Linear Algebra I:
Problem Set I

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Abstract

This work contains the solutions to the problem set I of Linear Algebra I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

Solution. Let u, v, w be a basis for a three dimensional vector space V. We show that the three vectors $u + v + w$, $v + w$, and w are linearly independent. Assume that

$$a_1(u + v + w) + a_2(v + w) + a_3(w) \quad = \quad 0.$$

Rearranging yields

$$(a_1)u + (a_1 + a_2)v + (a_1 + a_2 + a_3)w \quad = \quad 0.$$

As u, v, w form a basis, they are linearly independent. Hence, we have

$$\begin{aligned} a_1 &= 0 \\ a_1 + a_2 &= 0 \\ a_1 + a_2 + a_3 &= 0. \end{aligned}$$

Solving the system yields

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0. \end{aligned}$$

Hence, the three vectors, $u + v + w$, $v + w$ and w are linearly independent. Now, let $v \in V$. As u, v, w is a basis for v, v can be written as

$$v \quad = \quad c_1 u + c_2 v + c_3 w.$$

The above equality can be re-expressed as

$$v \quad = \quad c_1(u + v + w) + (c_2 - c_1)(v + w) + (c_3 - c_2)w.$$

Therefore v can be written as a linear combination of $u + w + v$, $v + w$ and w. Since v was arbitrary, we have shown that $u + v + w, v + w$ and w span V. Therefore, $u + v + w, v + w$ and w form a basis of V. $\square$

Question 2.

Solution.

Question 3.

Solution.

Question 4.

Solution. Let W_1 and W_2 be subspaces of V. First, assume that $W_1 \subseteq W_2$. Then, we have $W_1 \cup W_2 = W_2$. Since W_2 is a subspace, we have that $W_1 \cup W_2$ is a subspace. By symmetry, we also have that if $W_2 \subseteq W_1$, then $W_1 \cup W_2$ is a subspace.

Now, assume that $W_1 \cup W_2$ is a subspace. Suppose for sake of contradiction that $W_1 \nsubseteq W_2$ and $W_1 \nsubseteq W_2$. either $W_1 \setminus W_2 \neq \emptyset$ or Let $x \in W_1$ and $y \in W_1 \setminus W_2$.

Hence, we have shown that $W_1 \cup W_2$ is a subspace iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Question 5.

Solution.

Question 6.

Solution. Let V be a vector space and x_1 be a nonzero vector in V. Suppose that for all $l \in V'$, we have $l(x_1) = 0$

Question 7.

Solution. We wish to show that the annihilator Y^+ is a subspace of the dual V'. Let $l_1, l_2 \in Y^+$. Let $y \in Y$ and consider $l_1 + l_2$. Then, by definition of the annihilator, we have

$$\begin{aligned} l_1 + l_2(y) &= l_1(y) + l_2(y) \\ &= 0 \end{aligned}$$

Since y was arbitrary, we have

$$l_1 + l_2(y) = 0 \forall y \in Y.$$

Therefore, we have shown that Y^+ is a subspace of the dual V'.

Question 8.

Solution.