Real Variables: Problem Set X

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Abstract

This work contains solutions to the problem set X of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 13-41.

41. Let X be the linear space of all polynomials defined on $\mathbf{R}$. For $p \in X$, define $\|p\|$ to be the sum of the absolute values of the coefficients of p. Show that this is a norm on X. For each n, define $\psi_n : X \to \mathbf{R}$ by $\psi_n(p) = p^{(n)}(0)$. Use the properties of the sequence $\{\psi_n\}$ in $\mathcal{L}(X, \mathbf{R})$ to show that X is not a Banach space.

Solution. We first show that $\| \; \| : X \to \mathbb{R}$ given is a norm on X. First of all,

Question 2. Royden 14-18.

18. Let X be a normed linear space, ψ belong to X^*, and $\{\psi_n\}$ be in X^*. Show that if $\{\psi_n\}$ converges weak-$*$ to ψ, then

$$\|\psi\| \leq \limsup \|\psi_n\|.$$

Solution.

Question 3. Royden 14-23.

23. Let Y be a linear subspace of a normed linear space X and z be a vector in X. Show that

$$\operatorname{dist}(z, Y) = \sup \left\{\psi(z) \;\middle|\; \|\psi\| = 1, \psi = 0 \text{ on } Y\right\}.$$

Solution.

Question 4. Royden 15-12.

12. If Y is a linear subspace of a Banach space X, we define the *annihilator* $Y^{\perp}$ to be the subspace of X^* consisting of those $\psi \in X^*$ for which $\psi = 0$ on Y. If Y is a subspace of X^*, we define Y^0 to be the subspace of vectors in X for which $\psi(x) = 0$ for all $\psi \in Y$.
    (i) Show that $Y^{\perp}$ is a closed linear subspace of X^*.
    (ii) Show that $(Y^{\perp})^0 = \overline{Y}$.

Solution.