Real Variables: Problem Set XI

Youngduck Choi

Courant Institute of Mathematical Sciences New York University yc1104@nyu.edu

Abstract

This work contains solutions to the problem set XI of Real Variables 2015 at NYU.

1 Solutions

Question Royden 17-6.

6. Let (X, \mathcal{M}, μ) be a measure space and X_0 belong to \mathcal{M} . Define \mathcal{M}_0 to be the collection of sets in \mathcal{M} that are subsets of X_0 and μ_0 the restriction of μ to \mathcal{M}_0 . Show that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

Solution. We first show that (X_0, \mathscr{M}_0) is a measurable space. To this end, we must show that \mathscr{M}_0 is a σ -algebra of X_0 . As \emptyset and X_0 belong to \mathscr{M} , are subsets of X_0 , it follows that \emptyset and X_0 belong to \mathscr{M}_0 . Let $\{A_n\}_{n=1}^\infty$ be a countable collections of sets in \mathscr{M}_0 . As $A_n \subseteq X_0$ for all n, we have $\bigcup_{n=1}^\infty A_n \subseteq X_0$. Furthermore, as \mathscr{M} is a σ -algebra, and $A_n \in \mathscr{M}$ for all n, we also have $\bigcup_{n=1}^\infty A_n \in \mathscr{M}$. Hence, it follows that $\bigcup_{n=1}^\infty A_n \in \mathscr{M}_0$. Now, let A be a set, belonging to \mathscr{M}_0 . Then, as $X_0 \setminus A$ is a subset of X_0 , and X_0 and A belong to A, which gives $A_0 \setminus A \in \mathscr{M}_0$, we have $A_0 \setminus A = X_0 \setminus$

$$\sum_{n=1}^{\infty} \mu_0(A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$
$$= \mu(\bigcup_{n=1}^{\infty} A_n)$$
$$= \mu_0(\bigcup_{n=1}^{\infty} A_n).$$

Therefore, we have shown that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

Question Royden 17-15.

15. Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$, where α and β are real numbers. Show that

$$|\alpha \nu| = |\alpha| |\nu| \text{ and } |\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|,$$

where $\nu \le \mu$ means $\nu(E) \le \mu(E)$ for all measurable sets E.

Solution. Let (X, \mathcal{M}) be a measurable space. Let v_1 and v_2 be two finite signed measures on (X, \mathcal{M}) . Consider a set function $\alpha v_1 + \beta v_2$ on \mathcal{M} , for $\alpha, \beta \in \mathbb{R}$, which is defined by

$$\alpha v_1 + \beta v_2(E) = \alpha v_1(E) + \beta v_2(E),$$

for $E \in \mathcal{M}$. As v_1 and v_2 only take finite values, it also follows that $\alpha v_1 + \beta v_2$ also assumes only finite values, as an addition of two finite values are finite. Furthermore, it follows that

$$\alpha v_1 + \beta v_2(\emptyset) = \alpha v_1(\emptyset) + \beta v_2(\emptyset)$$

= 0 + 0 = 0.

as v_1 and v_2 are signed measures. Furthermore, the countable additivity property holds, as for any countable disjoint collection $\{E_k\}$ from \mathscr{M} , by linearity of limit, and countable additivity of v_1 and v_2 , it follows that

$$\alpha v_1 + \beta v_2(\bigcup_{k=1}^{\infty} E_k) = \alpha v_1(\bigcup_{k=1}^{\infty} E_k) + \beta v_2(\bigcup_{k=1}^{\infty} E_k)$$

$$= \alpha \sum_{k=1}^{\infty} v_1(E_k) + \beta \sum_{k=1}^{\infty} v_2(E_k)$$

$$= \sum_{k=1}^{\infty} \alpha v_1(E_k) + \beta v_2(E_k)$$

$$= \sum_{k=1}^{\infty} \alpha v_1 + \beta v_2(E_k).$$

Hence $\alpha v_1 + \beta v_2$ is a finite measure.

Let (X, \mathcal{M}) be a measurable space and v be a finite measure on the space. Consider $|\alpha v|$. It follows that for $E \in \mathcal{M}$, we have

$$|\alpha v(E)| = |\alpha||v(E)|.$$

Hence, $|\alpha v| = |\alpha||v|$. Now, let v_1 and v_2 be finite signed measures on (X, \mathcal{M}) . By the triangle inequality of reals, it follows that for any $E \in \mathcal{M}$,

$$|v_1 + v_2(E)| = |v_1(E) + v_2(E)|$$

 $\leq |v_1(E)| + |v_2(E)|.$

Hence, $|v_1 + v_2| \le |v_1| + |v_2|$.

Question Royden 17-17.

- 17. Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu + \nu |\mu \nu|)$ and $\mu \vee \nu = \mu + \nu \mu \wedge \nu$.
 - (i) Show that the signed measure $\mu \wedge \nu$ is smaller than μ and ν but larger than any other signed measure that is smaller than μ and ν .
 - (ii) Show that the signed measure $\mu \lor \nu$ is larger than μ and ν but smaller than any other measure that is larger than μ and ν .
 - (iii) If μ and ν are positive measures, show that they are mutually singular if and only if $\mu \wedge \nu = 0$.

Solution. (i) Let u and v be finite signed measures on (X, \mathcal{M}) . We wish to show that $u \wedge v \leq u$. By writing out $u \wedge v$ term, multiplying by 2, and re-arranging the terms trivially, we see that $u \wedge v \leq u$ is equivalent to $u - v \leq |u - v|$. As an absolute value of a real number is larger than the number itself, we see that $u \wedge v \leq u$ holds. By symmetry, we also see that $u \wedge v \leq v$ as well.

Now, let t be a signed measure such that $t \le u$ and $t \le v$. It follows that $t \le \min(u, v)$. Assume without loss of generality that $\min(u, v) = u$. It follows that

$$\frac{1}{2}(u+v|u-v|) = \frac{1}{2}(2u)$$
$$= u.$$

Therefore, $\min(u, v) = \frac{1}{2}(u + v - |u - v|)$. Hence, $t \le u \wedge v$.

- (ii) Let u and v be finite signed measures on (X, \mathcal{M}) . In the above solution, we have shown that $u \wedge v = \min(u, v)$. Hence, it follows that $u + v u \wedge v = \max(u, v)$. Therefore, we obtain that $u \vee v \geq u$ and $u \vee v \geq v$. Again, as for any t such that $t \geq u$ and $t \geq v$, it follows that $t \geq \max(u, v) = u \vee v$.
- (iii) Let that u and v are finite measures, and assume that $u \wedge v = \frac{1}{2}(u + v |u v|) = 0$. For any $E \in \mathcal{M}$, it follows that

$$u(E) + v(E) = |u(E) - v(E)|$$

Suppose that u(E) and v(E) are both strictly positive. It follows that $u(E) + v(E) > \max(u(E), v(E))$ and $|u(E) - v(E)| < \max(u(E), v(E))$, which yields a contradiction with the above equality. Hence, we have that at least one of u(E) or v(E) is 0. Assume without loss of generality that u(E) = 0. Now, with the same argument with respect to E^c , it follows that either $u(E^c)$ and $v(E^c)$ is 0. If $v(E^c) = 0$, then u and v are mutually singular. Now, consider the remaining case of $u(E^c) = 0$. By the additivity property of a signed measure, we have that u(X) = 0. Hence, u(X) = 0 and $v(\emptyset) = 0$. Therefore, we have shown that u and v are mutually singular.

Now, assume that u and v are mutually singular measures. Hence, there exists a disjoint measurable set A and B such that $A \cup B = X$ and u(A) = 0 and v(B) = 0. Let $E \in \mathcal{M}$. Then, it follows that

$$\begin{array}{rcl} u \wedge v(E) & = & u \wedge v(E \cap A) + u \wedge v(E \cap B) \\ & = & \frac{1}{2}(u + v - |u - v|)(E \cap A) + \frac{1}{2}(u + v - |u - v|)(E \cap B) \\ & = & 0 + 0, \end{array}$$

by the monotonicity property.

Question Royden 18-50.

50. Establish the uniqueness of the function f in the Radon-Nikodym Theorem.

Solution. Let (X, \mathcal{M}) be a measurable space, and u, v be σ -finite measures on the space with v << u. Let f and g be the non-negative measurable functions for which,

$$v(E) = \int_{E} f du$$
$$= \int_{E} g du,$$

for all $E \in \mathcal{M}$. By the linearity of integration for non-negative measures, we obtain that

$$\int_{E} f - g du = 0,$$

for all $E\in\mathcal{M}$. Now, we prove that if $\int_E f du=0$ for every measurable subset E of X, then f=0 a.e. on X. We prove the case for f non-negative and the result can be extended through an argument with f^+ and f^- decomposition. Define $A_n=\{x\in X\mid f(x)\geq \frac{1}{n}\}$. we have $\cup_n A_n=\{x\in X\mid f(x)>0\}$. Now, observe that by the assumption, $\int_{A_n} f du=0$ for all n and $\frac{1}{n}u(A_n)\leq \int_{A_n} f du=0$, which gives $u(A_n)=0$ for all n. Therefore, by the countable additivity, we obtain that $u(\cup_n A_n)=0$. Hence, we have shown that f=0 a.e. Therefore, we have that f-g from the Radon-Nikodym theorem, must be 0 a.e. on X. Hence, we have shown the uniqueness of Radon-Nikodym theorem.

Question Royden 18-54.

54. Let μ , ν , and λ be σ -finite measures on the measurable space (X, \mathcal{M}) .

(i) If $\nu \ll \mu$ and f is a nonnegative function on X that is measurable with respect to M, show that

$$\int_X f \ d\nu = \int_X f \left[\frac{d\nu}{d\mu} \right] d\mu.$$

(ii) If $\nu \ll \mu$ and $\lambda \ll \mu$, show that

$$\frac{d(\nu+\lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \text{ a.e. } [\mu].$$

(iii) If $\nu \ll \mu \ll \lambda$, show that

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} \text{ a.e. } [\lambda].$$

Solution. (i) By simple approximation theorem, we have $\{\psi_n\}$ simple that converges pointwise to f, by monotonoe convergence theorem, we have

$$\int_X f dv = \lim_{n \to \infty} \int_X \psi_n dv$$
$$\int_X f[\frac{dv}{du}] du = \lim_{n \to \infty} \psi_n [\frac{dv}{du}] du.$$

By explicitly writing out the integration formula for simple functions, we can show that $\int_X \psi_n dv = \int_X \psi_n \frac{dv}{du} du$ for all n. Hence, we have shown the claim.

(ii) As $v \ll u$ and $\lambda \ll u$, we have

$$v(E) = \int_{E} \left[\frac{dv}{du}\right] du$$
$$\lambda(E) = \int_{E} \left[\frac{d\lambda}{du}\right] du.$$

how in the 18-55 that $v + \lambda << u$ as well. It follows that

$$v + \lambda(E) = \int_{E} \left[\frac{d(v+\lambda)}{du}\right] du.$$

Combining with the above equality, we get

$$\int_E [\frac{dv}{du} + \frac{d\lambda}{du}] \quad = \quad \int_E [\frac{d(v+\lambda)}{du}] du.$$

By the lemma proven in the previous problem, we have that

$$\left[\frac{dv}{du} + \frac{d\lambda}{du}\right] = \left[\frac{d(v+\lambda)}{du}\right],$$

a.e.

(iii) This result directly follows from substitution to (i) and using the lemma from the previous problem like above.

Question Royden 18-55.

- 55. Let μ , ν , ν_1 , and ν_2 be measures on the measurable space (X, \mathcal{M}) .
 - (i) Show that if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.
 - (ii) Show that if ν_1 and ν_2 are singular with respect to μ , then, for any $\alpha \ge 0$, $\beta \ge 0$, so is the measure $\alpha \nu_1 + \beta \nu_2$.
 - (iii) Show that if ν_1 and ν_2 are absolutely continuous with respect to μ , then, for any $\alpha \ge 0$, $\beta \ge 0$, so is the measure $\alpha \nu_1 + \beta \nu_2$.
 - (iv) Prove the uniqueness assertion in the Lebesgue decomposition.

Solution. (i) Assume $v \perp u$, and v << u. It follows that there exists a pair of measurable sets A and B such that u(A) = 0 and v(B) = 0. By the absolute continuity of v with respect to u, it follows that v(A) = 0. By finite additivity of measure, we obtain that v(X) = 0. In other words, v is a zero measure.

(ii) Assume $u \perp v$. We show that $u \perp \alpha v$, for $\alpha \geq 0$ holds as well. As $u \perp v$, there exists A and B from \mathscr{M} such that u(A) = 0 and v(B) = 0 and $A \cup B = X$. Observe that $\alpha v(B) = 0$ as well. Hence, $u \perp \alpha v$. Now, assume $u \perp v_1$ and $u \perp v_2$. We show that $u \perp v_1 + v_2$. Let (A_1, B_1) and (A_2, B_2) be the pairs of sets that grant the mutual singularities. Then, observe that by finite additivity of measure, we have $u(A_1 \cup A_2) = 0$. Furthermore, observe that by monotonicity of measure, $v_1 + v_2(B_1 \cap B_2) = 0$. Since $A_1 \cup A_2 \cup (B_1 \cap B_2) = (A_1 \cup A_2 \cup B_1) \cap (A_1 \cup A_2 \cup B_2) = X \cap X = X$, we have that u and $v_1 + v_2$ are mutually singular. Hence, the claim is proven.

(iii) Assume v << u. Then, for any $E \in \mathcal{M}$ such that u(E)=0, we have v(E)=0. Since v(E)=0, we also have $\alpha v(E)=0$. Hence, $\alpha v << u$. Now, assume $v_1 << u$ and $v_2 << u$. It follows that for any $E \in \mathcal{M}$ such that u(E)=0, we have $v_1(E)=0$ and $v_2(E)=0$. Therefore, $v_1+v_2(E)=0$. Hence, the claim is proven.

(iv) Let v_0 and v_1 be the Lebesque decomposition measures, where v_0 is mutually singular and v_1 is absolutely continuous with respect to v_1 . Consider $v_0^{'}$ and $v_1^{'}$ with the same set-up. We have that $v_0 + v_1 = v_0^{'} + v_1^{'}$. It follows that

$$v_0 - v_0' = v_1 - v_1'.$$

Observe that $v_0 - v_0^{'}$ and $v_1 - v_1^{'}$ are singular and absolutely continuous with respect to u from (ii) and (iii). Hence, by (i), we have that $v_0 = v_0^{'}$ and $v_1 = v_0^{'}$. Therefore, we have shown that uniqueness of the decomposition.