Real Variables: Problem Set XI

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Abstract

This work contains solutions to the problem set XI of Real Variables 2015 at NYU.

1 Solutions

Question Royden 17-6.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and X_0 belong to $\mathcal{M}$. Define $\mathcal{M}_0$ to be the collection of sets in $\mathcal{M}$ that are subsets of X_0 and μ_0 the restriction of μ to $\mathcal{M}_0$. Show that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

Solution. We first show that $(X_0, \mathscr{M}_0)$ is a measurable space. To this end, we must show that $\mathscr{M}_0$ is a σ-algebra of X_0. As $\emptyset$ and X_0 belong to $\mathscr{M}$, are subsets of X_0, it follows that $\emptyset$ and X_0 belong to $\mathscr{M}_0$. Let $\{A_n\}_{n=1}^{\infty}$ be a countable collections of sets in $\mathscr{M}_0$. As $A_n \subseteq X_0$ for all n, we have $\bigcup_{n=1}^{\infty} A_n \subseteq X_0$. Furthermore, as $\mathscr{M}$ is a σ-algebra, and $A_n \in \mathscr{M}$ for all n, we also have $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}$. Hence, it follows that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}_0$. Now, let A be a set, belonging to $\mathscr{M}_0$. Then, as $X_0 \setminus A$ is a subset of X_0, and X_0 and A belong to M, which gives $X_0 \setminus A \in \mathscr{M}$, we have $X_0 \setminus A$ belongs to $\mathscr{M}_0$. Hence, we have shown that $\mathscr{M}_0$ is a σ-algebra, and $(X_0, \mathscr{M}_0)$ is a measurable space. Now, it remains to be shown that the restricted map μ_0 has the properties of a measure. First, observe that $\emptyset \in \mathscr{M}_0$ and $\mu_0(\emptyset) = \mu(\emptyset) = 0$. Now, let $\{A_n\}$ be a countable disjoint sets from $\mathscr{M}_0$. Since $A_n \in \mathscr{M}$ for all n, by the countable additivity of μ and the fact that $\mathscr{M}_0$ is a σ-algebra, it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu_0(A_n) &= \sum_{n=1}^{\infty} \mu(A_n) \\
&= \mu(\bigcup_{n=1}^{\infty} A_n) \\
&= \mu_0(\bigcup_{n=1}^{\infty} A_n).
\end{aligned}$$

Therefore, we have shown that $(X_0, \mathscr{M}_0, \mu_0)$ is a measure space. $\square$

Question Royden 17-15.

15. Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$, where α and β are real numbers. Show that

$$|\alpha\nu| = |\alpha||\nu| \text{ and } |\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|,$$

where $\nu \leq \mu$ means $\nu(E) \leq \mu(E)$ for all measurable sets E.

Solution. Let $(X, \mathscr{M})$ be a measurable space. Let v_1 and v_2 be two finite signed measures. Consider a set function $\alpha v_1 + \beta v_2$ on $\mathscr{M}$, for $\alpha, \beta \in \mathbb{R}$, which is defined by

$$\alpha v_1 + \beta v_2(E) \quad = \quad \alpha v_1(E) + \beta v_2(E),$$

for $E \in \mathscr{M}$. As v_1 and v_2 only take finite values, it also follows that $\alpha v_1 + \beta v_2$ also assumes only finite values, as an addition of two finite values are finite. Furthermore, it follows that

$$\begin{aligned} \alpha v_1 + \beta v_2(\emptyset) &= \alpha v_1(\emptyset) + \beta v_2(\emptyset) \\ &= 0 + 0 = 0, \end{aligned}$$

as v_1 and v_2 are signed measures.

Question Royden 17-17.

17. Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu+\nu-|\mu-\nu|)$ and $\mu \vee \nu = \mu+\nu-\mu \wedge \nu$.
    (i) Show that the signed measure $\mu \wedge \nu$ is smaller than μ and ν but larger than any other signed measure that is smaller than μ and ν.
    (ii) Show that the signed measure $\mu \vee \nu$ is larger than μ and ν but smaller than any other measure that is larger than μ and ν.
    (iii) If μ and ν are positive measures, show that they are mutually singular if and only if $\mu \wedge \nu = 0$.

Solution.

Question Royden 18-50.

50. Establish the uniqueness of the function f in the Radon-Nikodym Theorem.

Solution. Let $\mathcal{A}$ be an algebra of continuous real-valued functions on a compact Hausdorf space X that separates points. Show that either $\bar{\mathcal{A}} = C(X)$ or there is a point $x_0 \in X$ for which $\bar{\mathcal{A}} = \{f \in C(X) | f(x_0) = 0\}$.

To solve this problem, it suffices to prove that $\bar{A} = C(X)$ assuming $\bar{\mathcal{A}} \neq \{f \in C(X)|f(x_0) = 0\}$, i.e., $\forall x \in X, \exists f_x \in C(X)$, s.t. $f_x(x) = y_x \neq 0$. Then, the open interval $I_x = (y_x - \delta, y_x + \delta)$ where $0 < \delta < |y_x|$, is mapped to an open interval $O_x = f_x^{-1}(I_x) \subset X$ where $x \in O_x$. Therefore, $X \subseteq \cup_{x \in X} O_x$. Since the foregoing is an open cover of a compact space X, it contains a finite subcover $\cup_{i=1}^{n} O_{x_i}$. By construction, $0 \notin f_{x_i}(O_{x_i})$. Thus $g = \sum_{i=1}^{n} f_{x_i}^2$ is in A and takes strictly positive values on X.

Define $h : K \cup \{0\} \to R_+$ as follows

$$h(x) = \begin{cases} 1/x & \text{if } x \in K \\ 0 & \text{if } x = 0 \end{cases}$$

By Proposition 20, since X is compact and g is a continuous mapping, the range of g, $K = g(X)$, is compact, which in the case of a real-valued range means that it is closed and bounded. Furthermore, since K is closed and $0 \notin K$, it is not possible to have a sequence in K converging to 0. Therefore, $0 \notin \bar{K} = K$, which implies that $h(x)$ is continuous on $D = K \cup \{0\}$

Now, since $h \in C(K)$, by Stone-Weierstrass, given $\epsilon > 0$, there exists p_n, a polynomial, s.t.

$$|h(x) - p_n(x)| < \epsilon/2$$

for all $x \in K$. Since the above implies that $|p_n(0)| < \epsilon/2$

$$|h(x) - (p_n(x) - p_n(0))| \leq |h(x) - p_n(x)| + |p_n(0))| < \epsilon$$

where $p_n^*(x) = p_n(x) - p_n(0)$ (and thus $p_n^*(0) = 0$, which trivially guarantees the uniform convergence on $K \cup \{0\}$). Therefore, p_n^* is continuos on D, and $p_n^* \circ g \in A$. Since p_n^* converges uniformly to h and g is continuous and bounded (and consequently uniformly continuous on a compact set, $p_n^* \circ g \to 1/g$ uniformly. Therefore, $1/g \in \bar{A}$, which together with the fact that $g \in A$, implies that $1 \in \bar{A}$. This generates the family of constant functions in A. By the Stone-Weierstrass Approximation, $\bar{A}$ is dense in $C(X)$. Since $\bar{A}$ is closed, this implies that $\bar{A} = C(X)$.

Question Royden 18-54.

Solution.

54. Let μ, ν, and λ be σ-finite measures on the measurable space $(X, \mathcal{M})$.

(i) If $\nu \ll \mu$ and f is a nonnegative function on X that is measurable with respect to $\mathcal{M}$, show that

$$\int_X f\, d\nu = \int_X f \left[\frac{d\nu}{d\mu}\right] d\mu.$$

(ii) If $\nu \ll \mu$ and $\lambda \ll \mu$, show that

$$\frac{d(\nu+\lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \text{ a.e. } [\mu].$$

(iii) If $\nu \ll \mu \ll \lambda$, show that

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} \text{ a.e. } [\lambda].$$

Question Royden 18-55.

55. Let μ, ν, ν_1, and ν_2 be measures on the measurable space $(X, \mathcal{M})$.

(i) Show that if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

(ii) Show that if ν_1 and ν_2 are singular with respect to μ, then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.

(iii) Show that if ν_1 and ν_2 are absolutely continuous with respect to μ, then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.

(iv) Prove the uniqueness assertion in the Lebesgue decomposition.

Solution.