Real Variables:
Problem Set V

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Abstract

This work contains solutions to the problem set V of Real Variables 2015 at NYU.

1 Solutions

Question 6.10.

Solution. Let $x_1, x_2 \in (a, b)$, such that $x_1 < x_2$. Then, we have

$$\begin{aligned} f(x_1) &= \sum_{k=1}^{\infty} l((c_k, d_k) \cap (-\infty, x_1)) \\ f(x_2) &= \sum_{k=1}^{\infty} l((c_k, d_k) \cap (-\infty, x_2)). \end{aligned}$$

As $x_1 < x_2$, by the monotonicity property of the length function , we have

$$l((c_k, d_k) \cap (-\infty, x_1)) \quad \leq \quad l((c_k, d_k) \cap (-\infty, x_2)),$$

for all $k \geq 1$. It follows that $f(x_1) \leq f(x_2)$. Hence, f is increasing. We show that f fails to be differentiable at each point in E, which is a set of measure zero contained in the open interval (a, b). Let $x \in E$. Then, by the preceding problem, there exist a countable collection of open intervals contained in (a, b), $\{(c_k, d_k)\}_{k=1}^{\infty}$ such taht each point in E belongs to infinitely many intervals in the collection and $\sum_{k=1}^{\infty} d_k - c_k < \infty$. Let $\{(c_{k_i}, d_{k_i})\}_{i=1}^{\infty}$ be the sub-collection such that $x \in (c_{k_i}, d_{k_i})$ for all i. Then, there exist a finite sub-cover $\{(c_{k_i}, d_{k_i})\}_{i=1}^{n}$ that x belongs to. Since, n is finite, as intersection of finite open sets is open, there exists a_n such that

$$B(x, a_n) \in \cup_{k=1}^{n} (c_{k_i}, d_{k_i}),$$

such that (B, a_n) denotes the ball of radius a_n, centered at x. Observe that

$$\begin{aligned} f(x + a_n) - f(x) &\geq \sum_{i=1}^{n} l((c_{k_i}, d_{k_i}) \cap (x, x + a_n)) \\ &= n a_n. \end{aligned}$$

It follows that

$$\begin{aligned} \bar{D} f(x) &= \lim_{h \to 0} \sup_{0 < |t| \leq h} \{ \frac{f(x+t) - f(x)}{t} \} \\ &= \lim_{h \to 0} \sup_{0 < |t| \leq h} \frac{n a_n}{a_n} \\ &\geq n. \end{aligned}$$

Since n was arbitrary, we have that

$$\bar{D}f(x) = \infty,$$

which is not finite, and by definition, x is not differentiable at x. Therefore, f fails to be differntiable at each point in E. $\square$

Question 6.33.

Solution. Let $\{f_n\}$ be a sequence of real-valued functions on $[a, b]$ that converges pointwise on $[a, b]$ to the real-valued function f. We wish to show that $TV(f) \leq \liminf TV(f_n)$. Fix $P = \{x_0, ..., x_m\}$ be a partition of $[a, b]$. As $f_n \to f$ pointwise, we have

$$\begin{aligned} V(f, P) &= \sum_{k=0}^{m-1} |f(x_{k+1}) - f(x_k)| \\ &= \lim_{n\to\infty} \sum_{k=0}^{m-1} |f_n(x_{k+1}) - f_n(x_k)| \\ &= \lim_{n\to\infty} V(f_n, P). \end{aligned}$$

By the definition of total variation, it follows that

$$V(f_n, P) \leq TV(f_n),$$

for all n. Consequently, we obtain

$$V(f, P) \leq \liminf_{n\to\infty} TV(f_n),$$

and since P was arbitrary, we finally have that

$$TV(f) \leq \liminf_{n\to\infty} TV(f_n),$$

as desired. $\square$

Question 4. Royden 6.42.

Solution. Let f and g be real-valued functions, that are absolutely continuous functions on $[a, b]$. We wish to show that $f + g$ is absolutely continuous on $[a, b]$. Fix $\epsilon > 0$. As f and g are both absolutely continuous on $[a, b]$, there exist $\delta_f, \delta_g > 0$, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{k=1}^n$ in (a, b),

$$\begin{aligned} \sum_{k=1}^{n}[b_k - a_k] < \delta_f &\implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \frac{\epsilon}{2} \\ \sum_{k=1}^{n}[b_k - a_k] < \delta_g &\implies \sum_{k=1}^{n} |g_(b_k) - g(a_k)| < \frac{\epsilon}{2}. \end{aligned}$$

Define $\delta = \min(\delta_f, \delta_g)$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint open intervals in (a, b), such that $\sum_{k=1}^n [b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^{n} |f + g(b_k) - f + g(a_k)| &\leq \sum_{k=1}^{n} |f(b_k) - f(a_k)| + |g(b_k) - g(a_k)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we have shown that $f + g$ is absolutely continuous on $[a, b]$.

Let f be a real-valued function, that is absolutely continuous on $[a, b]$. We show that cf, for any $c \in \mathbb{R}$, is absolutely continuous on $[a, b]$. Let $c = 0$. Then $cf = 0$, which can trivially be shown to be absolutely continuous, as $f(c) = 0$ for any $c \in [a,b]$. Suppose $c \neq 0$. As f is absolutely continuous on $[a,b]$, there exists $\delta_f > 0$, such that for any finite disjoint open intervals $\{(a_k,b_k)\}_{k=1}^n$ in (a,b),

$$\sum_{k=1}^n |b_k - a_k| < \delta_f \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \frac{\epsilon}{|c|}.$$

Define $\delta = \delta_f$. Let $\{(a_k,b_k)\}_{k=1}^n$ be a finite disjoint open intervals in (a,b) such that $\sum_{k=1}^n [b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^n |cf(b_k) - cf(a_k)| &= |c| \sum_{k=1}^n |f(b_k) - f(a_k)| \\ &< |c| \frac{\epsilon}{|c|} = \epsilon. \end{aligned}$$

Since ϵ was arbitrary, combined with the $c = 0$ case, we have shown that cf, for any $c \in \mathbb{R}$, is absolutely continuous on $[a,b]$.

Let f be a real-valued function, that is absolutely continuous on $[a,b]$. We wish to show that f^2 is absolutely continuous on $[a,b]$. As f is absolutely continuous, f is continuous on $[a,b]$. Hence, by the Extreme Value Theorem, there exists M such that $|f| \leq M$ on $[a,b]$. Fix $\epsilon > 0$. As f is absolutely continuous on $[a,b]$, there exists $\delta_f > 0$, such that for any finite disjoint open intervals $\{(a_k,b_k)\}_{k=1}^n$ in (a,b),

$$\sum_{k=1}^n |b_k - a_k| < \delta_f \implies \sum_{k=1}^n |f(b_k) - f(a_k)| < \frac{\epsilon}{2M}.$$

Define $\delta = \delta_f$. Let $\{(a_k,b_k)\}_{k=1}^\infty$ be a finite disjoint open intervals in (a,b) such that $\sum_{k=1}^n [b_k - a_k] < \delta$. It follows that

$$\begin{aligned} \sum_{k=1}^n |f^2(b_k) - f^2(a_k)| &= \sum_{k=1}^n |f(b_k) - f(a_k)||f(b_k) + f(a_k)| \\ &\leq 2M \sum_{k=1}^n |f(b_k) - f(a_k)| \\ &< 2M \frac{\epsilon}{2M} = \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we have shown that f^2 is absolutely continuous on $[a,b]$.

Let f and g be real-valued functions, that are absolutely continuous on $[a,b]$. We wish to show that fg is absolutely continuous on $[a,b]$. Observe that

$$(f+g)^2 = f^2 + g^2 - 2fg,$$

which simplifies to

$$fg = -\frac{1}{2}((f+g)^2 + (-f^2) + (-g^2)).$$

As we have previously shown that a square of an absolutely continuous function is absolutely continuous, and a scalar multiple of an absolutely continuous function is absolutely continuous, we can deduce that fg is absolutely continuous on $[a,b]$. This completes the proof. $\square$

Question 4. 6.45.

Solution. Let f be a real-valued function, that is absolutely continuous on $\mathbb{R}$. Let g be a real-valued function, that is absolutely continuous and strictly monotone on $[a,b]$. Without the loss of generality, we assume that g is strictly increasing. We wish to show that $f \circ g$ is absolutely continuous on $[a,b]$. Fix $\epsilon > 0$. As f is absolutely continous on $\mathbb{R}$, it is also absolutely continuous on $[g(a), g(b)]$, which is a non-degenerate closed interval, as g is strictly increasing. there exists δ_f, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{n=1}^{\infty}$ in $(g(a), g(b))$,

$$\sum_{k=1}^{n}[b_k - a_k] < \delta_f \implies \sum_{k=1}^{n}|f(b_k) - f(a_k)| < \epsilon \ (*).$$

As g is absolutely continuous, there exists δ_g, such that for any finite disjoint open intervals $\{(a_k, b_k)\}_{n=1}^{\infty}$ in (a, b),

$$\sum_{k=1}^{n}[b_k - a_k] < \delta_g \implies \sum_{k=1}^{n}|g(b_k) - g(a_k)| < \delta_f.$$

Define $\delta = \delta_g$. Let $\{(a_k, b_k)\}_{k=1}^{n}$ be a finite djsjoint open intervals in (a, b) such that $\sum_{k=1}^{n}[b_k - a_k] < \delta_g$. It follows that $\sum_{k=1}^{n}[g(b_k) - g(a_k)] < \delta_f$. As g is strictly increasing, we observe that $\{(g(a_k), g(b_k))\}_{k=1}^{n}$ form a finite disjoint open intervals in $(g(a), g(b))$. Therefore, from $(*)$ it follows that

$$\sum_{k=1}^{n}|f \circ g(b_k) - f \circ g(a_k)| \quad < \quad \epsilon.$$

Since ϵ was arbitrary, we have shown that $f \circ g$ is absolutely continuous on $[a, b]$. $\square$

Question 6.55.

Solution. (ii) Assume that f is absolutely continuous. Let $P = \{x_0, ..., x_k\}$. Then, by the additivity over domain of integration, and the absolute continuity of f, we have

$$\begin{aligned} \int_a^b |f'(x)|dx &\geq \sum_{i=0}^{k-1} |\int_{x_i}^{x_{i+1}} f'(x)|dx \\ &= V(f, P). \end{aligned}$$

Since P was arbitrary, we have $\int_a^b |f'| \geq TV(f)$. Hence, with $\int_a^b |f'| \leq TV(f)$, we can conclude that

$$\int_a^b |f'| = TV(f).$$

Question 6.56.

Solution. Let g be strictly increasing an absolutely continuous on $[a, b]$.

(i) Let O be an open subset of (a, b). Then, O can be represented as a countable union of disjoint intervals in (a, b):

$$O \quad = \quad \cup_{k=1}^{\infty}(a_k, b_k),$$

and since g is strictly increasing, we have $\{(g(a_k), g(b_k))\}_{k=1}^{\infty}$ forms a collection of disjoint intervals, and

$$g(O) \quad = \quad \cup_{k=1}^{\infty}(g(a_k), g(b_k)).$$

Therefore, by the countable additivity of measure, it follows that

$$m(g(O)) \quad = \quad \sum_{k=1}^{\infty} g(b_k) - g(a_k).$$

On the other hand, by the countable additivity of integration, and as g is absolutely continuous, we have

$$\begin{aligned} \int_O g'(x)dx &= \sum_{k=1}^{\infty} \int_{g(a,k)}^{g(b_k)} g'(x)dx \\ &= \sum_{k=1}^{\infty} g(b_k) - g(a_k). \end{aligned}$$

Therefore, $m(g(O)) = \int_O g'(x)dx$, as desired.

(ii)

(iii) Let E be a meazuer zero set. We have previously shown that a continuous map carries a measure zero set to a measure zero set. Hence, $g(E)$ has measure zero. Furthermore, we have that an integral over a measure zero set is zero. Therfore, we have that $m(g(E)) = 0 = \int_E g'(x)dx$ as desired.

(iv) Let A be any measurable set of $[a, b]$. By the outer approximation of a measurable set via $G - \delta$ set, there exists $G - \delta$ set G such that $m(G \setminus A) = 0$ and $A \subseteq G$. Then, by the finite additivity of measure we have

$$\begin{aligned} m(g(A)) &= m(g(G \setminus A \cup A)) \\ &= m(g(G \setminus A)) + m(g(A)) \\ &= m(g(G)). \end{aligned}$$

On the other hand, by the additivity over domain property of integration and the fact that any integral on a measure zero set is zero, we have

$$\begin{aligned} \int_A g'(x)dx &= \int_A g'(x)dx + \int_{G\setminus A} g'(x)dx \\ &= \int_G g'(x)dx. \end{aligned}$$

By the preceding result with $G - \delta$ sets, $LHS = RHS$. Hence, we have $m(g(A)) = \int_A g'(x)dx$ for any measurable set of $[a, b]$.

(v) Let $c = g(a)$ and $d = g(b)$. We can write the simple function ψ as

$$\psi = \sum_{k=1}^{n} c_k \chi_{E_k}.$$

Then, by the countable additivity over domain property of integration, we have

$$\begin{aligned} \int_c^d \psi(y)dy &= \int_c^d \sum_{k=1}^{n} c_k \chi_{E_k}(y)dy \\ &= \sum_{k=1}^{n} c_k \int_c^d \chi_{E_k}(y)dy \\ &= \sum_{k=1}^{n} c_k m(E_k). \end{aligned}$$

Similarly, the RHS can be computed to be the same sum as desired.

(vi) Since f is non-negative integrable function on $[c, d]$, there exists a sequence of increasing simple functions $\{\phi_n\}$ such that $\phi_n \to f$ pointwise and $|\phi_n| \leq |f|$ on $[c, d]$ for all n. As f is integrable and dominates ϕ_n for all n, by the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_c^d \phi_n(y)dy = \int_c^d f(y)dy.$$

As g is strictly increasing and absolutely continuous, we have that g' is non-negative and integrable with $\int_c^d g'(x)dx = g(d) - g(c) < \infty$. Therefore, we have that $\phi_n(g)g'$ and $f(g)g'$ are both integrable and non-negative, and $\phi_n(g)g' \to f(g)g'$ pointwise and $f(g)g'$ dominates $\phi_n(g)g'$ for all n. Therefore, by the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_c^d \phi_n(g(x))g'(x)dx = \int_c^d f(g(x))g'(x)dx.$$

Since $\int_c^d \phi_n(y)dy = \int_c^d \phi_n(g(x))g'(x)dx$ for all n by the previous result, we have that

$$\int_c^d f(y)dy = \int_c^d f(g(x))g'(x)dx,$$

as desired.

(vii) As we have (vi), by setting $f = \chi_O$, we have

$$\int_c^d \chi_O(y)dy \quad = \quad \int_a^b \chi_O(g(x))g'(x)dx.$$

As χ_O is a characteristic function, we have

$$\begin{aligned} \int_c^d \chi_O(y)dy \quad &= \quad m(O) \\ \int_a^b \chi_O(g(x))g'(x)dx \quad &= \quad \int_O g'(x)dx. \end{aligned}$$

Hence, (i) holds as desired. $\square$