Linear Algebra I:
Problem Set II

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Abstract

This work contains the solutions to the problem set II of Linear Algebra I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

Solution.

Question 2.

Solution. Let A and B be square matrices. Assume that AB is invertible.

Question 3.

Solution. Let A be a square matrix. From the matrix multiplication rule, we have

$$(AA^T)_{ii} = \sum_k a_{ik} b_{ki},$$

where a_{ik} denotes the (i,k)th entry of the matrix A and b_{ki} denotes the (k,i)th entry of the matrix A^T. By definition of transpose, it follows that $a_{ik} = b_{ki}$, and we obtain

$$(AA^T)_{ii} = \sum_k a_{ik}^2.$$

Therefore, by definition of trace, we have

$$\text{tr}(AA^T) = \sum_{i,k} a_{ik}^2.$$

Since $a_{ik}^2 \geq 0$ for all i, k, it follows that

$$\text{tr}(AA^T) \geq 0.$$

□

Question 4.

Solution.

Question 5.

Solution.

Question 6.

Solution. We are given that $\det(A) = \det(A^T)$. Let λ be an eigenvalue of A. It follows that $\det(A - \lambda I) = 0$. Since $\det(A) = \det(A^T)$, we have

$$\begin{aligned} \det(A - \lambda I) &= \det((A - \lambda I)^T) \\ &= \det(A^T - \lambda I). \end{aligned}$$

Hence, $\det(A^T - \lambda I) = 0$ holds as well. Therefore, λ is an eigenvalue of A^T. Since λ was an arbitrary eigenvalue of A, we have shown that any eigenvalue of λ of A is also an eigenvalue of A^T.
$\square$