
Linear Algebra I: Problem Set II

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains the solutions to the problem set III of Linear Algebra I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

Solution. We assume that $n \geq 2$ and $n \leq j \leq 1$, in order for the $\det(M)$ and $\det(A)$ values to be well-defined. From the lemma 4 on pg.51 of Lax's Linear Algebra, we have that

$$\det(M) = \det(M_{11}),$$

where M_{11} denotes the $(n-1) \times (n-1)$ submatrix formed by entries m_{ij} , $i > 1, j > 1$. Observe that the same argument can be made on the M_{11} , to assert that $\det(M_{11}) = \det(M_{22})$. Using this argument exactly j -th time, we obtain that

$$\begin{aligned}\det(M) &= \det(M_{jj}) \\ &= \det(A),\end{aligned}$$

as desired. □

Question 2.

Solution. Let A and B be square matrices. Assume that AB is invertible. As a determinant of an invertible matrix is nonzero, it follows that $\det(AB) \neq 0$. Since $\det(AB) = \det(A)\det(B)$, we obtain $\det(A)\det(B) \neq 0$, and thus $\det(A) \neq 0$ and $\det(B) \neq 0$. As a matrix with non-zero determinant is invertible, we have that A and B are invertible. \square

Question 3.

Solution. Let A be a square matrix. From the matrix multiplication rule, we have

$$(AA^T)_{ii} = \sum_k a_{ik}b_{ki},$$

where a_{ik} denotes the (i, k) th entry of the matrix A and b_{ki} denotes the (k, i) th entry of the matrix A^T . By definition of transpose, it follows that $a_{ik} = b_{ki}$, and we obtain

$$(AA^T)_{ii} = \sum_k a_{ik}^2.$$

Therefore, by definition of trace, we have

$$\text{tr}(AA^T) = \sum_{i,k} a_{ik}^2.$$

Since $a_{ik}^2 \geq 0$ for all i, k , it follows that

$$\text{tr}(AA^T) \geq 0.$$

\square

Question 4.

Solution. Let λ be any eigenvalue of A and v be the corresponding eigenvector. It follows that

$$A^3v = \lambda^3v.$$

As $A^3 = I$, it follows that

$$v = \lambda^3v.$$

Hence, we have that $\lambda^3 = 1$. Therefore, we have shown that $\lambda^3 = 1$ is a necessary condition for λ being an eigenvalue of A . Therefore, from this, we deduce that the possible eigenvalues are 1 , $e^{i\frac{2\pi}{3}}$ and $e^{i\frac{4\pi}{3}}$. \square

Question 5.

Solution.

Question 6.

Solution. We are given that $\det(A) = \det(A^T)$. Let λ be an eigenvalue of A . It follows that $\det(A - \lambda I) = 0$. Since $\det(A) = \det(A^T)$, we have

$$\begin{aligned}\det(A - \lambda I) &= \det((A - \lambda I)^T) \\ &= \det(A^T - \lambda I).\end{aligned}$$

Hence, $\det(A^T - \lambda I) = 0$ holds as well. Therefore, λ is an eigenvalue of A^T . Since λ was an arbitrary eigenvalue of A , we have shown that any eigenvalue of λ of A is also an eigenvalue of A^T . \square