Real Variables:
Problem Set II

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Abstract

This work contains the solutions to the first problem set of Real Variables 2015.

1 Solutions

Question 4. Royden 3.5.

Solution. Assume that the function f is defined on a measurable domain E and has a property that $\{x \in E \mid f(x) > c\}$ is measurable for each rational number c. Let $r \in \mathbb{R} \setminus \mathbb{Q}$. Consider the set $\{x \in E \mid f(x) > r\}$. Notice that

$$\{x \in E \mid f(x) > c\} \quad = \quad \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c + \frac{1}{k}\}.$$

By the density of the rationals, we can choose a sequence of rationals, $\{c_k\}$ such that for each k, we have $c_k \in \mathbb{Q}$ and $c_k \in (c, c + \frac{1}{k})$. In particular, we have that

$$\{x \in E \mid f(x) > c\} \quad = \quad \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \geq c_k\}.$$

As $\{c_k\}$ is a rational sequence, $\{x \in E \mid f(x) \geq c_k\}$ is measurable for all k, and $\{x \in E \mid f(x) > c\}$ is measurable, as a countable union of measurable sets is measurable. Since r is an arbitrary irrational, we have shown that $\{x \in E \mid f(x) > a\}$ is measurable for any $a \in \mathbb{R}$. Therefore, f is measurable. $\square$

Question 5. Royden 3.7.

Solution. dd

Question 6. Royden 3.9.

Solution. dd