Real Variables:
Problem Set X

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Abstract

This work contains solutions to the problem set X of Real Variables 2015 at NYU.

1 Solutions

Question Royden 17-6.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and X_0 belong to $\mathcal{M}$. Define $\mathcal{M}_0$ to be the collection of sets in $\mathcal{M}$ that are subsets of X_0 and μ_0 the restriction of μ to $\mathcal{M}_0$. Show that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

Solution. We first show that $\| \, \| : X \to \mathbb{R}$ given is a norm on X. First of all,

Question Royden 17-15.

15. Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$, where α and β are real numbers. Show that

$$|\alpha\nu| = |\alpha||\nu| \text{ and } |\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|,$$

where $\nu \leq \mu$ means $\nu(E) \leq \mu(E)$ for all measurable sets E.

Solution.

Question Royden 17-17.

17. Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu+\nu-|\mu-\nu|)$ and $\mu \vee \nu = \mu+\nu-\mu \wedge \nu$.

    (i) Show that the signed measure $\mu \wedge \nu$ is smaller than μ and ν but larger than any other signed measure that is smaller than μ and ν.

    (ii) Show that the signed measure $\mu \vee \nu$ is larger than μ and ν but smaller than any other measure that is larger than μ and ν.

    (iii) If μ and ν are positive measures, show that they are mutually singular if and only if $\mu \wedge \nu = 0$.

Solution.

Question Royden 18-50.

50. Establish the uniqueness of the function f in the Radon-Nikodym Theorem.

Solution.

Question Royden 18-54.

54. Let μ, ν, and λ be σ-finite measures on the measurable space $(X, \mathcal{M})$.

(i) If $\nu \ll \mu$ and f is a nonnegative function on X that is measurable with respect to $\mathcal{M}$, show that

$$\int_X f\, d\nu = \int_X f \left[\frac{d\nu}{d\mu}\right] d\mu.$$

(ii) If $\nu \ll \mu$ and $\lambda \ll \mu$, show that

$$\frac{d(\nu + \lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \text{ a.e. } [\mu].$$

(iii) If $\nu \ll \mu \ll \lambda$, show that

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} \text{ a.e. } [\lambda].$$

Solution.

Question Royden 18-55.

55. Let μ, ν, ν_1, and ν_2 be measures on the measurable space $(X, \mathcal{M})$.

(i) Show that if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

(ii) Show that if ν_1 and ν_2 are singular with respect to μ, then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.

(iii) Show that if ν_1 and ν_2 are absolutely continuous with respect to μ, then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.

(iv) Prove the uniqueness assertion in the Lebesgue decomposition.

Solution.