Royden

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Abstract

This work contains the solutions to Royden's Real Variables.

1 Chapter II

Question Royden 2.1-1.

Solution. Let m be a set function defined for all sets in a $\sigma-$algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume that m is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Furthermore, assume that A and B are two sets in $\mathcal{A}$ with $A \subseteq B$. Given that m is countably additive over countable disjoint collections of sets in $\mathcal{A}$, we have

$$m(B) \quad = \quad m(A) + m(B \setminus A),$$

where $B \setminus A$ is a well-defined set with $A \subseteq B$ assumption, thus A and $B \setminus A$ forming a valid countable disjoint collections of sets whose union is B. With m being a set function with values in $[0, \infty]$, we obtain $m(B) = m(A) + r$, where r denotes some non-negative real value. Therefore, we finally get

$$m(A) \leq m(B).$$

Hence, we have shown that the given set function m has the monotonicity property.

Question Royden 2.1-2.

Solution. Let m be a set function defined for all sets in a $\sigma-$algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume that m is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Furthermore, assume that there exists a set A in the collection $\mathcal{A}$ such that $m(A) < \infty$. Using the countably additive property with a collection $\{A, \emptyset\}$, we obtain

$$m(A \cup \emptyset) \quad = \quad m(A) + m(\emptyset).$$

Substituting $A \cup \emptyset = A$ and subtracting $m(A)$ from both sides, granted with finiteness of $m(A)$, we get

$$m(\emptyset) \quad = \quad 0,$$

as desired. Hence, we have shown that if there is a set A in the collection $\mathcal{A}$ for whcih $m(A) < \infty$, then $m(\emptyset) = 0$.

Question Royden 2.1-3.

Solution.

Question Royden 2.1-6.

Solution. Let Q and A denote the set of rationals and irrationals in the interval $[0, 1]$ respectively. Consider a countable collection of sets $\{Q, A\}$. Since outer measure is countably subadditive, we have

$$m^*(Q \cup A) \quad \leq \quad m^*(Q) + m^*(A).$$

As Q is a countable set whose outer measure is zero and $Q \cup A = [0, 1]$ by construction, we obtain

$$m^*([0, 1]) \quad \leq \quad m^*(A).$$

As the outer measure of an interval is its length, we have

$$1 \quad \leq \quad m^*(A).$$

Using the monotonicity property of outer measure with $I \subset [0, 1]$, we also see

$$m^*(A) \quad \leq \quad 1,$$

thereby showing that $m^*(A) = 1$.

2 Chapter I

Question 1. Royden 1.1-1 (Distributive Property of Multiplicative Inverse in Reals).

Solution. Assume that $a \neq 0$ and $b \neq 0$. From the multiplicative identity axiom, we have that a multiplicative inverse exists for a and b individually, which we denote as a^{-1} and b^{-1} respectively. Now, consider the expression $(ab)(a^{-1}b^{-1})$, where ab denotes the product of a and b, and $a^{-1}b^{-1}$ denotes the product of a^{-1} and b^{-1}. From the commutativity of multiplication, we obtain

$$(ab)(a^{-1}b^{-1}) = (ab)(b^{-1}a^{-1}).$$

Using the associativity of multiplication and iteratively substituting $bb^{-1} = 1$ and $aa^{-1} = 1$, we have

$$(ab)(a^{-1}b^{-1}) = 1,$$

where 1 denotes the identity as usual. Hence, the product, $a^{-1}b^{-1}$ satisfies definition of multiplicative inverse with respect to the ab term whose multiplicative inverse can be denoted as $(ab)^{-1}$ by convention. Therefore, we obtain that

$$(ab)^{-1} = a^{-1}b^{-1},$$

as desired.

Question Royden 1.1-3.

Solution. Let E be a nonemepty set of real numbers.
($\Leftarrow$) Assume that E consists of a single point, which we denote as x. We claim that inf$E = x$ and sup$E = x$. As we have $x \leq x$, we see that x is an upper bound for E. Suppose that there exists an upper bound for E, a, that is smaller than x, namely $a < x$. This is a contradiction to the fact that a is an upper bound as it is required to have $x \leq a$ with $x \in E$. Hence, there does not exists any upper bound for E that is smaller than x. By definition of supremum, we have that sup$E = x$. By symmtry, we can see that inf$E = x$ as well. Therefore, inf$E =$ sup$E$.

($\Rightarrow$) Assume that inf$E =$ sup$E$. Given the assumption, let us denote the infimum and supremum for E as a single real number a. Then, by definition of infimum, any x in E, we have $a \leq x$. Furthermore, by definition of supremum, any x in E, we have $x \leq a$. The only real number that can satisfy the two given equality is a itself. We also know that a must be in E as E is a nonempty set of reals. Therefore, we have shown that $E = \{a\}$, and that E consists of a single point.