Royden

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Abstract

This work contains the solutions to Royden's Real Variables.

1 Solutions

Question 1. Royden 1.1-1 (Distributive Property of Multiplicative Inverse in Reals).

Solution. Assume that $a \neq 0$ and $b \neq 0$. From the multiplicative identity axiom, we have that a multiplicative inverse exists for a and b individually, which we denote as a^{-1} and b^{-1} respectively. Now, consider the expression $(ab)(a^{-1}b^{-1})$, where ab denotes the product of a and b, and $a^{-1}b^{-1}$ denotes the product of a^{-1} and b^{-1}. From the commutativity of multiplication, we obtain

$$(ab)(a^{-1}b^{-1}) = (ab)(b^{-1}a^{-1}).$$

Using the associativity of multiplication and iteratively substituting $bb^{-1} = 1$ and $aa^{-1} = 1$, we have

$$(ab)(a^{-1}b^{-1}) = 1,$$

where 1 denotes the identity as usual. Hence, the product, $a^{-1}b^{-1}$ satisfies definition of multiplicative inverse with respect to the ab term whose multiplicative inverse can be denoted as $(ab)^{-1}$ by convention. Therefore, we obtain that

$$(ab)^{-1} = a^{-1}b^{-1},$$

as desired.

Question Royden 1.1-3.

Solution. Let E be a nonempty set of real numbers.
($\Leftarrow$) Assume that E consists of a single point, which we denote as x. We claim that inf$E = x$ and sup$E = x$. As we have $x \leq x$, we see that x is an upper bound for E. Suppose that there exists an upper bound for E, a, that is smaller than x, namely $a < x$. This is a contradiction to the fact that a is an upper bound as it is required to have $x \leq a$ with $x \in E$. Hence, there does not exists any upper bound for E that is smaller than x. By definition of supremum, we have that sup$E = x$. By symmtry, we can see that inf$E = x$ as well. Therefore, inf$E =$ sup$E$.

($\Rightarrow$) Assume that inf$E =$ sup$E$. Given the assumption, let us denote the infimum and supremum for E as a single real number a. Then, by definition of infimum, any x in E, we have $a \leq x$. Furthermore, by definition of supremum, any x in E, we have $x \leq a$. The only real number that can satisfy the two given equality is a itself. We also know that a must be in E as E is a nonempty set of reals. Therefore, we have shown that $E = \{a\}$, and that E consits of a single point.