Linear Algebra II:
Problem Set III

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set III of Linear Algebra II 2016 at Courant Institute of Mathematical Sciences.

Question 1.

Exercise 1:
If $V = \mathbb{R}^2$ and $f \in V^*$ is given by $f(x, y) = f(xe_1 + ye_2) = 2x + y$. Consider the linear operation $T : V \to V$ such that $T(x, y) = (3x + 2y, x)$. Compute

1. $T^t(f)$;
2. Matrix $[T^t]_{\mathcal{X}^*}$ where $\mathcal{X} = \{e_1, e_2\}$ is the standard basis in $\mathbb{R}^2$ and $\mathcal{X}^* = \{e_1^*, e_2^*\}$ the dual basis;
3. Show $[T^t]_{\mathcal{X}^*} = [T]_{\mathcal{X}}^t$.

Solution. $\square$

Question 2.

Exercise 2 : If $W \subseteq V$ are vector spaces over $\mathbb{F}$, the **annihilator** W^0 of W is the subspace in the dual V^* consisting of all functionals $l : V \to \mathbb{F}$ that "kill" W

$$W^0 = \{l \in V^* : l|_W = 0\} = \{l \in V^* : l(w) = 0,\ for\ all\ w \in W\}$$

The point of the present exercise is to show that the annihilator W^0 can be identified in a completely natural way with the dual $(V/W)^*$ of the quotient space V/W.
Recall that the "quotient map" $\pi : V \to V/W$ is the $\mathbb{F}$-linear surjection $\pi(v) = v + W$.
Prove that

(a) Every $\tilde{l} \in (V/W)^*$ can be "pulled back" to a well-defined functional l on V, as follows

$$l(v) = \tilde{l}(\pi(v)),\ for\ all\ v \in V$$

Verify l is a linear functional in V^* and that l lies in the annihilator W^0.

Now write $\Phi : (V/W)^* \to W^0 \subseteq V^*$ for the map $\Phi(\tilde{l}) = l$ defined in (a).

(b) Prove that $\Phi : (V/W)^* \to V^*$ is a linear map

(c) Prove that $\Phi : (V/W)^* \to W^0$ is one-to-one. (Since Φ is linear this is, of course, equivalent to proving that $\tilde{l} = 0$ is the only functional such that $\Phi(\tilde{l}) = 0$ in V^*.)

(d) Prove that $\Phi : (V/W)^* \to W^0$ is surjective (i.e. every $l \in W^0$ is the Φ-image of some $\tilde{l} \in (V/W)^*$).

we have shown that there is the natural isomorphism Φ between the vector spaces $(V/W)^*$ and W^0.

Solution.

Question 3.

Exercise 3:

If A is an $n \times n$ matrix and $\mathbb{F}^n$ is given the standard inner product, prove that $L_{A^*} = (L_A)^*$ as operators on $\mathbb{F}^n$.

Solution.

Question 4.

Exercise 4: If a finite dimensional vector space is a direct sum $V = E \oplus F$ and P_E, $P_F = I - P_E$ are the associated projections onto E and F, prove that a linear operator $T : V \to V$ leaves both subspaces E, F invariant if and only if T commutes with P_E (hence also P_F).

Solution.

Question 5.

Exercise 5:
Let $W \subseteq V$ be finite dimensional vector spaces over $\mathbb{F}$. Let $\mathcal{X} = \{e_1, \cdots, e_m\}$ be any basis for the subspace W and let

$$\overline{\mathcal{X}} = \{\bar{f}_1, \cdots, \bar{f}_n\}$$

be any basis for the quotient space V/W. If we select any preimages $f_1, \cdots, f_n$ in V so that $\bar{f}_j = f_j + W$ in V/W for each j, prove that the $\{f_j\}$ are linearly independent vectors in V, and that

$$\{e_1, \cdots, e_m, f_1, \cdots, f_n\}$$

is always a basis for the vector space V.
Note: This is one way to prove that $dim(W) + dim(V/W) = dim(V)$.

Solution.

Question 6.

Exercise 6:
Prove that

1. Every $n \times n$ matrix can be written as a linear combination of matrices in $GL(n, \mathbb{F})$.
2. If a matrix A commutes with all matrices $B \in M(n, \mathbb{F})$, then A must be scalar, i.e. $A = diag(\lambda, \cdots, \lambda)$ for some $\lambda \in \mathbb{F}$.
3. A matrix A has a one-point similarity class if and only if

$$SAS^{-1} = A \; for \; all \; S \in GL(n, \mathbb{F})$$

Prove that this happens precisely when $A = \lambda I$ for some $\lambda \in \mathbb{F}$.

Hint: Part 2. is the most important fact; do this even if you have trouble with 1.; in 2. try some really simple choices for $B \in M(n, \mathbb{F})$ and see what $AB - BA = 0$ tells you about A.
Note: We have identified the center of the matrix algebra $M(n, \mathbb{F})$, the set of matrices A that commute with all $n \times n$ matrices. They are all scalar.

Solution.

Question 7.

Exercise 7:
If $dim(V) = 2$ and $T : V \to V$ is any linear operator, its characteristic polynomial $p_T(\lambda) \in \mathbb{F}[\lambda]$ has the form

$$p_T(\lambda) = \lambda^2 - Tr(T)\lambda + det(T)$$

Question: If $A, B \in M(2, \mathbb{F})$, i.e. is there an $S \in GL(2, \mathbb{F})$ such that $B = SAS^{-1}$?
Hint: Try some upper triangular 2×2 matrices.

Solution.