
Functional Analysis:

Problem Set III

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Abstract

This work contains solutions to the exercises of the problem set III.

Question 1.

3.5 Let E be a Banach space and let $K \subset E$ be a subset of E that is compact in the strong topology. Let (x_n) be a sequence in K such that $x_n \rightharpoonup x$ weakly $\sigma(E, E^*)$. Prove that $x_n \rightarrow x$ strongly.

[Hint: Argue by contradiction.]

Solution.

Suppose $x_n \not\rightarrow x$ strongly. Then, there exists $\epsilon > 0$ and $\{x_{n_k}\}$ such that

$$|x_{n_k} - x| > \epsilon \quad (1)$$

for all $k \geq 1$. By the compactness of K in strong topology, there exists a further subsequence $\{x_{n_{k_l}}\}$ such that

$$\lim_{l \rightarrow \infty} x_{n_{k_l}} = y$$

for some $y \in K$. From (1), $y \neq x$. Now, since convergence in strong topology implies convergence in weak topology, we have

$$x_{n_{k_l}} \rightarrow_{\text{weak}} y \quad \text{as} \quad l \rightarrow \infty.$$

From our assumption, however, $x_n \rightarrow_{\text{weak}} x$ as $n \rightarrow \infty$, so by Hausdorff property of weak topology $x_{n_{k_l}} \rightarrow_{\text{weak}} x$ as $l \rightarrow \infty$. This contradicts the uniqueness of limit property of weak topology, which also arises from Hausdorff property of weak topology. We have a contradiction, and we are done.

□

Question 2.

3.9 Let E be a Banach space; let $M \subset E$ be a linear subspace, and let $f_0 \in E^*$. Prove that there exists some $g_0 \in M^\perp$ such that

$$\inf_{g \in M^\perp} \|f_0 - g\| = \|f_0 - g_0\|.$$

Two methods are suggested:

1. Use Theorem 1.12.
2. Use the weak* topology $\sigma(E^*, E)$.

Solution.

Observe that

$$\begin{aligned} M^\perp &= \{g \in E^* : \langle g, x \rangle = 0 \forall x \in M\} \\ &= \bigcap_{x \in M} \{g \in E^* : \langle g, x \rangle = 0\} = \bigcap_{x \in M} J(x)^{-1}(0) \end{aligned}$$

where J is the natural embedding. Hence, M^\perp is weak-* closed. Choose $r_0 > 0$ such that

$$A := B(f_0, r_0) \cap M^\perp \neq \emptyset.$$

where $B(f_0, r_0)$ denotes closed ball of radius r_0 around f_0 . Since closed balls are weak-* closed, A is bounded and weak-* closed. Therefore, by Banach-Alaoglu, A is weak-* compact. Now, consider the map $\Phi : A \rightarrow \mathbb{R}$ defined by

$$g \mapsto \|f_0 - g\| \quad (g \in A).$$

Observe that

$$\{g \in A : \Phi(g) \leq \lambda\} = \{g \in A : \|f_0 - g\| \leq \lambda\} = B(f_0, \lambda) \cap A$$

which is weak-* closed for all $\lambda \in \mathbb{R}$. Hence, Φ is weak-* lsc, so there exists $g_0 \in A \subset M^\perp$ such that

$$\|f_0 - g_0\| = \inf_{g \in A} \|f_0 - g\| = \inf_{g \in M^\perp} \|f_0 - g\|$$

where the last equality holds by the choice of A .

In general, one should remark that Banach-Alaoglu implies that the dual norm is lsc with respect to the weak-* topology. \square

Question 3.

3.10 Let E and F be two Banach spaces. Let $T \in \mathcal{L}(E, F)$, so that $T^* \in \mathcal{L}(F^*, E^*)$. Prove that T^* is continuous from F^* equipped with $\sigma(F^*, F)$ into E^* equipped with $\sigma(E^*, E)$.

Solution.

Question 4.

3.14 Let E be a reflexive Banach space and let I be a set of indices. Consider a collection $(f_i)_{i \in I}$ in E^* and a collection $(\alpha_i)_{i \in I}$ in \mathbb{R} . Let $M > 0$.

Show that the following properties are equivalent:

- (A) $\left\{ \begin{array}{l} \text{There exists some } x \in E \text{ with } \|x\| \leq M \text{ such that } \langle f_i, x \rangle = \alpha_i \\ \text{for every } i \in I. \end{array} \right.$
- (B) $\left\{ \begin{array}{l} \text{One has } |\sum_{i \in J} \beta_i \alpha_i| \leq M \|\sum_{i \in J} \beta_i f_i\| \text{ for every collection } (\beta_i)_{i \in J} \\ \text{in } \mathbb{R} \text{ with } J \subset I, J \text{ finite.} \end{array} \right.$

Compare with Exercises 1.10, 1.11 and Lemma 3.3.

Solution.

$(A) \implies (B)$ is obvious. Fix J finite, and $\{\beta_i\}_{i \in J}$. Then, use definition of norm and (A) , we get (B) . For a moment, we assume the result of exercise 1.10 in Brezis. Suppose (B) is true. Then, by 1.10, there exists $\phi_0 \in E^{**}$ such that

$$\|\phi_0\| \leq M \quad \text{and} \quad \langle \phi_0, f_i \rangle = \alpha_i$$

for all $i \in I$. Then, by reflexivity of E , there exists $x_0 \in E$ such that

$$\|x_0\| \leq M \quad \text{and} \quad \langle f_i, x_0 \rangle = \alpha_i$$

for all $i \in I$. Hence, it suffices to prove the result of 1.10. In particular, we need $(B) \implies (A)$ direction. Let G be the vector space spanned by $\{f_i\}_{i \in I}$. Define $g : G \rightarrow \mathbb{R}$ by

$$g(x) = \sum_{i \in J} \beta_i \alpha_i$$

where $x = \sum_{i \in J} \beta_i f_i$. g is well-defined and bounded by assumption (B) . Now, extend g to the whole of E by corollary 1.2 of Hahn Banach, and we are done. \square

Question 5.

3.16 Let E be a Banach space.

1. Let (f_n) be a sequence in (E^*) such that for every $x \in E$, $\langle f_n, x \rangle$ converges to a limit. Prove that there exists some $f \in E^*$ such that $f_n \xrightarrow{*} f$ in $\sigma(E^*, E)$.
2. Assume here that E is reflexive. Let (x_n) be a sequence in E such that for every $f \in E^*$, $\langle f, x_n \rangle$ converges to a limit. Prove that there exists some $x \in E$ such that $x_n \rightarrow x$ in $\sigma(E, E^*)$.
3. Construct an example in a nonreflexive space E where the conclusion of 2 fails.
[Hint: Take $E = c_0$ (see Section 11.3) and $x_n = (1, 1, \dots, \frac{1}{n}, 0, 0, \dots)$.]

Solution.

(i) Let $f : E \rightarrow \mathbb{R}$ be defined by

$$\langle f, x \rangle = \lim_{n \rightarrow \infty} \langle f_n, x \rangle \quad (x \in E).$$

Then, f is linear, because by linearity of $\{f_n\}$,

$$\begin{aligned} \langle f, x + y \rangle &= \lim_{n \rightarrow \infty} \langle f_n, x + y \rangle = \lim_{n \rightarrow \infty} \langle f_n, x \rangle + \langle f_n, y \rangle \\ &= \lim_{n \rightarrow \infty} \langle f_n, x \rangle + \lim_{n \rightarrow \infty} \langle f_n, y \rangle = \langle f, x \rangle + \langle f, y \rangle \end{aligned}$$

for any $x, y \in E$ and

$$\langle f, \lambda x \rangle = \lim_{n \rightarrow \infty} \langle f_n, \lambda x \rangle = \lambda \lim_{n \rightarrow \infty} \langle f_n, x \rangle = \lambda \langle f, x \rangle$$

for any $\lambda \in \mathbb{R}$ and $x \in E$. Now, we prove the boundedness of f . By the pointwise convergence,

$$\sup_n |\langle f_n, x \rangle| < \infty$$

for all $x \in E$. Therefore, by uniform boundedness principle, there exists $C > 0$ such that

$$|\langle f_n, x \rangle| \leq C \|x\|$$

and hence

$$\begin{aligned} |\langle f, x \rangle| &\leq |\langle f_n, x \rangle| + |\langle f_n, x \rangle - \langle f, x \rangle| \\ &\leq C \|x\| + |\langle f_n, x \rangle - \langle f, x \rangle| \end{aligned}$$

for any $x \in E$ and $n \geq 1$. Now, letting $n \rightarrow \infty$ gives

$$|\langle f, x \rangle| \leq C \|x\|$$

for any $x \in E$. Therefore, $f \in E^*$ such that

$$\langle f_n, x \rangle \rightarrow \langle f, x \rangle \quad \text{as } n \rightarrow \infty$$

for any $x \in E$, which implies

$$f_n \rightarrow_{\text{weak-}^*} f \quad \text{as } n \rightarrow \infty.$$

(ii) Set $\Phi : E^* \rightarrow \mathbb{R}$ by

$$\Phi(f) = \lim_{n \rightarrow \infty} \langle f, x_n \rangle.$$

With uniform boundedness, and explicitly computing the limits as from above, $\Phi \in E^{**}$. Since the space is reflexive, there exists $x \in E$ such that $J(x) = \Phi$. Then, by choice,

$$\langle f, x_n \rangle \rightarrow \langle f, x \rangle \quad \text{as } n \rightarrow \infty.$$

therefore, $x_n \rightarrow_{\text{weak}} x$ as $n \rightarrow \infty$.

(iii) Let $E = C_0 \subset l^\infty$. Then, $C_0^* = l^1$, and $C_0^{**} = l^\infty \neq C_0$. Consider $\{x_n\}$ as in the hint. Let $u \in l^1$. Then,

$$\langle u, x_n \rangle = \sum_{k=1}^n u_k$$

for all $n \geq 1$, and hence

$$\lim_{n \rightarrow \infty} \langle u, x_n \rangle = \sum_{k=1}^{\infty} u_k$$

which converges as $u \in l^1$. Hence, the hypothesis of (ii) is satisfied, except for the fact that E is not reflexive. Note that the i -th projections $\{p_i\}$ are all trivially continuous. For x_n to converge weakly to x , it is necessary that

$$\lim_{n \rightarrow \infty} \langle p_i, x_n \rangle = x^i$$

and hence

$$x^i = 1$$

for all $i \geq 1$. As $x = (1, 1, 1, \dots) \notin C_0$, there cannot exist $x \in E$ such that $x_n \rightarrow_{\text{weakly}} x$. \square

Question 6.

3.21 Let E be a separable Banach space and let (f_n) be a bounded sequence in E^* . Prove directly—without using the metrizable of E^* —that there exists a subsequence (f_{n_k}) that converges in $\sigma(E^*, E)$.
[Hint: Use a diagonal process.]

Solution.

By 3.16-1, it suffices to obtain a subsequence of $\{f_n\}$ such that $\{f_n\}$ converge pointwise everywhere. As E is separable, there exists $\{a_i\}$, a dense countable subset of E . Since $\{f_n\}$ are bounded in E^* , $\{\langle f_n, a_1 \rangle\}$ is bounded in \mathbb{R} . Hence, we can choose a subsequence $\{n_k\}$, with relabeling $\{(1, k)\}$ such that

$$\lim_{k \rightarrow \infty} \langle f_{1,k}, a_1 \rangle \text{ exists.}$$

Now, with the fact that $\{\langle f_n, a_2 \rangle\}$ is bounded in \mathbb{R} , choose a further subsequence $\{n_{k_l}\}$ from $\{n_k\}$, with relabeling $\{(2, k)\}$ such that

$$\lim_{k \rightarrow \infty} \langle f_{2,k}, a_2 \rangle \text{ exists.}$$

Repeat this process inductively, so that we have chosen $f_{l,k}$ for all $l, k \in \mathbb{N}$. Then, consider $\{g_l\} = \{f_{l,l}\}$, which is the standard diagonal sequence. Then, by choice

$$\lim_{l \rightarrow \infty} \langle g_l, a_i \rangle \text{ exists}$$

for any $i \in \mathbb{N}$. Now, let $a \in E$, and $\epsilon > 0$. Choose a_i such that $\|a_i - a\| < \epsilon$. Then,

$$\begin{aligned} |\langle g_n, a \rangle - \langle g_m, a \rangle| &\leq |\langle g_n, a \rangle - \langle g_n, a_i \rangle| + |\langle g_m, a_i \rangle - \langle g_m, a \rangle| \\ &+ |\langle g_n, a_i \rangle - \langle g_m, a_i \rangle| \\ &\leq |\langle g_n, a - a_i \rangle| + |\langle g_m, a_i - a \rangle| \\ &+ |\langle g_n, a_i \rangle - \langle g_m, a_i \rangle| \end{aligned} \tag{2}$$

$$\leq 2C\epsilon + |\langle g_n, a_i \rangle - \langle g_m, a_i \rangle| \tag{3}$$

for all $n, m \geq 1$, where (2) holds by linearity, and (3) holds by the choice of a_i and C being the bound on the $\{f_n\}$ in the dual norm. Therefore,

$$|\langle g_n, a \rangle - \langle g_m, a \rangle| \leq (2C + 1)\epsilon$$

for all n, m large enough, and hence, we have shown that

$$\langle g_l, a \rangle \text{ converges to a limit as } l \rightarrow \infty$$

for any $a \in E$. Hence, we are done. \square