Real Variables:
Problem Set IX

Youngduck Choi
Courant Institute of Mathematical Sciences
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set IX of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 12-5.

5. **Suppose that a topological space X has the property that every continuous, bounded real-valued function on a closed subset has a continuous extension to all of X. Show that if X is Tychonoff, then it is normal.**

Solution. Assume that X is Tychonoff, and let A and B be non-empty disjoint closed subsets of X. Let $g : A \cup B \to \mathbb{R}$ such that such that $g(A) = a$ and $g(B) = b$. Observe that g is a real-valued function, that is continuous, bounded, on a closed subset of X. Therefore, by the given, there exists a continuous extension to all of X, which we denote as $g' : X \to \mathbb{R}$. Observe that as $(a - \frac{a+b}{2}, \frac{a+b}{2})$ is open in $\mathbb{R}$, by the continuity of g' we have $g'^{-1}((a - \frac{a+b}{2}, \frac{a+b}{2}))$ is open in X, which contains A. Likewise, $g'^{-1}((\frac{a+b}{2}, b + \frac{a+b}{2}))$ is open in X, which contains B. Notice that as g' is a function those two open sets are disjoint. Therefore, we have shown that A and B have neighborhoods that are disjoint. Since X is Tychonoff as well, X is normal. $\square$

Question 2. Royden 12-6.

6. **Let $(X, \mathcal{T})$ be a normal topological space and $\mathcal{F}$ the collection of continuous real-valued functions on X. Show that $\mathcal{T}$ is the weak topology induced by $\mathcal{F}$.**

Solution. Let $x \in X$. Consider a neighborhood $U_x \in \mathscr{T}$. It follows that $X \setminus U_x$ is closed in $\mathscr{T}$. As normal topological spaces are Tychnoff, and single points are closed in Tychnoff spaces, we have $\{x\}$ is closed in $\mathscr{T}$. Then, by the Urysohn's lemma, we have a continuous real-valued function $f : X \to [a, b]$ such that $f(\{x\}) = a$ and $f(X \setminus U_x) = b$. Note that $f \in \mathscr{F}$. Then, for a fixed ϵ such that $b - a > \epsilon > 0$, as $(a - \epsilon, a + \epsilon)$ is an open set in $\mathbb{R}$, we have $f^{-1}((a - \epsilon, a + \epsilon))$ is a basic open set of the weak-topology, as f is continuous and it's a finite intersection of the inverse image of an open set. Observe that as $f(X \setminus U_x) = b$, we have $f^{-1}((a - \epsilon, a + \epsilon)) \cap X \setminus U_x = \emptyset$. Hence $f^{-1}((a - \epsilon, a + \epsilon)) \subseteq U_x$. Therefore, we have found a basic open set of x in the weak topology contained in U_x. Hence, we have that the basis of weak-topology is a collection of open sets in $\mathscr{T}$, such that for each x and each neighborhood of x, U_x, there is an element of the basis of weak-topology, that is contained in U_x. Therefore, the basis of weak-topology, induced by $\mathscr{F}$ is a also basis of the strong topology. Hence, in this case, the strong topology $\mathscr{T}$ is the weak-topology induced by $\mathscr{F}$. □

Question 3. Royden 12-27.

27. For $f, g \in C[a, b]$, show that $f = g$ if and only if $\int_a^b x^n f(x)\,dx = \int_a^b x^n g(x)\,dx$ for all n.

Solution. Assume that $f = g$. Fix n. As $f, g \in C[a, b]$, $x^n \in C[a, b]$. and multiplication of continuous function is continuous, we have that $x^n f$ and $x^n g$ are continuous. As continuous functions on compact domain is integrable, by the linearity of integration, we have

$$\int_a^b x^n f(x)dx - \int_a^b x^n g(x)dx \quad = \quad \int_a^b x^n (f-g)(x)dx$$

As $f = g$, $f - g(x) = 0$ for all $x \in [a, b]$. It follows that

$$\int_a^b x^n f(x)dx - \int_a^b x^n g(x)dx \quad = \quad 0,$$

from which we obtain

$$\int_a^b x^n f(x)dx \quad = \quad \int_a^b x^n g(x)dx.$$

Since n was arbitrary, we have that the above equality holds for all n. Conversely, assume that $\int_a^b x^n f(x)dx = \int_a^b x^n g(x)dx$ for all n. By appealing to the linearity of integration, we see that

$$\int_a^b p(f-g)(x)dx = 0,$$

for any polynomial p defined on $[a, b]$. We claim that

$$\int_a^b (f-g)^2(x)dx = 0,$$

which will imply that $f = g$ almost everywhere immediately. By Weiestrass Approximation theorem, we can choose a sequence of polynomials p_n such that

$$|p_n - (f-g)| < \frac{1}{n}.$$

It follows that $\{p_n(f-g)\}$ converges to $(f-g)^2$ pointwise everywhere on $[a, b]$. As $|p_n - (f-g)| < 1$ for all n on $[a, b]$. As $f - g$ is a continuous function defined on a compact subset of $\mathbb{R}$, by the extreme value theorem, there exists $M > 0$ such that $|f - g| < M$ on $[a, b]$. It follows that $g(x) = M(M+1)$ on $[a, b]$ is integrable and dominates $\{p_n(f-g)\}$. Hence, by the Dominated Convergence theorem, we have

$$\int_a^b (f-g)^2(x)dx \quad = \quad \lim_{n\to\infty} \int_a^b p_n(f-g)(x)dx.$$

Since $\int_a^b p_n(f-g)(x)dx = 0$ for all n, it follows that

$$\int_a^b (f-g)^2(x)dx \quad = \quad 0.$$

Hence, we conclude that $f = g$ almost everywehre. As $f, g \in C[0, 1]$, and $f = g$ almost everywhere, it follows that $f = g$ everywhere. □

Question 4. Royden 12-35.

35. Let $\mathcal{A}$ be an algebra of continuous real-valued functions on a compact Hausdorff space X that separates points. Show that either $\overline{\mathcal{A}} = C(X)$ or there is a point $x_0 \in X$ for which $\overline{\mathcal{A}} = \{f \in C(X) \mid f(x_0) = 0\}$. (Hint: If $1 \in \overline{\mathcal{A}}$, we are done. Moreover, if for each $x \in X$ there is an $f \in \mathcal{A}$ with $f(x) \neq 0$, then there is a $g \in \mathcal{A}$ that is positive on X and this implies that $1 \in \overline{\mathcal{A}}$.)

Solution. Firstly, we show that if there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in A$, then $\bar{A} = \{f \in C(X) \mid f(x) = 0\}$. Let $f \in \bar{A}$. Then, there exists $\{f_n\}$ be a sequence of functions, chosen from A, such that it converges uniformly to f. Note that uniform convergence of functions preserve continuity. Therefore, $f \in C(x)$. As $f_n(x_0) = 0$ for all n, it follows that $f(x_0) = 0$. Hence, $\bar{A} \subseteq \{f \in C(x) \mid f(x) = 0\}$. Now, let $f \in C(X)$ such that $f(x_0) = 0$. We show that there exists a sequence $\{f_n\}$ from A, such that $f_n \to f$ uniformly.

Now, assume that for all $x \in X$, there exists $f \in A$, such that $f(x) \neq 0$. Consider a family of functions $\{f_x\}_{x \in X}$ such that each f_x satisfies $f_x(x) \neq 0$. Observe that as f_x are continuous, there exists $B(x, \delta_x)$ such that $f_x(B(x, \delta_x))$ is nonzero. Observe that $\{B(x, \delta_x)\}_{x \in X}$ is an open cover of X. As X is compact, there exists a finite sub-cover that covers X, we label that sub-cover as $\{B(x_n), \delta_{x_n}\}_{n=1}^{N}$. Consider a function g defined on X such that

$$g = \sum_{n=1}^{N} f_n^2.$$

Observe that g is stritly positive and in A, as it is a finite sum of squares of functions from A. By the extreme value theorem, there exists M such that $g < M$ on X. Let $l = \frac{1}{M} g$, which is still in A. Then, we have that $0 < l < 1$ on A. Consider a sequence of functions h_n on X defined by

$$h_n \quad = \quad \sum_{i=1}^{n} (-l^n)^i,$$

Observe that h_n converges uniformly to $\frac{1}{1 + l^n}$. Furthermore, l^n converges uniformly to 0 as $0 < l < 1$. Hence, we have that h_n converges uniformly to 1 and $1 \in \bar{A}$. Therefore, $\bar{A}$ is an algebra of continuous real-valued functions on X that separates points in X and contains the constant functions. Therefore, by Stone-Weierstrauss, $\bar{A}$ is dense in $C(X)$. Hence $\bar{A} = C(X)$. $\square$

Question 5. Royden 13-8.

8. A nonnegative real-valued function $\|\cdot\|$ defined on a vector space X is called a **pseudonorm** if $\|x+y\| \leq \|x\| + \|y\|$ and $\|\alpha x\| = |\alpha| \, \|x\|$. Define $x \cong y$, provided $\|x-y\| = 0$. Show that this is an equivalence relation. Define $X/_{\cong}$ to be the set of equivalence classes of X under $\cong$ and for $x \in X$ define $[x]$ to be the equivalence class of x. Show that $X/_{\cong}$ is a normed vector space if we define $\alpha[x] + \beta[y]$ to be the equivalence class of $\alpha x + \beta y$ and define $\|[x]\| = \|x\|$. Illustrate this procedure with $X = L^p[a, b]$, $1 \leq p < \infty$.

Solution. We show that the pseudo-norm relation is reflexive, symmetric, and transitive.

Let $x \in X$. It follows that

$$\|x - x\| = \|\theta\|,$$

where θ is the identity element of the linear space X. By definition of linear space, we have $\alpha \cdot \theta = \theta$ for all α. Hence, for some $\alpha > 1$, we have

$$\begin{aligned} \|\theta\| &= \|\alpha \cdot \theta\| \\ &= |\alpha|\|\theta\|. \end{aligned}$$

As $|a| > 0$, we have $|\theta| = 0$. Consequently, $\|x - x\| = 0$. It follows that for all $x \in X$, $x \equiv x$. The relation is reflexive.

Let $x, y \in X$ and $x \equiv y$. Observe that

$$\begin{aligned} \|x - y\| &= \|-1 \cdot (y - x)\| \\ &= |-1|\|y - x\| \\ &= \|y - x\|. \end{aligned}$$

As $x \equiv y$, which gives $\|x - y\| = 0$, it follows that $\|y - x\| = 0$ and $y \equiv x$. Hence, the relation is symmetric.

Let $x, y, z \in X$ and $x \equiv y$ and $y \equiv z$. By triangle inequality, it follows that

$$\begin{aligned} \|y - z\| &= \|(x - y) + (y - z)\| \\ &\leq \|x - y\| + \|y - z\| = 0 + 0 = 0. \end{aligned}$$

Hence, $\|y - z\| = 0$, and it follows that $x \equiv z$. Hence, the relation is symmetric. It follows that the pseudo-norm relation is an equivalence relation on the linear space X.

$\square$

We show that $X_{\equiv}$ is a normed vector space. The fact that given space is a vector space arises from the fact that X is a vector space and we define addition and scalar multiplication in terms of the corresponding operations on X. This can be trivially checked. We show that the defined relation is indeed a norm. Firstly, we check that the defined norm is well defined. Let $x, y \in X$, such that $x \equiv y$. It follows that $\|x - y\| = \|y - x\| = 0$. By triangle inequality, it follows that

$$\begin{aligned} \|x\| &= \|y + (x - y)\| \\ &\leq \|y\| + \|x - y\| \\ &= \|y\|, \\ \|y\| &= \|x + (y - x)\| \\ &\leq \|x\| + \|y - x\| \\ &= \|x\|, \end{aligned}$$

Hence, $\|x\| = \|y\|$, and it follows that $\|[x]\| = \|[y]\|$. The norm is well-defined. Now, observe that the non-negativity is satisfied, as the pseudo-norm is non-negative. By definition, it follows that $\|\alpha[x]\| = \|[\alpha x]\| = \|\alpha x\| = |\alpha|\|x\| = |\alpha|\|[x]\|$. Hence, homogeneity is satisfied. Again, by definition and the triangle inequality from the pseudo-norm, we have

$$\begin{aligned} \|[x] + [y]\| &= \|[x + y]\| \\ &= \|x + y\| \\ &\leq \|x\| + \|y\| \\ &= \|[x]\| + \|[y]\|. \end{aligned}$$

Hence, the triangle inequality is satisfied. We also have $\|[\theta]\| = \|\theta\| = 0$. Assume that $\|[x]\| = 0$. Suppose for sake of contradiction that $[x] =\neq [\theta]$. It follows that $0 \neq \|x - \theta\| = \|[x - \theta]\| = \|[x]\|$, which is a contradiction to $\|[x]\| = 0$. Therefore, $X_{\equiv}$ is a normed vector space. □

Question 6. Royden 13-34.

34. Let T be a linear operator from a normed linear space X to a finite-dimensional normed linear space Y. Show that T is continuous if and only if $\ker T$ is a closed subspace of X.

Solution. Assume that T is continuous. Observe that $\{\theta\}$, where θ is the identity element of the normed linear space Y, is closed, as a single point in a metric space is closed. Since T is continuous, we have $T^{-1}(\{\theta\}) = \ker T$ is closed. We now show that $\ker T$ is a subspace of X. Let $x, y \in \ker T$. By linearity it follows that

$$\begin{aligned} T(x + y) &= T(x) + T(y) = 0 + 0 = 0 \\ T(\alpha x) &= \alpha T(x) = \alpha 0 = 0. \end{aligned}$$

Hence, we have shown that $\ker T$ is a closed subspace of X.

□