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# Linear Algebra II: Problem Set I

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## Abstract

This work contains solutions to the problem set I of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

### Question 1.

**Exercise 1 (Uniqueness of Spectral Decompositions):** Suppose  $T : V \rightarrow V$  is diagonalizable on an arbitrary vector space (not necessarily an inner product space), so  $T = \sum_{i=1}^r \lambda_i P_{\lambda_i}$  where  $\text{sp}(T) = \{\lambda_1, \dots, \lambda_r\}$  and  $P_{\lambda_i}$  is the projection onto the  $\lambda_i$ -eigenspace. Now suppose  $T = \sum_{j=1}^s \mu_j Q_j$  is some other decomposition such that

$$Q_j^2 = Q_j \quad Q_j Q_k = Q_k Q_j = 0 \quad \text{if } j \neq k \quad \sum_{j=1}^s Q_j = I$$

Prove that

- (a)  $r = s$  and if the  $\mu_j$  are suitably relabeled we have  $\mu_i = \lambda_i$  for  $1 \leq i \leq r$ .
- (b)  $Q_i = P_{\lambda_i}$  for  $1 \leq i \leq r$ .

**Hint:** First show  $\{\mu_1, \dots, \mu_s\} \subseteq \{\lambda_1, \dots, \lambda_r\} = \text{sp}(T)$ ; then relabel.

### Solution.

**Question 2.**

**Exercise 2 :** Let  $V = C_c^\infty(\mathbb{R})$  be the space of real-valued functions  $f(t)$  on the real line that have continuous derivatives  $D^k f$  of all orders, and have “bounded support” – each  $f$  is zero off of some bounded interval (which is allowed to vary with  $f$ ). Because all such functions are “zero near  $\infty$ ” there is a well defined inner product

$$(f, h)_2 = \int_{-\infty}^{\infty} f(t) \overline{h(t)} dt$$

The derivative  $Df = df/dt$  is a linear operator on this infinite dimensional space.

1. Prove that the adjoint of  $D$  is *skew-adjoint*, with  $D^* = -D$ .
2. Prove that the second derivative  $D^2 = d^2/dt^2$  is *self-adjoint*.

**Hint:** Integration by parts.

**Solution. (a)** Let  $f, g \in V$ . By integration by parts, we have

$$\begin{aligned} \int_{-\infty}^{\infty} Df \overline{g} dt + \int_{-\infty}^{\infty} f \overline{Dg} dt &= [f \overline{g}]_{-\infty}^{\infty} \\ &= 0, \end{aligned}$$

as  $f, g$  have bounded supports. Therefore, it follows that

$$(Df, g) = -(f, Dg).$$

Since  $f, g$  were arbitrary, we have shown that  $D^* = -D$ .

**(b)** By (a), we have

$$\begin{aligned} (D^2 f, g) &= -(Df, Dg) \\ &= (f, D^2 g), \end{aligned}$$

for  $f, g \in V$ . Therefore,  $D^2$  is self-adjoint. □

**Question 3.****Exercise 3:**

If  $T : V \rightarrow V$  is a linear operator on an inner product space and  $\lambda \in \text{sp}(T)$ , prove that

1.  $E_{\bar{\lambda}}(T^*) = K(T^* - \bar{\lambda}I)$  is equal to  $R(T - \lambda I)^\perp$ .
2.  $\dim E_{\bar{\lambda}}(T^*) = \dim E_\lambda(T)$ .

**Solution. (1)** We first claim that for  $K(T^*) = R(T)^\perp$ . We prove it by showing the following series of equivalence.

$$\begin{aligned}
 w \in K(T^*) &\iff T^*w = 0 \\
 &\iff \langle v, T^*w \rangle = 0 \quad \forall v \in V \\
 &\iff \langle Tv, w \rangle = 0 \quad \forall v \in V \\
 &\iff w \in R(T)^\perp,
 \end{aligned}$$

as required. Now, by the elementary properties of adjoint, it follows that

$$\begin{aligned}
 (T - \lambda I)^* &= T^* - (\lambda I)^* \\
 &= T^* - \bar{\lambda}I^* \\
 &= T^* - \bar{\lambda}I.
 \end{aligned}$$

Therefore, combining the two results together, we have shown that

$$E_{\bar{\lambda}}(T^*) = R(T - \lambda I)^\perp$$

**(2)** By definition of eigenspace, and the rank-nullity theorem, we have

$$\begin{aligned}
 \dim E_\lambda(T) &= \dim N(T - \lambda I) \\
 &= \dim V - \dim R(T - \lambda I).
 \end{aligned}$$

As a subspace of a linear space forms a direct sum with its orthogonal complement, we have that  $\dim R(T - \lambda I) = \dim V - \dim R(T - \lambda I)^\perp$ . By (1), it follows that  $\dim R(T - \lambda I) = \dim V - \dim E_{\bar{\lambda}}(T^*)$ . Substituting to the result to the previous equation, we obtain

$$\dim E_\lambda(T) = \dim E_{\bar{\lambda}}(T^*),$$

as required. □

**Question 4.**

**Exercise 4:**

If  $m < n$  and the coordinate spaces  $\mathbb{K}^m$ ,  $\mathbb{K}^n$  are equipped with the standard inner products, consider the linear operator

$$T : \mathbb{K}^m \rightarrow \mathbb{K}^n \quad T(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0)$$

This is an isometry from  $\mathbb{K}^m$  into  $\mathbb{K}^n$ , with trivial kernel  $K(T) = (0)$  and range  $R(T) = \mathbb{K}^m \times (0)$  in  $\mathbb{K}^n = \mathbb{K}^m \oplus \mathbb{K}^{n-m}$ .

1. Provide an explicit description of the adjoint operator  $T^* : \mathbb{K}^n \rightarrow \mathbb{K}^m$  and determine  $K(T^*)$ ,  $R(T^*)$ .
2. Compute the matrices of  $[T]$  and  $[T^*]$  with respect to the standard orthonormal bases in  $\mathbb{K}^m, \mathbb{K}^n$ .
3. How is the action of  $T^*$  related to the subspaces  $K(T)$ ,  $R(T^*)$  in  $\mathbb{K}^m$  and  $R(T)$ ,  $K(T^*)$  in  $\mathbb{K}^n$ ? Can you give a geometric description of this action?

Unitary operators can be described in several different ways, each with its own advantages in applications.

**Solution. (1)** Let  $(e_i)$  and  $(f_j)$  denote the standard bases for  $\mathbb{K}^m$  and  $\mathbb{K}^n$  respectively. Let  $v \in \mathbb{K}^n$ . For  $i = 1, 2, \dots, m$ , it follows that

$$\begin{aligned} (T^*v)_i &= (T^*v, e_i) \\ &= (v, Te_i) \\ &= (v, f_i) \\ &= v_i. \end{aligned}$$

Therefore, we have shown that  $T^*$  is the projection onto the first  $m$  coordinates. It easily follows that  $K(T^*) = (0) \times \mathbb{K}^{n-m}$  and  $R(T^*) = \mathbb{K}^m$ .

(2)  $[T]$  can be written as  $[I_{m \times m}; 0_{n-m \times m}]$ , and  $[T^*]$  can be written as  $[I_{m \times m} 0_{m \times n-m}]$ .

(3) We can relate them as both pairs form a direct sum of  $\mathbb{K}^m$ , and  $\mathbb{K}^n$  respectively. Geometrically, we can describe this action as an orthogonal projection.

□

**Question 5.**

**Exercise 5:**

Let  $\mathfrak{X} = \{e_1, \dots, e_n\}$  be an arbitrary basis (not necessarily orthonormal) in a finite dimensional inner product space  $V$ .

- (a) Use induction on  $n$  to prove that there exist vectors  $\mathfrak{Y} = \{f_1, \dots, f_n\}$  such that  $(e_i, f_j) = \delta_{ij}$ .
- (b) Explain why the  $f_j$  are uniquely determined and a basis for  $V$ .

**Note:** If the initial basis  $\mathfrak{X}$  is orthonormal then  $f_i = e_i$  and the result trivial; we are interested in *arbitrary* bases in an inner product space.

**Solution.**

**Question 6.**

**Exercise 6 :**

Let  $V$  be an inner product space and  $T$  a linear operator that is diagonalizable in the ordinary sense, but not necessarily orthogonally diagonalizable. Prove that

- (a) The adjoint operator  $T^*$  is diagonalizable. What can you say about its eigenvalues and eigenspaces?
- (b) If  $T$  is *orthogonally* diagonalizable so is  $T^*$ .

**Hint:** If  $\{e_i\}$  diagonalizes  $T$  what does the “dual basis”  $\{f_j\}$  of Exercise 5 do for  $T^*$ ?

**Solution. (a)**

**(b)** Assume  $T$  is orthogonally diagonalizable. Then, there exists an orthonormal basis  $B$  such that  $[T]_B$  is diagonal. As  $B$  is orthonormal, we have that  $[T^*]_B = [T]_B^*$ . Since a conjugate transpose of a diagonal matrix is diagonal, we see that  $[T^*]_B$  is a diagonal matrix. Therefore, there exists an orthonormal basis, namely  $B$  such that the matrix representation of  $T^*$  with respect to the orthonormal basis is diagonal. Hence,  $T^*$  is diagonalizable.

□

**Question 7.**

**Exercise 7:** Here  $V = \mathbb{R}^3$  and  $f_1, f_2, f_3 \in V^*$  are the linear functionals  $f_k : V \rightarrow \mathbb{R}$  given by  $(**)$   $f_1(x, y, z) = x - 2y$ ,  $f_2(x, y, z) = x + y + z$  and  $f_3(x, y, z) = y - 3z$ ;

1. Prove  $\mathcal{Y} = \{f_1, f_2, f_3\}$  is a basis in  $V^*$ .
2. Find a basis  $\mathcal{X} = \{e'_1, e'_2, e'_3\} \subseteq V$  whose dual basis  $\mathcal{X}^*$  in  $V$  is equal to  $\mathcal{Y}$ .

**Solution. (1)** As the dimension of the dual space for a finite dimensional linear space equals the dimension of the linear space, it suffices to show that  $f_1, f_2$ , and  $f_3$  are linearly independent. Let  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ , such that  $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$ . By substitutions, it follows that

$$\begin{aligned}\lambda_1 f_1(x, y, z) + \lambda_2 f_2(x, y, z) + \lambda_3 f_3(x, y, z) &= \lambda_1(x - 2y) + \lambda_2(x + y + z) + \lambda_3(y - 3z) \\ &= (\lambda_1 + \lambda_2)x + (-2\lambda_1 + \lambda_2 + \lambda_3)y + (\lambda_2 - \lambda_3)z.\end{aligned}$$

Since the above equation holds for all  $x, y, z \in \mathbb{R}$  and  $\mathbb{R}$  is an integral domain, we obtain that

$$\begin{aligned}\lambda_1 + \lambda_2 &= 0 \\ -2\lambda_1 + \lambda_2 + \lambda_3 &= 0 \\ \lambda_2 - \lambda_3 &= 0.\end{aligned}$$

It immediately follows that  $\lambda_1 = \lambda_2 = \lambda_3 = 0$ . Therefore,  $\mathcal{Y}$  is linearly independent, thus a basis in  $V^*$ .

**(2)** We wish to find vectors  $v_1, v_2, v_3$