Functional Analysis: Problem Set I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

$\boxed{1.1}$ *Properties of the duality map.*

Let E be an n.v.s. The duality map F is defined for every $x \in E$ by

$$F(x) = \{f \in E^\star;\ \|f\| = \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}.$$

1. Prove that

$$F(x) = \{f \in E^\star;\ \|f\| \leq \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}$$

and deduce that $F(x)$ is nonempty, closed, and convex.

2. Prove that if $E^\star$ is strictly convex, then $F(x)$ contains a single point.
3. Prove that

$$F(x) = \left\{f \in E^\star;\ \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \geq \langle f, y - x\rangle \quad \forall y \in E\right\}.$$

4. Deduce that

$$\langle F(x) - F(y),\ x - y\rangle \geq 0 \quad \forall x, y \in E,$$

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and more precisely that

$$\langle f - g,\ x - y\rangle \geq 0 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

Show that, in fact,

$$\langle f - g,\ x - y\rangle \geq (\|x\| - \|y\|)^2 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

5. Assume again that $E^\star$ is strictly convex and let $x, y \in E$ be such that

$$\langle F(x) - F(y),\ x - y\rangle = 0.$$

Show that $Fx = Fy$.

Solution.
(1) The first set equality follows as

$$f \in E^* \quad \text{and} \quad < f, x > = ||x||^2 \implies ||f|| \geq ||x||,$$

because otherwise

$$| < f, x > | \quad = \quad ||x||^2 > ||f||||x||,$$

which is absurd. Now, by Corollary 1.3, it follows that $F(x)$ is non-empty.

We show that $F(x)$ is convex. Let $f, g \in F(x)$ and $t \in [0, 1]$. Then, it follows that

$$< tf + (1-t)g, x > \quad = \quad t < f, x > +(1-t) < g, x >= ||x||^2$$

and

$$||tf + (1-t)g|| \quad \leq \quad t||f|| + (1-t)||g|| \leq ||x||,$$

so $tf + (1-t)g \in F(x)$ and $F(x)$ is convex.

We show that $F(x)$ is closed. Let $f \in E^*$ such that there exists $\{f_n\} \subset F(x)$ with $f_n \to f$. As convergence in dual norm implies pointwise convergence, we have

$$||x||^2 =< f_n, x >\to< f, x > \quad \text{and} \quad < f, x >= ||x||^2.$$

Also, as $||f_n - f|| \to 0$, and by reverse-triangle inequality, we have

$$||f_n|| \to ||f|| \qquad \text{and} \qquad ||f|| \leq ||x||,$$

which shows that $f \in F(x)$, and consequently that $F(x)$ is closed.

(2)

Question 2.

$\boxed{1.2}$ Let E be a vector space of dimension n and let $(e_i)_{1\le i\le n}$ be a basis of E. Given $x \in E$, write $x = \sum_{i=1}^n x_i e_i$ with $x_i \in \mathbb{R}$; given $f \in E^\star$, set $f_i = \langle f, e_i \rangle$.

1. Consider on E the norm

$$\|x\|_1 = \sum_{i=1}^n |x_i|.$$

   (a) Compute explicitly, in terms of the f_i's, the dual norm $\|f\|_{E^\star}$ of $f \in E^\star$.
   (b) Determine explicitly the set $F(x)$ (duality map) for every $x \in E$.

2. Same questions but where E is provided with the norm

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

3. Same questions but where E is provided with the norm

$$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2},$$

   and more generally with the norm

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \quad \text{where } p \in (1,\infty).$$

Solution.

Question 3.

$\boxed{1.3}$ Let $E = \{u \in C([0,1];\mathbb{R});\, u(0) = 0\}$ with its usual norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Consider the linear functional

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$$f : u \in E \mapsto f(u) = \int_0^1 u(t)dt.$$

1. Show that $f \in E^\star$ and compute $\|f\|_{E^\star}$.
2. Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^\star}$?

Solution.
(1) By linearity of integration, it follows that f defined is linear. Since f is linear, it suffices to show continuity at 0. Fix $\epsilon > 0$. Then, it follows that, with $\delta = \frac{\epsilon}{2}$,

$$u \in B(0,\delta) \quad \implies \quad |\int_0^1 u(t)dt| \leq \int_0^1 |u(t)|dt \leq \delta < \epsilon.$$

Therefore f is continuous. Now, we compute its dual norm explicitly. Note that, for any $u \in E$,

$$|< f, u >| \quad = \quad |\int_0^1 u(t)dt| \leq \int_0^1 |u(t)|dt \leq ||u||,$$

so $||f|| \leq 1$. We now show the reverse inequality. Recall that

$$||f|| \quad = \quad \sup_{||u||=1} |< f, u >|$$

Fix $\epsilon > 0$. Set $u \in C[0,1]$ by

$$t \to \frac{1}{\epsilon} X_{[0,\epsilon]}(t) + X_{(\epsilon,1]}(t) \;\; (t \in [0,1])$$

Then, it follows that

$$< f, u > \quad = \quad \int_0^1 u(t)dt = 1 - \frac{\epsilon}{2}.$$

Therefore, it follows that $||f|| \geq 1$, and we have completed in showing that $||f|| = 1$. $\square$

(2)

Question 4.

$\boxed{1.4}$ Consider the space $E = c_0$ (sequences tending to zero) with its usual norm (see Section 11.3). For every element $u = (u_1, u_2, u_3, \dots)$ in E define

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n.$$

1. Check that f is a continuous linear functional on E and compute $\|f\|_{E^\star}$.
2. Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^\star}$?

Solution.
(1) Fix $u \in C_0$ such that $||u|| = \sup_n |u_n| = 1$, it follows that

$$|u_n| \quad \leq \quad 1$$

for all $n \geq 1$, so

$$|f(u)| \leq \quad \textstyle\sum_{n=1}^{\infty} |\frac{1}{2^n} u_n| = 1.$$

Therefore,

$$||f|| \quad = \quad \sup_{||u|=1} |f(u)| \leq 1.$$

Now, fix $\epsilon > 0$. Choose $N > 1$ such that

$$n \geq N \implies \sum_{k=1}^{n} \frac{1}{2^k} > 1 - \epsilon.$$

Set $u \in c_0$ as

$$u \quad = \quad 1 \ (n \leq N) \ \text{ and } \ u = 0 \ (n > N).$$

Then, $u \in c_0$, $||u|| = 1$, and $|f(u)| > 1 - \epsilon$. Therefore, it follows that

$$1 - \epsilon < ||f||$$

for any $\epsilon > 0$, so $||f|| \geq 1$, which combined with the previous estimate gives $||f|| = 1$. □

(2) Suppose for sake of contradiction that there exists $u \in c_0$, such that

$$||u|| = 1 \ \text{ and } \ f(u) = 1.$$

Choose $N > 1$ such that

$$n \geq N \quad \implies \quad u_n < \frac{1}{2}.$$

Then,

$$f(u) \quad < \quad \sum_{n=1}^{N-1} \frac{1}{2^n} u_n + \frac{1}{2} \sum_{n=N}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{N-1} \frac{1}{2^n} u_n + \frac{1}{2^{N-1}}.$$

Since $||u|| = 1$, continuing the above estimate gives

$$f(u) \quad < \quad 1 - \frac{1}{2^{N-1}} + \frac{1}{2^{N-1}} = 1,$$

which is absurd. □

Question 6.

$\boxed{1.6}$ Let E be an n.v.s. and let $H \subset E$ be a hyperplane. Let $V \subset E$ be an affine subspace containing H.

1. Prove that either $V = H$ or $V = E$.
2. Deduce that H is either closed or dense in E.

Solution.
ddd

Question 7.

$\boxed{1.7}$ Let E be an n.v.s. and let $C \subset E$ be convex.

1. Prove that $\overline{C}$ and $\operatorname{Int} C$ are convex.
2. Given $x \in C$ and $y \in \operatorname{Int} C$, show that $tx + (1-t)y \in \operatorname{Int} C \quad \forall t \in (0, 1)$.
3. Deduce that $\overline{C} = \overline{\operatorname{Int} C}$ whenever $\operatorname{Int} C \neq \emptyset$.

Solution.
(1) We first show that $\overline{C}$ is convex. Let $x, y \in \overline{C}$, and $t \in [0, 1]$. Choose, $\{x_n\}, \{y_n\} \subset C$ such that $x_n \to x$ and $y_n \to y$. By convexity of C, and linearity of limit, it follows that

$$\{tx_n + (1-t)y_n\} \subset C \quad \text{and} \quad tx_n + (1-t)y_n \to tx + (1-t)y.$$

Therefore, $tx + (1-t)y \in \overline{C}$, which proves the convexity of $\overline{C}$. We now show that $\int C$ is convex. Let $x, y \in \int C$, and $t \in [0, 1]$. By convexity of C,

$$tx + (1-t)y \in C$$

We now show that $\int C$ is convex. Let $x, y \in \mathrm{int}C$ and $t \in (0, 1)$.

(2) Suppose $x \in C$, $y \in \int C$, and $t \in (0, 1)$.

(3) It is trivial that $\overline{\mathrm{int}C} \subset \overline{C}$. Hence, it suffices to show that $\overline{C} \subset \overline{\int C}$.

Question 8.

$\boxed{1.8}$ Let E be an n.v.s. with norm $\| \;\|$. Let $C \subset E$ be an open convex set such that $0 \in C$. Let p denote the gauge of C (see Lemma 1.2).

1. Assuming C is symmetric (i.e., $-C = C$) and C is bounded, prove that p is a norm which is equivalent to $\| \;\|$.

2. Let $E = C([0,1]; \mathbb{R})$ with its usual norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Let

$$C = \left\{ u \in E;\ \int_0^1 |u(t)|^2 dt < 1 \right\}.$$

Check that C is convex and symmetric and that $0 \in C$. Is C bounded in E? Compute the gauge p of C and show that p is a norm on E. Is p equivalent to $\| \;\|$?

Solution.
(1) We first show that p is in fact a norm. By properties of any gauge of C, it suffices to show

$$p(x) = 0 \qquad \Longleftrightarrow \qquad x = 0.$$

If $x = 0$, then

$$\alpha > 0 \implies a^{-1}x = 0 \in C,$$

so $p(x) = 0$. Conversely, suppose that $p(x) = 0$. Firstly, let

$$I \quad = \quad \{\lambda > 0\,;\ \lambda^{-1}x \in C\}, \ \text{ and } \ 0 < \alpha < \beta.$$

We claim that

$$\beta \notin I \quad \implies \alpha \notin I.$$

We prove the contrapositive. Suppose $\alpha \in I$. Then, $\alpha^{-1}x \in C$. By convexity of C, it follows that

$$\frac{\beta^{-1}}{\alpha^{-1}}\alpha^{-1}x + (1 - \frac{\beta^{-1}}{\alpha^{-1}})0 \quad = \quad \beta^{-1}x \in C,$$

so $\beta \in I$. Therefore, to prove $p(x) > 0$, it suffices to show that there is a constant $k > 0$ such that $k^{-1}x \notin C$. Now, suppose for sake of contradiction that $x \neq 0$. Choose r large enough such that $C \subset B(r, 0)$ strictly. Then,

$$\frac{r}{||x||}x \in C \ \text{ and } \ 0 < \frac{||x||}{r} \in I,$$

which as discussed above implies that $p(x) > 0$. Hence, $x = 0$ as required.

(2) We first check convexity of C. Let $u, v \in C$ and $\lambda \in [0,1]$. Then,

$$\begin{aligned}
\int_0^1 |\lambda u + (1-\lambda)v|^2 dt \quad &\leq \quad \int_0^1 (\lambda|u| + (1-\lambda)|v|)^2 \\
&\leq \quad \lambda^2 \int_0^1 |u|^2 + 2\lambda(1-\lambda)\int_0^1 |u||v| + (1-\lambda)^2 \int_0^1 |v|^2 \\
&< \quad \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) = 1,
\end{aligned}$$

where the second last inequality holds by Cauchy-Schwarz. Now, 0 is clearly in C and C is symmetric, because

$$\int_0^1 |u(t)|^2 dt \quad = \quad \int_0^1 |-u(t)|^2 dt.$$

We claim that C is not bounded. Fix $r > 0$. Set

$$f \quad = \quad \sqrt{t} X_{[0,\frac{1}{2r}]} + (r - \sqrt{t}) X_{(\frac{1}{2r},\frac{1}{r}]}$$

We now compute the gauge p of C. For $u \in E$, it follows that

$$\begin{aligned} p(u) &= \inf\{\lambda > 0\,;\ \lambda^{-1} u \in C\} \\ &= \inf\{\lambda > 0\,;\ \lambda^{-2} \int_0^1 |u(t)|^2 dt < 1\} \\ &= \inf\{\lambda > 0\,;\ \int_0^1 |u(t)|^2 dt < \lambda^2\} \end{aligned}$$

Question 9.

$\boxed{1.9}$ *Hahn–Banach in finite-dimensional spaces.*

Let E be a finite-dimensional normed space. Let $C \subset E$ be a nonempty convex set such that $0 \notin C$. We claim that there always exists some hyperplane that separates C and $\{0\}$.

[Note that every hyperplane is closed (why?). The main point in this exercise is that no additional assumption on C is required.]

1. Let $(x_n)_{n\geq 1}$ be a countable subset of C that is dense in C (why does it exist?). For every n let

$$C_n = \text{conv}\{x_1, x_2, \ldots, x_n\} = \left\{x = \sum_{i=1}^n t_i x_i;\ t_i \geq 0\ \forall i \text{ and } \sum_{i=1}^n t_i = 1\right\}.$$

Check that C_n is compact and that $\bigcup_{n=1}^\infty C_n$ is dense in C.

2. Prove that there is some $f_n \in E^\star$ such that

$$\|f_n\| = 1 \text{ and } \langle f_n, x\rangle \geq 0 \quad \forall x \in C_n.$$

3. Deduce that there is some $f \in E^\star$ such that

$$\|f\| = 1 \text{ and } \langle f, x\rangle \geq 0 \quad \forall x \in C.$$

Conclude.

4. Let $A, B \subset E$ be nonempty disjoint convex sets. Prove that there exists some hyperplane H that separates A and B.

Solution.
We record two fundamental facts about finite dimensional spaces. First, linearity of a map on a finite dimensional space implies continuity. Second, every finite dimensional space is separable.

(1) Firstly, as $\{x_n\} \subset \bigcup_{n=1}^\infty C_n$, and $\{x_n\}$ is dense in C, $\bigcup_{n=1}^\infty C_n$ is dense in C. Now, consider

$$A = \left\{\lambda \in \mathbb{R}^n\ :\ \lambda_i \geq 0\ \forall i\ , \sum_i \lambda_i = 1\right\},$$

and

$$\Phi : \mathbb{R}^n \to E \ \text{ where } \ \lambda_i \mapsto \sum_i \lambda_i x_i.$$

It suffices to show that Φ is continuous, because A is a compact subset of $\mathbb{R}^n$, whose image is C_n. Φ, however, is trivially continuous, because it is linear.

(2) By the second geometric Hahn-Banach, applied with $A = \{0\}$ and $B = C_n$, there exists $f_n \in E^*$ not vanishing, such that

$$< f_n, x > \quad \geq \quad 0\ \forall x \in C_n.$$

By normalizing, we also obtain $||f_n|| = 1$.

(3) By compactness of the unit sphere in finite dimensional space, there exists $\{f_{n_k}\}$ such that

$$f_{n_k} \to f \ \text{ such that } \ ||f|| = 1.$$

Since uniform convergence implies pointwise convergence and $\{C_n\}$ are increasing, we have

$$||f|| = 1 \ \text{ and } \ < f, x > \geq 0\ \forall x \in \bigcup_n C_n,$$

which by density of C_k in C and continuity of f, gives

$$||f|| = 1 \ \text{ and } \ < f, x > \geq 0\ \forall x \in C,$$

as required.

(4) Set $C = A - B$. As $A \cap B = \emptyset$, we see that $0 \notin C$. We now show that C is still convex. Suppose $x, y \in C$ and $t \in [0,1]$. Then, there are $a_x, a_y \in A$ and $b_x, b_y \in B$ such that

$$x = a_x - b_x \text{ and } y = a_y - b_y.$$

Then, it follows that

$$tx + (1-t)y \quad = \quad t(a_x - b_x) + (1-t)(a_y - b_y) = (ta_x + (1-t)a_y) - (tb_x - (1-t)b_y) \in C,$$

where the last inclusion holds by convexity of A and B. Hence, C is a nonempty convex set such that $0 \notin C$. Apply (3) to C and $\{0\}$, then there is $f \in E^*$ such that

$$||f|| = 1 \text{ and } < f, x > \geq 0 \ \forall x \in C,$$

which implies that

$$< f, a - b > \geq \ 0 \quad \text{ and } \quad < f, a > \geq < f, b >,$$

for all $a \in A$ and $b \in B$. Therefore, there exists a hyperplane that separates A and B. We see that in finite dimensional space topological assumptions on A and B can be relaxed to obtain an existence of a separating hyperplane. □

Question 10.

$\boxed{1.10}$ Let E be an n.v.s. and let I be any set of indices. Fix a subset $(x_i)_{i\varepsilon I}$ in E and a subset $(\alpha_i)_{i\varepsilon I}$ in $\mathbb{R}$. Show that the following properties are equivalent:

(A) There exists some $f \in E^\star$ such that $\langle f, x_i \rangle = \alpha_i \quad \forall i \in I$.

(B) $\begin{cases} \text{There exists a constant } M \geq 0 \text{ such that for each finite subset} \\ J \subset I \text{ and for every choice of real numbers } (\beta_i)_{i \in J}\text{, we have} \\ \qquad \left|\sum_{i\in J} \beta_i \alpha_i\right| \leq M \left\|\sum_{i \in J} \beta_i x_i\right\|. \end{cases}$

Note that in the proof of (B) $\Rightarrow$ (A) one may find some $f \in E^\star$ with $\|f\|_{E^\star} \leq M$.
[**Hint:** Try first to define f on the linear space spanned by the $(x_i)_{i\varepsilon I}$.]

Solution.
ddd

Question 11.

$\boxed{1.11}$ Let E be an n.v.s. and let $M > 0$. Fix n elements $(f_1)_{1\le i\le n}$ in $E^\star$ and n real numbers $(\alpha_i)_{1\le i\le n}$. Prove that the following properties are equivalent:

$$\text{(A)} \qquad \begin{cases} \forall \varepsilon > 0 \ \ \exists x_\varepsilon \in E \text{ such that} \\ \|x_\varepsilon\| \le M + \varepsilon \text{ and } \langle f_i, x_\varepsilon\rangle = \alpha_i \quad \forall i = 1, 2, \dots, n. \end{cases}$$

$$\text{(B)} \qquad \left|\sum_{i=1}^n \beta_i \alpha_i\right| \le M \left\|\sum_{i=1}^n \beta_i f_i\right\| \quad \forall \beta_1, \beta_2, \dots, \beta_n \in \mathbb{R}.$$

[**Hint:** For the proof of (B) $\Rightarrow$ (A) consider first the case in which the f_i's are linearly independent and imitate the proof of Lemma 3.3.]

Compare Exercises 1.10, 1.11 and Lemma 3.3.

Solution.
ddd

Question 14.

$\boxed{1.14}$ Let $E = \ell^1$ (see Section 11.3) and consider the two sets

$$X = \{x = (x_n)_{n\geq 1} \in E;\ x_{2n} = 0\ \forall n \geq 1\}$$

and

$$Y = \left\{y = (y_n)_{n\geq 1} \in E;\ y_{2n} = \frac{1}{2^n} y_{2n-1}\ \forall n \geq 1\right\}.$$

1. Check that X and Y are closed linear spaces and that $\overline{X + Y} = E$.
2. Let $c \in E$ be defined by

$$\begin{cases} c_{2n-1} = 0 & \forall n \geq 1, \\ c_{2n} = \frac{1}{2^n} & \forall n \geq 1. \end{cases}$$

Check that $c \notin X + Y$.

3. Set $Z = X - c$ and check that $Y \cap Z = \emptyset$. Does there exist a closed hyperplane in E that separates Y and Z?
Compare with Theorem 1.7 and Exercise 1.9.
4. Same questions in $E = \ell^p$, $1 < p < \infty$, and in $E = c_0$.

Solution.
ddd

Question 16.

$\boxed{1.16}$ Let $E = \ell^1$, so that $E^\star = \ell^\infty$ (see Section 11.3). Consider $N = c_0$ as a closed subspace of $E^\star$.

Determine

$$N^\perp = \{x \in E;\ \langle f, x\rangle = 0 \quad \forall f \in N\}$$

and

$$N^{\perp\perp} = \{f \in E^\star;\ \langle f, x\rangle = 0 \quad \forall x \in N^\perp\}.$$

Check that $N^{\perp\perp} \neq N$.

Solution.

ddd

Question 17.

$\boxed{1.17}$ Let E be an n.v.s. and let $f \in E^{\star}$ with $f \neq 0$. Let M be the hyperplane $[f = 0]$.

1. Determine $M^{\perp}$.
2. Prove that for every $x \in E$, $\operatorname{dist}(x, M) = \inf_{y \in M} \|x - y\| = \frac{|\langle f, x \rangle|}{\|f\|}$.
   [Find a direct method or use Example 3 in Section 1.4.]
3. Assume now that $E = \{u \in C([0, 1]; \mathbb{R}); u(0) = 0\}$ and that

$$\langle f, u \rangle = \int_0^1 u(t)dt, \quad u \in E.$$

   Prove that $\operatorname{dist}(u, M) = |\int_0^1 u(t)dt| \ \forall u \in E$.
   Show that $\inf_{v \in M} \|u - v\|$ is never achieved for any $u \in E \backslash M$.

Solution.
ddd