Probabilistic Method:
Problem Set I

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Abstract

This work contains solutions to the problem set I of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. (-) Let $X_1, \ldots, X_n$ be independent random variables with $\Pr[X_i = +1] = \Pr[X_i = -1] = \frac{1}{2}$. Set $X = X_1 + \ldots + X_n$. Find $E[X^2]$ precisely. Find $E[X^4]$ precisely. [Idea: Expand and use linearity of expectation.]

Solution.

Question 2.

2. Find an asymptotic formula for

$$\sum_{k=n^{1/2}}^{2n^{1/2}} (n)_k n^{-k}$$

by parametrizing $k = cn^{1/2}$ and turning it into an integral which can be evaluated numerically. (You can leave it in the form of a definite integral if you wish.) (See Asymptopia, Chapter 4)

Solution.

Question 3.

3. Now we go to the complete sum by showing the edge effects are negligible.

   (a) Show

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} n^{-1/2}\sum_{k=1}^{\epsilon n^{1/2}}(n)_k n^{-k} = 0$$

   by using an appropriate upper bound for the addends.

   (b) (*) Show

$$\lim_{K\to\infty}\lim_{n\to\infty} n^{-1/2}\sum_{k=Kn^{1/2}}^{n}(n)_k n^{-k} = 0$$

   by using an appropriate upper bound for the addends.

   (c) Find an asymptotic formula for

$$\sum_{k=1}^{n}(n)_k n^{-k}$$

   by splitting it into the ranges $k < \epsilon n^{1/2}$, $\epsilon n^{1/2} \le k \le Kn^{1/2}$ and $Kn^{1/2} < k \le n$ and then taking appropriate limits. (You may assume the previous parts.)

Solution.

Question 4.

4. Prove, for $m = m(n)$ as large as you can, the existence of an $n \times n$ matrix A of zeroes and ones with m ones which does not contain a 3×3 submatrix of all ones. Use the alteration method: make each entry one with probability p and then for each such submatrix change a one to zero. When you optimize [using Calculus!] your final answer should be of the form $m \sim an^b$ for some reasonable a, b.

Solution. Consider a random $n \times n$ matrix, M_n, obtained by assigning each entry independently either 1 or 0, where the probability of assigning 1 is p. Let X be a random variable, which counts number of 1s in the matrix, and Y be a random variable, which counts number of 3×3 submatrices of all 1s. For any 3×3 submatrix S, let Y_S be the indicator random variable for the event for which the submatrix S has entries of all 1s, so that $Y = \sum Y_S$. By Linearity of Expectation, we have

$$E[Y] \quad = \quad \sum E[Y_S] = \binom{n}{3}^2 p^9.$$

Clearly, $E[X] = n^2 p$. Therefore, again by Linearity of Expectation, it follows that

$$E[X - Y] \quad = \quad n^2 p - \binom{n}{3}^2 p^9 = f(p).$$

Hence, there exists a random assignment, for which the number of 1s minus the number of 3×3 submatrices of 1 is at least $f(p)$. Fix such a coloring. Select one entry from each submatrix and change to 0. This leaves the matrix with at least $f(p)$ entries with 1.

We now optimize this result by maximizing $f(p)$ with respect to p. Observe that f is concave with respect to $p \in [0, 1]$. Solving for the local maxima by setting the first-order derivative equals to 0, we get that f is maximized at $p^* = (\frac{n^2}{9\binom{n}{3}^2})^{\frac{1}{8}} = (\frac{2}{(n-1)(n-2)})^{\frac{1}{4}}$. Substituting p^* back into $f(p)$, we obtain

$$\begin{aligned} f(p^*) \quad &= \quad n^2 (\frac{2}{(n-1)(n-2)})^{\frac{1}{4}} - \binom{n}{3}^2 \frac{n^2}{9\binom{n}{3}^2} (\frac{2}{(n-1)(n-2)})^{\frac{1}{4}} \\ &= \quad \frac{8}{9} n^2 (\frac{2}{(n-1)(n-2)})^{\frac{1}{4}} \\ &\sim \quad \frac{8}{9} 2^{\frac{1}{4}} n^{\frac{3}{2}}. \end{aligned}$$

Recall that $m(n)$ be the minimum number of 1 in $n \times n$ matrix, such that there must exist a 3×3 submatrix of all 1s. We have shown that $m(n) = \Omega(n^{\frac{3}{2}})$. $\square$

Question 5.

5. We are given $m = 2^{n-1}k$ sets, each of size n, in a universe Ω. Consider the following randomized algorithm for a 2-coloring: First color each point $v \in \Omega$ randomly. Now, for each monochromatic set e, select a random vertex $v \in e$. Each such selected v (regardless of how often it was selected) has its color (definitely, no probability here) flipped. Call the algorithm a failure if some set e originally had all or all but one vertex the same color and ended with all vertices that color. Find k as large as you can (as an asymptotic function of n) so that the failure probability is less than one. (Note that this, unfortunately, does not give us any result on $m(n)$ since there are other ways that a set e could end up monochromatic.)

Solution. Note that we use the langauge of $n-$uniform hypergraph, which is an equivalent problem as the problem under consideration. Before preceeding with the main part of the proof, we define an object, called a conflicting pair, in a two-coloring scheme. We say that an ordered pair of edges (e, f) a conflicting pair, if e is monochromatic with some color k, $f \setminus \{v\}$ is monochromatic with the color not k, and $v \in e \cap f$. Notice that the algorithm can potentially fail, if within the coloring scheme, it chooses v to flip its color. Hence, the existence of a conflicting pair is a necessary condition for a particular coloring to result in a failure with the given algorithm. Thereofre, if we consider the sample space of all possible random coloring of Ω, associated with a unifrom probability, by the monotonicity of probability, we have that the probability of the existence of a conflicting pair is an upper bound for the probability of the coloring resulting in a failure. Lastly, one should note that for a fixed ordered pair of edges, (e, f), we have

$$P(\{(e, f) \text{ is a conflicting pair }\}) \quad \leq \quad 2^{2-2n}$$

We now proceed with the main part of the proof. Color Ω randomly with two colors. By the above remark, and the repeated use of subadditivty of probability, we obtain

$$\begin{aligned} P(\{ \text{ colorings that the algorithm fails on}\}) &\leq P(\{ \text{ colorings that contain a conflicting pair}\}) \\ &\leq P(\bigcup_{(e,f)\in E\times E} \{(e, f) \text{ is a conflicting pair}\}) \\ &\leq \sum_{(e,f)\in E\times E} P(\{(e, f) \text{ is a conflicting pair}\}) \\ &\leq m^2 2^{2-2n}, \end{aligned}$$

where E denotes the edge set and the fact that the probability of having a conflicting pair coloring is at most $2 \cdot 2^{1-2n}$. If $m^2 2^{2-2n} < 1$, we can gurantee that the failure probability is less than 1. Substituting $2^{n-1}k$ for m, we have

$$(2^{n-1}k)^2 2^{2-2n} < 1,$$

which is equvialent to

$$dd$$

Question 6.

6. Set $X = \sum_{i=1}^{n} X_i$ where $X_i = \pm 1$ uniformly and independently. Bound $\Pr[X > \frac{n}{2}]$ as follows.

   (a) Find a closed form for $E[e^{\lambda X_i}]$.

   (b) Find a closed form for $E[e^{\lambda X}]$.

   (c) Use the Chernoff Bound $\Pr[X > a] < E[e^{\lambda X}]e^{-\lambda a}$ with $a = \frac{n}{2}$. Use Calculus (this gets a little messy to put in closed form, full points for numerical answers) to select the the optimal λ.

   (d) Compare this with the lower bound

$$\Pr[X \geq \frac{n}{2}] \geq \Pr[X = \frac{n}{2}] = 2^{-n}\binom{n}{\frac{3n}{4}}$$

   showing that the upper and lower bounds have the same main terms.

Solution.