Real Variables: Problem Set X

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Abstract

This work contains solutions to the problem set X of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 13-41.

41. Let X be the linear space of all polynomials defined on $\mathbf{R}$. For $p \in X$, define $\|p\|$ to be the sum of the absolute values of the coefficients of p. Show that this is a norm on X. For each n, define $\psi_n : X \to \mathbf{R}$ by $\psi_n(p) = p^{(n)}(0)$. Use the properties of the sequence $\{\psi_n\}$ in $\mathcal{L}(X, \mathbf{R})$ to show that X is not a Banach space.

Solution. We first show that $\| \ \| : X \to \mathbb{R}$ given is a norm on X. First of all, let $p = 0$. Then, $\|p\| = 0$. Now, let $p = \sum_{i=0}^{n} b_i x^i$, and assume that $\|p\| = 0$. It follows that $\sum_{i=0}^{n} |b_i| = 0$. As $|b_i| \geq 0$ for all i, we have that $b_i = 0$ for all i. Hence, $p = 0$. For proving the triangle inequality, let $p_1 = \sum_{i=0}^{n_1} b_i x^i$ and $p_2 = \sum_{i=0}^{n_2} c_i x^i$. Without the loss of generality, we assume that $n_1 \geq n_2$, and define $n = n_1$, $p_1 = \sum_{i=0}^{n} b_i x^i$ and $p_2 = \sum_{i=0}^{n} c_i x^i$, with $c_i = 0$ for $i > n_2$. By the triangle inequality of reals, it follows that

$$
\begin{aligned}
\|p_1 + p_2\| &= \|\sum_{i=0}^{n} (b_i + a_i) x^i\| \\
&= \sum_{i=0}^{n} |b_i + a_i| \\
&\leq \sum_{i=0}^{n} |b_i| + |a_i| \\
&= \sum_{i=0}^{n} |b_i| + \sum_{i=0}^{n} |c_i| = \|p_1\| + \|p_2\|.
\end{aligned}
$$

Now, let $p = \sum_{i=0}^{n} b_i x^i$, and $\alpha \in \mathbb{R}$. It follows that

$$\begin{aligned} \|\alpha p\| &= \|\alpha \sum_{i=0}^{n} b_i x^i\| \\ &= \|\sum_{i=0}^{n} \alpha b_i x^i\| \\ &= \sum_{i=0}^{n} |\alpha b_i| \\ &= |\alpha| \sum_{i=0}^{n} |b_i| = |\alpha| \|p\|. \end{aligned}$$

Hence, we have shown that $\| \ \|$ given is a norm.

Now, we first show that each operator ψ_n is bounded, thus continuous. Observe that we can represent an arbitrary polynomial p uniquely as , for some k, $p = \sum_{i=0}^{\infty} c_i x^i$, where $c_i = 0$ for $i \geq k$. Fix ψ_n. Observe that for any p, we have $|c_n| \leq \|p\|$. It follows that

$$\begin{aligned} |\psi_n(p)| &= |n! \cdot c_n| \\ &= |n!||c_n| \\ &\leq |n!|\|p\| \end{aligned}$$

Hence, ψ_n is bounded, thus continuous for any n. Note that by taking $p = x^n$, we obtain $n! \leq M$ for any bound M for ψ_n. Hence, it follows that $\|\psi_n\| = n!$. Again, for any polynomial p, observe that $\psi_n(p) = 0$ for $n > k$, where k denotes the degree of the polynomial p. Consequently, we have

$$\lim_{n \to \infty} \psi_n(p) = 0,$$

for any p. Therefore, if X is Banach, the conditions of the Banach-Saks-Steinhaus theorem is satisfied. However, as $\|\psi_n\| = n!$, $\{\psi_n\}$ cannot be uniformly bounded. This is a contradiction. X is not Banach.

Question 2. Royden 14-18.

18. Let X be a normed linear space, ψ belong to X^*, and $\{\psi_n\}$ be in X^*. Show that if $\{\psi_n\}$ converges weak-$*$ to ψ, then

$$\|\psi\| \leq \limsup \|\psi_n\|.$$

Solution. As $\{\psi_n\}$ is weak-* convergent to ψ, we have

$$\psi_n(x) \quad \to \quad \psi(x),$$

for all $x \in X$. Let $x \in X$. As $|\cdot|$ is continuous on $\mathbb{R}$, it follows that

$$\lim_{n\to\infty} |\psi_n(x)| \quad = \quad |\psi(x)|.$$

As $|\psi_n(x)| \leq ||\psi_n|| \cdot ||x||$,

$$\begin{aligned}
|\psi(x)| &= \lim_{n\to\infty} |\psi_n(x)| \\
&= \limsup_{n\to\infty} |\psi_n(x)| \\
&\leq \limsup_{n\to\infty} ||\psi_n|| \cdot ||x|| \\
&= ||x|| \limsup_{n\to\infty} ||\psi_n||.
\end{aligned}$$

Since $x \in X$ was arbitrary, it follows that

$$||\psi|| \quad \leq \quad \limsup_{n\to\infty} ||\psi_n||,$$

as desired. $\square$

Question 3. Royden 14-23.

23. Let Y be a linear subspace of a normed linear space X and z be a vector in X. Show that

$$\operatorname{dist}(z, Y) = \sup \left\{ \psi(z) \;\middle|\; \|\psi\| = 1, \psi = 0 \text{ on } Y \right\}.$$

Solution. I believe there is an error in this problem.

Question 4. Royden 15-12.

12. If Y is a linear subspace of a Banach space X, we define the *annihilator* $Y^{\perp}$ to be the subspace of X^* consisting of those $\psi \in X^*$ for which $\psi = 0$ on Y. If Y is a subspace of X^*, we define Y^0 to be the subspace of vectors in X for which $\psi(x) = 0$ for all $\psi \in Y$.
   (i) Show that $Y^{\perp}$ is a closed linear subspace of X^*.
   (ii) Show that $(Y^{\perp})^0 = \overline{Y}$.

Solution. (i) For $x \in X$, let $Y_x^{\perp}$ be defined by

$$Y_x^{\perp} = \{\psi \in X^* \mid \psi(x) = 0\}.$$

As $\psi \in X^*$, ψ is continuous, hence $Y_x^{\perp}$ is closed. Observe that

$$Y^{\perp} = \bigcap_{x \in Y} Y_x^{\perp}.$$

Each $Y_x^{\perp}$ is closed, since a limit function of continuous functions with respect to the operator norm, will preserve the property that 0 will be achieved at x. Since an arbitrary intersection of closed sets is closed, we have that $Y^{\perp}$ is closed linear subspace of X^*.

(ii) First, we show that $(Y^{\perp})^0$ is closed. Observe that

$$\begin{aligned}(Y^{\perp})^0 &= \bigcap_{\psi \in Y^{\perp}} \{x \in X \mid \psi(x) = 0\} \\ &= \bigcap_{\psi \in Y^{\perp}} \psi^{-1}(0).\end{aligned}$$

As $\psi^{-1}(0)$ is a pre-image of a single point, which is closed in a metric space, of a continuous function, and intersection of closed sets is closed, we have that $(Y^{\perp})^0$ is closed. By definition of $Y^{\perp}$, it follows that $Y \subseteq (Y^{\perp})^0$, and as $(Y^{\perp})^0$ is closed, we obtain $\overline{Y} \subseteq (Y^{\perp})^0$. Now, we show that $(Y^{\perp})^0 \subseteq \overline{Y}$ holds. It suffices to show that $X \setminus \overline{Y} \subseteq X \setminus (Y^{\perp})^0$ holds. Let $x \in X \setminus \overline{Y}$. Then, we know that there exists $\psi \in X^*$ such that $\psi(x) \neq 0$ and $\ker(\psi)$ contains Y. Hence, $x \notin (Y^{\perp})^0$. Therefore, we have shown that $Y = (Y^{\perp})^0$ as desired. $\square$