Probabilistic Method:
Problem Set I

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set I of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. (-) Let $X_1, \ldots, X_n$ be independent random variables with $\Pr[X_i = +1] = \Pr[X_i = -1] = \frac{1}{2}$. Set $X = X_1 + \ldots + X_n$. Find $E[X^2]$ precisely. Find $E[X^4]$ precisely. [Idea: Expand and use linearity of expectation.]

Solution.

Question 2.

2. Find an asymptotic formula for

$$\sum_{k=n^{1/2}}^{2n^{1/2}} (n)_k n^{-k}$$

by parametrizing $k = cn^{1/2}$ and turning it into an integral which can be evaluated numerically. (You can leave it in the form of a definite integral if you wish.) (See Asymptopia, Chapter 4)

Solution.

Question 3.

3. Now we go to the complete sum by showing the edge effects are negligible.

   (a) Show
   $$\lim_{\epsilon\to 0}\lim_{n\to\infty} n^{-1/2}\sum_{k=1}^{\epsilon n^{1/2}}(n)_k n^{-k}=0$$
   by using an appropriate upper bound for the addends.

   (b) (*) Show
   $$\lim_{K\to\infty}\lim_{n\to\infty} n^{-1/2}\sum_{k=Kn^{1/2}}^{n}(n)_k n^{-k}=0$$
   by using an appropriate upper bound for the addends.

   (c) Find an asymptotic formula for
   $$\sum_{k=1}^{n}(n)_k n^{-k}$$
   by splitting it into the ranges $k<\epsilon n^{1/2}$, $\epsilon n^{1/2}\le k\le Kn^{1/2}$ and $Kn^{1/2}<k\le n$ and then taking appropriate limits. (You may assume the previous parts.)

Solution.

Question 4.

4. Prove, for $m = m(n)$ as large as you can, the existence of an $n \times n$ matrix A of zeroes and ones with m ones which does not contain a 3×3 submatrix of all ones. Use the alteration method: make each entry one with probability p and then for each such submatrix change a one to zero. When you optimize [using Calculus!] your final answer should be of the form $m \sim an^b$ for some reasonable a, b.

Solution.

Question 5.

5. We are given $m = 2^{n-1}k$ sets, each of size n, in a universe Ω. Consider the following randomized algorithm for a 2-coloring: First color each point $v \in \Omega$ randomly. Now, for each monochromatic set e, select a random vertex $v \in e$. Each such selected v (regardless of how often it was selected) has its color (definitely, no probability here) flipped. Call the algorithm a failure if some set e originally had all or all but one vertex the same color and ended with all vertices that color. Find k as large as you can (as an asymptotic function of n) so that the failure probability is less than one. (Note that this, unfortunately, does not give us any result on $m(n)$ since there are other ways that a set e could end up monochromatic.)

Solution.

Question 6.

6. Set $X = \sum_{i=1}^{n} X_i$ where $X_i = \pm 1$ uniformly and independently. Bound $\Pr[X > \frac{n}{2}]$ as follows.

   (a) Find a closed form for $E[e^{\lambda X_i}]$.

   (b) Find a closed form for $E[e^{\lambda X}]$.

   (c) Use the Chernoff Bound $\Pr[X > a] < E[e^{\lambda X}]e^{-\lambda a}$ with $a = \frac{n}{2}$. Use Calculus (this gets a little messy to put in closed form, full points for numerical answers) to select the the optimal λ.

   (d) Compare this with the lower bound

$$\Pr[X \geq \frac{n}{2}] \geq \Pr[X = \frac{n}{2}] = 2^{-n}\binom{n}{\frac{3n}{4}}$$

   showing that the upper and lower bounds have the same main terms.

Solution.