Real Variables: Problem Set II

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Abstract

This work contains solutions to the problem set II of Real Variables 2015 at NYU.

1 Solutions

Question 3. Royden 2.29.

Solution. (i) Let X be any set of real numbers, and R be the relation defined by the rational equivalence. For $x \in X$, we have $x - x = 0$. Hence, the rational equivalence is reflexive. Let $(x, y) \in R$, then we have $x - y \in \mathbb{Q}$. As a negative of a rational number is rational, we have $y - x \in \mathbb{Q}$ and $(y, x) \in R$. Hence, the rational equivalence is symmetric. Let $(x, y), (y, z) \in R$. As a sum of two rationals is rational, we have $x - y + y - z$, which is $x - z$, is rational, and $(x, z) \in R$. Hence, the rational equivalence is transitive. Therefore, the rational equivalence is an equivalence relation.

(ii) The partition of $\mathbb{Q}$, induced by the rational equivalence, is simply $\{\mathbb{Q}\}$. Hence, $\{0\}$ is a set that consists of exactly one member of each equivalence class. Therefore, $\{0\}$ is an explicit choice set of the rational equivalence.

(iii) We define two numbers to be irrationally equivalent provided their difference is irrational. Let $x \in \mathbb{R}$. As $x - x = 0$ and 0 is a rational number, the relation defined fails to be reflexive. Hence, The relation is not an equivalence relation on $\mathbb{R}$. The same argument holds with $x \in \mathbb{Q}$, and the relation is not an equivalence relation on $\mathbb{Q}$ as well. $\square$

Question 3. Royden 2.38.

Solution. Let $f : [a, b] \to \mathbb{R}$ be Lipschitz with the associated Lipschitz constant c, and let $E_0 \in [a, b]$ such that $\mathrm{m}(E_0) = 0$. Fix $\epsilon > 0$. As $\mathrm{m}(E_0) = 0$, we have a countable collection of disjoint open intervals $\{I_k\}_{k=1}^{\infty}$ such that $E \subseteq \cup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \mathrm{m}(I_k) < \dfrac{\epsilon}{c}$. Since $E \subseteq \cup_{k=1}^{\infty} I_k$, we have $f(E_0) \subseteq \cup_{k=1}^{\infty} f(I_k)$. By the monotonicity of measure, and Lipshitz property of f, we obtain

$$\mathrm{m}(f(E_0)) \leq \sum_{k=1}^{\infty} \mathrm{m}(f(I_k)) \leq c \sum_{k=1}^{\infty} \mathrm{m}(I_k) = \epsilon.$$

Since ϵ is arbitrary, we have $\mathrm{m}(f(E_0)) = 0$. Therefore, we have shown that a Lipschitz function maps a set of zero measure on to a set of measure zero.

Question 3. Royden 3.1.

Solution. Let f and g are continuous functions on $[a, b]$. Assume that $f = g$ a.e. In other words, $f = g$ on $[a, b] \setminus E_0$, where $\mathrm{m}(E_0) = 0$. Let $x \in E_0$, and fix $\epsilon > 0$. By the continuity of f and g, we have δ_f and δ_g such that

$$\begin{aligned} |x-x'| < \delta_f \quad &\implies \quad |f(x)-f(x')| < \frac{\epsilon}{2} \\ |x-x'| < \delta_g \quad &\implies \quad |g(x)-g(x')| < \frac{\epsilon}{2} \end{aligned} \tag{1}$$

Now, consider the set $B(x, \min(\delta_f, \delta_g)) \cap [a, b]$, where B denotes a ball with a center and radius. As E_0 is a zero measure set, there exists x^* in $B(x, \min(\delta_f, \delta_g)) \cap [a, b]$ such that $f(x^*) = g(x^*)$. Furthermore, by (1), we have that $|f(x) - f(x^*)| < \frac{\epsilon}{2}$ and $|g(x) - g(x^*)| < \frac{\epsilon}{2}$. Consequently, by the trinagle inequality, we have

$$|f(x) - g(x)| \quad \leq \quad |f(x) - f(x^*)| + |g(x) - g(x^*)| + |f(x^*) - g(x^*)| = \epsilon.$$

Since ϵ was arbitrary, we have shown that for $x \in E_0$, we have $f(x) = g(x)$. Therefore, $f = g$ on $[a, b]$ holds. $\square$

Question 4. Royden 3.5.

Solution. Assume that the function f is defined on a measurable domain E and has a property that $\{x \in E \mid f(x) > c\}$ is measurable for each rational number c. Let $r \in \mathbb{R} \setminus \mathbb{Q}$. Consider the set $\{x \in E \mid f(x) > r\}$. Notice that

$$\{x \in E \mid f(x) > c\} \quad = \quad \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c + \frac{1}{k}\}.$$

By the density of the rationals, we can choose a sequence of rationals, $\{c_k\}$ such that for each k, we have $c_k \in \mathbb{Q}$ and $c_k \in (c, c + \frac{1}{k})$. In particular, we have that

$$\{x \in E \mid f(x) > c\} \quad = \quad \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c_k\}.$$

As $\{c_k\}$ is a rational sequence, $\{x \in E \mid f(x) \geq c_k\}$ is measurable for all k, and $\{x \in E \mid f(x) > c\}$ is measurable, as a countable union of measurable sets is measurable. Since r is an arbitrary irrational, we have shown that $\{x \in E \mid f(x) > a\}$ is measurable for any $a \in \mathbb{R}$. Therefore, f is measurable. $\square$

Question 5. Royden 3.7.

Solution. Let f be a function defined on a measurable set E. We wish to show that f is measurable if and only if an inverse image of any Borel set is measurable. We denote the Borel σ-algebra as $\mathscr{B}$.

Assume that an inverse image of any borel set is measurable. Then, as the (c, ∞) is a borel set for any c, we have that $f^{-1}((c, \infty))$, which can be re-written as $\{x \in E \mid f(x) > c\}$, is measurable for any c. This is precisely the definition of a measurable function. Hence, f is measurable.

Assume that f is measurable. Note that

$$\begin{aligned} \bigcup_{k=1}^{\infty} f^{-1}(E_k) \quad &= \quad f^{-1}(\bigcup_{k=1}^{\infty} E_k) \\ , \bigcap_{k=1}^{\infty} f^{-1}(E_k) \quad &= \quad f^{-1}(\bigcap_{k=1}^{\infty} E_k) \end{aligned}$$

Hence, $f^{-1}(B)$ is measurable for $B \in \mathscr{B}$.

Question 6. Royden 3.9.

Solution. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Let $E_0 = \{x \in E \mid \{f_n(x)\} \text{ converges}\}$. By the Cauchy Criterion of real sequences, we can re-characterize E_0 as follows:

$$\begin{aligned} E_0 &= \{x \in E \mid \forall K \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f_m(x)| < \frac{1}{K} \text{ for } n, m \geq N\} \\ &= \bigcap_{K=1}^{\infty} \bigcup_{N=1}^{\infty} \{x \in E \mid |f_n(x) - f_m(x)| < \frac{1}{K} \text{ for } n, m \geq N\}. \end{aligned}$$

We have that for a measurable function f and g, $|f - g|$ is measurable. Hence, $|f_n - f_m|$ is measurable. Consequently, $\{x \in E \mid |f_n(x) - f_m(x)| < \frac{1}{K} \text{ for } n, m \geq N\}$ is a measurable set for all K and N. Then, E_0 is a countable intersection of countable union of measurable sets, and thus is measurable. We have shown that E_0 is measurable. $\square$