Real Variables: Problem Set II

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Abstract

This work contains solutions to the problem set II of Real Variables 2015 at NYU.

1 Solutions

Question 3. Royden 2.29.

Solution. (i) Let X be any set of real numbers, and R be the relation defined by the rational equivalence. For $x \in X$, we have $x - x = 0$. Hence, the rational equivalence is reflexive. Let $(x, y) \in R$, then we have $x - y \in \mathbb{Q}$. As a negative of a rational number is rational, we have $y - x \in \mathbb{Q}$ and $(y, x) \in R$. Hence, the rational equivalence is symmetric. Let $(x, y), (y, z) \in R$. As a sum of two rationals is rational, we have $x - y + y - z$, which is $x - z$, is rational, and $(x, z) \in R$. Hence, the rational equivalence is transitive. Therefore, the rational equivalence is an equivalence relation.

(ii) The partition of $\mathbb{Q}$, induced by the rational equivalence, is simply $\{\mathbb{Q}\}$. Hence, $\{0\}$ is a set that consists of exactly one member of each equivalence class. Therefore, $\{0\}$ is an explicit choice set of the rational equivalence.

(iii) We define two numbers to be irrationally equivalent provided their difference is irrational. Let $x \in \mathbb{R}$. As $x - x = 0$ and 0 is a rational number, the relation defined fails to be reflexive. Hence, The relation is not an equivalence relation on $\mathbb{R}$. The same argument holds with $x \in \mathbb{Q}$, and the relation is not an equivalence relation on $\mathbb{Q}$ as well. $\square$

Question 3. Royden 2.38.

Solution. Let $f : [a, b] \to \mathbb{R}$ be Lipschitz with the associated Lipschitz constant c, and let $E_0 \in [a, b]$ such that $\mathrm{m}(E_0) = 0$. Fix $\epsilon > 0$. As $\mathrm{m}(E_0) = 0$, we have a countable collection of disjoint open intervals $\{I_k\}_{k=1}^{\infty}$ such that $E \subseteq \cup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \mathrm{m}(I_k) < \frac{\epsilon}{c}$. Since $E \subseteq \cup_{k=1}^{\infty} I_k$, we have $f(E_0) \subseteq \cup_{k=1}^{\infty} f(I_k)$. By the monotonicity of measure, and Lipshitz property of f, we obtain

$$\mathrm{m}(f(E_0)) \leq \sum_{k=1}^{\infty} \mathrm{m}(f(I_k)) \leq c \sum_{k=1}^{\infty} \mathrm{m}(I_k) = \epsilon.$$

Since ϵ is arbitrary, we have $\mathrm{m}(f(E_0)) = 0$. Therefore, we have shown that a Lipschitz function maps a set of zero measure on to a set of measure zero.

Now, let F be a $F - \sigma$ set in $[a, b]$, which we can express as $\bigcup_{k=1}^{\infty} F_k$, where F_k are closed sets in $[a, b]$. Consider the image of F, $f(\bigcup_{k=1}^{\infty} F_k)$. From the definition of a relation, we have $f(\bigcup_{k=1}^{\infty} F_k) = \bigcup_{k=1}^{\infty} f(F_k)$. Now, notice that F_k is compact for all k, as it is closed and bounded. As Lipschitz property of f implies the continuity of f, and continuity preserves compactness, we have that each $f(F_k)$ is compact for all k. Therefore, $f(F)$ is a countable union of closed set. Hence, we have shown that f carries $F-\sigma$ set to a $F-\sigma$ set.

Now, let E be a measurable set. From the inner approximation of measurable set by $F-\sigma$ sets, there exists a $F-\sigma$ set F such that $F \subseteq E$ and $m(E \setminus F) = 0$. Observe that $f(E) = f(E \setminus F) \cup f(F)$, and we have shown that $f(E \setminus F)$ and $f(F)$ are measurable, as they are respectively a measure zero set and $F-\sigma$ set. Since a union of finite collection of measurable sets is measurable, E is measurable. Therefore, f carries a measurable set to a measurable set.

□

Question 3. Royden 3.1.

Solution. Let f and g are continuous functions on $[a, b]$. Assume that $f = g$ a.e. In other words, $f = g$ on $[a, b] \setminus E_0$, where $\mathrm{m}(E_0) = 0$. Let $x \in E_0$, and fix $\epsilon > 0$. By the continuity of f and g, we have δ_f and δ_g such that

$$\begin{aligned} |x - x'| < \delta_f \quad &\implies \quad |f(x) - f(x')| < \frac{\epsilon}{2} \\ |x - x'| < \delta_g \quad &\implies \quad |g(x) - g(x')| < \frac{\epsilon}{2} \end{aligned} \tag{1}$$

Now, consider the set $B(x, \min(\delta_f, \delta_g)) \cap [a, b]$, where B denotes a ball with a center and radius. As E_0 is a zero measure set, there exists x^* in $B(x, \min(\delta_f, \delta_g)) \cap [a, b]$ such that $f(x^*) = g(x^*)$. Furthermore, by (1), we have that $|f(x) - f(x^*)| < \frac{\epsilon}{2}$ and $|g(x) - g(x^*)| < \frac{\epsilon}{2}$. Consequently, by the trinagle inequality, we have

$$|f(x) - g(x)| \quad \leq \quad |f(x) - f(x^*)| + |g(x) - g(x^*)| + |f(x^*) - g(x^*)| = \epsilon.$$

Since ϵ was arbitrary, we have shown that for $x \in E_0$, we have $f(x) = g(x)$. Therefore, $f = g$ on $[a, b]$ holds.

Now, consider the problem with a measurable domain E. We claim that the assertion need not hold. Consider functions f and g on a common domain $\mathbb{Z}$, defined by

$$f(x) = 0 \text{ for } x \in \mathbb{Z}, g(x) = 0 \text{ for } x \in \mathbb{Z} \setminus \{0\} \text{ and } g(0) = 1.$$

Observe that $\mathbb{Z}$ is measurable, f, g are continous, and $f = g$ almost everywhere, but $f(0) \neq g(0)$. Therefore, the assertion is not true for the case for E measurable. □

Question 4. Royden 3.5.

Solution. Assume that the function f is defined on a measurable domain E and has a property that $\{x \in E \mid f(x) > c\}$ is measurable for each rational number c. Let $r \in \mathbb{R} \setminus \mathbb{Q}$. Consider the set $\{x \in E \mid f(x) > r\}$. Notice that

$$\{x \in E \mid f(x) > c\} \quad = \quad \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c + \frac{1}{k}\}.$$

By the density of the rationals, we can choose a sequence of rationals, $\{c_k\}$ such that for each k, we have $c_k \in \mathbb{Q}$ and $c_k \in (c, c + \frac{1}{k})$. In particular, we have that

$$\{x \in E \mid f(x) > c\} \quad = \quad \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c_k\}.$$

As $\{c_k\}$ is a rational sequence, $\{x \in E \mid f(x) \geq c_k\}$ is measurable for all k, and $\{x \in E \mid f(x) > c\}$ is measurable, as a countable union of measurable sets is measurable. Since r is an arbitrary irrational, we have shown that $\{x \in E \mid f(x) > a\}$ is measurable for any $a \in \mathbb{R}$. Therefore, f is measurable. □

Question 5. Royden 3.7.

Solution. Let f be a function defined on a measurable set E. We wish to show that f is measurable if and only if an inverse image of any Borel set is measurable. We denote the Borel σ-algebra as $\mathscr{B}$.

Assume that an inverse image of any borel set is measurable. Then, as the (c, ∞) is a borel set for any c, we have that $f^{-1}((c,\infty))$, which can be re-written as $\{x \in E \mid f(x) > c\}$, is measurable for any c. This is precisely the definition of a measurable function. Hence, f is measurable.

Assume that f is measurable. Let B be a borel set. As f is measurable, we have that the collection $\{f^{-1}((c,\infty))\}_{c\in\mathbb{R}}$ forms a collection of measurable sets. Now, consider the σ-algebra genreated by the above collection, denoted by

$$\sigma(\{f^{-1}((c,\infty))\}_{c\in\mathbb{R}}).$$

As σ-algebra of measurable collection is a measurable collection itself, the above σ-algebra is a collection of measurable sets. Notice that the following identities hold: let $A_k \subseteq f(E)$ for all k. Then, from the definition of a relation, we obtain

$$\begin{aligned}
\bigcup_{k=1}^{\infty} f^{-1}(A_k) &= f^{-1}(\bigcup_{k=1}^{\infty} A_k), \\
\bigcap_{k=1}^{\infty} f^{-1}(A_k) &= f^{-1}(\bigcap_{k=1}^{\infty} A_k).
\end{aligned}$$

Therefore, by the above identity, we have

$$\sigma(\{f^{-1}((c,\infty))\}_{c\in\mathbb{R}}) = \bar{f}^{-1}(\sigma(\{(c,\infty)\}_{c\in\mathbb{R}})),$$

where $\bar{f}^{-1}$ denotes applying the inverse f^{-1} to each set in the collection pointwise. As we know that $\{\sigma((c,\infty)\}_{c\in\mathbb{R}}) = \mathscr{B}$, we obtain

$$\sigma(\{f^{-1}((c,\infty))\}_{c\in\mathbb{R}}) = \bar{f}^{-1}(\mathscr{B}).$$

As we have argued that the collection on the LHS is a collection of measurable sets, we have shown that $f^{-1}(B)$ is measurable for $B \in \mathscr{B}$. $\square$

Question 6. Royden 3.9.

Solution. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Let $E_0 = \{x \in E \mid \{f_n(x)\}$ converges$\}$. By the Cauchy Criterion of real sequences, we can re-characterize E_0 as follows:

$$\begin{aligned}
E_0 &= \{x \in E \mid \forall K \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that } |f_n(x) - f_m(x)| < \frac{1}{K} \text{ for } n, m \geq N\} \\
&= \bigcap_{K=1}^{\infty} \bigcup_{N=1}^{\infty} \{x \in E \mid |f_n(x) - f_m(x)| < \frac{1}{K} \text{ for } n, m \geq N\}.
\end{aligned}$$

We have that for a measurable function f and g, $|f - g|$ is measurable. Hence, $|f_n - f_m|$ is measurable. Consequently, $\{x \in E \mid |f_n(x) - f_m(x)| < \frac{1}{K}$ for $n, m \geq N\}$ is a measurable set for all K and N. Then, E_0 is a countable intersection of countable union of measurable sets, and thus is measurable. We have shown that E_0 is measurable. $\square$