Probabilistic Method:
Problem Set IV

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Abstract

This work contains solutions to the problem set IV of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. Here is a problem from work done by Roberto Oliveira, who received his Ph.D. under my supervision some years ago and is now at IMPA in his home town (lucky guy!) of Rio de Janiero. The exponential distribution Exp(a) is the positive distribution with density function $f(t) = ae^{-at}$. We let X_a denote this distribution. Calculus exercise: $E[X_a] = a^{-1}$. Now set $X = \sum_{i=1}^{\infty} X_{i^2}$ where the X_{i^2} are assumed to be mutually independent. (This is finite as $E[X] = \sum i^{-2}$ converges.) Our object will be to use Chernoff bounds to get an asymptotic upper bound for $\Pr[X < \epsilon]$ as $\epsilon \to 0^+$. (This is the kind of problem where it is very easy to be "too precise" and then get totally lost. Please follow the bounds indicated parenthetically below. You should still get a rigorous asymptotic upper bound – it turns out that the bound achieved is quite good.)

   (a) Find $E[e^{-\lambda X_a}]$ in closed form. (Do this precisely.)

   (b) Find $E[e^{-\lambda X}]$ as an infinite product. (Do this precisely.)

   (c) Setting $\lambda = K^2$ with K a positive integer bound $E[e^{-\lambda X}]$ from above by using the first K terms of the infinite product. (You should have a numerator and denominator product. For the numerator use Stirling's Formula. For the denominator observe that all the terms are between K^2 and $2K^2$ and lower bound the denominator product by taking each term as K^2.)

   (d) Now you will apply the Chernoff bound ($\lambda > 0$) $\Pr[X \leq \epsilon] \leq E[e^{-\lambda X}]e^{\lambda\epsilon}$. Using the upper bound just found find a value of K (as a function of ϵ) that yields a "good" upper bound on $\Pr[X < \epsilon]$. (Ignore the requirement that K needs to be integral. Also, in finding a good value of K look at just the main terms from Stirling (i.e., not the square root term) and find that K that does best there. Your final answer should be something like $e^{-1/\epsilon}$. Good luck!)

Solution.
(1) By elementary calculus, it follows that

$$\begin{aligned} E[e^{-\lambda X_i}] &= \int_0^\infty a e^{-(\lambda+a)t} dt \\ &= [-\frac{a}{\lambda+a} e^{-(\lambda+a)t}]_0^\infty = \frac{a}{\lambda+a}. \end{aligned}$$

(2) By (1) and mutual independence, it follows that

$$\begin{aligned} E[e^{-\lambda X}] &= E[e^{-\lambda \sum_{i=1}^\infty X_{i^2}}] = \prod_{i=1}^\infty E[e^{-\lambda X_{i^2}}] \\ &= \prod_{i=1}^\infty \frac{i^2}{\lambda^2 + i^2}. \end{aligned}$$

(3) Following the hint and using the Stirling's formula, we obtain

$$\begin{aligned} E[e^{-K^2 X}] &= \prod_{i=1}^\infty \frac{i^2}{K^2 + i^2} \\ &\leq \prod_{i=1}^K \frac{i^2}{K^2 + i^2} \leq \frac{(k!)^2}{(k^2)^k} \\ &\sim \frac{2\pi k}{e^{2k}} \end{aligned}$$

(4) Applying the Chernoff bound gives

$$P(X \geq \epsilon) \quad \leq \quad (2\pi k) e^{\epsilon k^2 - 2k}(1 + o(1)).$$

Since the quadratic is convex, we know the tightest bound is achieved at its minima, obtained by solving for $2\epsilon k - 2 = 0$. Therefore, we have that the tightest bound is achieved with $k = \frac{1}{\epsilon}$, which yields

$$P(X \geq \epsilon) \quad \leq \quad \frac{2\pi}{\epsilon} e^{-\frac{1}{\epsilon}}(1 + o(1)).$$

$\square$

Question 2.

2. Set $\Omega = [n] \times [n]$. Define a random set $C \subset \Omega$ by

$$\Pr[(x,y) \in C] = p = \frac{c}{n}$$

the events $(x,y) \in C$ mutually independent. A horozontal bond is a pair $(x,y),(x+1,y) \in C$ and a vertical bond is a pair $(x,y),(x,y+1) \in C$. Find the expected number of bonds. Use Janson's Inequality to bound the probability there are no bonds in both directions and find the limiting probability as $n \to \infty$.

Solution. Let A_i be the subset of Ω, where I is the index set of all adjacent vertex pairing in the grid. Let B_i be the event where $A_i \subset C$, X_i be the indicator random variable of B_i, and $X = \sum_{i \in I} X_i$, which counts the number of bonds in a given chosen subset of Ω. Counting the number of edges in the grid, it follows that

$$\begin{aligned} \mu &= \sum_{i \in I} E[X_i] = \sum_{i \in I} P(B_i) = 2n(n-1)p^2 = 2n(n-1)\frac{c^2}{n^2} \\ &\sim 2c^2. \end{aligned}$$

Now, we write $i \sim j$, if $|A_i \cap A_j| = 1$. We have that

$$\triangle = \sum_{i \sim j} P(B_i \wedge B_j).$$

Observe that $|A_i \cap A_j| = 1$ can happen in two ways: straight line or bent. The straight line corresponds to having three consecutively chosen nodes, and the bent line means one horizontal and one vertical bond. Observe that the bend ones can be bijected to each square excluding 1 node, and there are $n-2$ consecutive three nodes in each line. Therefore, we have that

$$\begin{aligned} \triangle &= \sum_{i \sim j} P(B_i \wedge B_j) = (2n(n-2) + 2(n-1)^2)p^3 \\ &= (2n(n-2) + 2(n-1)^2)\frac{c^3}{n} = o(1). \end{aligned}$$

Therefore, we have shown that $\mu \to 2c^2$ as $n \to \infty$ and $\triangle = o(1)$. Hence, by the Janson inequality(to be precise, a corollary of the inequality, which gives the identified case as a sufficient condition for the asserted convergence), we obtain that

$$P(\bigwedge_{i \in I} \overline{B_i}) \to e^{-2c^2},$$

as $n \to \infty$ where $\bigwedge_{i \in I} \overline{B_i}$ is the event where there is no bond in both directions. $\square$

Question 3.

5. Let $G \sim G(n,p)$ with $p = cn^{-2/3}$ and let v, w be two distinct fixed vertices of G. Use Janson's Inequality to find the limiting probability that v, w are *not* joined by a path of length three.

Solution. Fix n and fix $\{v,w\}$. Let A_i be the subset of the edges of the complete graph, where I is the index set of all paths of length three, joining v and w. Let B_i be the event where the random graph has all the edges of A_i, X_i be a indicator random variable of B_i, and $X = \sum_{i\in I} X_i$, which counts the number of paths of length three, joining v and w. The cardinality of I is $2\binom{n-2}{2}$, which can be seen by choosing an edge, not sharing a vertex with $\{v,w\}$ and choosing one of two orientations that the edge can connect to $\{v,w\}$. Therefore, it follows that

$$\begin{aligned}\mu &= \sum_{i\in I} E[X_i] = \sum_{i\in I} P(B_i) = 2\binom{n-2}{2}p^3 = 2\binom{n-2}{2}\frac{c^3}{n^2} \\ &\sim c^3.\end{aligned}$$

Now, we write $i \sim j$, if $i \neq j$ and $A_i \cap A_j \neq \emptyset$. Observe that as A_is are the edge sets that form a path of length three, joining u and v, we have the following equivalence:

$$i \neq j \text{ and } A_i \cap A_j = \emptyset \iff |A_i \cap A_j| = 1,$$

since $|A_i \cap A_j| = 2$ implies $A_i = A_j$. We now have that

$$\begin{aligned}\triangle &= \sum_{i\sim j} P(B_i \wedge B_j) = \sum_{|A_i\cap A_j|=1} P(B_i \wedge B_j) \\ &= 2\binom{n-2}{2}(1 + 2\binom{n-4}{1})p^5 \\ &\sim kn^{-\frac{1}{3}},\end{aligned}$$

where k is some constant, parametrized by c, as the cardinality of the ordered pair (i,j) satisfying $|A_i \cap A_j| = 1$, used above, can be seen by picking a point out side of the four vertices that that form A_i path, and choosing one out of two possible orientations or using four vertices and choosing the opposite orientation from itself. The above asymptotic immediately gives $\triangle = o(1)$. Therefore, we have shown that $\mu \to c^3$ as $n \to \infty$ and $\triangle = o(1)$. Hence, by Janson's inequality (the above condition is given as sufficient condition for the convergence below in the chapter), we obtain that

$$P(\bigwedge_{i\in I} \overline{B_i}) \to e^{-c^3},$$

as $n \to \infty$ where $\bigwedge_{i\in I} \overline{B_i}$ is the event where there is no path of length 3, joining v and w. $\square$