Real Variables: Problem Set IV

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Abstract

This work contains solutions to the problem set IV of Real Variables 2015 at NYU.

1 Solutions

Question 1. Royden 4.31.
Solution.

Question 2. Royden 4.44.
Solution.

Question 4. Royden 4.52.
Solution. (a) Consider the following family of functions:

$$\mathscr{F} = \{n\chi_{[0,\frac{1}{n}]}\}_{n=1}^{\infty}.$$

Observe that for each n, $n\chi_{[0,\frac{1}{n}]}$ is integrable and $\int_0^1 |n_\chi[0,\frac{1}{n}]| = 1$. The family $\mathscr{F}$, however, fails to be uniformly integrable. Fix $\epsilon = \dfrac{1}{2}$. Then, for any $\delta > 0$, by the Archimedean property of the reals, there exists n, such that $\dfrac{1}{n} < \delta$. Since the interval $[0, \dfrac{1}{n}]$ is measurable, has a measure smaller than δ, and $\int_0^{\frac{1}{n}} n\chi_{[0,\frac{1}{n}]} = 1 > \dfrac{1}{2}$, we have that $\mathscr{F}$ is not uniformly integrable. Hence, by a counter example, we have shown that under the given assumptions, the family of functions need not be uniformly integrable.

(b) We claim that $\mathscr{F}$ with the given assumption is uniformly integrable. Note that continuity implies integrability. Fix $\epsilon > 0$. Let $f \in \mathscr{F}$. Then, for any measurable set $E \subseteq [0,1]$ with $mE < \delta$ with, by using the $|f| \leq 1$ bound, we obtain

$$\begin{aligned}
\int_E f &\leq \int_E |f| \\
&\leq \int_E 1 \\
&= mE \\
&\leq \delta
\end{aligned}$$

By letting $\delta = \epsilon$, we have $\int_E f \leq \epsilon$. Since ϵ and f were arbitrary, we have shown that $\mathscr{F}$ is uniformly integrable.

(c) Let $\mathscr{F}$ be the family of functions f on $[0,1]$, each of which is integrable over $[0,1]$ and has $\int_a^b |f| \leq b-a$ for all $[a,b] \subseteq [0,1]$. We claim that $\mathscr{F}$ is uniformly integrable. Fix $\epsilon > 0$ and fix $f \in \mathscr{F}$. Let $A \subseteq [0,1]$ be a measurable set such that $mA < \delta$ By the outer approximation of measurable set by open sets, there exists an open set O such that $A \subseteq O$ and $m(O \backslash A) \leq \frac{\epsilon}{2}$. Observe that O can be written as a countable union of disjoint open intervals, which gives $O = \cup_{i=1}^{\infty}(a_i, b_i)$. From the monotonicity and excision property of measure, and countable additivity over domain property of integration, it follows that

$$\begin{aligned}
\int_A |f| &\leq \int_O |f| \\
&\leq \int_{\cup_{i=1}^{\infty}(a_i,b_i)} |f| \\
&= \sum_{i=1}^{\infty} \int_{(a_i,b_i)} |f| \\
&\leq \sum_{i=1}^{\infty} \int_{[a_i,b_i])} |f| \\
&\leq \sum_{i=1}^{\infty} b_i - a_i \\
&= mO \\
&= m(O \setminus A) + m(A) \\
&\leq \frac{\epsilon}{2} + \delta.
\end{aligned}$$

Define $\delta = \frac{\epsilon}{2}$ then, we have if A is measurable, and $mA < \delta$, then $\int_A |f| < \epsilon$. Since ϵ and f were arbitrary, we have that $\mathscr{F}$ is uniformly integrable. $\square$