Real Variables:
Problem Set I

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Abstract

This work contains the solutions to the first problem set of Real Variables 2015.

1 Solutions

Question 1. Royden 2.4. Counting Measure.

Solution. We wish to show that the counting measure, $c : \mathcal{P}(\mathbb{R}) \to [0, \infty]$, where $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$, is countably additive and translation invariant.

We first prove that it is countably additive. Let $\{E_k\}_{k=1}^{\infty}$ be a countable, disjoint collection of subsets of $\mathbb{R}$. If one of the set in the collection has infinite cardinality, then we have

$$\sum_{k=1}^{\infty} c(E_k) \quad = \quad \infty,$$

as $c(E_k) = \infty$ for some k. Notice that the union of the collection $\cup_{k=1}^{\infty} E_k$, also has infinite cardinality, as it has a subset with an infinite cardinality. Hence, by the definition of counting measure, we have $c(\cup_{k=1}^{\infty} E_k) = \infty$. Therefore, we have

$$c(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} c(E_k),$$

for the case under consideration. Now, assume that $c(E_k) < \infty$ for all k. There are two sub-cases now. First, assume that the series $\sum_{k=1}^{\infty} c(E_k)$ converges. In particular, we have that $\lim_{k \to \infty} c(E_k) = 0$. For some $\epsilon < 1$, we have an N such that $c(E_k) < \epsilon$ for all $k \geq N$. As the counting measure only takes an integer value or ∞, we obtain that $c(E_k) = 0$ and $E_k = \emptyset$ for all $k \geq N$. Furthermore, we get that

$$\begin{aligned} c(\cup_{k=1}^{\infty} E_k) &= c(\cup_{k=1}^{N} E_k), \\ \sum_{k=1}^{\infty} c(E_k) &= \sum_{k=1}^{N} c(E_k). \end{aligned}$$

As the finite additivity of counting measure trivially holds (can be shown with a simple inductive argument), $c(\cup_{k=1}^{N} E_k) = \sum_{k=1}^{N} c(E_k)$ holds, and thus we conclude that

$$c(\cup_{k=1}^{\infty} E_k) \quad = \quad \sum_{k=1}^{\infty} c(E_k).$$

Now, for the last case, assume that $\sum_{k=1}^{\infty} c(E_k) = \infty$. This gives that for any N, there exists $N' > N$, such that $c(E_{N'}) > 0$. Hence, there exists a subsequence of $\{E_k\}_{k\in\lambda}$, which only consists of disjoint nonempty subsets. Therefore, $\cup_{k=1}^{\infty} E_k$ contains a countably infinite sub-collection of nonempty disjoint sets. Hence, the set $\cup_{k=1}^{\infty} E_k$ must contain infinitely many members. Therefore, we have $c(\cup_{k=1}^{\infty} E_k) = \infty$, thus,

$$c(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} c(E_k).$$

As we covered all possible cases, this completes the proof for the countable additivity of counting measure. $\square$

We now prove the translation invariant property of counting measure. Consider a set E and its translation defined by $E + y = \{x + y | x \in E\}$. Notice that a map, $\phi : \mathbb{R} \to \mathbb{R}$ where $\phi(x) = x + y$, establishes a bijection between E and $E + y$. Hence, the cardinality of E and $E + y$ are equal. As the equal cardinality is a sufficient condition(in fact, strictly more) for the equal counting measure, we have shown that the counting measure is translation invariant. $\square$

Question 2. Royden 2.8.

Solution. Before we proceeding to the main part of the proof, we first prove the following lemma.

Lemma. Let $\{X_k\}_{k=1}^{n}$ be a finite collection of sets of real numbers. Then, we have that

$$\overline{\cup_{k=1}^{n} X_k} = \cup_{k=1}^{n} \overline{X_k},$$

where $\overline{X_k}$ denotes a closure of the set X_k.

Proof. Let X_1 and X_2 be sets of real numbers. Then, we have $X_1 \subseteq X_1 \cup X_2$ and $X_2 \subseteq X_1 \cup X_2$. Consequently, we have $\overline{X_1} \subseteq \overline{X_1 \cup X_2}$ and $\overline{X_2} \subseteq \overline{X_1 \cup X_2}$. Hence, we have $\overline{X_1} \cup \overline{X_2} = \overline{X_1 \cup X_2}$. This can be trivially extended to any finite n with an induction using set union, i.e. take $\cup_{k=1}^{n-1} X_k$ with X_n. Therefore, $\cup_{k=1}^{n} \overline{X_k} \subseteq \overline{\cup_{k=1}^{n} X_k}$.

Now, notice that $\overline{X_1} \cup \overline{X_2}$ is closed, as it is a finite union of closed sets. Since $X_1 \subseteq \overline{X_1}$ and $X_2 \subseteq \overline{X_2}$, we have $X_1 \cup X_2 \subseteq \overline{X_1} \cup \overline{X_2}$. As $\overline{X_1} \cup \overline{X_2}$ is closed, we obtain $\overline{X_1 \cup X_2} \subseteq \overline{X_1} \cup \overline{X_2}$. Again, this argument can be extended inductively for a finite n, by taking $\cup_{k=1}^{n-1} X_k$ and X_n. Therefore, $\overline{\cup_{k=1}^{n} X_k} \subseteq \cup_{k=1}^{n} \overline{X_k}$. This concludes that $\cup_{k=1}^{n} \overline{X_k} = \overline{\cup_{k=1}^{n} X_k}$ for n finite. One should note that this argument fails with n infinite, as we no longer can leverage the fact that finite union of closed sets is closed. $\square$

We now proceed to the main part of the proof. Let B be a set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^{n}$ be a finite collection of open intervals that cover B. As $B \subseteq \cup_{k=1}^{n} I_k$, we have $\overline{B} \subseteq \overline{\cup_{k=1}^{n} I_k}$. Furthermore, with n being finite, we obtain that $\overline{\cup_{k=1}^{n} I_k} = \cup_{k=1}^{n} \overline{I_k}$. Then, it follows from the monotonicity, and finite sub-additivity property that

$$m^*(\overline{B}) \leq m^*(\cup_{k=1}^{n} \overline{I_k}) \leq \sum_{k=1}^{n} m^*(\overline{I_k}). \tag{1}$$

In particular, we have $m^*(\overline{B}) = 1$, as $B = [0, 1]$, and $\sum_{i=1}^{n} m^*(\overline{I_k}) = \sum_{i=1}^{n} m^*(I_k)$, as the outer measure of an open interval and corresponding closed interval are equal. Substituting the two equalities into the (1) inequality, we obtain

$$\sum_{i=1}^{n} m^*(I_k) \geq 1,$$

as desired. $\square$

Question 3. Royden 2.14.

Solution. Let $m^*(E) > 0$. We wish to find a subset X of E such that $m^*(X) > 0$. Consider the countable collection of sets $\{(-M, M)\}_{M=1}^{\infty}$. Notice that, as $(-M, M)$ is bounded for some fixed M, $E \cap (-M, M)$ is a bounded subset of E. Furthermore, $E = \cup_{M=1}^{\infty} E \cap (-M, M)$. Then, by the countable sub-additivity of outer measure, we have

$$\sum_{M=1}^{\infty} m^*(E \cap (-M, M)) \geq m^*(E).$$

If $m^*(E \cap (-M, M)) = 0$ for all M, then we have the sum on the LHS equals 0, and obtain $0 > 0$, as $m^*(E) > 0$. This is a contradiction. Hence, there exists a M such that $m^*(E \cap (-M, M)) > 0$, and $E \cap (-M, M)$ is precisely the bounded subset of E with positive outer measure. We have shown that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure. $\square$

Question 4. Royden 2.15.

Solution. Let $m(E) < \infty$ and $\epsilon > 0$. We wish to show that E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ. First, assume that E is bounded. Then, there exists an interval $[-M, M]$ such that $E \subseteq [-M, M]$. By the Archimedean principle, there exists $N \in \mathbb{N}$ such that $\dfrac{2M}{N} < \epsilon$. Now, consider the following finite disjoint collection of sets:

$$\{[-M + \frac{2M}{N}(n-1), -M + \frac{2M}{N}n) \cap E\}_{n=1}^{N+1}.$$

Notice that $E = \cup_{n=1}^{N+1}[-M + \dfrac{2M}{N}(n-1), -M + \dfrac{2M}{N}n) \cap E$. Furthermore, as every interval is measurable and intersection of two measurable sets is measurable, each set in the collection is measurable. By the monotonicity property of measure, we have

$$\begin{aligned} m([-M + \frac{2M}{N}(n-1), -M + \frac{2M}{N}n) \cap E) &\leq m([-M + \frac{2M}{N}(n-1), -M + \frac{2M}{N}n)) \\ &= \frac{2M}{N} \\ &< \epsilon. \end{aligned}$$

Hence, we have found a finite disjoint collection of measurable sets, each of which has measure at most ϵ, whose union equals E. We have proven the statement for the E bounded case.

Now assume that E is unbounded. Notice that $E = \cup_{M=1}^{\infty} E \cap [-M, M]$. As $\{E \cap [-M, M]\}_{M=1}^{\infty}$ forms an ascending collection of measurable sets, by the continuity of measure, we have

$$m(E) = \lim_{M \to \infty} m(E \cap [-M, M]).$$

As $m(E) < \infty$, there exists an N such that

$$m(E) - \epsilon < m(E \cap [-N, N]) \leq m(E). \tag{2}$$

Note that, by the finite additivity of measure, we have

$$m(E \cap [-N, N]) + m(E \setminus [-N, N] = m(E). \tag{3}$$

Combining (2) and (3), we obtain that $m(E \setminus [-N, N]) \leq \epsilon$. Now, as $E \cap [-N, N]$ is bounded, there exists a finite collection of measurable sets $\{E_i\}_{i=1}^{K}$ such that $m(E_i) \leq \epsilon$ for $1 \leq i \leq k$, and $\cup_{i=1}^{K} E_i = E \cap [-N, N]$. Notice that $E = \cup_{i=1}^{K} E_i \cup (E \setminus [-N, N])$. In particular, the collection $\{\{E_i\}_{i=1}^{K}, E \setminus [-N, N]\}$ is a finite disjoint collection of measurable sets, whose union equals E, each of which has measure at most ϵ. This proves the statement for the E unbounded case, and we have completed the proof. $\square$

Question 5. Royden 2.17.

Solution. From the inner approximation by closed sets, and outer approximation by open sets, we have that E is measurable if and only if, for any $\epsilon > 0$, there exists a closed set F and an open set O, such that

$$E \subseteq O \text{ with } m^*(O \setminus E) < \frac{\epsilon}{2} \text{ and } F \subseteq E \text{ with } m^*(E \setminus F) < \frac{\epsilon}{2}.$$

Applying the sub-additivity property of outer measure with $O \setminus E$ and $E \setminus F$, we have

$$m^*(O \setminus F) < m^*(O \setminus E) + m^*(E \setminus F) < \epsilon.$$

Hence, E is measurable, if and only if, there exists an open set O and a closed set F for which $F \subseteq E \subseteq O$ and $m^*(O \setminus F) < \epsilon$. $\square$

Question 6. Royden 2.28.

Solution. Let $\{E_k\}_{k=1}^{\infty}$ be a countable, disjoint collection of measurable sets. By the finite additivity property, we have

$$m(\cup_{k=1}^{N} E_k) = \sum_{k=1}^{N} m(E_k),$$

for all N. Notice that $\{\cup_{k=1}^{N} E_k\}_{N=1}^{\infty}$ forms an ascending collection of measurable sets. Hence, by applying the continuity of measure to the ascending collection, $\{\cup_{k=1}^{N} E_k\}_{N=1}^{\infty}$, we have

$$m(\cup_{N=1}^{\infty} \cup_{k=1}^{N} E_k) \quad = \quad \lim_{N \to \infty} m(\cup_{k=1}^{N} E_k).$$

Simplifying the LHS and applying the finite additivity property to the RHS, we obtain

$$m(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$$

Since $\{E_k\}_{k=1}^{\infty}$ was chosen to be an arbitrary countable, disjoint collection of measurable sets, we have shown that finite additivty and continuity of measure implies countable additivity. $\square$