Real Variables:
Problem Set I

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Abstract

This work contains the solutions to the first problem set of Real Variables 2015.

1 Solutions

Question 1. Royden 2.4. Counting Measure.

Solution. We wish to show that the counting measure, $c : \mathcal{P}(\mathbb{R}) \to [0, \infty]$, where $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$, is countably additive and translation invariant.

First, we prove that it is countably additive. Let $\{E_k\}_{k=1}^{\infty}$ be a countable, disjoint collection of subsets of $\mathbb{R}$. If one of the set in the collection has infinite cardinality, then we have

$$\sum_{k=1}^{\infty} c(E_k) \quad = \quad \infty,$$

as $c(E_k) = \infty$ for some k. Notice that the union of the collection $\cup_{k=1}^{\infty} E_k$, also has infinite cardinality, as it has a subset with an infinite cardinality. Hence, by the definition of counting measure, we have $c(\cup_{k=1}^{\infty} E_k) = \infty$. Therefore, we have

$$c(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} c(E_k),$$

for the case under consideration. Now, assume that $c(E_k) < \infty$ for all k.

Question 2. Royden 2.8. Closure of Finite Union.

Solution. Let B be a set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that cover B. As $B \subseteq \cup_{k=1}^n I_k$, we have $\overline{B} \subseteq \overline{\cup_{k=1}^n I_k}$. Furthermore, with n being finite, we obtain that $\overline{\cup_{k=1}^n I_k} = \cup_{k=1}^n \overline{I_k}$. Then, it follows from the monotonicity, and finite sub-additivity property that

$$m^*(\overline{B}) \leq m^*(\cup_{k=1}^n \overline{I_k}) \leq \sum_{k=1}^n m^*(\overline{I_k}). \tag{1}$$

In particular, we have $m^*(\overline{B}) = 1$, as $B = [0, 1]$, and $\sum_{i=1}^n m^*(\overline{I_k}) = \sum_{i=1}^n m^*(I_k)$, as the outer measure of an open interval and corresponding closed interval are equal. Substituting the two equalities into the (1) inequality, we obtain

$$\sum_{i=1}^n m^*(I_k) \geq 1,$$

as desired. $\square$

Question 3. Royden 2.14.

Solution. Let $m^*(E) > 0$. We wish to find a subset X of E such that $m^*(X) > 0$. Consider the countable collection of sets $\{(-M, M)\}_{M=1}^\infty$. Notice that, as $(-M, M)$ is bounded for some fixed M, $E \cap (-M, M)$ is a bounded subset of E. Furthermore, $E = \cup_{M=1}^\infty E \cap (-M, M)$. Then, by the countable sub-additivity of outer measure, we have

$$\sum_{M=1}^\infty m^*(E \cap (-M, M)) \geq m^*(E).$$

If $m^*(E \cap (-M, M)) = 0$ for all M, then we have the sum on the LHS equals 0, and obtain $0 > 0$, as $m^*(E) > 0$. This is a contradiction. Hence, there exists a M such that $m^*(E \cap (-M, M)) > 0$, and $E \cap (-M, M)$ is precisely the bounded subset of E with positive outer measure. We have shown that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure. $\square$

Question 3. Royden 2.15.

Solution. Let $m(E) < \infty$ and $\epsilon > 0$. We wish to show that E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

Question 5. Royden 2.17.

Solution. Let E be a measurable set. Fix $\epsilon > 0$. Then, from inner approximation by closed sets, and outer approximation by open sets, there exists a closed set F and an open set O, such that

$$E \subseteq O \;\text{ with }\; m^*(O \setminus E) < \frac{\epsilon}{2} \;\text{ and }\; F \subseteq E \;\text{ with }\; m^*(E \setminus F) < \frac{\epsilon}{2}.$$

Applying the sub-additivity property of outer measure with $O \setminus E$ and $E \setminus F$, we have

$$m^*(O \setminus F) < m^*(O \setminus E) + m^*(E \setminus F) < \epsilon.$$

Hence, if E is measurable, then there exists an open set O and a closed set F for which $F \subseteq E \subseteq O$ and $m^*(E \setminus f) < \epsilon$.

Question 6. Royden 2.28.

Solution. Let $\{E_k\}_{k=1}^\infty$ be a countable, disjoint collection of measurable sets. By the finite additivity property, we have

$$m(\cup_{k=1}^N E_k) = \sum_{k=1}^N m(E_k),$$

for all N. Notice that $\{\cup_{k=1}^N E_k\}_{N=1}^\infty$ forms an ascending collection of measurable sets. Hence, by applying the continuity of measure to the ascending collection, $\{\cup_{k=1}^N E_k\}_{N=1}^\infty$, we have

$$m(\cup_{N=1}^\infty \cup_{k=1}^N E_k) \quad = \quad \lim_{N\to\infty} m(\cup_{k=1}^N E_k).$$

Simplifying the LHS and applying the finite additivity property to the RHS, we obtain

$$m(\cup_{k=1}^\infty E_k) = \sum_{k=1}^\infty m(E_k).$$

Since $\{E_k\}_{k=1}^\infty$ was chosen to be an arbitrary countable, disjoint collection of measurable sets, we have shown that finite additivty and continuity of measure implies countable additivity. $\square$

Question 7. Extra Problem. .

Solution.