
Probabilistic Method: Problem Set I

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Abstract

This work contains solutions to the problem set I of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Find $m = m(n)$ as large as you can so that the following holds: Let $A_1, \dots, A_m \subseteq \{1, \dots, 4n\}$ with all $|A_i| = n$. Then there exists a two coloring of $\{1, \dots, 4n\}$ such that none of the A_i are monochromatic. Use a random *equicoloring* of $\{1, \dots, 4n\}$. (That is, choose uniformly from the $\binom{4n}{2n}$ two colorings for which there are precisely $2n$ Red and precisely $2n$ Blue vertices.) Express your answer as an asymptotic function of n .

Solution. We consider a random equicoloring of $\{1, 2, \dots, 4n\}$. Formally, we consider a finite sample space of all possible equicoloring of $\{1, 2, \dots, 4n\}$, associated with a uniform probability, which assigns each outcome in the space with $\frac{1}{\binom{4n}{2n}}$. First, observe that the probability of the event, where A_i is monochromatic is given by

$$P(\{A_i \text{ is monochromatic}\}) = 2 \frac{\binom{3n}{n}}{\binom{4n}{2n}},$$

as there are $\binom{3n}{n}$ cases of coloring the rest of the graph, when A_i has a fixed monochromatic coloring. By the subadditivity of probability, we have

$$\begin{aligned} P\left(\bigcup_{i=1}^m \{A_i \text{ is monochromatic}\}\right) &\leq \sum_{i=1}^m P(\{A_i \text{ is monochromatic}\}) \\ &= m \cdot 2 \frac{\binom{3n}{n}}{\binom{4n}{2n}}. \end{aligned}$$

By DeMorgan's laws, we see that

$$\begin{aligned} \left(\bigcup_{i=1}^m \{A_i \text{ is monochromatic}\}\right)^c &= \bigcap_{i=1}^m \{A_i \text{ is monochromatic}\}^c \\ &= \{\text{No } A_i \text{ is monochromatic}\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, if we have $m < \frac{\binom{4n}{2n}}{2\binom{3n}{n}}$, then

$$P(\{\text{No } A_i \text{ is monochromatic}\}) > 0.$$

Now, we express this answer as an asymptotic function of n . Using Sterling's formula, we have

$$\begin{aligned} \frac{\binom{3n}{n}}{\binom{4n}{2n}} &= \frac{(3n)!(2n)!}{(n!(4n)!)} \\ &= \left(\frac{3n}{e}\right)^{3n} \frac{2n}{e} \end{aligned}$$

Question 2.

2. (-) Suppose $n \geq 2$ and let $A_1, \dots, A_m \subseteq \Omega$ all have size n . Suppose $m < 4^{n-1}$. Show that there is a coloring of Ω by 4 colors so that no A_i is monochromatic.

Solution. We consider a random vertex 4-coloring of Ω . Formally, we consider a finite sample space of all possible vertex 4-coloring of Ω , associated with a uniform probability, which assigns each outcome in the space with $\frac{1}{|\Omega|}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P\left(\bigcup_{i=1}^m \{A_i \text{ is monochromatic}\}\right) &\leq \sum_{i=1}^m P(\{A_i \text{ is monochromatic}\}) \\ &= m \cdot 4^{1-n}. \end{aligned}$$

As $m < 4^{n-1}$, it follows that

$$m \cdot 4^{1-n} < 4^{n-1} \cdot 4^{1-n} = 1,$$

which primarily grants us

$$P\left(\bigcup_{i=1}^m \{A_i \text{ is monochromatic}\}\right) < 1.$$

By DeMorgan's laws, we see that

$$\begin{aligned} \left(\bigcup_{i=1}^m \{A_i \text{ is monochromatic}\}\right)^c &= \bigcap_{i=1}^m \{A_i \text{ is monochromatic}\}^c \\ &= \{\text{No } A_i \text{ is monochromatic}\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P(\{\text{No } A_i \text{ is monochromatic}\}) > 0.$$

Hence, we have shown that there is a coloring of Ω by 4 colors such that no A_i is monochromatic.

□

Question 3.

3. (-) Suppose $n \geq 4$ and let $A_1, \dots, A_m \subseteq \Omega$ all have size n . Suppose $m < \frac{4^{n-1}}{3^n}$. Prove that there is a coloring of Ω by 4 colors so that in every A_i all 4 colors are represented.

Solution. We consider a random vertex 4-coloring of Ω . Formally, we consider a finite sample space of all possible vertex 4-coloring of Ω , associated with a uniform probability, which assigns each outcome in the space with $\frac{1}{|\Omega|}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P\left(\bigcup_{i=1}^m \{A_i \text{ has at most 3 colors}\}\right) &\leq \sum_{i=1}^m P(\{A_i \text{ has at most 3 colors}\}) \\ &\leq m \cdot 3^n 4^{-n+1}, \end{aligned}$$

as there are $\binom{4}{1} 3^n$ is an upper bound to ways to have coloring of at most 3 colors for n vertices. Since $m < 3^{-n} 4^{n-1}$, it follows that

$$m \cdot 3^n 4^{-n+1} < 3^{-n} 4^{n-1} 3^n 4^{-n+1} = 1,$$

which primarily grants us

$$P\left(\bigcup_{i=1}^m \{A_i \text{ has at most 3 colors}\}\right) < 1.$$

By DeMorgan's laws, we see that

$$\begin{aligned} \left(\bigcup_{i=1}^m \{A_i \text{ has at most 3 colors}\}\right)^c &= \bigcap_{i=1}^m \{A_i \text{ has at most 3 colors}\}^c \\ &= \{\text{Every } A_i \text{ has 4 colors}\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P(\{\text{Every } A_i \text{ has 4 colors}\}) > 0.$$

Hence, we have shown that there is a coloring of Ω by 4 colors such that every A_i has 4 colors. \square

Question 4.

4. The expected number of isolated trees [just take this as a fact] on k vertices in $G(n, p)$ is given by $f(n, k, p) := \binom{n}{k} k^{k-2} p^{k-1} (1-p)^B$ with $B = k(n-k) + \binom{k}{2} - k + 1$. Set $p = \frac{1}{n}$. Let c be a positive constant. Find (see Asymptopia, Chapter 5 or webnotes) for $\binom{n}{k}$ the asymptotics of $f(n, k, p)$ when $k \sim cn^{2/3}$. (*) Express the limit as $n \rightarrow \infty$ of the sum of $f(n, k, p)$ for $n^{2/3} \leq k < 2n^{2/3}$ as a definite integral and use a computer package to evaluate the integral numerically.

Solution.

Question 5.

5. (-) Consider Boolean expressions on atoms x_1, \dots, x_n . By a k -clause C we mean an expression of the form $y_{i_1} \vee \dots \vee y_{i_k}$ where each y_{i_j} is either x_{i_j} or \bar{x}_{i_j} . Prove a theorem of the following form [you fill in the $m = m(k)$] by the probabilistic method: For any m k -clauses

C_1, \dots, C_m there is a truth assignment such that $C_1 \wedge \dots \wedge C_m$ is satisfied.

Solution. We claim the following: Suppose $m < 2^k$. Then, there exists a truth assignment such that $\bigvee_{i=1}^m C_i$ is satisfied.

We consider a random truth assignment of atoms, $\{x_1, \dots, x_n\}$. Formally, we consider a finite sample space of all possible truth assignment of atoms, associated with a uniform probability, which assigns each outcome in the space with $\frac{1}{2^n}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P\left(\bigcup_{i=1}^m \{C_i \text{ is not satisfied}\}\right) &\leq \sum_{i=1}^m P(\{C_i \text{ is not satisfied}\}) \\ &\leq m \cdot 2^{-k}, \end{aligned}$$

as there is only one assignment, which assigns all false values to k variables, that makes C_i clause not satisfied. Since $m < 2^k$, it follows that

$$m \cdot 2^{-k} < 2^k 2^{-k} = 1,$$

which primarily grants us

$$P\left(\bigcup_{i=1}^m \{C_i \text{ is not satisfied}\}\right) < 1.$$

By DeMorgan's laws, we see that

$$\begin{aligned} \left(\bigcup_{i=1}^m \{C_i \text{ is not satisfied}\} \right)^c &= \bigcap_{i=1}^m \{C_i \text{ is not satisfied}\}^c \\ &= \bigcap_{i=1}^m \{C_i \text{ is satisfied}\} = \left\{ \bigvee_{i=1}^m C_i \text{ is satisfied} \right\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P\left(\bigvee_{i=1}^m C_i \text{ is satisfied}\right) > 0.$$

Hence, we have shown that there is a truth assignment such that $\bigvee_{i=1}^m C_i$ is satisfied, when $m < 2^k$.

□