Real Variables:
Problem Set XII

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Abstract

This work contains solutions to the problem set XII of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 7-10.

10. Show that in Hölder's Inequality there is equality if and only if there are constants α and β, not both zero, for which

$$\alpha|f|^p = \beta|g|^q \text{ a.e. on } E.$$

Solution.

Question 2. Royden 7-26.

26. **(The L^p Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise a.e. on E to f. For $1 \leq p < \infty$, suppose there is a function g in $L^p(E)$ such that for all n, $|f_n| \leq g$ a.e. on E. Prove that $\{f_n\} \to f$ in $L^p(E)$.**

Solution. First, we denote the set at which $\{f_n\}$ converges to f pointwise as E_0. Then, notice that for any $x \in E_0$ such that $|f_n(x)| \leq g(x)$, by the linearity of limit, and the continuity of absolute value, we obtain that $|f(x)| \leq g(x)$. Hence, we see that $|f| \leq g$ on E_0. It follows that

$$\begin{aligned} |f - f_n| &\leq |f| + |f_n| \\ &\leq 2g, \end{aligned}$$

on E_0. Raising both sides by p, we obtain

$$|f - f_n|^p \leq 2^p g^p,$$

on E_0. As sum and product of measurable functions is measurable, $\{f - f_n\}$ is a sequence of measurable functions that converge to 0 pointwise, and dominated by $2^p g^p$ everywhere on E_0. As $g \in L^P(E)$, we have that $2^p g^p$ is integrable. With the fact that $m(E \setminus E_0) = 0$, by the Lebesque dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \int_E |f - f_n|^p &= \lim_{n\to\infty} \int_{E_0} |f - f_n|^p \\ &= \int_{E_0} 0 \\ &= 0. \end{aligned}$$

Hence, $\{f_n\}$ is convergent to f in $L^P(E)$. $\square$

Question 3. Royden 19-5.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a Cauchy sequence in $L^\infty(X, \mu)$. Show that there is a measurable subset X_0 of X for which $\mu(X \sim X_0) = 0$ and for each $\epsilon > 0$, there is an index N for which
$$|f_n - f_m| \leq \epsilon \text{ on } X_0 \text{ for all } n, m \geq N.$$
Use this to show that $L^\infty(X, \mu)$ is complete.

Solution. Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(X, u)$. Then, for all $k \in \mathbb{N}$, there exists N_k such that
$$|f_i - f_j| < \frac{1}{k} \text{ on } X_{n,m} \text{ such that } u(X \sim X_{n,m}) = 0,$$
for $n, m \geq N_k$. Consider the following set:
$$X_0 = \bigcap_{k=1}^{\infty} \bigcup_{n,m \geq N_k} X_{n,m}.$$
By the DeMorgan's law, it follows that
$$\begin{aligned} X \sim X_0 &= X \sim \bigcap_{k=1}^{\infty} \bigcup_{n,m \geq N_k} X_{n,m} \\ &= \bigcup_{k=1}^{\infty} X \sim \bigcup_{n,m \geq N_k} X_{n,m} \\ &= \bigcup_{k=1}^{\infty} \bigcap_{n,m \geq N_k} X \sim X_{n,m}. \end{aligned}$$
By monotonicity of measure, we have
$$u(X \sim X_0) \leq u(\bigcup_{k=1}^{\infty} X \sim X_{N_k, N_k}).$$
By countable subadditivity of measure, we further get
$$\begin{aligned} u(X \sim X_0) &\leq \sum_{k=1}^{\infty} u(X \sim X_{N_k, N_k}) \\ &= 0. \end{aligned}$$
Now, fix $\epsilon > 0$. By Archemedian property of reals, there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. Observe that
$$X_0 \subseteq \bigcup_{n,m \geq N_k} X_{n,m}.$$
It follows that , for $x \in X_0$, $\{f_n(x)\}$ is Cauchy. By the completeness of reals, $\{f_n(x)\}$ is convergent. Define $f(x) = \lim_{n \to \infty} f_n(x)$ for $x \in X_0$. Again, fix $\epsilon > 0$. There exists N_k such that
$$|f_n - f_m| < \epsilon,$$
for $n, m \geq N_k$. Observe that for $x \in X_0$,
$$\lim_{m \to \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$
By linearity of limit, it follows that
$$|f_n(x) - f(x)| < \epsilon,$$
for $n \geq N_k$. Since ϵ and x was arbitrary we have.

Question 4. Royden 17-19.

19. Show that any measure that is induced by an outer measure is complete.

Solution. Let $u^* : 2^X \to [0, \infty]$ be an outer-measure, and let $(X, \mathscr{M}, u)$ be a measure space induced by u^*. Let $E \in \mathscr{M}$ and $u(E) = 0$. Let $S \subseteq E$, and $A \in \mathscr{M}$. Then, by finite monotonicity of u^*, we have $u^*(S) = 0$ and $u^*(S \cap A) = 0$. Again, using the finite monotonicity of u^*, we see that

$$\begin{aligned} u^*(A) &\geq u^*(A \cap S^c) + 0 \\ &= u^*(A \cap S^c) + u^*(A \cap S). \end{aligned}$$

Hence, $S \in \mathscr{M}$. We have shown that u is complete. $\square$

Question 5. Royden 17-29.

29. Show that a set function on a σ-algebra is a measure if and only if it is a premeasure.

Solution. Let $(X, \mathscr{M})$ be a measurable space. Let $u : \mathscr{M} \to [0, \infty]$ be a measure. By definition of measure we have, $u(\emptyset) = 0$. u is finitely additive and countably monotone, as any measure has finite additivity and countable monotonicity properties. Conversely, assume that u is a pre-measure. As u is a pre-measure and $\emptyset \in \mathscr{M})$, we have $u(\emptyset) = 0$. Now, let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collections chosen from $\mathscr{M}$. By finite additivity, and countable monotonicity of u, it follows that

$$\begin{aligned} \sum_{k=1}^{n} u(E_k) &= u(\bigcup_{k=1}^{n} E_k) \\ &\leq u(\bigcup_{k=1}^{\infty} E_k), \end{aligned}$$

for all n. Hence, by linearity of limits, we obtain

$$\sum_{k=1}^{\infty} u(E_k) \leq u(\bigcup_{k=1}^{\infty} E_k).$$

By finite additivity of u and the fact that $u(E_k) \geq 0$ for all k, we have

$$\begin{aligned} \sum_{k=1}^{\infty} u(E_k) &\geq \sum_{k=1}^{n} u(E_k) \\ &= u(\bigcup_{k=1}^{n} E_k), \end{aligned}$$

for all n. Again, by linearity of limits, we obtain that

$$\sum_{k=1}^{\infty} u(E_k) \geq u(\bigcup_{k=1}^{\infty} E_k).$$

Therefore, we have shown that

$$\sum_{k=1}^{\infty} u(E_k) = u(\bigcup_{k=1}^{\infty} E_k),$$

which shows that u is countably additive. Hence, u is a measure. The claim is true. $\square$

Question 6. Royden 17-36.

36. Let μ be a finite premeasure on an algebra $\mathcal{S}$, and μ^* the induced outer measure. Show that a subset E of X is μ^*-measurable if and only if for each $\epsilon > 0$ there is a set $A \in \mathcal{S}_\delta$, $A \subseteq E$, such that $\mu^*(E \sim A) < \epsilon$.

Solution. By the definition of set measurability by Caretheodory, E being u^* measurable is equivalent to E^c being u^* measurable. From Royden, we have that E is u^* measurable iff for any $\epsilon > 0$, there exists $A \in S_\sigma$ such that $A \subseteq S_\sigma$, and $u^*(A \setminus C) < \epsilon$.

We begin the main part of the proof. Assume E is u^* measurable. Fix $\epsilon > 0$. Then, by the established equivalence above, it follows that E^c is u^* measurable, and there exists $A \in S_\sigma$ such that $E^c \subseteq A$, and $u^*(A \setminus E^c) < \epsilon$. We claim that A^c is the set with the desired property. As $A \in S_\sigma$ there exists $\{O_n\}_{n=1}^\infty$ such that $A = \cup_{n=1}^\infty O_n$. By DeMorgan's law, we have $A^c = \cap_{n=1}^\infty {O_n}^c$. As, S is an algebra, ${O_n}^c$ is also in S, and it follows that $A^c \in S_\delta$. Furthermore, as $E^c \subseteq A$, it follows that $A^c \subseteq E$. Lastly, as $A \setminus E^c = E \setminus A^c$, we have $u^*(E \setminus A^c) < \epsilon$. Since ϵ was arbitrary, we have shown the forward implication.

Now, we prove the reverse implication. Assume that E from 2^X has the given property. Fix $\epsilon > 0$. Then, there exists $A \in S_\delta$ such that $A \subseteq E$, and $u^*(E \setminus A) < \epsilon$. Then, by taking the argument above in reverse, we obtain that $A^c \in S_\sigma$ such that $E^c \subseteq A^c$ and $u^*(A^c \setminus E^c) < \epsilon$. Since ϵ was arbitrary, we have shown that for any $\epsilon > 0$, there is a set $A \in S_\sigma$ such that $E^c \subseteq A$, such that $u^*(A \setminus E^c) < \epsilon$. By the established equivalence above, this implies that E^c is u^* measurable, and subsequently E is u^* measurable. $\square$