Real Variables: Problem Set VI

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Abstract

This work contains solutions to the problem set VI of Real Variables 2015 at NYU.

1 Solutions

Question 9.10.

Solution. Let $\{X_n, \rho_n\}_{n=1}^{\infty}$ be a countable collection of metric spaces. We now define $(\prod_{n=1}^{\infty} X_n, p_*) = (X, p_*)$ such that for $x, y \in X$,

$$p_*(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(x_n,y_n)}{1+p_n(x_n,y_n)}.$$

First, we can show that p_* is well-defined via comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, as $0 \leq \frac{p_n(x_n,y_n)}{1+p_n(x_n,y_n)} \leq 1$ for all n.

As $p_n(x_n,y_n) \geq 0$ for all n, we have $p_*(x,y) \geq 0$ for all $x, y \in X$. If $p_*(x,y) = 0$, then $p_n(x_n,y_n) = 0$ for all n. As each p_n is a metric space $x_n = y_n$ for all n. Therefore, $x = y$. If $x = y$, then $x_n = y_n$ for all n. As each p_n is a metirc space, $p_n(x_n,y_n) = 0$ for all n. Therefore, $p_*(x,y) = 0$.

Since $p_n(x_n,y_n) = p_n(y_n,x_n)$ for all n, for $x, y \in X$, we

$$\begin{aligned} p_*(x,y) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(x_n,y_n)}{1+p_n(x_n,y_n)} \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(y_n,x_n)}{1+p_n(y_n,x_n)} \\ &= p_*(y,x). \end{aligned}$$

Let $x, y, z \in X$. By the problem 6 and the triangle inequality of each metric space X_n, which gives $p_n(x_n,z_n) \leq p_n(x_n,y_n) + p_n(y_n,z_n)$ for each n, we have

$$\frac{p_n(x_n,z_n)}{1+p_n(x_n,z_n)} \leq \frac{p_n(x_n,y_n)}{1+p_n(x_n,y_n)} + \frac{p_n(y_n,z_n)}{1+p_n(y_n,z_n)},$$

for all n. Hence, we have

$$\sum_{n=1}^{\infty} \frac{p_n(x_n, z_n)}{1 + p_n(x_n, z_n)} \quad \leq \quad \sum_{n=1}^{\infty} \frac{p_n(x_n, y_n)}{1 + p_n(x_n, y_n)} + \frac{p_n(y_n, z_n)}{1 + p_n(y_n, z_n)},$$

which can be written as

$$p_*(x, z) \quad \leq \quad p_*(x, y) + p_*(y, z).$$

Therefore, we have shown that all required properties of a metric space hold for (X, p_*). (X, p_*) is a metric space. $\square$

Question 9.20.

Solution. Let E be a subset of a metric space X, and let int$E$ be the interior of E. We first show that int$E \subseteq E$. If $x \in X \setminus E$, then every ball of x contains a point in $X \setminus E$. Hence, $x \in E$. Therefore, int$E \subseteq E$.

Now, we wish to show that int$E$ is open. For the first case, assume that $E = $ int$E$. Let $x \in$ int$E$. Since x is an interior point of E, there exists an open ball $B(x, r)$ contained in E. Since $E = $ int$E$, the open ball $B(x, r)$ is contained in int$E$ as well. Hence, int$E$ is open in this case. For the remaining case, assume that $E \setminus$ int$E \neq \emptyset$. Let $x \in$ int$E$. Since x is an interior point of E, there exists an open ball $B(x, r)$ contained in E. Suppose that there exists $y \in B(x, r) \cap E \setminus$ int$E$. Then, we have $d(x, y) < r$. Consider $B(y, r - d(x, y))$, which is valid since $r - d(x, y) > 0$. By the triangle inequality, for any point $z \in B(y, r - d(x, y))$,

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< r. \end{aligned}$$

Hence, $B(y, r - d(x, y))$ is an open ball contained in $B(x, r)$, which is again contained E, which contradicts the fact that $y \in E \setminus$ int$E$. Hence, $B(x, r)$ is contained in int$E$. Therefore, int$E$ is open. As we covered all cases, int$E$ for any subset E of a metric space X is open.

Assume E is open. Let $x \in E$. As E is open, there exists an open ball around x contained in E. Therefore, $x \in$ int$E$. Hence, $E \subseteq$ int$E$. As we have int$E \subseteq E$ from above, we have shown that $E = $ int$E$.

Assume $E = $ int$E$. Let $x \in E$. Then, as $E = $ int$E$, $x \in$ int$E$. By the definition of interior point, there exists an open ball around x contained in E. Hence, E is open. $\square$

Question 9.32.

Solution. (a) We claim that f is $1-$Lipschitz. Let $x, y \in \mathbb{R}$. Then, by the triangle inequality, we have

$$\begin{aligned} \text{dist}(x, E) &\leq d(x, e) \\ &\leq d(x, y) + d(y, e), \end{aligned}$$

for all $e \in E$. By taking the inf over $e \in E$ on the RHS, we have

$$\text{dist}(x, E) \quad \leq \quad d(x, y) + \text{dist}(y, E).$$

Hence, it follows that

$$\text{dist}(x, E) - \text{dist}(y, E) \quad \leq \quad d(x, y).$$

By symmetry, we also obtain that

$$\text{dist}(y, E) - \text{dist}(x, E) \quad \leq \quad d(x, y).$$

Hence, it follows that

$$|\text{dist}(x, E) - \text{dist}(y, E)| \quad \leq \quad d(x, y).$$

Therefore, f is $1-$Lipshictz, thus continuous.

(b) Let $x \in \overline{E}$. By the definition of closure, there exists a sequence $\{x_n\}$ from E such that $x_n \to x$. Since dist is continuous, we have dist$(x_n, E) \to$ dist(x, E). As $x_n \in E$, it follows that dist$(x_n, E) = 0$ for all n. Therefore, we have dist$(x, E) = 0$. Hence, we obtain

$$\overline{E} \quad \subseteq \quad \{x \in X \mid \text{dist}(x, E) = 0\}.$$

Let $x \in \{x \in X \mid \text{dist}(x, E) = 0\}$. Then, for all n, there exists $y \in E$ such that $\rho(x, y) > \dfrac{1}{n}$, which we label as x_n. Then, $\{x_n\}$ is a sequence from E such that $x_n \to x$. Hence, $x \in \bar{E}$. Consequently, we obtain

$$\{x \in X \mid \text{dist}(x, E) = 0\} \quad = \quad \overline{E},$$

as desired.
□

Question 9.43.

Solution. Let $f : (X, \rho) \to (Y, \sigma)$, and $\{u_n\}$ and $\{v_n\}$ be any two sequences in X such that $\lim_{n\to\infty} \rho(u_n, v_n) = 0$. Assume that f is uniformly continuous. Fix $\epsilon > 0$. Let $\delta > 0$ be the real number, that responds to the ϵ-challenge of uniform continuity of f. Then, there exists an index N such that $\rho(u_n, v_n) < \delta$ for $n \geq N$. By uniform continuity of f, it follows that

$$\sigma(f(u_n), f(v_n)) \quad < \quad \epsilon,$$

for $n \geq N$. Since ϵ was arbitrary, we have

$$\lim_{n\to\infty} \sigma(f(u_n), f(v_n)) = 0,$$

as desired.

Conversely, assume that $\lim_{n\to\infty} \sigma(f(u_n), f(v_n)) = 0$ holds for any two sequences $\{u_n\}$ and $\{v_n\}$ in X such that $\lim_{n\to\infty} \rho(u_n, v_n) = 0$. Suppose for sake of contradiction of f is not uniform continuous. Then, there exists $\epsilon > 0$, such that for any $\delta > 0$, $\rho(u, v) < \delta$ does not imply $\sigma(f(u), f(v)) < \epsilon$. Then, we can construct two sequence $\{u_n\}$, $\{v_n\}$, such that $(u_n, v_n) < \dfrac{1}{n}$ and $\sigma(f(u_n), f(v_n)) \geq \epsilon$ for all n. It follows that $\rho(u_n, v_n) \to 0$, but $\sigma(f(u_n), f(v_n)) \nrightarrow 0$. This is a contradiction. Hence, f is uniformly continuous.
□

Question 9.72.

Solution. Assume $A \cap B \neq \emptyset$. Then, there exists $x \in A \cap B$. Since $\rho(x, x) = 0$, we have dist$(A, B) = 0$. By contrapositive, we have shown that if dist$(A, B) > 0$, then $A \cap B = \emptyset$.

Assume $A \cap B = \emptyset$. Recall that A is compact and B is closed.

Hence, dist$(A, B) > 0$. □

Question 9.77.

Solution. Let (X, d_X) and (Y, d_Y) be separable metric spaces. Consider the standard product metric on $X \times Y$. Then, there exist a countable dense subset D_X in X and countable dense subset D_Y in Y. Observe that $D_X \times D_Y$ is countable. We claim that $D_X \times D_Y$ is a dense subset in $X \times Y$. Consider an open nonempty subset O of $X \times Y$. Let $(x^*, y^*) \in O$. Since O is open, there exists $\epsilon > 0$, such that $B((x^*, y^*), \epsilon) \subseteq O$. Consider the set B_X defined by $\{(x, y^*) | d_X(x, x^*) < \dfrac{\epsilon}{2}\}$. Observe that by triangle inequality, B_X is contained in O, and by the desnity of D_X in X, there exists a point $x^D \in D_X \cap B_X$. Therefore, $X \times Y$ is separable. □