Linear Algebra I:
Problem Set I

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains the solutions to the problem set I of Linear Algebra I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

Solution. Let u, v, w be a basis for a three dimensional vector space V. We show that the three vectors $u + v + w$, $v + w$, and w are linearly independent. Assume that

$$a_1(u + v + w) + a_2(v + w) + a_3(w) \quad = \quad 0.$$

Rearranging yields

$$(a_1)u + (a_1 + a_2)v + (a_1 + a_2 + a_3)w \quad = \quad 0.$$

As u, v, w form a basis, they are linearly independent. Hence, we have

$$\begin{aligned} a_1 &= 0 \\ a_1 + a_2 &= 0 \\ a_1 + a_2 + a_3 &= 0. \end{aligned}$$

Solving the system yields

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0. \end{aligned}$$

Hence, the three vectors, $u + v + w$, $v + w$ and w are linearly independent. Now, let $v \in V$. As u, v, w is a basis for v, v can be written as

$$v \quad = \quad c_1 u + c_2 v + c_3 w.$$

The above equality can be re-expressed as

$$v \quad = \quad c_1(u + v + w) + (c_2 - c_1)(v + w) + (c_3 - c_2)w.$$

Therefore v can be written as a linear combination of $u + w + v$, $v + w$ and w. Since v was arbitrary, we have shown that $u + v + w, v + w$ and w span V. Therefore, $u + v + w, v + w$ and w form a basis of V. $\square$

Question 2.

Solution. First of all, we have that $W_1 \cap W_2$ is a subspace, proven in class. Since $W_1 \cap W_2 \subseteq W_1$, and W_1 is finite dimensional, we have that $\dim(W_1 \cap W_2) \leq \dim(W_1)$. Therefore, there exists a basis for $W_1 \cap W_2$, which can be written as

$$\{x_1, x_2, ..., x_n\},$$

where n denotes the dimension of $W_1 \cap W_2$. Now, we have from class that as W_1 is finite dimensional, and the set $\{x_1, ...x_n\}$ forms a set of linearly independent vectors in W_1 such that $n \leq \dim(W_1)$, we can extend the set to form a basis of W_1. Similarly, the set can be extended to W_2 by the same argument. Hence, we have

$$\begin{array}{c} \{x_1, ..., x_n, ..., y_{n+l}\} \\ \{x_1, ..., x_n, ..., z_{n+k}\} \end{array},$$

such that both sets are the basis of W_1 and W_2 respectively with $n+l$ and $n+k$ being their dimension sizes. Note that if $l = 0$, $k = 0$, then the extension is the original set itself, which is possible in cases like $W_1 = W_1 \cap W_2$. Now, consider the following concactanation from the above two sets:

$$B = \{x_1, ..., x_n, ..., y_{n+l}, ...z_{n+l+k}\}.$$

We now claim that the above set forms a basis of $W_1 + W_2$. First of all, it is trivial to see that the above set spans $W_1 + W_2$. Let $v \in W_1 + W_2$. By definition, we have $v = x + y$, such that $x \in W_1$ and $y \in W_2$. We have the basis of W_1 and W_2 included in the vector set, B. Therefore, x and y can be written as linear combinations of the vectors from B and the addition of those two, which is still a linear combination, results in v. Hence, the set B spans $W_1 + W_2$. Now, it remains to be shown that B is linearly independent. First of all, we have that $\{x_1, ...y_{n+l}\}$ is linearly independent, as it is the basis of W_1. Now, the new set $\{z_{n+1}, ...z_{n+k}\}$ must be linearly independent from $\{x_1, ..., y_{n+1}\}$, as otherwise one of z vector should have been included in the x set. Therefore, the set B is linearly independent. Hence, we have shown that B is a basis of $W_1 + W_2$, which results in $W_1 + W_2$ being a finite dimensional subspace of V. Now, notice that B has a dimenision of $n + l + k$, where n denotes the dimension of $W_1 \cap W_2$ and $n + l$ denotes the dimension of W_1 and $n + k$ denotes the dimension of W_2. Therefore, we have

$$\begin{aligned} dim(W_1 + W_2) &= n + l + k \\ &= (n + l) + (n + k) - n \\ &= dim(W_1) + dim(W_2) - dim(W_1 \cap W_2), \end{aligned}$$

which completes the proof. □

Question 3.

Solution. Consider the following two pairs of reals: $(1, 1)$ and $(2, 2)$. Then, by the given definition of addition, we have

$$\begin{aligned} (1, 1) + (2, 2) &= (5, 7) \\ (2, 2) + (1, 1) &= (4, 5). \end{aligned}$$

Hence, we have that $(1, 1) + (2, 2) \neq (2, 2) + (1, 1)$. The given addition fails to be commutative. Therefore, the given set of pairs do not form a vector space under the given definitions. □

Question 4.

Solution. Let W_1 and W_2 be subspaces of V. First, assume that $W_1 \subseteq W_2$. Then, we have $W_1 \cup W_2 = W_2$. Since W_2 is a subspace, we have that $W_1 \cup W_2$ is a subspace. By symmetry, we also have that if $W_2 \subseteq W_1$, then $W_1 \cup W_2$ is a subspace.

Now, assume that $W_1 \cup W_2$ is a subspace. Suppose for sake of contradiction that $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. Then, we have $u \in W_1 \setminus W_2$ and $v \in W_2 \setminus W_1$. Since $u, v \in W_1 \cup W_2$, and $W_1 \cup W_2$

is a subspace, we have $u + v \in W_1 \cup W_2$. Hence, $u + v$ is in either W_1 or W_2. If $u + v \in W_1$, then $u - u + v = w \in W_1$, which is a contradiction. If $u + v \in W_2$, then $u + v - v = u \in W_2$, which is a contradiction. Hence, in both cases we get contradictions, and we have that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. Therefore, we have shown that $W_1 \cup W_2$ is a subspace iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. $\square$

Question 5.

Solution. Let $p_1(x)$ and $p_2(x) \in P_1(\mathbb{R})$. First, note that polynomials are integrable, hence the integrals are well-defined. By the linearity of integration, we have

$$\begin{aligned}
f_1(p_1(x) + p_2(x)) &= \int_0^1 p_1(t) + p_2(t)dt \\
&= \int_0^1 p_1(t)dt + \int_0^1 p_2(t)dt \\
&= f_1(p_1(x)) + f_1(p_2(x)), \\
f_2(p_1(x) + p_2(x)) &= \int_1^2 p_1(t) + p_2(t)dt \\
&= \int_1^2 p_1(t)dt + \int_1^2 p_2(t)dt \\
&= f_2(p_1(x)) + f_2(p_2(x)).
\end{aligned}$$

For a scalar α and $p(x) \in P_1(\mathbb{R})$, by the linearity of integration again,

$$\begin{aligned}
f_1(\alpha p(x)) &= \int_0^1 \alpha p(t)dt \\
&= \alpha \int_0^1 p(t)dt \\
&= \alpha f_1(p(x)), \\
f_2(\alpha p(x)) &= \int_1^2 \alpha p(t)dt \\
&= \alpha \int_1^2 p(t)dt \\
&= \alpha f_2(p(x)).
\end{aligned}$$

Therefore, both f_1 and f_2 are linear functionals and are in the dual of $P_1(\mathbb{R})$.

We now claim that f_1 and f_2 are linearly independent. Assume that

$$a_1 f_1 + a_2 f_2 = 0,$$

which denotes an identically zero map, where a_1 and a_2 are the scalars. Substituting the integral yields

$$\begin{aligned}
a_1 \int_0^1 a + btdt + a_2 \int_1^2 a + btdt &= 0 \\
a_1(a + \frac{b}{2}) + a_2(2a + 2b) &= 0 \\
a(a_1 + 2a_2) + b(\frac{a_1}{2} + 2a_2) &= 0.
\end{aligned}$$

Since the above equation holds for all a, b we have that

$$\begin{aligned}
a_1 + 2a_2 &= 0 \\
\frac{a_1}{2} + 2a_2 &= 0
\end{aligned}$$

Solving the system gives $a_1, a_2 = 0$. Therefore, we have shown that f_1 and f_2 are linearly independent.

From the property of the dual space, we know that a dual space has the same dimension as the original space, which was proven in class. Hence, since $P_1(\mathbb{R})$ has dimension 2, the dual also has dimension 2. Now, as $\{f_1, f_2\}$ form a linear independent set with dimension exactly two, we have that $\{f_1, f_2\}$ forms a basis of the dual space. $\square$

Question 6.

Solution. Let V be a vector space and x_1 be a nonzero vector in V. Let V' be the dual of V. We argue that an identity map I, which is defined by $I(x) = x \ \forall x \in V$, is in V'. Let $x, y \in V$. Then, we have

$$\begin{aligned} I(x+y) &= x+y \\ &= I(x)+I(y). \end{aligned}$$

Since x, y were arbitrary, we $I(x+y) = I(x)+I(y)$ for all x, y. Now, let k be a scalar and $x \in V$. Then, we have

$$\begin{aligned} I(kx) &= kx \\ &= kI(x). \end{aligned}$$

Since k, x were arbitrary, we have $I(kx) = kI(x)$ for all $x \in V$ and all scalars. Therefore, I is linear and is in V'. Notice that $I(x_1) = x_1 \neq 0$. Hence, we have found a map where x_1 mapped to a non-zero element. $\square$

Question 7.

Solution. We wish to show that the annihilator $Y^\perp$ is a subspace of the dual V'. Let $l_1, l_2 \in Y^\perp$, and consider $l_1 + l_2$. Let $y \in Y$. Then, by definition of the annihilator, we have

$$\begin{aligned} l_1 + l_2(y) &= l_1(y) + l_2(y) \\ &= 0. \end{aligned}$$

Since y was arbitrary, we have

$$l_1 + l_2(y) = 0 \ \forall y \in Y.$$

Therefore, $l_1 + l_2$ is in $Y^\perp$.

Now, let $l \in Y^\perp$, and consider αl, where α is a scalar. Let $y \in Y$. Then, again by definition of the annihilator, we have

$$\begin{aligned} \alpha l(y) &= (\alpha) l(y) \\ &= (\alpha)(0) \\ &= 0. \end{aligned}$$

Since y was arbitrary, we have

$$\alpha l(y) = 0 \ \forall y \in Y.$$

Therefore, αl is in $Y^\perp$. As we have shown that $Y^\perp$ is closed under addition and scalar multiplication, we have shown that $Y^\perp$ is a subspace of the dual V'. $\square$

Question 8.

Solution. Let V be a finite dimensional vector space with two different bases $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_n\}$. Let $v \in V$ and denote the cordinates of v with respect to the x basis and with respect to the y basis respectively as

$$v = \sum_{i=1}^{n} a_i x_i \tag{1}$$

$$v = \sum_{i=1}^{n} b_i y_i. \tag{2}$$

Now, observe that each y_i vector from the y basis as a determined cordinates with respect to the x basis. We write them as follows:

$$y_i = \sum_{k=1}^{n} c_{ik} x_k,$$

for $1 \leq i \leq n$. Now substituting the above equality into (2), we obtain

$$v = \sum_{i=1}^{n} (b_i \sum_{k=1}^{n} c_{ik} x_k),$$

which can be re-written as

$$v = \sum_{k=1}^{n} (\sum_{i=1}^{n} b_i c_{ik}) x_k.$$

Hence, by matching the coefficients of the above equality to (1) and agreeing the dummy variables for indices, we have

$$a_i = \sum_{k=1}^{n} b_k c_{ki},$$

for $1 \leq i \leq n$. This deduction establishes the required relation as desired. $\square$