Probabilistic Method:
Problem Set I

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Abstract

This work contains solutions to the problem set I of Probabilistic Method 2016 at Courant Institute of Mathematical Sciences.

Question 1.

1. (-) Let $X_1, \ldots, X_n$ be independent random variables with $\Pr[X_i = +1] = \Pr[X_i = -1] = \frac{1}{2}$. Set $X = X_1 + \ldots + X_n$. Find $E[X^2]$ precisely. Find $E[X^4]$ precisely. [Idea: Expand and use linearity of expectation.]

Solution.

Question 2.

2. Find an asymptotic formula for

$$\sum_{k=n^{1/2}}^{2n^{1/2}} (n)_k n^{-k}$$

by parametrizing $k = cn^{1/2}$ and turning it into an integral which can be evaluated numerically. (You can leave it in the form of a definite integral if you wish.) (See Asymptopia, Chapter 4)

Solution.

Question 3.

3. Now we go to the complete sum by showing the edge effects are negligible.

   (a) Show

   $$\lim_{\epsilon\to 0}\lim_{n\to\infty} n^{-1/2}\sum_{k=1}^{\epsilon n^{1/2}} (n)_k n^{-k} = 0$$

   by using an appropriate upper bound for the addends.

   (b) (*) Show

   $$\lim_{K\to\infty}\lim_{n\to\infty} n^{-1/2}\sum_{k=Kn^{1/2}}^{n} (n)_k n^{-k} = 0$$

   by using an appropriate upper bound for the addends.

   (c) Find an asymptotic formula for

   $$\sum_{k=1}^{n} (n)_k n^{-k}$$

   by splitting it into the ranges $k < \epsilon n^{1/2}$, $\epsilon n^{1/2} \le k \le K n^{1/2}$ and $K n^{1/2} < k \le n$ and then taking appropriate limits. (You may assume the previous parts.)

Solution.

Question 4.

4. The expected number of isolated trees [just take this as a fact] on k vertices in $G(n,p)$ is given by $f(n,k,p) := \binom{n}{k}k^{k-2}p^{k-1}(1-p)^B$ with $B = k(n-k)+\binom{k}{2}-k+1$. Set $p=\frac{1}{n}$. Let c be a positive constant. Find (see Asymptopia, Chapter 5 or webnotes) for $\binom{n}{k}$) the asymptotics of $f(n,k,p)$ when $k \sim cn^{2/3}$. (*) Express the limit as $n \to \infty$ of the sum of $f(n,k,p)$ for $n^{2/3} \le k < 2n^{2/3}$ as a definite integral and use a computer package to evaluate the integral numerically.

Solution. First of all the problem can be found in the book, Asymptopia. We note that $k \sim cn^{\frac{2}{3}}$, thus $k = o(n^{\frac{3}{4}})$. Then, by the case 4 of the result in 5.1 Asymtopia, with Stirling's formula, we have

$$\begin{aligned}\binom{n}{k} &\sim e^{-\frac{k^2}{2n}}e^{-\frac{k^3}{6n^2}}\frac{n^k}{k!}\\ &\sim e^{-\frac{k^2}{2n}}e^{-\frac{k^3}{6n^2}}n^k\frac{e^k}{n^k\sqrt{2\pi k}}\end{aligned}$$

Now, by direct computation, it follows that

$$\begin{aligned}B &= k(n-k)+\binom{k}{2}-k+1\\ &= kn-\frac{1}{2}k^2-\frac{3}{2}+1\\ &= kn-\frac{1}{2}k^2+O(k),\end{aligned}$$

which then yields

$$ln[(1-p)^{k(n-k)+\binom{k}{2}-(k-1)}] = -k+\frac{k^2}{2n}+o(1).$$

Now, substituting the above into the first asyptomtic equivalence we have established, we have

$$f(n,k,p) \sim e^{-\frac{c^3}{6}}n^{-\frac{2}{3}}c^{-\frac{5}{2}}(2\pi)^{-\frac{1}{2}},$$

as required. $\square$

Question 5.

5. (-) Consider Boolean expressions on atoms $x_1, \ldots, x_n$. By a k-clause C we mean an expression of the form $y_{i_1} \vee \ldots \vee y_{i_k}$ where each y_{i_j} is either x_{i_j} or $\overline{x}_{i_j}$. Prove a theorem of the following form [you fill in the $m = m(k)$] by the probabilistic method: For any m k-clauses

$C_1, \ldots, C_m$ there is a truth assignment such that $C_1 \wedge \ldots \wedge C_m$ is satisfied.

Solution. We claim the following: Suppose $m < 2^k$. Then, there exists a truth assignment such that $\bigvee_{i=1}^{m} C_i$ is satisfied.

We consider a random truth assignment of atoms, $\{x_1, ...x_n\}$. Formally, we consider a finite sample space of all possible truth assignment of atoms, associated with a uniform probability, which assigns each outcome in the space with $\frac{1}{2^n}$ probability. By the subadditivity of probability, we have

$$\begin{aligned} P(\bigcup_{i=1}^{m}\{C_i \text{ is not satisfied }\}) &\leq \sum_{i=1}^{m} P(\{C_i \text{ is not satisfied }\}) \\ &\leq m \cdot 2^{-k}, \end{aligned}$$

as there is only one assignment, which assigns all false values to k variables, that makes C_i clause not satisfied. Since $m < 2^k$, it follows that

$$m \cdot 2^{-k} < 2^k 2^{-k} = 1,$$

which primarily grants us

$$P(\bigcup_{i=1}^{m}\{C_i \text{ is not satisfied }\}) < 1.$$

By DeMorgan's laws, we see that

$$
\begin{aligned}
(\bigcup_{i=1}^{m}\{C_i \text{ is not satisfied }\})^c &= \bigcap_{i=1}^{m}\{C_i \text{ is not satisfied }\}^c \\
&= \bigcap_{i=1}^{m}\{C_i \text{ is satisfied }\} = \{\bigvee C_i \text{ is satisfied }\}.
\end{aligned}
$$

Since the sum of an event and its complement event is 1 by one of the axioms of probability, we have shown that

$$
P(\{\bigvee_{i=1}^{m} C_i \text{ is satisfied }\}) \quad > \quad 0.
$$

Hence, we have shown that there is a truth assignment such that $\bigvee_{i=1}^{m} C_i$ is satisfied, when $m < 2^k$.

$\square$