Real Variables:
Problem Set II

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Abstract

This work contains solutions to the problem set III of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 3.20.
Solution.

$$\begin{aligned}
\chi_{A\cap B} &= \begin{cases} 1 & \text{if } x \in A \cap B \\ 0 & \text{if } x \notin A \cap B \end{cases} \\
\chi_{A\cup B} &= \begin{cases} 1 & \text{if } x \in A \cup B \\ 0 & \text{if } x \notin A \cup B \end{cases} \\
\chi_{A^c} &= \begin{cases} 1 & \text{if } x \in A^c \\ 0 & \text{if } x \notin A^c \end{cases}
\end{aligned}$$

Question 2. Royden 3.21.
Solution.

Question 3. Royden 3.27.
Solution. The Egoroff

Question 4. Royden 4.12.
Solution. Let f a bounded measurable function on a set of finite measure E. Assume g is bounded and $f = g$ a.e. on E. First, we note that g is measurable. Since both f and g are bounded measurable functions, we have $\int_E f$ and $\int_E g$ well-defined. Let $\epsilon > 0$, and $E_0 = \{x \in E \mid f(x) \neq g(x)\}$. Note that $\mathrm{m}(E_0) = 0$, as $f = g$ a.e. Consequently, $E \setminus E_0$ and E_0 are disjoint measurable sets. Then, by additivity over domain and linearity of integration, we have

$$\begin{aligned}
\left| \int_E f - \int_E g \right| &= \left| \int_{E\setminus E_0} f - \int_{E\setminus E_0} g + \int_{E_0} f - \int_{E_0} g \right| \\
&= \left| \int_{E\setminus E_0} f - g + \int_{E_0} f - g \right|.
\end{aligned}$$

As $f = g$ on $E \setminus E_0$, and $\int_{E\setminus E_0} f - g = 0$, and consequently,

$$\left|\int_E f - \int_E g\right| \quad = \quad \left|\int_{E_0} f - g\right|.$$

ddd,

$$\int_E f = \int_E g,$$

as desired. $\square$

Question 5. Royden 4.23.

Solution. Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ setting $f(x) = a_n$ if $n \leq x < n+1$. We first show that f is measurable. Let $c \in \mathbb{R}$. Then, we can express the pre-image set of c as

$$\{x \in E \mid f(x) = c\} \quad = \quad \bigcup_{k\in\lambda} [k, k+1),$$

where $\lambda = \{k \in \mathbb{N} \mid a_k = c\}$. Therefore, $\{x \in E \mid f(x) = c\}$ is measurable, as it is either an empty set or a countable union of collection of intervals of the form $[k, k+1)$, which are measurable. Since c was arbitrary, f is measurable. Then, by the additivity over domain of integration property, we have

$$\int_E f \quad = \quad \sum_{k=1}^{\infty} \int_{I_k} f,$$

where I_k denotes $[k, k+1)$. As $f(I_k) = a_k$ and by the linearity of integration, we obtain

$$\begin{aligned}
\int_E f \quad &= \quad \sum_{k=1}^{\infty} \int_{I_k} a_k \\
&= \quad \sum_{k=1}^{\infty} a_k \int_{I_k} 1 \\
&= \quad \sum_{k=1}^{\infty} a_k.
\end{aligned}$$

Therefore, we have shown that $\int_E f = \sum_{k=1}^{\infty} a_k$. $\square$

Question 6. Royden .

Solution.