Linear Algebra II:
Problem Set IV

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Abstract

This work contains solutions to the problem set IV of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Exercise 1:
Suppose $A \in M(n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If A is self-adjoint, so that

$$A^* = A \ (where \ (A^*)_{ij} = \overline{A_{ij}} \ for \ all \ i, j)$$

it is well known that there is a basis in $\mathbb{F}^n$ that diagonalizes the corresponding linear operator $L_A : \mathbb{F}^n \to \mathbb{F}^n$. (In fact, there is even a basis that is orthonormal with respect to the standard product in $\mathcal{F}^n$.)
Now, suppose that A is only symmetric, with

$$A^T = A \ (where \ (A^*)_{ij} = A_{ij} \ for \ all \ i, j)$$

When $\mathbb{F} = \mathbb{R}$, symmetry is the same thing as self-adjointness, and symmetriy $A^T = A$ suffices to guarantee orthonormal diagonalizability of L_A. What happens when $\mathbb{F} = \mathbb{C}$? If A is a symmetric is the same thing as self-adjointness, and symmetry $A^T = A$ suffices to guarantee orthonormal diagonalizability of L_A. What happens when $\mathbb{F} = \mathbb{C}$? If A is symmetric $A = A^T$ with complex entries, does L_A always have a diagonalizing basis? Prove or give a counterexample.

Solution.
Consider the following matrix:

$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

The matrix equals its transpose, but not its conjugate transpose. The characteristic polynomial of the matrix is $(1-\lambda)(-1-\lambda)+1 = \lambda^2$. Hence, we obtain that the eigenvalue of the matrix is 0 with algebraic multiplicity of 2. Now, observe that $E_{\lambda=0}(L_A) = \text{Null}(L_A - 0I) = \text{Null}(L_A)$. Since the nullspace of L_A is the span of row vectors of A, and $(1, i)$ and $(i, -1)$ are linearly dependent (take i and -1 as coefficients), we have that the geometric multiplictiy of 0 is 1. Therefore, the eigenspaces of L_A do not form a direct sum of $\mathbb{C}^2$. Therefore, by the Spectral Theorem, the matrix is not diagonalizable. $\square$

Question 2.

Exercise 2:
Let $\mathcal{X} = \{e_1, e_2\}$ be the standard basis in the complex inner product space $V = \mathbb{C}^2$ equipped with the usual Euclidean inner product $(a, b) = a_1\bar{b_1} + a_2\bar{b_2}$. Let $\mathcal{N} = \{f_1, f_2\}$ be the basis such that $f_1 = e_1$ and $f_2 = e_1 + e_2$ and define $T : V \to V$ to be the linear operator such that $T(f_1) = f_1$ and $T(f_2) = \frac{1}{2}f_2$. Obviously $\mathcal{N}$ diagonalizes T but the basis eigenvectors f_1, f_2 are not orthogonal.

1. Find $[T]_{\mathcal{X},\mathcal{X}}$ and $[e^{tT}]_{\mathcal{X},\mathcal{X}}$ for $t \in \mathbb{R}$.
2. Explain why T cannot be a self-adjoint operator ($T^* = T$).
3. Explain why there cannot be an ON basis in V that diagonalizes T.
4. Find the solution $X(t)$, $t \in \mathbb{R}$, of the vector-valued differential equation

$$\frac{dX}{dt} = A \cdot X(t) \ with \ X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   where $A = [T]_{\mathcal{X},\mathcal{X}}$.

Solution.
(a) With the given definition, we have

$$\begin{aligned} T(e_1) &= e_1, \\ T(e_2) &= T(f_2 - e_1) = T(f_2) = T(e_1) = \frac{1}{2}f_2 + e_1 = -\frac{1}{2}e_1 + \frac{1}{2}e_2. \end{aligned}$$

Therefore, it follows that

$$[T]_{X,X} = \begin{pmatrix} 1 & \frac{-1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

(b) We see that T has a matrix representation with respect to the standard basis that is not symmetric. Hence, T^* cannot be self-adjoint.

(c) Spectral theorem says that T is orthonormally diagonalizable, iff T is self-adjoint. Since T is not self-adjoint, it is not orthonormally diagonalizable.

(d) The solution will be $[e^{tT}]$ where t agrees with the initial data of $(1, 0)$.

Question 3.

Exercise 3:
In $V = \mathbb{R}^3$ equipped with the usual Euclidean inner product let $M = (\mathbb{R}f_3)^\perp$ where $f_3 = (1, -2, 3)$.

1. Give a formula $(y_1, y_2, y_3) = R(x_1, x_2, x_3)$ in terms of inner products for the linear operator $R : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects vectors across the plane M. Find the image of the particular vector $R(1, 1, -3)$.
2. Prove that R is an isometry, so that
$$||R(x) - R(y)|| = ||x - y||$$
and hence is a bijection $R : \mathbb{R}^3 \to \mathbb{R}^3$.
3. Find an ON basis $\{e_1, e_2\}$ for M.
4. What is the matrix $[R]_{\mathcal{N}\mathcal{N}}$ with respect to the ON basis $\mathcal{N} = \{e_1, e_2, e_3\}$ such that $e_3 = f_3/||f_3||$?
5. Is $R : V \to V$ orthogonally diagonalizable? Explain.

Solution. (a) The reflection formula can be given by subtracting the orthogonal projection twice from the original vector, which can be written as follows:

$$R(v) \quad = \quad v - 2\frac{v \cdot f_3}{f_3 \cdot f_3} f_3.$$

(b) From the above formula it follows that

$$\begin{aligned} ||R(x) - R(y)|| &= ||x - 2\frac{x \cdot f_3}{f_3 \cdot f_3} f_3 - y + 2\frac{y \cdot f_3}{f_3 \cdot f_3} f_3|| \\ &= ||x - y - 2\frac{(x - y) \cdot f_3}{f_3 \cdot f_3}|| \\ &= ||x - y||. \end{aligned}$$

(c) One can see that $(-1, 1, 1)$ has a 0 inner product with f_3 and $(1, 2, 1)$ also have a 0 inner product and they are linearly independent. Therefore, they form an orthonormal basis for M.

(e) As we have shown that R is an isometry in part b, and we know isometries are orthogonally diagnolizable, R is orthogonally diagonalizable.

□

Question 4.

Exercise 4:
Let $V = \mathbb{C}^2$ with the usual Euclidean inner product and let $T : V \to V$ be the linear operator whose matrix with respect to the standard ON basis $\mathcal{X} = \{e_1, e_2\}$ is

$$[T]_{\mathcal{X},\mathcal{X}} = \begin{pmatrix} 7 & \sqrt{3} \\ \sqrt{3} & 5 \end{pmatrix}$$

1. Determine the spectrum $sp_{\mathbb{C}}(T) = \{\lambda_1, \lambda_2\}$ and show T is orthogonally diagonalizable by finding an ON basis $\mathcal{N} = \{f_1, f_2\}$ of eigenvectors.
2. Explain why T is self-adjoint.

Solution. (1) The characteristic polynomial of T can be computed as $(7-\lambda)(5-\lambda) - 3 = \lambda^2 - 12\lambda + 32 = (\lambda - 8)(\lambda - 4)$. Therefore, we have that $\text{spec}_{\mathbb{C}}(T) = \{4, 8\}$. We see that the eigenvector associated with with 4 is $\frac{1}{2}(1, -\sqrt{3})$ and with 8 is $\frac{1}{2}(\sqrt{3}, 1)$, which are the ON basis of eigenvectors, that orthogonally diagonalizes T.

(2) When T is a linear operator over complex field, that is represented in a matrix form with respect to an orthonormal basis, we have that the matrix representation of T^*, with respect to the orthonormal basis, is the conjugate transpose of the matrix representation of T. As operators with the same matrix representation with respect to the same set of basis are in fact the same linear operator, we have that $T = T^*$, which is the condition for a linear operator to be self-adjoint.

$\square$

Question 4-continued.

If an operator is self-adjoint (or merely diagonalizable) it has a spectral decomposition

$$T = \sum_{\lambda \in sp(T)} \lambda P_\lambda$$

where $P_\lambda =$ (projection onto E_λ along $\oplus_{\nu \neq \lambda} E_\nu$) which describes T as a weighted sum of projections onto the eigenspaces.

3. Find the matrices $[P_{\lambda_k}]$ that describe the spectral projections with respect ti the diagonalizing basis $\mathcal{N}$.
   Hint: You only need to find one of these matrices because $P_{\lambda_1} + P_{\lambda_2} = I \Rightarrow P_{\lambda_2} = I - P_{\lambda_1}$.
4. Find the matrices $[P_{\lambda_k}]$ that describes the spectral projections with respect to the standard ON basis $\mathcal{X}$.
   Hint: Again, you only need to find one of these matrices.

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5. In terms of the spectral decomposition the square root of T is the operator given by
   $$\sqrt{T} = \sum_{\lambda \in sp(T)} \sqrt{\lambda} P_\lambda$$
   Find the matrix $[\sqrt{T}]$ that describes $\sqrt{T}$ with respect to the standard ON basis.

Solution.
(3) By Spectral Theorem, we have

$$P_\lambda = \prod_{u \neq \lambda} \frac{A - uI}{\lambda - u}.$$

Hence, it follows that

$$\begin{aligned} P_{\lambda=8} &= \frac{A - 4I}{8 - 4} \\ &= \frac{1}{4} \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \end{aligned}$$

□