

---

# Linear Algebra I:

## Problem Set I

---

Youngduck Choi  
CIMS  
New York University  
yc1104@nyu.edu

### Abstract

This work contains the solutions to the problem set I of Linear Algebra I 2015 at Courant Institute of Mathematical Sciences.

#### Question 1.

**Solution.** Let  $u, v, w$  be a basis for a three dimensional vector space  $V$ . We show that the three vectors  $u + v + w, v + w$ , and  $w$  are linearly independent. Assume that

$$a_1(u + v + w) + a_2(v + w) + a_3(w) = 0.$$

Rearranging yields

$$(a_1)u + (a_1 + a_2)v + (a_1 + a_2 + a_3)w = 0.$$

As  $u, v, w$  form a basis, they are linearly independent. Hence, we have

$$\begin{aligned} a_1 &= 0 \\ a_1 + a_2 &= 0 \\ a_1 + a_2 + a_3 &= 0. \end{aligned}$$

Solving the system yields

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0. \end{aligned}$$

Hence, the three vectors,  $u + v + w, v + w$  and  $w$  are linearly independent. Now, let  $v \in V$ . As  $u, v, w$  is a basis for  $V$ ,  $v$  can be written as

$$v = c_1u + c_2v + c_3w.$$

The above equality can be re-expressed as

$$v = c_1(u + v + w) + (c_2 - c_1)(v + w) + (c_3 - c_2)w.$$

Therefore  $v$  can be written as a linear combination of  $u + v + w, v + w$  and  $w$ . Since  $v$  was arbitrary, we have shown that  $u + v + w, v + w$  and  $w$  span  $V$ . Therefore,  $u + v + w, v + w$  and  $w$  form a basis of  $V$ .  $\square$

**Question 2.****Solution.****Question 3.****Solution.** Consider the following two pairs of reals:  $(1, 1)$  and  $(2, 2)$ . Then, by the given definition of addition, we have

$$\begin{aligned}(1, 1) + (2, 2) &= (5, 7) \\ (2, 2) + (1, 1) &= (4, 5).\end{aligned}$$

Hence, we have that  $(1, 1) + (2, 2) \neq (2, 2) + (1, 1)$ . The given addition fails to be commutative. Therefore, the given set of pairs do not form a vector space under the given definitions.  $\square$

**Question 4.****Solution.** Let  $W_1$  and  $W_2$  be subspaces of  $V$ . First, assume that  $W_1 \subseteq W_2$ . Then, we have  $W_1 \cup W_2 = W_2$ . Since  $W_2$  is a subspace, we have that  $W_1 \cup W_2$  is a subspace. By symmetry, we also have that if  $W_2 \subseteq W_1$ , then  $W_1 \cup W_2$  is a subspace.

Now, assume that  $W_1 \cup W_2$  is a subspace. Suppose for sake of contradiction that  $W_1 \not\subseteq W_2$  and  $W_2 \not\subseteq W_1$ . Then, we have  $u \in W_1 \setminus W_2$  and  $v \in W_2 \setminus W_1$ . Since  $u, v \in W_1 \cup W_2$ , and  $W_1 \cup W_2$  is a subspace, we have  $u + v \in W_1 \cup W_2$ . Hence,  $u + v$  is in either  $W_1$  or  $W_2$ . If  $u + v \in W_1$ , then  $u - u + v = v \in W_1$ , which is a contradiction. If  $u + v \in W_2$ , then  $u + v - v = u \in W_2$ , which is a contradiction. Hence, in both cases we get contradictions, and we have that  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ . Therefore, we have shown that  $W_1 \cup W_2$  is a subspace iff  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .  $\square$

**Question 5.****Solution.** Let  $p_1(x)$  and  $p_2(x) \in P_1(\mathbb{R})$ . First, note that polynomials are integrable, hence the integrals are well-defined. By the linearity of integration, we have

$$\begin{aligned}f_1(p_1(x) + p_2(x)) &= \int_0^1 p_1(t) + p_2(t) dt \\ &= \int_0^1 p_1(t) dt + \int_0^1 p_2(t) dt \\ &= f_1(p_1(x)) + f_1(p_2(x)), \\ f_2(p_1(x) + p_2(x)) &= \int_1^2 p_1(t) + p_2(t) dt \\ &= \int_1^2 p_1(t) dt + \int_1^2 p_2(t) dt \\ &= f_2(p_1(x)) + f_2(p_2(x)).\end{aligned}$$

For a scalar  $\alpha$  and  $p(x) \in P_1(\mathbb{R})$ , by the linearity of integration again,

$$\begin{aligned}f_1(\alpha p(x)) &= \int_0^1 \alpha p(t) dt \\ &= \alpha \int_0^1 p(t) dt \\ &= \alpha f_1(p(x)), \\ f_2(\alpha p(x)) &= \int_1^2 \alpha p(t) dt \\ &= \alpha \int_1^2 p(t) dt \\ &= \alpha f_2(p(x)).\end{aligned}$$

Therefore, both  $f_1$  and  $f_2$  are linear functionals and are in the dual of  $P_1(\mathbb{R})$ .

We now claim that  $f_1$  and  $f_2$  are linearly independent. Assume that

$$a_1 f_1 + a_2 f_2 = 0,$$

which denotes an identically zero map, where  $a_1$  and  $a_2$  are the scalars. Substituting the integral yields

$$\begin{aligned} a_1 \int_0^1 a + btdt + a_2 \int_1^2 a + btdt &= 0 \\ a_1(a + \frac{b}{2}) + a_2(2a + 2b) &= 0 \\ a(a_1 + 2a_2) + b(\frac{a_1}{2} + 2a_2) &= 0. \end{aligned}$$

Since the above equation holds for all  $a, b$  we have that

$$\begin{aligned} a_1 + 2a_2 &= 0 \\ \frac{a_1}{2} + 2a_2 &= 0 \end{aligned}$$

Solving the system gives  $a_1, a_2 = 0$ . Therefore, we have shown that  $f_1$  and  $f_2$  are linearly independent.

From the property of the dual space, we know that a dual space has the same dimension as the original space, which was proven in class. Hence, since  $P_1(\mathbb{R})$  has dimension 2, the dual also has dimension 2. Now, as  $\{f_1, f_2\}$  form a linear independent set with dimension exactly two, we have that  $\{f_1, f_2\}$  forms a basis of the dual space.  $\square$

### Question 6.

**Solution.** Let  $V$  be a vector space and  $x_1$  be a nonzero vector in  $V$ . Let  $V'$  be the dual of  $V$ . We argue that an identity map  $I$ , which is defined by  $I(x) = x \forall x \in V$ , is in  $V'$ . Let  $x, y \in V$ . Then, we have

$$\begin{aligned} I(x + y) &= x + y \\ &= I(x) + I(y). \end{aligned}$$

Since  $x, y$  were arbitrary, we  $I(x + y) = I(x) + I(y)$  for all  $x, y$ . Now, let  $k$  be a scalar and  $x \in V$ . Then, we have

$$\begin{aligned} I(kx) &= kx \\ &= kI(x). \end{aligned}$$

Since  $k, x$  were arbitrary, we have  $I(kx) = kI(x)$  for all  $x \in V$  and all scalars. Therefore,  $I$  is linear and is in  $V'$ . Notice that  $I(x_1) = x_1 \neq 0$ . Hence, we have found a map where  $x_1$  mapped to a non-zero element.  $\square$

### Question 7.

**Solution.** We wish to show that the annihilator  $Y^\perp$  is a subspace of the dual  $V'$ . Let  $l_1, l_2 \in Y^\perp$ , and consider  $l_1 + l_2$ . Let  $y \in Y$ . Then, by definition of the annihilator, we have

$$\begin{aligned} l_1 + l_2(y) &= l_1(y) + l_2(y) \\ &= 0. \end{aligned}$$

Since  $y$  was arbitrary, we have

$$l_1 + l_2(y) = 0 \quad \forall y \in Y.$$

Therefore,  $l_1 + l_2$  is in  $Y^\perp$ .

Now, let  $l \in Y^\perp$ , and consider  $\alpha l$ , where  $\alpha$  is a scalar. Let  $y \in Y$ . Then, again by definition of the annihilator, we have

$$\begin{aligned} \alpha l(y) &= (\alpha)l(y) \\ &= (\alpha)(0) \\ &= 0. \end{aligned}$$

Since  $y$  was arbitrary, we have

$$\alpha l(y) = 0 \quad \forall y \in Y.$$

Therefore,  $\alpha l$  is in  $Y^\perp$ . As we have shown that  $Y^\perp$  is closed under addition and scalar multiplication, we have shown that  $Y^\perp$  is a subspace of the dual  $V'$ .  $\square$

### Question 8.

**Solution.** Let  $V$  be a finite dimensional vector space with two different bases  $\{x_1, \dots, x_n\}$  and  $\{y_1, \dots, y_n\}$ . Let  $v \in V$  and denote the coordinates of  $v$  with respect to the  $x$  basis and with respect to the  $y$  basis respectively as

$$v = \sum_{i=1}^n a_i x_i \tag{1}$$

$$v = \sum_{i=1}^n b_i y_i. \tag{2}$$

Now, observe that each  $y_i$  vector from the  $y$  basis as a determined coordinates with respect to the  $x$  basis. We write them as follows:

$$y_i = \sum_{k=1}^n c_{ik} x_k,$$

for  $1 \leq i \leq n$ . Now substituting the above equality into (2), we obtain

$$v = \sum_{i=1}^n (b_i \sum_{k=1}^n c_{ik} x_k),$$

which can be re-written as

$$v = \sum_{k=1}^n \left( \sum_{i=1}^n b_i c_{ik} \right) x_k.$$

Hence, by matching the coefficients of the above equality to (1) and agreeing the dummy variables for indices, we have

$$a_i = \sum_{k=1}^n b_k c_{ki},$$

for  $1 \leq i \leq n$ . This deduction establishes the required relation as desired.  $\square$