Royden

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Abstract

This work contains the solutions to Royden's Real Variables.

1 Chapter II

Question Royden 2.1-1.

Solution. Let m be a set function defined for all sets in a $\sigma-$algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume that m is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Furthermore, assume that A and B are two sets in $\mathcal{A}$ with $A \subseteq B$. Given that m is countably additive over countable disjoint collections of sets in $\mathcal{A}$, we have

$$m(B) \quad = \quad m(A) + m(B \setminus A),$$

where $B \setminus A$ is a well-defined set with $A \subseteq B$ assumption, thus A and $B \setminus A$ forming a valid countable disjoint collections of sets whose union is B. With m being a set function with values in $[0, \infty]$, we obtain $m(B) = m(A) + r$, where r denotes some non-negative real value. Therefore, we finally get

$$m(A) \leq m(B).$$

Hence, we have shown that the given set function m has the monotonicity property.

Question Royden 2.1-2.

Solution. Let m be a set function defined for all sets in a $\sigma-$algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume that m is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Furthermore, assume that there exists a set A in the collection $\mathcal{A}$ such that $m(A) < \infty$. Using the countably additive property with a collection $\{A, \emptyset\}$, we obtain

$$m(A \cup \emptyset) \quad = \quad m(A) + m(\emptyset).$$

Substituting $A \cup \emptyset = A$ and subtracting $m(A)$ from both sides, granted with finiteness of $m(A)$, we get

$$m(\emptyset) \quad = \quad 0,$$

as desired. Hence, we have shown that if there is a set A in the collection $\mathcal{A}$ for whcih $m(A) < \infty$, then $m(\emptyset) = 0$.

Question Royden 2.1-3.

Solution.

Question Royden 2.1-5 (A Countable Set Has Outer Measure Zero).

Solution. We know that any countable set has outer measure zero. Using the fact that the outer measure of an interval is its length yields $m^*([0,1]) = 1$. Therefore, $[0,1]$ cannot countable.

Question : Royden 2.1-6.

Solution. Let Q and A denote the set of rationals and irrationals in the interval $[0,1]$ respectively. Consider a countable collection of sets $\{Q, A\}$. Since outer measure is countably subadditive, we have

$$m^*(Q \cup A) \quad \leq \quad m^*(Q) + m^*(A).$$

As Q is a countable set whose outer measure is zero and $Q \cup A = [0,1]$ by construction, we obtain

$$m^*([0,1]) \quad \leq \quad m^*(A).$$

As the outer measure of an interval is its length, we have

$$1 \quad \leq \quad m^*(A).$$

Using the monotonicity property of outer measure with $I \subset [0,1]$, we also see

$$m^*(A) \quad \leq \quad 1,$$

thereby showing that $m^*(A) = 1$.

Question : Royden 2.3-Proposition 4.

Solution. We want to show that any set of outer measure zero is measurable, which further implies that any countable set is measurable. Let the set E to have outer measure zero, $m^*(E) = 0$. Let A be any set. Since

$$A \cap E \subseteq E \;\text{ and }\; A \cap E^c \subseteq A,$$

by the monotonicity of outer measure, we obtain

$$m^*(A \cap E) \leq m^*(E) = 0 \;\text{ and }\; m^*(A \cap E^c) \leq m^*(A).$$

It is important to note that it suffices to show the above statement to show that the set E is measurable, as the inequality,

$$m^* \leq m^*(A \cap E) + m^*(A \cap E^c),$$

trivially holds with the finite subadditive property of outer measure.

Definition. A collection of subsets of $\mathbb{R}$ is an algebra, provided that it contains $\mathbb{R}$ and is closed with respect to the formation of complements and finite unions. A collection of subsets of $\mathbb{R}$ is an σ-algebra, provided that it contains $\mathbb{R}$ and is closed with respect to the formation of complements and countable unions.

Preposition. Algebra is closed with respect to the formation of finite intersection.

Proof. Let $\mathcal{A}$ be an algebra and $\{A_k\}_{k=1}^n$ be a finite collection of sets in $\mathcal{A}$. We want to show that $\cap_{k=1}^n A_k$ is in $\mathcal{A}$. Since algebra is closed with respect to the formation of finite union and complements, we have

$$\cup_{k=1}^n A_k^C \;\text{ is in } \mathcal{A}.$$

By applying De Morgan's identity and closedness of complements iteratively, we see that

$$\left(\cap_{k=1}^n A_k\right)^C \;\text{ is in } \mathcal{A} \quad\text{and}\quad \cap_{k=1}^n A_k \;\text{ is in } \mathcal{A},$$

thereby showing that an algebra is closed with respect to the formation of finite intersection. $\square$

Preposition. The collection of measurable sets is an algebra. The union of a countable collection of measurable sets is also the union of a countable disjoint collection of measurable sets.

Proof. Since we know that the union of a finite collection of measurable sets is measurable and the complement of a measurable set is measurable, we immediately see that the collection of measurable sets is an algebra. Now, consider the union of a countable collection of measurable sets, denoted as $\cup_{i=1}^{\infty} E_i$. Define a new countable collection of sets $\{A_i\}_{i=1}^{\infty}$ by

$$A_i \triangleq E_i \sim \cup_{k \neq i}^{\infty} E_k,$$

where each A_i is disjoint and indeed in the measurable sets as the construction of it respects the algebra. Furthermore, the union of $\{A_i\}$ collection is precisely the union of E_i as well. Therefore, we have proven the preposition. □

Preposition. The union of a countable collection of measurable sets is measurable. Thus, the collection of measurable sets is an σ-algebra.

Proof. □

2 Chapter I

Question : Royden 1.1-1 (Distributive Property of Multiplicative Inverse in Reals).

Solution. Assume that $a \neq 0$ and $b \neq 0$. From the multiplicative identity axiom, we have that a multiplicative inverse exists for a and b individually, which we denote as a^{-1} and b^{-1} respectively. Now, consider the expression $(ab)(a^{-1}b^{-1})$, where ab denotes the product of a and b, and $a^{-1}b^{-1}$ denotes the product of a^{-1} and b^{-1}. From the commutativity of multiplication, we obtain

$$(ab)(a^{-1}b^{-1}) = (ab)(b^{-1}a^{-1}).$$

Using the associativity of multiplication and iteratively substituting $bb^{-1} = 1$ and $aa^{-1} = 1$, we have

$$(ab)(a^{-1}b^{-1}) = 1,$$

where 1 denotes the identity as usual. Hence, the product, $a^{-1}b^{-1}$ satisfies definition of multiplicative inverse with respect to the ab term whose multiplicative inverse can be denoted as $(ab)^{-1}$ by convention. Therefore, we obtain that

$$(ab)^{-1} = a^{-1}b^{-1},$$

as desired.

Question Royden 1.1-3.

Solution. Let E be a nonempty set of real numbers.
($\Leftarrow$) Assume that E consists of a single point, which we denote as x. We claim that inf$E = x$ and sup$E = x$. As we have $x \leq x$, we see that x is an upper bound for E. Suppose that there exists an upper bound for E, a, that is smaller than x, namely $a < x$. This is a contradiction to the fact that a is an upper bound as it is required to have $x \leq a$ with $x \in E$. Hence, there does not exists any upper bound for E that is smaller than x. By definition of supremum, we have that sup$E = x$. By symmtry, we can see that inf$E = x$ as well. Therefore, inf$E$ = sup$E$.

($\Rightarrow$) Assume that inf$E$ = sup$E$. Given the assumption, let us denote the infimum and supremum for E as a single real number a. Then, by definition of infimum, any x in E, we have $a \leq x$. Furthermore, by definition of supremum, any x in E, we have $x \leq a$. The only real number that can satisfy the two given equality is a itself. We also know that a must be in E as E is a nonempty set of reals. Therefore, we have shown that $E = \{a\}$, and that E consists of a single point.

Preposition 1-8. The intersection of any finite collection of open sets is open; and the union of any finite collection of closed sets is closed.

Proof. Let $\{A_k\}_{k=1}^n$ be a finite collection of open sets. We want to show that $\cap_{k=1}^n A_k$ is open. Consider $x \in \cap_{k=1}^n A_k$. Since each A_k is open, there exist r_k for $1 \leq k \leq n$ such that $B(x, r_k) \subseteq A_k$. Let $r^* = \min_{1 \leq k \leq n}(A_k)$. Then, $B(x, r^*) \in A_k$ for $1 \leq k \leq n$. Hence, $B(x, r^*) \subseteq \cap_{k=1}^n A_k$. Therefore, there exists $r > 0$ such that $B(x, r)$ for all points in $\cap_{k=1}^n A_k$ and by definition of open sets, $\cap_{k=1}^n A_k$ is open. We have shown that the intersection of any finite collection of open sets is open.

Let $\{B_k\}_{k=1}^n$ be a finite collection of closed sets. We want to show that $\cup_{k=1}^n B_k$ is closed, which is equivalent to $(\cup_{k=1}^n B_k)^C$ being open. By DeMorgan's identity, we have that

$$(\cup_{k=1}^n B_k)^C \quad = \quad \cap_{k=1}^n {B_k}^C.$$

Notice that as B_k is closed, ${B_k}^C$ is open. Then, the RHS expression gives us that the set under consideration is indeed a finite intersection of open sets, which we have shown to be open. Therefore, the union of any finite collection of closed sets is closed. $\square$