Real Variables: Problem Set XI

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Abstract

This work contains solutions to the problem set XI of Real Variables 2015 at NYU.

1 Solutions

Question Royden 17-6.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and X_0 belong to $\mathcal{M}$. Define $\mathcal{M}_0$ to be the collection of sets in $\mathcal{M}$ that are subsets of X_0 and μ_0 the restriction of μ to $\mathcal{M}_0$. Show that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

Solution. We first show that $(X_0, \mathscr{M}_0)$ is a measurable space. To this end, we must show that $\mathscr{M}_0$ is a σ-algebra of X_0. As $\emptyset$ and X_0 belong to $\mathscr{M}$, are subsets of X_0, it follows that $\emptyset$ and X_0 belong to $\mathscr{M}_0$. Let $\{A_n\}_{n=1}^{\infty}$ be a countable collections of sets in $\mathscr{M}_0$. As $A_n \subseteq X_0$ for all n, we have $\bigcup_{n=1}^{\infty} A_n \subseteq X_0$. Furthermore, as $\mathscr{M}$ is a σ-algebra, and $A_n \in \mathscr{M}$ for all n, we also have $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}$. Hence, it follows that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}_0$. Now, let A be a set, belonging to $\mathscr{M}_0$. Then, as $X_0 \setminus A$ is a subset of X_0, and X_0 and A belong to M, which gives $X_0 \setminus A \in \mathscr{M}$, we have $X_0 \setminus A$ belongs to $\mathscr{M}_0$. Hence, we have shown that $\mathscr{M}_0$ is a σ-algebra, and $(X_0, \mathscr{M}_0)$ is a measurable space. Now, it remains to be shown that the restricted map μ_0 has the properties of a measure. First, observe that $\emptyset \in \mathscr{M}_0$ and $\mu_0(\emptyset) = \mu(\emptyset) = 0$. Now, let $\{A_n\}$ be a countable disjoint sets from $\mathscr{M}_0$. Since $A_n \in \mathscr{M}$ for all n, by the countable additivity of μ and the fact that $\mathscr{M}_0$ is a σ-algebra, it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu_0(A_n) &= \sum_{n=1}^{\infty} \mu(A_n) \\
&= \mu(\bigcup_{n=1}^{\infty} A_n) \\
&= \mu_0(\bigcup_{n=1}^{\infty} A_n).
\end{aligned}$$

Therefore, we have shown that $(X_0, \mathscr{M}_0, \mu_0)$ is a measure space. $\square$

Question Royden 17-15.

15. Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$, where α and β are real numbers. Show that

$$|\alpha\nu| = |\alpha||\nu| \text{ and } |\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|,$$

where $\nu \leq \mu$ means $\nu(E) \leq \mu(E)$ for all measurable sets E.

Solution. Let $(X, \mathscr{M})$ be a measurable space. Let v_1 and v_2 be two finite signed measures. Consider a set function $\alpha v_1 + \beta v_2$ on $\mathscr{M}$, for $\alpha, \beta \in \mathbb{R}$, which is defined by

$$\alpha v_1 + \beta v_2(E) \quad = \quad \alpha v_1(E) + \beta v_2(E),$$

for $E \in \mathscr{M}$. As v_1 and v_2 only take finite values, it also follows that $\alpha v_1 + \beta v_2$ also assumes only finite values, as an addition of two finite values are finite. Furthermore, it follows that

$$\begin{aligned} \alpha v_1 + \beta v_2(\emptyset) &= \alpha v_1(\emptyset) + \beta v_2(\emptyset) \\ &= 0 + 0 = 0, \end{aligned}$$

as v_1 and v_2 are signed measures.

Question Royden 17-17.

17. Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu+\nu-|\mu-\nu|)$ and $\mu \vee \nu = \mu+\nu-\mu \wedge \nu$.
    (i) Show that the signed measure $\mu \wedge \nu$ is smaller than μ and ν but larger than any other signed measure that is smaller than μ and ν.
    (ii) Show that the signed measure $\mu \vee \nu$ is larger than μ and ν but smaller than any other measure that is larger than μ and ν.
    (iii) If μ and ν are positive measures, show that they are mutually singular if and only if $\mu \wedge \nu = 0$.

Solution.

Question Royden 18-50.

50. Establish the uniqueness of the function f in the Radon-Nikodym Theorem.

Solution.

Question Royden 18-54.

54. Let μ, ν, and λ be σ-finite measures on the measurable space $(X, \mathcal{M})$.

(i) If $\nu \ll \mu$ and f is a nonnegative function on X that is measurable with respect to $\mathcal{M}$, show that

$$\int_X f \, d\nu = \int_X f \left[\frac{d\nu}{d\mu}\right] d\mu.$$

(ii) If $\nu \ll \mu$ and $\lambda \ll \mu$, show that

$$\frac{d(\nu + \lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \text{ a.e. } [\mu].$$

(iii) If $\nu \ll \mu \ll \lambda$, show that

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} \text{ a.e. } [\lambda].$$

Solution.

Question Royden 18-55.

55. Let μ, ν, ν_1, and ν_2 be measures on the measurable space $(X, \mathcal{M})$.

(i) Show that if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

(ii) Show that if ν_1 and ν_2 are singular with respect to μ, then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.

(iii) Show that if ν_1 and ν_2 are absolutely continuous with respect to μ, then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.

(iv) Prove the uniqueness assertion in the Lebesgue decomposition.

Solution.