ||S_N f - \sigma_N f||_\infty + ||\sigma_N f - f||_\infty,$$

for all $N \in \mathbb{Z}^+$. As $f \in C(\mathbb{T})$, we have that $||\sigma_N f - f||_\infty \to 0$ as $N \to \infty$. Therefore, by the linearity of limit, it suffices to show that $||S_N f - \sigma_N f||_\infty \to 0$ as $N \to \infty$. By definition of S_N and σ_N, triangle inequality, and Cauchy-Schwarz, we obtain

$$\begin{aligned}
||S_N f - \sigma_N f||_\infty = \quad &\leq \quad \sum_{n=-N}^{N} \frac{|n|}{N} |\hat{f}(n)| \\
&\leq \quad \sum_{n=-M}^{M} \frac{|n||\hat{f}(n)|}{N} + (\sum_{N \geq |n| > M} \frac{|n|}{N^2})^{\frac{1}{2}} (\sum_{N \geq |n| > M} |n||\hat{f}(n)|^2)^{\frac{1}{2}}, \\
&\leq \quad \sum_{n=-M}^{M} \frac{|n||\hat{f}(n)|}{N} + 2(\sum_{N \geq |n| > M} |n||\hat{f}(n)|^2)^{\frac{1}{2}},
\end{aligned}$$

for any $N > M$. Taking $\limsup$ with respect to N on both sides, we get

$$\limsup_{N \to \infty} ||S_N f - \sigma_N f||_\infty \quad \leq \quad 2(\sum_{|n| > M} |n||\hat{f}(n)|^2)^{\frac{1}{2}},$$

As $f \in H^{\frac{1}{2}}(\mathbb{T})$, taking the limit as $M \to \infty$ gives

$$\limsup_{N \to \infty} ||S_N f - \sigma_N f||_\infty \quad \leq \quad 0$$

Hence, we have shown that $||S_N f - \sigma_N f||_\infty \to 0$ as $N \to \infty$ as desired. $\square$

Question 4.

4. Let $0 < \alpha < 1$. Note by a theorem we have seen in class (which one?) that $f \in C^\alpha(\mathbb{T})$ implies $\hat{f}(n) = O(|n|^{-\alpha})$. Then, note that the exponent in this decay estimate cannot be improved by showing that the function

$$F(x) = \sum_{m=1}^{\infty} \frac{1}{3^{m\alpha}} \cos(2\pi 3^m x)$$

belongs to $C^\alpha(\mathbb{T})$. Also solve Exercise 1.9 in Muscalu & Schlag.

Solution.
A theorem that gives this result of $f \in C^\alpha(\mathbb{T}) \implies \hat{f}(n) = O(n^{-\alpha})$ is recorded in section 1.4.4, pg.18 of Schleg. $\square$

Question 5.

5. Draw a minimal Venn diagram that shows all possible intersections of the sets below:

$$C(\mathbb{T}), \quad A(\mathbb{T}), \quad C^{2/3}(\mathbb{T}), \quad H^{1/2}(\mathbb{T}), \quad U(\mathbb{T}) := \{f : S_N f \to f \text{ uniformly}\}.$$

Your diagram should not have any redundancy or ambiguity, i.e., if $A \cap B = \emptyset$, $A \subset B$, or $A \neq B$, this should be visible and indicated. Give an example (or show the existence) of a function in each region of intersection.

Solution. $\square$