Real Variables:
Problem Set IV

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Abstract

This work contains solutions to the problem set IV of Real Variables 2015 at NYU.

1 Solutions

Question 4. Royden 6.42.

Solution. Let f and g be absolutely continuous functions on $[a, b]$. We wish to show that $f + g$ is absolutely continuous. Fix $\epsilon > 0$. Let $\{(a_k, b_k)\}_{k=1}^n$ be an arbitrary finite disjoint open intervals in (a, b). As f and g are both absolutely continuous, there exist $\delta_f, \delta_g > 0$, such that

$$\sum_{k=1}^{n}[b_k - a_k] < \delta_f \implies \sum_{k=1}^{n}|f(b_k) - f(a_k)| < \frac{\epsilon}{2}$$

$$\sum_{k=1}^{n}[b_k - a_k] < \delta_g \implies \sum_{k=1}^{n}|g_(b_k) - g(a_k)| < \frac{\epsilon}{2}.$$

Define $\delta = \min(\delta_f, \delta_g)$, and suppose that $\sum_{k=1}^{n}[b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^{n}|f + g(b_k) - f + g(a_k)| &\leq \sum_{k=1}^{n}|f(b_k) - f(a_k)| + |g(b_k) - g(a_k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ and $\{(a_k, b_k)\}_{k=1}^n$ were arbitrary, we have shown that $f + g$ is absolutely continuous.

Let f be an absolutely continuous function on $[a, b]$. We show that cf, for any $c \in \mathbb{R}$, is absolutely continuous. Let $c = 0$. Then $cf = 0$, which can trivially be shown to be absolutely continuous, as $f(c) = 0$ for any $c \in [a, b]$ Suppose $c \neq 0$. Fix $\epsilon > 0$. Let $\{(a_k, b_k)\}_{k=1}^n$ be an arbitrary finite disjoint open intervals in (a, b). As f is absolutely continuous, there exists $\delta_f > 0$, such that

$$\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \frac{\epsilon}{|c|}.$$

Define $\delta = \delta_f$, and suppose that $\sum_{k=1}^{n}[b_k - a_k] < \delta$ It follows that

$$\begin{aligned} \sum_{k=1}^{n}|cf(b_k) - cf(a_k)| &= |c|\sum_{k=1}^{n}|f(b_k) - f(a_k)| \\ &\leq |c|\frac{\epsilon}{|c|} = \epsilon. \end{aligned}$$

Since ϵ and $\{(a_k, b_k)\}_{k=1}^n$ were arbitrary, combined with $c = 0$ case, we have shown that cf is absolutely continuous.

Let f be an absolutely continuous function on $[a, b]$. We first show that f^2 is absolutely continuous. As f is absolutely continuous, f is continuous on $[a, b]$. Hence, by the Extreme Value Theorem, there exists M such that $|f| \le M$ on $[a, b]$. Fix $\epsilon > 0$. Let $\{(a_k, b_k)\}_{k=1}^n$ be an arbitrary finite disjoint open intervals in (a, b). As f is absolutely continuous, there exists $\delta_f > 0$, such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \frac{\epsilon}{2M}.$$

Define $\delta = \delta_f$, and suppose that $\sum_{k=1}^n [b_k - a_k] < \delta$ It follows that

$$\begin{aligned}
\sum_{k=1}^{n} |f^2(b_k) - f^2(a_k)| &= \sum_{k=1}^{n} |f(b_k) - f(a_k)||f(b_k) + f(a_k)| \\
&\le 2M \sum_{k=1}^{n} |f(b_k) - f(a_k)| \\
&\le 2M \frac{\epsilon}{2M} = \epsilon.
\end{aligned}$$

Since ϵ and $\{(a_k, b_k)\}_{k=1}^n$ were arbitrary, we have shown that f^2 is absolutely continuous.

Let f and g be an absolutely continuous function on $[a, b]$. We wish to show that fg is absolutely continuous. Observe that

$$(f + g)^2 \quad = \quad f^2 + g^2 - 2fg,$$

which simplifies to

$$fg = -\frac{1}{2}((f + g)^2 + (-f^2) + (-g^2)).$$

As we have previously shown that a square of an absolutely continuous function is absolutely continuous, and a scalar multiple of an absolutely continuous function is absolutely continuous, we can deduce that fg is absolutely continuous. This completes the proof. $\square$