Linear Algebra II: Problem Set I

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Abstract

This work contains solutions to the problem set I of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Exercise 1.
Let V_N be the restrictions to $[0,1]$ of polynomials $f \in \mathbb{C}[x]$ having degree $\leq N$. Give this $(N+1)$-dimensional space of $\mathcal{C}[0,1]$ the usual L^2 inner product $(f,h)_2 = \int_0^1 f(t)\overline{h(t)}\,dt$ inherited from the larger space of continuous functions. Let $D : V_N \to V_N$ be the differentiation operator

$$D(a_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N) = a_1 + 2a_2\,t + 3a_3\,t^2 + \cdots + Na_n\,t^{N-1}$$

1. Compute the L^2-inner product $(f,h)_2$ in terms of the coefficients a_k, b_k that determine f and h.
2. Is D a self-adjoint operator? Skew-adjoint?

Solution. (1) By expressing f, h in terms of the coefficients a_k, b_k that determine f and g, exploiting the fact that the complex conjugate of the product is the product of the conjugate, using the differentiation of complex polynomials, we obtain

$$\begin{aligned}
(f,g)_2 &= \int_0^1 (\sum_{i=0}^N a_i t^i)\overline{(\sum_{i=0}^N b_i t^i)}dt \\
&= \int_0^1 (\sum_{i=0}^N a_i t^i)(\sum_{i=0}^N \overline{b_i} t^i)dt \\
&= \int_0^1 \sum_{0\leq i,j\leq N} a_i\overline{b_j}t^{i+j}\,dt \\
&= \left[\sum_{0\leq i,j\leq N} \frac{a_i\overline{b_j}}{i+j+1}t^{i+j+1}\right]_0^1 \\
&= \sum_{0\leq i,j\leq N} \frac{a_i\overline{b_j}}{i+j+1}.
\end{aligned}$$

(2) By writing down the matrix representation of the differentiation operator, using the standard polynomial basis, $\{1, x, x^2, ..., x^n\}$, we see that the matrix does not equal its conjugate transpose, and also does not equal the negative of the conjugate transpose. Hence, it is not self-adjoint, and not skew-joint. $\square$

Question 2.

Exercise 2:
Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be L_A for the matrix

$$A = A^* = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

in M(2.$\mathbb{C}$). Determine the eigenvalues in $\mathbb{C}$ and the eigenspaces, and exhibit an orthonormal basis $\mathfrak{Y} = \{\mathbf{f}_1, \mathbf{f}_2\}$ that diagonalizes T.

Solution. Firstly, the characteristic equation of the matrix A is given by

$$(1-\lambda)(2-\lambda) - 1 \quad = \quad 0,$$

which can equivalently written as

$$\lambda^2 - 3\lambda + 1 \quad = \quad 0.$$

Using the quadratic formula, we obtain that $\dfrac{3 \pm \sqrt{5}}{2}$ are the eigenvalues of the matrix. Now, we determine the respective eigenspaces. Recall that we can characterize the eigenspace as Null$(A - \lambda I)$. Hence, for $\lambda = \dfrac{3+\sqrt{5}}{2}$, we have

$$\begin{aligned} \text{Null}(A - \lambda I) &= \text{Null}\Bigg(\begin{pmatrix} 1 - \dfrac{3+\sqrt{5}}{2} & -1 \\ -1 & 2 - \dfrac{3+\sqrt{5}}{2} \end{pmatrix}\Bigg) \\ &= \text{Null}\Bigg(\begin{pmatrix} \dfrac{-1-\sqrt{5}}{2} & -1 \\ -1 & \dfrac{1-\sqrt{5}}{2} \end{pmatrix}\Bigg) \\ &= \text{Span}\Bigg(\begin{pmatrix} 1 \\ \dfrac{-1-\sqrt{5}}{2} \end{pmatrix}\Bigg), \end{aligned}$$

where the last spanning vector is chosen via inspection. Analogously, $\lambda = \dfrac{3-\sqrt{5}}{2}$, we have

$$\begin{aligned} \text{Null}(A - \lambda I) &= \text{Null}\Bigg(\begin{pmatrix} 1 - \dfrac{3-\sqrt{5}}{2} & -1 \\ -1 & 2 - \dfrac{3-\sqrt{5}}{2} \end{pmatrix}\Bigg) \\ &= \text{Null}\Bigg(\begin{pmatrix} \dfrac{-1+\sqrt{5}}{2} & -1 \\ -1 & \dfrac{1+\sqrt{5}}{2} \end{pmatrix}\Bigg) \\ &= \text{Span}\Bigg(\begin{pmatrix} 1 \\ \dfrac{-1+\sqrt{5}}{2} \end{pmatrix}\Bigg). \end{aligned}$$

Now, with the eigenspaces computed, we can simply normalize each spanning vector, and obtain the orthonormal basis that diagnoalizes T, which turn out to be $\frac{1}{N_1}(1, \frac{-1-\sqrt{5}}{2})$ and $\frac{1}{N_2}(1, \frac{-1-\sqrt{5}}{2})$, where N_1 and N_2 are the corresponding normalization scalar. $\square$

Question 3.

Exercise 3:
Prove that a normal operator $T : V \to V$ on a finite dimensional inner product space over $\mathbb{C}$ is self adjoint if and only if its spectrum is real: $\mathrm{sp}_{\mathbb{C}}(T) \subseteq \mathbb{R} + i0$.

Solution. Let T be a normal operator on a finite dimensional inner product space. By Complex Spectral theorem, we obtain that T has a diagonal matrix with respect to some orthonormal basis, which we denote as $M(T)$. By definition, T is self-adjoint iff $M(T) = M(T)^*$, where $M(T)^*$ denotes the conjugate transpose of $M(T)$. Since, $M(T)$ is diagonal, we have that $M(T) = M(T)^*$ iff diagonal entries are real. Since we know that the diagonal entries of a diagonal matrix is the specturm, we have that the diagonal entries of $M(T)$ is real iff all of its eigenvalues are real. By the chain of equivalence obtained, we are done. $\square$

Question 4.

Exercise 4:
If T is diagonalizable over $\mathbb{R}$ or $\mathbb{C}$, prove that

$$e^T = \sum_{\lambda \in \mathrm{sp}(T)} e^{\lambda} P_{\lambda}$$

is the same as the linear operator given by the exponential series

$$e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k$$

Solution. By definition of exponential operator and the given hint, we have

$$\begin{aligned} e^T &= \sum_{k=0}^{\infty} \frac{1}{k!} T^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\lambda \in \mathrm{sp}(T)} \lambda^k P_{\lambda} \\ &= \sum_{\lambda \in \mathrm{sp}(T)} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} P_{\lambda} \\ &= \sum_{\lambda \in \mathrm{sp}(T)} e^{\lambda} P_{\lambda}, \end{aligned}$$

as $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$ is a well-known identity in analysis, and the exchange of the summand is justified by the original convergence. $\square$

Question 5.

Exercise 5:
Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be the operator $T = L_A$ for

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

Explain why T is self-adjoint with respect to the standard inner product $(z, w) = z_1\overline{w_1} + z_2\overline{w_2}$ on $\mathbb{C}^2$. Then determine

(a) The spectrum $\mathrm{sp}_{\mathbb{C}}(T) = \{\lambda_1, \lambda_2\}$;

(b) The eigenspaces $E_\lambda(T)$ and find an orthonormal basis $\{f_1, f_2\}$ in $\mathbb{C}^2$ that diagonalize T. Then

(c) Find a unitary matrix $U^*U = I$ such that

$$UAU^* = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\mathrm{sp}(T) = \{\lambda_1, \lambda_2\}$.

Solution. Since L_A typically denotes the matrix of an operator, with respect to the standard basis, we have that the basis that the matrix is induced by is orthonormal. In that case, we know that the matrix of the adjoint operator, with respect to the same orthonormal basis, is the conjugate transpose of the original matrix. The given matrix equals its conjugate transpose. Hence, we conclude that it is self-adjoint.

(a) Firstly, the characteristic equation of the matrix A is given by

$$(2-\lambda)(4-\lambda) - 9 \quad = \quad 0,$$

which can equivalently written as

$$\lambda^2 - 6\lambda - 1 \quad = \quad 0,$$

Using the quadratic formula, we obtain that $3 \pm \sqrt{10}$ are the eigenvalues of the matrix. The specturm of A is just the set formed by those two eigenvalues.

(b)

Question 6.

Exercise 6 (Uniquess of Spectral Decompositions): Suppose $T : V \to V$ is diagonalizable on an arbitrary vector space (not necesarily an inner product space), so $T = \sum_{i=1}^{r} \lambda_i P_{\lambda_i}$ where $\mathrm{sp}(T) = \{\lambda_1, \ldots, \lambda_r\}$ and P_{λ_i} is the projection onto the λ_i-eigenspace. Now suppose $T = \sum_{j=1}^{s} \mu_j Q_j$ is some other decomposition such that

$$Q_j^2 = Q_j \qquad Q_j Q_k = Q_k Q_j = 0 \quad \text{if } j \neq k \qquad \sum_{j=1}^{s} Q_j = I$$

Prove that

(a) $r = s$ and if the μ_j are suitably relabeled we have $\mu_i = \lambda_i$ for $1 \leq i \leq r$.

(b) $Q_i = P_{\lambda_i}$ for $1 \leq i \leq r$.

Hint: First show $\{\mu_1, \ldots, \mu_s\} \subseteq \{\lambda_1, \ldots, \lambda_r\} = \mathrm{sp}(T)$; then relabel.

Solution.