Real Variables: Problem Set V

Youngduck Choi
Courant Institute of Mathematical Sciences
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set V of Real Variables 2015 at NYU.

1 Solutions

Question 6.33.

Solution. Let $\{f_n\}$ be a sequence of real-valued functions on $[a,b]$ that converges pointwise on $[a,b]$ to the real-valued function f. We wish to show that $TV(f) \leq \liminf TV(f_n)$. Fix $\epsilon > 0$. Let $P = \{x_0, ..., x_m\}$ be a partition of $[a,b]$. By the triangle inequality, it follows that

$$\begin{aligned}
V(f,P) &= \sum_{k=0}^{m-1} |f(x_{k+1}) - f(x_k)| \\
&= \sum_{k=0}^{m-1} |f(x_{k+1}) + f_n(x_{k+1}) - f_n(x_{k+1}) - f(x_k) + f_n(x_k) - f_n(x_k)| \\
&\leq \sum_{k=0}^{m-1} |f(x_{k+1}) - f_n(x_{k+1})| + |f_n(x_{k+1}) - f_n(x_k)| + |f(x_k) - f_n(x_k)| \\
&\leq V(f_n) + \sum_{k=1}^{m} |f(x_k) - f_n(x_k)| + \sum_{k=0}^{m-1} |f(x_k) - f_n(x_k)|,
\end{aligned}$$

for any n. Define $N = \max(N_0, ..., N_k)$, where $N_i (0 \leq i \leq k)$ corresponds to the convergence index for $\frac{\epsilon}{2m}$ at x_i. Then, it follows that

$$V(f,p) - \epsilon \quad \leq \quad V(f_n, p)$$

for $n \geq N$. As p was arbitrary, we can take the supremum over p on both sides, and obtain

$$TV(f,p) - \epsilon \quad \leq \quad TV(f_n, p),$$

for $n \geq N$. As ϵ was arbitrary, we obtain that

$$TV(f,p) \quad \leq \quad \inf_{n \geq N} TV(f_n, p).$$

Now as $N \to \infty$, by the linearity of limit,

$$TV(f,p) \quad \leq \quad \liminf_{N \to \infty} TV(f_n, p),$$

as desired. $\square$

Question 4. Royden 6.42.

Solution. Let f and g be absolutely continuous functions on $[a, b]$. We wish to show that $f + g$ is absolutely continuous. Fix $\epsilon > 0$. Let $\{(a_k, b_k)\}_{k=1}^n$ be an arbitrary finite disjoint open intervals in (a, b). As f and g are both absolutely continuous, there exist $\delta_f, \delta_g > 0$, such that

$$\sum_{k=1}^{n}[b_k - a_k] < \delta_f \implies \sum_{k=1}^{n}|f(b_k) - f(a_k)| < \frac{\epsilon}{2}$$
$$\sum_{k=1}^{n}[b_k - a_k] < \delta_g \implies \sum_{k=1}^{n}|g_(b_k) - g(a_k)| < \frac{\epsilon}{2}.$$

Define $\delta = \min(\delta_f, \delta_g)$, and suppose that $\sum_{k=1}^{n}[b_k - a_k] < \delta$. It follows that

$$\begin{aligned}\sum_{k=1}^{n}|f + g(b_k) - f + g(a_k)| &\leq \sum_{k=1}^{n}|f(b_k) - f(a_k)| + |g(b_k) - g(a_k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Since ϵ and $\{(a_k, b_k)\}_{k=1}^n$ were arbitrary, we have shown that $f + g$ is absolutely continuous.

Let f be an absolutely continuous function on $[a, b]$. We show that cf, for any $c \in \mathbb{R}$, is absolutely continuous. Let $c = 0$. Then $cf = 0$, which can trivially be shown to be absolutely continuous, as $f(c) = 0$ for any $c \in [a, b]$ Suppose $c \neq 0$. Fix $\epsilon > 0$. Let $\{(a_k, b_k)\}_{k=1}^n$ be an arbitrary finite disjoint open intervals in (a, b). As f is absolutely continuous, there exists $\delta_f > 0$, such that

$$\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \frac{\epsilon}{|c|}.$$

Define $\delta = \delta_f$, and suppose that $\sum_{k=1}^{n}[b_k - a_k] < \delta$ It follows that

$$\begin{aligned}\sum_{k=1}^{n}|cf(b_k) - cf(a_k)| &= |c|\sum_{k=1}^{n}|f(b_k) - f(a_k)| \\ &\leq |c|\frac{\epsilon}{|c|} = \epsilon.\end{aligned}$$

Since ϵ and $\{(a_k, b_k)\}_{k=1}^n$ were arbitrary, combined with $c = 0$ case, we have shown that cf is absolutely continuous.

Let f be an absolutely continuous function on $[a, b]$. We first show that f^2 is absolutely continuous. As f is absolutely continuous, f is continuous on $[a, b]$. Hence, by the Extreme Value Theorem, there exists M such that $|f| \leq M$ on $[a, b]$. Fix $\epsilon > 0$. Let $\{(a_k, b_k)\}_{k=1}^n$ be an arbitrary finite disjoint open intervals in (a, b). As f is absolutely continuous, there exists $\delta_f > 0$, such that

$$\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \frac{\epsilon}{2M}.$$

Define $\delta = \delta_f$, and suppose that $\sum_{k=1}^{n}[b_k - a_k] < \delta$ It follows that

$$\begin{aligned}\sum_{k=1}^{n}|f^2(b_k) - f^2(a_k)| &= \sum_{k=1}^{n}|f(b_k) - f(a_k)||f(b_k) + f(a_k)| \\ &\leq 2M\sum_{k=1}^{n}|f(b_k) - f(a_k)| \\ &\leq 2M\frac{\epsilon}{2M} = \epsilon.\end{aligned}$$

Since ϵ and $\{(a_k, b_k)\}_{k=1}^n$ were arbitrary, we have shown that f^2 is absolutely continuous.

Let f and g be an absolutely continuous function on $[a, b]$. We wish to show that fg is absolutely continuous. Observe that

$$(f+g)^2 \quad = \quad f^2 + g^2 - 2fg,$$

which simplifies to

$$fg = -\frac{1}{2}((f+g)^2 + (-f^2) + (-g^2)).$$

As we have previously shown that a square of an absolutely continuous function is absolutely continuous, and a scalar multiple of an absolutely continuous function is absolutely continuous, we can deduce that fg is absolutely continuous. This completes the proof. $\square$

Question 4. 6.45.

Solution. Let f be an absolutely continuous on $\mathbb{R}$, and g be an absolutely continuous function, which is strictly monotone on $[a, b]$. We wish to show that $f \circ g$ is absolutely continuous. Fix $\epsilon > 0$. Let $\{(a_k, b_k)\}_{k=1}^n$ be an arbitrary disjoint open intervals in (a, b). As g is absolutely continuous, there exists δ_g such that

$$\sum_{k=1}^{n} [b_k - a_k] < \delta_g \implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$