Real Variables:
Problem Set VI

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Abstract

This work contains solutions to the problem set VI of Real Variables 2015 at NYU.

1 Solutions

Question 9.10.

Solution. Let $\{X_n, \rho_n\}_{n=1}^{\infty}$ be a countable collection of metric spaces. We now define $(\prod_{n=1}^{\infty} X_n, p_*) = (X, p_*)$ such that for $x, y \in X$,

$$p_*(x,y) \quad = \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(x_n, y_n)}{1 + p_n(x_n, y_n)}.$$

First, we can show that p_* is well-defined via comparison test with the series $\sum_{n=1}^{\infty} \dfrac{1}{2^n}$, as $0 \leq \dfrac{p_n(x_n, y_n)}{1 + p_n(x_n, y_n)} \leq 1$ for all n.

As $p_n(x_n, y_n) \geq 0$ for all n, we have $p_*(x, y) \geq 0$ for all $x, y \in X$. If $p_*(x, y) = 0$, then $p_n(x_n, y_n) = 0$ for all n. As each p_n is a metric space $x_n = y_n$ for all n. Therefore, $x = y$. If $x = y$, then $x_n = y_n$ for all n. As each p_n is a metirc space, $p_n(x_n, y_n) = 0$ for all n. Therefore, $p_*(x, y) = 0$.

Since $p_n(x_n, y_n) = p_n(y_n, x_n)$ for all n, for $x, y \in X$, we

$$\begin{aligned} p_*(x,y) \quad &= \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(x_n, y_n)}{1 + p_n(x_n, y_n)} \\ &= \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(y_n, x_n)}{1 + p_n(y_n, x_n)} \\ &= \quad p_*(y, x). \end{aligned}$$

Let $x, y, z \in X$. By the problem 6 and the triangle inequality of each metric space X_n, we have

$$\frac{p_n(x_n, z_n)}{1 + p_n(x_n, z_n)} \leq \frac{p_n(x_n, y_n)}{1 + p_n(x_n, y_n)} + \frac{p_n(y_n, z_n)}{1 + p_n(y_n, z_n)}$$

Question 9.20.

Solution.

Question 9.32.
Solution.