
Real Variables: Problem Set X

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Abstract

This work contains solutions to the problem set X of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 13-41.

41. Let X be the linear space of all polynomials defined on \mathbf{R} . For $p \in X$, define $\|p\|$ to be the sum of the absolute values of the coefficients of p . Show that this is a norm on X . For each n , define $\psi_n : X \rightarrow \mathbf{R}$ by $\psi_n(p) = p^{(n)}(0)$. Use the properties of the sequence $\{\psi_n\}$ in $\mathcal{L}(X, \mathbf{R})$ to show that X is not a Banach space.

Solution. We first show that $\|\cdot\| : X \rightarrow \mathbb{R}$ given is a norm on X . First of all, let $p = 0$. Then, $\|p\| = 0$. Now, let $p = \sum_{i=0}^n b_i x^i$, and assume that $\|p\| = 0$. It follows that $\sum_{i=0}^n |b_i| = 0$. As $|b_i| \geq 0$ for all i , we have that $b_i = 0$ for all i . Hence, $p = 0$. For proving the triangle inequality, let $p_1 = \sum_{i=0}^{n_1} b_i x^i$ and $p_2 = \sum_{i=0}^{n_2} c_i x^i$. Without the loss of generality, we assume that $n_1 \geq n_2$, and define $n = n_1$, $p_1 = \sum_{i=0}^n b_i x^i$ and $p_2 = \sum_{i=0}^n c_i x^i$, with $c_i = 0$ for $i > n_2$. By the triangle inequality of reals, it follows that

$$\begin{aligned}\|p_1 + p_2\| &= \left\| \sum_{i=0}^n (b_i + c_i) x^i \right\| \\ &= \sum_{i=0}^n |b_i + c_i| \\ &\leq \sum_{i=0}^n |b_i| + |c_i| \\ &= \sum_{i=0}^n |b_i| + \sum_{i=0}^n |c_i| = \|p_1\| + \|p_2\|.\end{aligned}$$

Now, let $p = \sum_{i=0}^n b_i x^i$, and $\alpha \in \mathbb{R}$. It follows that

$$\begin{aligned}\|\alpha p\| &= \left\| \alpha \sum_{i=0}^n b_i x^i \right\| \\ &= \left\| \sum_{i=0}^n \alpha b_i x^i \right\| \\ &= \sum_{i=0}^n |\alpha b_i| \\ &= |\alpha| \sum_{i=0}^n |b_i| = |\alpha| \|p\|.\end{aligned}$$

Hence, we have shown that $\|\cdot\|$ given is a norm.

Question 2. Royden 14-18.

18. Let X be a normed linear space, ψ belong to X^* , and $\{\psi_n\}$ be in X^* . Show that if $\{\psi_n\}$ converges weak-* to ψ , then

$$\|\psi\| \leq \limsup \|\psi_n\|.$$

Solution. As $\{\psi_n\}$ is weak-* convergent to ψ , we have

$$\psi_n(x) \rightarrow \psi(x),$$

for all $x \in X$. Let $x \in X$. As $|\cdot|$ is continuous on \mathbb{R} , it follows that

$$\lim_{n \rightarrow \infty} |\psi_n(x)| = |\psi(x)|.$$

As $|\psi_n(x)| \leq \|\psi_n\| \cdot \|x\|$,

$$\begin{aligned} |\psi(x)| &= \lim_{n \rightarrow \infty} |\psi_n(x)| \\ &= \limsup_{n \rightarrow \infty} |\psi_n(x)| \\ &\leq \limsup_{n \rightarrow \infty} \|\psi_n\| \cdot \|x\| \\ &= \|x\| \limsup_{n \rightarrow \infty} \|\psi_n\|. \end{aligned}$$

Since $x \in X$ was arbitrary, it follows that

$$\|\psi\| \leq \limsup_{n \rightarrow \infty} \|\psi_n\|,$$

as desired. □

Question 3. Royden 14-23.

23. Let Y be a linear subspace of a normed linear space X and z be a vector in X . Show that

$$\text{dist}(z, Y) = \sup \{ \psi(z) \mid \|\psi\| = 1, \psi = 0 \text{ on } Y \}.$$

Solution. Consider a functional $p : X \rightarrow [0, \infty)$ be defined by

$$p = \begin{cases} 0, & \text{if } x \in Y \\ \|x\|, & \text{otherwise.} \end{cases}$$

We first show that p is positively homogeneous. Let $\lambda > 0$. Let $x \in Y$, then as Y is a linear subspace of X , $\lambda x \in Y$. It follows that

$$\begin{aligned} p(\lambda x) &= 0 \\ &= \lambda p(x). \end{aligned}$$

Let $x \notin Y$. It follows that $\lambda x \notin Y$, as otherwise we get a contradiction that $x \in Y$ from the linear subspace property of Y . It follows that

$$\begin{aligned} p(\lambda x) &= \|\lambda x\| \\ &= |\lambda| \|x\| \\ &= \lambda \|x\| \\ &= \lambda p(x). \end{aligned}$$

Hence, we have shown that p is positively homogeneous. Now, we show that p is sub-additive. Let $x, y \in Y$. Then, it follows that

$$\begin{aligned} p(x + y) &= 0 \\ &= p(x) + p(y). \end{aligned}$$

Hence, $p(x + y) \leq p(x) + p(y)$ holds. Let $x, y \notin Y$. Then, by the triangle inequality of norm, it follows that

$$\begin{aligned} p(x + y) &= \|x + y\| \\ &\leq \|x\| + \|y\| \\ &= p(x) + p(y). \end{aligned}$$

Let $x \in Y$ and $y \notin Y$. It follows that

$$\begin{aligned} p(x + y) &= \|x + y\| \\ &\leq \|x\| + \|y\| \\ &= 0 \end{aligned}$$

Question 4. Royden 15-12.

12. If Y is a linear subspace of a Banach space X , we define the *annihilator* Y^\perp to be the subspace of X^* consisting of those $\psi \in X^*$ for which $\psi = 0$ on Y . If Y is a subspace of X^* , we define Y^0 to be the subspace of vectors in X for which $\psi(x) = 0$ for all $\psi \in Y$.
- (i) Show that Y^\perp is a closed linear subspace of X^* .
 - (ii) Show that $(Y^\perp)^0 = \overline{Y}$.

Solution.