Real Variables:
Problem Set III

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Abstract

This work contains solutions to the problem set III of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 3.20.
Solution.

$$\chi_{A\cap B} = \begin{cases} 1 & \text{if } x \in A \cap B \\ 0 & \text{if } x \notin A \cap B \end{cases}$$

$$\chi_{A\cup B} = \begin{cases} 1 & \text{if } x \in A \cup B \\ 0 & \text{if } x \notin A \cup B \end{cases}$$

$$\chi_{A^c} = \begin{cases} 1 & \text{if } x \in A^c \\ 0 & \text{if } x \notin A^c \end{cases}$$

Question 2. Royden 3.21.
Solution.

Question 3. Royden 3.27.
Solution. The Egoroff

Question 4. Royden 4.12.
Solution. Let f a bounded measurable function on a set of finite measure E. Assume g is bounded and $f = g$ a.e. on E. First, as g is a function that equals a measurable function a.e., we have that g is measurable. Since both f and g are bounded measurable functions, we have $\int_E f$ and $\int_E g$ terms well-defined. Let $E_0 = \{x \in E \mid f(x) \neq g(x)\}$. Note that $\mathrm{m}(E_0) = 0$, as $f = g$ a.e. Consequently, $E \setminus E_0$ and E_0 are disjoint measurable sets. Then, by additivity over domain and linearity of integration, we have

$$\begin{aligned} \left| \int_E f - \int_E g \right| &= \left| \int_{E\setminus E_0} f - \int_{E\setminus E_0} g + \int_{E_0} f - \int_{E_0} g \right| \\ &= \left| \int_{E\setminus E_0} f - g + \int_{E_0} f - g \right|. \end{aligned}$$

As $f = g$ on $E \setminus E_0$, we have

$$\begin{aligned} \left| \int_E f - \int_E g \right| &= \left| \int_{E_0} f - g \right| \\ &\leq \int_{E_0} |f - g| . \end{aligned}$$

As both f and g are bounded, there exists M such that $|f - g| \leq M$ on E_0. Hence, we have

$$\begin{aligned} \left| \int_E f - \int_E g \right| &\leq M \cdot \mathrm{m}(E_0) \\ &\leq 0. \end{aligned}$$

Therefore, we have $\int_E f = \int_E g$ as desired. $\square$

Question 5. Royden 4.23.

Solution. Let $\{a_n\}$ be a sequence of non-negative real numbers. Let f be a function on $E = [1, \infty)$, defined by setting $f(x) = a_n$ if $n \leq x < n + 1$. Then, consider the following sequence of functions of nonnegative real numbers $\{f_n\}$ defined on E such that

$$f_n = \sum_{k=1}^{n} a_k \chi_{I_k},$$

where I_k denotes the characteristic function of an interval $[k, k+1)$. Notice that $\{f_n\}$ is increasing, and converges to f pointwise everywhere on E. Hence, by the Monotone Convergence Theorem, we have

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

As the integral on the RHS is a simple function with n values, we have

$$\int_E f = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \mathrm{m}(I_k).$$

By noting that $\mathrm{m}(I_k) = 1$ for all k and subsuming the limit into the summation, we finally obtain

$$\int_E f = \sum_{k=1}^{\infty} a_k,$$

as desired. $\square$

Question 6. Royden .

Solution.