Multivariable Analysis: Problem Set II

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Abstract

This work contains solutions to the problem set II of Multivariable Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

8. An infinite series $\mathbf{x}_1 + \mathbf{x}_2 + \cdots$ converges *absolutely* if the series of nonnegative numbers $|\mathbf{x}_1| + |\mathbf{x}_2| + \cdots$ converges. Prove that any absolutely convergent infinite series is convergent. [*Hint*: Show that the sequence $[\mathbf{s}_m]$ of partial sums is Cauchy.]

Solution.

Question 2.

6. (*Subsequences.*) Let $[\mathbf{x}_m]$ be a sequence, and $\mathbf{y}_l = \mathbf{x}_{m_l}$ for $l = 1, 2, \ldots,$ where $m_1 < m_2 < \ldots$. Then $[\mathbf{y}_l]$ is called a *subsequence* of $[\mathbf{x}_m]$.

(a) Show that any bounded sequence in E^n has a convergent subsequence.

(b) A set S is called *sequentially compact* if: any bounded sequence $[\mathbf{x}_m]$, with $\mathbf{x}_m \in S$ for $m = 1, 2, \ldots,$ has a subsequence $[\mathbf{y}_l]$ such that $\mathbf{y}_l \to \mathbf{y}_0$ as $l \to \infty$, $\mathbf{y}_0 \in S$. Show that a nonempty set $S \subset E^n$ is sequentially compact if and only if S is closed and bounded.

Solution.

Question 3.

8. (*Uniform continuity.*) A transformation $\mathbf{f}$ is *uniformly continuous* on $S \subset E^n$ if given $\varepsilon > 0$ there exists $\delta > 0$ (depending only on ε) such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$ for every $\mathbf{x}, \mathbf{y} \in S$ with $|\mathbf{x} - \mathbf{y}| < \delta$. Show that if S is closed and bounded then every $\mathbf{f}$ continuous on S is uniformly continuous on S. [*Hint*: If not, then there exists $\varepsilon > 0$ and for $m = 1, 2, \ldots, \mathbf{x}_m, \mathbf{y}_m \in S$ such that $|\mathbf{f}(\mathbf{x}_m) - \mathbf{f}(\mathbf{y}_m)| \geq \varepsilon$ and $|\mathbf{x}_m - \mathbf{y}_m| \leq 1/m$. Let $\mathbf{x}_0$ be an accumulation point of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$. Show that the continuity of $\mathbf{f}$ at $\mathbf{x}_0$ is contradicted.]

Solution.

Question 4.

6. (*Indiscrete spaces.*) Let S be any set, and let every $p \in S$ have exactly one "neighborhood," namely, S itself; that is, each $\mathscr{U}_p$ consists of the set S only.

(a) Verify Axioms (1) through (4).

(b) Show that the only open sets are S and the empty set.

(c) Show that any real valued function continuous on S is constant.

Solution.

Question 5.

12. Let S be as in Example 3. Show that:

(a) S is a closed set.

(b) There is no path in S joining $(0, 0)$ and any point of S_2.

(c) S is a connected set.

Solution.

Question 6.

5. Let A, B be nonempty subsets of E^n, and let $d = \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{x} \in B, \mathbf{y} \in A\}$.

(a) Show that $d > 0$ if A is closed, B is compact, and $A \cap B$ is empty. [*Hint*: Problem 4.]

(b) Give an example of closed sets A, B such that $A \cap B$ is empty but $d = 0$.

Solution.

Question 7.

7. A topological space S_0 is called a *Hausdorff* space if S_0 has the property that for every $p, q \in S_0$ $(p \neq q)$ there exist a neighborhood U of p and a neighborhood V of q such that $U \cap V$ is empty.
   (a) Show that any metric space is a Hausdorff space.
   (b) Show that any compact set $S \subset S_0$ is closed, if S_0 is a Hausdorff space.
   (c) Let f be continuous and univalent from a compact space S onto a Hausdorff space T. Show that f^{-1} is continuous from T onto S. [*Hint*: Show that $(f^{-1})^{-1}(B)$ is closed if B is closed.]

Solution.

Question 8.

4. Let $f(x) = \sum_{k=1}^{\infty} (\sin kx)/k^2$. Use Theorem 2.11 to show that f is continuous on E^1.

Solution.

Question 9.

5. A *seminorm* on E^n is a real valued function f satisfying: $f(\mathbf{x}) \geq 0$ for every $\mathbf{x}$; $f(c\mathbf{x}) = |c| f(\mathbf{x})$ for every c and $\mathbf{x}$; and $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ for every $\mathbf{x}$ and $\mathbf{y}$.

(a) Let f be a seminorm and $K = \{\mathbf{x} : f(\mathbf{x}) \leq 1\}$. Show that K is closed and satisfies Properties (ii) through (iv). Show that K is compact if and only if f is a norm. [*Hint*: First prove that f is continuous.]

(b) Conversely, let K be any closed set satisfying Properties (ii) through (iv). Let $f(\mathbf{x}) = 0$ if $\mathbf{x} = \mathbf{0}$ or if the line through $\mathbf{0}$ and $\mathbf{x}$ is contained in K. Otherwise, let

$$f(\mathbf{x}) = \frac{1}{\max\{t : t\mathbf{x} \in K\}}$$

as in (2.5). Show that f is a seminorm.

(c) Let $n = 3$ and $f(x, y, z) = |x| + 2|y|$. Sketch K and show that f is a seminorm.

Solution.