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# Functional Analysis:

## Problem Set I

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### Abstract

This work contains solutions to the exercises of the problem set I.

#### Question 1.

**1.1** *Properties of the duality map.*

Let  $E$  be an n.v.s. The duality map  $F$  is defined for every  $x \in E$  by

$$F(x) = \{f \in E^*; \|f\| = \|x\| \text{ and } \langle f, x \rangle = \|x\|^2\}.$$

1. Prove that

$$F(x) = \{f \in E^*; \|f\| \leq \|x\| \text{ and } \langle f, x \rangle = \|x\|^2\}$$

and deduce that  $F(x)$  is nonempty, closed, and convex.

2. Prove that if  $E^*$  is strictly convex, then  $F(x)$  contains a single point.

3. Prove that

$$F(x) = \left\{ f \in E^*; \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \geq \langle f, y - x \rangle \quad \forall y \in E \right\}.$$

4. Deduce that

$$\langle F(x) - F(y), x - y \rangle \geq 0 \quad \forall x, y \in E,$$

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20 1 The Hahn–Banach Theorems. Introduction to the Theory of Conjugate Convex Functions

and more precisely that

$$\langle f - g, x - y \rangle \geq 0 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

Show that, in fact,

$$\langle f - g, x - y \rangle \geq (\|x\| - \|y\|)^2 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

5. Assume again that  $E^*$  is strictly convex and let  $x, y \in E$  be such that

$$\langle F(x) - F(y), x - y \rangle = 0.$$

Show that  $Fx = Fy$ .

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**Solution.**

(1) The first set equality follows as

$$f \in E^* \quad \text{and} \quad \langle f, x \rangle = \|x\|^2 \implies \|f\| \geq \|x\|,$$

because otherwise

$$|\langle f, x \rangle| = \|x\|^2 > \|f\|\|x\|,$$

which is absurd. Now, by Corollary 1.3, it follows that  $F(x)$  is non-empty.

We show that  $F(x)$  is convex. Let  $f, g \in F(x)$  and  $t \in [0, 1]$ . Then, it follows that

$$\langle tf + (1-t)g, x \rangle = t\langle f, x \rangle + (1-t)\langle g, x \rangle = \|x\|^2$$

and

$$\|tf + (1-t)g\| \leq t\|f\| + (1-t)\|g\| \leq \|x\|,$$

so  $tf + (1-t)g \in F(x)$  and  $F(x)$  is convex.

We show that  $F(x)$  is closed. Let  $f \in E^*$  such that there exists  $\{f_n\} \subset F(x)$  with  $f_n \rightarrow f$ . As convergence in dual norm implies pointwise convergence, we have

$$\|x\|^2 = \langle f_n, x \rangle \rightarrow \langle f, x \rangle \quad \text{and} \quad \langle f, x \rangle = \|x\|^2.$$

Also, as  $\|f_n - f\| \rightarrow 0$ , and by reverse-triangle inequality, we have

$$\|f_n\| \rightarrow \|f\| \quad \text{and} \quad \|f\| \leq \|x\|,$$

which shows that  $f \in F(x)$ , and consequently that  $F(x)$  is closed.

(2)

## Question 2.

**1.2** Let  $E$  be a vector space of dimension  $n$  and let  $(e_i)_{1 \leq i \leq n}$  be a basis of  $E$ . Given  $x \in E$ , write  $x = \sum_{i=1}^n x_i e_i$  with  $x_i \in \mathbb{R}$ ; given  $f \in E^*$ , set  $f_i = \langle f, e_i \rangle$ .

1. Consider on  $E$  the norm

$$\|x\|_1 = \sum_{i=1}^n |x_i|.$$

- (a) Compute explicitly, in terms of the  $f_i$ 's, the dual norm  $\|f\|_{E^*}$  of  $f \in E^*$ .  
(b) Determine explicitly the set  $F(x)$  (duality map) for every  $x \in E$ .

2. Same questions but where  $E$  is provided with the norm

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

3. Same questions but where  $E$  is provided with the norm

$$\|x\|_2 = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2},$$

and more generally with the norm

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}, \quad \text{where } p \in (1, \infty).$$

## Solution.

**Question 3.**

1.3 Let  $E = \{u \in C([0, 1]; \mathbb{R}); u(0) = 0\}$  with its usual norm

$$\|u\| = \max_{t \in [0, 1]} |u(t)|.$$

Consider the linear functional



1.4 Exercises for Chapter 1

21

$$f : u \in E \mapsto f(u) = \int_0^1 u(t) dt.$$

1. Show that  $f \in E^*$  and compute  $\|f\|_{E^*}$ .
2. Can one find some  $u \in E$  such that  $\|u\| = 1$  and  $f(u) = \|f\|_{E^*}$ ?

**Solution.**

(1) By linearity of integration, it follows that  $f$  defined is linear. Since  $f$  is linear, it suffices to show continuity at 0. Fix  $\epsilon > 0$ . Then, it follows that, with  $\delta = \frac{\epsilon}{2}$ ,

$$u \in B(0, \delta) \implies \left| \int_0^1 u(t) dt \right| \leq \int_0^1 |u(t)| dt \leq \delta < \epsilon.$$

Therefore  $f$  is continuous. Now, we compute its dual norm explicitly. Note that, for any  $u \in E$ ,

$$| \langle f, u \rangle | = \left| \int_0^1 u(t) dt \right| \leq \int_0^1 |u(t)| dt \leq \|u\|,$$

so  $\|f\| \leq 1$ . We now show the reverse inequality. Recall that

$$\|f\| = \sup_{\|u\|=1} | \langle f, u \rangle |$$

Fix  $\epsilon > 0$ . Set  $u \in C[0, 1]$  by

$$t \mapsto \frac{1}{\epsilon} X_{[0, \epsilon]}(t) + X_{(\epsilon, 1]}(t) \quad (t \in [0, 1])$$

Then, it follows that

$$\langle f, u \rangle = \int_0^1 u(t) dt = 1 - \frac{\epsilon}{2}.$$

Therefore, it follows that  $\|f\| \geq 1$ , and we have completed in showing that  $\|f\| = 1$ . □

(2)

**Question 4.**

**1.4** Consider the space  $E = c_0$  (sequences tending to zero) with its usual norm (see Section 11.3). For every element  $u = (u_1, u_2, u_3, \dots)$  in  $E$  define

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n.$$

1. Check that  $f$  is a continuous linear functional on  $E$  and compute  $\|f\|_{E^*}$ .
2. Can one find some  $u \in E$  such that  $\|u\| = 1$  and  $f(u) = \|f\|_{E^*}$ ?

**Solution.**

**(1)** Fix  $u \in C_0$  such that  $\|u\| = \sup_n |u_n| = 1$ , it follows that

$$|u_n| \leq 1$$

for all  $n \geq 1$ , so

$$|f(u)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} |u_n| = 1.$$

Therefore,

$$\|f\| = \sup_{\|u\|=1} |f(u)| \leq 1.$$

Now, fix  $\epsilon > 0$ . Choose  $N > 1$  such that

$$n \geq N \implies \sum_{k=1}^n \frac{1}{2^k} > 1 - \epsilon.$$

Set  $u \in c_0$  as

$$u_n = 1 \ (n \leq N) \text{ and } u_n = 0 \ (n > N).$$

Then,  $u \in c_0$ ,  $\|u\| = 1$ , and  $|f(u)| > 1 - \epsilon$ . Therefore, it follows that

$$1 - \epsilon < \|f\|$$

for any  $\epsilon > 0$ , so  $\|f\| \geq 1$ , which combined with the previous estimate gives  $\|f\| = 1$ . □

**(2)** Suppose for sake of contradiction that there exists  $u \in c_0$ , such that

$$\|u\| = 1 \text{ and } f(u) = 1.$$

Choose  $N > 1$  such that

$$n \geq N \implies u_n < \frac{1}{2}.$$

Then,

$$f(u) < \sum_{n=1}^{N-1} \frac{1}{2^n} u_n + \frac{1}{2} \sum_{n=N}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{N-1} \frac{1}{2^n} u_n + \frac{1}{2^{N-1}}.$$

Since  $\|u\| = 1$ , continuing the above estimate gives

$$f(u) < 1 - \frac{1}{2^{N-1}} + \frac{1}{2^{N-1}} = 1,$$

which is absurd. □

**Question 5.**

1.5 Let  $E$  be an infinite-dimensional n.v.s.

1. Prove (using Zorn's lemma) that there exists an algebraic basis  $(e_i)_{i \in I}$  in  $E$  such that  $\|e_i\| = 1 \ \forall i \in I$ .

Recall that an algebraic basis (or Hamel basis) is a subset  $(e_i)_{i \in I}$  in  $E$  such that every  $x \in E$  may be written uniquely as

$$x = \sum_{i \in J} x_i e_i \text{ with } J \subset I, J \text{ finite.}$$

2. Construct a linear functional  $f : E \rightarrow \mathbb{R}$  that is not continuous.
3. Assuming in addition that  $E$  is a Banach space, prove that  $I$  is not countable.

[**Hint:** Use Baire category theorem (Theorem 2.1).]

**Solution.**

(1) Consider subsets of  $E$  that only contain linearly independent vectors, denoted by  $\mathcal{L}$ . We impose the order by the usual set inclusion. Then, it is clear that  $\mathcal{L}$  is inductive, as for any totally ordered subset  $\mathcal{T} \subset \mathcal{L}$ , it follows that  $\bigcup_{T \in \mathcal{T}} T \in \mathcal{L}$  and is an upper bound of  $\mathcal{T}$ . Hence, there exists a maximal element of  $\mathcal{L}$ ,  $\mathcal{A}$ . We claim that  $\mathcal{A}$  is an algebraic basis. Normalize each vector in  $\mathcal{A}$ , then we are done.

(2) Choose an normalized algebraic basis  $\{e_i\}_{i \in I}$ , and choose a countable subset and re-index them by  $\mathbb{N}$ , so that  $C = \{e_n\}_{n \in \mathbb{N}} \subset \{e_i\}_{i \in I}$ . Define  $f : E \rightarrow \mathbb{R}$  by

$$e_n \mapsto n \ (n \in \mathbb{N}),$$

and

$$e_i \mapsto 0 \ (i \notin C),$$

with the extension given by

$$x = \sum_J x_j e_j \mapsto \sum_J x_j f(e_j) \ (x \in E)$$

where  $J$  is given by the unique basis representation given by the algebraic basis. It is clear that  $f$  is linear and  $\sup_{\|x\|=1} |f(x)|$  is not bounded.

(3)

**Question 6.**

1.6 Let  $E$  be an n.v.s. and let  $H \subset E$  be a hyperplane. Let  $V \subset E$  be an affine subspace containing  $H$ .

1. Prove that either  $V = H$  or  $V = E$ .
2. Deduce that  $H$  is either closed or dense in  $E$ .

**Solution.**

ddd

**Question 7.**

1.7 Let  $E$  be an n.v.s. and let  $C \subset E$  be convex.

1. Prove that  $\overline{C}$  and  $\text{Int } C$  are convex.
2. Given  $x \in C$  and  $y \in \text{Int } C$ , show that  $tx + (1 - t)y \in \text{Int } C \quad \forall t \in (0, 1)$ .
3. Deduce that  $\overline{C} = \overline{\text{Int } C}$  whenever  $\text{Int } C \neq \emptyset$ .

**Solution.**

(1) We first show that  $\overline{C}$  is convex. Let  $x, y \in \overline{C}$ , and  $t \in [0, 1]$ . Choose,  $\{x_n\}, \{y_n\} \subset C$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . By convexity of  $C$ , and linearity of limit, it follows that

$$\{tx_n + (1 - t)y_n\} \subset C \quad \text{and} \quad tx_n + (1 - t)y_n \rightarrow tx + (1 - t)y.$$

Therefore,  $tx + (1 - t)y \in \overline{C}$ , which proves the convexity of  $\overline{C}$ . We now show that  $\text{Int } C$  is convex. Let  $x, y \in \text{Int } C$ , and  $t \in [0, 1]$ . By convexity of  $C$ ,

$$tx + (1 - t)y \in C$$

We now show that  $\text{Int } C$  is convex. Let  $x, y \in \text{Int } C$  and  $t \in (0, 1)$ .

(2) Suppose  $x \in C, y \in \text{Int } C$ , and  $t \in (0, 1)$ .

(3) It is trivial that  $\overline{\text{Int } C} \subset \overline{C}$ . Hence, it suffices to show that  $\overline{C} \subset \overline{\text{Int } C}$ .



**Question 8.**

**1.8** Let  $E$  be an n.v.s. with norm  $\| \cdot \|$ . Let  $C \subset E$  be an open convex set such that  $0 \in C$ . Let  $p$  denote the gauge of  $C$  (see Lemma 1.2).

1. Assuming  $C$  is symmetric (i.e.,  $-C = C$ ) and  $C$  is bounded, prove that  $p$  is a norm which is equivalent to  $\| \cdot \|$ .

22 1 The Hahn–Banach Theorems. Introduction to the Theory of Conjugate Convex Functions

2. Let  $E = C([0, 1]; \mathbb{R})$  with its usual norm

$$\|u\| = \max_{t \in [0, 1]} |u(t)|.$$

Let

$$C = \left\{ u \in E; \int_0^1 |u(t)|^2 dt < 1 \right\}.$$

Check that  $C$  is convex and symmetric and that  $0 \in C$ . Is  $C$  bounded in  $E$ ? Compute the gauge  $p$  of  $C$  and show that  $p$  is a norm on  $E$ . Is  $p$  equivalent to  $\| \cdot \|$ ?

**Solution.**

- (1) We first show that  $p$  is in fact a norm. By properties of any gauge of  $C$ , it suffices to show

$$p(x) = 0 \iff x = 0.$$

If  $x = 0$ , then

$$\alpha > 0 \implies \alpha^{-1}x = 0 \in C,$$

so  $p(x) = 0$ . Conversely, suppose that  $p(x) = 0$ . Firstly, let

$$I = \{ \lambda > 0; \lambda^{-1}x \in C \}, \text{ and } 0 < \alpha < \beta.$$

We claim that

$$\beta \notin I \implies \alpha \notin I.$$

We prove the contrapositive. Suppose  $\alpha \in I$ . Then,  $\alpha^{-1}x \in C$ . By convexity of  $C$ , it follows that

$$\frac{\beta^{-1}}{\alpha^{-1}}\alpha^{-1}x + (1 - \frac{\beta^{-1}}{\alpha^{-1}})0 = \beta^{-1}x \in C,$$

so  $\beta \in I$ . Therefore, to prove  $p(x) > 0$ , it suffices to show that there is a constant  $k > 0$  such that  $k^{-1}x \notin C$ . Now, suppose for sake of contradiction that  $x \neq 0$ . Choose  $r$  large enough such that  $C \subset B(r, 0)$  strictly. Then,

$$\frac{r}{\|x\|}x \in C \text{ and } 0 < \frac{\|x\|}{r} \in I,$$

which as discussed above implies that  $p(x) > 0$ . Hence,  $x = 0$  as required.

- (2) We first check convexity of  $C$ . Let  $u, v \in C$  and  $\lambda \in [0, 1]$ . Then,

$$\begin{aligned} \int_0^1 |\lambda u + (1 - \lambda)v|^2 dt &\leq \int_0^1 (\lambda|u| + (1 - \lambda)|v|)^2 \\ &\leq \lambda^2 \int_0^1 |u|^2 + 2\lambda(1 - \lambda) \int_0^1 |u||v| + (1 - \lambda)^2 \int_0^1 |v|^2 \\ &< \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda) = 1, \end{aligned}$$

where the second last inequality holds by Cauchy-Schwarz. Now, 0 is clearly in  $C$  and  $C$  is symmetric, because

$$\int_0^1 |u(t)|^2 dt = \int_0^1 |-u(t)|^2 dt.$$

We claim that  $C$  is not bounded. Fix  $r > 0$ . Set

$$f = \sqrt{t}X_{[0, \frac{1}{2r}]} + (r - \sqrt{t})X_{(\frac{1}{2r}, \frac{1}{r}]}$$

We now compute the gauge  $p$  of  $C$ . For  $u \in E$ , it follows that

$$\begin{aligned} p(u) &= \inf\{\lambda > 0 ; \lambda^{-1}u \in C\} \\ &= \inf\{\lambda > 0 ; \lambda^{-2} \int_0^1 |u(t)|^2 dt < 1\} \\ &= \inf\{\lambda > 0 ; \int_0^1 |u(t)|^2 dt < \lambda^2\} \end{aligned}$$

**Question 9.**

**1.9** *Hahn-Banach in finite-dimensional spaces.*

Let  $E$  be a finite-dimensional normed space. Let  $C \subset E$  be a nonempty convex set such that  $0 \notin C$ . We claim that there always exists some hyperplane that separates  $C$  and  $\{0\}$ .

[Note that every hyperplane is closed (why?). The main point in this exercise is that no additional assumption on  $C$  is required.]

1. Let  $(x_n)_{n \geq 1}$  be a countable subset of  $C$  that is dense in  $C$  (why does it exist?). For every  $n$  let

$$C_n = \text{conv}\{x_1, x_2, \dots, x_n\} = \left\{ x = \sum_{i=1}^n t_i x_i; t_i \geq 0 \forall i \text{ and } \sum_{i=1}^n t_i = 1 \right\}.$$

Check that  $C_n$  is compact and that  $\bigcup_{n=1}^{\infty} C_n$  is dense in  $C$ .

2. Prove that there is some  $f_n \in E^*$  such that

$$\|f_n\| = 1 \text{ and } \langle f_n, x \rangle \geq 0 \quad \forall x \in C_n.$$

3. Deduce that there is some  $f \in E^*$  such that

$$\|f\| = 1 \text{ and } \langle f, x \rangle \geq 0 \quad \forall x \in C.$$

Conclude.

4. Let  $A, B \subset E$  be nonempty disjoint convex sets. Prove that there exists some hyperplane  $H$  that separates  $A$  and  $B$ .

**Solution.**

We record two fundamental facts about finite dimensional spaces. First, linearity of a map on a finite dimensional space implies continuity. Second, every finite dimensional space is separable.

(1) Firstly, as  $\{x_n\} \subset \bigcup_{n=1}^{\infty} C_n$ , and  $\{x_n\}$  is dense in  $C$ ,  $\bigcup_{n=1}^{\infty} C_n$  is dense in  $C$ . Now, consider

$$A = \left\{ \lambda \in \mathbb{R}^n : \lambda_i \geq 0 \forall i, \sum_i \lambda_i = 1 \right\},$$

and

$$\Phi : \mathbb{R}^n \rightarrow E \text{ where } \lambda_i \mapsto \sum_i \lambda_i x_i.$$

It suffices to show that  $\Phi$  is continuous, because  $A$  is a compact subset of  $\mathbb{R}^n$ , whose image is  $C_n$ .  $\Phi$ , however, is trivially continuous, because it is linear.

(2) By the second geometric Hahn-Banach, applied with  $A = \{0\}$  and  $B = C_n$ , there exists  $f_n \in E^*$  not vanishing, such that

$$\langle f_n, x \rangle \geq 0 \quad \forall x \in C_n.$$

By normalizing, we also obtain  $\|f_n\| = 1$ .

(3) By compactness of the unit sphere in finite dimensional space, there exists  $\{f_{n_k}\}$  such that

$$f_{n_k} \rightarrow f \text{ such that } \|f\| = 1.$$

Since uniform convergence implies pointwise convergence and  $\{C_n\}$  are increasing, we have

$$\|f\| = 1 \text{ and } \langle f, x \rangle \geq 0 \quad \forall x \in \bigcup_n C_n,$$

which by density of  $C_k$  in  $C$  and continuity of  $f$ , gives

$$\|f\| = 1 \text{ and } \langle f, x \rangle \geq 0 \quad \forall x \in C,$$

as required.

(4) Set  $C = A - B$ . As  $A \cap B = \emptyset$ , we see that  $0 \notin C$ . We now show that  $C$  is still convex. Suppose  $x, y \in C$  and  $t \in [0, 1]$ . Then, there are  $a_x, a_y \in A$  and  $b_x, b_y \in B$  such that

$$x = a_x - b_x \quad \text{and} \quad y = a_y - b_y.$$

Then, it follows that

$$tx + (1 - t)y = t(a_x - b_x) + (1 - t)(a_y - b_y) = (ta_x + (1 - t)a_y) - (tb_x + (1 - t)b_y) \in C,$$

where the last inclusion holds by convexity of  $A$  and  $B$ . Hence,  $C$  is a nonempty convex set such that  $0 \notin C$ . Apply (3) to  $C$  and  $\{0\}$ , then there is  $f \in E^*$  such that

$$\|f\| = 1 \quad \text{and} \quad \langle f, x \rangle \geq 0 \quad \forall x \in C,$$

which implies that

$$\langle f, a - b \rangle \geq 0 \quad \text{and} \quad \langle f, a \rangle \geq \langle f, b \rangle,$$

for all  $a \in A$  and  $b \in B$ . Therefore, there exists a hyperplane that separates  $A$  and  $B$ . We see that in finite dimensional space topological assumptions on  $A$  and  $B$  can be relaxed to obtain an existence of a separating hyperplane.  $\square$

**Question 10.**

1.10 Let  $E$  be an n.v.s. and let  $I$  be any set of indices. Fix a subset  $(x_i)_{i \in I}$  in  $E$  and a subset  $(\alpha_i)_{i \in I}$  in  $\mathbb{R}$ . Show that the following properties are equivalent:

- (A) There exists some  $f \in E^*$  such that  $\langle f, x_i \rangle = \alpha_i \quad \forall i \in I$ .
- (B)  $\left\{ \begin{array}{l} \text{There exists a constant } M \geq 0 \text{ such that for each finite subset} \\ J \subset I \text{ and for every choice of real numbers } (\beta_i)_{i \in J}, \text{ we have} \\ \left| \sum_{i \in J} \beta_i \alpha_i \right| \leq M \left\| \sum_{i \in J} \beta_i x_i \right\|. \end{array} \right.$

Note that in the proof of (B)  $\Rightarrow$  (A) one may find some  $f \in E^*$  with  $\|f\|_{E^*} \leq M$ .  
**[Hint:** Try first to define  $f$  on the linear space spanned by the  $(x_i)_{i \in I}$ .]

**Solution.**  
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**Question 11.**

**1.11** Let  $E$  be an n.v.s. and let  $M > 0$ . Fix  $n$  elements  $(f_i)_{1 \leq i \leq n}$  in  $E^*$  and  $n$  real numbers  $(\alpha_i)_{1 \leq i \leq n}$ . Prove that the following properties are equivalent:

- (A) 
$$\begin{cases} \forall \varepsilon > 0 \exists x_\varepsilon \in E \text{ such that} \\ \|x_\varepsilon\| \leq M + \varepsilon \text{ and } \langle f_i, x_\varepsilon \rangle = \alpha_i \quad \forall i = 1, 2, \dots, n. \end{cases}$$
- (B) 
$$\left| \sum_{i=1}^n \beta_i \alpha_i \right| \leq M \left\| \sum_{i=1}^n \beta_i f_i \right\| \quad \forall \beta_1, \beta_2, \dots, \beta_n \in \mathbb{R}.$$

**[Hint:** For the proof of (B)  $\Rightarrow$  (A) consider first the case in which the  $f_i$ 's are linearly independent and imitate the proof of Lemma 3.3.]

Compare Exercises 1.10, 1.11 and Lemma 3.3.

**Solution.**

ddd

**Question 12.**

**1.12** Let  $E$  be a vector space. Fix  $n$  linear functionals  $(f_i)_{1 \leq i \leq n}$  on  $E$  and  $n$  real numbers  $(\alpha_i)_{1 \leq i \leq n}$ . Prove that the following properties are equivalent:

(A) There exists some  $x \in E$  such that  $f_i(x) = \alpha_i \quad \forall i = 1, 2, \dots, n$ .

(B)  $\left\{ \begin{array}{l} \text{For any choice of real numbers } \beta_1, \beta_2, \dots, \beta_n \text{ such that} \\ \sum_{i=1}^n \beta_i f_i = 0, \text{ one also has } \sum_{i=1}^n \beta_i \alpha_i = 0. \end{array} \right.$

**Solution.**

(A)  $\implies$  (B) is trivial. Choose  $x \in E$  with the condition in (A). Then,

$$0 = \sum_{i=1}^n \beta_i f_i(x) = \sum_{i=1}^n \beta_i \alpha_i,$$

for any  $\beta_1, \dots, \beta_n \in \mathbb{R}$ . We now show  $(A) \implies (B)$ . Fix  $x \in E$ . Then, there exists  $1 \leq i^* \leq n$  such that  $f_{i^*}(x) \neq \alpha_{i^*}$ . Choose  $\beta_k = 0$  if  $k \neq i^*$  and  $\beta_k = 1$  if  $k = i^*$ . Then, we obtain

$$0 = \sum_{i=1}^n \beta_i f_i(x) = \beta_{i^*} \alpha_{i^*},$$

**Question 14.**

1.14 Let  $E = \ell^1$  (see Section 11.3) and consider the two sets

$$X = \{x = (x_n)_{n \geq 1} \in E; x_{2n} = 0 \forall n \geq 1\}$$

and

$$Y = \left\{ y = (y_n)_{n \geq 1} \in E; y_{2n} = \frac{1}{2^n} y_{2n-1} \forall n \geq 1 \right\}.$$

1. Check that  $X$  and  $Y$  are closed linear spaces and that  $\overline{X + Y} = E$ .
2. Let  $c \in E$  be defined by

24 1 The Hahn–Banach Theorems. Introduction to the Theory of Conjugate Convex Functions

$$\begin{cases} c_{2n-1} = 0 & \forall n \geq 1, \\ c_{2n} = \frac{1}{2^n} & \forall n \geq 1. \end{cases}$$

Check that  $c \notin X + Y$ .

3. Set  $Z = X - c$  and check that  $Y \cap Z = \emptyset$ . Does there exist a closed hyperplane in  $E$  that separates  $Y$  and  $Z$ ?  
Compare with Theorem 1.7 and Exercise 1.9.
4. Same questions in  $E = \ell^p$ ,  $1 < p < \infty$ , and in  $E = c_0$ .

**Solution.**

(1) Let  $x \in \ell^1$  such that there is  $\{x_n\} \subset X$  with  $x_n \rightarrow x$ . Then,

$$\sum_k |(x)_k - (x_n)_k| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which implies

$$(x_n)_k \rightarrow (x)_k \text{ as } n \rightarrow \infty$$

for any  $k \geq 1$ . Since,  $(x_n)_{2k} = 0$  for any  $n, k \geq 1$ , it follows that  $x_{2k} = 0$  for any  $k \geq 1$  and  $x \in X$ , which shows that  $X$  is closed.

Now, we check that  $Y$  is closed. Let  $y \in \ell^1$  such that there is  $\{y_n\} \subset Y$  with  $y_n \rightarrow y$ . Then,

$$\sum_k |(y)_k - (y_n)_k| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

which implies

$$(y_n)_k \rightarrow (y)_k \text{ as } n \rightarrow \infty$$

for any  $k \geq 1$ . Since

$$\frac{1}{2^k} (y_n)_{2k-1} = (y_n)_{2k},$$

for all  $k, n \geq 1$ , it follows that

$$\lim_n (y_n)_{2k} = \lim_n \frac{1}{2^k} (y_n)_{2k-1} = \frac{1}{2^k} \lim_n (y_n)_{2k-1} = \frac{1}{2^k} (y)_{2k-1},$$

for all  $k \geq 1$ , so  $y \in Y$ , and  $Y$  is closed.



Now, we show that  $\overline{X + Y} = E$ . Let  $z \in E$ .

(2) Suppose that  $c \in X + Y$ , then it follows that there exists  $x \in X$  and  $y \in Y$ , such that, for any  $n \in \mathbb{N}$ ,

$$\frac{1}{2^n} = \frac{1}{2^n} y_{2n-1},$$

which implies

$$y_{2n-1} = 1,$$

which contradicts  $y \in l^1$ . So,  $x \notin X + Y$ .

(3)

(4)

**Question 16.**

1.16 Let  $E = \ell^1$ , so that  $E^* = \ell^\infty$  (see Section 11.3). Consider  $N = c_0$  as a closed subspace of  $E^*$ .

Determine

$$N^\perp = \{x \in E; \langle f, x \rangle = 0 \quad \forall f \in N\}$$

and

$$N^{\perp\perp} = \{f \in E^*; \langle f, x \rangle = 0 \quad \forall x \in N^\perp\}.$$

Check that  $N^{\perp\perp} \neq N$ .

**Solution.**

ddd

**Question 17.**

1.17 Let  $E$  be an n.v.s. and let  $f \in E^*$  with  $f \neq 0$ . Let  $M$  be the hyperplane  $[f = 0]$ .

1. Determine  $M^\perp$ .
2. Prove that for every  $x \in E$ ,  $\text{dist}(x, M) = \inf_{y \in M} \|x - y\| = \frac{|\langle f, x \rangle|}{\|f\|}$ .  
[Find a direct method or use Example 3 in Section 1.4.]
3. Assume now that  $E = \{u \in C([0, 1]; \mathbb{R}); u(0) = 0\}$  and that

$$\langle f, u \rangle = \int_0^1 u(t) dt, \quad u \in E.$$

Prove that  $\text{dist}(u, M) = |\int_0^1 u(t) dt| \quad \forall u \in E$ .

Show that  $\inf_{v \in M} \|u - v\|$  is never achieved for any  $u \in E \setminus M$ .

**Solution.**

ddd