Real Variables:
Problem Set V

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Abstract

This work contains solutions to the problem set V of Real Variables 2015 at NYU.

1 Solutions

Question 6.10.

Solution. Let $x_1, x_2 \in (a,b)$, such that $x_1 < x_2$. Then, we have

$$\begin{aligned} f(x_1) &= \sum_{k=1}^{\infty} l((c_k, d_k) \cap (-\infty, x_1)) \\ f(x_2) &= \sum_{k=1}^{\infty} l((c_k, d_k) \cap (-\infty, x_2)). \end{aligned}$$

As $x_1 < x_2$, by the monotonicity property of the length function , we have

$$l((c_k, d_k) \cap (-\infty, x_1)) \quad \leq \quad l((c_k, d_k) \cap (-\infty, x_2)),$$

for all $k \geq 1$. It follows that $f(x_1) \leq f(x_2)$. Hence, f is increasing. We show that f fails to be differentiable at each point in E, which is a set of measure zero contained in the open interval (a,b). Let $x \in E$. Then, by the preceding problem, there exist a countable collection of open intervals contained in (a,b), $\{(c_k, d_k)\}_{k=1}^{\infty}$ such taht each point in E belongs to infinitely many intervals in the collection and $\sum_{k=1}^{\infty} d_k - c_k < \infty$. Let $\{(c_{k_i}, d_{k_i})\}_{i=1}^{\infty}$ be the sub-collection such that $x \in (c_{k_i}, d_{k_i})$ for all i. Then, there exist a finite sub-cover $\{(c_{k_i}, d_{k_i})\}_{i=1}^{n}$ that x belongs to. Since, n is finite, as intersection of finite open sets is open, there exists a_n such that

$$B(x, a_n) \in \cup_{k=1}^{n} (c_{k_i}, d_{k_i}),$$

such that (B, a_n) denotes the ball of radius a_n, centered at x. Observe that

$$\begin{aligned} f(x + a_n) - f(x) &\geq \sum_{i=1}^{n} l((c_{k_i}, d_{k_i}) \cap (x, x + a_n)) \\ &= n a_n. \end{aligned}$$

It follows that

$$\begin{aligned} \bar{D} f(x) &= \lim_{h \to 0} \sup_{0 < |t| \leq h} \{ \frac{f(x+t) - f(x)}{t} \} \\ &= \lim_{h \to 0} \sup_{0 < |t| \leq h} \frac{n a_n}{a_n} \\ &\geq m. \end{aligned}$$

Since m was arbitrary, we have that

$$\bar{D}f(x) = \infty,$$

which is not finite, and by definition, x is not differentiable at x. Therefore, f fails to be differntiable at each point in E. $\square$

Question 6.33.

Solution. Let $\{f_n\}$ be a sequence of real-valued functions on $[a, b]$ that converges pointwise on $[a, b]$ to the real-valued function f. We wish to show that $TV(f) \leq \liminf TV(f_n)$. Fix $P = \{x_0, ..., x_m\}$ be a partition of $[a, b]$. As $f_n \to f$ pointwise, we have

$$\begin{aligned} V(f, P) &= \sum_{k=0}^{m-1} |f(x_{k+1}) - f(x_k)| \\ &= \lim_{n\to\infty} \sum_{k=0}^{m-1} |f_n(x_{k+1}) - f_n(x_k)| \\ &= \lim_{n\to\infty} V(f_n, P). \end{aligned}$$

By the definition of total variation, it follows that

$$V(f_n, P) \quad \leq \quad TV(f_n),$$

for all n. Consequently, we obtain

$$V(f, P) \quad \leq \quad \liminf_{n\to\infty} TV(f_n),$$

and since P was arbitrary, we finally have that

$$TV(f) \quad \leq \quad \liminf_{n\to\infty} TV(f_n),$$

as desired. $\square$

Question 4. Royden 6.42.

Solution. Let f and g be real-valued functions, that are absolutely continuous functions on $[a, b]$. We wish to show that $f + g$ is absolutely continuous on $[a, b]$. Fix $\epsilon > 0$. As f and g are both absolutely continuous on $[a, b]$, there exist $\delta_f, \delta_g > 0$, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{k=1}^n$ in (a, b),

$$\begin{aligned} \sum_{k=1}^{n} [b_k - a_k] < \delta_f &\implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \frac{\epsilon}{2} \\ \sum_{k=1}^{n} [b_k - a_k] < \delta_g &\implies \sum_{k=1}^{n} |g_(b_k) - g(a_k)| < \frac{\epsilon}{2}. \end{aligned}$$

Define $\delta = \min(\delta_f, \delta_g)$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint open intervals in (a, b), such that $\sum_{k=1}^n [b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^{n} |f + g(b_k) - f + g(a_k)| &\leq \sum_{k=1}^{n} |f(b_k) - f(a_k)| + |g(b_k) - g(a_k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we have shown that $f + g$ is absolutely continuous on $[a, b]$.

Let f be a real-valued function, that is absolutely continuous on $[a, b]$. We show that cf, for any $c \in \mathbb{R}$, is absolutely continuous on $[a, b]$. Let $c = 0$. Then $cf = 0$, which can trivially be shown to be absolutely continuous, as $f(c) = 0$ for any $c \in [a, b]$. Suppose $c \neq 0$. As f is absolutely continuous on $[a, b]$, there exists $\delta_f > 0$, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{k=1}^n$ in (a, b),

$$\sum_{k=1}^{n} |b_k - a_k| < \delta_f \implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \frac{\epsilon}{|c|}.$$

Define $\delta = \delta_f$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint open intervals in (a, b) such that $\sum_{k=1}^n [b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^{n} |cf(b_k) - cf(a_k)| &= |c| \sum_{k=1}^{n} |f(b_k) - f(a_k)| \\ &< |c| \frac{\epsilon}{|c|} = \epsilon. \end{aligned}$$

Since ϵ was arbitrary, combined with the $c = 0$ case, we have shown that cf, for any $c \in \mathbb{R}$, is absolutely continuous on $[a, b]$.

Let f be a real-valued function, that is absolutely continuous on $[a, b]$. We wish to show that f^2 is absolutely continuous on $[a, b]$. As f is absolutely continuous, f is continuous on $[a, b]$. Hence, by the Extreme Value Theorem, there exists M such that $|f| \leq M$ on $[a, b]$. Fix $\epsilon > 0$. As f is absolutely continuous on $[a, b]$, there exists $\delta_f > 0$, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{k=1}^n$ in (a, b),

$$\sum_{k=1}^{n} |b_k - a_k| < \delta_f \implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \frac{\epsilon}{2M}.$$

Define $\delta = \delta_f$. Let $\{(a_k, b_k)\}_{k=1}^\infty$ be a finite disjoint open intervals in (a, b) such that $\sum_{k=1}^n [b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^{n} |f^2(b_k) - f^2(a_k)| &= \sum_{k=1}^{n} |f(b_k) - f(a_k)||f(b_k) + f(a_k)| \\ &\leq 2M \sum_{k=1}^{n} |f(b_k) - f(a_k)| \\ &< 2M \frac{\epsilon}{2M} = \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we have shown that f^2 is absolutely continuous on $[a, b]$.

Let f and g be real-valued functions, that are absolutely continuous on $[a, b]$. We wish to show that fg is absolutely continuous on $[a, b]$. Observe that

$$(f + g)^2 = f^2 + g^2 - 2fg,$$

which simplifies to

$$fg = -\frac{1}{2}((f + g)^2 + (-f^2) + (-g^2)).$$

As we have previously shown that a square of an absolutely continuous function is absolutely continuous, and a scalar multiple of an absolutely continuous function is absolutely continuous, we can deduce that fg is absolutely continuous on $[a, b]$. This completes the proof. $\square$

Question 4. 6.45.

Solution. Let f be a real-valued function, that is absolutely continuous on $\mathbb{R}$. Let g be a real-valued function, that is absolutely continuous and strictly monotone on $[a, b]$. Without the loss of generality, we assume that g is strictly increasing. We wish to show that $f \circ g$ is absolutely continuous on $[a, b]$. Fix $\epsilon > 0$. As f is absolutely continous on $\mathbb{R}$, it is also absolutely continuous on $[g(a), g(b)]$, which is a non-degenerate closed interval, as g is strictly increasing. there exists δ_f, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{n=1}^{\infty}$ in $(g(a), g(b))$,

$$\sum_{k=1}^{n}[b_k - a_k] < \delta_f \implies \sum_{k=1}^{n}|f(b_k) - f(a_k)| < \epsilon \ (*).$$

As g is absolutely continuous, there exists δ_g, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{n=1}^{\infty}$ in (a, b),

$$\sum_{k=1}^{n}[b_k - a_k] < \delta_g \implies \sum_{k=1}^{n}|g(b_k) - g(a_k)| < \delta_f.$$

Define $\delta = \delta_g$. Let $\{(a_k, b_k)\}_{k=1}^{n}$ be a finite djsjoint open intervals in (a, b) such that $\sum_{k=1}^{n}[b_k - a_k] < \delta_g$. It follows that $\sum_{k=1}^{n}[g(b_k) - g(a_k)] < \delta_f$. As g is strictly increasing, we observe that $\{(g(a_k), g(b_k))\}_{k=1}^{n}$ form a finite disjoint open intervals in $(g(a), g(b))$. Therefore, from $(*)$ it follows that

$$\sum_{k=1}^{n}|f \circ g(b_k) - f \circ g(a_k)| \quad < \quad \epsilon.$$

Since ϵ was arbitrary, we have shown that $f \circ g$ is absolutely continuous on $[a, b]$. $\square$

Question.

Solution.