Real Variables:
Problem Set IV

Youngduck Choi
Courant Institute of Mathematical Sciences
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the problem set IV of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 4.31.

Solution. Let f and g be absolutely continuous functions on $[a, b]$. We wish to show that $f + g$ is absolutely continuous. Fix $\epsilon > 0$. Let $\{(a_k, b_k)\}_{k=1}^{n}$ be an arbitrary finite disjoint open intervals in (a, b). As f and g are both absolutely continuous, there exist $\delta_f, \delta_g > 0$, such that

$$\sum_{k=1}^{n}[b_k - a_k] < \delta_f \implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \frac{\epsilon}{2}$$
$$\sum_{k=1}^{n}[b_k - a_k] < \delta_g \implies \sum_{k=1}^{n} |g_(b_k) - g(a_k)| < \frac{\epsilon}{2}.$$

Define $\delta = \min(\delta_f, \delta_g)$, and suppose that $\sum_{k=1}^{n}[b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^{n} |f + g(b_k) - f + g(a_k)| &\leq \sum_{k=1}^{n} |f(b_k) - f(a_k)| + |g(b_k) - g(a_k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ and $\{(a_k, b_k)\}$ were arbitrary, we have shown that $f + g$ is absolutely continuous.

Let f and g be absolutely continuous functions on $[a, b]$.