Real Variables:
Problem Set IV

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Abstract

This work contains solutions to the problem set IV of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 4.31.

Solution. Let f be a measurable function on E, which can be expressed as $f = g + h$ on E, where g is finite and integrable over E and h is nonnegative. Let $f = g_1 + h_1$ and $f = g_2 + h_2$ satisfying the given properties of g and h respectively. We wish to show that

$$\int_E g_1 + \int_E h_1 = \int_E g_2 + \int_E h_2.$$

Assume that h_1 and h_2 are integrable. Then, by the linearity of integration, we have

$$\begin{aligned} \int_E g_1 + \int_E h_1 &= \int_E g_1 + h_1 \\ &= \int_E f \\ &= \int_E g_2 + h_2 \\ &= \int_E g_2 + \int_E h_2, \end{aligned}$$

as desired. Now, consider the remaining case of at least one of h not being integrable. Without loss of generality, assume that $\int_E h_1 = \infty$. Since $g_1 + h_1 = g_2 + h_2$, we have

$$\begin{aligned} h_2 &= h_1 + g_1 - g_2 \\ &= h_1 + (g_1 - g_2)^+ - (g_1 - g_2)^- \\ &\geq h_1 - (g_1 - g_2)^-, \end{aligned}$$

with $(g_1 - g_2)^+$ and $(g_1 - g_2)^-$ being properly defined by the finiteness assumption on the gs. Since h_2, h_1 and $(g_1 - g_2)^-$ are all non-negative measurable functions, by the monotonicity and linearity of integration of non-negative measurable functions, we have

$$\begin{aligned} \int_E h_2 &\geq \int_E h_1 - (g_1 - g_2)^- \\ &= \int_E h_1 - \int_E (g_1 - g_2)^-. \end{aligned}$$

From the linearity of general integrable functions, we have that $g_1 - g_2$ is integrable. Consequently, $(g_1 - g_2)^-$ is integrable as well. It follows that

$$\begin{aligned} \left| \int_E (g_1 - g_2)^- \right| &\leq \int_E |(g_1 - g_2)^-| \\ &< \infty. \end{aligned}$$

Therefore, we obtain that

$$\int_E h_1 - \int_E (g_1 - g_2)^- = \infty,$$

which combined with the established inequality of $\int_E h_2 \geq \int_E h_1 - \int_E (g_1 - g_2)^-$ yields

$$\int_E h_2 = \infty.$$

Hence, we have

$$\begin{aligned} \int_E g_1 + \int_E h_1 &= \infty \\ &= \int_E g_2 + \int_E h_2, \end{aligned}$$

as g_1 and g_2 are integrable. This completes the proof. $\square$

Question 2. Royden 4.44.

Solution. Let f be integrable over $\mathbb{R}$ and $\epsilon > 0$.

(i) First, we prove the given property for f nonnegative. Assume $f \geq 0$. Since f is integrable, thus measuralbe, by the Simple Approximation Theorem, there exists a sequence of increasing simple functions $\{\phi_n\}$ on $\mathbb{R}$ which converges pointwise on $\mathbb{R}$ to f, such that

$$|\phi_n| \leq |f| \text{ on } \mathbb{R},$$

for all n. Now, define a new sequence of simple function by

$$\psi_n = max\{0, \phi_n\} \cdot \chi_{[-n,n]}.$$

Observe that $\{\psi_n\}$ is an increasing sequence of simple functions on $\mathbb{R}$, which has finite support and is non-negative, that converges to f pointwise. By the Monotone convergnece theorem, there exists N such that

$$|\int_{\mathbb{R}} f - \int_{\mathbb{R}} \psi_n| < \epsilon,$$

for $n \geq N$. By the linearity of integration and the fact that $\psi_n \leq f$ for all n, we have

$$\begin{aligned} \epsilon &> |\int_{\mathbb{R}} f - \int_{\mathbb{R}} \psi_n| \\ &= |\int_{\mathbb{R}} f - \psi_n| \\ &= \int_{\mathbb{R}} |f - \psi_n|, \end{aligned}$$

for $n \geq N$. Therefore, we have found a function with the desired property, namely ψ_n.

Now, we lift the non-negativity constraint. By the definition of integral, we have

$$\int_{\mathbb{R}} f = \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-.$$

Since f is integrable, f^+ and f^- are integrable and from the previous result, we have simple functions ψ^+ and ψ^- with finite support such that

$$\int_{\mathbb{R}} |f^+ - \psi^+| < \frac{\epsilon}{2}$$
$$\int_{\mathbb{R}} |f^- - \psi^-| < \frac{\epsilon}{2}.$$

Observe that $\psi^+ - \psi^-$ is simple and has finite support as well. Now, by the triangle inequality and monotonicity of integration, it follows that

$$\begin{aligned}
\int_{\mathbb{R}} |f - (\psi^+ - \psi^-)| &= \int_{\mathbb{R}} |f^+ - f^- - \psi^+ + \psi^-| \\
&\leq \int_{\mathbb{R}} |f^+ - \psi^+| + |f^- - \psi^-| \\
&= \int_{\mathbb{R}} |f^+ - \psi^+| + \int_{\mathbb{R}} |f^- - \psi^-| \\
&< \epsilon.
\end{aligned}$$

Therefore, $\psi^+ - \psi^-$ is the construction of the function with the desired property. We have shown that there is a simple function η on $\mathbb{R}$ which has a finite support and $\int_R |f - \eta| < \epsilon$.

(ii) From the result of (i), there exists a simple function η on $\mathbb{R}$ with finite support such that $\int_{\mathbb{R}} |f - \eta| < \dfrac{\epsilon}{4}$. We denote the support of η as S. Since η is simple, it is measuarble and bounded. Let M be the max of upper bound of $|\eta|$ and 1. Since $mS < \infty$, there exists a sufficiently large close bounded interval $I = [-N, N]$, such that $m(S \setminus I) < \dfrac{\epsilon}{4M}$. Furthermore, by the result from $3 - 18$, as η is measurable and bounded, there exists a step function s and a measurable subset F of I such that

$$|s - \eta| < \frac{\epsilon}{4m(I)M} \text{ on } F \text{ and } m(I \setminus F) < \frac{\epsilon}{4m(I)M}.$$

It follows that

$$\begin{aligned}
\int_{\mathbb{R}} |f - s| &= \int_{\mathbb{R}} |f - \eta + \eta - s| \\
&\leq \int_{\mathbb{R}} |f - \eta| + |\eta - s| \\
&= \int_{\mathbb{R}} |f - \eta| + \int_{\mathbb{R}} |\eta - s| \\
&= \int_{\mathbb{R}} |f - \eta| + \int_{I} |\eta - s| + \int_{S \setminus I} |\eta - s| \\
&= \int_{\mathbb{R}} |f - \eta| + \int_{I \setminus F} |\eta - s| + \int_{F} |\eta - s| + \int_{S \setminus I} |\eta - s|.
\end{aligned}$$

As $s = 0$ outside of I, we have that

$$\int_{\mathbb{R}} |f - s| \leq \int_{\mathbb{R}} |f - \eta| + \int_{I \setminus F} |\eta - s| + \int_{F} |\eta - s| + \int_{S \setminus I} |\eta|.$$

From constructions, we have the following bounds on each term on the RHS of the inequality:

$$\begin{aligned}
\int_{\mathbb{R}} |f - s| &\leq \frac{\epsilon}{4} \\
\int_{I \setminus F} |\eta - s| &\leq M \frac{\epsilon}{4m(I)M} \leq \frac{\epsilon}{4} \\
\int_{F} |\eta - s| &\leq \int_{I} |\eta - s| \leq \frac{\epsilon}{4m(I)M} m(I) \leq \frac{\epsilon}{4} \\
\int_{S \setminus I} |\eta| &\leq \frac{\epsilon}{4M} M = \frac{\epsilon}{4}.
\end{aligned}$$

Hence, it follows that

$$\int_{\mathbb{R}} |f - s| \leq \epsilon,$$

as desired. $\square$

(iii)

Question 2. Royden 4.47.

Solution. Let g be integrable over $\mathbb{R}$.

(i) Let $k \in \mathbb{R}$. Assume that g non-negative. Let $E_n = [-n, n]$ and $E_n - k = [-n-k, n-k]$. By the definition of integration of non-negative functions, it follows that

$$\begin{aligned}
\int_{E_n} g(x)dx &= sup\{\int_{E_n} h(x)dx \mid h \text{ bounded, measurable, of finite support and} \\
&\quad 0 \leq h(x) \leq g(x) \ \text{ for } \ x \in E_n\} \\
&= sup\{\int_{E_n - k} h(x+k)dx \mid h \text{ bounded, measurable, of finite support and} \\
&\quad 0 \leq h(x+k) \leq g(x+k) \text{ for } \ x \in E_n - k\} \\
&= \int_{E_n - k} g(x+k)dx.
\end{aligned}$$

Since the general integral is defined as the sum of non-negative integrals, for integrable functions, the result trivially generalizes to an integrable function. From this point on, we drop the non-negativity assumption on g and assume that g is integrable. Notice that $\{E_n\}$ and $\{E_n - k\}$ form ascending countable collection of measurable subsets of $\mathbb{R}$, with $\cup_{n=1}^{\infty} E_n = \cup_{n=1}^{\infty} E_n - k = \mathbb{R}$. Hence, by the continuity of integration, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} g(x)dx &= \lim_{n\to\infty} \int_{E_n} g(x)dx \\
\int_{\mathbb{R}} g(x+k)dx &= \lim_{n\to\infty} \int_{E_n - k} g(x+k)dx.
\end{aligned}$$

Since $\int_{E_n} g(x)dx = \int_{E_n - k} g(x+k)dx$ for all n, it follow that

$$\int_{\mathbb{R}} g(x)dx = \int_{\mathbb{R}} g(x+k)dx,$$

as desired. $\square$

(ii) Fix $\epsilon > 0$. Since g is a bounded function, there exists M such that $g \leq M$. Assume that f is a continous function that vanishes outside of a bounded set. Let I be a sufficiently large closed interval of the form $[-N, N]$ such that the bounded set is contained in I. Then, as f is continuous on a closed, bounded interval I, it is uniformly continous on I. Hence, there exists $\delta > 0$, such that

$$|t| < \delta \implies |f(x) - f(x+t)| \frac{\epsilon}{Mm(I)}$$

It follows that

$$\begin{aligned}
\lim_{t\to 0} \int_{\mathbb{R}} |g(x)[f(x) - f(x+t)]| &= \lim_{t\to 0} |\int_I g(x)[f(x) - f(x+t)]| \\
&\leq \lim_{t\to 0} \int_I |g(x)[f(x) - f(x+t)]| \\
&= \lim_{t\to 0} \int_I |g(x)||[f(x) - f(x+t)]| \\
&\leq \lim_{t\to 0} \int_I |g(x)| \frac{\epsilon}{Mm(I)} \\
&\leq m(I)M \frac{\epsilon}{Mm(I)} \\
&= \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we have that

$$\lim_{t\to 0}\int_{\mathbb{R}} g(x)[f(x)-f(x+t)] \quad = \quad 0$$

Now, assume that f is not continous and vanishes outside of a boudned set. As f is integrable over $\mathbb{R}$, from the approximation property (iii) of 4.44, there exists a continous function h on $\mathbb{R}$ such that it vanishes outside a bounded set and $\int_{\mathbb{R}}|f-h| < \dfrac{\epsilon}{2M}$. As we have proven the result for a continous function that vanishes outside of a bounded set, we obtain

$$\lim_{t\to 0}\int_{\mathbb{R}} g(x)[h(x)-h(x+t)] \quad = \quad 0.$$

With the above limit being 0, it follows that

$$\begin{aligned}
\lim_{t\to 0}|\int_{\mathbb{R}} g(x)[f(x)-f(x+t)]| &= \lim_{t\to 0}|\int_{\mathbb{R}} g(x)[f(x)-f(x+t)]-0| \\
&= \lim_{t\to 0}|\int_{\mathbb{R}} g(x)[f(x)-f(x+t)]-\int_{\mathbb{R}} g(x)[h(x)-h(x+t)]|,
\end{aligned}$$

provided that the limit exists. The above equality can be simplified via linearity of integration as follows:

$$\begin{aligned}
\lim_{t\to 0}|\int_{\mathbb{R}} g(x)[f(x)-f(x+t)]| &= \lim_{t\to 0}|\int_{\mathbb{R}} g(x)[f(x)-f(x+t)-h(x)+h(x+t)]| \\
&\le \lim_{t\to 0}\int_{\mathbb{R}} |g(x)[f(x)-h(x)]+g(x)[f(x+t)-h(x+t)]| \\
&\le \lim_{t\to 0}\int_{\mathbb{R}} |g(x)||[f(x)-h(x)]|+|g(x)||[f(x+t)-h(x+t)]| \\
&\le \lim_{t\to 0}\int_{\mathbb{R}} M\int_{\mathbb{R}}|f(x)-h(x)|+M\int_{\mathbb{R}}|f(x+t)-h(x+t)| \\
&\le 2M\frac{\epsilon}{2M}=\epsilon
\end{aligned}$$

Since ϵ was arbitrary, we have

$$\lim_{t\to 0}\int_{\mathbb{R}} g(x)[f(x)-f(x+t)] \quad = \quad 0,$$

as desired. $\square$

Question 4. Royden 4.52.

Solution. (a) Consider the following family of functions:

$$\mathscr{F}=\{n\chi_{[0,\frac{1}{n}]}\}_{n=1}^{\infty}.$$

Observe that for each n, $n\chi_{[0,\frac{1}{n}]}$ is integrable and $\int_0^1|n_\chi[0,\frac{1}{n}]|=1$. The family $\mathscr{F}$, however, fails to be uniformly integrable. Fix $\epsilon=\dfrac{1}{2}$. Then, for any $\delta>0$, by the Archimedean property of the reals, there exists n, such that $\dfrac{1}{n}<\delta$. Since the interval $[0,\dfrac{1}{n}]$ is measurable, has a measure smaller than δ, and $\int_0^{\frac{1}{n}} n\chi_{[0,\frac{1}{n}]}=1>\dfrac{1}{2}$, we have that $\mathscr{F}$ is not uniformly integrable. Hence, by a counter example, we have shown that under the given assumptions, the family of functions need not be uniformly integrable.

(b) We claim that $\mathscr{F}$ with the given assumption is uniformly integrable. Note that continuity implies integrability. Fix $\epsilon>0$. Let $f\in\mathscr{F}$. Then, for any measurable set $E\subseteq[0,1]$ with $mE<\delta$ with, by using the $|f| \leq 1$ bound, we obtain

$$\begin{aligned}
\int_E f &\leq \int_E |f| \\
&\leq \int_E 1 \\
&= mE \\
&\leq \delta
\end{aligned}$$

By letting $\delta = \epsilon$, we have $\int_E f \leq \epsilon$. Since ϵ and f were arbitrary, we have shown that $\mathscr{F}$ is uniformly integrable.

(c) Let $\mathscr{F}$ be the family of functions f on $[0,1]$, each of which is integrable over $[0,1]$ and has $\int_a^b |f| \leq b - a$ for all $[a,b] \subseteq [0,1]$. We claim that $\mathscr{F}$ is uniformly integrable. Fix $\epsilon > 0$ and fix $f \in \mathscr{F}$. Let $A \subseteq [0,1]$ be a measurable set such that $mA < \delta$ By the outer approximation of measurable set by open sets, there exists an open set O such that $A \subseteq O$ and $m(O \backslash A) \leq \dfrac{\epsilon}{2}$. Observe that O can be written as a countable union of disjoint open intervals, which gives $O = \cup_{i=1}^{\infty}(a_i, b_i)$. From the monotonicity and excision property of measure, and countable additivity over domain property of integration, it follows that

$$\begin{aligned}
\int_A |f| &\leq \int_O |f| \\
&\leq \int_{\cup_{i=1}^{\infty}(a_i,b_i)} |f| \\
&= \sum_{i=1}^{\infty} \int_{(a_i,b_i)} |f| \\
&\leq \sum_{i=1}^{\infty} \int_{[a_i,b_i]} |f| \\
&\leq \sum_{i=1}^{\infty} b_i - a_i \\
&= mO \\
&= m(O \setminus A) + m(A) \\
&\leq \frac{\epsilon}{2} + \delta.
\end{aligned}$$

Define $\delta = \dfrac{\epsilon}{2}$ then, we have if A is measurable, and $mA < \delta$, then $\int_A |f| < \epsilon$. Since ϵ and f were arbitrary, we have that $\mathscr{F}$ is uniformly integrable. $\square$

Question 5. 5-11.

Solution. Assume that E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable on E for which f and each f_n is finite a.e.

We first show that the if implication holds by proving its contrapositive. Assume that f_n does not converge to f in measure. This implies that there exists $\delta > 0$ and $\eta > 0$ such that

$$m\{x \in E \mid |f_n(x) - f(x)| > \eta\} > \delta,$$

infinitely often in the sequence of f_n. Choose $\{f_{n_k}\}$ such that

$$m\{x \in E \mid |f_{n_k}(x) - f(x)| > \eta\} > \delta,$$

for all k. Hence, there is a non measure-zero set, on which $|f_{n_k} - f| > \delta$ for all k. Hence, any subsequence of $\{f_{n_k}\}$ cannot converge pointwise a.e. on E. Therefore, there exists a subsequence of $\{f_n\}$ who does not have a further subsequence that converges to f pointwise a.e. on E, which completes the proof.

Now, we prove the only if implication. Assume that $f_n \to f$ in measure on E. Fix $\eta > 0$ and $\epsilon > 0$. Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$. Then, by the definition of convergence in measure, there exists N such that

$$m\{x \in E \mid |f_n(x) - f(x)| < \eta\} < \epsilon,$$

for $n \geq N$. Note that there exists K, such that $n_K \geq N$. Since the inequality holds for all $n \geq N$, we have

$$m\{x \in E \mid |f_{n_k}(x) - f(x)| < \eta\} < \epsilon,$$

for $k \geq K$. Since η and ϵ were arbtirary, we have that f_{n_k} converges to f in measure by definition. Then, by the Riesz theorem, we have that there exists a further subseuqence of $\{f_{n_k}\}$ that converges pointwise a.e. on E to f. Since the subsequence was arbitrary, we have shown that every subsequence has a further subseuqence that converges to pointwise to f a.e. on E. $\square$

Question 6. 5-13.

Solution. Suppose Cauchy in measure. Define E_j given by the hint. Since E_j is summable, by Borel Cantelli, measure of points that occur infinitely often is measure zero. Thus, off a set of measure zero f_{n_j} is Cauchy by 5-11. By the completness of the reals, the f_{n_j} converges to a function a.e. This shows that any sequence of functions that is Cauchy in measure has a subsequence that converges pointwise a.e. Observe that every subsequence of a sequence that is Cauchy in meausre is also Cauchy in measure. Thus, we have shown that every sequence has a further subsequence that converge pointwise. $\square$