Real Variables:
Problem Set V

Youngduck Choi
Courant Institute of Mathematical Sciences
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set V of Real Variables 2015 at NYU.

1 Solutions

Question 6.10.

Solution. Let $x_1, x_2 \in (a, b)$, such that $x_1 < x_2$. Then, we have

$$\begin{aligned} f(x_1) &= \sum_{k=1}^{\infty} l((c_k, d_k) \cap (-\infty, x_1)) \\ f(x_2) &= \sum_{k=1}^{\infty} l((c_k, d_k) \cap (-\infty, x_2)). \end{aligned}$$

As $x_1 < x_2$, by the monotonicity property of the length function , we have

$$l((c_k, d_k) \cap (-\infty, x_1)) \quad \leq \quad l((c_k, d_k) \cap (-\infty, x_2)),$$

for all $k \geq 1$. It follows that $f(x_1) \leq f(x_2)$. Hence, f is increasing. Now, we show that f fails to be differentiable at each point in E. Let $x \in E$. Then,

Question 6.33.

Solution. Let $\{f_n\}$ be a sequence of real-valued functions on $[a, b]$ that converges pointwise on $[a, b]$ to the real-valued function f. We wish to show that $TV(f) \leq \liminf TV(f_n)$. Fix $\epsilon > 0$. Let $P = \{x_0, ..., x_m\}$ be a partition of $[a, b]$. By the triangle inequality, it follows that

$$\begin{aligned} V(f, P) &= \sum_{k=0}^{m-1} |f(x_{k+1}) - f(x_k)| \\ &= \sum_{k=0}^{m-1} |f(x_{k+1}) + f_n(x_{k+1}) - f_n(x_{k+1}) - f(x_k) + f_n(x_k) - f_n(x_k)| \\ &\leq \sum_{k=0}^{m-1} |f(x_{k+1}) - f_n(x_{k+1})| + |f_n(x_{k+1}) - f_n(x_k)| + |f(x_k) - f_n(x_k)| \\ &\leq V(f_n, P) + \sum_{k=1}^{m} |f(x_k) - f_n(x_k)| + \sum_{k=0}^{m-1} |f(x_k) - f_n(x_k)|, \\ &\leq TV(f_n) + \sum_{k=1}^{m} |f(x_k) - f_n(x_k)| + \sum_{k=0}^{m-1} |f(x_k) - f_n(x_k)|, \end{aligned}$$

for any n. Define $N = \max(N_0, ..., N_k)$, where $N_i (0 \leq i \leq k)$ corresponds to the convergence index for $\frac{\epsilon}{2m}$ at x_i. Then, it follows that

$$V(f, P) - \epsilon \quad \leq \quad TV(f_n),$$

for $n \geq N$. Consequently, we have that

$$V(f, P) - \epsilon \quad \leq \quad \liminf_{n \to \infty} TV(f_n),$$

Since ϵ was arbitrary, we have

$$V(f, P) \quad \leq \quad \liminf_{n \to \infty} TV(f_n).$$

Since P was arbitrary, we obtain that

$$TV(f) \quad \leq \quad \liminf_{n \to \infty} TV(f_n),$$

as desired. $\square$

Question 4. Royden 6.42.

Solution. Let f and g be real-valued functions, that are absolutely continuous functions on $[a, b]$. We wish to show that $f + g$ is absolutely continuous on $[a, b]$. Fix $\epsilon > 0$. As f and g are both absolutely continuous on $[a, b]$, there exist $\delta_f, \delta_g > 0$, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{k=1}^n$ in (a, b),

$$\sum_{k=1}^n [b_k - a_k] < \delta_f \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \frac{\epsilon}{2}$$
$$\sum_{k=1}^n [b_k - a_k] < \delta_g \implies \sum_{k=1}^n |g_(b_k) - g(a_k)| < \frac{\epsilon}{2}.$$

Define $\delta = \min(\delta_f, \delta_g)$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint open intervals in (a, b), such that $\sum_{k=1}^n [b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^n |f + g(b_k) - f + g(a_k)| &\leq \sum_{k=1}^n |f(b_k) - f(a_k)| + |g(b_k) - g(a_k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we have shown that $f + g$ is absolutely continuous on $[a, b]$.

Let f be a real-valued function, that is absolutely continuous on $[a, b]$. We show that cf, for any $c \in \mathbb{R}$, is absolutely continuous on $[a, b]$. Let $c = 0$. Then $cf = 0$, which can trivially be shown to be absolutely continuous, as $f(c) = 0$ for any $c \in [a, b]$. Suppose $c \neq 0$. As f is absolutely continuous on $[a, b]$, there exists $\delta_f > 0$, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{k=1}^n$ in (a, b),

$$\sum_{k=1}^n |b_k - a_k| < \delta_f \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \frac{\epsilon}{|c|}.$$

Define $\delta = \delta_f$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint open intervals in (a, b) such that $\sum_{k=1}^n [b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^n |cf(b_k) - cf(a_k)| &= |c| \sum_{k=1}^n |f(b_k) - f(a_k)| \\ &< |c| \frac{\epsilon}{|c|} = \epsilon. \end{aligned}$$

Since ϵ was arbitrary, combined with the $c = 0$ case, we have shown that cf, for any $c \in \mathbb{R}$, is absolutely continuous on $[a, b]$.

Let f be a real-valued function, that is absolutely continuous on $[a, b]$. We wish to show that f^2 is absolutely continuous on $[a, b]$. As f is absolutely continuous, f is continuous on $[a, b]$. Hence, by the Extreme Value Theorem, there exists M such that $|f| \leq M$ on $[a, b]$. Fix $\epsilon > 0$. As f is absolutely continuous on $[a, b]$, there exists $\delta_f > 0$, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{k=1}^n$ in (a, b),

$$\sum_{k=1}^{n} |b_k - a_k| < \delta_f \implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \frac{\epsilon}{2M}.$$

Define $\delta = \delta_f$. Let $\{(a_k, b_k)\}_{k=1}^\infty$ be a finite disjoint open intervals in (a, b) such that $\sum_{k=1}^n [b_k - a_k] < \delta$. It follows that

$$\begin{aligned}
\sum_{k=1}^{n} |f^2(b_k) - f^2(a_k)| &= \sum_{k=1}^{n} |f(b_k) - f(a_k)||f(b_k) + f(a_k)| \\
&\leq 2M \sum_{k=1}^{n} |f(b_k) - f(a_k)| \\
&< 2M \frac{\epsilon}{2M} = \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we have shown that f^2 is absolutely continuous on $[a, b]$.

Let f and g be real-valued functions, that are absolutely continuous on $[a, b]$. We wish to show that fg is absolutely continuous on $[a, b]$. Observe that

$$(f + g)^2 = f^2 + g^2 - 2fg,$$

which simplifies to

$$fg = -\frac{1}{2}((f + g)^2 + (-f^2) + (-g^2)).$$

As we have previously shown that a square of an absolutely continuous function is absolutely continuous, and a scalar multiple of an absolutely continuous function is absolutely continuous, we can deduce that fg is absolutely continuous on $[a, b]$. This completes the proof. □

Question 4. 6.45.

Solution. Let f be a real-valued function, that is absolutely continuous on $\mathbb{R}$. Let g be a real-valued function, that is absolutely continuous and strictly monotone on $[a, b]$. Without the loss of generality, we assume that g is strictly increasing. We wish to show that $f \circ g$ is absolutely continuous on $[a, b]$. Fix $\epsilon > 0$. As f is absolutely continous on $\mathbb{R}$, it is also absolutely continuous on $[g(a), g(b)]$, which is a non-degenerate closed interval, as g is strictly increasing. there exists δ_f, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{n=1}^\infty$ in $(g(a), g(b))$,

$$\sum_{k=1}^{n} [b_k - a_k] < \delta_f \implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon \ (*).$$

As g is absolutely continuous, there exists δ_g, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{n=1}^\infty$ in (a, b),

$$\sum_{k=1}^{n} [b_k - a_k] < \delta_g \implies \sum_{k=1}^{n} |g(b_k) - g(a_k)| < \delta_f.$$

Define $\delta = \delta_g$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite djsjoint open intervals in (a, b) such that $\sum_{k=1}^n [b_k - a_k] < \delta_g$. It follows that $\sum_{k=1}^n [g(b_k) - g(a_k)] < \delta_f$. As g is strictly increasing, we observe that $\{(g(a_k), g(b_k))\}_{k=1}^n$ form a finite disjoint open intervals in $(g(a), g(b))$. Therefore, from $(*)$ it follows that

$$\sum_{k=1}^{n} |f \circ g(b_k) - f \circ g(a_k)| < \epsilon.$$

Since ϵ was arbitrary, we have shown that $f \circ g$ is absolutely continuous on $[a, b]$. □