Linear Algebra I:
Problem Set II

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Abstract

This work contains the solutions to the problem set III of Linear Algebra I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

Solution. We assume that $n \geq 2$ and $n \leq j \leq 1$, in order for the $\det(M)$ and $\det(A)$ values to be well-defined. From the lemma 4 on pg.51 of Lax's Linear Algebra, which has the proof, we have that

$$\det(M) = \det(M_{11}),$$

where M_{11} denotes the $(n-1) \times (n-1)$ submatrix formed by entries m_{ij}, $i > 1, j > 1$. Observe that the same argument can be made on the M_{11}, to assert that $\det(M_{11}) = \det(M_{22})$. Using this argument exactly j-th time, we obtain that

$$\begin{aligned} \det(M) &= \det(M_{jj}) \\ &= \det(A), \end{aligned}$$

as desired. $\square$

Question 2.

Solution. Let A and B be square matrices. Assume that AB is invertible. As a determinant of an invertible matrix is nonzero, it follows that $\det(AB) \neq 0$. Since $\det(AB) = \det(A)\det(B)$, we obtain $\det(A)\det(B) \neq 0$, and thus $\det(A) \neq 0$ and $\det(B) \neq 0$. As a matrix with non-zero determinant is invertible, we have that A and B are invertible. $\square$

Question 3.

Solution. Let A be a square matrix. From the matrix multiplication rule, we have

$$(AA^T)_{ii} \quad = \quad \sum_k a_{ik} b_{ki},$$

where a_{ik} denotes the (i,k)th entry of the matrix A and b_{ki} denotes the (k,i)th entry of the matrix A^T. By definition of transpose, it follows that $a_{ik} = b_{ki}$, and we obtain

$$(AA^T)_{ii} \quad = \quad \sum_k a_{ik}^2.$$

Therefore, by definition of trace, we have

$$\text{tr}(AA^T) \quad = \quad \sum_{i,k} a_{ik}^2.$$

Since $a_{ik}^2 \geq 0$ for all i, k, it follows that

$$\text{tr}(AA^T) \quad \geq \quad 0.$$

$\square$

Question 4.

Solution. Let λ be any eigenvalue of A and v be the corresponding eigenvector. It follows that

$$A^3 v = \lambda^3 v.$$

As $A^3 = I$, it follows that

$$v = \lambda^3 v.$$

Hence, we have that $\lambda^3 = 1$. Therefore, we have shown that $\lambda^3 = 1$ is a necessary condition for λ being an eigenvalue of A. Therefore, from this, we deduce that the possible eigenvalues are 1, $e^{i\frac{2\pi}{3}}$ and $e^{i\frac{4\pi}{3}}$. □

Question 5.

Solution. Let A be a diagonalizable matrix, and D be the diagonal matrix, that is similar to A. From the theorem 12 from Lax's Linear Algebra in pg.73, which has a proof, we have that A and D have the same eigenvalues. Since the diagonal entires are precisely the set of eigenvalues for diagonal matrices, and the A matrix can only have one possible eigenvalue, D also can only have one posible eigenvalue, which we denote as λ. Hence, it follows that

$$\begin{aligned} D &= \lambda I \\ A &= SDS^{-1} \\ &= SS^{-1} \\ &= \lambda SS^{-1} \\ &= \lambda I. \end{aligned}$$

Hence, we have shown that A is a multiple of the identity matrix. □

Question 6.

Solution. We are given that $\det(A) = \det(A^T)$. Let λ be an eigenvalue of A. It follows that $\det(A - \lambda I) = 0$. Since $\det(A) = \det(A^T)$, we have

$$\begin{aligned} \det(A - \lambda I) &= \det((A - \lambda I)^T) \\ &= \det(A^T - \lambda I). \end{aligned}$$

Hence, $\det(A^T - \lambda I) = 0$ holds as well. Therefore, λ is an eigenvalue of A^T. Since λ was an arbitrary eigenvalue of A, we have shown that any eigenvalue of λ of A is also an eigenvalue of A^T. □