Linear Algebra II: Problem Set I

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Abstract

This work contains solutions to the problem set I of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Exercise 1.
Let V_N be the restrictions to $[0,1]$ of polynomials $f \in \mathbb{C}[x]$ having degree $\leq N$. Give this $(N+1)$-dimensional space of $\mathcal{C}[0,1]$ the usual L^2 inner product $(f,h)_2 = \int_0^1 f(t)\overline{h(t)}\,dt$ inherited from the larger space of continuous functions. Let $D : V_N \to V_N$ be the differentiation operator

$$D(a_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N) = a_1 + 2a_2\, t + 3a_3\, t^2 + \cdots + N a_n\, t^{N-1}$$

1. Compute the L^2-inner product $(f,h)_2$ in terms of the coefficients a_k, b_k that determine f and h.
2. Is D a self-adjoint operator? Skew-adjoint?

Solution. (a) By expressing f,h in terms of the coefficients a_k, b_k that determine f and g, exploiting the fact that the complex conjugate of the product is the product of the conjugate, using the differentiation of complex polynomials, we obtain

$$\begin{aligned}
(f,g)_2 &= \int_0^1 (\sum_{i=0}^N a_i t^i)\overline{(\sum_{i=0}^N b_i t^i)}dt \\
&= \int_0^1 (\sum_{i=0}^N a_i t^i)(\sum_{i=0}^N \overline{b_i} t^i)dt \\
&= \int_0^1 \sum_{0\leq i,j\leq N} a_i \overline{b_j} t^{i+j}\,dt \\
&= \left[\sum_{0\leq i,j\leq N} \frac{a_i\overline{b_j}}{i+j+1} t^{i+j+1} \right]_0^1 \\
&= \sum_{0\leq i,j\leq N} \frac{a_i\overline{b_j}}{i+j+1}.
\end{aligned}$$

(b) We are given $D : V_N \to V_N$ such that