Multivariable Analysis: Problem Set I

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Abstract

This work contains solutions to the problem set I of Multivariable Analysis 2016 at Courant Institute of Mathematical Sciences.

Question 1.

11. Show that K is a convex set by directly applying the definition. Sketch K in the cases $n = 1, 2, 3$.

(a) $K = \{\mathbf{x} : |x^1| + \cdots + |x^n| \leq 1\}$.

(b) $K = \{\mathbf{x} = c^1\mathbf{v}_1 + \cdots + c^n\mathbf{v}_n,\ 0 \leq c^i \leq 1 \text{ for } i = 1, \ldots, n\}$, where $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for E^n. This is the *n-parallelepiped* spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with $\mathbf{0}$ as a vertex.

Solution. (a) Let $K = \{x : \sum_{i=1}^n |x_i| \leq 1\}$, $a, b \in K$, and $t \in [0, 1]$. Consider $l = ta + (1-t)b$ and its l_1 quantity, $\sum_{i=1}^n |ta_i + (1-t)b_i|$. By the triangle inequality, it follows that

$$\begin{aligned}\sum_{i=1}^n |ta_i + (1-t)b_i| &\leq \sum_{i=1}^n |ta_i| + |(1-t)b_i| \\ &= |t|\sum_{i=1}^n |a_i| + |1-t|\sum_{i=1}^n |b_i| \\ &\leq |t| + |1-t| = 1.\end{aligned}$$

Hence, $l \in K$. Since x, y, t were arbitrary, we have shown that K is convex.

(b) Let $K = \{x = \sum_{i=1}^n c_i v_i | \ 0 \leq c_i \leq 1 \text{ for } i = 1, 2, ..., n\}$, $a = \sum_{i=1}^n a_i v_i, b = \sum_{i=1}^n b_i v_i \in K$, and $t \in [0, 1]$. Consider $l = ta + (1-t)b$. It follows that

$$\begin{aligned}ta + (1-t)b &= t\sum_{i=1}^n a_i v_i + (1-t)\sum_{i=1}^n v_i \\ &= \sum_{i=1}^n (ta_i + (1-t)b_i)v_i.\end{aligned}$$

As $t, a_i, b_i, 1-t$ are all non-negative, we have $0 \leq ta_i + (1-t)b_i$. As $0 \leq a_i, b_i \leq 1$, we obtain $ta_i + (1-t)b_i \leq t + 1 - t = 1$, which combined with the previous inequality gives $0 \leq ta_i + (1-t)b_i \leq 1$. Hence, $l \in K$. Since x, y, t were arbitrary, we have shown that K is convex.

$\square$

Question 2.

10. Show that:

(a) $\text{int}(A \cup B) \supset (\text{int } A) \cup (\text{int } B)$.

(b) $\text{cl}(A \cap B) \subset (\text{cl } A) \cap (\text{cl } B)$.

Give examples in which $=$ does not hold.

Solution. (a) Let $x \in \text{Int}A \cup \text{Int}B$. It follows that there exists a neighborhood of x contained in A or exists a neighborhood of x contained in B, respectively denoted as U_A and U_B. If we have the existence of U_A, it follows that $x \in U_A \subset A \subset A \cup B$. Likewise, if we have the existence of U_B, it follows that $x \in U_B \subset B \subset A \cup B$. Therefore, we have x is an interior point of $A \cup B$. Since x was arbitrary, we have shown that $\text{Int}A \cup \text{Int}B \subset \text{Int}(A \cup B)$. We now show that the equality does not hold, by providing a counter example. Let $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$. Then, $\text{int}\mathbb{R} = \mathbb{R}$ and $\text{int}\mathbb{Q} = \emptyset$ and $\text{int}\mathbb{R} \setminus \mathbb{Q} = \emptyset$. Since $\mathbb{R} \not\subset \emptyset$, we have shown that the equality does not hold. □

(b) Firstly, taking complements on both sides shows that the given inclusion is equivalent to

$$(\text{cl}A \cap \text{cl}B)^c \quad \subset \quad (\text{cl}(A \cap B))^c.$$

By DeMorgan's laws, we have that the above inclusion is equivalent to

$$(\text{cl}A)^c \cup (\text{cl}B)^c \quad \subset \quad (\text{cl}(A \cap B))^c.$$

Using the fact that $(\text{cl}A)^c = \text{int}(A^c)$, for any arbitrary set $A \subset E^n$, the above inclusion is equivalent to

$$\text{int}A^c \cup \text{int}B^c \quad \subset \quad \text{int}((A \cap B)^c).$$

Again, by DeMorgan's laws, the above inclusion is equivalent to

$$\text{int}A^c \cup \text{int}B^c \quad \subset \quad \text{int}(A^c \cup B^c).$$

As we have shown the part (a) to be true, we are done. We now show that the equality does not hold, by providing a counter example. Let $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$. Then, $\text{cl}(A \cap B) = \emptyset$ and $\text{cl}(A) \cap \text{cl}(B) = \mathbb{R}$. Since $\mathbb{R} \not\subset \emptyset$, we have shown that the equality does not hold. □

Question 3.

12. Let A and B be convex subsets of E^n. The *join* of A and B is the set of all $\mathbf{x}$ such that $\mathbf{x}$ lies on a line segment with one endpoint in A and the other in B. Show that the join of A and B is a convex set.

Solution. Let join(A, B) be the join of A and B, $\lambda \in [0, 1]$, and $x_1, x_2 \in$ join(A, B). Consider $\lambda x_1 + (1 - \lambda)x_2$. It follows that

$$\lambda x_1 + (1 - \lambda)x_2 \quad = \quad \lambda(t_1 a_1 + (1 - t_1)b_1) + (1 - \lambda)(t_2 a_2 + (1 - t_2)b_2),$$

for some $a_1, a_2 \in A$, $b_1, b_2 \in B$, and $t_1, t_2 \in [0, 1]$.

Question 4.

4. Let f be a function from S into T. Show that, for any subsets A and B of S:
(a) $f(A \cup B) = f(A) \cup f(B)$.
(b) $f(A \cap B) \subset f(A) \cap f(B)$.
(c) If f is univalent, then $f(A \cap B) = f(A) \cap f(B)$.

Solution. We first establish a trivial, yet central lemma: if $A \subset B$, then $f(A) \subset f(B)$. We proceed with a short proof of it for sake of completeness. Let $y \in f(A)$. Then, there exists $x \in A$ such that $f(x) = y$. As $A \subset B$, we have $x \in B$, and $f(x) = y \in f(B)$. hence, the lemma is true.

(a) Firstly, by the lemma, we have that $f(A) \subset f(A \cup B)$ and $f(B) \subset f(A \cup B)$. It follows that $f(A) \cup f(B) \subset f(A \cup B)$. Conversely, let $y \in f(A \cup B)$. Then, there exists $x \in A \cup B$ such that $f(x) = y$. If $x \in A$, then $f(x) = y \in f(A)$. Likewise if $x \in B$, then $f(x) = y \in f(B)$. Hence, $y \in f(A) \cup f(B)$. Since y was arbitrary, it follows that $f(A \cup B) \subset f(A) \cup f(B)$. This completes the proof of $f(A \cup B) = f(A) \cup f(B)$.

(b) By the lemma, we have

$$f(A \cap B) \subset f(A) \quad \text{and} \quad f(A \cap B) \subset f(B).$$

It follows that $f(A \cap B) \subset f(A) \cap f(B)$.

(c) With the additional assumption of injectivity, we wish to establish $f(A) \cap f(B) \subset f(A \cap B)$. This will suffice, as we have established the reverse inclusion for a general function in part (b). Let $y \in f(A) \cap f(B)$. Then, by the injectivity of f, there exists a unique $x \in S$ such that $f(x) = y$. As $y \in f(A)$ and $y \in f(B)$, it follows that $x \in A \cap B$. Hence, $f(x) = y \in f(A \cap B)$. We have shown that if f is injective, we have $f(A \cap B) = f(A) \cap f(B)$. $\square$

Question 5.

3. **Show that if $\mathbf{y}_0 = \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$, then $|\mathbf{y}_0| = \lim_{\mathbf{x}\to\mathbf{x}_0} |\mathbf{f}(\mathbf{x})|$. Prove that the converse holds if $\mathbf{y}_0 = \mathbf{0}$.**

Solution. Before proceeding to the main part of the proof, we prove the following lemma, which is a mere consequence of the triangle inequality: Let $x, y \in E^n$. Then, $||x| - |y|| < |x - y|$. The proof is as follows: Let $x, y \in E^n$. Then, by the triangle inequality, we obtain

$$\begin{aligned} |x| &= |x - y + y| \\ &\leq |x - y| + |y|. \end{aligned}$$

By subtracting both sides with $|y|$, we have $|x| - |y| \leq |x - y|$. By symmetry, we further obtain $|y| - |x| \leq |x - y|$, thereby showing that $||x| - |y|| < |x - y|$ as required.

We now proceed with the main part of the proof. Assume that $y_0 = \lim_{x\to x_0} f(x)$. Then, by definition, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - y_0| < \epsilon$ whenever $0 < |x - x_0| < \delta$. Fix $\epsilon > 0$. Let $\delta > 0$ be chosen such that $|f(x) - y_0| < \epsilon$ whenever $0 < |x - x_0| < \delta$, which exists by the assumption. But, by the lemma, whenever $0 < |x - x_0| < \delta$, we have

$$||f(x)| - |y_0|| \leq |f(x) - y_0| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have shown that $|y_0| = \lim_{x\to x_0} |f(x)|$.

Now assume that $0 = \lim_{x\to x_0} |f(x)|$. Then, by definition, for every $\epsilon > 0$, there exists $\delta > 0$ such that $||f(x)| - 0| < \epsilon$ whenever $0 < |x - x_0| < \delta$. The above implies that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon$ whenever $0 < |x - x_0| < \delta$. This is precisely, $y_0 = 0 = \lim_{x\to x_0} f(x)$. Hence, we are done.

$\square$