ProbLimI:
Pset I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

$\boxed{1.1}$ *Properties of the duality map.*

Let E be an n.v.s. The duality map F is defined for every $x \in E$ by

$$F(x) = \{f \in E^\star;\ \|f\| = \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}.$$

1. Prove that

$$F(x) = \{f \in E^\star;\ \|f\| \le \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}$$

and deduce that $F(x)$ is nonempty, closed, and convex.

2. Prove that if $E^\star$ is strictly convex, then $F(x)$ contains a single point.
3. Prove that

$$F(x) = \left\{f \in E^\star;\ \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \ge \langle f, y - x\rangle \quad \forall y \in E\right\}.$$

4. Deduce that

$$\langle F(x) - F(y),\ x - y\rangle \ge 0 \quad \forall x, y \in E,$$

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and more precisely that

$$\langle f - g,\ x - y\rangle \ge 0 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

Show that, in fact,

$$\langle f - g,\ x - y\rangle \ge (\|x\| - \|y\|)^2 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

5. Assume again that $E^\star$ is strictly convex and let $x, y \in E$ be such that

$$\langle F(x) - F(y),\ x - y\rangle = 0.$$

Show that $Fx = Fy$.

Solution.
(1) The first set equality follows as

$$f \in E^* \quad \text{and} \quad < f, x > = ||x||^2 \implies ||f|| \geq ||x||,$$

because otherwise

$$| < f, x > | \quad = \quad ||x||^2 > ||f|| ||x||,$$

which is absurd. Now, by Corollary 1.3, it follows that $F(x)$ is non-empty.

We show that $F(x)$ is convex. Let $f, g \in F(x)$ and $t \in [0, 1]$. Then, it follows that

$$< tf + (1-t)g, x > \quad = \quad t < f, x > + (1-t) < g, x >= ||x||^2$$

and

$$||tf + (1-t)g|| \quad \leq \quad t||f|| + (1-t)||g|| \leq ||x||,$$

so $tf + (1-t)g \in F(x)$ and $F(x)$ is convex.

We show that $F(x)$ is closed. Let $f \in E^*$ such that there exists $\{f_n\} \subset F(x)$ with $f_n \to f$. As convergence in dual norm implies pointwise convergence, we have

$$||x||^2 = < f_n, x > \to < f, x > \quad \text{and} \quad < f, x >= ||x||^2.$$

Also, as $||f_n - f|| \to 0$, and by reverse-triangle inequality, we have

$$||f_n|| \to ||f|| \qquad \text{and} \qquad ||f|| \leq ||x||,$$

which shows that $f \in F(x)$, and consequently that $F(x)$ is closed.

(2)

Question 2.

$\boxed{1.2}$ Let E be a vector space of dimension n and let $(e_i)_{1\le i\le n}$ be a basis of E. Given $x \in E$, write $x = \sum_{i=1}^{n} x_i e_i$ with $x_i \in \mathbb{R}$; given $f \in E^\star$, set $f_i = \langle f, e_i\rangle$.

1. Consider on E the norm
$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$
   (a) Compute explicitly, in terms of the f_i's, the dual norm $\|f\|_{E^\star}$ of $f \in E^\star$.
   (b) Determine explicitly the set $F(x)$ (duality map) for every $x \in E$.
2. Same questions but where E is provided with the norm
$$\|x\|_\infty = \max_{1\le i\le n} |x_i|.$$
3. Same questions but where E is provided with the norm
$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2},$$
and more generally with the norm
$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad \text{where } p \in (1, \infty).$$

Solution.

Question 3.

$\boxed{1.3}$ Let $E = \{u \in C([0,1]; \mathbb{R});\, u(0) = 0\}$ with its usual norm
$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$
Consider the linear functional

$$f : u \in E \mapsto f(u) = \int_0^1 u(t)\,dt.$$

1. Show that $f \in E^\star$ and compute $\|f\|_{E^\star}$.
2. Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^\star}$?

Solution.

Question 4.

$\boxed{1.4}$ Consider the space $E = c_0$ (sequences tending to zero) with its usual norm (see Section 11.3). For every element $u = (u_1, u_2, u_3, \dots)$ in E define

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n.$$

1. Check that f is a continuous linear functional on E and compute $\|f\|_{E^\star}$.
2. Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^\star}$?

Solution.
(1) Fix $u \in C_0$ such that $||u|| = \sup_n |u_n| = 1$, it follows that

$$|u_n| \quad \leq \quad 1$$

for all $n \geq 1$, so

$$|f(u)| \leq \quad \textstyle\sum_{n=1}^{\infty} |\frac{1}{2^n} u_n| = 1.$$

Therefore,

$$||f|| \quad = \quad \sup_{||u||=1} |f(u)| \leq 1.$$

Now, fix $\epsilon > 0$. Choose $N > 1$ such that

$$n \geq N \implies \sum_{k=1}^{n} \frac{1}{2^k} > 1 - \epsilon.$$

Set $u \in c_0$ as

$$u \quad = \quad 1 \ (n \leq N) \ \text{ and } \ u = 0 \ (n > N).$$

Then, $u \in c_0$, $||u|| = 1$, and $|f(u)| > 1 - \epsilon$. Therefore, it follows that

$$1 - \epsilon < ||f||$$

for any $\epsilon > 0$, so $||f|| \geq 1$, which combined with the previous estimate gives $||f|| = 1$. □

(2) Suppose for sake of contradiction that there exists $u \in c_0$, such that

$$||u|| = 1 \ \text{ and } \ f(u) = 1.$$

Choose $N > 1$ such that

$$n \geq N \quad \implies \quad u_n < \frac{1}{2}.$$

Then,

$$f(u) \quad < \quad \sum_{n=1}^{N-1} \frac{1}{2^n} u_n + \frac{1}{2} \sum_{n=N}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{N-1} \frac{1}{2^n} u_n + \frac{1}{2^{N-1}}.$$

Since $||u|| = 1$, continuing the above estimate gives

$$f(u) \quad < \quad 1 - \frac{1}{2^{N-1}} + \frac{1}{2^{N-1}} = 1,$$

which is absurd. □

Question 6.

$\boxed{1.6}$ Let E be an n.v.s. and let $H \subset E$ be a hyperplane. Let $V \subset E$ be an affine subspace containing H.

1. Prove that either $V = H$ or $V = E$.
2. Deduce that H is either closed or dense in E.

Solution.
We first show that $\overline{C}$ is convex. Let $x, y \in \overline{C}$, and $t \in [0, 1]$. Choose, $\{x_n\}, \{y_n\} \subset C$ such that $x_n \to x$ and $y_n \to y$. By convexity of C, and linearity of limit, it follows that

$$\{tx_n + (1-t)y_n\} \subset C \quad \text{and} \quad tx_n + (1-t)y_n \to tx + (1-t)y.$$

Therefore, $tx + (1-t)y \in C$, which proves the convexity of $\overline{C}$.

Question 7.

$\boxed{1.7}$ Let E be an n.v.s. and let $C \subset E$ be convex.

1. Prove that $\overline{C}$ and $\operatorname{Int} C$ are convex.
2. Given $x \in C$ and $y \in \operatorname{Int} C$, show that $tx + (1-t)y \in \operatorname{Int} C \quad \forall t \in (0, 1)$.
3. Deduce that $\overline{C} = \overline{\operatorname{Int} C}$ whenever $\operatorname{Int} C \neq \emptyset$.

Solution.
(1) We first show that $\overline{C}$ is convex. Let $x, y \in \overline{C}$, and $t \in [0,1]$. Choose, $\{x_n\}, \{y_n\} \subset C$ such that $x_n \to x$ and $y_n \to y$. By convexity of C, and linearity of limit, it follows that

$$\{tx_n + (1-t)y_n\} \subset C \quad \text{and} \quad tx_n + (1-t)y_n \to tx + (1-t)y.$$

Therefore, $tx + (1-t)y \in \overline{C}$, which proves the convexity of $\overline{C}$. We now show that $\int C$ is convex. Let $x, y \in \int C$, and $t \in [0,1]$. By convexity of C,

$$tx + (1-t)y \in C$$

We now show that $\int C$ is convex. Let $x, y \in \text{int}C$ and $t \in (0,1)$.

(2) Suppose $x \in C$, $y \in \int C$, and $t \in (0,1)$.

(3) It is trivial that $\overline{\text{int}C} \subset \overline{C}$. Hence, it suffices to show that $\overline{C} \subset \overline{\int C}$.

Question 8.

$\boxed{1.8}$ Let E be an n.v.s. with norm $\| \ \|$. Let $C \subset E$ be an open convex set such that $0 \in C$. Let p denote the gauge of C (see Lemma 1.2).

1. Assuming C is symmetric (i.e., $-C = C$) and C is bounded, prove that p is a norm which is equivalent to $\| \ \|$.

2. Let $E = C([0,1]; \ \mathbb{R})$ with its usual norm

$$\|u\| = \max_{t \in [0,1]} |u(t)|.$$

Let

$$C = \left\{ u \in E; \ \int_0^1 |u(t)|^2 dt < 1 \right\}.$$

Check that C is convex and symmetric and that $0 \in C$. Is C bounded in E? Compute the gauge p of C and show that p is a norm on E. Is p equivalent to $\| \ \|$?

Solution.
(1) We first show that p is in fact a norm. By properties of any gauge of C, it suffices to show

$$p(x) = 0 \qquad \Longleftrightarrow \qquad x = 0.$$

If $x = 0$, then

$$\alpha > 0 \implies a^{-1}x = 0 \in C,$$

so $p(x) = 0$. Conversely, suppose that $p(x) = 0$. Firstly, let

$$I \quad = \quad \{\lambda > 0\,;\ \lambda^{-1}x \in C\}, \ \text{ and } \ 0 < \alpha < \beta.$$

We claim that

$$\beta \notin I \quad \implies \alpha \notin I.$$

We prove the contrapositive. Suppose $\alpha \in I$. Then, $\alpha^{-1}x \in C$. By convexity of C, it follows that

$$\frac{\beta^{-1}}{\alpha^{-1}}\alpha^{-1}x + (1 - \frac{\beta^{-1}}{\alpha^{-1}})0 \quad = \quad \beta^{-1}x \in C,$$

so $\beta \in I$. Therefore, to prove $p(x) > 0$, it suffices to show that there is a constant $k > 0$ such that $k^{-1}x \notin C$. Now, suppose for sake of contradiction that $x \neq 0$. Choose r large enough such that $C \subset B(r, 0)$ strictly. Then,

$$\frac{r}{||x||}x \in C \ \text{ and } \ 0 < \frac{||x||}{r} \in I,$$

which as discussed above implies that $p(x) > 0$. Hence, $x = 0$ as required.

(2) We claim that C is not bounded. Fix $r > 0$. Set

$$f \quad = \quad \sqrt{t}X_{[0, \frac{1}{2r}]} + (r - \sqrt{t})X_{(\frac{1}{2r}, \frac{1}{r}]}$$

Question 9.

$\boxed{1.9}$ *Hahn–Banach in finite-dimensional spaces.*

Let E be a finite-dimensional normed space. Let $C \subset E$ be a nonempty convex set such that $0 \notin C$. We claim that there always exists some hyperplane that separates C and $\{0\}$.

[Note that every hyperplane is closed (why?). The main point in this exercise is that no additional assumption on C is required.]

1. Let $(x_n)_{n\geq 1}$ be a countable subset of C that is dense in C (why does it exist?). For every n let

$$C_n = \operatorname{conv}\{x_1, x_2, \ldots, x_n\} = \left\{ x = \sum_{i=1}^{n} t_i x_i;\ t_i \geq 0\ \forall i \text{ and } \sum_{i=1}^{n} t_i = 1 \right\}.$$

   Check that C_n is compact and that $\bigcup_{n=1}^{\infty} C_n$ is dense in C.
2. Prove that there is some $f_n \in E^{\star}$ such that

$$\|f_n\| = 1 \text{ and } \langle f_n, x\rangle \geq 0 \quad \forall x \in C_n.$$

3. Deduce that there is some $f \in E^{\star}$ such that

$$\|f\| = 1 \text{ and } \langle f, x\rangle \geq 0 \quad \forall x \in C.$$

   Conclude.
4. Let $A, B \subset E$ be nonempty disjoint convex sets. Prove that there exists some hyperplane H that separates A and B.

Solution.

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Question 10.

$\boxed{1.10}$ Let E be an n.v.s. and let I be any set of indices. Fix a subset $(x_i)_{i\varepsilon I}$ in E and a subset $(\alpha_i)_{i\varepsilon I}$ in $\mathbb{R}$. Show that the following properties are equivalent:

(A) There exists some $f \in E^\star$ such that $\langle f, x_i \rangle = \alpha_i \quad \forall i \in I$.

(B) $\begin{cases} \text{There exists a constant } M \geq 0 \text{ such that for each finite subset} \\ J \subset I \text{ and for every choice of real numbers } (\beta_i)_{i \in J}\text{, we have} \\ \qquad \left| \sum\limits_{i \in J} \beta_i \alpha_i \right| \leq M \left\| \sum\limits_{i \in J} \beta_i x_i \right\|. \end{cases}$

Note that in the proof of (B) $\Rightarrow$ (A) one may find some $f \in E^\star$ with $\|f\|_{E^\star} \leq M$.
[**Hint:** Try first to define f on the linear space spanned by the $(x_i)_{i\varepsilon I}$.]

Solution.
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Question 11.

$\boxed{1.11}$ Let E be an n.v.s. and let $M > 0$. Fix n elements $(f_1)_{1\le i\le n}$ in $E^\star$ and n real numbers $(\alpha_i)_{1\le i\le n}$. Prove that the following properties are equivalent:

(A) $$\begin{cases} \forall \varepsilon > 0 \;\; \exists x_\varepsilon \in E \text{ such that} \\ \|x_\varepsilon\| \le M + \varepsilon \text{ and } \langle f_i, x_\varepsilon\rangle = \alpha_i \quad \forall i = 1, 2, \dots, n. \end{cases}$$

(B) $$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \le M \left\|\sum_{i=1}^{n} \beta_i f_i\right\| \quad \forall \beta_1, \beta_2, \dots, \beta_n \in \mathbb{R}.$$

[**Hint:** For the proof of (B) $\Rightarrow$ (A) consider first the case in which the f_i's are linearly independent and imitate the proof of Lemma 3.3.]

Compare Exercises 1.10, 1.11 and Lemma 3.3.

Solution.

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