Probabilistic Method: Problem Set VI

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Abstract

This work contains solutions to the problem set VI of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 2.

2. Consider a Galton-Watson process beginning with root Eve in which each node independently has number of children given by a Poisson distribution with mean c. Find the probabilities of the follwing events. (Use the nice fact that if a node has $Po(c)$ children and each child has a given property A with independent probability z then the node has $Po(cz)$ children with property A.)

   (a) Eve has precisely two children.

   (b) Eve has no children with precisely two children.

   (c) Eve has no children that have no children that have no children.

Solution.
(a) As the number of children for each node is given by the Poisson distribution, we have that the probability of Eve having precisely two children given by

$$P(\{\text{ Eve has precisely two children }\}) = e^{-c}\frac{c^2}{2!}.$$

(b) From the above computation, we see that the event of each child having precisely two children has independent probability of $e^{-c}\frac{c^2}{2!}$. Therefore, using the useful fact, we have

$$\begin{aligned} P(\{\text{ Eve has no children with precisely two children }\}) &= e^{-cz} \\ &= e^{-ce^{-c}\frac{c^2}{2!}}. \end{aligned}$$

(c) We proceed by the same method. We again see that the event of each child having no children has independent probability of e^{-c}. Therefore, by the useful fact, we have

$$P(\{\text{ A node has no children that hve no children }\}) = e^{-ce^{-c}}.$$

Using the same argument once more, we obtain

$$P(\{\text{ Eve has no children that have no children that have no children}\}) = e^{-ce^{-ce^{-c}}}.$$

$\square$

Question 2.

3. Consider a Galton-Watson process beginning with root Eve in which each node independently has number of children given by a Binomial distribution with parameters m, p.

   (a) Find an equation, as in the Poisson case, for $y = \Pr[T = \infty]$.

   (b) Show that this equation has only the solution $y = 0$ when $mp < 1$ and two solutions when $mp > 1$.

   (c) (*) Show that this equation has only the solution $y = 0$ when $mp = 1$ and $m \neq 1$.

   (d) Let $c > 1$. Let $y = y(m, p)$ denote the nonzero solution. Show that $y(m, p) \to y(c)$ when $m \to \infty$ and $mp \to c$.

Solution. **(a)** Let $y = Pr[T = \infty]$, and $z = Pr[T < \infty] = 1 - y$. By partitioning the space with the number of children that Eve has, and using the fact that the sub-tree must be finite, we obtain

$$\begin{aligned} z &= \sum_{i=0}^{m} \binom{m}{i} p^i (1-p)^{m-i} z^i \\ &= (pz + 1 - p)^m, \end{aligned}$$

thus

$$1 - y = [1 - py]^m.$$

(b) Now, we view the problem of finding solutions to the above equation as finding intersections of the graphs $f(y) = 1 - y$ and $g(y) = (1 - py)^m$ with p, m as parameters within the domain $y \in [0, 1]$. Observe that for any p, m, we trivially have $y = 0$ as a solution. Furthermore, observe that $g'(y) = -mp(1 - py)^{m-1}$, and $g''(y) = m(m - 1)p^2(1 - py)^{m-2}$. As p and y are both probabilities, $1 - py$ is always non-negative, and we have that g is convex. Hence, we can deduce that there are at most one more solution, other than $y = 0$. Now, as $g'(0) = -mp$ and $f'(0) = -1$, if $mp < 1$, it follows that $f'(0) < g'(0)$. This implies that there will be no other intersection. Similarly, if $mp > 1$, we have $f'(0) > g'(0)$, and as $f(1) < g(1)$. By intermediate value theorem, we must have a crossing between f and g. Therefore, if $mp > 1$, there are two solutions.

(c) Now, $mp = 1$ with $m \neq 1$, which implies that $p \neq 1$. From the above analysis, we have that $g'(0) = -mp$ and this cases shows that f is a tangent line of g precisely at 0. Again, by convexity, the graph cannot intersect at any other point.

(d) Let $y(c)$ be the unique positive y with $e^{-cy} = 1 - y$, and let $f_m(y) = (1 - py)^m$. With $p = \frac{c}{m}$, we have

$$1 - y = \lim_{m \to \infty} (1 - \frac{cy}{m})^m = e^{-cy},$$

by elementary analysis. As $y(c)$ is the solution to $1 - y = e^{-cy}$, it follows that for m large enough, we have $(1 - \frac{cy}{m})^m - y(c)$ is small. Hence, we have shown that the solution of $1 - y = (1 - py)^m$ goes to $y(c)$ in the limit. □