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# Royden

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## Abstract

This work contains the solutions to Royden's Real Variables.

## 1 Chapter III

**Theorem 3. Egoroff's Theorem.**

## 2 Chapter II

**Question Royden 2.1-1.**

**Solution.** Let  $m$  be a set function defined for all sets in a  $\sigma$ -algebra  $\mathcal{A}$  with values in  $[0, \infty]$ . Assume that  $m$  is countably additive over countable disjoint collections of sets in  $\mathcal{A}$ . Furthermore, assume that  $A$  and  $B$  are two sets in  $\mathcal{A}$  with  $A \subseteq B$ . Given that  $m$  is countably additive over countable disjoint collections of sets in  $\mathcal{A}$ , we have

$$m(B) = m(A) + m(B \setminus A),$$

where  $B \setminus A$  is a well-defined set with  $A \subseteq B$  assumption, thus  $A$  and  $B \setminus A$  forming a valid countable disjoint collections of sets whose union is  $B$ . With  $m$  being a set function with values in  $[0, \infty]$ , we obtain  $m(B) = m(A) + r$ , where  $r$  denotes some non-negative real value. Therefore, we finally get

$$m(A) \leq m(B).$$

Hence, we have shown that the given set function  $m$  has the monotonicity property.

**Question Royden 2.1-2.**

**Solution.** Let  $m$  be a set function defined for all sets in a  $\sigma$ -algebra  $\mathcal{A}$  with values in  $[0, \infty]$ . Assume that  $m$  is countably additive over countable disjoint collections of sets in  $\mathcal{A}$ . Furthermore, assume that there exists a set  $A$  in the collection  $\mathcal{A}$  such that  $m(A) < \infty$ . Using the countably additive property with a collection  $\{A, \emptyset\}$ , we obtain

$$m(A \cup \emptyset) = m(A) + m(\emptyset).$$

Substituting  $A \cup \emptyset = A$  and subtracting  $m(A)$  from both sides, granted with finiteness of  $m(A)$ , we get

$$m(\emptyset) = 0,$$

as desired. Hence, we have shown that if there is a set  $A$  in the collection  $\mathcal{A}$  for which  $m(A) < \infty$ , then  $m(\emptyset) = 0$ .

**Question Royden 2.1-3.****Solution.****Question Royden 2.1-5 (A Countable Set Has Outer Measure Zero).****Solution.** We know that any countable set has outer measure zero. Using the fact that the outer measure of an interval is its length yields  $m^*([0, 1]) = 1$ . Therefore,  $[0, 1]$  cannot be countable.**Question : Royden 2.1-6.****Solution.** Let  $Q$  and  $A$  denote the set of rationals and irrationals in the interval  $[0, 1]$  respectively. Consider a countable collection of sets  $\{Q, A\}$ . Since outer measure is countably subadditive, we have

$$m^*(Q \cup A) \leq m^*(Q) + m^*(A).$$

As  $Q$  is a countable set whose outer measure is zero and  $Q \cup A = [0, 1]$  by construction, we obtain

$$m^*([0, 1]) \leq m^*(A).$$

As the outer measure of an interval is its length, we have

$$1 \leq m^*(A).$$

Using the monotonicity property of outer measure with  $A \subset [0, 1]$ , we also see

$$m^*(A) \leq 1,$$

thereby showing that  $m^*(A) = 1$ .**Question : Royden 2.3-Proposition 4.****Solution.** We want to show that any set of outer measure zero is measurable, which further implies that any countable set is measurable. Let the set  $E$  have outer measure zero,  $m^*(E) = 0$ . Let  $A$  be any set. Since

$$A \cap E \subseteq E \text{ and } A \cap E^c \subseteq A,$$

by the monotonicity of outer measure, we obtain

$$m^*(A \cap E) \leq m^*(E) = 0 \text{ and } m^*(A \cap E^c) \leq m^*(A).$$

It is important to note that it suffices to show the above statement to show that the set  $E$  is measurable, as the inequality,

$$m^* \leq m^*(A \cap E) + m^*(A \cap E^c),$$

trivially holds with the finite subadditive property of outer measure.

**Definition.** A collection of subsets of  $\mathbb{R}$  is an algebra, provided that it contains  $\mathbb{R}$  and is closed with respect to the formation of complements and finite unions. A collection of subsets of  $\mathbb{R}$  is a  $\sigma$ -algebra, provided that it contains  $\mathbb{R}$  and is closed with respect to the formation of complements and countable unions.**Proposition.** Algebra is closed with respect to the formation of finite intersection.*Proof.* Let  $\mathcal{A}$  be an algebra and  $\{A_k\}_{k=1}^n$  be a finite collection of sets in  $\mathcal{A}$ . We want to show that  $\cap_{k=1}^n A_k$  is in  $\mathcal{A}$ . Since algebra is closed with respect to the formation of finite union and complements, we have

$$\cup_{k=1}^n A_k^C \text{ is in } \mathcal{A}.$$

By applying De Morgan's identity and closedness of complements iteratively, we see that

$$(\cap_{k=1}^n A_k)^C \text{ is in } \mathcal{A} \text{ and } \cap_{k=1}^n A_k \text{ is in } \mathcal{A},$$

thereby showing that an algebra is closed with respect to the formation of finite intersection.  $\square$

**Proposition.** The collection of measurable sets is an algebra. The union of a countable collection of measurable sets is also the union of a countable disjoint collection of measurable sets.

*Proof.* Since we know that the union of a finite collection of measurable sets is measurable and the complement of a measurable set is measurable, we immediately see that the collection of measurable sets is an algebra. Now, consider the union of a countable collection of measurable sets, denoted as  $\bigcup_{i=1}^{\infty} E_i$ . Define a new countable collection of sets  $\{A_i\}_{i=1}^{\infty}$  by

$$A_i \triangleq E_i \sim \bigcup_{k \neq i}^{\infty} E_k,$$

where each  $A_i$  is disjoint and indeed in the measurable sets as the construction of it respects the algebra. Furthermore, the union of  $\{A_i\}$  collection is precisely the union of  $E_i$  as well. Therefore, we have proven the proposition.  $\square$

**Proposition.** The union of a countable collection of measurable sets is measurable. Thus, the collection of measurable sets is an  $\sigma$ -algebra.

*Proof.*  $\square$

### Question 2.6-29: Equivalence relation.

**Solution.** (i) We want to show that the rational equivalence defines an equivalence relation on any set. Let  $A$  be a set and  $R$  be a relation set induced by the rational equivalence. For  $x \in A$ ,  $x - x = 0$  is clearly a rational. Hence, the relation is reflexive. For  $(x, y) \in R$ , we have that  $x - y$  is a rational. As a negation of a rational is a rational,  $(y, x) \in R$ . Thus, the relation is symmetric. For  $(x, y), (y, z) \in R$ , we have that  $x - y$  and  $y - z$  are rationals. Notice that negating  $y - z$  term and subtracting it from  $x - y$  gives  $x - z$ . Since rationals are closed under subtraction, we have that  $x - z$  is a rational and  $(x, z) \in R$ . Therefore, the relation is transitive, thereby showing that the rational equivalence defines an equivalence relation on any set.

(ii) Irrationals

(iii) The irrationally equivalent as a relation fails to be reflexive on both  $\mathbb{R}$  and  $\mathbb{Q}$ , as 0 is not a irrational number.

## 3 Chapter I

### Question : Royden 1.1-1 (Distributive Property of Multiplicative Inverse in Reals).

**Solution.** Assume that  $a \neq 0$  and  $b \neq 0$ . From the multiplicative identity axiom, we have that a multiplicative inverse exists for  $a$  and  $b$  individually, which we denote as  $a^{-1}$  and  $b^{-1}$  respectively. Now, consider the expression  $(ab)(a^{-1}b^{-1})$ , where  $ab$  denotes the product of  $a$  and  $b$ , and  $a^{-1}b^{-1}$  denotes the product of  $a^{-1}$  and  $b^{-1}$ . From the commutativity of multiplication, we obtain

$$(ab)(a^{-1}b^{-1}) = (ab)(b^{-1}a^{-1}).$$

Using the associativity of multiplication and iteratively substituting  $bb^{-1} = 1$  and  $aa^{-1} = 1$ , we have

$$(ab)(a^{-1}b^{-1}) = 1,$$

where 1 denotes the identity as usual. Hence, the product,  $a^{-1}b^{-1}$  satisfies definition of multiplicative inverse with respect to the  $ab$  term whose multiplicative inverse can be denoted as  $(ab)^{-1}$  by convention. Therefore, we obtain that

$$(ab)^{-1} = a^{-1}b^{-1},$$

as desired.

**Question Royden 1.1-3.**

**Solution.** Let  $E$  be a nonempty set of real numbers.

( $\Leftarrow$ ) Assume that  $E$  consists of a single point, which we denote as  $x$ . We claim that  $\inf E = x$  and  $\sup E = x$ . As we have  $x \leq x$ , we see that  $x$  is an upper bound for  $E$ . Suppose that there exists an upper bound for  $E$ ,  $a$ , that is smaller than  $x$ , namely  $a < x$ . This is a contradiction to the fact that  $a$  is an upper bound as it is required to have  $x \leq a$  with  $x \in E$ . Hence, there does not exist any upper bound for  $E$  that is smaller than  $x$ . By definition of supremum, we have that  $\sup E = x$ . By symmetry, we can see that  $\inf E = x$  as well. Therefore,  $\inf E = \sup E$ .

( $\Rightarrow$ ) Assume that  $\inf E = \sup E$ . Given the assumption, let us denote the infimum and supremum for  $E$  as a single real number  $a$ . Then, by definition of infimum, any  $x$  in  $E$ , we have  $a \leq x$ . Furthermore, by definition of supremum, any  $x$  in  $E$ , we have  $x \leq a$ . The only real number that can satisfy the two given equality is  $a$  itself. We also know that  $a$  must be in  $E$  as  $E$  is a nonempty set of reals. Therefore, we have shown that  $E = \{a\}$ , and that  $E$  consists of a single point.

**Proposition 1-8 Variant.** The intersection of any finite collection of open sets is open; and the union of any finite collection of closed sets is closed.

*Proof.* Let  $\{A_k\}_{k=1}^n$  be a finite collection of open sets. We want to show that  $\cap_{k=1}^n A_k$  is open. Consider  $x \in \cap_{k=1}^n A_k$ . Since each  $A_k$  is open, there exist  $r_k$  for  $1 \leq k \leq n$  such that  $B(x, r_k) \subseteq A_k$ . Let  $r^* = \min_{1 \leq k \leq n} (r_k)$ . Then,  $B(x, r^*) \subseteq A_k$  for  $1 \leq k \leq n$ . Hence,  $B(x, r^*) \subseteq \cap_{k=1}^n A_k$ .

Therefore, there exists  $r > 0$  such that  $B(x, r)$  for all points in  $\cap_{k=1}^n A_k$  and by definition of open sets,  $\cap_{k=1}^n A_k$  is open. We have shown that the intersection of any finite collection of open sets is open.

Let  $\{B_k\}_{k=1}^n$  be a finite collection of closed sets. We want to show that  $\cup_{k=1}^n B_k$  is closed, which is equivalent to  $(\cup_{k=1}^n B_k)^C$  being open. By DeMorgan's identity, we have that

$$(\cup_{k=1}^n B_k)^C = \cap_{k=1}^n B_k^C.$$

Notice that as  $B_k$  is closed,  $B_k^C$  is open. Then, the RHS expression gives us that the set under consideration is indeed a finite intersection of open sets, which we have shown to be open. Therefore, the union of any finite collection of closed sets is closed.

Note that the result does not hold true for the countable collection of open sets. The minimum can converge 0, breaking the above proof by having  $r^*$  no longer well defined. For example, let  $A_k$  be an open interval  $(-\frac{1}{k}, \frac{1}{k})$ . Taking the intersection of  $A_k$ s indexed by naturals, we have

$$\cap_{k=1}^{\infty} A_k = \{0\},$$

thereby yielding an intersection that is not-open as a singleton-set is not open in reals. Later on, we will see that the intersection of any countable of open sets is measurable, but this result tells us that the intersection might not necessarily open.  $\square$

**Theorem The Heine-Borel Theorem.**

*Proof.* We first consider the case of a closed and bounded interval. Let  $F$  be a closed, bounded set, denoted as  $[a, b]$ , and  $\mathcal{F}$  be an open cover of  $F$ . Now, define a set  $E$  as follows:

$$\{x \in F \mid [a, x] \text{ can be covered by a finite collection of sets in } \mathcal{F}\}.$$

Note that  $a \in E$ , as  $\mathcal{F}$  is an open cover, there exists an open set in  $\mathcal{F}$  that contains  $a$  and the open set itself forms a finite cover for  $[a, a]$ . Therefore, we have that  $E$  is non-empty, bounded set of reals, which grants us an existence of supremum by the completeness of the reals; define  $c = \sup E$ .  $\square$

**Theorem The Nested Set Theorem.**

*Proof.* Granted the Heine-Borel Theorem, we prove the nested set theorem. Let  $\{F_n\}_{n=1}^\infty$  be a descending countable collection of nonempty, bounded and closed sets. Suppose for sake of contradiction that  $\bigcap_{i=1}^\infty F_i = \emptyset$ . Taking the complements and applying DeMorgan's identity to both sides, we obtain

$$\bigcup_{i=1}^\infty F_i^C = \mathbb{R}.$$

As each  $F_i^C$  is an open set, by the Heine-Borel Theorem, we have

$$\bigcup_{n \in N} F_i^C = \mathbb{R},$$

where  $N$  denotes a finite index set. As  $\{F_n\}$  is descending, and  $N$  is finite, we have

$$F_{n^*}^C = \mathbb{R},$$

where  $n^*$  denotes the maximum index from the index set  $N$ . Taking complements on both sides, we get  $F_{n^*} = \emptyset$ , which contradicts the fact that  $\{F_n\}$  is a collection of nonempty sets. Therefore,  $\bigcap_{i=1}^\infty F_i \neq \emptyset$ . Hence, we have proven that the intersection of descending countable collection of nonempty, bounded, closed sets is nonempty.  $\square$

**Theorem 1-15 The Monotone Convergence Criterion for Real Sequences.**

*Proof.* Let  $\{a_n\}$  be an increasing sequence. Assume that the sequence is bounded as well. Consider a set  $S = \{a_n | n \in \mathbb{N}\}$ . Since  $S$  is bounded, by the completeness axiom, there exists a supremum of  $S$ , which we denote as  $a$ . Now, we claim that

$$\lim_{n \rightarrow \infty} a_n = a.$$

Let  $\epsilon > 0$ . Then, as  $a$  is the supremum of  $S$ ,  $a - \epsilon$  is not an upper bound of  $S$ . Hence, there exists an index  $n^*$  such that  $a - a_{n^*} < \epsilon$ . As the sequence is increasing, for  $m \geq n^*$ , we have  $a - a_m < \epsilon$ . Therefore, we have shown that for any  $\epsilon > 0$ , there exists an index  $n$  such that for  $m \geq n$ ,  $|a - a_m| < \epsilon$ , which is precisely the definition of convergence given. The decreasing sequence case can also be proven in a similar vein. Hence, we have shown that if a sequence is bounded and monotone, then it is also convergent.  $\square$

**Theorem 1-16 The Bolzano-Weiestrass Theorem.**

*Proof.* Let  $\{a_n\}$  be a sequence of real numbers. Assume that the sequence is bounded.  $\square$

**4 Results: Chapter III**

**Definition.** All the functions considered in this chapter take values in the extended real numbers, that is, the set  $\mathbb{R} \cup \{\pm\infty\}$ . A property is said to hold **almost everywhere** (abbreviated a.e.) on a measurable set  $E$  provided it holds on  $E \sim E_0$ , where  $E_0$  is a subset of  $E$  for which  $m(E_0) = 0$ .

**Proposition 3.1 : Definition of Measurable functions.**

Let the function  $f$  have a measurable domain  $E$ . Then, the following statements are equivalent:

- (i) For each real number  $c$ , the set  $\{x \in E | f(x) > c\}$  is measurable.
- (ii) For each real number  $c$ , the set  $\{x \in E | f(x) \geq c\}$  is measurable.
- (iii) For each real number  $c$ , the set  $\{x \in E | f(x) < c\}$  is measurable.
- (iv) For each real number  $c$ , the set  $\{x \in E | f(x) \leq c\}$  is measurable.

**Definition.** An extended real-valued function  $f$  is defined on  $E$  is said to be **Lebesgue measurable**, or simply **measurable**, provided its domain  $E$  is measurable and it satisfies one of the four statements of Proposition 3.1.

**Proposition 3.2 : The Inverse Image of Open is Measurable Implies  $f$  Measurable.**

Let the function  $f$  be defined on a measurable set  $E$ . Then  $f$  is measurable if and only if for each open set  $\mathcal{O}$ , the inverse image of  $\mathcal{O}$  under  $f$ ,  $f^{-1}(\mathcal{O}) = \{x \in E | f(x) \in \mathcal{O}\}$ , is measurable.

**Proposition 3.3 : Continuity and Measurability.**

A real-valued function that is continuous on its measurable domain is measurable.

**Proposition 3.4.**

A monotone function that is defined on an interval is measurable.

**Proposition 3.5 : .**

Let  $f$  be an extended real-valued function on  $E$ .

(i) If  $f$  is measurable on  $E$  and  $f = g$  a.e. on  $E$ , then  $g$  is measurable on  $E$ .

(ii) For a measurable subset  $D$  of  $E$ ,  $f$  is measurable on  $E$ , if and only if the restrictions of  $f$  to  $D$  and  $E \sim D$  are measurable.

**Theorem 3.6.** Let  $f$  and  $g$  be measurable functions on  $E$  that are finite a.e. on  $E$ .

(Linearity) For any  $\alpha$  and  $\beta$ ,

$$\alpha f + \beta g \text{ is measurable on } E.$$

(Product)

$$fg \text{ is measurable on } E.$$

**Remark.** Many of the properties of functions considered in elementary analysis, including continuity and differentiability, are preserved under the operation of composition of functions. However, the composition of measurable functions may not be measurable.

**Proposition 3.7 : Composition and Measurability.**

Let  $g$  be a measurable real-valued function defined on  $E$  and  $f$  a continuous real-valued function defined on all of  $\mathbb{R}$ . Then the composition  $f \circ g$  is a measurable function on  $E$ .

**Proposition.** For a finite family  $\{f_k\}_{k=1}^n$  of measurable functions with common domain  $E$ , the functions  $\max\{f_1, \dots, f_n\}$  and  $\min\{f_1, \dots, f_n\}$  also are measurable.

**Remark.** For a function  $f$  defined on  $E$ , we have the associated functions  $|f|$ ,  $f^+$ ,  $f^-$  defined on  $E$  by

$$|f|(x) = \max\{f(x), -f(x)\}, f^+(x) = \max\{f(x), 0\}, f^-(x) = \max\{-f(x), 0\}.$$

If  $f$  is measurable on  $E$ , then, by the preceding proposition, so are the functions  $|f|$ ,  $f^+$ , and  $f^-$ . This will be important when we study integration since the expression of  $f$  as the difference of two nonnegative functions,

$$f = f^+ - f^- \text{ on } E,$$

plays an important part in defining the Lebesgue integral.

**Proposition.** Let  $\{f_n\}$  be a sequence of measurable functions on  $E$  that converges pointwise a.e. on  $E$  to the function  $f$ . Then,  $f$  is measurable.

**Definition.** If  $A$  is any set, the **characteristic function** of  $A$ ,  $\mathcal{X}_A$ , is the function on  $\mathbb{R}$  defined by

$$\mathcal{X}_A = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

It is clear that the function  $\mathcal{X}_A$  is measurable if and only if the set  $A$  is measurable. Thus, the existence of a non-measurable set implies the existence of a non-measurable function. Linear combinations of characteristic functions of measurable sets play a role in Lebesgue integration similar to that played by step functions in Riemann integration, and so we name these functions.

**Definition.** A real-valued function  $\varphi$  defined on a measurable set  $E$  is called **simple** provided it is measurable and takes only a finite number of values.

We emphasize that a simple function only takes real values. Linear combinations and products of simple functions are simple since each of them takes on only a finite number of values. If  $\varphi$  is simple, has domain  $E$  and takes the distinct values  $c_1, \dots, c_n$ , then

$$\varphi = \sum_{k=1}^n c_k \cdot \mathcal{X}_{E_k} \text{ on } E, \text{ where } E_k = \{x \in E \mid \varphi(x) = c_k\}.$$

This particular expression of  $\varphi$  as a linear combination of characteristic functions is called the **canonical representation of the simple function**  $\varphi$ .

**Lemma : The Simple Approximation Lemma.** Let  $f$  be a measurable real-valued function on  $E$ . Assume  $f$  is bounded on  $E$ , that is, there is an  $M \geq 0$  for which  $|f| \leq M$  on  $E$ . Then, for each