Probabilistic Method:
Problem Set V

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Abstract

This work contains solutions to the problem set V of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Let H have vertices $\{A, B, C, D, E\}$ and be the complete graph on $\{A, B, C, D\}$ and the edges $\{E, A\}$, $\{E, B\}$. For $\alpha > 0$ let $f(n, \alpha)$ denote the probability that $G(n, p)$ does not contain a copy of H when $p = n^{-\alpha}$. Here we will give $f(n, \alpha)$ up to a constant in the exponent.

   (a) For $t = 2, 3, 4, 5$ find the subgraph H_t of H on t vertices with the maximal number of edges and find e_t, the number of edges of H_t.

   (b) Show that H is strictly balanced. What is the threshold function for containing a copy of H? Henceforth, restrict to α so that $p = n^{-\alpha}$ is bigger than that threshold.

   (c) Let LB_t denote the lower bound, from Janson's Inequality, on the probability that $G(n, p)$ does not contain a copy of H_t. Set LB equal the maximum of LB_t, $t = 2, 3, 4, 5$. Find LB as a function of α – there will be three ranges (some graph paper will help!) of α at which different t give the maximum.

   (d) Find μ, Δ of the upper bound of the Extended Janson's Inequality. Show that the Δ breaks into a finite number of ranges depending on which addend predominates.

   (e) Combining the lower and upper bounds above get a result of the form:
   If $0 < \alpha < \kappa_1$ the $f(n, p) = \exp[-\Theta(n^{\gamma_1 + \gamma_2 \alpha})]$.
   If $\kappa_1 < \alpha < \kappa_2$ the $f(n, p) = \exp[-\Theta(n^{\gamma_3 + \gamma_4 \alpha})]$.
   If $\kappa_2 < \alpha < \kappa_3$ the $f(n, p) = \exp[-\Theta(n^{\gamma_5 + \gamma_6 \alpha})]$.
   If $\alpha > \kappa_3$ then $f(n, p) \sim 1$. Here the κs and γs will be nice rational numbers.

Solution.
(a) We have $e_2 = 1$, $e_3 = 3$, $e_4 = 6$, and $e_5 = 8$.

(b) $n^{-\frac{5}{8}}$ is the threshold function as $\frac{1}{2}, 1, \frac{6}{4}$ is less than $\frac{8}{5}$, which also reveals that H is strictly balanced.

(c) We have

$$P(\text{ G has no } K_i) \leq (1 - p^{f_i})^{\binom{n}{i}} \sim \exp(-\Theta(n^i - f_i \alpha)),$$

where $f_i = 1, 3, 6, 8$.

(d)

(e)

Question 2.

2. In $G(n, p)$ with $p = c/n$ let X be the number of *isolated* triangles. Let $\mu = E[X]$. In an earlier assignment you calculated the limiting value of μ.

   (a) For $r \geq 1$ give an *exact* formula for $S^{(r)} = E[\binom{X}{r}]$. (Hint: There is only one "picture" for r isolated triangles!)

[redacted]

   (b) For r and c fixed find the limiting (in n) value of $S^{(r)}$.

   (c) Use Brun's Sieve to deduce the limiting value of $\Pr[X = 0]$.

Solution. **(a)** By counting, we know that there are $\frac{1}{r!}\prod_{i=0}^{r-1}\binom{n-3}{3}$ cases of isolated triangles. Hence, we have the exact formula as

$$E\binom{X}{r}] \quad = \quad \frac{1}{r!}\prod_{i=0}^{r-1}\binom{n-3i}{3}(1-p)^{3r(n-3r)+9\binom{r}{2}}.$$

(b) As $(1-p)^{3r(n-3r)+9\binom{r}{2}} \sim (1-p)^{r(3n-9)}$, we have

$$E[X] \sim \frac{u^r}{r!}.$$

(c) By Brun's Sieve, we have that the limiting value of $P(X = 0)$ is $e^{\frac{c^3 e^{-3c}}{6}}$.

□

Question 3.

3. `The Coupon Collector Problem:` Set $m = n \ln n + cn$ where c is a constant. (Don't worry about integrality.) Let f be a random function from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. Call $j \in \{1, \ldots, n\}$ *missed* if there is no $i \in \{1, \ldots, m\}$ with $f(i) = j$. Let X be the number of missed $j \in \{1, \ldots, n\}$.

   (a) Find $E[X]$ precisely.

   (b) Find the limiting value of $E[X]$.

   (c) For $r \geq 2$ find $E[\binom{X}{r}]$ precisely.

   (d) For fixed $r \geq 2$ and c find the limiting value of $E[\binom{X}{r}]$.

   (e) Apply Brun's Sieve to find the limiting value of $\Pr[X = 0]$.

Solution. (a) For, $1 \leq i \leq n$, let B_i denote the event, where i is missed, and X_i be the indicator random variable for B_i. Let $X = \sum_{i=1}^{n} X_i$. By Linearity of Expectation, it follows that

$$\begin{aligned} E[X] &= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P(B_i) \\ &= n(\frac{n-1}{n})^m = n(\frac{n-1}{n})^{n\ln(n)+cn}. \end{aligned}$$

(b) With an argument involving Taylor series, which was used in class previously, we know that $n(1 - \frac{1}{n})^m \; ne^{-\frac{m}{n}}$. Hence, it follows that $E[X] \sim e^{-c}$.

(c) By counting, we have

$$E[\binom{X}{r}] = \binom{n}{r}(1 - \frac{r}{n})^m.$$

(d) Similar to (b), we have

$$E[\binom{X}{r}] \sim \frac{1}{r!}(ne^{-\frac{m}{n}})^r \sim \frac{\mu^r}{r!}.$$

(e) By Brun's Sieve, we have $P(X = 0) \to e^{-e^{-c}}$.

□

Question 4.

4. In $G \sim G(n,p)$ let X denote the number of isolated edges – i.e., the number of v, w adjacent to each other and no other vertices.

   (a) Find $E[X]$ precisely.

   (b) Give an explicit parameterization $p = f_1(n) + cf_2(n)$ so that $E[X] \to g(c)$ where $g(c)$ will be an explicit continuous function with $\lim_{c\to-\infty} g(c) = 0$ and $\lim_{c\to+\infty} g(c) = +\infty$. (When X is the number of isolated vertices the parametrization $p = \frac{\ln n}{n} + \frac{c}{n}$ was given in class. This is similar, though the answers are not the same.)

   (c) With the above parametrization set $\mu := E[X] \sim g(c)$. Use the Brun's Sieve method to show that X approaches a Poisson Distribution with mean μ.

   (d) Put everything together to make a statement analogous to the isolated vertices statement of the form: If $p =$ blah blah blah then the probability that G has no isolated edges is yadda yadda yadda.

Solution. (a) Let B_i denote the event, where ith edge from the index set of edges I, is an isolated edge for $i = 1, 2..., \binom{n}{2}$. Let X_i be the indicator random variable for B_i, and $X = \sum_{i\in I} X_i$. By linearity of expectation, it follows that

$$E[X] \;=\; \sum_{i\in I} E[X_i] = \binom{n}{2} p(1-p)^{2(n-2)},$$

as $E[X_i] = P(X_i) = p(1-p)^{2(n-2)}$ with $2(n-2)$ edges required not to be present.

(b) Consider the following parametrization:

$$p \;=\; \frac{\ln(n)}{2n} + \frac{\ln(\ln(n))}{2n} + \frac{c}{n}.$$

Then, it follows that

$$E[X] \sim \frac{n^2}{2}\ln(n)2ne^{-\ln(n)-\ln(\ln(n))-c} = \frac{e^{-c}}{4}.$$

(c) By counting, we have

$$\begin{aligned} E[\binom{n}{r}] &= \frac{1}{r!}\prod_{i=0}^{r-1}\binom{n-2i}{2}p^r(1-p)^{2r(n-2)-2r(n-1)} \\ &\sim \binom{n}{2}^r \frac{1}{r!}(p(1-p)^{2(n-2)})^r \sim \frac{\mu^r}{r!} \end{aligned}$$

$S^{(r)} \sim \dfrac{u^r}{r!}$. Hence, X approaches Poisson in the limit.

(d) If $p = \dfrac{\ln(n)}{2n} + \dfrac{\ln(\ln(n)}{2n} + \dfrac{c}{n}$, then the probability that G has no isolated edges is $e^{\frac{-e^{-c}}{4}} + o(1)$.

$\square$