Functional Analysis:
Problem Set III

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the exercises of the problem set III.

Question 1.

$\boxed{3.5}$ Let E be a Banach space and let $K \subset E$ be a subset of E that is compact in the strong topology. Let (x_n) be a sequence in K such that $x_n \rightharpoonup x$ weakly $\sigma(E, E^\star)$. Prove that $x_n \to x$ strongly.

[**Hint**: Argue by contradiction.]

Solution.
Suppose $x_n \not\to x$ strongly. Then, there exists $\epsilon > 0$ and $\{x_{n_k}\}$ such that

$$|x_{n_k} - x| > \epsilon \tag{1}$$

for all $k \geq 1$. By the compactness of K in strong topology, there exists a further subsequence $\{x_{n_{k_l}}\}$ such that

$$\lim_{l \to \infty} x_{n_{k_l}} = y$$

for some $y \in K$. From (1), $y \neq x$. Now, since convergence in strong topology implies convergence in weak topology, we have

$$x_{n_{k_l}} \to_{\text{weak}} y \qquad \text{as} \qquad l \to \infty.$$

From our assumption, however, $x_n \to_{\text{weak}} x$ as $n \to \infty$, so by Hausdroff property of weak topology $x_{n_{k_l}} \to_{\text{weak}} x$ as $l \to \infty$. This contradicts the uniquness of limit property of weak topology, which also arises from Hausdorff property of weak topology. We have a contradiction, and we are done.

$\square$

Question 2.

$\boxed{3.9}$ Let E be a Banach space; let $M \subset E$ be a linear subspace, and let $f_0 \in E^\star$. Prove that there exists some $g_0 \in M^\perp$ such that

$$\inf_{g \in M^\perp} \|f_0 - g\| = \|f_0 - g_0\|.$$

Two methods are suggested:

1. Use Theorem 1.12.
2. Use the weak* topology $\sigma(E^\star, E)$.

Solution.
Observe that

$$\begin{aligned} M^\perp &= \{g \in E^* :< g, x >= 0\, \forall x \in M\} \\ &= \bigcap_{x \in M} \{g \in E^* :< g, x >= 0\} = \bigcap_{x \in M} J(x)^{-1}(0) \end{aligned}$$

where J is the natural embedding. Hence, $M^\perp$ is weak-* closed. Now, choose $\{g_n\} \subset M^\perp$ such that

$$||f_0 - g_n|| \to \inf_{g \in M^\perp} ||f_0 - g|| \quad \text{as} \quad n \to \infty. \tag{2}$$

Fix $\epsilon > 0$. From above,

$$||g_n|| \quad \leq \quad ||f_0|| + ||f_0 - g_n|| \leq ||f_0|| + c + \epsilon$$

for n large enough. Therefore, $\{g_n\}$ is bounded in the dual norm. Now, consider $A = \overline{\{g_n\}}^{\text{weak-*}} \subset M^\perp$, where the second inclusion follows from the weak-* closure of $M^\perp$.

We now claim that any bounded $B \subset E^*$ is weak-* precompact. Choose $\lambda > 0$ such that $||b|| \leq \lambda$ for all $b \in B$. Then,

$$\frac{1}{\lambda} B \subset B_{E^*} \quad \text{and} \quad \overline{\frac{1}{\lambda} B}^{\text{weak}-*} \subset B_{E^*}$$

since B_{E^*} is compact in weak-* and since weak-* is Hausdorff, which implies that it is closed. As closed subset of compact set is compact and $x \mapsto \frac{1}{\lambda} x$ is a homeomorphism, $\frac{1}{\lambda} B$ is pre-compact, and B is precompact.

From the above result, A is weak-* compact. Now, consider the map $\Phi : A \to \mathbb{R}$ defined by

$$g \quad \mapsto \quad ||f_0 - g|| \quad (g \in A).$$

By lower semi-continuity of dual norm with respect to weak-* topology, Φ is lower semi-continuous as well. Hence, there exists $g_0 \in A$ such that

$$||f_0 - g_0|| \quad = \quad \inf_{g \in A} ||f_0 - g|| = \inf_{g \in M^\perp} ||f_0 - g||$$

where the last equality holds, by (2).

Question 3.

$\boxed{3.10}$ Let E and F be two Banach spaces. Let $T \in \mathscr{L}(E, F)$, so that $T^\star \in \mathscr{L}(F^\star, E^\star)$. Prove that $T^\star$ is continuous from $F^\star$ equipped with $\sigma(F^\star, F)$ into $E^\star$ equipped with $\sigma(E^\star, E)$.

Solution.

Question 4.

$\boxed{3.14}$ Let E be a reflexive Banach space and let I be a set of indices. Consider a collection $(f_i)_{i \in I}$ in $E^\star$ and a collection $(\alpha_i)_{i \in I}$ in $\mathbb{R}$. Let $M > 0$.

Show that the following properties are equivalent:

(A) $\begin{cases} \text{There exists some } x \in E \text{ with } \|x\| \leq M \text{ such that } \langle f_i, x \rangle = \alpha_i \\ \text{for every } i \in I. \end{cases}$

(B) $\begin{cases} \text{One has } |\sum_{i \in J} \beta_i \alpha_i| \leq M \| \sum_{i \in J} \beta_i f_i \| \text{ for every collection } (\beta_i)_{i \in J} \\ \text{in } \mathbb{R} \text{ with } J \subset I, J \text{ finite.} \end{cases}$

Compare with Exercises 1.10, 1.11 and Lemma 3.3.

Solution.
$(A) \implies (B)$ is obvious. For a moment, we assume the result of exercise 1.10 in Brezis. Suppose (B) is true. Then, by 1.10, there exists $\phi_0 \in E^{**}$ such that

$$||f|| \leq M \qquad \text{and} \qquad < \phi_0, f_i >= \alpha_i$$

for all $i \in I$. Then, by reflexivity of E, there exists $x_0 \in E$ such that

$$||x_0|| \leq M \qquad \text{and} \qquad < f, x_0 >= \alpha_i$$

for all $i \in I$. Hence, it suffices to prove the result of 1.10. In particular, we need $(B) \implies (A)$ direction. Let G be the vector space spanned by $\{x_i\}_{i \in I}$. Define $g : G \to \mathbb{R}$ by

$$g(x) \quad = \quad \sum_{i \in J} \beta_i \alpha_i$$

where $x = \sum_{i \in J} \beta_i x_i$. g is well-defined and bounded by assumption (B). Now, extend g to the whole of E by corollary 1.2 of Hahn Banach, and we are done. $\square$

Question 5.

$\boxed{3.16}$ Let E be a Banach space.

1. Let (f_n) be a sequence in $(E^\star)$ such that for every $x \in E$, $\langle f_n, x\rangle$ converges to a limit. Prove that there exists some $f \in E^\star$ such that $f_n \overset{\star}{\rightharpoonup} f$ in $\sigma(E^\star, E)$.
2. Assume here that E is reflexive. Let (x_n) be a sequence in E such that for every $f \in E^\star$, $\langle f, x_n\rangle$ converges to a limit. Prove that there exists some $x \in E$ such that $x_n \rightharpoonup x$ in $\sigma(E, E^\star)$.
3. Construct an example in a nonreflexive space E where the conclusion of 2 fails. [**Hint**: Take $E = c_0$ (see Section 11.3) and $x_n = (1, 1, \dots, \underset{(n)}{1}, 0, 0, \dots)$.]

Solution.
(i) Let $f : E \to \mathbb{R}$ be defined by

$$< f, x > \;=\; \lim_{n\to\infty} < f_n, x > \quad (x \in E).$$

Then, f is linear, because by linearty of $\{f_n\}$,

$$\begin{aligned} < f, x+y > &= \lim_{n\to\infty} < f_n, x+y > = \lim_{n\to\infty} < f_n, x > + < f_n, y > \\ &= \lim_{n\to\infty} < f_n, x > + \lim_{n\to\infty} < f_n, y > = < f, x > + < f, y > \end{aligned}$$

for any $x, y \in E$ and

$$< f, \lambda x > \;=\; \lim_{n\to\infty} < f_n, \lambda x > = \lambda \lim_{n\to\infty} < f_N, x > = \lambda < f, x >$$

for any $\lambda \in \mathbb{R}$ and $x \in E$. Now, we prove the boundedness of f. By the pointwise convergence,

$$\sup_n | < f_n, x > | < \infty$$

for all $x \in E$. Therefore, by uniform boundedness principle, there exists $C > 0$ such that

$$| < f_n, x > | \;\leq\; C||x||$$

and hence

$$\begin{aligned} | < f, x > | &\leq | < f_n, x > | + | < f_n, x > - < f, x > | \\ &\leq C||x|| + | < f_n, x > - < f, x > | \end{aligned}$$

for any $x \in E$ and $n \geq 1$. Now, letting $n \to \infty$ gives

$$| < f, x > | \;\leq\; C||x||$$

for any $x \in E$. Therefore, $f \in E^*$ such that

$$< f_n, x > \to < f, x > \quad \text{as} \quad n \to \infty$$

for any $x \in E$, which implies

$$f_n \to_{\text{weak}-*} f \quad \text{as} \quad n \to \infty.$$

(ii)

Question 6.

$\boxed{3.21}$ Let E be a separable Banach space and let (f_n) be a bounded sequence in $E^\star$. Prove directly—without using the metrizability of $E^\star$—that there exists a subsequence (f_{n_k}) that converges in $\sigma(E^\star, E)$.

[**Hint**: Use a diagonal process.]

Solution.
By 3.16-1, it suffices to obtain a subsequence of $\{f_n\}$ such that $\{f_n\}$ converge pointwise everywhere. As E is separable, there exists $\{a_i\}$, a dense countable subset of E. Since $\{f_n\}$ are bounded in E^*, $\{< f_n, a_1 >\}$ is bounded in $\mathbb{R}$. Hence, we can choose a subsequence $\{n_k\}$, with relabeling $\{(1,k)\}$ such that

$$\lim_{k\to\infty} < f_{1,k}, a_1 > \quad \text{exists.}$$

Now, with the fact that $\{< f_n, a_2 >\}$ is bounded in $\mathbb{R}$, choose a further subsequence $\{n_{k_l}\}$ from $\{n_k\}$, with relabeling $\{(2,k)\}$ such that

$$\lim_{k\to\infty} < f_{2,k}, a_2 > \quad \text{exists.}$$

Repeat this process inductively, so that we have chosen $f_{l,k}$ for all $l, k \in \mathbb{N}$. Then, consider $\{g_l\} = \{f_{l,l}\}$, which is the standard diagonal sequence. Then, by choice

$$\lim_{l\to\infty} < g_l, a_i > \quad \text{exists}$$

for any $i \in \mathbb{N}$. Now, let $a \in E$, and $\epsilon > 0$. Choose a_i such that $||a_i - a|| < \epsilon$. Then,

$$\begin{aligned}
| < g_n, a > - < g_m, a > | \quad &\le \quad | < g_n, a > - < g_n, a_i > | + | < g_m, a_i > - < g_m, a > | \\
&+ \quad | < g_n, a_i > - < g_m, a_i > | \\
&\le \quad | < g_n, a - a_i > | + | < g_m, a_i - a > | \qquad (3) \\
&+ \quad | < g_n, a_i > - < g_m, a_i > | \\
&\le \quad 2C\epsilon + | < g_n, a_i > - < g_m, a_i > | \qquad (4)
\end{aligned}$$

for all $n, m \ge 1$, where (3) holds by linearity, and (4) holds by the choice of a_i and C being the bound on the $\{f_n\}$ in the dual norm. Therefore,

$$| < g_n, a > - < g_m, a > | \quad \le \quad (2C + 1)\epsilon$$

for all n, m large enough, and hence, we have shown that

$$< g_l, a > \quad \text{converges to a limit} \quad \text{as} \quad l \to \infty$$

for any $a \in E$. Hence, we are done. $\square$