Real Variables:
Problem Set III

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Abstract

This work contains solutions to the problem set III of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 3.20.

Solution. Let A and B be any sets. The LHS of the first equation can be written as

$$\chi_{A \cap B} = \begin{cases} 1 & \text{if } x \in A \cap B \\ 0 & \text{if } x \notin A \cap B. \end{cases}$$

By noting that the product of has to be of the form, $1 \cdot 1$, to yield 1, the RHS of the second equation can be written as

$$\chi_A \chi_B = \begin{cases} 1 & \text{if } x \in A \text{ and if } x \in B \\ 0 & \text{otherwise,} \end{cases}$$

which is precisely the LHS as $x \in A$ and $x \in B$ is the definition of $x \in A \cap B$. Now, the LHS of the second equation can be written as

$$\chi_{A \cup B} = \begin{cases} 1 & \text{if } x \in A \cup B \\ 0 & \text{if } x \notin A \cup B. \end{cases}$$

The RHS of the second equation can be written as

$$\chi_A + \chi_B - \chi_A \cdot \chi_B = \begin{cases} 1 + 1 - 1 \cdot 1 = 1 & \text{if } x \in A, x \in B \\ 1 + 0 - 1 \cdot 0 = 1 & \text{if } x \in A, x \notin B \\ 0 + 1 - 0 \cdot 1 = 1 & \text{if } x \notin A, x \in B \\ 0 + 0 - 0 \cdot 0 = 0 & \text{if } x \notin A, x \notin B, \end{cases}$$

which can be simplified to

$$\chi_A + \chi_B - \chi_A \cdot \chi_B = \begin{cases} 1 & \text{if } x \in A \cup B \\ 0 & \text{if } x \notin A \cup B \end{cases},$$

as desired. The LHS of the third equation can be written as

$$\chi_{A^c} = \begin{cases} 1 & \text{if } x \in A^c \\ 0 & \text{if } x \notin A^c \end{cases}$$
,

as desired. The RHS of the third equation can be written as

$$1 - \chi_A \quad = \quad \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases},$$

which is precisely the LHS, as $x \notin A$ is equivalent to $x \in A^c$ by definition. Hence, we have shown the three given equalities. $\square$

Question 2. Royden 3.21.

Solution. Let $\{f_n\}$ be a sequence of measurable functions with common domain E. Consider the function $\sup\{f_n\}$, which we will denote as s. Let $c \in \mathbb{R}$. We wish to show that $\{x \in E \mid s(x) > c\}$ is measurable. By the definition of supremum, we have that $s(x) > c$ iff there exists n such that $f_n(x) > c$. Hence, it follows that

$$\{x \in E \mid s(x) > c\} \quad = \quad \bigcup_{n=1}^{\infty} \{x \in E \mid f_n(x) > c\}.$$

Since the RHS is a countable collection of measurable sets, the set $\{x \in E | s(x) > c\}$ is measurable. Since c was arbitrary, s is measurable. The $\inf$ case can be shown analogously.

Now, consider the $\limsup\{f_n\}$ case. Observe that $\limsup_{n\to\infty} f_n = \inf_n \sup_{m \geq n} f_n$. Consequently, as we have shown that $\sup\{f_n\}$ and $\inf\{f_n\}$ are measurable functions, we have that $\limsup\{f_n\}$ is measurable. The $\liminf$ case can be shown analogously. $\square$

Question 3. Royden 3.27.

Solution. Let $f = 1$ on $[0, \infty)$. Define $f_n = \chi_{[0,n]}$ for all n. Then, we have that $f_n \to f$ pointwise everywhere. Suppose for sake of contradiction that there exists a closed set F such that $m([0, \infty) \setminus F) < \epsilon$ and $f_n \to f$ uniformly on F. F is unbounded, as otherwise $m([0, \infty) \setminus F) > \infty$, which is a contradiction. Since $f_n \to f$ uniformly on F, there exists N such that $f_n = f$ on F. As F is unbounded, there exists $x \in F \setminus [0, N]$. Since $f_N(x) = 0$ and $f(x) = 1$, this is a contradiction with $f_n = f$ on F. Therefore, we have shown that the conclusion of Egoroff can fail without the finiteness assumption on the measure of domain. $\square$

Question 4. Royden 4.12.

Solution. Let f a bounded measurable function on a set of finite measure E. Assume g is bounded and $f = g$ a.e. on E. First, as g is a function that equals a measurable function a.e., we have that g is measurable. Since both f and g are bounded measurable functions, we have $\int_E f$ and $\int_E g$ terms well-defined. Let $E_0 = \{x \in E \mid f(x) \neq g(x)\}$. Note that $\mathrm{m}(E_0) = 0$, as $f = g$ a.e. Consequently, $E \setminus E_0$ and E_0 are disjoint measurable sets. Then, by additivity over domain and linearity of integration, we have

$$\begin{aligned} \left| \int_E f - \int_E g \right| &= \left| \int_{E \setminus E_0} f - \int_{E \setminus E_0} g + \int_{E_0} f - \int_{E_0} g \right| \\ &= \left| \int_{E \setminus E_0} f - g + \int_{E_0} f - g \right|. \end{aligned}$$

As $f = g$ on $E \setminus E_0$, we have

$$\begin{aligned} \left| \int_E f - \int_E g \right| &= \left| \int_{E_0} f - g \right| \\ &\leq \int_{E_0} |f - g|. \end{aligned}$$

As both f and g are bounded, there exists M such that $|f-g| \leq M$ on E_0. Hence, we have

$$\begin{aligned} \left| \int_E f - \int_E g \right| &\leq M \cdot \mathrm{m}(E_0) \\ &\leq 0. \end{aligned}$$

Therefore, we have $\int_E f = \int_E g$ as desired. $\square$

Question 5. Royden 4.23.

Solution. Let $\{a_n\}$ be a sequence of non-negative real numbers. Let f be a function on $E = [1, \infty)$, defined by setting $f(x) = a_n$ if $n \leq x < n+1$. Then, consider the following sequence of functions of nonnegative real numbers $\{f_n\}$ defined on E such that

$$f_n = \sum_{k=1}^{n} a_k \chi_{I_k},$$

where I_k denotes the characteristic function of an interval $[k, k+1)$. Notice that $\{f_n\}$ is increasing, and converges to f pointwise everywhere on E. Hence, by the Monotone Convergence Theorem, we have

$$\int_E f = \lim_{n \to \infty} \int_E f_n.$$

As the integral on the RHS is a simple function with n values, we have

$$\int_E f = \lim_{n \to \infty} \sum_{k=1}^{n} a_k \mathrm{m}(I_k).$$

By noting that $\mathrm{m}(I_k) = 1$ for all k and subsuming the limit into the summation, we finally obtain

$$\int_E f = \sum_{k=1}^{\infty} a_k,$$

as desired. $\square$

Question 6. Royden 4.28.

Solution. Let f be integrable over E and C a measurable subset of E. We wish to show that $\int_C f = \int_E f \cdot \chi_C$. First, observe that $f \cdot \chi_C$ is measurable. Furthermore, we have $|f \cdot \chi_C| \leq f$ on E. Hence, by the integral comparison test, we have that $f \cdot \chi_C$ is integrable over E. It follows that

$$\int_E f \cdot \chi_C = \int_E (f \cdot \chi_C)^+ - \int_E (f \cdot \chi_C)^-.$$

By the additivity over domain of integration for nonnegative measurable functions, we have

$$\begin{aligned} \int_E f \cdot \chi_C &= \int_{E \setminus C} (f \cdot \chi_C)^+ + \int_C (f \cdot \chi_C)^+ \\ &- \int_{E \setminus C} (f \cdot \chi_C)^- - \int_C (f \cdot \chi_C)^-. \end{aligned}$$

We can write $(f \cdot \chi_C)^+$ and $(f \cdot \chi_C)^-$ explicitly as follow:

$$\begin{aligned} (f \cdot \chi_C)^+ &= \begin{cases} \max(f(x), 0) & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases} \\ (f \cdot \chi_C)^- &= \begin{cases} \max(-f(x), 0) & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases} \end{aligned}$$

Hence, the above integral can be simplified to

$$\int_E f \cdot \chi_C = \int_C (f \cdot \chi_C)^+ - \int_C (f \cdot \chi_C)^-,$$

which simplifies to

$$\int_E f \cdot \chi_C = \int_C (f \cdot \chi_C),$$

as desired. $\square$