Real Variables: Problem Set I

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Abstract

This work contains the solutions to the first problem set of Real Variables 2015.

1 Solutions

Question 1. Royden 2.4. Counting Measure.

Solution. We wish to show that the counting measure, $c : \mathcal{P}(\mathbb{R}) \to [0, \infty]$, where $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$, is countably additive and translation invariant.

We first prove that it is countably additive. Let $\{E_k\}_{k=1}^{\infty}$ be a countable, disjoint collection of subsets of $\mathbb{R}$. If one of the set in the collection has infinite cardinality, then we have

$$\sum_{k=1}^{\infty} c(E_k) \quad = \quad \infty,$$

as $c(E_k) = \infty$ for some k. Notice that the union of the collection $\cup_{k=1}^{\infty} E_k$, also has infinite cardinality, as it has a subset with an infinite cardinality. Hence, by the definition of counting measure, we have $c(\cup_{k=1}^{\infty} E_k) = \infty$. Therefore, we have

$$c(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} c(E_k),$$

for the case under consideration. Now, assume that $c(E_k) < \infty$ for all k. There are two sub-cases now. First, assume that the series $\sum_{k=1}^{\infty} c(E_k)$ converges. In particular, we have that $\lim_{k \to \infty} c(E_k) = 0$. For some $\epsilon < 1$, we have an N such that $c(E_k) < \epsilon$ for all $k \geq N$. As the counting measure only takes an integer value or ∞, we obtain that $c(E_k) = 0$ and $E_k = \emptyset$ for all $k \geq N$. Furthermore, we get that

$$\begin{aligned} c(\cup_{k=1}^{\infty} E_k) &= c(\cup_{k=1}^{N} E_k), \\ \sum_{k=1}^{\infty} c(E_k) &= \sum_{k=1}^{N} c(E_k). \end{aligned}$$

As the finite additivity of counting measure trivially holds, $c(\cup_{k=1}^{N} E_k) = \sum_{k=1}^{N} c(E_k)$ holds, and thus we conclude that

$$c(\cup_{k=1}^{\infty} E_k) \quad = \quad \sum_{k=1}^{\infty} c(E_k).$$

Now, for the last case, assume that $\sum_{k=1}^{\infty} c(E_k) = \infty$.

Question 2. Royden 2.8. Closure of Finite Union.

Solution. Before we proceeding to the main part of the proof, we first prove the following lemma.

Lemma. Let $\{X_k\}_{k=1}^{n}$ be a finite collection of sets of real numbers. Then, we have that

$$\overline{\cup_{k=1}^{n} X_k} = \cup_{k=1}^{n} \overline{X_k},$$

where $\overline{X_k}$ denotes a closure of the set X_k.

Proof. □

Let B be a set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^{n}$ be a finite collection of open intervals that cover B. As $B \subseteq \cup_{k=1}^{n} I_k$, we have $\overline{B} \subseteq \overline{\cup_{k=1}^{n} I_k}$. Furthermore, with n being finite, we obtain that $\overline{\cup_{k=1}^{n} I_k} = \cup_{k=1}^{n} \overline{I_k}$. Then, it follows from the monotonicity, and finite sub-additivity property that

$$m^*(\overline{B}) \leq m^*(\cup_{k=1}^{n} \overline{I_k}) \leq \sum_{k=1}^{n} m^*(\overline{I_k}). \tag{1}$$

In particular, we have $m^*(\overline{B}) = 1$, as $B = [0, 1]$, and $\sum_{i=1}^{n} m^*(\overline{I_k}) = \sum_{i=1}^{n} m^*(I_k)$, as the outer measure of an open interval and corresponding closed interval are equal. Substituting the two equalities into the (1) inequality, we obtain

$$\sum_{i=1}^{n} m^*(I_k) \geq 1,$$

as desired. □

Question 3. Royden 2.14.

Solution. Let $m^*(E) > 0$. We wish to find a subset X of E such that $m^*(X) > 0$. Consider the countable collection of sets $\{(-M, M)\}_{M=1}^{\infty}$. Notice that, as $(-M, M)$ is bounded for some fixed M, $E \cap (-M, M)$ is a bounded subset of E. Furthermore, $E = \cup_{M=1}^{\infty} E \cap (-M, M)$. Then, by the countable sub-additivity of outer measure, we have

$$\sum_{M=1}^{\infty} m^*(E \cap (-M, M)) \geq m^*(E).$$

If $m^*(E \cap (-M, M)) = 0$ for all M, then we have the sum on the LHS equals 0, and obtain $0 > 0$, as $m^*(E) > 0$. This is a contradiction. Hence, there exists a M such that $m^*(E \cap (-M, M)) > 0$, and $E \cap (-M, M)$ is precisely the bounded subset of E with positive outer measure. We have shown that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure. □

Question 4. Royden 2.15.

Solution. Let $m(E) < \infty$ and $\epsilon > 0$. We wish to show that E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ. First, assume that E is bounded. Then, there exists an interval $[-M, M]$ such that $E \subseteq [-M, M]$. By the Archimedean principle, there exists $N \in \mathbb{N}$ such that $\frac{2M}{N} < \epsilon$. Now, consider the following finite disjoint collection of sets:

$$\{[-M + \frac{2M}{N}(n-1), -M + \frac{2M}{N}n) \cap E\}_{n=1}^{N+1}.$$

Notice that $E = \cup_{n=1}^{N+1}[-M + \frac{2M}{N}(n-1), -M + \frac{2M}{N}n) \cap E$. Furthermore, as every interval is measurable and intersection of two measurable sets is measurable, each set in the collection is measurable. By the monotonicity property of measure, we have

$$\begin{aligned} m([-M + \frac{2M}{N}(n-1), -M + \frac{2M}{N}n) \cap E) &\leq m([-M + \frac{2M}{N}(n-1), -M + \frac{2M}{N}n)) \\ &= \frac{2M}{N} \\ &< \epsilon. \end{aligned}$$

Hence, we have found a finite disjoint collection of measurable sets, each of which has measure at most ϵ, whose union equals E. Hence, we have proven that if $m(E) < \infty$, and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ for the E bounded case.

Now assume that E is unbounded. Consider the following disjoint collection of sets:

$$\{[M-1, M) \cap E\}_{M=-\infty}^{\infty}.$$

Notice that $E = \cup_{M=-\infty}^{\infty}[M-1, M) \cap E$. Furthermore, from the same reasoning as above, each set in the collection is measurable. By the countable additivity, we have

$$m(E) = \sum_{M=-\infty}^{\infty} m([M-1, M) \cap E).$$

Question 5. Royden 2.17.

Solution. Let E be a measurable set. Fix $\epsilon > 0$. Then, from inner approximation by closed sets, and outer approximation by open sets, there exists a closed set F and an open set O, such that

$$E \subseteq O \text{ with } m^*(O \setminus E) < \frac{\epsilon}{2} \text{ and } F \subseteq E \text{ with } m^*(E \setminus F) < \frac{\epsilon}{2}.$$

Applying the sub-additivity property of outer measure with $O \setminus E$ and $E \setminus F$, we have

$$m^*(O \setminus F) < m^*(O \setminus E) + m^*(E \setminus F) < \epsilon.$$

Hence, if E is measurable, then there exists an open set O and a closed set F for which $F \subseteq E \subseteq O$ and $m^*(E \setminus f) < \epsilon$.

Question 6. Royden 2.28.

Solution. Let $\{E_k\}_{k=1}^{\infty}$ be a countable, disjoint collection of measurable sets. By the finite additivity property, we have

$$m(\cup_{k=1}^{N} E_k) = \sum_{k=1}^{N} m(E_k),$$

for all N. Notice that $\{\cup_{k=1}^{N} E_k\}_{N=1}^{\infty}$ forms an ascending collection of measurable sets. Hence, by applying the continuity of measure to the ascending collection, $\{\cup_{k=1}^{N} E_k\}_{N=1}^{\infty}$, we have

$$m(\cup_{N=1}^{\infty} \cup_{k=1}^{N} E_k) = \lim_{N \to \infty} m(\cup_{k=1}^{N} E_k).$$

Simplifying the LHS and applying the finite additivity property to the RHS, we obtain

$$m(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$$

Since $\{E_k\}_{k=1}^{\infty}$ was chosen to be an arbitrary countable, disjoint collection of measurable sets, we have shown that finite additivty and continuity of measure implies countable additivity. $\square$