
Functional Analysis:

Problem Set I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

Problem 1. Let Y be a closed subspace of a normed vector space E . Show that the dual of E/Y is isometrically isomorphic to Y^\perp .

Solution.

First, define a map $\Phi : (E/Y)^* \rightarrow Y^\perp$ naturally by

$$f \mapsto (x \mapsto f([x]))$$

where $[x] \in E/Y$. The map on the RHS is clearly linear and vanishes on Y , because for any $y \in Y$, $f([y]) = 0$ by linearity. The map is bounded as well, since

$$|f([x])| \leq \|f\| \|x\| \leq \|f\| \|x\| \quad (*)$$

for any $x \in E$. Now, we show that Φ is an isometry. It follows that

$$\|\Phi(f)\| = \sup_{\|x\|=1} |\langle \Phi(f), x \rangle| = \sup_{\|x\|=1} |f([x])|$$

Hence, combined with (*), it suffices to show that

$$\sup_{\|x\|=1} |f([x])| \leq \sup_{\|[x]\|=1} |f([x])|,$$

but this follows, since Y is closed, so we can choose $\{y_n\} \subset Y$ such that $\limsup_{n \rightarrow \infty} \|x + y_n\| \leq 1$. Now, we show that Φ is surjective. Let $l \in Y^\perp$. Then, define $f : E/Y \rightarrow \mathbb{R}$ by

$$[x] \mapsto l(x).$$

The map is well-defined, since for any $x' \in [x]$ such that $x' = x + y$, $l([x']) = l(x + y) = l(x)$, and the map inherits linearity and boundedness from l . Now, it follows that $\Phi(f) = (x \mapsto f([x]) = (x \mapsto l(x)) = l$, and we are done.

□

Question 2.

Problem 2. (Duality–Min-Max Principle) Let E be a normed vector space, and G a linear subspace of E . For any $x \in E$, define $m(x) = \inf_{y \in G} \|x - y\|$. Show that $m(x) =$

$M(x)$, where $M(x) = \max_{\|f\|_{E^*} \leq 1, f=0 \text{ on } G} |\langle f, x \rangle|$.

Similarly, for any $g \in E^*$, we define $\|g\|_G = \sup\{|g(y)| : y \in G, \|y\| \leq 1\}$. Then $\|g\|_G = \min\{\|g - h\|_{E^*} : h \in G^\perp\}$.

Solution.

Fix $x \in E$. For any $y \in G$, and $f \in E^*$ such that $\|f\| \leq 1$ with $f = 0$ on G ,

$$|\langle f, x \rangle| = |\langle f, x - y \rangle| \leq \|f\| \|x - y\| = \|x - y\|,$$

so

$$m(x) \geq \sup_{\|f\| \leq 1; f=0 \text{ on } G} |\langle f, x \rangle|.$$

Hence, it suffices to show that there exists $f \in E^*$ such that $\|f\| \leq 1$ and $f = 0$ on G with $|\langle f, x \rangle| = m(x)$. We can further assume without loss of generality that $x \in E \setminus G$, because if $x \in G$, both sides are trivially 0. Now, define a map $g : G \oplus \mathbb{R}x \rightarrow \mathbb{R}$ by $y + \lambda x \mapsto \lambda m(x)$. Then, we claim that g is linear. If $z_1 = y_1 + \lambda_1 x$ and $z_2 = y_2 + \lambda_2 x$, then

$$g(z_1) + g(z_2) = (\lambda_1 + \lambda_2)x = g(z_1 + z_2).$$

If $z = y + \lambda x$ and $\gamma \in \mathbb{R}$, then

$$g(\gamma z) = g(\gamma y + \gamma \lambda x) = \gamma \lambda m(x) = \gamma g(z).$$

Note that $g(x) = m(x)$. Now, in view of Hahn-Banach, we certainly need a Minkowski functional that bounds g on its domain. We show that m in fact is the Minkowski functional. Firstly,

$$m(\lambda v) = \inf_{y \in G} \|\lambda v - y\| = |\lambda| \inf_{y \in G} \|v - \frac{y}{\lambda}\| = |\lambda| m(v)$$

for any $\lambda \in \mathbb{R}$ and $v \in E$, because $y \mapsto \frac{y}{\lambda}$ is a bijection from G to G itself for any $\lambda \in \mathbb{R}$. Secondly,

$$m(u + v) = \inf_{y \in G} \|u + v - y\| \leq \inf_{y \in G} \|u - \frac{1}{2}y\| + \inf_{y \in G} \|v - \frac{1}{2}y\| = m(u) + m(v)$$

for any $u, v \in E$. where the last equality holds by the same reasoning as above. Finally,

$$g(z) = g(y + \lambda x) = \lambda m(x) \leq |\lambda| m(x) = m(y + \lambda x) = m(z)$$

for any $z = y + \lambda x \in G$. Therefore, by Hahn-Banach, we can extend g to the entire domain E and call it f . Since

$$f(z) \leq m(z) \leq \|z\|,$$

for any $z \in E$, it follows that $\|f\| \leq 1$, $f = 0$ on G and $|\langle f, x \rangle| = 0$ as required. \square

The second part follows similarly. Observe that

$$\|g\|_G = \sup_{\|y\| \leq 1; y \in G} |\langle g, y \rangle| = \sup_{\|y\| \leq 1; y \in G} |\langle g - h, y \rangle| \leq \|g - h\|_{E^*}$$

for any $g \in E^*$ and $h \in G^\perp$, so

$$\|g\|_G \leq \inf_{h \in G^\perp} \|g - h\|_{E^*}$$

for any $g \in E^*$. Now, g restricted to G is linear and bounded, so by a corollary of Hahn-Banach, there exists f such that $\|f\|_{E^*} = \|g\|_G$. Set $h = g - f$. Since $f = g$ on G , $h \in G^\perp$, and $\|g - h\|_{E^*} = \|f\|_{E^*} = \|g\|_G$, so we are done. \square

Question 3.

Problem 3. Let E be a normed vector space.

- (i) If Y is a closed proper subspace of E , then there is $x \in E$ such that $\|x\| = 1$ and $\|x - y\| > \frac{1}{2}$ for any $y \in Y$.
- (ii) If E is of infinite dimension, then the unit ball $B_1 = \{x \in E : \|x\| \leq 1\}$ is never compact in strong topology.

Solution.

(i) We prove the following generalization, known as the Riesz Lemma: for each $\epsilon > 0$, there exists $x \in E$ such that $\|x - y\| \geq 1 - \epsilon$, for any $y \in Y$.

Let $0 < \epsilon < 1$. Let $x \in E \setminus Y$. As Y is closed,

$$d := \text{dist}(x, Y) > 0.$$

Choose y^* in Y such that

$$d \leq \|x - y^*\| \leq \frac{d}{1 - \epsilon}. \quad (1)$$

Set $x^* = \frac{x - y^*}{\|x - y^*\|}$. Clearly, $\|x^*\| = 1$, and, for any $y \in Y$,

$$\begin{aligned} \|x^* - y\| &= \left\| \frac{x - y^*}{\|x - y^*\|} - y \right\| = \frac{1}{\|x - y^*\|} \|x - (y^* + y\|x - y^*\|)\| \\ &\geq \frac{d}{\|x - y^*\|} \leq 1 - \epsilon, \end{aligned}$$

where the last inequality follows from (1), and we are done. \square

(ii) We proceed to construct a sequence $\{x_n\} \subset B_1$ such that there is no convergent subsequence, which shows that B_1 is not compact in strong topology through sequential characterization of compactness (strong topology is trivially metrizable).

Choose any $x \in E$ such that $\|x\| = 1$ and set $x_1 = x$. Then, for any n , using (i), choose x_n such that

$$\|x_n\| = 1 \text{ and } \|x_n - y\| > \frac{1}{2},$$

for any $y \in \text{span}(x_1, \dots, x_{n-1})$, where the validity comes from the fact that any finite dimensional subspace is a proper, closed subspace of an infinite dimensional space. Then, it is clear that $\{x_n\}$ has no convergent subsequence, because for any $n \geq 1$, there exists $k, l \geq n$ with $k \neq l$, such that $\|x_k - x_l\| > \frac{1}{2}$. Since being cauchy is a necessary condition for being convergent, we are done. \square

Question 4.

Problem 4. Let $L^\infty[0, 1]$ be the space of bounded, Lebesgue measurable functions on $[0, 1]$. We define $l(f) = \int_0^1 f(t)dt$. Then l is a positive, linear, continuous functional on $L^\infty[0, 1]$. Here l is called positive if $l(f_2) \geq l(f_1)$ whenever $f_2 \geq f_1$. Define, for any bounded real-valued function g , $p(g) = \inf\{l(f) : g \leq f \in L^\infty[0, 1]\}$. Show that
 (i) p is a positive homogeneous, subadditive and $p(g) \leq 0$ whenever $g \leq 0$. Moreover $p(f) = l(f)$ if $f \in L^\infty[0, 1]$.
 (ii) l can be extended to a positive linear functional on the space of all bounded functions.

Solution.

(i) Let g be a bounded real-valued function and $\lambda > 0$. Then, by linearity of Lebesgue integration,

$$\begin{aligned} p(\lambda g) &= \inf\{l(h) : \lambda g \leq h \in L^\infty\} \\ \lambda p(g) &= \inf\{l(\lambda h) : g \leq h \in L^\infty\}. \end{aligned}$$

We claim that

$$A := \{l(h) : \lambda g \leq h \in L^\infty\} = \{l(\lambda h) : g \leq h \in L^\infty\} =: B$$

If $\lambda g \leq h \in L^\infty$, then $g \leq \frac{h}{\lambda} \in L^\infty$, so $l(\lambda \frac{h}{\lambda}) = l(h) \in B$. Conversely, if $g \leq h \in L^\infty$ then, $\lambda g \leq \lambda h \in L^\infty$, so $l(\lambda h) \in A$. Hence, $p(\lambda g) = \lambda p(g)$.

We now show that p is sub-additive. Let f, g be bounded real functions. Then, for any $h_1, h_2 \in L^\infty$ such that $f \leq h_1$ and $g \leq h_2$,

$$f + g \leq h_1 + h_2 \in L^\infty,$$

so, again by linearity of integration,

$$p(f + g) \leq l(h_1 + h_2) = l(h_1) + l(h_2).$$

Taking infs for h_1 , then h_2 , gives

$$p(f + g) \leq p(f) + p(g),$$

as required.

For any f, g bounded real-valued functions,

$$\begin{aligned} p(f + g) &= \inf\{l(f + g) : f + g \leq h, h \in L^\infty[0, 1]\} \\ &= \inf\{l(f) + l(g) : f + g \leq h, h \in L^\infty[0, 1]\}. \end{aligned}$$

Suppose $g \leq 0$. Then, as $0 \in L^\infty[0, 1]$ and $l(0) = 0$, by definition, $p(g) \leq 0$.

We show that $p(f) = l(f)$ if $f \in L^\infty[0, 1]$. For all $h \in L^\infty[0, 1]$ such that $f \leq h$, then, by monotonicity of Lebesgue integration, $l(f) \leq l(h)$. Since $f \leq f$ trivially, it follows that $p(f) = l(f)$. \square

(ii) Now, as $l = p$ on $L^\infty[0, 1]$, by Hahn-Banach, l can be extended to the entire space of bounded real-valued functions. This shows that we can make sense of integration for any bounded functions in a weaker sense, sacrificing some nice properties, such as countable additivity and so on (probably if such properties hold, then it will contradict existence of non-measurable sets by considering appropriate indicators). \square

Question 5.

Problem 5. Let E be a normed vector space. The norm of E is called uniformly convex if $\|\frac{x+y}{2}\| \leq 1 - \varepsilon(\|x-y\|)$ for all $x, y \in \{z \in E : \|z\| = 1\}$. Here $\varepsilon(r)$ is an increasing and positive function defined for $r > 0$ such that $\lim_{r \rightarrow 0^+} \varepsilon(r) = 0$. Let K be a closed, convex subset in a Banach space E with uniformly convex norm. Prove that for any $x \in E$, there is a unique $y \in K$ such that $\|x - y\| = \inf\{\|x - z\| : z \in K\}$. (Hint: Assume $x = 0$ not in K , and let $\{y_n\} \subset K$ be a minimizing sequence, then consider $x_n = \frac{y_n}{\|y_n\|}$ and $\frac{x_m + x_n}{2}$.)

Solution.

Without loss of generality, assume $x = 0$ and $x \notin K$. Let $\{y_n\}$ be the minimizing sequence. As E is Banach, and K is closed, it suffices to show that $\{y_n\}$ is Cauchy. Let $d = \lim_{n \rightarrow \infty} \|y_n\|$. We claim that

$$\|\frac{x_n + x_m}{2}\| \rightarrow 1 \quad \text{as } n, m \rightarrow \infty.$$

For each $n, m \geq 1$,

$$\begin{aligned} \|\frac{x_n + x_m}{2}\| &= \frac{1}{2} \frac{\|y_m\| + \|y_n\|}{\|y_n\| \|y_m\|} \|\frac{\|y_m\| y_n}{\|y_n\| + \|y_m\|} + \frac{\|y_n\| y_m}{\|y_n\| + \|y_m\|}\| \\ &\leq \frac{1}{2} \frac{\|y_m\| + \|y_n\|}{\|y_n\| \|y_m\|} d \end{aligned}$$

by convexity of K . The RHS goes to 1 as $n, m \rightarrow \infty$ via $d = \lim_{n \rightarrow \infty} \|y_n\|$ condition, and $\|\frac{x_n + x_m}{2}\| \geq 1$. We have shown the claimed limit. Now, by uniform convexity, $\{x_n\}$ is Cauchy. Observe that

$$\|y_n - y_m\| \leq \|y_n\| \|x_n - x_m\| + \| \|y_n\| - \|y_m\| \| \|x_m\|$$

for each $n, m \geq 1$. Since the RHS goes to 0 as $n, m \rightarrow \infty$, it implies that $\{y_n\}$ is Cauchy and we are done. \square

Question 6.

Problem 6. Let E be a normed vector space, and O be a bounded open set in E such that it is convex and symmetric with respect to $0 \in O$ (i.e., $x \in O \Rightarrow -x \in O$). Then show that the Minkowski functional associated with O introduces a new norm of E .

(ii) Is this new norm equivalent to the original norm $\|\cdot\|$.

(iii) If E is reflexive with respect to the original norm, does it imply then E is also reflexive with respect to the new norm?

Solution.

By ordinary properties of Minkowski functionals, it suffices to show that for any $x \in E$ and $\lambda \in \mathbb{R}$,

$$p(\lambda x) = |\lambda|p(x) \quad \text{and} \quad p(x) = 0 \implies x = 0.$$

We first prove the absolute homogeneity of p . Now, if $\lambda \geq 0$, then $p(\lambda x) = \lambda p(x)$ by positive homogeneity of Minkowski functionals. Now, if $\lambda \leq 0$, then, by symmetry, and positive homogeneity again, we obtain

$$p(\lambda x) = p(-\lambda x) = -\lambda p(x),$$

which completes the proof of absolute homogeneity.

Now, observe that, for any $0 < \alpha < \beta$, and $x \in E$,

$$\alpha^{-1}x \in C \implies \beta^{-1}x \in C, \quad (**)$$

because by convexity

$$(1 - \frac{\beta^{-1}}{\alpha^{-1}})0 + \frac{\beta^{-1}}{\alpha^{-1}}\alpha^{-1}x = \beta^{-1}x \in C.$$

Let $x \in E$ such that $p(x) = 0$. Then, by the above discussion, it follows that

$$\alpha^{-1}x \in C \quad (*),$$

for any $\alpha \in (0, \infty)$. Suppose $x \neq 0$, and let $r > 0$ large enough that $C \subset B(0, r)$. Then, it follows that, from (*), $\frac{r}{\|x\|}x \in C$, which contradicts the fact that $C \subset B(0, r)$. Therefore, $x = 0$ and we are done. \square

(i) The new norm is equivalent with the original norm. Choose $r > 0$ such that $\overline{B(0, r)} \subset O$. Then,

$$\frac{r}{\|x\|} \in B(0, r) \subset O$$

and

$$rp(x) \leq \|x\|$$

for any $x \in E$. For the other direction, choose $R > 0$ such that $O \subset B(0, R)$ with $\partial B(0, R) \cap O = \emptyset$. Then,

$$\frac{R}{\|x\|}x \notin O$$

for any $x \in E$. By the contrapositive of (**), it follows that

$$\|x\| \leq Rp(x)$$

for any $x \in E$. Hence, the claimed norm equivalence holds.

(ii) Reflexive implies that the space is Banach (so there is no confusion with definition).

We prove the following claim: Let E be Banach space that is reflexive. For any normed linear space, defined by an equivalent norm on E , is reflexive.

Let Q be a norm on E such that E is reflexive, and P be a norm that is equivalent to Q . Since the topology induced by Q and P are the same, we have that $E_Q^* = E_P^*$, which we denote as E^* . Now, we claim that the dual norms Q^* and P^* on E^* , induced by Q and P are again equivalent. From the norm equivalence and definition of dual norm,

$$|f(x)| \leq Q^*(f)Q(x)$$

and

$$|f(x)| \leq Q^*(f)CP(x)$$

for some $C > 0$, and for any $f \in E^*$ and $x \in E$. Therefore,

$$P^*(f) \leq CQ^*(f)$$

for any $f \in E^*$. The other direction can be shown similarly, so we have shown that the induced dual norms are again equivalent. Now, by the same argument as above, we see that $E_Q^{**} = E_P^{**}$, from which it follows that $J_Q = J_P$. Therefore, by reflexivity assumption, we have

$$E_P^{**} = E_Q^{**} = J_Q(E) = J_P(E)$$

which shows that (E, P) is reflexive as required. Therefore, we have shown that the new norm is reflexive, whenever the original norm is reflexive. \square

Question 7.

Problem 7. Let E be the space of bounded Lebesgue measurable functions on $[a, b]$. Find a sequence $\{f_n\} \subset E^*$ such that $f_n(x) \rightarrow 0$ for all $x \in E$ and $\|f_n\|_{E^*} = 1$. Is E separable, that is, does E have a countable dense set?

Solution.

Consider $\{f_n\}$ defined by

$$g \mapsto \int_{\mathbb{T}} g(t) e^{-2\pi i n t} dt = \hat{g}(n) \quad (g \in L^\infty(\mathbb{T}))$$

for each $n \geq 1$. As $L^\infty(\mathbb{T}) \subset L^1(\mathbb{T})$, by Riemann-Lebesgue lemma, for any $g \in L^\infty(\mathbb{T})$,

$$f_n(g) = \hat{g}(n) \rightarrow 0.$$

For any $n \geq 1$, $g \in L^\infty(\mathbb{T})$ with $\|g\| = 1$,

$$|\hat{g}(n)| \leq \int_{\mathbb{T}} |g| \leq \|g\|_\infty = 1,$$

Now, for any $n \geq 1$, take $g = e^{2\pi i n t}$ to see

$$|\hat{g}(n)| = \left| \int_{\mathbb{T}} 1 dt \right| = 1,$$

so, combined with the above estimate, we have

$$\|f_n\| = 1,$$

as required.

It is a well-known fact that L^∞ is not separable, given that the ambient space is not finite. Instead of appealing to this general result, we provide a construction of an uncountable family of functions in $L^\infty[a, b]$ with $a < b$ such that the distance is at least 1 apart between any two elements to contradict the separability assumption. Take $[a, b]$ such that $a < b$. Fix $x_0 \in (a, b)$ and consider $\mathcal{A} = \{1_{B(x_0, r)}\}$ where $0 < r \leq \min(x_0 - a, b - x_0)$. Then, the family is uncountable, but for any $f, g \in \mathcal{A}$,

$$\|f - g\|_\infty = 1.$$

Now, suppose the space is separable, hence there exists a countable dense subset C . Now, observe that

$$\bigcup_{x \in \mathcal{A}} B(x, \frac{1}{2}) \cap C \subset C,$$

but the left hand side is uncountable, because it's an uncountable disjoint union of countable sets. Therefore, $L^\infty[a, b]$ is not separable. \square