Linear Algebra II: Problem Set III

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set III of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Exercise 1:
If $V = \mathbb{R}^2$ and $f \in V^*$ is given by $f(x, y) = f(xe_1 + ye_2) = 2x + y$. Consider the linear operation $T : V \to V$ such that $T(x, y) = (3x + 2y, x)$. Compute

1. $T^t(f)$;
2. Matrix $[T^t]_{\mathcal{X}^*}$ where $\mathcal{X} = \{e_1, e_2\}$ is the standard basis in $\mathbb{R}^2$ and $\mathcal{X}^* = \{e_1^*, e_2^*\}$ the dual basis;
3. Show $[T^t]_{\mathcal{X}^*} = [T]^t_{\mathcal{X}}$.

Solution.
(1) By the definition of a transpose map, it follows that

$$\begin{aligned} T^t(f)(x, y) &= fT(x, y) \\ &= f(3x + 2y, x) = 7x + 4y. \end{aligned}$$

(2) From the Theorem 2.25, it follows that the matrix of a transpose map with respect to the standard basis is the transpose of a matrix with respect to the standard basis. We compute the matrix in the next section.

(3) We have

$$\begin{aligned} T(1, 0) &= (3, 1) = 3(1, 0) + 1(0, 1) \\ T(0, 1) &= (2, 0) = 2(1, 0) + 0(0, 0). \end{aligned}$$

It follows that

$$[T]_{\mathscr{X}} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix},$$

which consequently gives

$$[T]^t_{\mathscr{X}} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}.$$

Hence, we have shown that $[T^t]_{\mathscr{X}^*} = [T]^t_{\mathscr{X}}$, which is to be expected from the Theorem 2.25, pg.121, in Friedberg. $\square$

Question 2.

Exercise 2 : If $W \subseteq V$ are vector spaces over $\mathbb{F}$, the **annihilator** W^0 of W is the subspace in the dual V^* consisting of all functionals $l : V \to \mathbb{F}$ that "kill" W

$$W^0 = \{l \in V^* : l|_W = 0\} = \{l \in V^* : l(w) = 0,\ for\ all\ w \in W\}$$

The point of the present exercise is to show that the annihilator W^0 can be identified in a completely natural way with the dual $(V/W)^*$ of the quotient space V/W.
Recall that the "quotient map" $\pi : V \to V/W$ is the $\mathbb{F}$-linear surjection $\pi(v) = v + W$.
Prove that

(a) Every $\tilde{l} \in (V/W)^*$ can be "pulled back" to a well-defined functional l on V, as follows

$$l(v) = \tilde{l}(\pi(v)),\ for\ all\ v \in V$$

Verify l is a linear functional in V^* and that l lies in the annihilator W^0.

Now write $\Phi : (V/W)^* \to W^0 \subseteq V^*$ for the map $\Phi(\tilde{l}) = l$ defined in (a).

(b) Prove that $\Phi : (V/W)^* \to V^*$ is a linear map

(c) Prove that $\Phi : (V/W)^* \to W^0$ is one-to-one. (Since Φ is linear this is, of course, equivalent to proving that $\tilde{l} = 0$ is the only functional such that $\Phi(\tilde{l}) = 0$ in V^*.)

(d) Prove that $\Phi : (V/W)^* \to W^0$ is surjective (i.e. every $l \in W^0$ is the Φ-image of some $\tilde{l} \in (V/W)^*$).

we have shown that there is the natural isomorphism Φ between the vector spaces $(V/W)^*$ and W^0.

Solution.
(a) We first show that l is a linear functional. Let $v_1, v_2 \in V$. By the linearity of $\tilde{l}$ and a fact about quotient space $v_1 + v_2 + W = (v_1 + W) + (v_2 + W)$, it follows that

$$\begin{aligned}
l(v_1 + v_2) &= \tilde{l}(\pi(v_1 + v_2)) \\
&= \tilde{l}(v_1 + v_2 + W) \\
&= \tilde{l}((v_1 + W) + (v_2 + W)) \\
&= \tilde{l}(\pi(v_1) + \pi(v_2)) \\
&= \tilde{l}(\pi(v_1)) + \tilde{l}(\pi(v_2)) \\
&= l(v_1) + l(v_2).
\end{aligned}$$

Let $\alpha \in \mathbb{F}$. Similarly, we have

$$\begin{aligned}
l(\alpha v) &= \tilde{l}(\pi(\alpha v_1)) \\
&= \tilde{l}(\alpha v_1 + W) \\
&= \alpha \tilde{l}(v_1 + W) \\
&= \alpha l(v).
\end{aligned}$$

Observe that for $w \in W$, as any linear functional sends origin to 0, we have

$$\begin{aligned}
l(w) &= \tilde{l}(\pi(w)) \\
&= \tilde{l}(W) \\
&= 0.
\end{aligned}$$

Therefore, we have shown that $l \in V^*$ and l lies in the annihilator W^0.

(b) For any $v \in V$, it follows that

$$\begin{aligned} \Phi(\tilde{l}_1 + \tilde{l}_2)(v) &= \tilde{l}_1 + \tilde{l}_2(\pi(v)) \\ &= \tilde{l}_1(\pi(v)) + \tilde{l}_2(\pi(v)) \\ &= \Phi(\tilde{l}_1)(v) + \Phi(\tilde{l}_2)(v), \end{aligned}$$

which gives $\Phi(\tilde{l}_1 + \tilde{l}_2) = \Phi(\tilde{l}_1) + \Phi(\tilde{l}_2)$. Similarly, we have, for any $v \in V$,

$$\begin{aligned} \Phi(\alpha\tilde{l})(v) &= \alpha\tilde{l}(\pi v) \\ &= \alpha\Phi(\tilde{l}). \end{aligned}$$

Therefore, Φ is linear.

(c) Let $\tilde{l} \in (V \setminus W)^*$ such that $\Phi(\tilde{l}) = 0$. Then, for $v \in V$, it follows that $\tilde{l}(\pi(v)) = 0$, thus $\tilde{l}(v + W) = 0$. Therefore, $\tilde{l}$ is zero for any coset, hence $\tilde{l} = 0$. Therefore, Φ is injective.

(d) Let $l \in W^0$. Now, let $\{v_\lambda\}$ be the coset representatives of left cosets of W in V. Define $\tilde{l} \in (V \setminus W)^*$ by $\tilde{l}([v_\lambda]) = l(v_\lambda)$. Then, by definition of Φ, we have that $\Phi(\tilde{l}) = l$. Hence, we have shown that Φ is surjective.

$\square$

Question 3.

Exercise 3:

If A is an $n \times n$ matrix and $\mathbb{F}^n$ is given the standard inner product, prove that $L_{A^*} = (L_A)^*$ as operators on $\mathbb{F}^n$.

Solution. Let B be the standard basis of $\mathbb{F}^n$. It follows that $[L_A]_B = A$, and $L_{A^*}]_B = A^*$. Since B is orthonormal basis of $\mathbb{F}^n$, we have that $[(L_A)^*]_B = [L_A]_B^* = A^*$. Hence, we obtain $[L_{A^*}]_B = [(L_A)^*]_B$, thus $L_{A^*} = (L_A)^*$ as required. $\square$

Question 4.

Exercise 4: If a finite dimensional vector space is a direct sum $V = E \oplus F$ and P_E, $P_F = I - P_E$ are the associated projections onto E and F, prove that a linear operator $T : V \to V$ leaves both subspaces E, F invariant if and only if T commutes with P_E (hence also P_F).

Solution. Assume that $T(E) \subset E$ and $T(F) \subset F$. Fix $x \in V$. As E and F form a direct sum, we have $x = e + f$ for unique e, f from E and F respectively. By the invariance of the subspaces with respect to the T operator, and the linearity of T and P_E, it follows that

$$\begin{aligned} TP_E(x) &= T(e) \\ P_ET(x) &= P_E(T(e) + T(f)) = P_E(T(e)) + P_E(T(f)) = T(e). \end{aligned}$$

Hence, it follows that $TP_E = P_ET$.

Assume that $TP_E = P_ET$. It follows that $TP_E(E) = P_ET(E)$. Since $P_E(E) = E$, we have $T(E) = P_E(T(E))$. As P_E is a projection onto E, this implies that $T(E) \subset E$. Hence, T leaves E, and F invariant. $\square$

Question 5.

Exercise 5:
Let $W \subseteq V$ be finite dimensional vector spaces over $\mathbb{F}$. Let $\mathcal{X} = \{e_1, \cdots, e_m\}$ be any basis for the subspace W and let

$$\overline{\mathcal{X}} = \{\bar{f}_1, \cdots, \bar{f}_n\}$$

be any basis for the quotient space V/W. If we select any preimages $f_1, \cdots, f_n$ in V so that $\bar{f}_j = f_j + W$ in V/W for each j, prove that the $\{f_j\}$ are linearly independent vectors in V, and that

$$\{e_1, \cdots, e_m, f_1, \cdots, f_n\}$$

is always a basis for the vector space V.
Note: This is one way to prove that $dim(W) + dim(V/W) = dim(V)$.

Solution. Let $\pi : V \to V \setminus W$ be the canonical projection map. We first show that $\{f_1, ..., f_n\}$ is linearly independent. Assume that $\sum_{i=1}^{n} c_i f_i = 0$. By the linearity of π, it follows that

$$\begin{aligned} \pi(\sum_{i=1}^{n} c_i f_i) &= \sum_{i=1}^{N} c_i \pi(f_i) \\ &= \sum_{i=1}^{n} c_i \tilde{f}_i. \end{aligned}$$

Since $\pi(0) = W$, we obtain

$$W = \sum_{i=1}^{n} c_i \tilde{f}_i,$$

and by the linear independence of $\tilde{f}_i$s, we have that $c_1 = ... = c_n = 0$. Hence, $\{f_1, ..., f_n\}$ is linearly independent. Now, as $\{e_i\}$ and $\{f_j\}$ are linearly independent from each other, it follows that $\{e_1, ..., f_n\}$. We now show that the set spans V. Fix $v \in V$. Take $v + W$. Since $\{\tilde{f}_j\}$ are basis of the quotient space, we have that $v + W$ can be expressed as a linear combination of $\{\tilde{f}_j\}$. Now, taking the pre-image it follows that $v - \sum c_j f_j \in W$. Therefore, Using the $\{e_i\}$ we can express $v - \sum c_j f_j$ as a linear combination of them. Hence, this shows that an arbitrary element can be spanned by $\{e_1, ... f_n\}$. $\square$

Question 6.

Exercise 6:
Prove that

1. Every $n \times n$ matrix can be written as a linear combination of matrices in $GL(n, \mathbb{F})$.
2. If a matrix A commutes with all matrices $B \in M(n, \mathbb{F})$, then A must be scalar, i.e. $A = diag(\lambda, \cdots, \lambda)$ for some $\lambda \in \mathbb{F}$.
3. A matrix A has a one-point similarity class if and only if

$$SAS^{-1} = A \ for \ all \ S \in GL(n, \mathbb{F})$$

Prove that this happens precisely when $A = \lambda I$ for some $\lambda \in \mathbb{F}$.

Hint: Part 2. is the most important fact; do this even if you have trouble with 1.; in 2. try some really simple choices for $B \in M(n, \mathbb{F})$ and see what $AB - BA = 0$ tells you about A.
Note: We have identified the center of the matrix algebra $M(n, \mathbb{F})$, the set of matrices A that commute with all $n \times n$ matrices. They are all scalar.

Solution. (a) We must show that $GL(n, \mathbb{F})$ spans $M(n, \mathbb{F})$.

(b) Let $A \in M(n, \mathbb{F})$. Assume that $AB = BA$ for any $B \in M(n, \mathbb{F})$. Let E_{ij} be a matrix, where its 1 at (i, j) and 0 elsewhere. Observe that

$$A = \sum_{1 \le k,l \le n} a_{kl} E_{kl} \quad \text{and} \quad E_{ij} E_{kl} = \delta_{jk} E_{il}, \tag{1}$$

for $1 \le i, j, k, l \le n$. As A commutes with any matrix, and by (1), it follows that

$$\begin{aligned} 0 &= AE_{ij} - E_{ij}A \\ &= (\sum_{1 \le k,l \le n} a_{kl} E_{kl}) E_{ij} - E_{ij} (\sum_{1 \le k.l \le n} a_{kl} E_{kl}) \\ &= \sum_{k=1}^{n} a_{ki} E_{kj} - \sum_{l=1}^{n} a_{jl} E_{il} \end{aligned}$$

(c) Let $A = \lambda I$. It follows that for any $S \in A$,

$$\begin{aligned} SAS^{-1} &= S\lambda I S^{-1} \\ &= \lambda S I S^{-1} \\ &= \lambda I = A. \end{aligned}$$

Now, let $A \in M(n, \mathbb{F})$ and assume that $SAS^{-1} = A$ for all $S \in GL(n, \mathbb{F})$. Equivalently, we have $SA = AS$ for all $S \in GL(n, \mathbb{F})$. Now consider a matrix $B \in M(n, \mathbb{F}$. By (a), we have that $B = \sum_{i=1}^{n} c_i S_i$, for some c_i and S_i from $GL(n, \mathbb{F})$. It follows that

$$\begin{aligned} AB &= A(\sum_{i=1}^{n} c_i S_i) = c_i \sum_{i=1}^{n} AS_i \\ &= c_i \sum_{i=1}^{n} S_i A = BA. \end{aligned}$$

Therefore, by (b), we have that $A = \lambda I$ for some $\lambda \in \mathbb{F}$. $\square$

Question 7.

Exercise 7:
If $dim(V) = 2$ and $T : V \to V$ is any linear operator, its characteristic polynomial $p_T(\lambda) \in \mathbb{F}[\lambda]$ has the form

$$p_T(\lambda) = \lambda^2 - Tr(T)\lambda + det(T)$$

Question: If $A, B \in M(2, \mathbb{F})$, i.e. is there an $S \in GL(2, \mathbb{F})$ such that $B = SAS^{-1}$?
Hint: Try some upper triangular 2×2 matrices.

Solution. It is a well-known result that one can determine the equivalence class of an $n \times n$ matrix by looking at its real jordan form. In the case of $M(2, \mathbb{R})$, the forms are as follows:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \text{ and } \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where the implication is that two matrices are similar if and only if they have the same real jordan form. Hence, in the two by two case, we have identified 3 equvialence classes of matrix similarity. $\square$