Real Variables:
Problem Set X

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Abstract

This work contains solutions to the problem set X of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 13-41.

41. Let X be the linear space of all polynomials defined on $\mathbf{R}$. For $p \in X$, define $\|p\|$ to be the sum of the absolute values of the coefficients of p. Show that this is a norm on X. For each n, define $\psi_n : X \to \mathbf{R}$ by $\psi_n(p) = p^{(n)}(0)$. Use the properties of the sequence $\{\psi_n\}$ in $\mathcal{L}(X, \mathbf{R})$ to show that X is not a Banach space.

Solution. We first show that $\| \ \| : X \to \mathbb{R}$ given is a norm on X. First of all, let $p = 0$. Then, $\|p\| = 0$. Now, let $p = \sum_{i=0}^{n} b_i x^i$, and assume that $\|p\| = 0$. It follows that $\sum_{i=0}^{n} |b_i| = 0$. As $|b_i| \geq 0$ for all i, we have that $b_i = 0$ for all i. Hence, $p = 0$. For proving the triangle inequality, let $p_1 = \sum_{i=0}^{n_1} b_i x^i$ and $p_2 = \sum_{i=0}^{n_2} c_i x^i$. Without the loss of generality, we assume that $n_1 \geq n_2$, and define $n = n_1$, $p_1 = \sum_{i=0}^{n} b_i x^i$ and $p_2 = \sum_{i=0}^{n} c_i x^i$, with $c_i = 0$ for $i > n_2$. By the triangle inequality of reals, it follows that

$$
\begin{aligned}
\|p_1 + p_2\| &= \|\sum_{i=0}^{n} (b_i + a_i) x^i\| \\
&= \sum_{i=0}^{n} |b_i + a_i| \\
&\leq \sum_{i=0}^{n} |b_i| + |a_i| \\
&= \sum_{i=0}^{n} |b_i| + \sum_{i=0}^{n} |c_i| = \|p_1\| + \|p_2\|.
\end{aligned}
$$

Now, let $p = \sum_{i=0}^{n} b_i x^i$, and $\alpha \in \mathbb{R}$. It follows that

$$
\begin{aligned}
\|\alpha p\| &= \|\alpha \sum_{i=0}^{n} b_i x^i\| \\
&= \|\sum_{i=0}^{n} \alpha b_i x^i\| \\
&= \sum_{i=0}^{n} |\alpha b_i| \\
&= |\alpha| \sum_{i=0}^{n} |b_i| = |\alpha| \|p\|.
\end{aligned}
$$

Hence, we have shown that $\| \ \|$ given is a norm.

Now, we first show that each operator ψ_n is bounded, thus continuous. Observe that we can represent an arbitrary polynomial p uniquely as , for some k, $p = \sum_{i=0}^{\infty} c_i x^i$, where $c_i = 0$ for $i \geq k$. Fix ψ_n. Observe that for any p, we have $|c_n| \leq \|p\|$. It follows that

$$
\begin{aligned}
|\psi_n(p)| &= |n! \cdot c_n| \\
&= |n!||c_n| \\
&\leq |n!|\|p\|
\end{aligned}
$$

Hence, ψ_n is bounded, thus continuous for any n. Note that by taking $p = x^n$, we obtain $n! \leq M$ for any bound M for ψ_n. Hence, it follows that $\|\psi_n\| = n!$. Again, for any polynomial p, observe that $\psi_n(p) = 0$ for $n > k$, where k denotes the degree of the polynomial p. Consequently, we have

$$
\lim_{n \to \infty} \psi_n(p) \quad = \quad 0,
$$

for any p. Therefore, if X is Banach, the conditions of the Banach-Saks-Steinhaus theorem is satisfied. However, as $\|\psi_n\| = n!$, $\{\psi_n\}$ cannot be uniformly bounded. This is a contradiction. X is not Banach.

Question 2. Royden 14-18.

18. Let X be a normed linear space, ψ belong to X^*, and $\{\psi_n\}$ be in X^*. Show that if $\{\psi_n\}$ converges weak-$*$ to ψ, then

$$\|\psi\| \leq \limsup \|\psi_n\|.$$

Solution. As $\{\psi_n\}$ is weak-* convergent to ψ, we have

$$\psi_n(x) \quad \to \quad \psi(x),$$

for all $x \in X$. Let $x \in X$. As $|\cdot|$ is continuous on $\mathbb{R}$, it follows that

$$\lim_{n\to\infty} |\psi_n(x)| \quad = \quad |\psi(x)|.$$

As $|\psi_n(x)| \leq ||\psi_n|| \cdot ||x||$,

$$\begin{aligned} |\psi(x)| &= \lim_{n\to\infty} |\psi_n(x)| \\ &= \limsup_{n\to\infty} |\psi_n(x)| \\ &\leq \limsup_{n\to\infty} ||\psi_n|| \cdot ||x|| \\ &= ||x|| \limsup_{n\to\infty} ||\psi_n||. \end{aligned}$$

Since $x \in X$ was arbitrary, it follows that

$$||\psi|| \quad \leq \quad \limsup_{n\to\infty} ||\psi_n||,$$

as desired. $\square$

Question 3. Royden 14-23.

23. Let Y be a linear subspace of a normed linear space X and z be a vector in X. Show that

$$\operatorname{dist}(z, Y) = \sup\left\{\psi(z) \mid \|\psi\| = 1, \psi = 0 \text{ on } Y\right\}.$$

Solution. Consider a functional $p : X \to [0, \infty)$ be defined by

$$p = \begin{cases} 0, & \text{if } x \in Y \\ ||x||, & \text{otherwise.} \end{cases}$$

We first show that p is positively homogeneous. Let $\lambda > 0$. Let $x \in Y$, then as Y is a linear subspace of X, $\lambda x \in Y$. It follows that

$$\begin{aligned} p(\lambda x) &= 0 \\ &= \lambda p(x). \end{aligned}$$

Let $x \notin Y$. It follows that $\lambda x \notin Y$, as otherwise we get a contradiction that $x \in Y$ from the linear subspace property of Y. It follows that

$$\begin{aligned} p(\lambda x) &= ||\lambda x|| \\ &= |\lambda|||x|| \\ &= \lambda||x|| \\ &= \lambda p(x). \end{aligned}$$

Hence, we have shown that p is positively homogeneous. Now, we show that p is sub-additive. Let $x, y \in Y$. Then, it follows that

$$\begin{aligned} p(x+y) &= 0 \\ &= p(x) + p(y). \end{aligned}$$

Hence, $p(x+y) \leq p(x) + p(y)$ holds. Let $x, y \notin Y$. Then, by the triangle inequality of norm, it follows that

$$\begin{aligned} p(x+y) &= ||x+y|| \\ &\leq ||x| + ||y|| \\ &= p(x) + p(y). \end{aligned}$$

Let $x \in Y$ and $y \notin Y$. It follows that

$$\begin{aligned} p(x+y) &= ||x+y|| \\ &\leq ||x|| + ||y|| \\ &= 0 \end{aligned}$$

Question 4. Royden 15-12.

12. If Y is a linear subspace of a Banach space X, we define the *annihilator* $Y^{\perp}$ to be the subspace of X^* consisting of those $\psi \in X^*$ for which $\psi = 0$ on Y. If Y is a subspace of X^*, we define Y^0 to be the subspace of vectors in X for which $\psi(x) = 0$ for all $\psi \in Y$.
    (i) Show that $Y^{\perp}$ is a closed linear subspace of X^*.
    (ii) Show that $(Y^{\perp})^0 = \overline{Y}$.

Solution.