Real Variables:
Problem Set XII

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Abstract

This work contains solutions to the problem set XII of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 7-10.

10. Show that in Hölder's Inequality there is equality if and only if there are constants α and β, not both zero, for which

$$\alpha|f|^p = \beta|g|^q \text{ a.e. on } E.$$

Solution.

Question 2. Royden 7-26.

26. (The L^p Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise a.e. on E to f. For $1 \leq p < \infty$, suppose there is a function g in $L^p(E)$ such that for all n, $|f_n| \leq g$ a.e. on E. Prove that $\{f_n\} \to f$ in $L^p(E)$.

Solution.

Question 3. Royden 19-5.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a Cauchy sequence in $L^\infty(X, \mu)$. Show that there is a measurable subset X_0 of X for which $\mu(X \sim X_0) = 0$ and for each $\epsilon > 0$, there is an index N for which

$$|f_n - f_m| \leq \epsilon \text{ on } X_0 \text{ for all } n, m \geq N.$$

Use this to show that $L^\infty(X, \mu)$ is complete.

Solution.

Question 4. Royden 17-19.

19. Show that any measure that is induced by an outer measure is complete.

Solution. Let $u^* : 2^X \to [0, \infty]$ be an outer-measure, and let $(X, \mathscr{M}, u)$ be a measure space induced by u^*. Let $E \in \mathscr{M}$ and $u(E) = 0$. Let $S \subseteq E$, and $A \in \mathscr{M}$. Then, by finite monotonicity of u^*, we have $u^*(S) = 0$ and $u^*(S \cap A) = 0$. Again, using the finite monotonicity of u^*, we see that

$$\begin{aligned} u^*(A) &\geq u^*(A \cap S^c) + 0 \\ &= u^*(A \cap S^c) + u^*(A \cap S). \end{aligned}$$

Hence, $S \in \mathscr{M}$. We have shown that u is complete. $\square$

Question 5. Royden 17-29.

29. Show that a set function on a σ-algebra is a measure if and only if it is a premeasure.

Solution. Let $(X, \mathscr{M})$ be a measurable space. Let $u : \mathscr{M} \to [0, \infty]$ be a measure. By definition of measure we have, $u(\emptyset) = 0$. u is finitely additive and countably monotone, as any measure has finite additivity and countable monotonicity properties. Conversely, assume that u is a pre-measure. As u is a pre-measure and $\emptyset \in \mathscr{M})$, we have $u(\emptyset) = 0$. Now, let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collections chosen from $\mathscr{M}$. By finite additivity, and countable monotonicity of u, it follows that

$$\begin{aligned} \sum_{k=1}^{n} u(E_k) &= u(\bigcup_{k=1}^{n} E_k) \\ &\leq u(\bigcup_{k=1}^{\infty} E_k), \end{aligned}$$

for all n. Hence, by linearity of limits, we obtain

$$\sum_{k=1}^{\infty} u(E_k) \leq u(\bigcup_{k=1}^{\infty} E_k).$$

By finite additivity of u and the fact that $u(E_k) \geq 0$ for all k, we have

$$\begin{aligned} \sum_{k=1}^{\infty} u(E_k) &\geq \sum_{k=1}^{n} u(E_k) \\ &= u(\bigcup_{k=1}^{n} E_k), \end{aligned}$$

for all n. Again, by linearity of limits, we obtain that

$$\sum_{k=1}^{\infty} u(E_k) \geq u(\bigcup_{k=1}^{\infty} E_k).$$

Therefore, we have shown that

$$\sum_{k=1}^{\infty} u(E_k) = u(\bigcup_{k=1}^{\infty} E_k),$$

which shows that u is countably additive. Hence, u is a measure. The claim is true. $\square$

Question 6. Royden 17-36.

36. Let μ be a finite premeasure on an algebra $\mathcal{S}$, and μ^* the induced outer measure. Show that a subset E of X is μ^*-measurable if and only if for each $\epsilon > 0$ there is a set $A \in \mathcal{S}_\delta$, $A \subseteq E$, such that $\mu^*(E \sim A) < \epsilon$.

Solution. Let $u^* : 2^X \to [0, \infty]$ be an outer-measure, and let $(X, \mathscr{M}, u)$ be a measure space induced by u^*. Let $E \in \mathscr{M}$ and $u(E) = 0$.