
Real Variables: Problem Set XII

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Abstract

This work contains solutions to the problem set XII of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 7-10.

10. Show that in Hölder's Inequality there is equality if and only if there are constants α and β , not both zero, for which

$$\alpha|f|^p = \beta|g|^q \text{ a.e. on } E.$$

Solution.

Question 2. Royden 7-26.

26. (The L^p Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise a.e. on E to f . For $1 \leq p < \infty$, suppose there is a function g in $L^p(E)$ such that for all n , $|f_n| \leq g$ a.e. on E . Prove that $\{f_n\} \rightarrow f$ in $L^p(E)$.

Solution. First, we denote the set at which $\{f_n\}$ converges to f pointwise as E_0 . Then, notice that for any $x \in E_0$ such that $|f_n(x)| \leq g(x)$, by the linearity of limit, and the continuity of absolute value, we obtain that $|f(x)| \leq g(x)$. Hence, we see that $|f| \leq g$ on E_0 . It follows that

$$\begin{aligned} |f - f_n| &\leq |f| + |f_n| \\ &\leq 2g, \end{aligned}$$

on E_0 . Raising both sides by p , we obtain

$$|f - f_n|^p \leq 2^p g^p,$$

on E_0 . As sum and product of measurable functions is measurable, $\{f - f_n\}$ is a sequence of measurable functions that converge to 0 pointwise, and dominated by $2^p g^p$ everywhere on E_0 . As $g \in L^p(E)$, we have that $2^p g^p$ is integrable. With the fact that $m(E \setminus E_0) = 0$, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_E |f - f_n|^p &= \lim_{n \rightarrow \infty} \int_{E_0} |f - f_n|^p \\ &= \int_{E_0} 0 \\ &= 0. \end{aligned}$$

Hence, $\{f_n\}$ is convergent to f in $L^p(E)$. □

Question 3. Royden 19-5.

5. Let (X, \mathcal{M}, μ) be a measure space and $\{f_n\}$ a Cauchy sequence in $L^\infty(X, \mu)$. Show that there is a measurable subset X_0 of X for which $\mu(X \setminus X_0) = 0$ and for each $\epsilon > 0$, there is an index N for which

$$|f_n - f_m| \leq \epsilon \text{ on } X_0 \text{ for all } n, m \geq N.$$

Use this to show that $L^\infty(X, \mu)$ is complete.

Solution. Let $\{f_k\}$ be a Cauchy sequence in $L^\infty(X, \mu)$. Hence, there exists k_n such that $\|f_i - f_j\|_\infty < \frac{1}{n}$ for $i, j \geq k_n$.

Question 4. Royden 17-19.

19. Show that any measure that is induced by an outer measure is complete.

Solution. Let $u^* : 2^X \rightarrow [0, \infty]$ be an outer-measure, and let (X, \mathcal{M}, u) be a measure space induced by u^* . Let $E \in \mathcal{M}$ and $u(E) = 0$. Let $S \subseteq E$, and $A \in \mathcal{M}$. Then, by finite monotonicity of u^* , we have $u^*(S) = 0$ and $u^*(S \cap A) = 0$. Again, using the finite monotonicity of u^* , we see that

$$\begin{aligned} u^*(A) &\geq u^*(A \cap S^c) + 0 \\ &= u^*(A \cap S^c) + u^*(A \cap S). \end{aligned}$$

Hence, $S \in \mathcal{M}$. We have shown that u is complete. □

Question 5. Royden 17-29.

29. Show that a set function on a σ -algebra is a measure if and only if it is a premeasure.

Solution. Let (X, \mathcal{M}) be a measurable space. Let $u : \mathcal{M} \rightarrow [0, \infty]$ be a measure. By definition of measure we have, $u(\emptyset) = 0$. u is finitely additive and countably monotone, as any measure has finite additivity and countable monotonicity properties. Conversely, assume that u is a pre-measure. As u is a pre-measure and $\emptyset \in \mathcal{M}$, we have $u(\emptyset) = 0$. Now, let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collections chosen from \mathcal{M} . By finite additivity, and countable monotonicity of u , it follows that

$$\begin{aligned} \sum_{k=1}^n u(E_k) &= u\left(\bigcup_{k=1}^n E_k\right) \\ &\leq u\left(\bigcup_{k=1}^{\infty} E_k\right), \end{aligned}$$

for all n . Hence, by linearity of limits, we obtain

$$\sum_{k=1}^{\infty} u(E_k) \leq u\left(\bigcup_{k=1}^{\infty} E_k\right).$$

By finite additivity of u and the fact that $u(E_k) \geq 0$ for all k , we have

$$\begin{aligned} \sum_{k=1}^{\infty} u(E_k) &\geq \sum_{k=1}^n u(E_k) \\ &= u\left(\bigcup_{k=1}^n E_k\right), \end{aligned}$$

for all n . Again, by linearity of limits, we obtain that

$$\sum_{k=1}^{\infty} u(E_k) \geq u\left(\bigcup_{k=1}^{\infty} E_k\right).$$

Therefore, we have shown that

$$\sum_{k=1}^{\infty} u(E_k) = u\left(\bigcup_{k=1}^{\infty} E_k\right),$$

which shows that u is countably additive. Hence, u is a measure. The claim is true. \square

Question 6. Royden 17-36.

36. Let μ be a finite premeasure on an algebra S , and μ^* the induced outer measure. Show that a subset E of X is μ^* -measurable if and only if for each $\epsilon > 0$ there is a set $A \in S_\delta$, $A \subseteq E$, such that $\mu^*(E \setminus A) < \epsilon$.

Solution.