Probabilistic Method:
Problem Set I

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Abstract

This work contains solutions to the problem set I of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Find $m = m(n)$ as large as you can so that the following holds: Let $A_1, \ldots, A_m \subseteq \{1, \ldots, 4n\}$ with all $|A_i| = n$. Then there exists a two coloring of $\{1, \ldots, 4n\}$ such that none of the A_i are monochromatic. Use a random *equicoloring* of $\{1, \ldots, 4n\}$. (That is, choose uniformly from the $\binom{4n}{2n}$ two colorings for which there are precisely $2n$ Red and precisely $2n$ Blue vertices.) Express your answer as an asymptotic function of n.

Solution.

Question 2.

2. (-) Suppose $n \geq 2$ and let $A_1, \ldots, A_m \subseteq \Omega$ all have size n. Suppose $m < 4^{n-1}$. Show that there is a coloring of Ω by 4 colors so that no A_i is monochromatic.

Solution. We consider a random vertex 4-coloring of Ω. Formally, we consider a finite sample space of all possible vertex 4-coloring of Ω, associated with uniform probability, which assigns each outcome in the space with $\frac{1}{|\Omega|}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P(\bigcup_{i=1}^{m}\{A_i \text{ is monochromatic }\}) &\leq \sum_{i=1}^{m} P(\{A_i \text{ is monochromatic}\}) \\ &= m \cdot 4^{1-n}. \end{aligned}$$

As $m < 4^{n-1}$, it follows that

$$m \cdot 4^{1-n} \quad < \quad 4^{n-1} \cdot 4^{1-n} = 1,$$

which primarily grants us

$$P(\bigcup_{i=1}^{m}\{A_i \text{ is monochromatic }\}) < 1.$$

By DeMorgan's laws, we see that

$$\begin{aligned} (\bigcup_{i=1}^{m}\{A_i \text{ is monochromatic }\})^c &= \bigcap_{i=1}^{m}\{A_i \text{ is monochromatic }\}^c \\ &= \{\text{No } A_i \text{ is monochromatic}\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P(\{\text{No } A_i \text{ is monochromatic}\}) \quad > \quad 0.$$

Hence, we have shown that there is a coloring of Ω by 4 colors such that no A_i is monochromatic.
$\square$

Question 3.

3. (-) Suppose $n \geq 4$ and let $A_1, \ldots, A_m \subseteq \Omega$ all have size n. Suppose $m < \frac{4^{n-1}}{3^n}$. Prove that there is a coloring of Ω by 4 colors so that in every A_i all 4 colors are represented.

Solution. We consider a random vertex 4-coloring of Ω. Formally, we consider a finite sample space of all possible vertex 4-coloring of Ω, associated with uniform probability, which assigns each outcome in the space with $\frac{1}{|\Omega|}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P(\bigcup_{i=1}^{m}\{A_i \text{ has at most 3 colors }\}) &\leq \sum_{i=1}^{m} P(\{A_i \text{ has at most 3 colors }\}) \\ &\leq m \cdot 3^n 4^{-n+1}, \end{aligned}$$

as there are $\binom{4}{1}3^n$ is an upper bound to ways to have coloring of at most 3 colors for n vertices. Since $m < 3^{-n}4^{n-1}$, it follows that

$$m \cdot 3^n 4^{-n+1} \quad < \quad 3^{-n}4^{n-1}3^n 4^{-n+1} = 1,$$

which primarily grants us

$$P(\bigcup_{i=1}^{m}\{A_i \text{ has at most 3 colors }\}) < 1.$$

By DeMorgan's laws, we see that

$$\begin{aligned} (\bigcup_{i=1}^{m}\{A_i \text{ has at most 3 colors }\})^c &= \bigcap_{i=1}^{m}\{A_i \text{ has at most 3 colors }\}^c \\ &= \{\text{Every } A_i \text{ has 4 colors}\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P(\{\text{Every } A_i \text{ has 4 colors }\}) \quad > \quad 0.$$

Hence, we have shown that there is a coloring of Ω by 4 colors such that every A_i has 4 colors. $\square$

Question 4.

4. The expected number of isolated trees [just take this as a fact] on k vertices in $G(n,p)$ is given by $f(n,k,p) := \binom{n}{k} k^{k-2} p^{k-1} (1-p)^B$ with $B = k(n-k) + \binom{k}{2} - k + 1$. Set $p = \frac{1}{n}$. Let c be a positive constant. Find (see Asymptopia, Chapter 5 or webnotes) for $\binom{n}{k}$) the asymptotics of $f(n,k,p)$ when $k \sim cn^{2/3}$. (*) Express the limit as $n \to \infty$ of the sum of $f(n,k,p)$ for $n^{2/3} \leq k < 2n^{2/3}$ as a definite integral and use a computer package to evaluate the integral numerically.

Solution.

Question 5.

5. (-) Consider Boolean expressions on atoms $x_1, \ldots, x_n$. By a k-clause C we mean an expression of the form $y_{i_1} \vee \ldots \vee y_{i_k}$ where each y_{i_j} is either x_{i_j} or $\overline{x}_{i_j}$. Prove a theorem of the following form [you fill in the $m = m(k)$] by the probabilistic method: For any m k-clauses

$C_1, \ldots, C_m$ there is a truth assignment such that $C_1 \wedge \ldots \wedge C_m$ is satisfied.

Solution. We claim the following: Suppose $m < 2^k$. Then, there exists a truth assignment such that $\bigvee_{i=1}^{m} C_i$ is satisfied.

We consider a random truth assignment of atoms, $\{x_1, ...x_n\}$. Formally, we consider a finite sample space of all possible truth assignment of atoms, associated with uniform probability, which assigns each outcome in the space with $\frac{1}{2^n}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P(\bigcup_{i=1}^{m}\{C_i \text{ is not satisfied }\}) &\leq \sum_{i=1}^{m} P(\{C_i \text{ is not satisfied }\}) \\ &\leq m \cdot 2^{-k}, \end{aligned}$$

as there is only one assignment, which assigns all false values to k variables, that makes C_i clause not satisfied. Since $m < 2^k$, it follows that

$$m \cdot 2^{-k} < 2^k 2^{-k} = 1,$$

which primarily grants us

$$P(\bigcup_{i=1}^{m}\{C_i \text{ is not satisfied }\}) < 1.$$

By DeMorgan's laws, we see that

$$\begin{aligned}
(\bigcup_{i=1}^{m}\{C_i \text{ is not satisfied }\})^c &= \bigcap_{i=1}^{m}\{C_i \text{ is not satisfied }\}^c \\
&= \bigcap_{i=1}^{m}\{C_i \text{ is satisfied }\} = \{\bigvee C_i \text{ is satisfied }\}.
\end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P(\{\bigvee_{i=1}^{m} C_i \text{ is satisfied }\}) \quad > \quad 0.$$

Hence, we have shown that there is a truth assignment such that $\bigvee_{i=1}^{m} C_i$ is satisfied, when $m < 2^k$.

$\square$

Question 6.

6. Formula (13) on the n choose k notes (on the web) is applied with $c = \frac{1}{2}$ to give the asymptotics of the middle binomial coefficient. Here we want to extend this to binomial coefficients near the middle.

   (a) (-) Give the Taylor Series for the Entropy function $H(c)$ around $c = \frac{1}{2}$ (set $c = \frac{1}{2} + x$ for convenience) out to the quadratic term with error $O(x^3)$.

   (b) Apply (13) to the asymptotics of $\binom{n}{k}$ where $k = \frac{n}{2} + u$ and $u = o(n)$, getting the answer in terms of the entropy function $H(k/n)$.

   (c) Use the quadratic approximation of the Entropy function you derived above to get an asymptotic formula for $\binom{n}{k}$ when $k = \frac{n}{2}+u$ is sufficiently close to $\frac{n}{2}$. (You should get a rather (joke!) normal result.) To clarify: you are being asked to find a *scaling* which with be a simple function $g(n)$ such that the sum of $\binom{n}{k}$ over $k \leq \frac{n}{2} + \lambda g(n)$ is a well known function of λ. You probably already know the answer via Central Limit Theorem but you are here asked to derive that answer through these asymptotics.

Solution.