Real Variables: Problem Set IV

Youngduck Choi
Courant Institute of Mathematical Sciences
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set IV of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 4.31.

Solution. Let f be a measurable function on E, which can be expressed as $f = g + h$ on E, where g is finite and integrable over E and h is nonnegative. Let $f = g_1 + h_1$ and $f = g_2 + h_2$ satisfying the given properties of g and h respectively. We wish to show that

$$\int_E g_1 + \int_E h_1 = \int_E g_2 + \int_E h_2.$$

Assume that h_1 and h_2 are integrable. Then, by the linearity of integration, we have

$$\begin{aligned}\int_E g_1 + \int_E h_1 &= \int_E g_1 + h_1 \\ &= \int_E f \\ &= \int_E g_2 + h_2 \\ &= \int_E g_2 + \int_E h_2,\end{aligned}$$

as desired. Now, consider the remaining case of at least one of h not being integrable. Without loss of generality, assume that $\int_E h_1 = \infty$. Since $g_1 + h_1 = g_2 + h_2$, we have

$$\begin{aligned}h_2 &= h_1 + g_1 - g_2 \\ &= h_1 + (g_1 - g_2)^+ - (g_1 - g_2)^- \\ &\geq h_1 - (g_1 - g_2)^-,\end{aligned}$$

with $(g_1 - g_2)^+$ and $(g_1 - g_2)^-$ being properly defined by the finiteness assumption on the gs. Since h_2, h_1 and $(g_1 - g_2)^-$ are all non-negative measurable functions, by the monotonicity and linearity of integration of non-negative measurable functions, we have

$$\begin{aligned}\int_E h_2 &\geq \int_E h_1 - (g_1 - g_2)^- \\ &= \int_E h_1 - \int_E (g_1 - g_2)^-.\end{aligned}$$

From the linearity of general integrable functions, we have that $g_1 - g_2$ is integrable. Consequently, $(g_1 - g_2)^-$ is integrable as well. It follows that

$$\begin{aligned} \left| \int_E (g_1 - g_2)^- \right| &\leq \int_E |(g_1 - g_2)^-| \\ &< \infty. \end{aligned}$$

Therefore, we obtain that

$$\int_E h_1 - \int_E (g_1 - g_2)^- = \infty,$$

which combined with the established inequality of $\int_E h_2 \geq \int_E h_1 - \int_E (g_1 - g_2)^-$ yields

$$\int_E h_2 = \infty.$$

Hence, we have

$$\begin{aligned} \int_E g_1 + \int_E h_1 &= \infty \\ &= \int_E g_2 + \int_E h_2, \end{aligned}$$

as g_1 and g_2 are integrable. This completes the proof. $\square$

Question 2. Royden 4.44.

Solution. Let f be integrable over $\mathbb{R}$ and $\epsilon > 0$.

(i) First, we prove the given property for f nonnegative. Assume $f \geq 0$. Since f is integrable, thus measuralbe, by the Simple Approximation Theorem, there exists a sequence of increasing simple functions $\{\phi_n\}$ on $\mathbb{R}$ which converges pointwise on $\mathbb{R}$ to f, such that

$$|\phi_n| \leq |f| \text{ on } \mathbb{R},$$

for all n. Now, define a new sequence of simple function by

$$\psi_n = max\{0, \phi_n\} \cdot \chi_{[-n,n]}.$$

Observe that $\{\psi_n\}$ is an increasing sequence of simple functions on $\mathbb{R}$, which has finite support and is non-negative, that converges to f pointwise. By the Monotone convergnece theorem, there exists N such that

$$|\int_{\mathbb{R}} f - \int_{\mathbb{R}} \psi_n| < \epsilon,$$

for $n \geq N$. By the linearity of integration and the fact that $\psi_n \leq f$ for all n, we have

$$\begin{aligned} \epsilon &> |\int_{\mathbb{R}} f - \int_{\mathbb{R}} \psi_n| \\ &= |\int_{\mathbb{R}} f - \psi_n| \\ &= \int_{\mathbb{R}} |f - \psi_n|, \end{aligned}$$

for $n \geq N$. Therefore, we have found a function with the desired property, namely ψ_n.

Now, we lift the non-negativity constraint. By the definition of integral, we have

$$\int_{\mathbb{R}} f = \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-.$$

Since f is integrable, f^+ and f^- are integrable and from the previous result, we have simple functions ψ^+ and ψ^- with finite support such that

$$\int_{\mathbb{R}} |f^+ - \psi^+| < \frac{\epsilon}{2}$$
$$\int_{\mathbb{R}} |f^- - \psi^-| < \frac{\epsilon}{2}.$$

Observe that $\psi^+ - \psi^-$ is simple and has finite support as well. Now, by the triangle inequality and monotonicity of integration, it follows that

$$\begin{aligned}\int_{\mathbb{R}} |f - (\psi^+ - \psi^-)| &= \int_{\mathbb{R}} |f^+ - f^- - \psi^+ + \psi^-| \\ &\leq \int_{\mathbb{R}} |f^+ - \psi^+| + |f^- - \psi^-| \\ &= \int_{\mathbb{R}} |f^+ - \psi^+| + \int_{\mathbb{R}} |f^- - \psi^-| \\ &< \epsilon.\end{aligned}$$

Therefore, $\psi^+ - \psi^-$ is the construction of the function with the desired property. We have shown that there is a simple function η on $\mathbb{R}$ which has a finite support and $\int_R |f - \eta| < \epsilon$.

(ii) From the result of (i), there exists a simple function η on $\mathbb{R}$ with finite support such that $|f - \eta| < \epsilon$.

(iii)

Question 2. Royden 4.47.

Solution. Let g be integrable over $\mathbb{R}$.
(i) Let $k \in \mathbb{R}$. Assume that g non-negative. Let $E_n = [-n, n]$ and $E_n - k = [-n-k, n-k]$. By the definition of integration of non-negative functions, it follows that

$$\begin{aligned}\int_{E_n} g(x)dx &= sup\{\int_{E_n} h(x)dx \mid h \text{ bounded, measurable, of finite support and} \\ &\quad 0 \leq h(x) \leq g(x) \text{ for } x \in E_n\} \\ &= sup\{\int_{E_n - k} h(x+k)dx \mid h \text{ bounded, measurable, of finite support and} \\ &\quad 0 \leq h(x+k) \leq g(x+k) \text{ for } x \in E_n - k\} \\ &= \int_{E_n - k} g(x+k)dx.\end{aligned}$$

Since the general integral is defined as the sum of non-negative integrals, for integrable functions, the result trivially generalizes to an integrable function. From this point on, we drop the non-negativity assumption on g and assume that g is integrable. Notice that $\{E_n\}$ and $\{E_n - k\}$ form ascending countable collection of measurable subsets of $\mathbb{R}$, with $\cup_{n=1}^{\infty} E_n = \cup_{n=1}^{\infty} E_n - k = \mathbb{R}$. Hence, by the continuity of integration, we obtain

$$\begin{aligned}\int_{\mathbb{R}} g(x)dx &= \lim_{n\to\infty} \int_{E_n} g(x)dx \\ \int_{\mathbb{R}} g(x+k)dx &= \lim_{n\to\infty} \int_{E_n - k} g(x+k)dx.\end{aligned}$$

Since $\int_{E_n} g(x)dx = \int_{E_n - k} g(x+k)dx$ for all n, it follow that

$$\int_{\mathbb{R}} g(x)dx = \int_{\mathbb{R}} g(x+k)dx,$$

as desired. $\square$

(ii) Fix $\epsilon > 0$. Since g is a bounded function, there exists M such that $g \leq M$. Assume that f is a continous function that vanishes outside of a bounded set. Let I be a sufficiently large closed interval of the form $[-N, N]$ such that the bounded set is contained in I. Then, as f is continuous on a closed, bounded interval I, it is uniformly continous on I. Hence, there exists $\delta > 0$, such that

$$|t| < \delta \implies |f(x) - f(x+t)| \frac{\epsilon}{Mm(I)}$$

It follows that

$$\begin{aligned}
\lim_{t\to 0} \int_{\mathbb{R}} |g(x)[f(x) - f(x+t)]| &= \lim_{t\to 0} |\int_I g(x)[f(x) - f(x+t)]| \\
&\leq \lim_{t\to 0} \int_I |g(x)[f(x) - f(x+t)]| \\
&= \lim_{t\to 0} \int_I |g(x)||[f(x) - f(x+t)]| \\
&\leq \lim_{t\to 0} \int_I |g(x)| \frac{\epsilon}{Mm(I)} \\
&\leq m(I) M \frac{\epsilon}{Mm(I)} \\
&= \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we have that

$$\lim_{t\to 0} \int_{\mathbb{R}} g(x)[f(x) - f(x+t)] = 0$$

Now, assume that f is not continous and vanishes outside of a boudned set. As f is integrable over $\mathbb{R}$, from the approximation property (iii) of 4.44, there exists a continous function h on $\mathbb{R}$ such that it vanishes outside a bounded set and $\int_{\mathbb{R}} |f - h| < \dfrac{\epsilon}{2M}$. As we have proven the result for a continous function that vanishes outside of a bounded set, we obtain

$$\lim_{t\to 0} \int_{\mathbb{R}} g(x)[h(x) - h(x+t)] = 0.$$

With the above limit being 0, it follows that

$$\begin{aligned}
|\lim_{t\to 0} \int_{\mathbb{R}} g(x)[f(x) - f(x+t)]| &= |\lim_{t\to 0} \int_{\mathbb{R}} g(x)[f(x) - f(x+t)] - 0| \\
&= |\lim_{t\to 0} \int_{\mathbb{R}} g(x)[f(x) - f(x+t)] - \lim_{t\to 0} \int_{\mathbb{R}} g(x)[h(x) - h(x+t)]|,
\end{aligned}$$

provided that the limit exists. The above equality can be simplified via linearity of integration as follows:

$$\begin{aligned}
|\lim_{t\to 0} \int_{\mathbb{R}} g(x)[f(x) - f(x+t)]| &= |\lim_{t\to 0} \int_{\mathbb{R}} g(x)[f(x) - f(x+t)] - \lim_{t\to 0} \int_{\mathbb{R}} g(x)[h(x) - h(x+t)]| \\
&= |\lim_{t\to 0} \int_{\mathbb{R}} g(x)[f(x) - f(x+t) - h(x) + h(x+t)]| \\
&\leq \lim_{t\to 0} \int_{\mathbb{R}} |g(x)[f(x) - h(x)] + g(x)[f(x+t) - h(x+t)]| \\
&\leq \lim_{t\to 0} \int_{\mathbb{R}} |g(x)||[f(x) - h(x)]| + |g(x)||[f(x+t) - h(x+t)]| \\
&\leq \lim_{t\to 0} \int_{\mathbb{R}} M \int_{\mathbb{R}} |f(x) - h(x)| + M \int_{\mathbb{R}} |f(x+t) - h(x+t)| \\
&\leq 2M \frac{\epsilon}{2M} = \epsilon
\end{aligned}$$

Since ϵ was arbitrary, we have

$$\lim_{t \to 0} \int_{\mathbb{R}} g(x)[f(x) - f(x+t)] \quad = \quad 0,$$

as desired. $\square$

Question 4. Royden 4.52.

Solution. (a) Consider the following family of functions:

$$\mathscr{F} = \{n\chi_{[0,\frac{1}{n}]}\}_{n=1}^{\infty}.$$

Observe that for each n, $n\chi_{[0,\frac{1}{n}]}$ is integrable and $\int_0^1 |n_\chi[0, \frac{1}{n}]| = 1$. The family $\mathscr{F}$, however, fails to be uniformly integrable. Fix $\epsilon = \dfrac{1}{2}$. Then, for any $\delta > 0$, by the Archimedean property of the reals, there exists n, such that $\dfrac{1}{n} < \delta$. Since the interval $[0, \dfrac{1}{n}]$ is measurable, has a measure smaller than δ, and $\int_0^{\frac{1}{n}} n\chi_{[0,\frac{1}{n}]} = 1 > \dfrac{1}{2}$, we have that $\mathscr{F}$ is not uniformly integrable. Hence, by a counter example, we have shown that under the given assumptions, the family of functions need not be uniformly integrable.

(b) We claim that $\mathscr{F}$ with the given assumption is uniformly integrable. Note that continuity implies integrability. Fix $\epsilon > 0$. Let $f \in \mathscr{F}$. Then, for any measurable set $E \subseteq [0,1]$ with $mE < \delta$ with, by using the $|f| \leq 1$ bound, we obtain

$$\begin{aligned}
\int_E f \quad &\leq \quad \int_E |f| \\
&\leq \quad \int_E 1 \\
&= \quad mE \\
&\leq \quad \delta
\end{aligned}$$

By letting $\delta = \epsilon$, we have $\int_E f \leq \epsilon$. Since ϵ and f were arbitrary, we have shown that $\mathscr{F}$ is uniformly integrable.

(c) Let $\mathscr{F}$ be the family of functions f on $[0,1]$, each of which is integrable over $[0,1]$ and has $\int_a^b |f| \leq b - a$ for all $[a,b] \subseteq [0,1]$. We claim that $\mathscr{F}$ is uniformly integrable. Fix $\epsilon > 0$ and fix $f \in \mathscr{F}$. Let $A \subseteq [0,1]$ be a measurable set such that $mA < \delta$ By the outer approximation of measurable set by open sets, there exists an open set O such that $A \subseteq O$ and $m(O \backslash A) \leq \dfrac{\epsilon}{2}$. Observe that O can be written as a countable union of disjoint open intervals, which gives $O = \cup_{i=1}^{\infty}(a_i, b_i)$. From the monotonicity and excision property of measure, and countable additivity over domain property of integration, it follows that

$$\begin{aligned}
\int_A |f| &\le \int_O |f| \\
&\le \int_{\cup_{i=1}^{\infty}(a_i,b_i)} |f| \\
&= \sum_{i=1}^{\infty}\int_{(a_i,b_i)} |f| \\
&\le \sum_{i=1}^{\infty}\int_{[a_i,b_i]} |f| \\
&\le \sum_{i=1}^{\infty} b_i - a_i \\
&= mO \\
&= m(O \setminus A) + m(A) \\
&\le \frac{\epsilon}{2} + \delta.
\end{aligned}$$

Define $\delta = \dfrac{\epsilon}{2}$ then, we have if A is measurable, and $mA < \delta$, then $\int_A |f| < \epsilon$. Since ϵ and f were arbitrary, we have that $\mathscr{F}$ is uniformly integrable. $\square$

Question 5. 5-11.

Solution. Assume that E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable on E for which f and each f_n is finite a.e.

We first show that the if implication holds by proving its contrapositive. Assume that f_n does not converge to f in measure. This implies that there exists $\delta > 0$ and $\eta > 0$ such that

$$m\{x \in E \mid |f_n(x) - f(x)| > \eta\} > \delta,$$

infinitely often in the sequence of f_n. Choose $\{f_{n_k}\}$ such that

$$m\{x \in E \mid |f_{n_k}(x) - f(x)| > \eta\} > \delta,$$

for all k. Hence, there is a non measure-zero set, on which $|f_{n_k} - f| > \delta$ for all k. Hence, any subsequence of $\{f_{n_k}\}$ cannot converge pointwise a.e. on E. Therefore, there exists a subsequence of $\{f_n\}$ who does not have a further subsequence that converges to f pointwise a.e. on E, which completes the proof.

Now, we prove the only if implication. Assume that $f_n \to f$ in measure on E. Hence, we have that for all $\eta > 0$,

$$\lim_{n\to\infty} m\{x \in E \mid f_n(x) - f(x)| > \eta\} \quad \to \quad 0.$$

Question 6. 5-13.

Solution. dd