
Real Variables: Problem Set XI

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Abstract

This work contains solutions to the problem set XI of Real Variables 2015 at NYU.

1 Solutions

Question Royden 17-6.

6. Let (X, \mathcal{M}, μ) be a measure space and X_0 belong to \mathcal{M} . Define \mathcal{M}_0 to be the collection of sets in \mathcal{M} that are subsets of X_0 and μ_0 the restriction of μ to \mathcal{M}_0 . Show that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

Solution. We first show that (X_0, \mathcal{M}_0) is a measurable space. To this end, we must show that \mathcal{M}_0 is a σ -algebra of X_0 . As \emptyset and X_0 belong to \mathcal{M} , are subsets of X_0 , it follows that \emptyset and X_0 belong to \mathcal{M}_0 . Let $\{A_n\}_{n=1}^{\infty}$ be a countable collections of sets in \mathcal{M}_0 . As $A_n \subseteq X_0$ for all n , we have $\bigcup_{n=1}^{\infty} A_n \subseteq X_0$. Furthermore, as \mathcal{M} is a σ -algebra, and $A_n \in \mathcal{M}$ for all n , we also have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$. Hence, it follows that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}_0$. Now, let A be a set, belonging to \mathcal{M}_0 . Then, as $X_0 \setminus A$ is a subset of X_0 , and X_0 and A belong to \mathcal{M} , which gives $X_0 \setminus A \in \mathcal{M}$, we have $X_0 \setminus A$ belongs to \mathcal{M}_0 . Hence, we have shown that \mathcal{M}_0 is a σ -algebra, and (X_0, \mathcal{M}_0) is a measurable space. Now, it remains to be shown that the restricted map μ_0 has the properties of a measure. First, observe that $\emptyset \in \mathcal{M}_0$ and $\mu_0(\emptyset) = \mu(\emptyset) = 0$. Now, let $\{A_n\}$ be a countable disjoint sets from \mathcal{M}_0 . Since $A_n \in \mathcal{M}$ for all n , by the countable additivity of μ and the fact that \mathcal{M}_0 is a σ -algebra, it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \mu_0(A_n) &= \sum_{n=1}^{\infty} \mu(A_n) \\ &= \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \\ &= \mu_0\left(\bigcup_{n=1}^{\infty} A_n\right). \end{aligned}$$

Therefore, we have shown that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space. □

Question Royden 17-15.

15. Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$, where α and β are real numbers. Show that

$$|\alpha\nu| = |\alpha||\nu| \text{ and } |\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|,$$

where $\nu \leq \mu$ means $\nu(E) \leq \mu(E)$ for all measurable sets E .

Solution. Let (X, \mathcal{M}) be a measurable space. Let ν_1 and ν_2 be two finite signed measures on (X, \mathcal{M}) . Consider a set function $\alpha\nu_1 + \beta\nu_2$ on \mathcal{M} , for $\alpha, \beta \in \mathbb{R}$, which is defined by

$$\alpha\nu_1 + \beta\nu_2(E) = \alpha\nu_1(E) + \beta\nu_2(E),$$

for $E \in \mathcal{M}$. As ν_1 and ν_2 only take finite values, it also follows that $\alpha\nu_1 + \beta\nu_2$ also assumes only finite values, as an addition of two finite values are finite. Furthermore, it follows that

$$\begin{aligned} \alpha\nu_1 + \beta\nu_2(\emptyset) &= \alpha\nu_1(\emptyset) + \beta\nu_2(\emptyset) \\ &= 0 + 0 = 0, \end{aligned}$$

as ν_1 and ν_2 are signed measures. Furthermore, the countable additivity property holds, as for any countable disjoint collection $\{E_k\}$ from \mathcal{M} , by linearity of limit, and countable additivity of ν_1 and ν_2 , it follows that

$$\begin{aligned} \alpha\nu_1 + \beta\nu_2\left(\bigcup_{k=1}^{\infty} E_k\right) &= \alpha\nu_1\left(\bigcup_{k=1}^{\infty} E_k\right) + \beta\nu_2\left(\bigcup_{k=1}^{\infty} E_k\right) \\ &= \alpha \sum_{k=1}^{\infty} \nu_1(E_k) + \beta \sum_{k=1}^{\infty} \nu_2(E_k) \\ &= \sum_{k=1}^{\infty} \alpha\nu_1(E_k) + \beta\nu_2(E_k) \\ &= \sum_{k=1}^{\infty} \alpha\nu_1 + \beta\nu_2(E_k). \end{aligned}$$

Hence $\alpha\nu_1 + \beta\nu_2$ is a finite measure.

Let (X, \mathcal{M}) be a measurable space and ν be a finite measure on the space. Consider $|\alpha\nu|$. It follows that for $E \in \mathcal{M}$, we have

$$|\alpha\nu(E)| = |\alpha||\nu(E)|.$$

Hence, $|\alpha\nu| = |\alpha||\nu|$. Now, let ν_1 and ν_2 be finite signed measures on (X, \mathcal{M}) . By the triangle inequality of reals, it follows that for any $E \in \mathcal{M}$,

$$\begin{aligned} |\nu_1 + \nu_2(E)| &= |\nu_1(E) + \nu_2(E)| \\ &\leq |\nu_1(E)| + |\nu_2(E)|. \end{aligned}$$

Hence, $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. □

Question Royden 17-17.

17. Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$ and $\mu \vee \nu = \mu + \nu - \mu \wedge \nu$.
- (i) Show that the signed measure $\mu \wedge \nu$ is smaller than μ and ν but larger than any other signed measure that is smaller than μ and ν .
 - (ii) Show that the signed measure $\mu \vee \nu$ is larger than μ and ν but smaller than any other measure that is larger than μ and ν .
 - (iii) If μ and ν are positive measures, show that they are mutually singular if and only if $\mu \wedge \nu = 0$.

Solution. (i) Let u and v be finite signed measures on (X, \mathcal{M}) . We wish to show that $u \wedge v \leq u$. By writing out $u \wedge v$ term, multiplying by 2, and re-arranging the terms trivially, we see that $u \wedge v \leq u$ is equivalent to $u - v \leq |u - v|$. As an absolute value of a real number is larger than the number itself, we see that $u \wedge v \leq u$ holds. By symmetry, we also see that $u \wedge v \leq v$ as well.

Now, let t be a signed measure such that $t \leq u$ and $t \leq v$. It follows that $t \leq \min(u, v)$. Assume without loss of generality that $\min(u, v) = u$. It follows that

$$\begin{aligned} \frac{1}{2}(u + v|u - v|) &= \frac{1}{2}(2u) \\ &= u. \end{aligned}$$

Therefore, $\min(u, v) = \frac{1}{2}(u + v - |u - v|)$. Hence, $t \leq u \wedge v$.

(ii) Let u and v be finite signed measures on (X, \mathcal{M}) . In the above solution, we have shown that $u \wedge v = \min(u, v)$. Hence, it follows that $u + v - u \wedge v = \max(u, v)$. Therefore, we obtain that $u \vee v \geq u$ and $u \vee v \geq v$. Again, as for any t such that $t \geq u$ and $t \geq v$, it follows that $t \geq \max(u, v) = u \vee v$.

(iii) Let that u and v are finite measures, and assume that $u \wedge v = \frac{1}{2}(u + v - |u - v|) = 0$. For any $E \in \mathcal{M}$, it follows that

$$u(E) + v(E) = |u(E) - v(E)|$$

Suppose that $u(E)$ and $v(E)$ are both strictly positive. It follows that $u(E) + v(E) > \max(u(E), v(E))$ and $|u(E) - v(E)| < \max(u(E), v(E))$, which yields a contradiction with the above equality. Hence, we have that at least one of $u(E)$ or $v(E)$ is 0. Assume without loss of generality that $u(E) = 0$. Now, with the same argument with respect to E^c , it follows that either $u(E^c)$ and $v(E^c)$ is 0. If $v(E^c) = 0$, then u and v are mutually singular. Now, consider the remaining case of $u(E^c) = 0$. By the additivity property of a signed measure, we have that $u(X) = 0$. Hence, $u(X) = 0$ and $v(\emptyset) = 0$. Therefore, we have shown that u and v are mutually singular.

Now, assume that u and v are mutually singular measures. Hence, there exists a disjoint measurable set A and B such that $A \cup B = X$ and $u(A) = 0$ and $v(B) = 0$. Let $E \in \mathcal{M}$. Then, it follows that

□

Question Royden 18-50.

50. Establish the uniqueness of the function f in the Radon-Nikodym Theorem.

Solution. Let (X, \mathcal{M}) be a measurable space, and u, v be σ -finite measures on the space with $v \ll u$. Let f and g be the non-negative measurable functions for which,

$$\begin{aligned} v(E) &= \int_E f du \\ &= \int_E g du, \end{aligned}$$

for all $E \in \mathcal{M}$. By the linearity of integration for non-negative measures, we obtain that

$$\int_E f - g du = 0,$$

for all $E \in \mathcal{M}$. Now, we prove that if $\int_E f du = 0$ for every measurable subset E of X , then $f = 0$ a.e. on X . We prove the case for f non-negative and the result can be extended through the f^+ and f^- decomposition.

Question Royden 18-54.

54. Let μ , ν , and λ be σ -finite measures on the measurable space (X, \mathcal{M}) .

- (i) If $\nu \ll \mu$ and f is a nonnegative function on X that is measurable with respect to \mathcal{M} , show that

$$\int_X f \, d\nu = \int_X f \left[\frac{d\nu}{d\mu} \right] d\mu.$$

- (ii) If $\nu \ll \mu$ and $\lambda \ll \mu$, show that

$$\frac{d(\nu + \lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \text{ a.e. } [\mu].$$

- (iii) If $\nu \ll \mu \ll \lambda$, show that

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda} \text{ a.e. } [\lambda].$$

Solution.

Question Royden 18-55.

55. Let μ , ν , ν_1 , and ν_2 be measures on the measurable space (X, \mathcal{M}) .

- (i) Show that if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.
- (ii) Show that if ν_1 and ν_2 are singular with respect to μ , then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.
- (iii) Show that if ν_1 and ν_2 are absolutely continuous with respect to μ , then, for any $\alpha \geq 0, \beta \geq 0$, so is the measure $\alpha\nu_1 + \beta\nu_2$.
- (iv) Prove the uniqueness assertion in the Lebesgue decomposition.

Solution. (i) Assume $\nu \perp \mu$, and $\nu \ll \mu$. It follows that there exists a pair of measurable sets A and B such that $\mu(A) = 0$ and $\nu(B) = 0$. By the absolute continuity of ν with respect to μ , it follows that $\nu(A) = 0$. By finite additivity of measure, we obtain that $\nu(X) = 0$. In other words, ν is a zero measure.

(ii) Assume $\mu \perp \nu$. We show that $\mu \perp \alpha\nu$, for $\alpha \geq 0$ holds as well. As $\mu \perp \nu$, there exists A and B from \mathcal{M} such that $\mu(A) = 0$ and $\nu(B) = 0$ and $A \cup B = X$. Observe that $\alpha\nu(B) = 0$ as well. Hence, $\mu \perp \alpha\nu$. Now, assume $\mu \perp \nu_1$ and $\mu \perp \nu_2$. We show that $\mu \perp \nu_1 + \nu_2$. Let (A_1, B_1) and (A_2, B_2) be the pairs of sets that grant the mutual singularities. Then, observe that by finite additivity of measure, we have $\mu(A_1 \cup A_2) = 0$. Furthermore, observe that by monotonicity of measure, $\nu_1 + \nu_2(B_1 \cap B_2) = 0$. Since $A_1 \cup A_2 \cup (B_1 \cap B_2) = (A_1 \cup A_2 \cup B_1) \cap (A_1 \cup A_2 \cup B_2) = X \cap X = X$, we have that μ and $\nu_1 + \nu_2$ are mutually singular. Hence, the claim is proven.

(iii) Assume $\nu \ll \mu$. Then, for any $E \in \mathcal{M}$ such that $\mu(E) = 0$, we have $\nu(E) = 0$. Since $\nu(E) = 0$, we also have $\alpha\nu(E) = 0$. Hence, $\alpha\nu \ll \mu$. Now, assume $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$. It follows that for any $E \in \mathcal{M}$ such that $\mu(E) = 0$, we have $\nu_1(E) = 0$ and $\nu_2(E) = 0$. Therefore, $\nu_1 + \nu_2(E) = 0$. Hence, the claim is proven.

(iv) Let ν_0 and ν_1 be the Lebesgue decomposition measures, where ν_0 is mutually singular and ν_1 is absolutely continuous with respect to ν . Consider ν'_0 and ν'_1 with the same set-up. We have that $\nu_0 + \nu_1 = \nu'_0 + \nu'_1$. It follows that

$$\nu_0 - \nu'_0 = \nu'_1 - \nu_1.$$

Observe that $\nu_0 - \nu'_0$ and $\nu_1 - \nu'_1$ are singular and absolutely continuous with respect to ν from (ii) and (iii). Hence, by (i), we have that $\nu_0 = \nu'_0$ and $\nu_1 = \nu'_1$. Therefore, we have shown that uniqueness of the decomposition. \square