Real Variables:
Problem Set X

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Abstract

This work contains solutions to the problem set X of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 13-41.

41. Let X be the linear space of all polynomials defined on $\mathbf{R}$. For $p \in X$, define $\|p\|$ to be the sum of the absolute values of the coefficients of p. Show that this is a norm on X. For each n, define $\psi_n : X \to \mathbf{R}$ by $\psi_n(p) = p^{(n)}(0)$. Use the properties of the sequence $\{\psi_n\}$ in $\mathcal{L}(X, \mathbf{R})$ to show that X is not a Banach space.

Solution. We first show that $\| \, \| : X \to \mathbb{R}$ given is a norm on X. First of all,

Question 2. Royden 14-18.

18. Let X be a normed linear space, ψ belong to X^*, and $\{\psi_n\}$ be in X^*. Show that if $\{\psi_n\}$ converges weak-$*$ to ψ, then

$$\|\psi\| \leq \limsup \|\psi_n\|.$$

Solution. As $\{\psi_n\}$ is weak-* convergent to ψ, we have

$$\psi_n(x) \quad \to \quad \psi(x),$$

for all $x \in X$. Let $x \in X$. As $|\cdot|$ is continuous on $\mathbb{R}$, it follows that

$$\lim_{n\to\infty} |\psi_n(x)| \quad = \quad |\psi(x)|.$$

As $|\psi_n(x)| \leq ||\psi_n|| \cdot ||x||$,

$$\begin{aligned}
|\psi(x)| \quad &= \quad \lim_{n\to\infty} |\psi_n(x)| \\
&= \quad \limsup_{n\to\infty} |\psi_n(x)| \\
&\leq \quad \limsup_{n\to\infty} ||\psi_n|| \cdot ||x|| \\
&= \quad ||x|| \limsup_{n\to\infty} ||\psi_n||.
\end{aligned}$$

Since $x \in X$ was arbitrary, it follows that

$$||\psi|| \quad \leq \quad \limsup_{n\to\infty} ||\psi_n||,$$

as desired. $\square$

Question 3. Royden 14-23.

23. Let Y be a linear subspace of a normed linear space X and z be a vector in X. Show that

$$\operatorname{dist}(z, Y) = \sup \left\{ \psi(z) \mid \|\psi\| = 1, \psi = 0 \text{ on } Y \right\}.$$

Solution. Consider a functional $p : X \to [0, \infty)$ be defined by

$$p = \begin{cases} 0, & \text{if } x \in Y \\ ||x||, & \text{otherwise.} \end{cases}$$

We first show that p is positively homogeneous. Let $\lambda > 0$. Let $x \in Y$, then as Y is a linear subspace of X, $\lambda x \in Y$. It follows that

$$\begin{aligned} p(\lambda x) &= 0 \\ &= \lambda p(x). \end{aligned}$$

Let $x \notin Y$. It follows that $\lambda x \notin Y$, as otherwise we get a contradiction that $x \in Y$ from the linear subspace property of Y. It follows that

$$\begin{aligned} p(\lambda x) &= ||\lambda x|| \\ &= |\lambda|||x|| \\ &= \lambda||x|| \\ &= \lambda p(x). \end{aligned}$$

Hence, we have shown that p is positively homogeneous. Now, we show that p is sub-additive. Let $x, y \in Y$. Then, it follows that

$$\begin{aligned} p(x+y) &= 0 \\ &= p(x) + p(y). \end{aligned}$$

Hence, $p(x+y) \leq p(x) + p(y)$ holds. Let $x, y \notin Y$. Then, by the triangle inequality of norm, it follows that

$$\begin{aligned} p(x+y) &= ||x+y|| \\ &\leq ||x| + ||y|| \\ &= p(x) + p(y). \end{aligned}$$

Let $x \in Y$ and $y \notin Y$. It follows that

$$\begin{aligned} p(x+y) &= ||x+y|| \\ &\leq ||x|| + ||y|| \\ &= 0 \end{aligned}$$

Question 4. Royden 15-12.

12. If Y is a linear subspace of a Banach space X, we define the *annihilator* $Y^{\perp}$ to be the subspace of X^* consisting of those $\psi \in X^*$ for which $\psi = 0$ on Y. If Y is a subspace of X^*, we define Y^0 to be the subspace of vectors in X for which $\psi(x) = 0$ for all $\psi \in Y$.
    (i) Show that $Y^{\perp}$ is a closed linear subspace of X^*.
    (ii) Show that $(Y^{\perp})^0 = \overline{Y}$.

Solution. *Let $\mathcal{A}$ be an algebra of continuous real-valued functions on a compact Hausdorf space X that separates points. Show that either $\bar{\mathcal{A}} = C(X)$ or there is a point $x_0 \in X$ for which $\bar{\mathcal{A}} = \{f \in C(X) | f(x_0) = 0\}$.*

To solve this problem, it suffices to prove that $\bar{A} = C(X)$ assuming $\bar{\mathcal{A}} \neq \{f \in C(X) | f(x_0) = 0\}$, i.e., $\forall x \in X, \exists f_x \in C(X)$, s.t. $f_x(x) = y_x \neq 0$. Then, the open interval $I_x = (y_x - \delta, y_x + \delta)$ where $0 < \delta < |y_x|$, is mapped to an open interval $O_x = f_x^{-1}(I_x) \subset X$ where $x \in O_x$. Therefore, $X \subseteq \cup_{x \in X} O_x$. Since the foregoing is an open cover of a compact space X, it contains a finite subcover $\cup_{i=1}^{n} O_{x_i}$. By construction, $0 \notin f_{x_i}(O_{x_i})$. Thus $g = \sum_{i=1}^{n} f_{x_i}^2$ is in A and takes strictly positive values on X.

Define $h : K \cup \{0\} \to R_+$ as follows

$$h(x) = \begin{cases} 1/x & \text{if } x \in K \\ 0 & \text{if } x = 0 \end{cases}$$

By Proposition 20, since X is compact and g is a continuous mapping, the range of g, $K = g(X)$, is compact, which in the case of a real-valued range means that it is closed and bounded. Furthermore, since K is closed and $0 \notin K$, it is not possible to have a sequence in K converging to 0. Therefore, $0 \notin \bar{K} = K$, which implies that $h(x)$ is continuous on $D = K \cup \{0\}$

Now, since $h \in C(K)$, by Stone-Weierstrass, given $\epsilon > 0$, there exists p_n, a polynomial, s.t.

$$|h(x) - p_n(x)| < \epsilon/2$$

for all $x \in K$. Since the above implies that $|p_n(0)| < \epsilon/2$

$$|h(x) - (p_n(x) - p_n(0))| \leq |h(x) - p_n(x)| + |p_n(0))| < \epsilon$$

where $p_n^*(x) = p_n(x) - p_n(0)$ (and thus $p_n^*(0) = 0$, which trivially guarantees the uniform convergence on $K \cup \{0\}$). Therefore, p_n^* is continuos on D, and $p_n^* \circ g \in A$. Since p_n^* converges uniformly to h and g is continuous and bounded (and consequently uniformly continuous on a compact set, $p_n^* \circ g \to 1/g$ uniformly. Therefore, $1/g \in \bar{A}$, which together with the fact that $g \in A$, implies that $1 \in \bar{A}$. This generates the family of constant functions in A. By the Stone-Weierstrass Approximation, $\bar{A}$ is dense in $C(X)$. Since $\bar{A}$ is closed, this implies that $\bar{A} = C(X)$.