# Real Variables:
# Problem Set IX

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## Abstract

This work contains solutions to the problem set IX of Real Variables 2015 at NYU.

## 1 Solutions

**Question 1. Royden 12-5.**

5. Suppose that a topological space $X$ has the property that every continuous, bounded real-valued function on a closed subset has a continuous extension to all of $X$. Show that if $X$ is Tychonoff, then it is normal.

**Solution.** Consider

$$\mathscr{S} = \{f^{-1}(O) \mid f \text{ is continuous, and } O \text{ is open in} \mathbb{R}\}.$$

**Question 2. Royden 12-6.**

6. Let $(X, \mathcal{T})$ be a normal topological space and $\mathcal{F}$ the collection of continuous real-valued functions on $X$. Show that $\mathcal{T}$ is the weak topology induced by $\mathcal{F}$.

**Solution.** Consider

$$\mathscr{S} = \{f^{-1}(O) \mid f \text{ is continuous, and } O \text{ is open in} \mathbb{R}\}.$$

**Question 3. Royden 12-27.**

27. For $f, g \in C[a, b]$, show that $f = g$ if and only if $\int_a^b x^n f(x)\,dx = \int_a^b x^n g(x)\,dx$ for all $n$.

**Solution.** Consider

$$\mathscr{S} = \{f^{-1}(O) \mid f \text{ is continuous, and } O \text{ is open in} \mathbb{R}\}.$$

**Question 4. Royden 12-35.**

35. Let $\mathcal{A}$ be an algebra of continuous real-valued functions on a compact Hausdorff space $X$ that separates points. Show that either $\overline{\mathcal{A}} = C(X)$ or there is a point $x_0 \in X$ for which $\overline{\mathcal{A}} = \{f \in C(X) \mid f(x_0) = 0\}$. (Hint: If $1 \in \overline{\mathcal{A}}$, we are done. Moreover, if for each $x \in X$ there is an $f \in \mathcal{A}$ with $f(x) \neq 0$, then there is a $g \in \mathcal{A}$ that is positive on $X$ and this implies that $1 \in \overline{\mathcal{A}}$.)

**Solution.** Consider

$$\mathscr{S} \quad = \quad \{f^{-1}(O) \mid f \text{ is continuous, and } O \text{ is open in} \mathbb{R}\}.$$

**Question 5. Royden 13-8.**

8. A nonnegative real-valued function $\|\cdot\|$ defined on a vector space $X$ is called a **pseudonorm** if $\|x+y\| \leq \|x\| + \|y\|$ and $\|\alpha x\| = |\alpha|\,\|x\|$. Define $x \cong y$, provided $\|x - y\| = 0$. Show that this is an equivalence relation. Define $X/_{\cong}$ to be the set of equivalence classes of $X$ under $\cong$ and for $x \in X$ define $[x]$ to be the equivalence class of $x$. Show that $X/_{\cong}$ is a normed vector space if we define $\alpha[x] + \beta[y]$ to be the equivalence class of $\alpha x + \beta y$ and define $\|[x]\| = \|x\|$. Illustrate this procedure with $X = L^p[a, b]$, $1 \leq p < \infty$.

**Solution.** We show that the relation is reflexive, symmetric, and transitive.

Let $x \in X$. It follows that

$$\|x - x\| = \|\theta\|,$$

where $\theta$ is the identity element of the linear space $X$. By definition of linear space, we have $\alpha \cdot \theta = \theta$ for all $\alpha$. Hence, for some $\alpha > 1$, we have

$$\begin{aligned} \|\theta\| &= \|\alpha \cdot \theta\| \\ &= |\alpha| \|\theta\|. \end{aligned}$$

As $|a| > 0$, we have $|\theta| = 0$. Consequently, $\|x - x\| = 0$. It follows that for all $x \in X$, $x \equiv x$. The relation is reflexive.

Let $x, y \in X$ and $x \equiv y$. Observe that

$$\begin{aligned} \|x - y\| &= \|-1 \cdot (y - x)\| \\ &= |-1| \|y - x\| \\ &= \|y - x\|. \end{aligned}$$

As $x \equiv y$, which gives $\|x - y\| = 0$, it follows that $\|y - x\| = 0$ and $y \equiv x$. Hence, the relation is symmetric.

Let $x, y, z \in X$ and $x \equiv y$ and $y \equiv z$. By triangle inequality, it follows that

$$\begin{aligned} \|y - z\| &= \|(x - y) + (y - z)\| \\ &\leq \|x - y\| + \|y - z\| = 0 + 0 = 0. \end{aligned}$$

Hence, $\|y - z\| = 0$, and it follows that $x \equiv z$. Hence, the relation is symmetric.

We show that $X_{\equiv}$ is a normed vector space. Firstly, we check that the defined norm is well defined. Let $x, y \in X$, such that $x \equiv y$. It follows that $\|x - y\| = 0$. Hence, $\|x\| = \|y\|$, and it follows that $\|[x]\| = \|[y]\|$. The norm is well-defined.

**Question 6. Royden 13-33.**

**Solution.** Consider

$$\mathscr{S} \quad = \quad \{f^{-1}(O) \mid f \text{ is continuous, and } O \text{ is open in} \mathbb{R}\}.$$

33. Let $X$ be a linear subspace of $C[0, 1]$ that is closed as a subset of $L^2[0, 1]$.Verify the following assertions to show that $X$ has finite dimension. The sequence $\{f_n\}$ belongs to $X$.

(i) Show that $X$ is a closed subspace of $C[0, 1]$.

(ii) Show that there is a constant $M \geq 0$ such that for all $f \in X$ we have $\|f\|_2 \leq \|f\|_\infty$ and $\|f\|_\infty \leq M \cdot \|f\|_2$.

(iii) Show that for each $y \in [0, 1]$, there is a function $k_y$ in $L^2$ such that for each $f \in X$ we have $f(y) = \int_0^1 k_y(x) f(x)\, dx$.

(iv) Show that if $\{f_n\} \to f$ weakly in $L^2$, then $\{f_n\} \to f$ pointwise on $[0,1]$.

(v) Show $\{f_n\} \to f$ weakly in $L^2$, then $\{f_n\}$ is bounded (in what sense?), and hence $\{f_n\} \to f$ strongly in $L^2$ by the Lebesgue Dominated Convergence Theorem.

(vi) Conclude that $X$, when normed by $\|\cdot\|_2$, has a compact closed unit ball and therefore, by Riesz's Theorem, is finite dimensional.