Linear Algebra II: Problem Set I

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set I of Multivariable Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

11. Show that K is a convex set by directly applying the definition. Sketch K in the cases $n = 1, 2, 3$.

(a) $K = \{\mathbf{x} : |x^1| + \cdots + |x^n| \leq 1\}$.

(b) $K = \{\mathbf{x} = c^1\mathbf{v}_1 + \cdots + c^n\mathbf{v}_n,\ 0 \leq c^i \leq 1 \text{ for } i = 1, \ldots, n\}$, where $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for E^n. This is the *n-parallelepiped* spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with $\mathbf{0}$ as a vertex.

Solution.

Question 2.

10. Show that:

(a) $\operatorname{int}(A \cup B) \supset (\operatorname{int} A) \cup (\operatorname{int} B)$.

(b) $\operatorname{cl}(A \cap B) \subset (\operatorname{cl} A) \cap (\operatorname{cl} B)$.

Give examples in which $=$ does not hold.

Solution. **(a)** Let $x \in \text{Int}A \cup \text{Int}B$. It follows that there exists a neighborhood of x contained in A or exists a neighborhood of x contained in B, respectively denoted as U_A and U_B. If we have the existence of U_A, it follows that $x \in U_A \subset A \subset A \cup B$. Likewise, if we have the existence of U_B, it follows that $x \in U_B \subset B \subset A \cup B$. Therefore, we have x is an interior point of $A \cup B$. Since x was arbitrary, we have shown that $\text{Int}A \cup \text{Int}B \subset \text{Int}(A \cup B)$. We now show that the equality does not hold, by providing a counter example. Let $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$. Then, $\text{int}\mathbb{R} = \mathbb{R}$ and $\text{int}\mathbb{Q} = \emptyset$ and $\text{int}\mathbb{R} \setminus \mathbb{Q} = \emptyset$. Since $\mathbb{R} \not\subset \emptyset$, we have shown that the equality does not hold. $\square$

(b)

Question 3.

12. Let A and B be convex subsets of E^n. The *join* of A and B is the set of all $\mathbf{x}$ such that $\mathbf{x}$ lies on a line segment with one endpoint in A and the other in B. Show that the join of A and B is a convex set.

Solution.

Question 4.

4. Let f be a function from S into T. Show that, for any subsets A and B of S:

(a) $f(A \cup B) = f(A) \cup f(B)$.

(b) $f(A \cap B) \subset f(A) \cap f(B)$.

(c) If f is univalent, then $f(A \cap B) = f(A) \cap f(B)$.

Solution. We first establish a trivial, yet central lemma: if $A \subset B$, then $f(A) \subset f(B)$. We proceed with a short proof of it for sake of completeness. Let $y \in f(A)$. Then, there exists $x \in A$ such that $f(x) = y$. As $A \subset B$, we have $x \in B$, and $f(x) = y \in f(B)$. hence, the lemma is true.

(a) Firstly, by the lemma, we have that $f(A) \subset f(A \cup B)$ and $f(B) \subset f(A \cup B)$. It follows that $f(A) \cup f(B) \subset f(A \cup B)$. Conversely, let $y \in f(A \cup B)$. Then, there exists $x \in A \cup B$ such that $f(x) = y$. If $x \in A$, then $f(x) = y \in f(A)$. Likewise if $x \in B$, then $f(x) = y \in f(B)$. Hence, $y \in f(A) \cup f(B)$. Since y was arbitrary, it follows that $f(A \cup B) \subset f(A) \cup f(B)$. This completes the proof of $f(A \cup B) = f(A) \cup f(B)$.

(b) By the lemma, we have

$$f(A \cap B) \subset f(A) \quad \text{and} \quad f(A \cap B) \subset f(B).$$

It follows that $f(A \cap B) \subset f(A) \cap f(B)$.

(c) With the additional assumption of injectivity, we wish to establish $f(A) \cap f(B) \subset f(A \cap B)$. This will suffice, as we have established the reverse inclusion for a general function in part (b). Let $y \in f(A) \cap f(B)$. Then, by the injectivity of f, there exists a unique $x \in S$ such that $f(x) = y$. As $y \in f(A)$ and $y \in f(B)$, it follows that $x \in A \cap B$. Hence, $f(x) = y \in f(A \cap B)$. We have shown that if f is injective, we have $f(A \cap B) = f(A) \cap f(B)$. $\square$

Question 5.

3. Show that if $\mathbf{y}_0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x})$, then $|\mathbf{y}_0| = \lim_{\mathbf{x} \to \mathbf{x}_0} |\mathbf{f}(\mathbf{x})|$. Prove that the converse holds if $\mathbf{y}_0 = \mathbf{0}$.

Solution.