Real Variables:
Problem Set I

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Abstract

This work contains the solutions to the first problem set of Real Variables 2015.

1 Solutions

Question 1. Royden 2.4. Counting Measure.

Solution. We wish to show that the counting measure, $c : \mathcal{P}(\mathbb{R}) \to [0, \infty]$, where $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$, is countably additive and translation invariant.

First, we prove that it is countably additive. Let $\{E_k\}_{k=1}^{\infty}$ be a countable, disjoint collection of subsets of $\mathbb{R}$. If one of the set in the collection has infinite cardinality, then we have

$$\sum_{k=1}^{\infty} c(E_k) \quad = \quad \infty,$$

as $c(E_k) = \infty$ for some k. Notice that the union of the collection $\cup_{k=1}^{\infty} E_k$, also has infinite cardinality, as it has a subset with an infinite cardinality. Hence, by the definition of counting measure, we have $c(\cup_{k=1}^{\infty} E_k) = \infty$. Therefore, we have

$$c(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} c(E_k),$$

for the case under consideration. Now, assume that $c(E_k) < \infty$ for all k.

Question 2. Royden 2.8.

Solution. Let B be a set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^{n}$ be a finite collection of open intervals that cover B.