Real Variables:
Problem Set XII

Youngduck Choi
Courant Institute of Mathematical Sciences
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set XII of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 20-10.

10. Let h and g be integrable functions on X and Y, and define $f(x, y) = h(x)g(y)$. Show that f is integrable on $X \times Y$ with respect to the product measure, then

$$\int_{X\times Y} f \, d(\mu \times \nu) = \int_X h \, d\mu \int_Y g \, d\nu.$$

(Note: We do not need to assume that μ and ν are σ-finite.)

Solution.

Question 2. Royden 20-34.

34. Let f be a nonnegative function that is integrable over $\mathbf{R}$ with respect to μ_1. Show that

$$\mu_2\left\{(x, y) \in \mathbf{R}^2 \;\middle|\; 0 \le y \le f(x)\right\} = \mu_2\left\{(x, y) \in \mathbf{R}^2 \;\middle|\; 0 < y < f(x)\right\} = \int_{\mathbf{R}} f(x)\, dx.$$

For each $t \ge 0$, define $\varphi(t) = \mu_1\{x \in \mathbf{R} \mid f(x) \ge t\}$. Show that φ is a decreasing function and

$$\int_0^\infty \varphi(t)\, d\mu_1(t) = \int_{\mathbf{R}} f(x)\, d\mu_1(x).$$

Solution.

Question 3. Royden 20-44.

44. If the Borel measure μ is absolutely continuous with respect to Lebesgue measure, show that its Radon-Nikodym derivative is the derivative of its cumulative distribution function.

Solution.

Question 4. Royden 21-49.

49. Let X be a compact Hausdorff space. Show that the Jordan Decomposition Theorem for signed Borel measures on $\mathcal{B}(X)$ follows from the Riesz Representation Theorem for the dual of $C(X)$ and Proposition 12.

Solution.