Linear Algebra I:
Problem Set I

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Abstract

This work contains the solutions to the problem set I of Linear Algebra I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

Solution. Let u, v, w be a basis for a three dimensional vector space V. We show that the three vectors $u + v + w$, $v + w$, and w are linearly independent. Assume that

$$a_1(u + v + w) + a_2(v + w) + a_3(w) \quad = \quad 0.$$

Rearranging yields

$$(a_1)u + (a_1 + a_2)v + (a_1 + a_2 + a_3)w \quad = \quad 0.$$

As u, v, w form a basis, they are linearly independent. Hence, we have

$$\begin{aligned} a_1 &= 0 \\ a_1 + a_2 &= 0 \\ a_1 + a_2 + a_3 &= 0. \end{aligned}$$

Solving the system yields

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0. \end{aligned}$$

Hence, the three vectors, $u + v + w$, $v + w$ and w are linearly independent. Now, let $v \in V$. As u, v, w is a basis for v, v can be written as

$$v \quad = \quad c_1 u + c_2 v + c_3 w.$$

The above equality can be re-expressed as

$$v \quad = \quad c_1(u + v + w) + (c_2 - c_1)(v + w) + (c_3 - c_2)w.$$

Therefore v can be written as a linear combination of $u + w + v$, $v + w$ and w. Since v was arbitrary, we have shown that $u + v + w, v + w$ and w span V. Therefore, $u + v + w, v + w$ and w form a basis of V. $\quad\square$

Question 2.

Solution.

Question 3.

Solution. Consider the following two pairs of reals: $(1,1)$ and $(2,2)$. Then, by the given definition of addition, we have

$$\begin{aligned}(1,1)+(2,2) &= (5,7)\\ (2,2)+(1,1) &= (4,5).\end{aligned}$$

Hence, we have that $(1,1)+(2,2) \neq (2,2)+(1,1)$. The given addition fails to be commutative. Therefore, the given set of pairs do not form a vector space under the given definitions. $\square$

Question 4.

Solution. Let W_1 and W_2 be subspaces of V. First, assume that $W_1 \subseteq W_2$. Then, we have $W_1 \cup W_2 = W_2$. Since W_2 is a subspace, we have that $W_1 \cup W_2$ is a subspace. By symmetry, we also have that if $W_2 \subseteq W_1$, then $W_1 \cup W_2$ is a subspace.

Now, assume that $W_1 \cup W_2$ is a subspace. Suppose for sake of contradiction that $W_1 \nsubseteq W_2$ and $W_1 \nsubseteq W_2$. either $W_1 \setminus W_2 \neq \emptyset$ or Let $x \in W_1$ and $y \in W_1 \setminus W_2$.

Hence, we have shown that $W_1 \cup W_2$ is a subspace iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Question 5.

Solution. Let $p_1(x)$ and $p_2(x) \in P_1(\mathbb{R})$. First, note that polynomials are integrable, hence the integrals are well-defined. By the linearity of integration, we have

$$\begin{aligned}f_1(p_1(x)+p_2(x)) &= \int_0^1 p_1(t)+p_2(t)dt\\ &= \int_0^1 p_1(t)dt + \int_0^1 p_2(t)dt\\ &= f_1(p_1(x)) + f_1(p_2(x)),\\ f_2(p_1(x)+p_2(x)) &= \int_1^2 p_1(t)+p_2(t)dt\\ &= \int_1^2 p_1(t)dt + \int_1^2 p_2(t)dt\\ &= f_2(p_1(x)) + f_2(p_2(x)).\end{aligned}$$

For a scalar α and $p(x) \in P_1(\mathbb{R})$, by the linearity of integration again,

$$\begin{aligned}f_1(\alpha p(x)) &= \int_0^1 \alpha p(t)dt\\ &= \alpha \int_0^1 p(t)dt\\ &= \alpha f_1(p(x)),\\ f_2(\alpha p(x)) &= \int_1^2 \alpha p(t)dt\\ &= \alpha \int_1^2 p(t)dt\\ &= \alpha f_2(p(x)).\end{aligned}$$

Therefore, both f_1 and f_2 are linear functionals and are in the dual of $P_1(\mathbb{R})$.

We now claim that f_1 and f_2 are linearly independent. Assume that

$$a_1 f_1 + a_2 f_2 = 0,$$

Hence $a_1, a_2 = 0$. Therefore, we have shown that

Since we know that $dim P_1(\mathbb{R}) = 2$, and th

Question 6.

Solution. Let V be a vector space and x_1 be a nonzero vector in V. Let V' be the dual of V. We argue that an identity map I, which is defined by $I(x) = x\,\forall x \in V$, is in V'. Let $x, y \in V$. Then, we have

$$\begin{aligned} I(x+y) &= x+y \\ &= I(x) + I(y). \end{aligned}$$

Since x, y were arbitrary, we $I(x+y) = I(x) + I(y)$ for all x, y. Now, let k be a scalar and $x \in V$. Then, we have

$$\begin{aligned} I(kx) &= kx \\ &= kI(x). \end{aligned}$$

Since k, x were arbitrary, we have $I(kx) = kI(x)$ for all $x \in V$ and all scalars. Therefore, I is linear and is in V'. Notice that $I(x_1) = x_1 \neq 0$. Hence, we have found a map where x_1 mapped to a non-zero element. $\square$

Question 7.

Solution. We wish to show that the annihilator $Y^\perp$ is a subspace of the dual V'. Let $l_1, l_2 \in Y^\perp$, and consider $l_1 + l_2$. Let $y \in Y$. Then, by definition of the annihilator, we have

$$\begin{aligned} l_1 + l_2(y) &= l_1(y) + l_2(y) \\ &= 0. \end{aligned}$$

Since y was arbitrary, we have

$$l_1 + l_2(y) = 0 \ \forall y \in Y.$$

Therefore, $l_1 + l_2$ is in $Y^\perp$.

Now, let $l \in Y^\perp$, and consider αl, where α is a scalar. Let $y \in Y$. Then, again by definition of the annihilator, we have

$$\begin{aligned} \alpha l(y) &= (\alpha) l(y) \\ &= (\alpha)(0) \\ &= 0. \end{aligned}$$

Since y was arbitrary, we have

$$\alpha l(y) = 0 \ \forall y \in Y.$$

Therefore, αl is in $Y^\perp$. As we have shown that $Y^\perp$ is closed under addition and scalar multiplication, we have shown that $Y^\perp$ is a subspace of the dual V'. $\square$

Question 8.

Solution. Let V be a finite dimensional vector space with two different bases $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_n\}$. Let $v \in V$ and denote the cordinates of v with respect to the x basis and with respect to the y basis respectively as

$$
\begin{aligned}
v &= \sum_{i=1}^{n} a_i x_i \qquad (1) \\
v &= \sum_{i=1}^{n} b_i y_i. \qquad (2)
\end{aligned}
$$

Now, observe that each y_i vector from the y basis as a determined cordinates with respect to the x basis. We write them as follows:

$$y_i = \sum_{k=1}^{n} c_{ik} x_k,$$

for $1 \leq i \leq n$. Now substituting the above equality into (2), we obtain

$$v = \sum_{i=1}^{n} (b_i \sum_{k=1}^{n} c_{ik} x_k),$$

which can be re-written as

$$v = \sum_{k=1}^{n} (\sum_{i=1}^{n} b_i c_{ik}) x_k.$$

Hence, by matching the coefficients of the above equality to (1) and agreeing the dummy variables for indices, we have

$$a_i = \sum_{k=1}^{n} b_k c_{ki},$$

for $1 \leq i \leq n$. This deduction establishes the required relation as desired. $\square$