ProbLimI:
Pset I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

$\boxed{1.1}$ *Properties of the duality map.*

Let E be an n.v.s. The duality map F is defined for every $x \in E$ by

$$F(x) = \{f \in E^\star;\ \|f\| = \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}.$$

1. Prove that

$$F(x) = \{f \in E^\star;\ \|f\| \le \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}$$

and deduce that $F(x)$ is nonempty, closed, and convex.

2. Prove that if $E^\star$ is strictly convex, then $F(x)$ contains a single point.
3. Prove that

$$F(x) = \left\{f \in E^\star;\ \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \ge \langle f, y - x\rangle \quad \forall y \in E\right\}.$$

4. Deduce that

$$\langle F(x) - F(y),\ x - y\rangle \ge 0 \quad \forall x, y \in E,$$

20 1 The Hahn–Banach Theorems. Introduction to the Theory of Conjugate Convex Functions

and more precisely that

$$\langle f - g,\ x - y\rangle \ge 0 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

Show that, in fact,

$$\langle f - g,\ x - y\rangle \ge (\|x\| - \|y\|)^2 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

5. Assume again that $E^\star$ is strictly convex and let $x, y \in E$ be such that

$$\langle F(x) - F(y),\ x - y\rangle = 0.$$

Show that $Fx = Fy$.

Solution.
(1) The first set equality follows as

$$f \in E^* \quad \text{and} \quad < f, x > = ||x||^2 \implies ||f|| \geq ||x||,$$

because otherwise

$$| < f, x > | \quad = \quad ||x||^2 > ||f|| ||x||,$$

which is absurd. Now, by Corollary 1.3, it follows that $F(x)$ is non-empty.

We show that $F(x)$ is convex. Let $f, g \in F(x)$ and $t \in [0, 1]$. Then, it follows that

$$< tf + (1 - t)g, x > \quad = \quad t < f, x > +(1 - t) < g, x >= ||x||^2$$

and

$$||tf + (1 - t)g|| \quad \leq \quad t||f|| + (1 - t)||g|| \leq ||x||,$$

so $tf + (1 - t)g \in F(x)$ and $F(x)$ is convex.

We show that $F(x)$ is closed. Let $f \in E^*$ such that there exists $\{f_n\} \subset F(x)$ with $f_n \to f$. As convergence in dual norm implies pointwise convergence, we have

$$||x||^2 =< f_n, x >\to< f, x > \quad \text{and} \quad < f, x >= ||x||^2.$$

Also, as $||f_n - f|| \to 0$, and by reverse-triangle inequality, we have

$$||f_n|| \to ||f|| \quad \text{and} \quad ||f|| \leq ||x||,$$

which shows that $f \in F(x)$, and consequently that $F(x)$ is closed.

(2)

Question 2.

$\boxed{1.2}$ Let E be a vector space of dimension n and let $(e_i)_{1\le i\le n}$ be a basis of E. Given $x \in E$, write $x = \sum_{i=1}^{n} x_i e_i$ with $x_i \in \mathbb{R}$; given $f \in E^\star$, set $f_i = \langle f, e_i\rangle$.

1. Consider on E the norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

   (a) Compute explicitly, in terms of the f_i's, the dual norm $\|f\|_{E^\star}$ of $f \in E^\star$.
   (b) Determine explicitly the set $F(x)$ (duality map) for every $x \in E$.

2. Same questions but where E is provided with the norm

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

3. Same questions but where E is provided with the norm

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2},$$

   and more generally with the norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad \text{where } p \in (1, \infty).$$

Solution.

Question 3.

$\boxed{1.3}$ Let $E = \{u \in C([0,1];\mathbb{R});\ u(0) = 0\}$ with its usual norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Consider the linear functional

$$f : u \in E \mapsto f(u) = \int_0^1 u(t)dt.$$

1. Show that $f \in E^\star$ and compute $\|f\|_{E^\star}$.
2. Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^\star}$?

Solution.

Question 4.

$\boxed{1.4}$ Consider the space $E = c_0$ (sequences tending to zero) with its usual norm (see Section 11.3). For every element $u = (u_1, u_2, u_3, \dots)$ in E define

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n.$$

1. Check that f is a continuous linear functional on E and compute $\|f\|_{E^\star}$.
2. Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^\star}$?

Solution.

(2) Suppose for sake of contradiction that there exists $u \in c_0$, such that

$$||u|| = 1 \text{ and } f(u) = 1.$$

Choose $N > 1$ such that

$$n \geq N \quad \implies \quad u_n < \frac{1}{2}.$$

Then,

$$f(u) \quad < \quad \sum_{n=1}^{N-1} \frac{1}{2^n} u_n + \frac{1}{2} \sum_{n=N}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{N-1} \frac{1}{2^n} u_n + \frac{1}{2^{N+1}}.$$