Linear Algebra I:
Problem Set II

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Abstract

This work contains the solutions to the problem set II of Linear Algebra I 2015 at Courant Institute of Mathematical Sciences.

Question 1.

Solution. By the definition of the map O, and the fact that $0x = 0$ for all $x \in X$, as X is a linear space, we have

$$\begin{aligned} O(ax) &= 0 \\ &= a0 \\ &= aO(x), \end{aligned}$$

for all $x \in X$. Furthermore, again by the definition of the map O, it follows that

$$\begin{aligned} O(x+y) &= 0 \\ &= 0+0 \\ &= O(x)+O(y), \end{aligned}$$

for all $x, y \in X$. Therefore, we have shown that O is linear.

Let A be another map such that $A \circ A = O$. Note that this implicitly gives that the linear space X under consideration is a non-trivial one, as otherwise the only map available is O. Suppose for sake of contradiction that there exists an inverse map of A, denoted by A^{-1}. Note that by the existence of an inverse map, A and A^{-1} are bijective maps, and thus the domain and range of these maps are all X. Observe that $O \circ A^{-1} = O$. By using the associativity of map composition, $O = A \circ A$ and $O = O \circ A^{-1}$, we have

$$\begin{aligned} I &= A \circ A^{-1} \\ &= A \circ I \circ A^{-1} \\ &= (A \circ A) \circ A^{-1} \circ A^{-1} \\ &= (O \circ A^{-1}) \circ A^{-1} \\ &= O \circ A^{-1} \\ &= O. \end{aligned}$$

We have reached a conclusion that $I = O$. As discussed before, the linear space X is nontrivial. Hence, $I = O$ is a contradiction. Consequently, A does not have an inverse map. $\square$

Question 2.

Solution. Let $x \in X$. Consider $[A,[B,C]] + [C,[A,B]] + B,[C,A]](x)$. Given the definition of the commutator, it follows that

$$\begin{aligned} [A,[B,C]](x) &= [A, B \circ C - C \circ B](x) \\ &= (A \circ (B \circ C - C \circ B) - (B \circ C - C \circ B) \circ A)(x) \\ &= (A \circ B \circ C - A \circ C \circ B - B \circ C \circ A + C \circ B \circ A)(x). \end{aligned}$$

By the associativity of map composition and the linearity of the maps, which is called "Composition is distributive with respect to the addition of linear maps" in Lax, it follows that

$$\begin{aligned} [A[B,C]](x) &= (A \circ B \circ C)(x) - (A \circ C \circ B)(x) - (B \circ C \circ A)(x) + (C \circ B \circ A)(x) \\ &= A(B(C(x))) - A(C(B(x))) - B(C(A(x))) + C(B(A(x))). \end{aligned}$$

By symmetry, it follows that

$$[C,[A,B]](x) = C(A(B(x))) - C(B(A(x))) - A(B(C(x))) + B(A(C(x))),$$

and

$$[B,[C,A]](x) = B(C(A(x))) - B(A(C(x))) - C(A(B(x))) + A(C(B(x)))$$

.

It follows that

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]](x) = 0.$$

As x was arbitrary, we have shown that $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = O$. □

Question 3.

Solution. Let T be a linear map, and $T^{'}$ be a transpose map of T. Assume that T and $T^{'}$ are both invertible. The statement that we want to show asserts that

$$(T^{-1})^{'} = (T^{'})^{-1},$$

which states that $(T^{-1})^{'}$ is the inverse of $T^{'}$. By the definition of inverse, it suffices to show that

$$\begin{aligned} T^{'}(T^{-1})^{'} &= I^{'}, \\ (T^{-1})^{'}T^{'} &= I^{'}. \end{aligned}$$

We have previously shown in class that for any linear map $(ST)^{'} = T^{'}S^{'}$. It follows that

$$\begin{aligned} T^{'}(T^{-1})^{'} &= (T^{-1}T)^{'} \\ &= I^{'}, \\ (T^{-1})^{'}T^{'} &= (TT^{-1})^{'} \\ &= I^{'}. \end{aligned}$$

□

Question 4.

Solution. We have that the dimension of the range space of T is 1. Let r be the vector that spans the range space. Extend the set $\{r\}$ with linearly independent vectors to obtain the basis set that spans X entire space. Denote this set as $\{r, v_1, ..., v_{n-1}\}$. Observe that $T(v_i) = 0$ for all i as they belong to the null-space. Let $x \in X$. Then, there exists a set of scalars such that

$$x = a_1 r + a_2 v_1 ... + a_n v_{n-1}.$$

Consider $T(x)$. As $\{r\}$ spans the range space, there exists a scalar c such that $T(x) = cr$. It follows that

$$tr = T(a_1 r + a_2 v_1 ... + a_n v_{n-1}),$$

which by linearity of T and the property of null-space mentioned above, can be simplified as,

$$\begin{aligned} cr &= rT(a_1) + a_2 T(v_1) ... + a_n T(v_{n-1}), \\ &= rT(a_1). \end{aligned}$$

Hence, we have $c = T(a_1)$. It follows that

$$\begin{aligned} T^2(x) &= T(T(x)) \\ &= T(cr) \\ &= cT(r) \\ &= cT(x). \end{aligned}$$

We have shown that $T^2 = cT$. Now, assume that $c \neq 1$. Consider the linear map $I + \dfrac{1}{1-c}T$. By the linearity of maps, which allows the compositions to distribute, it follows that

$$\begin{aligned} (I - T) \circ (I + \frac{1}{1-c}T) &= I \circ I + I \circ \frac{1}{1-c}T - T \circ I - T \circ \frac{1}{1-c}T \\ &= I + \frac{1}{1-c}T - T - \frac{1}{1-c}T^2 \\ &= I + \frac{1}{1-c}T - T - \frac{c}{1-c}T \\ &= I. \end{aligned}$$

Hence, we have shown that if $c \neq 1$, $I - T$ is invertible. $\square$

Question 5.

Solution. Let A and B be a 2×2 matrix such that

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}. \end{aligned}$$

Observe that both A and B are nonzero matrices and

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\square$