Real Variables: Problem Set IX

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Abstract

This work contains solutions to the problem set IX of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 12-5.

5. **Suppose that a topological space X has the property that every continuous, bounded real-valued function on a closed subset has a continuous extension to all of X. Show that if X is Tychonoff, then it is normal.**

Solution. Assume that X is Tychonoff, and let A and B be non-empty disjoint closed subsets of X. Let $g : A \cup B \to \mathbb{R}$ such that such that $g(A) = a$ and $g(B) = b$. Observe that g is a real-valued function, that is continuous, bounded, on a closed subset of X. Therefore, by the given, there exists a continuous extension to all of X, which we denote as $g' : X \to \mathbb{R}$. Observe that as $(a - \frac{a+b}{2}, \frac{a+b}{2})$ is open in $\mathbb{R}$, by the continuity of g' we have $g'^{-1}((a - \frac{a+b}{2}, \frac{a+b}{2}))$ is open in X, which contains A. Likewise, $g'^{-1}((\frac{a+b}{2}, b + \frac{a+b}{2}))$ is open in X, which contains B. Notice that as g' is a function those two open sets are disjoint. Therefore, we have shown that A and B have neighborhoods that are disjoint. Since X is Tychonoff as well, X is normal. $\square$

Question 2. Royden 12-6.

6. **Let $(X, \mathcal{T})$ be a normal topological space and $\mathcal{F}$ the collection of continuous real-valued functions on X. Show that $\mathcal{T}$ is the weak topology induced by $\mathcal{F}$.**

Solution. Let $x \in X$. Consider a neighborhood $U_x \in \mathscr{T}$. It follows that $X \setminus U_x$ is closed in $\mathscr{T}$. As normal topological spaces are Tychnoff, and single points are closed in Tychnoff spaces, we have $\{x\}$ is closed in $\mathscr{T}$. Then, by the Urysohn's lemma, we have a continuous real-valued function $f : X \to [a, b]$ such that $f(\{x\}) = a$ and $f(X \setminus U_x) = b$. Note that $f \in \mathscr{F}$. Then, for a fixed ϵ such that $b - a > \epsilon > 0$, as $(a - \epsilon, a + \epsilon)$ is an open set in $\mathbb{R}$, we have $f^{-1}((a - \epsilon, a + \epsilon))$ is a basic open set of the weak-topology, as f is continuous and it's a finite intersection of the inverse image of an open set. Observe that as $f(X \setminus U_x) = b$, we have $f^{-1}((a - \epsilon, a + \epsilon)) \cap X \setminus U_x = \emptyset$. Hence $f^{-1}((a - \epsilon, a + \epsilon)) \subseteq U_x$. Therefore, we have found a basic open set of x in the weak topology contained in U_x. Hence, we have that the basis of weak-topology is a collection of open sets in $\mathscr{T}$, such that for each x and each neighborhood of x, U_x, there is an element of the basis of weak-topology, that is contained in U_x. Therefore, the basis of weak-topology, induced by $\mathscr{F}$ is a also basis of the strong topology. Hence, in this case, the strong topology $\mathscr{T}$ is the weak-topology induced by $\mathscr{F}$. $\square$

Question 3. Royden 12-27.

27. For $f, g \in C[a, b]$, show that $f = g$ if and only if $\int_a^b x^n f(x)\,dx = \int_a^b x^n g(x)\,dx$ for all n.

Solution. Assume that $f = g$. Fix n. As $f, g \in C[a, b]$, $x^n \in C[a, b]$. and multiplication of continuous function is continuous, we have that $x^n f$ and $x^n g$ are continuous. As continuous functions on compact domain is integrable, by the linearity of integration, we have

$$\int_a^b x^n f(x)dx - \int_a^b x^n g(x)dx \quad = \quad \int_a^b x^n (f-g)(x)dx$$

As $f = g$, $f - g(x) = 0$ for all $x \in [a, b]$. It follows that

$$\int_a^b x^n f(x)dx - \int_a^b x^n g(x)dx \quad = \quad 0,$$

from which we obtain

$$\int_a^b x^n f(x)dx \quad = \quad \int_a^b x^n g(x)dx.$$

Since n was arbitrary, we have that the above equality holds for all n. Conversely, assume that $\int_a^b x^n f(x)dx = \int_a^b x^n g(x)dx$ for all n. By appealing to the linearity of integration, we see that

$$\int_a^b p(f-g)(x)dx = 0,$$

for any polynomial p defined on $[a, b]$. We claim that

$$\int_a^b (f-g)^2(x)dx = 0,$$

which will imply that $f = g$ almost everywhere immediately. By Weiestrass Approximation theorem, we can choose a sequence of polynomials p_n such that

$$|p_n - (f-g)| < \frac{1}{n}.$$

It follows that $\{p_n(f-g)\}$ converges to $(f-g)^2$ pointwise everywhere on $[a, b]$. As $|p_n - (f-g)| < 1$ for all n on $[a, b]$. As $f - g$ is a continuous function defined on a compact subset of $\mathbb{R}$, by the extreme value theorem, there exists $M > 0$ such that $|f - g| < M$ on $[a, b]$. It follows that $g(x) = M(M+1)$ on $[a, b]$ is integrable and dominates $\{p_n(f-g)\}$. Hence, by the Dominated Convergence theorem, we have

$$\int_a^b (f-g)^2(x)dx \quad = \quad \lim_{n\to\infty} \int_a^b p_n(f-g)(x)dx.$$

Since $\int_a^b p_n(f-g)(x)dx = 0$ for all n, it follows that

$$\int_a^b (f-g)^2(x)dx \quad = \quad 0.$$

Hence, we conclude that $f = g$ almost everywehre. As $f, g \in C[0, 1]$, and $f = g$ almost everywhere, it follows that $f = g$ everywhere. $\square$

Question 4. Royden 12-35.

35. Let $\mathcal{A}$ be an algebra of continuous real-valued functions on a compact Hausdorff space X that separates points. Show that either $\overline{\mathcal{A}} = C(X)$ or there is a point $x_0 \in X$ for which $\overline{\mathcal{A}} = \{f \in C(X) \mid f(x_0) = 0\}$. (Hint: If $1 \in \overline{\mathcal{A}}$, we are done. Moreover, if for each $x \in X$ there is an $f \in \mathcal{A}$ with $f(x) \neq 0$, then there is a $g \in \mathcal{A}$ that is positive on X and this implies that $1 \in \overline{\mathcal{A}}$.)

Solution. Consider

$$\mathscr{S} \quad = \quad \{f^{-1}(O) \mid f \text{ is continuous, and } O \text{ is open in} \mathbb{R}\}.$$

Question 5. Royden 13-8.

8. A nonnegative real-valued function $\|\cdot\|$ defined on a vector space X is called a **pseudonorm** if $\|x + y\| \leq \|x\| + \|y\|$ and $\|\alpha x\| = |\alpha|\,\|x\|$. Define $x \cong y$, provided $\|x - y\| = 0$. Show that this is an equivalence relation. Define $X/_{\cong}$ to be the set of equivalence classes of X under $\cong$ and for $x \in X$ define $[x]$ to be the equivalence class of x. Show that $X/_{\cong}$ is a normed vector space if we define $\alpha[x] + \beta[y]$ to be the equivalence class of $\alpha x + \beta y$ and define $\|[x]\| = \|x\|$. Illustrate this procedure with $X = L^p[a, b]$, $1 \leq p < \infty$.

Solution. We show that the pseudo-norm relation is reflexive, symmetric, and transitive.

Let $x \in X$. It follows that

$$\|x - x\| = \|\theta\|,$$

where θ is the identity element of the linear space X. By definition of linear space, we have $\alpha \cdot \theta = \theta$ for all α. Hence, for some $\alpha > 1$, we have

$$\begin{aligned} \|\theta\| &= \|\alpha \cdot \theta\| \\ &= |\alpha|\|\theta\|. \end{aligned}$$

As $|a| > 0$, we have $|\theta| = 0$. Consequently, $\|x - x\| = 0$. It follows that for all $x \in X$, $x \equiv x$. The relation is reflexive.

Let $x, y \in X$ and $x \equiv y$. Observe that

$$\begin{aligned} \|x - y\| &= \|-1 \cdot (y - x)\| \\ &= |-1|\|y - x\| \\ &= \|y - x\|. \end{aligned}$$

As $x \equiv y$, which gives $\|x - y\| = 0$, it follows that $\|y - x\| = 0$ and $y \equiv x$. Hence, the relation is symmetric.

Let $x, y, z \in X$ and $x \equiv y$ and $y \equiv z$. By triangle inequality, it follows that

$$\begin{aligned} \|y - z\| &= \|(x - y) + (y - z)\| \\ &\leq \|x - y\| + \|y - z\| = 0 + 0 = 0. \end{aligned}$$

Hence, $\|y - z\| = 0$, and it follows that $x \equiv z$. Hence, the relation is symmetric. It follows that the pseudo-norm relation is an equivalence relation on the linear space X.

$\square$

We show that $X_{\equiv}$ is a normed vector space. Firstly, we check that the defined norm is well defined. Let $x, y \in X$, such that $x \equiv y$. It follows that $\|x - y\| = 0$. Hence, $\|x\| = \|y\|$, and it follows that $\|[x]\| = \|[y]\|$. The norm is well-defined.

33. Let X be a linear subspace of $C[0, 1]$ that is closed as a subset of $L^2[0, 1]$.Verify the following assertions to show that X has finite dimension. The sequence $\{f_n\}$ belongs to X.

(i) Show that X is a closed subspace of $C[0, 1]$.

(ii) Show that there is a constant $M \geq 0$ such that for all $f \in X$ we have $\|f\|_2 \leq \|f\|_\infty$ and $\|f\|_\infty \leq M \cdot \|f\|_2$.

(iii) Show that for each $y \in [0, 1]$, there is a function k_y in L^2 such that for each $f \in X$ we have $f(y) = \int_0^1 k_y(x) f(x)\, dx$.

(iv) Show that if $\{f_n\} \to f$ weakly in L^2, then $\{f_n\} \to f$ pointwise on $[0,1]$.

(v) Show $\{f_n\} \to f$ weakly in L^2, then $\{f_n\}$ is bounded (in what sense?), and hence $\{f_n\} \to f$ strongly in L^2 by the Lebesgue Dominated Convergence Theorem.

(vi) Conclude that X, when normed by $\|\cdot\|_2$, has a compact closed unit ball and therefore, by Riesz's Theorem, is finite dimensional.

Question 6. Royden 13-33.

Solution. Consider

$$\mathscr{S} \quad = \quad \{f^{-1}(O) \mid f \text{ is continuous, and } O \text{ is open in} \mathbb{R}\}.$$