Multivariable Analysis:
Problem Set II

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Abstract

This work contains solutions to the problem set II of Multivariable Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

8. An infinite series $\mathbf{x}_1 + \mathbf{x}_2 + \cdots$ converges *absolutely* if the series of nonnegative numbers $|\mathbf{x}_1| + |\mathbf{x}_2| + \cdots$ converges. Prove that any absolutely convergent infinite series is convergent. [*Hint*: Show that the sequence $[\mathbf{s}_m]$ of partial sums is Cauchy.]

Solution. Fix $\epsilon > 0$. As the series converges absolutely, we have that $\{a_n = \sum_{i=1}^{n} |x_i|\}$ converges, hence is Cauchy. As $\{a_n\}$ is Cauchy, there exists an index N such that

$$\begin{aligned}\sum_{i=n}^{m} |x_i| &= |a_m - a_n| \\ &< \epsilon,\end{aligned}$$

for $m \geq n \geq N$. Observe that for $m \geq n \geq N$, by the triangle inequality and the above inequality, we have

$$\begin{aligned}|\sum_{i=1}^{m} x_i - \sum_{i=1}^{n} x_i| &= |\sum_{i=n}^{m} x_i| \\ &\leq \sum_{i=m}^{n} |x_i| < \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this shows that $\{\sum_{i=1}^{n} x_i\}$ is Cauchy. Since the sequence is drawn from a Euclidean space, we have shown that it is convergent.

$\square$

Question 2.

6. (*Subsequences.*) Let $[\mathbf{x}_m]$ be a sequence, and $\mathbf{y}_l = \mathbf{x}_{m_l}$ for $l = 1, 2, \dots,$ where $m_1 < m_2 < \dots$. Then $[\mathbf{y}_l]$ is called a *subsequence* of $[\mathbf{x}_m]$.

(a) Show that any bounded sequence in E^n has a convergent subsequence.

(b) A set S is called *sequentially compact* if: any bounded sequence $[\mathbf{x}_m]$, with $\mathbf{x}_m \in S$ for $m = 1, 2, \dots,$ has a subsequence $[\mathbf{y}_l]$ such that $\mathbf{y}_l \to \mathbf{y}_0$ as $l \to \infty$, $\mathbf{y}_0 \in S$. Show that a nonempty set $S \subset E^n$ is sequentially compact if and only if S is closed and bounded.

Solution. **(a)** Let $\{x^k\}$ be a bounded sequence in E^n. It follows that the sequences formed by each component are bounded as well, as otherwise it would contradict the boundedness of the original sequence in E^n. Now, consider the sequence of reals from the first component $\{x_1^k\}$. By Bolzano-Weiestrass theorem, we have that there exists a convergent subsequence $\{x_1^{k_i}\}$. Now, consider the sequence of reals from the second component $\{x_2^k\}$ and form a subsequence using the subsequence indices from the convergent subsequence from the first component, which we denote as $\{x_2^{k_i}\}$. Now, by Bolzano-Weiestrass theorem, once again, we get a convergent subsequence of the second component sequence, with a property that it is also a subsequence of the convergent subsequence from the first component sequence. We do the above construction inductively until we get a convergent subsequence for the nth component's convergent subsequence, whose indices we denote as k_l. By construction, it follows that $\{x_i^{k_l}\}$ is a convergent sequence for $i = 1, 2.., n$, and they are subsequences of $\{x_i^k\}$ respectively. By preposition 2.7, pg.38, we have the sequence $\{x_l^k\}$ converges, as each of its component sequence converges. Hence, we have constructed a convergent subsequence of $\{x^k\}$. Therefore, we have shown that a bounded sequence in E^n has a convergent subsequence. $\square$

(b)

Question 3.

8. (*Uniform continuity.*) A transformation $\mathbf{f}$ is *uniformly continuous* on $S \subset E^n$ if given $\varepsilon > 0$ there exists $\delta > 0$ (depending only on ε) such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$ for every $\mathbf{x}, \mathbf{y} \in S$ with $|\mathbf{x} - \mathbf{y}| < \delta$. Show that if S is closed and bounded then every $\mathbf{f}$ continuous on S is uniformly continuous on S. [*Hint*: If not, then there exists $\varepsilon > 0$ and for $m = 1, 2, \ldots, \mathbf{x}_m, \mathbf{y}_m \in S$ such that $|\mathbf{f}(\mathbf{x}_m) - \mathbf{f}(\mathbf{y}_m)| \geq \varepsilon$ and $|\mathbf{x}_m - \mathbf{y}_m| \leq 1/m$. Let $\mathbf{x}_0$ be an accumulation point of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$. Show that the continuity of $\mathbf{f}$ at $\mathbf{x}_0$ is contradicted.]

Solution. Let S be a closed and bounded set in $\mathbb{R}^n$ and f be a continuous transformation. on S. We know that a closed bounded set in $\mathbb{R}^n$ is compact. Therefore, we prove the following more general theorem.

Theorem. *Let $f : X \to Y$, such that f is continuous, X, Y are metric spaces, and X is compact. Then, f is uniformly continuous.*

Proof. Fix $\epsilon > 0$. As f is continuous on X, for any $x \in X$, there exists $\delta_x > 0$ that corresponds to the $\frac{\epsilon}{2}$-challenge. Then, we have

$$X = \bigcup_{x \in X} B(x, \delta_x).$$

Now, observe that the sets in the RHS form an open cover of X. Since X is compact, the open cover has a finite sub-cover. Thus, we can write X as follows:

$$X = \bigcup_{i=1}^{n} B(x_i, \delta_{x_i}),$$

where x_i are from X and δ_{x_i} are the values that correspond to the $\frac{\epsilon}{2}$ challenge at x_i. Now, let $\delta = \dfrac{\min_{i=1,2..,n}(\delta_{x_i})}{2}$. We claim that δ corresponds to the ϵ-challenge of uniform continuity. Let $x, y \in X$, such that $d(x, y) < \delta$. It follows that there exists $x_i \in X$, such that $x, y \in B(x_i, \delta_{x_i})$. By the triangle inequality, and the continuity of f at x_i, we have

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(x_i) + f(x_i) - f(y)| \\ &\leq |f(x) - f(x_i)| + |f(x_i) - f(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, δ corresponds to the ϵ-challenge of uniform continuity of f. Since $\epsilon > 0$ was arbitrary, we have shown that f is uniformly continuous. ☐

As a corollary, it follows that if S is closed and bounded, then every continuous function f is uniformly continuous on S. ☐

Question 4.

6. (*Indiscrete spaces.*) Let S be any set, and let every $p \in S$ have exactly one "neighborhood," namely, S itself; that is, each $\mathscr{U}_p$ consists of the set S only.

(a) Verify Axioms (1) through (4).

(b) Show that the only open sets are S and the empty set.

(c) Show that any real valued function continuous on S is constant.

Solution. (a) For any point $p \in S$, and we have defined S as a neighborhood of p. Hence, there is a neighborhood of p. The axiom (1) is satisfied.

Let $p \in S$. We have that the only neighborhood of p is S. Since $p \in S$, the axiom (2) is satisfied.

Let $p \in S$, U_1 and U_2 be neighborhoods of p. Since S is the only neighborhood of p, we have $U_1 = U_2 = U_1 \cap U_2 = S$. Since $S \subset S$, the axiom (3) is satisfied.

Let $p \in S$, U be a neighborhood of p, and $q \in U$. We have $U = S$ and S is a neighborhood of q by definition. Since $S \subset S$, the axiom (4) is satisfied.

(b) By the 4th axiom, we have that any neighborhood is an open set. Hence, S is open. $\emptyset$ is open, because the statement of open holds vacuously. Now, let A be a nonempty subset of S such that $A \neq S$. Since A is nonempty, there exists a point $p \in A$, and by definition of the topology, p has S as a neighborhood. Since $A \neq S$, $S \not\subset A$, and we have that p is not interior to A. Hence, A is not open. We have shown that S and $\emptyset$ are the only open sets.

(c) Let $f : S \to \mathbb{R}$ be continuous with respect to the indiscrete topology. By the corollary $2.6.2$ in Fleming, pg.53, we have that $\{p : f(p) > c\}$ is open for any $c \in \mathbb{R}$. Assume that f is not a constant function. Then, it follows that there exists $p_1 \neq p_2 \in S$ such that $f(p_1) \neq f(p_2)$. Since $f(p_1) \neq f(p_2)$, we have either $f(p_1) > f(p_2)$ or $f(p_1) < f(p_2)$. As the cases are symmetric, assume without loss of generality that $f(p_1) > f(p_2)$. It follows that $f(p_1) > \dfrac{f(p_1) + f(p_2)}{2} > f(p_2)$. Now, consider $A = \{p : f(p) > \dfrac{f(p_1) + f(p_2)}{2}\}$. We have that $p_1 \in A$ and $p_2 \notin A$. Therefore, we have that A is nonempty and $A \neq S$. By the corollary, we have that A is open, but we have previously shown that S and $\emptyset$ are the only open sets. Hence, we have reached a contradiction and f must be a constant function.

$\square$

Question 5.

12. Let S be as in Example 3. Show that:

(a) S is a closed set.

(b) There is no path in S joining $(0, 0)$ and any point of S_2.

(c) S is a connected set.

Solution.

Question 6.

5. Let A, B be nonempty subsets of E^n, and let $d = \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{x} \in B, \mathbf{y} \in A\}$.

(a) Show that $d > 0$ if A is closed, B is compact, and $A \cap B$ is empty. [*Hint*: Problem 4.]

(b) Give an example of closed sets A, B such that $A \cap B$ is empty but $d = 0$.

Solution. (a) Assume A is closed, B is compact, and $A \cap B = \emptyset$.

(b)

Question 7.

7. A topological space S_0 is called a *Hausdorff* space if S_0 has the property that for every $p, q \in S_0$ $(p \neq q)$ there exist a neighborhood U of p and a neighborhood V of q such that $U \cap V$ is empty.
   (a) Show that any metric space is a Hausdorff space.
   (b) Show that any compact set $S \subset S_0$ is closed, if S_0 is a Hausdorff space.
   (c) Let f be continuous and univalent from a compact space S onto a Hausdorff space T. Show that f^{-1} is continuous from T onto S. [*Hint*: Show that $(f^{-1})^{-1}(B)$ is closed if B is closed.]

Solution. (a) Let (X, d) be a metric space, with $p_1 \neq p_2 \in X$. By one of the axioms of metric spaces, we have that $d(p_1, p_2) > 0$. Let $\delta = \dfrac{d(p_1, p_2)}{2}$, and consider $B_1 = B(p_1, \delta)$ and $B_2 = B(p_2, \delta)$. We claim that $B_1 \cap B_2 = \emptyset$. Suppose that there exists $p \in X$ such that $p \in B_1 \cap B_2$. It follows that $d(p, p_1) < \delta$ and $d(p, p_2) < \delta$. By the triangle inequality, we have

$$\begin{aligned} d(p_1, p_2) &\leq d(p_1, p) + d(p, p_2) \\ &< \delta + \delta = d(p_1, p_2). \end{aligned}$$

Hence, we have shown that $d(p_1, p_2) < d(p_1, p_2)$, which is a contradiction. Therefore, $B_1 \cap B_2 = \emptyset$. Since p_1 and p_2 were arbitrary two distinct points from X, we have shown that a metric space is Hausdorff. □

(b) Let S_0 be a Hausdorff space, and S be a compact subset of S_0. Let $x \in S_0 \setminus S$. Now, for any $s \in S$, by Hausdorff property of S_0, there exists a neighborhood of x, N_x, and a neighborhood of s, N_s, such that $N_x \cap N_s = \emptyset$. Observe that

$$S \subset \bigcup_{s \in S} N_s.$$

As the RHS is an open cover of S, by compactness of S, there exists a finite sub-cover $\{N_{s_i}\}_{i=1}^n$ such that

$$S \subset \bigcup_{i=1}^n N_{s_i},$$

with the corresponding neighborhood of x, selected via Hausdorff property denoted as $\{N_{x_i}\}_{i=1}^n$. As an intersection of finite collection of open sets is open, we have that $\bigcap_{i=1}^n N_{x_i}$ is open. Furthermore, it is a neighborhood of x, that is disjoint from $\bigcup_{i=1}^n N_{s_i}$, thus from S as well. Since x was chosen arbitrarily from $S_0 \setminus S$, we have shown that $S_0 \setminus S$ is open. Hence, S is closed. □

(c) We first prove a simple central lemma:

Lemma 7.c. Closedness implies Compactness in Compact Space. *Let X be a compact topological space, and A be a closed subset of X. Then, A is compact.*

Proof. Let $\{O_\lambda\}$ be an open cover of A. As A is closed, $X \setminus A$ is open, and we have that $\{O_\lambda\}$ with $X \setminus A$ is an open cover of X. By compactness of X, there exists a finite subcover of the open cover that covers X. Remove $X \setminus A$ if its in the finite subcover. Since we only removed $X \setminus A$, the finite subcover still covers A. Also, it is a finite subcover of the original open cover of A. Hence, we have shown that A is compact. □

We wish to show that f^{-1} is continuous. We know that a function is continuous iff an inverse image of a closed set is closed. Hence, it suffices to show that for a closed subset B of S, $(f^{-1})^{-1}(B)$ is closed. Note that $(f^{-1})^{-1}(B) = f(B)$. Let B be a closed subset of S. By the established lemma, as B is closed, B is compact. By the theorem 2.10 on pg.61 in Fleming, since f is continuous, $f(B)$ compact. We have shown that a compact subset is closed in Hausdroff space in part (b). Therefore, $f(B)$ is closed. We have shown that f^{-1} is continuous. □

Question 8.

4. Let $f(x) = \sum_{k=1}^{\infty} (\sin kx)/k^2$. Use Theorem 2.11 to show that f is continuous on E^1.

Solution. To begin with, we note that the function is well-defined as for a fixed $x \in \mathbb{R}$, $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}$ converges absolutely. The absolute convergence of the series can be shown through a comparison test with $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Now, we denote the nth partial sum function as f_n. Observe that $\{f_n\}$ forms a sequence of continuous functions, as $\sin(kx)/k^2$ is continuous for all $k \in \mathbb{N}$ and a sum of two continuous function is continuous. By Theorem 2.11, it suffices to show that $\{f_n\}$ converges uniformly to $f = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}$, i.e. $||f - f_n|| \to 0$ in the sup-norm. As we have shown that the space of continuous functions with respect to sup-norm is complete, it again suffices to show that the partial sums as a sequence form a Cauchy sequence.

Fix $\epsilon > 0$. Observe that

$$\begin{aligned} |f_n - f_m| &= \left| \sum_{k=1}^{n} \frac{\sin(kx)}{k^2} - \sum_{k=1}^{m} \frac{\sin(kx)}{k^2} \right| \\ &= \left| \sum_{k=m}^{n} \frac{\sin(k)}{k^2} \right| \\ &\leq \sum_{k=m}^{n} \left| \frac{\sin(k)}{k^2} \right| \\ &\leq \sum_{k=m}^{n} \frac{1}{k^2} \end{aligned}$$

$n \leq m$.

Question 9.

5. A *seminorm* on E^n is a real valued function f satisfying: $f(\mathbf{x}) \geq 0$ for every $\mathbf{x}$; $f(c\mathbf{x}) = |c| f(\mathbf{x})$ for every c and $\mathbf{x}$; and $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ for every $\mathbf{x}$ and $\mathbf{y}$.

(a) Let f be a seminorm and $K = \{\mathbf{x} : f(\mathbf{x}) \leq 1\}$. Show that K is closed and satisfies Properties (ii) through (iv). Show that K is compact if and only if f is a norm. [*Hint*: First prove that f is continuous.]

(b) Conversely, let K be any closed set satisfying Properties (ii) through (iv). Let $f(\mathbf{x}) = 0$ if $\mathbf{x} = \mathbf{0}$ or if the line through $\mathbf{0}$ and $\mathbf{x}$ is contained in K. Otherwise, let

$$f(\mathbf{x}) = \frac{1}{\max\{t : t\mathbf{x} \in K\}}$$

as in (2.5). Show that f is a seminorm.

(c) Let $n = 3$ and $f(x, y, z) = |x| + 2|y|$. Sketch K and show that f is a seminorm.

Solution.