Linear Algebra II: Problem Set I

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set I of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Exercise 1.
Let V_N be the restrictions to $[0,1]$ of polynomials $f \in \mathbb{C}[x]$ having degree $\leq N$. Give this $(N+1)$-dimensional space of $\mathcal{C}[0,1]$ the usual L^2 inner product $(f,h)_2 = \int_0^1 f(t)\overline{h(t)}\,dt$ inherited from the larger space of continuous functions. Let $D : V_N \to V_N$ be the differentiation operator

$$D(a_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N) = a_1 + 2a_2\,t + 3a_3\,t^2 + \cdots + Na_n\,t^{N-1}$$

1. Compute the L^2-inner product $(f,h)_2$ in terms of the coefficients a_k, b_k that determine f and h.
2. Is D a self-adjoint operator? Skew-adjoint?

Solution. (1) By expressing f, h in terms of the coefficients a_k, b_k that determine f and g, exploiting the fact that the complex conjugate of the product is the product of the conjugate, using the differentiation of complex polynomials, we obtain

$$\begin{aligned}
(f,g)_2 &= \int_0^1 (\sum_{i=0}^{N} a_i t^i)\overline{(\sum_{i=0}^{N} b_i t^i)}dt \\
&= \int_0^1 (\sum_{i=0}^{N} a_i t^i)(\sum_{i=0}^{N} \overline{b_i} t^i)dt \\
&= \int_0^1 \sum_{0 \leq i,j \leq N} a_i \overline{b_j} t^{i+j}\,dt \\
&= \left[\sum_{0 \leq i,j \leq N} \frac{a_i \overline{b_j}}{i+j+1} t^{i+j+1} \right]_0^1 \\
&= \sum_{0 \leq i,j \leq N} \frac{a_i \overline{b_j}}{i+j+1}.
\end{aligned}$$

(2) By writing down the matrix representation of the differentiation operator, using the standard polynomial basis, $\{1, x, x^2, ..., x^n\}$, we see that the matrix does not equal its conjugate transpose, and also does not equal the negative of the conjugate transpose. Hence, it is not self-adjoint, and not skew-joint. $\square$

Question 2.

Exercise 2:
Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be L_A for the matrix

$$A = A^* = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

in M(2.$\mathbb{C}$). Determine the eigenvalues in $\mathbb{C}$ and the eigenspaces, and exhibit an orthonormal basis $\mathfrak{Y} = \{\mathbf{f}_1, \mathbf{f}_2\}$ that diagonalizes T.

Solution. Firstly, the characteristic equation of the matrix A is given by

$$(1-\lambda)(2-\lambda) - 1 \quad = \quad 0,$$

which can equivalently written as

$$\lambda^2 - 3\lambda + 1 \quad = \quad 0.$$

Using the quadratic formula, we obtain that $\dfrac{3 \pm \sqrt{5}}{2}$ are the eigenvalues of the matrix. Now, we determine the respective eigenspaces. Recall that we can characterize the eigenspace as Null$(A - \lambda I)$. Hence, for $\lambda = \dfrac{3+\sqrt{5}}{2}$, we have

$$\begin{aligned}
\text{Null}(A - \lambda I) \quad &= \quad \text{Null}\Big(\begin{pmatrix} 1 - \dfrac{3+\sqrt{5}}{2} & -1 \\ -1 & 2 - \dfrac{3+\sqrt{5}}{2} \end{pmatrix}\Big) \\
&= \quad \text{Null}\Big(\begin{pmatrix} \dfrac{-1+\sqrt{5}}{2} & -1 \\ -1 & \dfrac{1+\sqrt{5}}{2} \end{pmatrix}\Big) \\
&= \quad \text{span}\Big(\begin{pmatrix} 1 \\ \dfrac{-1+\sqrt{5}}{2} \end{pmatrix}\Big),
\end{aligned}$$

where the last spanning vector is chosen via inspection. Analogously, $\lambda = \dfrac{3-\sqrt{5}}{2}$, we have

$$\begin{aligned}
\text{Null}(A - \lambda I) \quad &= \quad \text{Null}\Big(\begin{pmatrix} 1 - \dfrac{3-\sqrt{5}}{2} & -1 \\ -1 & 2 - \dfrac{3-\sqrt{5}}{2} \end{pmatrix}\Big) \\
&= \quad \text{Null}\Big(\begin{pmatrix} \dfrac{-1-\sqrt{5}}{2} & -1 \\ -1 & \dfrac{1-\sqrt{5}}{2} \end{pmatrix}\Big) \\
&= \quad \text{span}\Big(\begin{pmatrix} 1 \\ \dfrac{-1-\sqrt{5}}{2} \end{pmatrix}\Big).
\end{aligned}$$

Exercise 3:
Prove that a normal operator $T : V \to V$ on a finite dimensional inner product space over $\mathbb{C}$ is self adjoint if and only if its spectrum is real: $\mathrm{sp}_{\mathbb{C}}(T) \subseteq \mathbb{R} + i0$.

Question 3.

Solution. Let T be a normal operator on a finite dimensional inner product space. By Complex Spectral theorem, we obtain that T has a diagonal matrix with respect to some orthonormal basis, which we denote as $M(T)$. By definition, T is self-adjoint iff $M(T) = M(T)^*$, where $M(T)^*$ denotes the conjugate transpose of $M(T)$. Since, $M(T)$ is diagonal, we have that $M(T) = M(T)^*$ iff diagonal entries are real. Since we know that the diagonal entries of a diagonal matrix is the specturm, we have that the diagonal entries of $M(T)$ is real iff all of its eigenvalues are real. By the chain of equivalence obtained, we are done. □

Question 4.

Exercise 4:
If T is diagonalizable over $\mathbb{R}$ or $\mathbb{C}$, prove that

$$e^T = \sum_{\lambda \in \mathrm{sp}(T)} e^{\lambda} P_{\lambda}$$

is the same as the linear operator given by the exponential series

$$e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k$$

Solution.

Question 5.

Exercise 5:
Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be the operator $T = L_A$ for

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

Explain why T is self-adjoint with respect to the standard inner product $(z, w) = z_1\overline{w_1} + z_2\overline{w_2}$ on $\mathbb{C}^2$. Then determine

(a) The spectrum $\mathrm{sp}_{\mathbb{C}}(T) = \{\lambda_1, \lambda_2\}$;

(b) The eigenspaces $E_\lambda(T)$ and find an orthonormal basis $\{f_1, f_2\}$ in $\mathbb{C}^2$ that diagonalize T. Then

(c) Find a unitary matrix $U^*U = I$ such that

$$UAU^* = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\mathrm{sp}(T) = \{\lambda_1, \lambda_2\}$.

Solution.

Question 6.

Exercise 6 (Uniquess of Spectral Decompositions): Suppose $T : V \to V$ is diagonalizable on an arbitrary vector space (not necesarily an inner product space), so $T = \sum_{i=1}^{r} \lambda_i P_{\lambda_i}$ where $\mathrm{sp}(T) = \{\lambda_1, \ldots, \lambda_r\}$ and P_{λ_i} is the projection onto the λ_i-eigenspace. Now suppose $T = \sum_{j=1}^{s} \mu_j Q_j$ is some other decomposition such that

$$Q_j^2 = Q_j \qquad Q_j Q_k = Q_k Q_j = 0 \quad \text{if } j \neq k \qquad \sum_{j=1}^{s} Q_j = I$$

Prove that

(a) $r = s$ and if the μ_j are suitably relabeled we have $\mu_i = \lambda_i$ for $1 \leq i \leq r$.

(b) $Q_i = P_{\lambda_i}$ for $1 \leq i \leq r$.

Hint: First show $\{\mu_1, \ldots, \mu_s\} \subseteq \{\lambda_1, \ldots, \lambda_r\} = \mathrm{sp}(T)$; then relabel.

Solution.