Real Variables:
Problem Set XI

Youngduck Choi
Courant Institute of Mathematical Sciences
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set XI of Real Variables 2015 at NYU.

1 Solutions

Question Royden 17-6.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and X_0 belong to $\mathcal{M}$. Define $\mathcal{M}_0$ to be the collection of sets in $\mathcal{M}$ that are subsets of X_0 and μ_0 the restriction of μ to $\mathcal{M}_0$. Show that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

Solution. We first show that $(X_0, \mathscr{M}_0)$ is a measurable space. To this end, we must show that $\mathscr{M}_0$ is a σ-algebra of X_0. As $\emptyset$ and X_0 belong to $\mathscr{M}$, are subsets of X_0, it follows that $\emptyset$ and X_0 belong to $\mathscr{M}_0$. Let $\{A_n\}_{n=1}^{\infty}$ be a countable collections of sets in $\mathscr{M}_0$. As $A_n \subseteq X_0$ for all n, we have $\bigcup_{n=1}^{\infty} A_n \subseteq X_0$. Furthermore, as $\mathscr{M}$ is a σ-algebra, and $A_n \in \mathscr{M}$ for all n, we also have $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}$. Hence, it follows that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}_0$. Now, let A be a set, belonging to $\mathscr{M}_0$. Then, as $X_0 \setminus A$ is a subset of X_0, and X_0 and A belong to M, which gives $X_0 \setminus A \in \mathscr{M}$, we have $X_0 \setminus A$ belongs to $\mathscr{M}_0$. Hence, we have shown that $\mathscr{M}_0$ is a σ-algebra, and $(X_0, \mathscr{M}_0)$ is a measurable space. Now, it remains to be shown that the restricted map μ_0 has the properties of a measure. First, observe that $\emptyset \in \mathscr{M}_0$ and $\mu_0(\emptyset) = \mu(\emptyset) = 0$. Now, let $\{A_n\}$ be a countable disjoint sets from $\mathscr{M}_0$. Since $A_n \in \mathscr{M}$ for all n, by the countable additivity of μ and the fact that $\mathscr{M}_0$ is a σ-algebra, it follows that

$$
\begin{aligned}
\sum_{n=1}^{\infty} \mu_0(A_n) &= \sum_{n=1}^{\infty} \mu(A_n) \\
&= \mu(\bigcup_{n=1}^{\infty} A_n) \\
&= \mu_0(\bigcup_{n=1}^{\infty} A_n).
\end{aligned}
$$

Therefore, we have shown that $(X_0, \mathscr{M}_0, \mu_0)$ is a measure space. $\square$

Question Royden 17-15.

15. Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$, where α and β are real numbers. Show that

$$|\alpha\nu| = |\alpha||\nu| \text{ and } |\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|,$$

where $\nu \leq \mu$ means $\nu(E) \leq \mu(E)$ for all measurable sets E.

Solution. Let $(X, \mathscr{M})$ be a measurable space. Let v_1 and v_2 be two finite signed measures on $(X, \mathscr{M})$. Consider a set function $\alpha v_1 + \beta v_2$ on $\mathscr{M}$, for $\alpha, \beta \in \mathbb{R}$, which is defined by

$$\alpha v_1 + \beta v_2(E) \quad = \quad \alpha v_1(E) + \beta v_2(E),$$

for $E \in \mathscr{M}$. As v_1 and v_2 only take finite values, it also follows that $\alpha v_1 + \beta v_2$ also assumes only finite values, as an addition of two finite values are finite. Furthermore, it follows that

$$\begin{aligned} \alpha v_1 + \beta v_2(\emptyset) &= \alpha v_1(\emptyset) + \beta v_2(\emptyset) \\ &= 0 + 0 = 0, \end{aligned}$$

as v_1 and v_2 are signed measures. Furthermore, the countable additivity property holds, as for any countable disjoint collection $\{E_k\}$ from $\mathscr{M}$, by linearity of limit, and countable additivity of v_1 and v_2, it follows that

$$\begin{aligned} \alpha v_1 + \beta v_2(\bigcup_{k=1}^{\infty} E_k) &= \alpha v_1(\bigcup_{k=1}^{\infty} E_k) + \beta v_2(\bigcup_{k=1}^{\infty} E_k) \\ &= \alpha \sum_{k=1}^{\infty} v_1(E_k) + \beta \sum_{k=1}^{\infty} v_2(E_k) \\ &= \sum_{k=1}^{\infty} \alpha v_1(E_k) + \beta v_2(E_k) \\ &= \sum_{k=1}^{\infty} \alpha v_1 + \beta v_2(E_k). \end{aligned}$$

Hence $\alpha v_1 + \beta v_2$ is a finite measure.

Let $(X, \mathscr{M})$ be a measurable space and v be a finite measure on the space. Consider $|\alpha v|$. It follows that for $E \in \mathscr{M}$, we have

$$|\alpha v(E)| \quad = \quad |\alpha||v(E)|.$$

Hence, $|\alpha v| = |\alpha||v|$. Now, let v_1 and v_2 be finite signed measures on $(X, \mathscr{M})$. By the triangle inequality of reals, it follows that for any $E \in \mathscr{M}$,

$$\begin{aligned} |v_1 + v_2(E)| &= |v_1(E) + v_2(E)| \\ &\leq |v_1(E)| + |v_2(E)|. \end{aligned}$$

Hence, $|v_1 + v_2| \leq |v_1| + |v_2|$. $\square$

Question Royden 17-17.

17. Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu+\nu-|\mu-\nu|)$ and $\mu \vee \nu = \mu+\nu-\mu \wedge \nu$.
    (i) Show that the signed measure $\mu \wedge \nu$ is smaller than μ and ν but larger than any other signed measure that is smaller than μ and ν.
    (ii) Show that the signed measure $\mu \vee \nu$ is larger than μ and ν but smaller than any other measure that is larger than μ and ν.
    (iii) If μ and ν are positive measures, show that they are mutually singular if and only if $\mu \wedge \nu = 0$.

Solution. (i) Let u and v be finite signed measures on $(X, \mathscr{M})$. We wish to show that $u \wedge v \leq u$. By writing out $u \wedge v$ term, multiplying by 2, and re-arranging the terms trivially, we see that $u \wedge v \leq u$ is equivalent to $u - v \leq |u - v|$. As an absolute value of a real number is larger than the number itself, we see that $u \wedge v \leq u$ holds. By symmetry, we also see that $u \wedge v \leq v$ as well.

Now, let t be a signed measure such that $t \leq u$ and $t \leq v$. It follows that $t \leq \min(u, v)$. Assume without loss of generality that $\min(u, v) = u$. It follows that

$$\begin{aligned} \frac{1}{2}(u + v|u - v|) &= \frac{1}{2}(2u) \\ &= u. \end{aligned}$$

Therefore, $\min(u, v) = \dfrac{1}{2}(u + v - |u - v|)$. Hence, $t \leq u \wedge v$.

(ii) Let u and v be finite signed measures on $(X, \mathscr{M})$. In the above solution, we have shown that $u \wedge v = \min(u, v)$. Hence, it follows that $u + v - u \wedge v = \max(u, v)$. Therefore, we obtain that $u \vee v \geq u$ and $u \vee v \geq v$. Again, as for any t such that $t \geq u$ and $t \geq v$, it follows that $t \geq \max(u, v) = u \vee v$.

(iii) Let that u and v are finite measures, and assume that $u \wedge v = \dfrac{1}{2}(u + v - |u - v|) = 0$. For any $E \in \mathscr{M}$, it follows that

$$u(E) + v(E) = |u(E) - v(E)|$$

Suppose that $u(E)$ and $v(E)$ are both strictly positive. It follows that $u(E) + v(E) > \max(u(E), v(E))$ and $|u(E) - v(E) < \max(u(E), v(E))$, which yields a contradiction with the above equality. Hence, we have that at least one of $u(E)$ or $v(E)$ is 0. Assume without loss of generality that $u(E) = 0$. Now, with the same argument with respect to E^c, it follows that either $u(E^c)$ and $v(E^c)$ is 0. If $v(E^c) = 0$, then u and v are mutually singular. Now, consider the remaining case of $u(E^c) = 0$. By the additivity property of a signed measure, we have that $u(X) = 0$. Hence, $u(X) = 0$ and $v(\emptyset) = 0$. Therefore, we have shown that u and v are mutually singular.

Now, assume that u and v are mutually singular measures. Hence, there exists a disjoint measurable set A and B such that $A \cup B = X$ and $u(A) = 0$ and $v(B) = 0$. Let $E \in \mathscr{M}$. Then, it follows that

$$\begin{aligned} u \wedge v(E) &= u \wedge v(E \cap A) + u \wedge v(E \cap B) \\ &= \frac{1}{2}(u + v - |u - v|)(E \cap A) + \frac{1}{2}(u + v - |u - v|)(E \cap B) \\ &= 0 + 0, \end{aligned}$$

by the monotonicity property.

$\square$

Question Royden 18-50.

50. Establish the uniqueness of the function f in the Radon-Nikodym Theorem.

Solution. Let $(X, \mathscr{M})$ be a measurable space, and u, v be σ-finite measures on the space with $v << u$. Let f and g be the non-negative measurable functions for which,

$$\begin{aligned} v(E) &= \int_E f du \\ &= \int_E g du, \end{aligned}$$

for all $E \in \mathscr{M}$. By the linearity of integration for non-negative measures, we obtain that

$$\int_E f - g du = 0,$$

for all $E \in \mathscr{M}$. Now, we prove that if $\int_E f du = 0$ for every measurable subset E of X, then $f = 0$ a.e. on X. We prove the case for f non-negative and the result can be extended through an argument with f^+ and f^- decomposition. Define $A_n = \{x \in X \mid f(x) \geq \frac{1}{n}\}$. we have $\cup_n A_n = \{x \in X \mid f(x) > 0\}$. Now, observe that by the assumption, $\int_{A_n} f du = 0$ for all n and $\frac{1}{n} u(A_n) \leq \int_{A_n} f du = 0$, which gives $u(A_n) = 0$ for all n. Therefore, by the countable additivity, we obtain that $u(\cup_n A_n) = 0$. Hence, we have shown that $f = 0$ a.e. Therefore, we have that $f - g$ from the Radon-Nikodym theorem, must be 0 a.e. on X. Hence, we have shown the uniqueness of Radon-Nikodym theorem. $\square$

Question Royden 18-54.

54. Let μ, ν, and λ be σ-finite measures on the measurable space $(X, \mathcal{M})$.

(i) If $\nu \ll \mu$ and f is a nonnegative function on X that is measurable with respect to $\mathcal{M}$, show that

$$\int_X f \, d\nu = \int_X f \left[\frac{d\nu}{d\mu}\right] d\mu.$$

(ii) If $\nu \ll \mu$ and $\lambda \ll \mu$, show that

$$\frac{d(\nu+\lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \text{ a.e. } [\mu].$$

(iii) If $\nu \ll \mu \ll \lambda$, show that

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} \text{ a.e. } [\lambda].$$

Solution.

Question Royden 18-55.

55. Let μ, ν, ν_1, and ν_2 be measures on the measurable space $(X, \mathcal{M})$.
   (i) Show that if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.
   (ii) Show that if ν_1 and ν_2 are singular with respect to μ, then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.
   (iii) Show that if ν_1 and ν_2 are absolutely continuous with respect to μ, then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.
   (iv) Prove the uniqueness assertion in the Lebesgue decomposition.

Solution. (i) Assume $v \perp u$, and $v << u$. It follows that there exists a pair of measurable sets A and B such that $u(A) = 0$ and $v(B) = 0$. By the absolute continuity of v with respect to u, it follows that $v(A) = 0$. By finite additivity of measure, we obtain that $v(X) = 0$. In other words, v is a zero measure.

(ii) Assume $u \perp v$. We show that $u \perp \alpha v$, for $\alpha \geq 0$ holds as well. As $u \perp v$, there exists A and B from $\mathscr{M}$ such that $u(A) = 0$ and $v(B) = 0$ and $A \cup B = X$. Observe that $\alpha v(B) = 0$ as well. Hence, $u \perp \alpha v$. Now, assume $u \perp v_1$ and $u \perp v_2$. We show that $u \perp v_1 + v_2$. Let (A_1, B_1) and (A_2, B_2) be the pairs of sets that grant the mutual singularities. Then, observe that by finite additivity of measure, we have $u(A_1 \cup A_2) = 0$. Furthermore, observe that by monotonicity of measure, $v_1 + v_2(B_1 \cap B_2) = 0$. Since $A_1 \cup A_2 \cup (B_1 \cap B_2) = (A_1 \cup A_2 \cup B_1) \cap (A_1 \cup A_2 \cup B_2) = X \cap X = X$, we have that u and $v_1 + v_2$ are mutually singular. Hence, the claim is proven.

(iii) Assume $v << u$. Then, for any $E \in \mathscr{M}$ such that $u(E) = 0$, we have $v(E) = 0$. Since $v(E) = 0$, we also have $\alpha v(E) = 0$. Hence, $\alpha v << u$. Now, assume $v_1 << u$ and $v_2 << u$. It follows that for any $E \in \mathscr{M}$ such that $u(E) = 0$, we have $v_1(E) = 0$ and $v_2(E) = 0$. Therefore, $v_1 + v_2(E) = 0$. Hence, the claim is proven.

(iv) Let v_0 and v_1 be the Lebesque decomposition measures, where v_0 is mutually singular and v_1 is absolutely continuous with respect to v_1. Consider $v_0^{'}$ and $v_1^{'}$ with the same set-up. We have that $v_0 + v_1 = v_0^{'} + v_1^{'}$. It follows that

$$v_0 - v_0^{'} \quad = \quad v_1 - v_1^{'}.$$

Observe that $v_0 - v_0^{'}$ and $v_1 - v_1^{'}$ are singular and absolutely continuous with respect to u from (ii) and (iii). Hence, by (i), we have that $v_0 = v_0^{'}$ and $v_1 = v_0^{'}$. Therefore, we have shown that uniqueness of the decomposition. $\square$