Putnam Compendium

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Abstract

The work contains the solutions to Putnam problems.

1 Solutions

Question 2012 A-1.

Solution. Assume without loss of generality that $d_1 \leq d_2 \leq ... \leq d_n$. Suppose for sake of contradiction that there does not exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle. By the property of an acute traingle we have (PATCH)

$$d_{i+2}^2 \geq d_{i+1}^2 + d_i^2 \tag{1}$$

for all i such that $1 \leq i \leq 10$. We first claim that

$$d_i^2 \geq F_i d_1^2$$

holds for all $1 \leq i \leq 12$ where F_i is a ith Fibonacci number with $F_1 = 1$ and $F_2 = 1$. We proceed to prove the claim by strong induction. Base case of the induction trivially holds as $F_1 = 1$ yielding

$$d_1^2 \geq d_1^2.$$

Now assume that the statement holds true from 1 to i. From (1) we obtain

$$d_{i+1}^2 \geq d_i^2 + d_{i-1}^2.$$

With the inductive hypothesis we can lower bound the RHS as

$$d_i^2 + d_{i-1}^2 \geq F_i d_1^2 + F_{i-1} d_1^2. \tag{2}$$

Factoring and substituting the Fibonacci recurrence to RHS of (2) we obtain

$$d_i^2 + d_{i-1}^2 \geq F_{i+1} d_1^2.$$

Hence, we finally get that

$$d_{i+1}^2 \geq F_{i+1} d_i^2$$

which completes the induction. Hence the claim $d_i^2 \leq d_1^2$ holds true for all i such that $1 \leq i \leq 12$. Notice that for $i = 12$ case we have $F_1 2 = 144$ yielding

$$d_{i+1}^2 \geq 144 d_1^2.$$

As the numbers are chosen from the open interval $(1, 12)$ we have that d_{i+1}^2 is strictly less than 144, but $144 d_1^2$ is strictly greater than 144, which is a contradiction. Therefore, we have shown given any $d_1, d_2, ..., d_{12}$ chosen from $(1, 11)$, there exists three distinct indices i, j, k such that d_i, d_j, d_k form side lenghts of an acute traingle. $\square$

Remark.

Question 2012 A-2.

Solution. Assume that for every x and y in S there exists z in S such that $x * z = y$. Assume that a, b, c are in S and $a * c = b * c$ holds. Let d be the element such that $d = a * c = b * c$ holds. From the assumption we can deduce that there exist e and f in S such that

$$\begin{aligned} d * e &= a \\ d * f &= b \end{aligned}$$

Question 2008 A-1.

Solution. We claim that $g(x) = f(x, 0)$ satisfies the given properties. Substituting $(x, y, z) = (0, 0, 0)$ into the functional equations yields

$$f(0,0) + f(0,0) + f(0,0) = 3f(0,0) = 0$$

which gives that $f(0,0) = 0$. Substituting $(x, y, z) = (0, 0, 0)$ we obtain

$$f(x,0) + f(0,0) + f(0,x) = 0;$$

hence, $f(x, 0) = -f(0, x)$. Substituing $(x, y, z) = (x, y, 0)$ gives

$$f(x,y) + f(y,0) + f(0,x) = 0.$$

Rearranging and substituting $f(x, 0) = -f(0, x)$ results in

$$f(x,y) = f(x,0) - f(y,0)$$

Let $g(x) = f(x, 0)$ which is a well-defined function $g : \mathbb{R} \to \mathbb{R}$, we see that

$$f(x,y) = g(x) - g(y)$$

as desired. Hence, we have shown that there exists a function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = f(x, 0)$ in particular, that $f(x, y) = g(x) - g(y)$ is satisfied.

Question 2010 A-1.

Solution. We claim that $k = \lceil \frac{n}{2} \rceil$. For n odd, the lower bound can be acheived by the partition

$$\{n\}, \{1, n-1\}, \{2, n-2\}, ...$$

Since $n - 1$ is even, the partition is well defined. Now for the case where

$$\{n, 1\}, \{n-1, 2\}, \{n-2, 3\}, ...$$

Now we show that such partition strategy is indeed optimal for all n. Let us denote the sum of each box of the optmal strategy as S. Since the number n itself belongs to one of the boxes we obtain a lower bound, $S \geq n$. As the partition strategy for n odd case yields $S = n$ and n is indeed a lower bound on S, we have shown that the strategy is optiaml for n odd case. For n even we show that S has to be lower bounded by $n + 1$. Suppose that $S = n$. A paritition that can acheived S must have the following configuration

Question 2010 B-1.

Solution. Suppose that such sequence exists. Suppose that $0 \leq a_i^2 \leq 1$ for i. Then we get that

$$\sum_{i=1}^{\infty} a_i^4 \leq \sum_{i=1}^{\infty} a_i^2.$$

Since $\sum_{i=1}^{\infty} a_i^k = k$, by substitution we get $4 \leq 2$, which is a contradiction. Hence, there exists an index k such that $a_k^2 > 1$. Now, for large enough m, we have $a_k^{2m} > 2m$, which is a contradicts the condition of the sequence. Therefore, there does not exist an infinite sequence of real numbers such that

$$a_1^m + a_2^m ... = m$$

for every positive integer m.

Question 2001 A-1.

Solution. Consider a set S and a binary operation $*$ and S is closed under $*$. Assume that $(a*b)*a = b$ for all $a, b \in S$. Substituting $(b*a)$ into the assumption, we obtain

$$((b*a)*b)*(b*a) = b.$$

From $(b*a)*b = a$, we can re-write the above equality as

$$a*(b*a) = b,$$

as desired.

Question 2001 A-2.

Solution. Let us denote the probability of having odd number of heads after i tosses as P_i. Then, for $i \geq 2$ we can express P_i as

$$P_i = (1 - C_i)P_{i-1} + C_i(1 - P_{i-1}),$$

where C_i is a probability of flipping a head for the ith coin. Simplifying the above equality yields

$$P_i = (1 - 2C_i)P_{i-1} + C_i.$$

Substituting $C_i = \dfrac{1}{2i+1}$ into the last equality, we obtain

$$P_i = P_{i-1} + \frac{1}{2i+1} - 2$$