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# Real Variables: Problem Set V

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## Abstract

This work contains solutions to the problem set V of Real Variables 2015 at NYU.

## 1 Solutions

### Question 6.33.

**Solution.** Let  $\{f_n\}$  be a sequence of real-valued functions on  $[a, b]$  that converges pointwise on  $[a, b]$  to the real-valued function  $f$ . We wish to show that  $TV(f) \leq \liminf TV(f_n)$ . Fix  $\epsilon > 0$ . Let  $P = \{x_0, \dots, x_m\}$  be a partition of  $[a, b]$ . By the triangle inequality, it follows that

$$\begin{aligned} V(f, P) &= \sum_{k=0}^{m-1} |f(x_{k+1}) - f(x_k)| \\ &= \sum_{k=0}^{m-1} |f(x_{k+1}) + f_n(x_{k+1}) - f_n(x_{k+1}) - f(x_k) + f_n(x_k) - f_n(x_k)| \\ &\leq \sum_{k=0}^{m-1} |f(x_{k+1}) - f_n(x_{k+1})| + |f_n(x_{k+1}) - f_n(x_k)| + |f(x_k) - f_n(x_k)| \\ &\leq V(f_n, P) + \sum_{k=1}^m |f(x_k) - f_n(x_k)| + \sum_{k=0}^{m-1} |f(x_k) - f_n(x_k)|, \end{aligned}$$

for any  $n$ . Define  $N = \max(N_0, \dots, N_k)$ , where  $N_i$  ( $0 \leq i \leq k$ ) corresponds to the convergence index for  $\frac{\epsilon}{2m}$  at  $x_i$ . Then, it follows that

$$\begin{aligned} V(f, P) &\leq V(f_n, P) \\ &\leq TV(f_n) \end{aligned}$$

for  $n \geq N$ . Consequently

$$V(f, P) \leq \liminf_{n \rightarrow \infty} TV(f_n),$$

Since  $P$  was arbitrary, we obtain that

$$TV(f) \leq \liminf_{n \rightarrow \infty} TV(f_n),$$

as desired.  $\square$

### Question 4. Royden 6.42.

**Solution.** Let  $f$  and  $g$  be real-valued functions, that are absolutely continuous functions on  $[a, b]$ . We wish to show that  $f + g$  is absolutely continuous on  $[a, b]$ . Fix  $\epsilon > 0$ . As  $f$  and  $g$  are both absolutely continuous on  $[a, b]$ , there exist  $\delta_f, \delta_g > 0$ , such that for any finite disjoint open intervals  $\{(a_k, b_k)\}_{k=1}^n$  in  $(a, b)$ ,

$$\begin{aligned}\sum_{k=1}^n [b_k - a_k] < \delta_f &\implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \frac{\epsilon}{2} \\ \sum_{k=1}^n [b_k - a_k] < \delta_g &\implies \sum_{k=1}^n |g(b_k) - g(a_k)| < \frac{\epsilon}{2}.\end{aligned}$$

Define  $\delta = \min(\delta_f, \delta_g)$ . Let  $\{(a_k, b_k)\}_{k=1}^n$  be a finite disjoint open intervals in  $(a, b)$ , such that  $\sum_{k=1}^n [b_k - a_k] < \delta$ . It follows that

$$\begin{aligned}\sum_{k=1}^n |f + g(b_k) - f + g(a_k)| &\leq \sum_{k=1}^n |f(b_k) - f(a_k)| + |g(b_k) - g(a_k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Since  $\epsilon$  was arbitrary, we have shown that  $f + g$  is absolutely continuous on  $[a, b]$ .

Let  $f$  be a real-valued function, that is absolutely continuous on  $[a, b]$ . We show that  $cf$ , for any  $c \in \mathbb{R}$ , is absolutely continuous on  $[a, b]$ . Let  $c = 0$ . Then  $cf = 0$ , which can trivially be shown to be absolutely continuous, as  $f(c) = 0$  for any  $c \in [a, b]$ . Suppose  $c \neq 0$ . As  $f$  is absolutely continuous on  $[a, b]$ , there exists  $\delta_f > 0$ , such that for any finite disjoint open intervals  $\{(a_k, b_k)\}_{k=1}^n$  in  $(a, b)$ ,

$$\sum_{k=1}^n |b_k - a_k| < \delta_f \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \frac{\epsilon}{|c|}.$$

Define  $\delta = \delta_f$ . Let  $\{(a_k, b_k)\}_{k=1}^n$  be a finite disjoint open intervals in  $(a, b)$  such that  $\sum_{k=1}^n [b_k - a_k] < \delta$ . It follows that

$$\begin{aligned}\sum_{k=1}^n |cf(b_k) - cf(a_k)| &= |c| \sum_{k=1}^n |f(b_k) - f(a_k)| \\ &< |c| \frac{\epsilon}{|c|} = \epsilon.\end{aligned}$$

Since  $\epsilon$  was arbitrary, combined with the  $c = 0$  case, we have shown that  $cf$ , for any  $c \in \mathbb{R}$ , is absolutely continuous on  $[a, b]$ .

Let  $f$  be a real-valued function, that is absolutely continuous on  $[a, b]$ . We wish to show that  $f^2$  is absolutely continuous on  $[a, b]$ . As  $f$  is absolutely continuous,  $f$  is continuous on  $[a, b]$ . Hence, by the Extreme Value Theorem, there exists  $M$  such that  $|f| \leq M$  on  $[a, b]$ . Fix  $\epsilon > 0$ . As  $f$  is absolutely continuous on  $[a, b]$ , there exists  $\delta_f > 0$ , such that for any finite disjoint open intervals  $\{(a_k, b_k)\}_{k=1}^n$  in  $(a, b)$ ,

$$\sum_{k=1}^n |b_k - a_k| < \delta_f \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \frac{\epsilon}{2M}.$$

Define  $\delta = \delta_f$ . Let  $\{(a_k, b_k)\}_{k=1}^n$  be a finite disjoint open intervals in  $(a, b)$  such that  $\sum_{k=1}^n [b_k - a_k] < \delta$ . It follows that

$$\begin{aligned}\sum_{k=1}^n |f^2(b_k) - f^2(a_k)| &= \sum_{k=1}^n |f(b_k) - f(a_k)| |f(b_k) + f(a_k)| \\ &\leq 2M \sum_{k=1}^n |f(b_k) - f(a_k)| \\ &< 2M \frac{\epsilon}{2M} = \epsilon.\end{aligned}$$

Since  $\epsilon$  was arbitrary, we have shown that  $f^2$  is absolutely continuous on  $[a, b]$ .

Let  $f$  and  $g$  be real-valued functions, that are absolutely continuous on  $[a, b]$ . We wish to show that  $fg$  is absolutely continuous on  $[a, b]$ . Observe that

$$(f + g)^2 = f^2 + g^2 + 2fg,$$

which simplifies to

$$fg = -\frac{1}{2}((f + g)^2 + (-f^2) + (-g^2)).$$

As we have previously shown that a square of an absolutely continuous function is absolutely continuous, and a scalar multiple of an absolutely continuous function is absolutely continuous, we can deduce that  $fg$  is absolutely continuous on  $[a, b]$ . This completes the proof.  $\square$

#### Question 4. 6.45.

**Solution.** Let  $f$  be a real-valued function, that is absolutely continuous on  $\mathbb{R}$ . Let  $g$  be a real-valued function, that is absolutely continuous and strictly monotone on  $[a, b]$ . Without the loss of generality, we assume that  $g$  is strictly increasing. We wish to show that  $f \circ g$  is absolutely continuous on  $[a, b]$ . Fix  $\epsilon > 0$ . As  $f$  is absolutely continuous on  $\mathbb{R}$ , it is also absolutely continuous on  $[g(a), g(b)]$ , which is a non-degenerate closed interval, as  $g$  is strictly increasing. there exists  $\delta_f$ , such that for any finite disjoint open intervals  $\{(a_k, b_k)\}_{k=1}^\infty$  in  $(g(a), g(b))$ ,

$$\sum_{k=1}^n [b_k - a_k] < \delta_f \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon \quad (*).$$

As  $g$  is absolutely continuous, there exists  $\delta_g$ , such that for any finite disjoint open intervals  $\{(a_k, b_k)\}_{k=1}^\infty$  in  $(a, b)$ ,

$$\sum_{k=1}^n [b_k - a_k] < \delta_g \implies \sum_{k=1}^n |g(b_k) - g(a_k)| < \delta_f.$$

Define  $\delta = \delta_g$ . Let  $\{(a_k, b_k)\}_{k=1}^n$  be a finite disjoint open intervals in  $(a, b)$  such that  $\sum_{k=1}^n [b_k - a_k] < \delta_g$ . It follows that  $\sum_{k=1}^n [g(b_k) - g(a_k)] < \delta_f$ . As  $g$  is strictly increasing, we observe that  $\{(g(a_k), g(b_k))\}_{k=1}^n$  form a finite disjoint open intervals in  $(g(a), g(b))$ . Therefore, from (\*) it follows that

$$\sum_{k=1}^n |f \circ g(b_k) - f \circ g(a_k)| < \epsilon.$$

Since  $\epsilon$  was arbitrary, we have shown that  $f \circ g$  is absolutely continuous on  $[a, b]$ .  $\square$