Functional Analysis:
Problem Set I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

Problem 1. Let Y be a closed subspace of a normed vector space E. Show that the dual of E/Y is isometrically isomorphic to $Y^{\perp}$.

Solution.
First, define a map $\Phi : Y^{\perp} \to (E \setminus Y)^*$ naturally by

$$f \quad \mapsto \quad ([x] \mapsto f(x))$$

where $[x] \in E \setminus Y$. The map is well-defined, because for any $[x] = [x']$,

$$x' = x + y \text{ for some } y \in Y \text{ and } f(x') = f(x) + f(y) = f(x).$$

We now claim that Φ is a surjective isometry. By definition, for any $f \in Y^{\perp}$,

$$\begin{aligned} ||\Phi(f)|| &= \sup_{||[x]||=1} | < \Phi(f), [x] > | = \sup_{||[x]||=1} | < f, x > | \\ &= \sup_{\inf_{y \in Y} ||x-y||=1} | < f, x > | \end{aligned}$$

For sake of completeness, we note that if both pre-image and image are Banach, then a surjective isometry is a isometric isomorphism. This is a direct consequence of open mapping theorem. First, isometry implies injectivity: for $T \in \mathscr{L}(E, F)$ and $x, y \in E$, if $T(x) = T(y)$, then $0 = ||T(x - y)|| = ||x - y||$, so $x = y$. Therefore, T^{-1} is well-defined as a map. Now, by open mapping theorem, we see that for any O open in E, $(T^{-1})^{-1}(O) = T(O)$ is open in F. Therefore, T^{-1} is continuous as required.

It remains to be shown that $Y^{\perp}$ and $E \setminus Y$ are Banach. Suppose $f \in E^*$ such that there exists $\{f_n\} \subset Y^{\perp}$ with $f_n \to f$. It suffices to show that for any $yinY$, we have $< f, y >= 0$. But this is true, because norm convergence implies pointwise convergence and for all $n \geq 1$, $< f_n, x >= 0$. Now, $E \setminus Y$ is Banach with respect to the quotient norm, because

Question 2.

Problem 2. (Duality–Min-Max Principle) Let E be a normed vector space, and G a linear subspace of E. For any $x \in E$, define $m(x) = \inf_{y \in G} |x - y|$. Show that $m(x) = M(x)$, where $M(x) = \max_{\|f\|_{E^*} \leq 1, f=0 \text{ on } G} |\langle f, x \rangle|$.

Similarly, for any $g \in E^*$, we define $\|g\|_G = \sup\{|g(y)| : y \in G, \quad \|y\| \leq 1\}$. Then $\|g\|_G = \min\{\|g - h\|_{E^*} : h \in G^{\perp}\}$.

Solution.

Question 3.

Problem 3. Let E be a normed vector space.
(i) If Y is a closed proper subspace of E, then there is $x \in E$ such that $\|x\| = 1$ and $\|x - y\| > \frac{1}{2}$ for any $y \in Y$.
(ii) If E is of infinite dimension, then the unit ball $B_1 = \{x \in E : \|x\| \leq 1\}$ is never compact in strong topology.

Solution.
(i) We prove the following generalization, known as the Riesz Lemma: for each $\epsilon > 0$, there exists $x \in E$ such that $||x - y|| \geq 1 - \epsilon$, for any $y \in Y$.

Let $0 < \epsilon < 1$. Let $x \in E \setminus Y$. As Y is closed,

$$d \quad := \quad \text{dist}(x, Y) > 0.$$

Choose y^* in Y such that

$$d \leq ||x - y^*|| \leq \frac{d}{1 - \epsilon}. \quad (1)$$

Set $x^* = \dfrac{x - y^*}{||x - y^*||}$. Clearly, $||x^*|| = 1$, and, for any $y \in Y$,

$$\begin{aligned} ||x^* - y|| \quad &= \quad ||\frac{x - y^*}{||x - y^*||} - y|| = \frac{1}{||x - y^*||}||x - (y^* + y||x - y||^*)|| \\ &\geq \quad \frac{d}{||x - y^*||} = 1 - \epsilon, \end{aligned}$$

where the last inequality follows from (1), and we are done.

(ii) We proceed to construct a sequence $\{x_n\} \subset B_1$ such that there is no convergent subsequence, which shows that B_1 is not compact in strong topology through sequential characterization of compactness (strong topology is trivially metrizable).

Choose any $x \in E$ such that $||x|| = 1$ and set $x_1 = x$. Then, for any n, using (i), choose x_n such that

$$||x_n|| = 1 \text{ and } ||x_n - y|| > \frac{1}{2},$$

for any $y \in \text{span}(x_1, ..., x_{n-1})$, where the validity comes from the fact that any finite dimensional subspace is a proper, closed subspace of an infinite dimensional space. Then, it is clear that $\{x_n\}$ has no convergent subsequence, because for any $n \geq 1$, there exists $k, l \geq n$ with $k \neq l$, such that $||x_k - x_l|| > \dfrac{1}{2}$. Since being cauchy is a necessary condition for being convergent, we are done. $\square$

Question 4.

Problem 4. Let $L^\infty[0,1]$ be the space of bounded, Lebesgue measurable functions on $[0,1]$. We define $l(f) = \int_0^1 f(t)\mathrm{d}t$. Then l is a positive, linear, continuous functional on $L^\infty[0,1]$. Here l is called positive if $l(f_2) \geq l(f_1)$ whenever $f_2 \geq f_1$. Define, for any bounded real-valued function g, $p(g) = \inf\{l(f) : g \leq f \in L^\infty[0,1]\}$. Show that
(i) p is a positive homogeneous, subadditive and $p(g) \leq 0$ whenever $g \leq 0$. Moreover $p(f) = l(f)$ if $f \in L^\infty[0,1]$.
(ii) l can be extended to a positive linear functional on the space of all bounded functions.

Solution.
(i) Let g be a bounded real-valued function and $\lambda > 0$. Then, by linearity of Lebesgue integration,

$$\begin{aligned} p(\lambda g) &= \inf\{l(h) : \lambda g \leq h \in L^\infty\} \\ \lambda p(g) &= \inf\{l(\lambda h) : g \leq h \in L^\infty\}. \end{aligned}$$

We claim that

$$A := \{l(h) : \lambda g \leq h \in L^\infty\} \quad = \quad \{l(\lambda h) : g \leq h \in L^\infty\} =: B$$

If $\lambda g \leq h \in L^\infty$, then $g \leq \dfrac{h}{\lambda} \in L^\infty$, so $l(\lambda \dfrac{h}{\lambda}) = l(\lambda) \in B$. Conversely, if $g \leq h \in L^\infty$ then, $\lambda g \leq \lambda h \in L^\infty$, so $l(\lambda h) \in A$. Hence, $p(\lambda g) = \lambda p(g)$.

We now show that p is sub-additive. Let f, g be bounded real functions. Then, for any $h_1, h_2 \in L^\infty$ such that $f \leq h_1$ and $g \leq h_2$,

$$f + g \quad \leq \quad h_1 + h_2 \in L^\infty,$$

so, again by linearity of integration,

$$p(f+g) \quad \leq \quad l(h_1 + h_2) = l(h_1) + l(h_2).$$

Taking infs for h_1, then h_2, gives

$$p(f+g) \quad \leq \quad p(f) + p(g),$$

as required.

For any f, g bounded real-valued functions,

$$\begin{aligned} p(f+g) &= \inf\{l(f+g) : f+g \leq h, h \in L^\infty[0,1]\} \\ &= \inf\{l(f) + l(g) : f+g \leq h, h \in L^\infty[0,1]\}. \end{aligned}$$

Suppose $g \leq 0$. Then, as $0 \in L^\infty[0,1]$ and $l(0) = 0$, by definition, $p(g) \leq 0$.

We show that $p(f) = l(f)$ if $f \in L^\infty[0,1]$. For all $h \in L^\infty[0,1]$ such that $f \leq h$, then, by monotonicity of Lebesgue integration, $l(f) \leq l(h)$. Since $f \leq f$ trivially, it follows that $p(f) = l(f)$. $\square$

(ii) Now, as $l = p$ on $L^\infty[0,1]$, by Hahn-Banach, l can be extended to the entire space of bounded real-valued functions. This shows that we can make sense of integration for any bounded functions in a weaker sense, sacrificing some nice properties, such as countable additivity and so on(probably if such properties hold, then it will contradict existence of non-measurable sets by considering appropriate indicators). $\square$

Question 5.

Problem 5. Let E be a normed vector space. The norm of E is called uniformly convex if $\|\frac{x+y}{2}\| \leq 1 - \varepsilon(\|x-y\|)$ for all $x, y \in \{z \in E : \|z\| = 1\}$. Here $\varepsilon(r)$ is an increasing and positive function defined for $r > 0$ such that $\lim_{r \to 0^+} \varepsilon(r) = 0$. Let K be a closed, convex subset in a Banach space E with uniformly convex norm. Prove that for any $x \in E$, there is a unique $y \in K$ such that $\|x - y\| = \inf\{\|x - z\| : z \in K\}$. (Hint: Assume $x = 0$ not in K, and let $\{y_n\} \subset K$ be a minimizing sequence, then consider $x_n = \frac{y_n}{\|y_n\|}$ and $\frac{x_m + x_n}{2}$.)

Solution.

Question 6.

Problem 6. Let E be a vector space with a metric, and O be a bounded open set in E such that it is convex and symmetric with respect to $\underline{0} \in O$ (i.e., $x \in O \Rightarrow -x \in O$). Then show that the Minkowski functional associated with O introduces a norm of E.

Solution.
By ordinary properties of Minkowski functionals, it suffices to show that for any $x \in E$ and $\lambda \in \mathbb{R}$,

$$p(\lambda x) = |\lambda| p(x) \quad \text{and} \quad p(x) = 0 \implies x = 0.$$

We first prove the absolute homogeneity of p. Now, if $\lambda \geq 0$, then $p(\lambda x) = \lambda p(x)$ by positive homogeneity of Minkowski functionals. Now, if $\lambda \geq 0$, then, by symmetry, and positive homogeneity again, we obtain

$$p(\lambda x) = p(-\lambda x) = -\lambda p(x),$$

which completes the proof of absolute homogeneity.

Now, observe that, for any $0 < \alpha < \beta$, and $x \in E$,

$$\alpha^{-1} x \in C \quad \implies \beta^{-1} x \in C,$$

because by convexity

$$(1 - \frac{\beta^{-1}}{\alpha^{-1}})0 + \frac{\beta^{-1}}{\alpha^{-1}} \alpha^{-1} x = \beta^{-1} x \in C.$$

Let $x \in E$ such that $p(x) = 0$, Then, by the above discussion, it follows that

$$\alpha^{-1} x \quad \in \quad C \quad (*),$$

for any $\alpha \in (0, \infty)$. Suppose $x \neq 0$, and let $r > 0$ large enough that $C \subset B(0, r)$. Then, it follows that, from $(*)$, $\frac{r}{||x||} x \in C$, which contradicts the fact that $C \subset B(0, r)$. Therefore, $x = 0$ and we are done. $\square$

Question 7.

Problem 7. Let E be the space of bounded Lebesque measurable functions on [a,b]. Find a sequence $\{f_n\} \subset E^*$ such that $f_n(x) \to 0$ for all $x \in E$ and $\|f_n\|_{E^*} = 1$. Is E separable, that is, does E has a countable dense set?

Solution.

It is a well-known fact that L^∞ is not separable. The proof can be found in the Brezis textbook chapter 4.