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# Linear Algebra II: Problem Set IV

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## Abstract

This work contains solutions to the problem set IV of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

### Question 1.

#### Exercise 1:

Suppose  $A \in M(n, \mathbb{F})$  with  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . If  $A$  is self-adjoint, so that

$$A^* = A \text{ (where } (A^*)_{ij} = \overline{A_{ji}} \text{ for all } i, j)$$

it is well known that there is a basis in  $\mathbb{F}^n$  that diagonalizes the corresponding linear operator  $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^n$ . (In fact, there is even a basis that is orthonormal with respect to the standard product in  $\mathbb{F}^n$ .)

Now, suppose that  $A$  is only symmetric, with

$$A^T = A \text{ (where } (A^T)_{ij} = A_{ji} \text{ for all } i, j)$$

When  $\mathbb{F} = \mathbb{R}$ , symmetry is the same thing as self-adjointness, and symmetry  $A^T = A$  suffices to guarantee orthonormal diagonalizability of  $L_A$ . What happens when  $\mathbb{F} = \mathbb{C}$ ? If  $A$  is a symmetric is the same thing as self-adjointness, and symmetry  $A^T = A$  suffices to guarantee orthonormal diagonalizability of  $L_A$ . What happens when  $\mathbb{F} = \mathbb{C}$ ? If  $A$  is symmetric  $A = A^T$  with complex entries, does  $L_A$  always have a diagonalizing basis? Prove or give a counterexample.

#### Solution.

Consider the following matrix:

$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

The matrix equals its transpose, but not its conjugate transpose. The characteristic polynomial of the matrix is  $(1 - \lambda)(-1 - \lambda) + 1 = \lambda^2$ . Hence, we obtain that the eigenvalue of the matrix is 0 with algebraic multiplicity of 2. Now, observe that  $E_{\lambda=0}(L_A) = \text{Null}(L_A - 0I) = \text{Null}(L_A)$ . Since the nullspace of  $L_A$  is the span of row vectors of  $A$ , and  $(1, i)$  and  $(i, -1)$  are linearly dependent (take  $i$  and  $-1$  as coefficients), we have that the geometric multiplicity of 0 is 1. Therefore, the eigenspaces of  $L_A$  do not form a direct sum of  $\mathbb{C}^2$ . Therefore, by the Spectral Theorem, the matrix is not diagonalizable.  $\square$

**Question 2.****Exercise 2:**

Let  $\mathcal{X} = \{e_1, e_2\}$  be the standard basis in the complex inner product space  $V = \mathbb{C}^2$  equipped with the usual Euclidean inner product  $(a, b) = a_1\bar{b}_1 + a_2\bar{b}_2$ . Let  $\mathcal{N} = \{f_1, f_2\}$  be the basis such that  $f_1 = e_1$  and  $f_2 = e_1 + e_2$  and define  $T : V \rightarrow V$  to be the linear operator such that  $T(f_1) = f_1$  and  $T(f_2) = \frac{1}{2}f_2$ . Obviously  $\mathcal{N}$  diagonalizes  $T$  but the basis eigenvectors  $f_1, f_2$  are not orthogonal.

1. Find  $[T]_{\mathcal{X}, \mathcal{X}}$  and  $[e^{tT}]_{\mathcal{X}, \mathcal{X}}$  for  $t \in \mathbb{R}$ .
2. Explain why  $T$  cannot be a self-adjoint operator ( $T^* = T$ ).
3. Explain why there cannot be an ON basis in  $V$  that diagonalizes  $T$ .
4. Find the solution  $X(t)$ ,  $t \in \mathbb{R}$ , of the vector-valued differential equation

$$\frac{dX}{dt} = A \cdot X(t) \text{ with } X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where  $A = [T]_{\mathcal{X}, \mathcal{X}}$ .

**Solution.**

(a) With the given definition, we have

$$\begin{aligned} T(e_1) &= e_1, \\ T(e_2) &= T(f_2 - e_1) = T(f_2) = T(e_1) = \frac{1}{2}f_2 + e_1 = -\frac{1}{2}e_1 + \frac{1}{2}e_2. \end{aligned}$$

Therefore, it follows that

$$[T]_{\mathcal{X}, \mathcal{X}} = \begin{pmatrix} 1 & \frac{-1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

(b) We see that  $T$  has a matrix representation with respect to the standard basis that is not symmetric. Hence,  $T^*$  cannot be self-adjoint.

(c) Spectral theorem says that  $T$  is orthonormally diagonalizable, iff  $T$  is self-adjoint. Since  $T$  is not self-adjoint, it is not orthonormally diagonalizable.

(d) The solution will be  $[e^{tT}]$  where  $t$  agrees with the initial data of  $(1, 0)$ .

**Question 3.**

**Exercise 3:**

In  $V = \mathbb{R}^3$  equipped with the usual Euclidean inner product let  $M = (\mathbb{R}f_3)^\perp$  where  $f_3 = (1, -2, 3)$ .

1. Give a formula  $(y_1, y_2, y_3) = R(x_1, x_2, x_3)$  in terms of inner products for the linear operator  $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  that reflects vectors across the plane  $M$ . Find the image of the particular vector  $R(1, 1, -3)$ .
2. Prove that  $R$  is an isometry, so that

$$\|R(x) - R(y)\| = \|x - y\|$$

and hence is a bijection  $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ .

3. Find an ON basis  $\{e_1, e_2\}$  for  $M$ .
4. What is the matrix  $[R]_{\mathcal{N}, \mathcal{N}}$  with respect to the ON basis  $\mathcal{N} = \{e_1, e_2, e_3\}$  such that  $e_3 = f_3/\|f_3\|$ ?
5. Is  $R : V \rightarrow V$  orthogonally diagonalizable? Explain.

**Solution. (a)** The reflection formula can be given by subtracting the orthogonal projection twice from the original vector, which can be written as follows:

$$R(v) = v - 2 \frac{v \cdot f_3}{f_3 \cdot f_3} f_3.$$

**(b)** From the above formula it follows that

$$\begin{aligned} \|R(x) - R(y)\| &= \left\| x - 2 \frac{x \cdot f_3}{f_3 \cdot f_3} f_3 - y + 2 \frac{y \cdot f_3}{f_3 \cdot f_3} f_3 \right\| \\ &= \left\| x - y - 2 \frac{(x - y) \cdot f_3}{f_3 \cdot f_3} f_3 \right\| \\ &= \|x - y\|. \end{aligned}$$

**(c)** One can see that  $(-1, 1, 1)$  has a 0 inner product with  $f_3$  and  $(1, 2, 1)$  also have a 0 inner product and they are linearly independent. Therefore, they form an orthonormal basis for  $M$ .

**(e)** We have that

□

**Question 4.**

**Exercise 4:**

Let  $V = \mathbb{C}^2$  with the usual Euclidean inner product and let  $T : V \rightarrow V$  be the linear operator whose matrix with respect to the standard ON basis  $\mathcal{X} = \{e_1, e_2\}$  is

$$[T]_{\mathcal{X}, \mathcal{X}} = \begin{pmatrix} 7 & \sqrt{3} \\ \sqrt{3} & 5 \end{pmatrix}$$

1. Determine the spectrum  $sp_{\mathbb{C}}(T) = \{\lambda_1, \lambda_2\}$  and show  $T$  is orthogonally diagonalizable by finding an ON basis  $\mathcal{N} = \{f_1, f_2\}$  of eigenvectors.
2. Explain why  $T$  is self-adjoint.

**Solution. (1)** The characteristic polynomial of  $T$  can be computed as  $(7 - \lambda)(5 - \lambda) - 3 = \lambda^2 - 12\lambda + 32 = (\lambda - 8)(\lambda - 4)$ . Therefore, we have that  $\text{spec}_{\mathbb{C}}(T) = \{4, 8\}$ . We see that the eigenvector associated with 4 is  $\frac{1}{2}(1, -\sqrt{3})$  and with 8 is  $\frac{1}{2}(\sqrt{3}, 1)$ , which are the ON basis of eigenvectors, that orthogonally diagonalizes  $T$ .

**(2)** When  $T$  is a linear operator over complex field, that is represented in a matrix form with respect to an orthonormal basis, we have that the matrix representation of  $T^*$ , with respect to the orthonormal basis, is the conjugate transpose of the matrix representation of  $T$ . As operators with the same matrix representation with respect to the same set of basis are in fact the same linear operator, we have that  $T = T^*$ , which is the condition for a linear operator to be self-adjoint.

□

**Question 4-continued.**

If an operator is self-adjoint (or merely diagonalizable) it has a spectral decomposition

$$T = \sum_{\lambda \in sp(T)} \lambda P_\lambda$$

where  $P_\lambda =$  (projection onto  $E_\lambda$  along  $\oplus_{\nu \neq \lambda} E_\nu$ ) which describes  $T$  as a weighted sum of projections onto the eigenspaces.

3. Find the matrices  $[P_{\lambda_k}]$  that describe the spectral projections with respect to the diagonalizing basis  $\mathcal{N}$ .

Hint: You only need to find one of these matrices because  $P_{\lambda_1} + P_{\lambda_2} = I \Rightarrow P_{\lambda_2} = I - P_{\lambda_1}$ .

4. Find the matrices  $[P_{\lambda_k}]$  that describes the spectral projections with respect to the standard ON basis  $\mathcal{X}$ .

Hint: Again, you only need to find one of these matrices.

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5. In terms of the spectral decomposition the square root of  $T$  is the operator given by

$$\sqrt{T} = \sum_{\lambda \in sp(T)} \sqrt{\lambda} P_\lambda$$

Find the matrix  $[\sqrt{T}]$  that describes  $\sqrt{T}$  with respect to the standard ON basis.

**Solution.**

(3) By Spectral Theorem, we have

$$P_\lambda = \prod_{u \neq \lambda} \frac{A - uI}{\lambda - u}.$$

Hence, it follows that

$$\begin{aligned} P_{\lambda=8} &= \frac{A - 4I}{8 - 4} \\ &= \frac{1}{4} \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \end{aligned}$$

□