Probabilistic Method:
Problem Set I

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Abstract

This work contains solutions to the problem set I of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Find $m = m(n)$ as large as you can so that the following holds: Let $A_1, \ldots, A_m \subseteq \{1, \ldots, 4n\}$ with all $|A_i| = n$. Then there exists a two coloring of $\{1, \ldots, 4n\}$ such that none of the A_i are monochromatic. Use a random *equicoloring* of $\{1, \ldots, 4n\}$. (That is, choose uniformly from the $\binom{4n}{2n}$ two colorings for which there are precisely $2n$ Red and precisely $2n$ Blue vertices.) Express your answer as an asymptotic function of n.

Solution. We consider a random equicoloring of $\{1, 2, ..., 4n\}$. Formally, we consider a finite sample space of all possible equicoloring of $\{1, 2, ..., 4n\}$, associated with a uniform probability, which assigns each outcome in the space with $\frac{1}{\binom{4n}{2n}}$. First, observe that the probability of the event, where A_i is monochromatic is given by

$$P(\{A_i \text{ is monochromatic}\}) \quad = \quad 2\frac{\binom{3n}{n}}{\binom{4n}{2n}},$$

as there are $\binom{3n}{n}$ cases of coloring the rest of the graph , when A_i has a fixed monochromatic coloring. By the subadditivity of probability, we have

$$\begin{aligned} P(\bigcup_{i=1}^{m}\{A_i \text{ is monochromatic }\}) \quad &\leq \quad \sum_{i=1}^{m} P(\{A_i \text{ is monochromatic}\}) \\ &= \quad m \cdot 2\frac{\binom{3n}{n}}{\binom{4n}{2n}}. \end{aligned}$$

By DeMorgan's laws, we see that

$$\begin{aligned} (\bigcup_{i=1}^{m}\{A_i \text{ is monochromatic }\})^c \quad &= \quad \bigcap_{i=1}^{m}\{A_i \text{ is monochromatic }\}^c \\ &= \quad \{\text{No } A_i \text{ is monochromatic}\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, if we have $m < \frac{\binom{4n}{2n}}{2\binom{3n}{n}}$, then

$$P(\{\text{No } A_i \text{ is monochromatic}\}) \quad > \quad 0.$$

Now, we express this answer as an asymptotic function of n. Using Sterling's formula, we have

$$\begin{aligned} 2\frac{\binom{3n}{n}}{\binom{4n}{2n}} &= 2\frac{(3n)!(2n)!}{(n!(4n)!} \\ &= 2\frac{(\frac{3n}{e})^{3n}(\frac{2n}{e})^{2n}\sqrt{2\pi(3n)}\sqrt{2\pi(2n)}}{(\frac{n}{e})^{n}(\frac{4n}{e})^{4n}\sqrt{2\pi(n)}\sqrt{2\pi(4n)}}(1+o(1)) \\ &= \sqrt{6}(\frac{3}{4})^{3n}(1+o(1)), \end{aligned}$$

as required. It follows that the main asymptotic term on the upper bound of m is the $(\frac{4}{3})^{3n}$ term, which is an exponential function of n. $\square$

Question 2.

2. (-) Suppose $n \geq 2$ and let $A_1, \ldots, A_m \subseteq \Omega$ all have size n. Suppose $m < 4^{n-1}$. Show that there is a coloring of Ω by 4 colors so that no A_i is monochromatic.

Solution. We consider a random vertex 4-coloring of Ω. Formally, we consider a finite sample space of all possible vertex 4-coloring of Ω, associated with a uniform probability, which assigns each outcome in the space with $\frac{1}{|\Omega|}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P(\bigcup_{i=1}^{m}\{A_i \text{ is monochromatic }\}) &\leq \sum_{i=1}^{m} P(\{A_i \text{ is monochromatic}\}) \\ &= m \cdot 4^{1-n}. \end{aligned}$$

As $m < 4^{n-1}$, it follows that

$$m \cdot 4^{1-n} \quad < \quad 4^{n-1} \cdot 4^{1-n} = 1,$$

which primarily grants us

$$P(\bigcup_{i=1}^{m}\{A_i \text{ is monochromatic }\}) < 1.$$

By DeMorgan's laws, we see that

$$\begin{aligned} (\bigcup_{i=1}^{m}\{A_i \text{ is monochromatic }\})^c &= \bigcap_{i=1}^{m}\{A_i \text{ is monochromatic }\}^c \\ &= \{\text{No } A_i \text{ is monochromatic}\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P(\{\text{No } A_i \text{ is monochromatic}\}) \quad > \quad 0.$$

Hence, we have shown that there is a coloring of Ω by 4 colors such that no A_i is monochromatic. $\square$

Question 3.

3. (-) Suppose $n \geq 4$ and let $A_1, \ldots, A_m \subseteq \Omega$ all have size n. Suppose $m < \frac{4^{n-1}}{3^n}$. Prove that there is a coloring of Ω by 4 colors so that in every A_i all 4 colors are represented.

Solution. We consider a random vertex 4-coloring of Ω. Formally, we consider a finite sample space of all possible vertex 4-coloring of Ω, associated with a uniform probability, which assigns each outcome in the space with $\frac{1}{|\Omega|}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P(\bigcup_{i=1}^{m}\{A_i \text{ has at most 3 colors }\}) &\leq \sum_{i=1}^{m} P(\{A_i \text{ has at most 3 colors }\}) \\ &\leq m \cdot 3^n 4^{-n+1}, \end{aligned}$$

as there are $\binom{4}{1}3^n$ is an upper bound to ways to have coloring of at most 3 colors for n vertices. Since $m < 3^{-n}4^{n-1}$, it follows that

$$m \cdot 3^n 4^{-n+1} \quad < \quad 3^{-n}4^{n-1}3^n 4^{-n+1} = 1,$$

which primarily grants us

$$P(\bigcup_{i=1}^{m}\{A_i \text{ has at most 3 colors }\}) < 1.$$

By DeMorgan's laws, we see that

$$\begin{aligned} (\bigcup_{i=1}^{m}\{A_i \text{ has at most 3 colors }\})^c &= \bigcap_{i=1}^{m}\{A_i \text{ has at most 3 colors }\}^c \\ &= \{\text{Every } A_i \text{ has 4 colors}\}. \end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P(\{\text{Every } A_i \text{ has 4 colors }\}) \quad > \quad 0.$$

Hence, we have shown that there is a coloring of Ω by 4 colors such that every A_i has 4 colors. $\square$

Question 4.

4. The expected number of isolated trees [just take this as a fact] on k vertices in $G(n,p)$ is given by $f(n,k,p) := \binom{n}{k}k^{k-2}p^{k-1}(1-p)^B$ with $B = k(n-k)+\binom{k}{2}-k+1$. Set $p = \frac{1}{n}$. Let c be a positive constant. Find (see Asymptopia, Chapter 5 or webnotes) for $\binom{n}{k}$) the asymptotics of $f(n,k,p)$ when $k \sim cn^{2/3}$. (*) Express the limit as $n \to \infty$ of the sum of $f(n,k,p)$ for $n^{2/3} \le k < 2n^{2/3}$ as a definite integral and use a computer package to evaluate the integral numerically.

Solution. First of all the problem can be found in the book, Asymptopia. We note that $k \sim cn^{\frac{2}{3}}$, thus $k = o(n^{\frac{3}{4}})$. Then, by the case 4 of the result in 5.1 Asymtopia, with Stirling's formula, we have

$$\begin{aligned}\binom{n}{k} &\sim e^{-\frac{k^2}{2n}}e^{-\frac{k^3}{6n^2}}\frac{n^k}{k!}\\ &\sim e^{-\frac{k^2}{2n}}e^{-\frac{k^3}{6n^2}}n^k\frac{e^k}{n^k\sqrt{2\pi k}}\end{aligned}$$

Now, by direct computation, it follows that

$$\begin{aligned}B &= k(n-k)+\binom{k}{2}-k+1\\ &= kn-\frac{1}{2}k^2-\frac{3}{2}+1\\ &= kn-\frac{1}{2}k^2+O(k),\end{aligned}$$

which then yields

$$ln[(1-p)^{k(n-k)+\binom{k}{2}-(k-1)}] = -k+\frac{k^2}{2n}+o(1).$$

Now, substituting the above into the first asyptomtic equivalence we have established, we have

$$f(n,k,p) \sim e^{-\frac{c^3}{6}}n^{-\frac{2}{3}}c^{-\frac{5}{2}}(2\pi)^{-\frac{1}{2}},$$

as required. $\square$

Question 5.

5. (-) Consider Boolean expressions on atoms $x_1, \ldots, x_n$. By a k-clause C we mean an expression of the form $y_{i_1} \vee \ldots \vee y_{i_k}$ where each y_{i_j} is either x_{i_j} or $\overline{x}_{i_j}$. Prove a theorem of the following form [you fill in the $m = m(k)$] by the probabilistic method: For any m k-clauses

$C_1, \ldots, C_m$ there is a truth assignment such that $C_1 \wedge \ldots \wedge C_m$ is satisfied.

Solution. We claim the following: Suppose $m < 2^k$. Then, there exists a truth assignment such that $\bigvee_{i=1}^m C_i$ is satisfied.

We consider a random truth assignment of atoms, $\{x_1, ...x_n\}$. Formally, we consider a finite sample space of all possible truth assignment of atoms, associated with a uniform probability, which assigns each outcome in the space with $\frac{1}{2^n}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P(\bigcup_{i=1}^m \{C_i \text{ is not satisfied }\}) &\leq \sum_{i=1}^m P(\{C_i \text{ is not satisfied }\}) \\ &\leq m \cdot 2^{-k}, \end{aligned}$$

as there is only one assignment, which assigns all false values to k variables, that makes C_i clause not satisfied. Since $m < 2^k$, it follows that

$$m \cdot 2^{-k} < 2^k 2^{-k} = 1,$$

which primarily grants us

$$P(\bigcup_{i=1}^m \{C_i \text{ is not satisfied }\}) < 1.$$

By DeMorgan's laws, we see that

$$\begin{aligned}
(\bigcup_{i=1}^{m}\{C_i \text{ is not satisfied }\})^c &= \bigcap_{i=1}^{m}\{C_i \text{ is not satisfied }\}^c \\
&= \bigcap_{i=1}^{m}\{C_i \text{ is satisfied }\} = \{\bigvee C_i \text{ is satisfied }\}.
\end{aligned}$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$P(\{\bigvee_{i=1}^{m} C_i \text{ is satisfied }\}) \quad > \quad 0.$$

Hence, we have shown that there is a truth assignment such that $\bigvee_{i=1}^{m} C_i$ is satisfied, when $m < 2^k$.

$\square$