Probabilistic Method:
Problem Set I

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set I of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. (-) Let $X_1, \ldots, X_n$ be independent random variables with $\Pr[X_i = +1] = \Pr[X_i = -1] = \frac{1}{2}$. Set $X = X_1 + \ldots + X_n$. Find $E[X^2]$ precisely. Find $E[X^4]$ precisely. [Idea: Expand and use linearity of expectation.]

Solution. By expanding the terms, using linearity of expectation, we have

$$\begin{aligned} E[X^2] &= E[(\sum_{i=1}^{n} X_i)^2] \\ &= E[\sum_{i=1}^{n} X_i^2] + 2 \sum_{1 \le i < j \le n} X_i X_j] \\ &= \sum_{i=1}^{n} E[X_i^2] + 2 \sum_{1 \le i < j \le n} E[X_i X_j]. \end{aligned}$$

Observe that

$$\begin{aligned} E[X_i^2] &= 1^2 \frac{1}{2} + (-1)^2 \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

From the independence assumption, we have

$$E[X_i X_j] = E[X_i] E[X_j] = 0,$$

for $i \neq j$. Therefore, by substitution to the previous equality, we have

$$E[X^2] = n.$$

Now, for the $E[X^4]$, we note that the terms with $E[X_i]$ terms will all be 0. Therefore, we directly obtain

$$
\begin{aligned}
E[X^4] &= \sum_{i=1}^{n} E[X_i]^4 + \binom{4}{2} E[\sum_{1 \le i \le j \le n} X_i^2 X_j^2] \\
&= n + \binom{4}{2}\binom{n}{2} = 3n^2 - 2n.
\end{aligned}
$$

□

Question 2.

2. Find an asymptotic formula for

$$\sum_{k=n^{1/2}}^{2n^{1/2}} (n)_k n^{-k}$$

by parametrizing $k = cn^{1/2}$ and turning it into an integral which can be evaluated numerically. (You can leave it in the form of a definite integral if you wish.) (See Asymptopia, Chapter 4)

Solution. By definition, we have

$$
\begin{aligned}
(n)_k k^{-k} &= \frac{n!}{(n-k)!} n^{-k} \\
&= \binom{n}{k} k! n^{-k}.
\end{aligned}
$$

As we are given $k \sim c\sqrt{n}$, from Asymtopia, we have

$$\binom{n}{k} \sim e^{-\frac{c^2}{2}} \frac{n^k}{k!}.$$

By the Stirling's formula, it immediately follows that

$$(n)_k n^{-k} \sim e^{-\frac{c^2}{2}} \frac{n^k}{k!} k! n^{-k} = e^{-\frac{c^2}{2}}.$$

Using the integration given in Asymtopia, we have

$$
\begin{aligned}
\sum_{k=\sqrt{n}}^{2\sqrt{n}} (n)_k n^{-k} &\sim \int_1^2 \sqrt{n} e^{-\frac{c^2}{2}} dc \\
&\sim 0.34\sqrt{n},
\end{aligned}
$$

as required. □

Question 3.

3. Now we go to the complete sum by showing the edge effects are negligible.

   (a) Show
   $$\lim_{\epsilon\to 0}\lim_{n\to\infty} n^{-1/2}\sum_{k=1}^{\epsilon n^{1/2}}(n)_k n^{-k}=0$$
   by using an appropriate upper bound for the addends.

   (b) (*) Show
   $$\lim_{K\to\infty}\lim_{n\to\infty} n^{-1/2}\sum_{k=Kn^{1/2}}^{n}(n)_k n^{-k}=0$$
   by using an appropriate upper bound for the addends.

   (c) Find an asymptotic formula for
   $$\sum_{k=1}^{n}(n)_k n^{-k}$$
   by splitting it into the ranges $k<\epsilon n^{1/2}$, $\epsilon n^{1/2}\le k\le Kn^{1/2}$ and $Kn^{1/2}<k\le n$ and then taking appropriate limits. (You may assume the previous parts.)

Solution. (a) By definition of $(n)_k$, it follows that

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{k=1}^{\epsilon\sqrt{n}}(n)_k n^{-k} &\le \frac{1}{\sqrt{n}}\sum_{k=1}^{\epsilon\sqrt{n}}\frac{n...(n-k+1)}{n...n}\\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{\epsilon\sqrt{n}}1=\epsilon,\end{aligned}$$

which goes to 0 as $\epsilon\to 0$.

(b) By taking the natural log, we have

$$\begin{aligned}\ln(\frac{(n)_k}{n^k}) &= \sum_{i=1}^{k-1}(1-\frac{i}{n})\\ &\le \sum_{i=1}^{k-1}(-\frac{i}{n})\\ &= -\frac{k(k-1)}{2n}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{k=k\sqrt{n}}^{n}\frac{(n)_k}{n^k} &\le \int_{k\sqrt{n}}^{\infty}e^{\frac{-k^2}{4n}}\,dk\\ &\le \sqrt{n}\int_{k-1}^{\infty}e^{\frac{-c^2}{4}}\,dc,\end{aligned}$$

when $k = c\sqrt{n}$. The upper bound goes to 0 as k goes to ∞. Therefore, by the squeeze theorem (the term is trivially nonnegative), we have shown the desired claim.

(c) Using the additivity of summation, and by the problem (2), we obtain

$$\begin{aligned}\sum_{k=1}^{n} \frac{(n)_k}{n^k} &= \sum_{k=1}^{\epsilon\sqrt{n}} \frac{(n)_k}{n^k} + \sum_{k=\epsilon\sqrt{n}}^{k\sqrt{n}} \frac{(n)_k}{n^k} + \sum_{k=k\sqrt{n}}^{n} \frac{(n)_k}{n^k} \\ &\sim \sum_{k=1}^{\epsilon\sqrt{n}} \frac{(n)_k}{n^k} + \sqrt{n}\int_{\epsilon}^{k} e^{-\frac{c^2}{2}}\, dc + \sum_{k=k\sqrt{n}}^{n} \frac{(n)_k}{n^k}.\end{aligned}$$

By part (a) and (b), as $\epsilon \to 0$ and $k \to \infty$, we have the first term and the thrid term going to 0. Therefore, the desired limit is

$$\sqrt{n}\int_{0}^{\infty} e^{-c^2} 2dc \quad \sim \quad 1.25\sqrt{n},$$

as required. $\square$

Question 4.

4. Prove, for $m = m(n)$ as large as you can, the existence of an $n \times n$ matrix A of zeroes and ones with m ones which does not contain a 3×3 submatrix of all ones. Use the alteration method: make each entry one with probability p and then for each such submatrix change a one to zero. When you optimize [using Calculus!] your final answer should be of the form $m \sim an^b$ for some reasonable a, b.

Solution. Consider a random $n \times n$ matrix, M_n, obtained by assigning each entry independently either 1 or 0, where the probability of assigning 1 is p. Let X be a random variable, which counts number of 1s in the matrix, and Y be a random variable, which counts number of 3×3 submatrices of all 1s. For any 3×3 submatrix S, let Y_S be the indicator random variable for the event for which the submatrix S has entries of all 1s, so that $Y = \sum Y_S$. By Linearity of Expectation, we have

$$E[Y] \quad = \quad \sum E[Y_S] = \binom{n}{3}^2 p^9.$$

Clearly, $E[X] = n^2 p$. Therefore, again by Linearity of Expectation, it follows that

$$E[X - Y] \quad = \quad n^2 p - \binom{n}{3}^2 p^9 = f(p).$$

Hence, there exists a random assignment, for which the number of 1s minus the number of 3×3 submatrices of 1 is at least $f(p)$. Fix such a coloring. Select one entry from each submatrix and change to 0. This leaves the matrix with at least $f(p)$ entries with 1.

We now optimize this result by maximizing $f(p)$ with respect to p. Observe that f is concave with respect to $p \in [0, 1]$. Solving for the local maxima by setting the first-order derivative equals to 0, we get that f is maximized at $p^* = (\frac{n^2}{9\binom{n}{3}^2})^{\frac{1}{8}} = (\frac{2}{(n-1)(n-2)})^{\frac{1}{4}}$. Substituting p^* back into $f(p)$, we obtain

$$\begin{aligned} f(p^*) &= n^2 (\frac{2}{(n-1)(n-2)})^{\frac{1}{4}} - \binom{n}{3}^2 \frac{n^2}{9\binom{n}{3}^2} (\frac{2}{(n-1)(n-2)})^{\frac{1}{4}} \\ &= \frac{8}{9} n^2 (\frac{2}{(n-1)(n-2)})^{\frac{1}{4}} \\ &\sim \frac{8}{9} 2^{\frac{1}{4}} n^{\frac{3}{2}}. \end{aligned}$$

Recall that $m(n)$ be the minimum number of 1 in $n \times n$ matrix, such that there must exist a 3×3 submatrix of all 1s. We have shown that $m(n) = \Omega(n^{\frac{3}{2}})$. □

Question 5.

5. We are given $m = 2^{n-1}k$ sets, each of size n, in a universe Ω. Consider the following randomized algorithm for a 2-coloring: First color each point $v \in \Omega$ randomly. Now, for each monochromatic set e, select a random vertex $v \in e$. Each such selected v (regardless of how often it was selected) has its color (definitely, no probability here) flipped. Call the algorithm a failure if some set e originally had all or all but one vertex the same color and ended with all vertices that color. Find k as large as you can (as an asymptotic function of n) so that the failure probability is less than one. (Note that this, unfortunately, does not give us any result on $m(n)$ since there are other ways that a set e could end up monochromatic.)

Solution. Note that we use the langauge of $n-$uniform hypergraph, which is an equivalent problem to the problem under consideration. Before proceeding with the main part of the proof, we define an object, called a conflicting pair, and peacebreaker, in a two-coloring scheme.

Definition 5.1. An ordered pair of edges (e, f) is said to be a **conflicting pair** , if e is monochromatic with some color k, and for some $v \in e \cap f$, $f \setminus \{v\}$ is monochromatic with the color not k. We call the vertex v the **peacebreaker** of (e, f).

We now consider the randomized algorithm. Fix (e, f) an ordered pair of edges from the hypergraph. Consider an event, where after the initial coloring, (e, f) is a conflicting pair with a peacebreaker v, and at the second stage chooses the peacebreaker v to flip its color, exactly when it reviews e. We denote such event as $T_{(e,f)}$. We first note that if e and f have a number of common vertices not equal to 1, we have that $P(T_{(e,f)}) = 0$, as no matter what coloring is given to e and f, they cannot be a conflicting pair. Now, for the case where e, f exactly have one vertex in common, the probability of $T_{(e,f)}$ is given by

$$\begin{aligned} P(T_{(e,f)}) &= P(\{(e,f) \text{ is a conflicting pair after the i.c.}\}) \\ &\cdot \ P(\{v \text{ is chosen at the review of } e\} | \{(e,f) \text{ is a conflicting pair after the i.c.}\}) \\ &= 2^{2-2n} n^{-1}, \end{aligned}$$

where i.c. denotes the initial coloring, prior to the second round of flipping colors. Therefore, $P(T_{(e,f)})$ in general can be upper-bounded as follows:

$$P(T_{(e,f)}) \leq 2^{2-2n} n^{-1},$$

with equality being achieved when e and f share exactly one vertex.

Lemma 5.2. Failure implies $\bigcup_{(\mathbf{e},\mathbf{f}) \in \mathbf{E} \times \mathbf{E}} \mathbf{T}_{(\mathbf{e},\mathbf{f})}$. *The algorithm fails, only if there exists a conflicting pair (e, f) with its peacebreaker v chosen at the second stage at the review of e.*

Proof. Observe that any monochromatic edge after the initial coloring cannot cause a failure, because at least one of its vertex will be chosen to flip the color at the second stage. Hence, failure occurs, if only if there exists an edge, having all but one color monochromatic, end up having a monochromatic coloring with the majority color. Now, for any edge, having all but one color monochromatic, will end up having the monochromatic coloring, only if there exists a monochromatic edge containing the one vertex with the opposite color and the vertex is chosen to be fliped at the second stage of the algorithm, as the algorithm reviews the monochromatic edge. This is precisely the identification of a conflicting pair with its peacebreaker. Hence, we have proven the claim. $\square$

By the established lemma, we obtain

$$\{\text{ failure}\} \implies \bigcup_{(e,f)\in E\times E} T_{(e,f)}.$$

As there are m^2 ordered pairs of edges in the hypergraph, by the above equality, and the subadditivity of probability, we obtain

$$\begin{aligned} P(\{\text{ failure }\}) &\leq P(\bigcup_{(e,f)\in E\times E} T_{(e,f)}) \\ &\leq \sum_{(e,f)\in E\times E} P(T_{(e,f)}) \\ &\leq m^2 2^{2-2n} n^{-1}, \end{aligned}$$

where E denotes the edge set. If $m^2 2^{2-2n} n^{-1} < 1$, we can ensure that the failure probability is less than 1. Substituting $2^{n-1}k$ for m, we have

$$(2^{n-1}k)^2 2^{2-2n} n^{-1} < 1,$$

which is equivalent to $k < \sqrt{n}$. Hence, by taking $k = \lfloor\sqrt{n}\rfloor - 1$, we can have a failure probability less than 1. We have shown that $k = \Omega(\sqrt{n})$. One should note that this bound is higher by a factor of $\ln(n)$ compared to the state of the art lower bound, established Kozik et al. As the note mentions, this is not a surprise, since the "failure" defined through this procedure is not inclusive enough to capture all possible ways an edge e could end up monochromatic. $\square$

Question 6.

6. Set $X = \sum_{i=1}^{n} X_i$ where $X_i = \pm 1$ uniformly and independently. Bound $\Pr[X > \frac{n}{2}]$ as follows.

   (a) Find a closed form for $E[e^{\lambda X_i}]$.

   (b) Find a closed form for $E[e^{\lambda X}]$.

   (c) Use the Chernoff Bound $\Pr[X > a] < E[e^{\lambda X}]e^{-\lambda a}$ with $a = \frac{n}{2}$. Use Calculus (this gets a little messy to put in closed form, full points for numerical answers) to select the the optimal λ.

   (d) Compare this with the lower bound

$$\Pr[X \geq \frac{n}{2}] \geq \Pr[X = \frac{n}{2}] = 2^{-n}\binom{n}{\frac{3n}{4}}$$

   showing that the upper and lower bounds have the same main terms.

Solution. (a) By definition of expectation, it follows that

$$E[e^{\lambda x_i}] \quad = \quad \frac{e^{\lambda} + e^{-\lambda}}{2} = \cosh(\lambda).$$

(b) By using the fact that expectation of product of independent random variables is a product of expectations and (b), we have

$$\begin{aligned} E[e^{\lambda X}] &= \prod_{i=1}^{n} E[e^{\lambda X_i}] \\ &= (\cosh(\lambda))^n. \end{aligned}$$

(c) By the use of Chernoff bound, we obtain

$$P(X > \frac{n}{2}) < (\frac{e^{\frac{\lambda}{2}} + e^{-\frac{3\lambda}{2}}}{2})^n.$$

Let $f(\lambda) = e^{\frac{\lambda}{2}} + e^{-\frac{3\lambda}{2}}$. By taking the first order derivative and setting it equal to 0, we obtain $e^{\frac{\lambda}{2}} = 3e^{-\frac{3\lambda}{2}}$. Solving this equality, we get the optimal λ, λ^*, is $\frac{\ln(3)}{2}$. Plugging it back in, we get that $f(\lambda^*) = 3^{\frac{1}{4}+\frac{3}{4}}$. Hence, it follows that

$$P(X > \frac{n}{2}) < (\frac{2}{3^{\frac{3}{4}}})^n,$$

as required.

(d) Using the Stirling's formula, we have

$$\begin{aligned}
P(X \geq \frac{n}{2}) &\leq P(X = \frac{n}{2}) \\
&= 2^{-n}\binom{n}{\frac{3n}{4}} \\
&= 2^{-n}\frac{n!}{(\frac{3n}{4})!(\frac{n}{4})!} \\
&\sim 2^{-n}\frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\sqrt{2\pi\frac{3n}{4}}(\frac{3n}{4e})^{\frac{3n}{4}}\sqrt{2\pi\frac{n}{4}}(\frac{n}{4e})^{\frac{n}{4}}} \\
&= \frac{2\sqrt{2}}{\sqrt{3\pi n}}(\frac{2}{3^{\frac{3}{4}}})^n.
\end{aligned}$$

The main terms are the same for the upper bound and the lower bound as remarked. □