Probabilistic Method: Problem Set I

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set I of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Find $m = m(n)$ as large as you can so that the following holds: Let $A_1, \ldots, A_m \subseteq \{1, \ldots, 4n\}$ with all $|A_i| = n$. Then there exists a two coloring of $\{1, \ldots, 4n\}$ such that none of the A_i are monochromatic. Use a random *equicoloring* of $\{1, \ldots, 4n\}$. (That is, choose uniformly from the $\binom{4n}{2n}$ two colorings for which there are precisely $2n$ Red and precisely $2n$ Blue vertices.) Express your answer as an asymptotic function of n.

Solution.

Question 2.

2. (-) Suppose $n \geq 2$ and let $A_1, \ldots, A_m \subseteq \Omega$ all have size n. Suppose $m < 4^{n-1}$. Show that there is a coloring of Ω by 4 colors so that no A_i is monochromatic.

Solution. We consider a random vertex 4-coloring of Ω. Formally, we consider a finite sample space of all possible vertex 4-coloring of Ω, associated with uniform probability, which assigns each outcome in the space with $\frac{1}{|\Omega|}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P(\bigvee_{i=1}^{n}\{A_i \text{ is monochromatic }\}) &\leq \sum_{i=1}^{m} P(\{A_i \text{ is monochromatic}\}) \\ &= m \cdot 4^{1-n}. \end{aligned}$$

As $m < 4^{n-1}$ and $n \leq 2$, it follows that

$$m \cdot 4^{1-n} \quad < \quad 4^{n-1} \cdot 4^{1-n} = 1,$$

which primarily grants us

$$P(\bigvee_{i=1}^{n}\{A_i \text{ is monochromatic }\}) < 1.$$

By DeMorgan's law, we see that

$$\begin{aligned} (\bigvee_{i=1}^{n}\{A_i \text{ is monochromatic }\})^c &= \bigwedge_{i=1}^{n}\{A_i \text{ is monochromatic }\}^c \\ &= \{\text{No } A_i \text{ is monochromatic}\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P(\{\text{No } A_i \text{ is monochromatic}\}) \quad > \quad 0.$$

In particular, the event, where no A_i is monochromatic, has positive probability. Hence, we have shown that there is a coloring of Ω by 4 colors such that no A_i is monochromatic. $\square$

Question 3.

3. (-) Suppose $n \geq 4$ and let $A_1, \ldots, A_m \subseteq \Omega$ all have size n. Suppose $m < \frac{4^{n-1}}{3^n}$. Prove that there is a coloring of Ω by 4 colors so that in every A_i all 4 colors are represented.

Solution. By union bound, we have

$$\begin{aligned} P(\bigvee_{i=1}^{n}\{A_i \text{ has at most 3 colors }\}) &\leq \sum_{i=1}^{m} P(\{A_i \text{ has at most 3 colors }\}) \\ &= m \cdot 3^{\binom{n}{2}} 4^{-\binom{n}{2}}. \end{aligned}$$

which primarily grants us

$$P(\bigvee_{i=1}^{n}\{A_i \text{ is monochromatic }\}) < 1.$$

Therefore, the complement event of $\bigvee_{i=1}^{n}\{A_i \text{ is monochromatic}\}$, namely the event where no A_i is monochromatic, has positive probability. Hence, we have shown that there is a coloring of Ω by 4 colors so that no A_i is monochromatic. $\square$

Question 4.
Solution.

4. The expected number of isolated trees [just take this as a fact] on k vertices in $G(n,p)$ is given by $f(n,k,p) := \binom{n}{k} k^{k-2} p^{k-1} (1-p)^B$ with $B = k(n-k) + \binom{k}{2} - k + 1$. Set $p = \frac{1}{n}$. Let c be a positive constant. Find (see Asymptopia, Chapter 5 or webnotes) for $\binom{n}{k}$) the asymptotics of $f(n,k,p)$ when $k \sim cn^{2/3}$. (*) Express the limit as $n \to \infty$ of the sum of $f(n,k,p)$ for $n^{2/3} \leq k < 2n^{2/3}$ as a definite integral and use a computer package to evaluate the integral numerically.

Question 5.

5. (-) Consider Boolean expressions on atoms $x_1, \ldots, x_n$. By a k-clause C we mean an expression of the form $y_{i_1} \vee \ldots \vee y_{i_k}$ where each y_{i_j} is either x_{i_j} or $\overline{x}_{i_j}$. Prove a theorem of the following form [you fill in the $m = m(k)$] by the probabilistic method: For any m k-clauses

$C_1, \ldots, C_m$ there is a truth assignment such that $C_1 \wedge \ldots \wedge C_m$ is satisfied.

Solution.

Question 6.

6. Formula (13) on the n choose k notes (on the web) is applied with $c = \frac{1}{2}$ to give the asymptotics of the middle binomial coefficient. Here we want to extend this to binomial coefficients near the middle.

   (a) (-) Give the Taylor Series for the Entropy function $H(c)$ around $c = \frac{1}{2}$ (set $c = \frac{1}{2} + x$ for convenience) out to the quadratic term with error $O(x^3)$.

   (b) Apply (13) to the asymptotics of $\binom{n}{k}$ where $k = \frac{n}{2} + u$ and $u = o(n)$, getting the answer in terms of the entropy function $H(k/n)$.

   (c) Use the quadratic approximation of the Entropy function you derived above to get an asymptotic formula for $\binom{n}{k}$ when $k = \frac{n}{2} + u$ is sufficiently close to $\frac{n}{2}$. (You should get a rather (joke!) normal result.) To clarify: you are being asked to find a *scaling* which with be a simple function $g(n)$ such that the sum of $\binom{n}{k}$ over $k \leq \frac{n}{2} + \lambda g(n)$ is a well known function of λ. You probably already know the answer via Central Limit Theorem but you are here asked to derive that answer through these asymptotics.

Solution.