Linear Algebra II:
Problem Set I

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Abstract

This work contains solutions to the problem set I of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Exercise 1 (Uniquess of Spectral Decompositions): Suppose $T : V \to V$ is diagonalizable on an arbitrary vector space (not necesarily an inner product space), so $T = \sum_{i=1}^{r} \lambda_i P_{\lambda_i}$ where $\mathrm{sp}(T) = \{\lambda_1, \ldots, \lambda_r\}$ and P_{λ_i} is the projection onto the λ_i-eigenspace. Now suppose $T = \sum_{j=1}^{s} \mu_j Q_j$ is some other decomposition such that

$$Q_j^2 = Q_j \qquad Q_j Q_k = Q_k Q_j = 0 \quad \text{if } j \neq k \qquad \sum_{j=1}^{s} Q_j = I$$

Prove that

(a) $r = s$ and if the μ_j are suitably relabeled we have $\mu_i = \lambda_i$ for $1 \leq i \leq r$.

(b) $Q_i = P_{\lambda_i}$ for $1 \leq i \leq r$.

Hint: First show $\{\mu_1, \ldots, \mu_s\} \subseteq \{\lambda_1, \ldots, \lambda_r\} = \mathrm{sp}(T)$; then relabel.

Solution. (**(a)** Firstly, note that u_j is an eigenvalue of T. To see this, let $v \neq 0 \in Q_j$. Then, as $Q_j v = v$, it follows that

$$\begin{aligned} Tv &= \sum_{i=1}^{s} u_i Q_i v \\ &= \sum_{i=1}^{s} u_i Q_i Q_j v \\ &= u_j Q_j^2 v \\ &= u_j v. \end{aligned}$$

Therefore, we have shown that $\mathrm{Im}(Q_j) \subset \mathrm{Im}(P_{u_j})$. Next, suppose w is an arbitrary eigenvector of T, It follows that $w = \sum_{i=1}^{s} Q_i w$, which expresses w as a sum of zero-vectors and eigenvectors. But, since eigenvectors corresponding to distinct eigenvalues are linearly independent, the only possibility is that $w \in \mathrm{Im}(Q_i) = \mathrm{Im}(P_{\lambda_i})$ for some i. Hence, we have shown that

$$\{u_1, .., u_r\} = \mathrm{sp}(T).$$

(b) We have shown that $\mathrm{Im}(Q_i) = \mathrm{Im}(P_{\lambda_i})$ for all i after relabeling. As they are both projections, the proof of uniqueness will be established, if we show that they possess the same kernel. By assumption, we have

$$\begin{aligned} \ker(P_{\lambda_i}) &= \sum_{j \neq i} \mathrm{Im}(P_{\lambda_j}) \\ &= \sum_{j \neq i} \mathrm{Im}(Q_j). \end{aligned}$$

Let $v \in \ker(P_{\lambda_i})$. Then, it follows that

$$v = \sum_{j \neq i} Q_j x_j$$

for $x_j \in V$ with $j \neq i$. Furthermore, we have

$$Q_i v = Q_i \sum_{j \neq i} Q_j x_j = 0.$$

Converse holds true with the same logic. $\square$

Question 2.

Exercise 2 : Let $V = \mathcal{C}_c^\infty(\mathbb{R})$ be the space of real-valued functions $f(t)$ on the real line that have continuous derivatives $D^k f$ of all orders, and have "bounded support" – each f is zero off of some bounded interval (which is allowed to vary with f). Because all such functions are "zero near ∞" there is a well defined inner product

$$(f,h)_2 = \int_{-\infty}^{\infty} f(t)\overline{h(t)}\,dt$$

The derivative $Df = df/dt$ is a linear operator on this infinite dimensional space.

1. Prove that the adjoint of D is *skew-adjoint*, with $D^* = -D$.
2. Prove that the second derivative $D^2 = d^2/dt^2$ is *self-adjoint*.

Hint: Integration by parts.

Solution. **(a)** Let $f, g \in V$. By integration by parts, we have

$$\begin{aligned} \int_{-\infty}^{\infty} Df\overline{g}dt + \int_{-\infty}^{\infty} f\overline{Dg}dt &= [f\overline{g}]_{-\infty}^{\infty} \\ &= 0, \end{aligned}$$

as f, g have bounded supports. Therefore, it follows that

$$(Df, g) = -(f, Dg).$$

Since f, g were arbitrary, we have shown that $D^* = D$.

(b) By (a), we have

$$\begin{aligned} (D^2 f, g) &= -(Df, Dg) \\ &= (f, D^2 g), \end{aligned}$$

for $f, g \in V$. Therefore, D^2 is self-adjoint. $\square$

Question 3.

Exercise 3:
If $T : V \to V$ is a linear operator on an inner product space and $\lambda \in \text{sp}(T)$, prove that

1. $E_{\overline{\lambda}}(T^*) = K(T^* - \overline{\lambda}I)$ is equal to $R(T - \lambda I)^{\perp}$.
2. dim $E_{\overline{\lambda}}(T^*) =$ dim $E_{\lambda}(T)$.

Solution. (1) We first claim that for $K(T^*) = R(T)^{\perp}$. We prove it by showing the following series of equivalence.

$$\begin{aligned} w \in K(T^*) &\iff T^*w = 0 \\ &\iff < v, T^*w >= 0 \quad \forall v \in V \\ &\iff < Tv, w >= 0 \quad \forall v \in V \\ &\iff w \in R(T)^{\perp}, \end{aligned}$$

as required. Now, by the elementary properties of adjoint, it follows that

$$\begin{aligned} (T - \lambda I)^* &= T^* - (\lambda I)^* \\ &= T^* - \overline{\lambda} I^* \\ &= T^* - \overline{\lambda} I. \end{aligned}$$

Therefore, combining the two results together, we have shown that

$$E_{\overline{\lambda}}(T^*) = R(T - \lambda I)^{\perp}$$

(2) By definition of eigenspace, and the rank-nullity theorem, we have

$$\begin{aligned} \text{dim} E_{\lambda}(T) &= \text{dim} N(T - \lambda I) \\ &= \text{dim} V - \text{dim} R(T - \lambda I). \end{aligned}$$

As a subspace of a linear space forms a direct sum with its orthogonal complement, we have that $\text{dim} R(T - \lambda I) = \text{dim} V - \text{dim} R(T - \lambda I)^{\perp}$. By (1), it follows that $\text{dim} R(T - \lambda I) = \text{dim} V - \text{dim} E_{\overline{\lambda}}(T^*)$. Substituting to the result to the previous equation, we obtain

$$\text{dim} E_{\lambda}(T) = \text{dim} E_{\overline{\lambda}}(T^*),$$

as required. $\square$

Question 4.

Exercise 4:
If $m < n$ and the coordinate spaces $\mathbb{K}^m$, $\mathbb{K}^n$ are equipped with the standard inner products, consider the linear operator

$$T : \mathbb{K}^m \to \mathbb{K}^n \qquad T(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0)$$

This is an isometry from $\mathbb{K}^m$ into $\mathbb{K}^n$, with trivial kernel $K(T) = (0)$ and range $R(T) = \mathbb{K}^m \times (\mathbf{0})$ in $\mathbb{K}^n = \mathbb{K}^m \oplus \mathbb{K}^{n-m}$.

1. Provide an explicit description of the adjoint operator $T^* : \mathbb{K}^n \to \mathbb{K}^m$ and determine $K(T^*), R(T^*)$.
2. Compute the matrices of $[T]$ and $[T^*]$ with respect to the standard orthonormal bases in $\mathbb{K}^m, \mathbb{K}^n$.
3. How is the action of T^* related to the subspaces $K(T), R(T^*)$ in $\mathbb{K}^m$ and $R(T), K(T^*)$ in $\mathbb{K}^n$? Can you give a geometric description of this action?

Unitary operators can be described in several different ways, each with its own advantages in applications.

Solution. **(1)** Let (e_i) and (f_j) denote the standard bases for $\mathbb{K}^m$ and $\mathbb{K}^n$ respectively. Let $v \in \mathbb{K}^n$. For $i = 1, 2, ..., m$, it follows that

$$\begin{aligned} (T^*v)_i &= (T^*v, e_i) \\ &= (v, Te_i) \\ &= (v, f_i) \\ &= v_i. \end{aligned}$$

Therefore, we have shown that T^* is the projection onto the first m coordinates. It easily follows that $K(T^*) = (0) \times \mathbb{K}^{n-m}$ and $R(T^*) = \mathbb{K}^m$.

(2) $[T]$ can be written as $[I_{m\times m}; 0_{n-m\times m}]$, and $[T^*]$ can be written as $[I_{m\times m} 0_{m\times n-m}]$.

(3) We can relate them as both pairs form a direct sum of $\mathbb{K}^m$, and $\mathbb{K}^n$ respectively. Geometrically, we can describe this action as an orthogonal projection.

□

Question 5.

Exercise 5:
Let $\mathfrak{X} = \{e_1, \ldots, e_n\}$ be an arbitrary basis (not necessarily orthonormal) in a finite dimensional inner product space V.

(a) Use induction on n to prove that there exist vectors $\mathfrak{Y} = \{f_1, \ldots, f_n\}$ such that $(e_i, f_j) = \delta_{ij}$.

(b) Explain why the f_j are uniquely determined and a basis for V.

Note: If the initial basis $\mathfrak{X}$ is orthonormal then $f_i = e_i$ and the result trivial; we are interested in *arbitrary* bases in an inner product space.

Solution.

Question 6.

Exercise 6 :
Let V be an inner product space and T a linear operator that is diagonalizable in the ordinary sense, but not necessarily orthogonally diagonalizable. Prove that

(a) The adjoint operator T^* is diagonalizable. What can you say about its eigenvalues and eigenspaces?

(b) If T is *orthogonally* diagonalizable so is T^*.

Hint: If $\{e_i\}$ diagonalizes T what does the "dual basis" $\{f_j\}$ of Exercise 5 do for T^*?

Solution. **(a)** As T^* is diagonalizable, we can express V as a direct sum of eigenspaces, which we denote as $E_\lambda(T^*)$ for $\lambda \in sp(T^*)$. The same logic aapplies to T being diagonalizable with $\lambda \in sp(T)$. Hence, they will be conjugate transpose of each other, with the corrpsponding eigenspaces.

(b) Assume T is orthogonally diagonalizable. Then, there exists an orthonormal basis B such that $[T]_B$ is diagonal. As B is orthonormal, we have that $[T^*]_B = [T]_B^*$. Since a conjugate transpose of a diagonal matrix is diagonal, we see that $[T^*]_B$ is a diagonal matrix. Therefore, there exists an orthonormal basis, namely B such that the matrix representation of T^* with respect to the orthonormal basis is diagonal. Hence, T^* is diagonalizable.

□

Question 7.

Exercise 7: Here $V = \mathbb{R}^3$ and $f_1, f_2, f_3 \in V^*$ are the linear functionals $f_k : V \to \mathbb{R}$ given by $(**)$ $f_1(x, y, z) = x - 2y,\ f_2(x, y, z) = x + y + z$ and $f_3(x, y, z) = y - 3z$;

1. Prove $\mathcal{Y} = \{f_1, f_2, f_3\}$ is a basis in V^*.
2. Find a basis $\mathcal{X} = \{e'_1, e'_2, e'_3\} \subseteq V$ whose dual basis $\mathcal{X}^*$ in V is equal to $\mathcal{Y}$.

Solution. (1) As the dimension of the dual space for a finite dimensional linear space equals the dimension of the linear space, it suffices to show that f_1, f_2, and f_3 are linearly independent. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, such that $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$. By substitutions, it follows that

$$\begin{aligned}\lambda_1 f_1(x, y, z) + \lambda_2 f_2(x, y, z) + \lambda_3 f_3(x, y, z) &= \lambda_1(x - 2y) + \lambda_2(x + y + z) + \lambda_3(y - 3z) \\ &= (\lambda_1 + \lambda_2)x + (-2\lambda_1 + \lambda_2 + \lambda_3)y + (\lambda_2 - \lambda_3)z.\end{aligned}$$

Since the above equation holds for all $x, y, z \in \mathbb{R}$ and $\mathbb{R}$ is an integral domain, we obtain that

$$\begin{aligned}\lambda_1 + \lambda_2 &= 0 \\ -2\lambda_1 + \lambda_2 + \lambda_3 &= 0 \\ \lambda_2 - \lambda_3 &= 0.\end{aligned}$$

It immediately follows that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Therefore, $\mathscr{Y}$ is linearly independent, thus a basis in V^*.

(2) We wish to find vectors $v_i = (x_i, y_i, z_i)$, for $1 \leq i \leq 3$, such that $f_i(v_j) = \delta_{ij}$. For v_1 we have

$$\begin{aligned}x_1 - 2y_1 &= 0 \\ x_1 + y_1 + z_1 &= 0 \\ y_1 - 3z_1 &= 0.\end{aligned}$$

Solving the system, we obtain $x_1 = \dfrac{4}{10}$, $y_1 = \dfrac{-3}{10}$, and $z_1 = \dfrac{-1}{10}$. The rest can be computed in the exact same manner. □