
Linear Algebra I:

Problem Set I

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Abstract

This work contains the solutions to the problem set I of Linear Algebra I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

Solution. Let u, v, w be a basis for a three dimensional vector space V . We show that the three vectors $u + v + w, v + w$, and w are linearly independent. Assume that

$$a_1(u + v + w) + a_2(v + w) + a_3(w) = 0.$$

Rearranging yields

$$(a_1)u + (a_1 + a_2)v + (a_1 + a_2 + a_3)w = 0.$$

As u, v, w form a basis, they are linearly independent. Hence, we have

$$\begin{aligned} a_1 &= 0 \\ a_1 + a_2 &= 0 \\ a_1 + a_2 + a_3 &= 0. \end{aligned}$$

Solving the system yields

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0. \end{aligned}$$

Hence, the three vectors, $u + v + w, v + w$ and w are linearly independent. Now, let $v \in V$. As u, v, w is a basis for V , v can be written as

$$v = c_1u + c_2v + c_3w.$$

The above equality can be re-expressed as

$$v = c_1(u + v + w) + (c_2 - c_1)(v + w) + (c_3 - c_2)w.$$

Therefore v can be written as a linear combination of $u + v + w, v + w$ and w . Since v was arbitrary, we have shown that $u + v + w, v + w$ and w span V . Therefore, $u + v + w, v + w$ and w form a basis of V . \square

Question 2.

Solution. Let W_1 and W_2 be finite dimensional vector spaces. We first claim that $W_1 \cap W_2$ is a finite dimensional vector space. Let $\{x_1, \dots, x_k\}$ be a basis for $W_1 \cap W_2$. Since, $W_1 \cap W_2 \subseteq W_1$ and $W_1 \cap W_2 \subseteq W_2$, from the lemma proven previously in class, we can extend the basis as follows:

$$\begin{aligned} &\{x_1, \dots, x_k, y_{k+1}, \dots, y_m\} \\ &\{x_1, \dots, x_k, z_{k+1}, \dots, z_n\} \end{aligned}$$

Question 3.

Solution. Consider the following two pairs of reals: $(1, 1)$ and $(2, 2)$. Then, by the given definition of addition, we have

$$\begin{aligned} (1, 1) + (2, 2) &= (5, 7) \\ (2, 2) + (1, 1) &= (4, 5). \end{aligned}$$

Hence, we have that $(1, 1) + (2, 2) \neq (2, 2) + (1, 1)$. The given addition fails to be commutative. Therefore, the given set of pairs do not form a vector space under the given definitions. \square

Question 4.

Solution. Let W_1 and W_2 be subspaces of V . First, assume that $W_1 \subseteq W_2$. Then, we have $W_1 \cup W_2 = W_2$. Since W_2 is a subspace, we have that $W_1 \cup W_2$ is a subspace. By symmetry, we also have that if $W_2 \subseteq W_1$, then $W_1 \cup W_2$ is a subspace.

Now, assume that $W_1 \cup W_2$ is a subspace. Suppose for sake of contradiction that $W_1 \not\subseteq W_2$ and $W_1 \not\subseteq W_2$. either $W_1 \setminus W_2 \neq \emptyset$ or Let $x \in W_1$ and $y \in W_1 \setminus W_2$.

Hence, we have shown that $W_1 \cup W_2$ is a subspace iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Question 5.

Solution.

Question 6.

Solution. Let V be a vector space and x_1 be a nonzero vector in V . Let V' be the dual of V . We argue that an identity map I , which is defined by $I(x) = x \forall x \in V$, is in V' . Let $x, y \in V$. Then, we have

$$\begin{aligned} I(x + y) &= x + y \\ &= I(x) + I(y). \end{aligned}$$

Since x, y were arbitrary, we $I(x + y) = I(x) + I(y)$ for all x, y . Now, let k be a scalar and $x \in V$. Then, we have

$$\begin{aligned} I(kx) &= kx \\ &= kI(x). \end{aligned}$$

Since k, x were arbitrary, we have $I(kx) = kI(x)$ for all $x \in V$ and all scalars. Therefore, I is linear and is in V' . Notice that $I(x_1) = x_1 \neq 0$. Hence, we have found a map where x_1 mapped to a non-zero element. \square

Question 7.

Solution. We wish to show that the annihilator Y^\perp is a subspace of the dual V' . Let $l_1, l_2 \in Y^\perp$, and consider $l_1 + l_2$. Let $y \in Y$. Then, by definition of the annihilator, we have

$$\begin{aligned} l_1 + l_2(y) &= l_1(y) + l_2(y) \\ &= 0. \end{aligned}$$

Since y was arbitrary, we have

$$l_1 + l_2(y) = 0 \quad \forall y \in Y.$$

Therefore, $l_1 + l_2$ is in Y^\perp .

Now, let $l \in Y^\perp$, and consider αl , where α is a scalar. Let $y \in Y$. Then, again by definition of the annihilator, we have

$$\begin{aligned} \alpha l(y) &= (\alpha)l(y) \\ &= (\alpha)(0) \\ &= 0. \end{aligned}$$

Since y was arbitrary, we have

$$\alpha l(y) = 0 \quad \forall y \in Y.$$

Therefore, αl is in Y^\perp . As we have shown that Y^\perp is closed under addition and scalar multiplication, we have shown that Y^\perp is a subspace of the dual V' . \square

Question 8.

Solution. Let V be a finite dimensional vector space with two different bases $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$. Let $v \in V$ and denote the coordinates of v with respect to the x basis and with respect to the y basis respectively as

$$v = \sum_{i=1}^n a_i x_i \tag{1}$$

$$v = \sum_{i=1}^n b_i y_i. \tag{2}$$

Now, observe that each y_i vector from the y basis has a determined coordinates with respect to the x basis. We write them as follows:

$$y_i = \sum_{k=1}^n c_{ik} x_k,$$

for $1 \leq i \leq n$. Now substituting the above equality into (2), we obtain

$$v = \sum_{i=1}^n (b_i \sum_{k=1}^n c_{ik} x_k),$$

which can be re-written as

$$v = \sum_{k=1}^n (\sum_{i=1}^n b_i c_{ik}) x_k.$$

Hence, by matching the coefficients of the above equality to (1) and agreeing the dummy variables for indices, we have

$$a_i = \sum_{k=1}^n b_k c_{ki},$$

for $1 \leq i \leq n$. This deduction establishes the required relation as desired. \square