Real Variables:
Problem Set VIII

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Abstract

This work contains solutions to the problem set VIII of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 11-30.

30. For topological spaces X and Y, let the mapping $f: X \to Y$ be one-to-one and onto. Show that the following assertions are equivalent.
    (i) f is a homeomorphism of X onto Y.
    (ii) A subset E of X is open in X if and only if $f(E)$ is open in Y.
    (iii) A subset E of X is closed in X if and only if $f(E)$ is closed in Y.
    (iv) The image of the closure of a set is the closure of the image, that is, for each subset A of X, $f(\overline{A}) = \overline{f(A)}$.

Solution. Assume (i). Let E be a subset of X. Assume that E is open in X. As f^{-1} is continuous, $f(E)$ is open. Conversely, assume that $f(E)$ is open. as f is continuous, E is open in X. Therefore, (ii) holds. Assume that (ii) holds. Then, for any open set O in X, $f(O)$ is open. As $f = {f^{-1}}^{-1}$, f^{-1} is continuous. Let O be an open set in Y. As f is surjective, there exists a subset E of X that $f(E) = O$. By (ii), E is open. Hence, $f^{-1}(O)$ is open. Hence f is continuous. f is a homeomorphism. Therefore, (i) and (ii) are equivalent.

Assume (ii), and let E be a subset of X. E being closed is equivalent to $X \setminus E$ being open. Since f is bijective $f(X) = Y$, and by (ii), $X \setminus E$ being open is equivalent to $f(X \setminus E)$, which equals, $Y \setminus E$, being open. This is again equivalent to E being closed. Hence (ii) and (iii) are equivalent.

So far, we have shown that (i), (ii), and (iii) are equivalent. We now show that (i) implies (iv) and (iv) implies (iii). Assume (i). We claim that $f(\bar{A}) \subseteq \overline{f(A)}$. Let $y \in f(\bar{A})$. Then, as f is bijective, there exists an unique $x \in \bar{A}$, such that $f(x) = y$. Since f is homeomorphic, it is continuous. By continuity of f at x, for any neighborhood O of y, there exists a neighborhood U of x, such that $f(U) \subseteq O$. As $x \in \bar{A}$, $U \cap A \neq \emptyset$, and $f(U) \cap f(A) \neq \emptyset$. Since $f(U) \subseteq O$, $f(A) \cap O \neq \emptyset$. Hence, $y \in \overline{f(A)}$. We now claim that $\overline{f(A)} \subseteq f(\bar{A})$. Let $x \in \overline{f(A)}$. As f^{-1} is bijective, there exists an unique $y \in X$ such that $y = f^{-1}(x)$ By the continuity of f^{-1} at x, for any neighborhood O of y, there exists a neighborhood U of x such that $f^{-1}(U) \subseteq O$. As $x \in \overline{f(A)}$, $U \cap f(A) \neq \emptyset$, and $f^{-1}(U) \cap A \neq \emptyset$. Since $f^{-1}(U) \subseteq O$, we have $O \cap A \neq \emptyset$. Hence, $y \in \overline{A}$. Hence, $x \in f(\overline{A})$. Therefore, we have shown that $f(\bar{A}) = \overline{f(A)}$.

Assume (iv). Let E be a subset of X. Assume that E is closed in X. Then, $E = \overline{E}$. Consequently, by (iv), it follows that $f(E) = f(\overline{E}) = \overline{f(E)}$. Since, $f(E) = \overline{f(E)}$, $f(E)$ is closed. Now, assume that $f(E)$ is closed. Then, by (iv), it follows that $f(E) = \overline{f(E)} = f(\overline{E})$. Since f is injective, $E = \overline{E}$ holds. Therefore, (iv) implies (iii).

We have shown that all four statements are equivalent. □

Question 2. Royden 11-34.

34. Suppose that a topological space X has the property that every continuous real-valued function on X takes a minimum value. Show that any topological space that is homeomorphic to X also possesses this property.

Solution. Let Y be a topological space that is homeomorphic to X. Let g be a continuous real-valued function, defined on Y. As X and Y are homeomorhpic, there exists a continuous bijection from X to Y, which we denote as ϕ. Observe that $g \circ \phi$ is a real-valued function, defined on X. Since ϕ and g are continuous, and composition of continuous maps is continuous, we have $g \circ \phi$ is continuous on X. Therefore, by the given, $g \circ \phi(X)$ attains a minimum value. Since $\phi(X) = Y$, we have $g \circ \phi(X) = g(Y)$. Hence, $g(Y)$ attains a minimum value. As g was considered to be an arbitrary, continuous function, all continuous function on Y attains a minimum value. □

Question 3. Royden 11-44.

44. Let $(X, \mathcal{T})$ be a topological space.

(i) Prove that if $(X, \mathcal{T})$ is compact, then $(X, \mathcal{T}_1)$ is compact for any topology $\mathcal{T}_1$ weaker than $\mathcal{T}$.

(ii) Show that if $(X, \mathcal{T})$ is Hausdorff, then $(X, \mathcal{T}_2)$ is Hausdorff for any topology $\mathcal{T}_2$ stronger than $\mathcal{T}$.

(iii) Show that if $(X, \mathcal{T})$ is compact and Hausdorff, then any strictly weaker topology is not Hausdorff and any strictly stronger topology is not compact.

Solution. Before preceeding to the main problem, we state the following preposition:
Let $(X, \mathscr{T})$ be a topological space, and $(X, \mathscr{T}_1)$ be a topological space such that $\mathscr{T}_1$ is weaker or stronger than $\mathscr{T}$. Let E be a subset of X. Then, the subspace topology of $\mathscr{T}_1$ on E is still weaker or stronger than the subspace topology of $\mathscr{T}$ on E.

By definition of subspace topology, we have

$$\mathscr{T}_E = \{E \cap U \mid U \in \mathscr{T}\}$$
$$\mathscr{T}_{1E} = \{E \cap U \mid U \in \mathscr{T}_1\}.$$

Let $A \in \mathscr{T}_{1E}$. Then, $A = E \cap U$ for some $U \in \mathscr{T}_1$. As $\mathscr{T}_1 \subseteq \mathscr{T}$, $A = E \cap U$, and $U \in \mathscr{T}$. Hence, $A \in \mathscr{T}_E$. the subpsace topology of $\mathscr{T}_1$ on E is still weaker than the subspace toplogy of $\mathscr{T}_1$ on E. The result for stronger relation can be proven in the same way.

(i) Let $\mathscr{T}_1$ be a topology for X, that is weaker than $\mathscr{T}$. It follows that $\mathscr{T}_1 \subseteq \mathscr{T}$. Let E be a subset of X, and $\{O_\lambda\}_{\lambda \in \Lambda}$ be an open cover of E in $(X, \mathscr{T}_1)$. As $\mathscr{T}_1 \subseteq \mathscr{T}$, the considered open cover is also an open cover in $(X, \mathscr{T})$. By compactness of $(X, \mathscr{T})$, there exists a finite sub-collection of the open cover, that covers E. Hence, $(X, \mathscr{T}_1)$ is compact. □

(ii) Let $\mathscr{T}_2$ be a topology for X, that is stronger than $\mathscr{T}$. It follows that $\mathscr{T} \subseteq \mathscr{T}_2$. If $|X| < 2$, X with any topology is trivially Hausdorff. Hence, we only consider the remaining case of $|X| \geq 2$. Let $x, y \in X$ such that $x \neq y$. As $(X, \mathscr{T})$ is Hausdorff, there exists a neighborhood of x, and a neighborhood of y, that are disjoint, which we denote as U and V respectively. As $\mathscr{T} \subseteq \mathscr{T}_2$, U and V are also open in $(X, \mathscr{T}_2)$. Hence, U is a neighborhood of x, and V is a neighborhood of y in $(X, \mathscr{T}_2)$. Moreover, U and V are disjoint. Hence, $(X, \mathscr{T}_2)$ is Hausdorff. $\square$

(iii) Let $\mathscr{T}_1$ be a topology for X, that is strictly weaker than $\mathscr{T}$. It follows that there exists a subset E of X such that it is open in $(X, \mathscr{T})$, but not open in $(X, \mathscr{T}_1)$. Furthermore, $X \setminus E$ is closed in $(X, \mathscr{T})$, but not closed in $(X \mathscr{T}_1)$. As $(X, \mathscr{T})$ is compact and $X \setminus E$ is closed in $(X, \mathscr{T})$, $(X \setminus E, \mathscr{T}_{X \setminus E})$, where $\mathscr{T}_{X \setminus E}$ denotes the standard subspace topology with respect to $X \setminus E$, is compact. By the stated preposition on the top, $(X \setminus E, \mathscr{T}_{X \setminus E}$ is weaker than $(X \setminus E, \mathscr{T}_{1X \setminus E})$. Therefore, by (i), $(X \setminus E, \mathscr{T}_{X \setminus E})$ is compact. Suppose for sake of contradiction that $(X, \mathscr{T}_1)$ is Hausdorff. It implies that $X \setminus E$ is closed in $(X, \mathscr{T})$, which is a contradiction. Hence, $(X, \mathscr{T}_1)$ is not Hausdorff. $\square$

Let $\mathscr{T}_2$ be a topology for X, that is strictly stronger than $\mathscr{T}$. It follows that there exists a subset E of X such that it is open in $(X, \mathscr{T}_2)$, but not open in $(X, \mathscr{T})$. Furthermore, $X \setminus E$ is closed in $(X, \mathscr{T}_2)$, but not closed in $(X, \mathscr{T})$. Suppose for sake of contradiction that $(X, \mathscr{T}_2)$ is compact. Then, as $X \setminus E$ is closed in $\mathscr{T}_2$, $X \setminus E$ is compact as a subspace topology of $\mathscr{T}_2$ on $X \setminus E$. As $\mathscr{T}$ is weaker than $\mathscr{T}_2$, by the stated preposition on the top, $X \setminus E$ is compact in $(X, \mathscr{T})$. As $(X \mathscr{T})$ is Hausdorff and compact now, $X \setminus E$ is closed in $(X, \mathscr{T})$, which is a contradiction. Hence, $(X, \mathscr{T}_2)$ is not compact. $\square$

Question 4. Royden 11-46.

46. (Dini's Theorem) Let $\{f_n\}$ be a sequence of continuous real-valued functions on a countably compact space X. Suppose that for each $x \in X$, the sequence $\{f_n(x)\}$ decreases monotonically to zero. Show that $\{f_n\}$ converges to zero uniformly.

Solution. Fix $\epsilon > 0$. Define X_n by

$$X_n = \{x \in X \mid f_n(x) < \epsilon\},$$

for all $n \in \mathbb{N}$. As $\{f_n(x)\}$ decreases to 0 monotonically for all $x \in X$, we have $\bigcup_{n=1}^{\infty} X_n = X$, and $\{X_n\}$ is ascending. Re-writing X_ns in terms of pre-images gives

$$X = \bigcup_{n=1}^{\infty} {f_n}^{-1}(B(0, \epsilon)).$$

As f_n is continuous for all n, each $f_n^{-1}(B(0, \epsilon))$ is open. Therefore, $\{f_n^{-1}(B(0, \epsilon))\}$ is a countable open cover of X. As X is countably compact and, there exists a finite subcover of the open cover, yielding

$$X = \bigcup_{i=1}^{K} \{{f_{n_i}}^{-1}(B(0, \epsilon)).$$

Since the pre-images form an ascending collection, we have

$$\begin{aligned} X &= f_{n_K}^{-1}(B(0, \epsilon)) \\ &= \{x \in X \mid f_{n_K}(x) < \epsilon\}. \end{aligned}$$

Again as $\{f_n(x)\}$ decreases 0 monotonically for all $x \in X$, it follows that

$$X = \{x \in X \mid f_j(x) < \epsilon \text{ for } j \geq n_K\}.$$

Since ϵ was arbitrary, $\{f_n\}$ converges to 0 uniformly. $\square$

Question 4. Royden 12-16.

16. Consider the countable collection of metric spaces $\{(X_n, \rho_n)\}_{n=1}^{\infty}$. For the Cartesian product of these sets $X = \prod_{n=1}^{\infty} X_n$, define $\rho\colon X \times X \to \mathbf{R}$ by

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{\rho_n(x_n, y_n)}{2^n[1 + \rho_n(x_n, y_n)]}.$$

[1] It is convenient here to call an open set $\mathcal{O}$ set of the form $\mathcal{O} = \Pi_{\lambda \in \Lambda} \mathcal{O}_\lambda$, where each $\mathcal{O}_\lambda$ is an open subset of X_λ and $\mathcal{O}_\lambda = X_\lambda$ except for one λ, a *subbasic set* and the finite intersection of such sets a *basic set.*

Section 12.3 The Stone-Weierstrass Theorem **247**

Use the preceding problem to show that ρ is a metric on $X = \prod_{n=1}^{\infty} X_n$ which induces the product topology on X, where each X_n has the topology induced by the metric ρ_n.

Solution. Let $\mathscr{T}_n$ be the topology on X_n, induced by the metric p_n on X_n. Note that by the problem 15, the metric $p_n{}^*$ on X_n, which is defined by $p_n{}^*(x, y) = \dfrac{p(x, y)}{1 + p(x, y)}$, also induce $\mathscr{T}_n$ as well. Let $\mathscr{T}$ be the product topology defined on $\{(X_n, T_n)\}$. Let $(X, \mathscr{S})$ be the topological space, induced by the metric space (X, p), which we have shown indeed to be a metric space in the previous problem set. We claim that $\mathscr{T} = \mathscr{S}$.

Let $U \in \mathscr{T}$. Then,

Question 6. Royden 12-20.

20. **Provide a direct proof of the assertion that if X is compact and I is a closed, bounded interval, then $X \times I$ is compact. (Hint: Let $\mathcal{U}$ be an open covering of $X \times I$, and consider the smallest value of $t \in I$ such that for each $t' < t$ the set $X \times [0, t']$ can be covered by a finite number of sets in $\mathcal{U}$. Use the compactness of X to show that $X \times [0, t]$ can also be covered by a finite number of sets in $\mathcal{U}$ and that if $t < 1$, then for some $t'' > t$, $X \times [0, t'']$ can be covered by a finite number of sets in $\mathcal{U}$.)**

Solution. The direct proof of this problem, which does not appeal to the Tychonoff theorem, will resemble the proof of Heine-Borel theorem in $\mathbb{R}$. Let X be a compact topological space, and I be a closed, bounded interval. Consider $X \times I$, which can be written as $X \times [a, b]$. Let $\mathscr{U}$ be the open cover of $X \times [a, b]$. Define a set S by

$$S = \{t \in [a, b] \mid X \times [a, t] \text{ can be covered a finite number of the sets of } \mathscr{U}\}.$$

We first show that $a \in S$, and S is nonempty. Consider $t = a$, which corresponds to $X \times \{a\}$. Since S is nonempty, and is bounded above by b, by the completness of $\mathbb{R}$, E has a supremum. Let $c = \sup E$. Since $\mathscr{U}$ is an open cover of $X \times [a, b]$, and $c \in [a, b]$ there exists O su

be the smallest value such that $X \times [a, t']$ for any $t' \in [a, t)$ can be covered by a finite number of sets in $\mathscr{U}$.