Linear Algebra II: Problem Set IV

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Abstract

This work contains solutions to the problem set V of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Exercise 1:
Explain why every linear operator $T : V \to V$ on a finite dimensional vector space over $\mathbb{C}$ always has an eigenvalue.

Solution.
We know that a scalar is an eigenvalue of an operator iff it is a root to the operator's characteristic polynomial. By the Fundemental Theorem of Algebra, we know that any polynomial over $\mathbb{C}$ splits. Therefore, we have that any operator over $\mathbb{C}$ has an eigenvalue. $\square$

Question 2.

Exercise 2:
Does the concept of "self adjoint make sense for linear operators $T : V \to V$ on a finite dimensional vector space over $\mathbb{C}$ if the space V does not come equipped with an inner product?
Here's what we mean: given $T : V \to V$, suppose there is a basis $\mathcal{X}$ such that $A = [T]_{\mathcal{X}}$ is a self-adjoint matrix, in the sense that $A^* = A$. If $\mathcal{N}$ is any other basis does the matrix $B = [T]_{\mathcal{N}}$ always have this property? Prove or provide a counter example.
Note: If the answer is no, then "self-adjointness" is not a coordinate independent property of linear operators on complex vector spaces. An adjoint operator T^* can only be defined when V is equipped with the extra structure of an inner product.

Solution.
No. The definition of adjoint operator requires inner product structure to begin with. Therefore, we cannot discuss whether or not an operator equals its adjoint, without the inner product structure. As a remark, we know that the transpose of a matrix representation of a linear operator equals the matrix representation of its adjoint, when the chosen basis is orthonormal. $\square$

Question 3.

Exercise 3:
Let W be the linear span of three independent vectors $\{v_1, v_2, v_3\}$ in a finite dimensional vector space V.

1. If $V = \mathbb{C}^5$ and $v_1 = (1, 1, 2, 0, 3)$, $v_2 = (3, 2, 1, 5, -1)$, $v_3 = (2, 1, 0, 2, -1)$. Find an explicit choice of 2 vectors e_{i_1}, e_{i_2} from the standard basis $\{e_1, \cdots, e_5\}$ such that $\{v_1, v_2, v_3, e_{i_1}, e_{i_2}\}$ is a basis for $\mathbb{C}^5$.
2. If $\{f_1, \cdots, f_n\}$ is a particular basis in V, prove that one can always create a basis for V by augmenting the v_i with $n - 3$ vectors $f_{i_1}, \cdots, f_{n-3}$ selected from the given basis.
3. From 2., explain why the cosets $\{\overline{f_{i_1}}, \cdots, \overline{f_{i_{n-3}}}\}$ in the quotient space V/W are a basis for V/W.

Solution. (a)

(b)

(c) As the quotient space has a dimension $\dim(V) - \dim(W)$, which can be shown using the fundamental theorem of linear algebra and the fact that the quotient map has W as the kernel, it suffices to check that the cosets are linearly independent. Let $c_{i_1}\overline{f_{i_1}} + ... + c_{i_n}\overline{f_{i_{n-3}}} = W$. It follows that $c_{i_1}f_{i_1} + ... + c_{i_n}f_{i_{n-3}} \in W$. Using the basis of W, we obtain

$$c_{i_1}f_{i_1} + ... + c_{i_n}f_{i_{n-3}} \quad = \quad d_1v_1 + d_2v_2 + d_3v_3,$$

which immediately gives

$$c_{i_1}f_{i_1} + ... + c_{i_n}f_{i_{n-3}} - d_1v_1 - d_2v_2 - d_3v_3 \quad = \quad 0.$$

By the independence of the basis set, we obtain that $c_{i_1} = ... = c_{i_{n-3}} = 0$. Therefore, the cosets form a basis of the quotient space.

$\square$

Question 4.

Exercise 4: Let $T : V \to V$ be an arbitrary linear map and W a T-invariant subspace. We say that vectors $e_1, \cdots, e_m$ in V are:

1. **Independent (mod W)** if their images $\overline{e_1}, \cdots, \overline{e_m}$ in V/W are linearly indepen-

dent. Since $\sum_i c_i \overline{e_i} = 0$ in V/W if and only if $\sum_i c_i e_i \in W$ in V, that means:

$$\sum_{i=1}^{m} c_i e_i \in W \;\Rightarrow\; c_1 = \cdots = c_m = 0 \qquad (c_i \in \mathbb{F})$$

2. **Span V (mod W)** if $\mathbb{F}\text{-span}\{\overline{e_i}\} = V/W$, which means: given $v \in V$, there are $c_i \in \mathbb{F}$ such that $v - \sum_i c_i e_i \in W$, or $\overline{v} = \sum_{i=0} c_i \overline{e_i}$ in V/W.
3. **A basis for V (mod W)** if the images $\{\overline{e_i}\}$ are a basis in V/W (if and only if 1. and 2. hold.

Now, let $W \subseteq \mathbb{R}^5$ be the solution set of system

$$\begin{cases} x_1 + x_3 &= 0 \\ x_1 - x_4 &= 0 \end{cases}$$

and let $\{e_i\}$ be the standard basis in $V = \mathbb{R}^5$.

1. Find vectors v_1, v_2 that are a basis for V (mod W).
2. Is $\mathfrak{X} = \{e_1, e_2, e_3, v_1, v_2\}$ a basis for V where v_1, v_2 are the vectors in (1.)?
3. Find a basis $\{f_1, f_2, f_3\}$ for the subspace W.

Solution. $\square$

Question 5.

Exercise 5
Let

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$$

and $\mathbb{F} = \mathbb{R}$.

1. If possible find a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.
2. If successful in finding a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution.
(a) The eigenvalues are given by the roots of the following equation:

$$\det \begin{pmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix} = 0,$$

which can be simplified to

$$-(\lambda-1)(\lambda-2)(\lambda-3) = 0.$$

Therefore, we have that $\lambda = 1, 2, 3$ are the eigenvalues of the given matrix A. With the given eigenvalues, we can solve a homogenous system of equation given by $A - \lambda I = 0$ to obtain the respective eigenvectors. Doings so yields $(-1,-1,1)$, $(-1,1,0)$, and $(-1,0,1)$.

(b) Using the information above, we have

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

as required. $\square$