Real Variables:
Problem Set VIII

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Abstract

This work contains solutions to the problem set VIII of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 11-30.

30. For topological spaces X and Y, let the mapping $f: X \to Y$ be one-to-one and onto. Show that the following assertions are equivalent.
 (i) f is a homeomorphism of X onto Y.
 (ii) A subset E of X is open in X if and only if $f(E)$ is open in Y.
 (iii) A subset E of X is closed in X if and only if $f(E)$ is closed in Y.
 (iv) The image of the closure of a set is the closure of the image, that is, for each subset A of X, $f(\overline{A}) = \overline{f(A)}$.

Solution. Assume (i). We claim that $f(\bar{A}) \subseteq \overline{f(A)}$. Let $y \in f(\bar{A})$. Then, there exists $x \in \bar{A}$, such that $f(x) = y$. Since f is homeomorphic, it is continuous. By continuity of f at x, for any neighborhood O of y, there exists a neighborhood U of x, such that $f(U) \subseteq O$. As $x \in \bar{A}$, $U \cap A \neq \emptyset$, and $f(U) \cap f(A) \neq \emptyset$. Since $f(U) \subseteq O$, $f(A) \cap O \neq \emptyset$. Hence, $y \in \overline{f(A)}$. We now claim that $\overline{f(A)} \subseteq f(\bar{A})$. Let $y \in \overline{f(A)}$.

Assume (ii), and let E be a closed subset of X. We have $X \setminus E$ is open. By (ii), $f(X \setminus E)$ is open. As f is surjective, we have $f(X) = Y$. It follows that

$$\begin{aligned} f(X \setminus E) &= f(X) \setminus f(E) \\ &= Y \setminus f(E). \end{aligned}$$

Since $Y \setminus f(E)$ is open, $f(E)$ is closed.

Question 2. Royden 11-34.

34. Suppose that a topological space X has the property that every continuous real-valued function on X takes a minimum value. Show that any topological space that is homeomorphic to X also possesses this property.

Solution. Let Y be a topological space that is homeomorphic to X, and $\phi : X \to Y$ be a bijective map such that ϕ^{-1} is continuous. Let g be a continuous real-valued function, defined on Y. Consider $g(Y)$. We wish to show that $\inf_{y \in Y} g(y) \in g(Y)$.

Question 3. Royden 11-44.

44. Let $(X, \mathcal{T})$ be a topological space.

(i) Prove that if $(X, \mathcal{T})$ is compact, then $(X, \mathcal{T}_1)$ is compact for any topology $\mathcal{T}_1$ weaker than $\mathcal{T}$.

(ii) Show that if $(X, \mathcal{T})$ is Hausdorff, then $(X, \mathcal{T}_2)$ is Hausdorff for any topology $\mathcal{T}_2$ stronger than $\mathcal{T}$.

(iii) Show that if $(X, \mathcal{T})$ is compact and Hausdorff, then any strictly weaker topology is not Hausdorff and any strictly stronger topology is not compact.

Solution. **(i)** Let $\mathscr{T}_1$ be a topology for X, that is weaker than $\mathscr{T}$. It follows that $\mathscr{T}_1 \subseteq \mathscr{T}$. Let E be a subset of X, and $\{O_\lambda\}_{\lambda \in \Lambda}$ be an open cover of E in $(X, \mathscr{T}_1)$. As $\mathscr{T}_1 \subseteq \mathscr{T}$, the considered open cover is also an open cover in $(X, \mathscr{T})$. By compactness of $(X, \mathscr{T})$, there exists a finite subcollection of the open cover, that covers E. Hence, $(X, \mathscr{T}_1)$ is compact. $\square$

(ii) Let $\mathscr{T}_2$ be a topology for X, that is stronger than $\mathscr{T}$. It follows that $\mathscr{T} \subseteq \mathscr{T}_2$. If $|X| < 2$, X with any topology is trivially Hausdorff. Hence, we only consider the remaining case of $|X| \geq 2$. Let $x, y \in X$ such that $x \neq y$. As $(X, \mathscr{T})$ is Hausdorff, there exists a neighborhood of x, and a neighborhood of y, that are disjoint, which we denote as U and V respectively. As $\mathscr{T} \subseteq \mathscr{T}_2$, U and V are also open in $(X, \mathscr{T}_2)$. Hence, U is a neighborhood of x, and V is a neighborhood of y in $(X, \mathscr{T}_2)$. Moreover, U and V are disjoint. Hence, $(X, \mathscr{T}_2)$ is Hausdorff. $\square$

(iii) Let $\mathscr{T}_1$ be a topology for X, that is strictly weaker than $\mathscr{T}$. It follows that there exists a subset E of X such that it is open in $(X, \mathscr{T})$, but not open in $(X, \mathscr{T}_1)$. Furthermore, $X \setminus E$ is closed in $(X, \mathscr{T})$, but not closed in $(X \mathscr{T}_1)$. As $(X, \mathscr{T})$ is compact, $X \setminus E$ is compact as a subspace. Since $\mathscr{T}_1$ is weaker than $\mathscr{T}$, $X \setminus E$ is compact. Suppose for sake of contradiction that $(X, \mathscr{T}_1)$ is Hausdorff. It implies that $X \setminus E$ is closed in $(X, \mathscr{T})$, which is a contradiction. Hence, $(X, \mathscr{T}_1)$ is not Hausdorff. $\square$

Let $\mathscr{T}_2$ be a topology for X, that is strictly stronger than $\mathscr{T}$. It follows that there exists a subset E of X such that it is open in $(X, \mathscr{T}_2)$, but not open in $(X, \mathscr{T})$. Furthermore, $X \setminus E$ is closed in $(X, \mathscr{T}_2)$, but not closed in $(X, \mathscr{T})$. Suppose for sake of contradiction that $(X, \mathscr{T}_2)$ is compact. Then, as $X \setminus E$ is closed in $(X, \mathscr{T}_2)$ is compact. As $\mathscr{T}$ is weaker than $\mathscr{T}_2$, $X \setminus E$ is compact in $(X, \mathscr{T})$. As $(X \mathscr{T})$ is Hausdorff, $X \setminus E$ is closed in $(X, \mathscr{T})$, which is a contradiction. Hence, $(X, \mathscr{T}_2)$ is not compact. $\square$

Question 4. Royden 11-46.

46. (Dini's Theorem) Let $\{f_n\}$ be a sequence of continuous real-valued functions on a countably compact space X. Suppose that for each $x \in X$, the sequence $\{f_n(x)\}$ decreases monotonically to zero. Show that $\{f_n\}$ converges to zero uniformly.

Solution.

Question 4. Royden 12-16.

16. Consider the countable collection of metric spaces $\{(X_n, \rho_n)\}_{n=1}^{\infty}$. For the Cartesian product of these sets $X = \prod_{n=1}^{\infty} X_n$, define $\rho \colon X \times X \to \mathbf{R}$ by

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{\rho_n(x_n, y_n)}{2^n[1 + \rho_n(x_n, y_n)]}.$$

[1] It is convenient here to call an open set $\mathcal{O}$ set of the form $\mathcal{O} = \Pi_{\lambda \in \Lambda} \mathcal{O}_\lambda$, where each $\mathcal{O}_\lambda$ is an open subset of X_λ and $\mathcal{O}_\lambda = X_\lambda$ except for one λ, a *subbasic set* and the finite intersection of such sets a *basic set*.

Use the preceding problem to show that ρ is a metric on $X = \prod_{n=1}^{\infty} X_n$ which induces the product topology on X, where each X_n has the topology induced by the metric ρ_n.

Solution.

Question 6. Royden 12-20.

20. Provide a direct proof of the assertion that if X is compact and I is a closed, bounded interval, then $X \times I$ is compact. (Hint: Let $\mathcal{U}$ be an open covering of $X \times I$, and consider the smallest value of $t \in I$ such that for each $t' < t$ the set $X \times [0, t']$ can be covered by a finite number of sets in $\mathcal{U}$. Use the compactness of X to show that $X \times [0, t]$ can also be covered by a finite number of sets in $\mathcal{U}$ and that if $t < 1$, then for some $t'' > t$, $X \times [0, t'']$ can be covered by a finite number of sets in $\mathcal{U}$.)

Solution.