Functional Analysis:
Problem Set I

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

Problem 1. Let Y be a closed subspace of a normed vector space E. Show that the dual of E/Y is isometrically isomorphic to $Y^{\perp}$.

Solution.
First, define a map $\Phi : (E \setminus Y)^* \to Y^{\perp}$ naturally by

$$f \quad \mapsto \quad (x \mapsto f([x]))$$

where $[x] \in E \setminus Y$. The map on the RHS is clearly linear and vanishes on Y, because for any $y \in Y$, $f([y]) = 0$ by linearity. The map is bounded as well, since

$$|f([x])| \quad \leq \quad ||f||||[x]|| \leq ||f||||x|| \;\; (*)$$

for any $x \in E$. Now, we show that Φ is an isometry. It follows that

$$||\Phi(f)|| \quad = \quad \sup_{||x||=1} | < \Phi(f), x > | = \sup_{||x||=1} |f([x])|$$

Hence, combined with $(*)$, it suffices to show that

$$\sup_{||x||=1} |f([x])| \quad \leq \quad \sup_{||[x]||=1} |f([x])|,$$

but this follows, since Y is closed, so we can choose $\{y_n\} \subset Y$ such that $\limsup_{n\to\infty} ||x+y_n|| \leq 1$. Now, we show that Φ is surjective. Let $l \in Y^{\perp}$. Then, define $f : E \setminus Y \to \mathbb{R}$ by

$$[x] \quad \mapsto \quad l(x).$$

The map is well-defined, since for any $x' \in [x]$ such that $x' = x + y$, $l([x']) = l(x + y) = l(x)$. and the map inherits linearity and boundedness from l. Now, it follows that $\Phi(f) = (x \mapsto f([x]) = (x \mapsto l(x)) = l$, and we are done.

$\square$

Question 2.

Problem 2. (Duality–Min-Max Principle) Let E be a normed vector space, and G a linear subspace of E. For any $x \in E$, define $m(x) = \inf_{y\in G} |x-y|$. Show that $m(x) = M(x)$, where $M(x) = \max_{\|f\|_{E^*}\le 1, f=0 \text{ on } G} |\langle f, x\rangle|$.

Similarly, for any $g \in E^*$, we define $\|g\|_G = \sup\{|g(y)| : y \in G, \quad \|y\| \le 1\}$. Then $\|g\|_G = \min\{\|g-h\|_{E^*} : h \in G^\perp\}$.

Solution.
Fix $x \in E$. For any $y \in G$, and $f \in E^*$ such that $||f|| \le 1$ with $f = 0$ on G,

$$|< f, x >| \quad = \quad |< f, x-y >| \le ||f||||x-y|| = ||x-y||,$$

so

$$m(x) \quad \ge \quad \sup_{|f||\le 1; f=0 \text{ on } G} |< f, x >|.$$

Hence, it suffices to show that there exists $f \in E^*$ such that $||f|| \le 1$ and $f = 0$ on G with $|< f, x >| = m(x)$. We can further assume without loss of generality that $x \in E \setminus G$, because if $x \in G$, both sides are trivially 0. Now, define a map $g : G \oplus \mathbb{R}x \to \mathbb{R}$ by $y + \lambda x \mapsto \lambda m(x)$. Then, we claim that g is linear. If $z_1 = y_1 + \lambda_1 x$ and $z_2 = y_2 + \lambda_2 x$, then

$$g(z_1) + g(z_2) = (\lambda_1 + \lambda_2)x = g(z_1 + z_2).$$

If $z = y + \lambda x$ and $\gamma \in \mathbb{R}$, then

$$g(\gamma z) = g(\gamma y + \gamma\lambda x) = \gamma\lambda m(x) = \gamma g(z).$$

Note that $g(x) = m(x)$. Now, in view of Hahn-Banach, we certainly need a Minkowski functional that bounds g on its domain. We show that m in fact is the Minkowski functional. Firstly,

$$m(\lambda v) \quad = \quad \inf_{y\in G} ||\lambda v - y|| = |\lambda| \inf_{y\in G} ||v - \frac{y}{\lambda}|| = |\lambda| m(v)$$

for any $\lambda \in \mathbb{R}$ and $v \in E$, because $y \mapsto \dfrac{y}{\lambda}$ is a bijection from G to G itself for any $\lambda \in \mathbb{R}$. Secondly,

$$m(u+v) \quad = \quad \inf_{y\in G} ||u+v-y|| \le \inf_{y\in G} ||u - \frac{1}{2}y|| + \inf_{y\in G} ||v - \frac{1}{2}y|| = m(u) + m(v)$$

for any $u, v \in E$. where the last equality holds by the same reasoning as above. Finally,

$$g(z) \quad = \quad g(y + \lambda x) = \lambda m(x) \le |\lambda| m(x) = m(y + \lambda x) = m(z)$$

for any $z = y + \lambda x \in G$. Therefore, by Hahn-Banach, we can extend g to the entire domain E and call it f. Since

$$f(z) \le m(z) \le ||z||,$$

for any $z \in E$, it follows that $||f|| \le 1$, $f = 0$ on G and $|< f, x >| = m(x)$ as required. $\square$

The second part follows similarly. Observe that

$$||g||_G = \sup_{||y||\le 1; y\in G} |< g, y >| = \sup_{||y||\le 1; y\in G} |< g-h, x >| \le ||g-h||_{E^*}$$

for any $g \in E^*$ and $h \in G^\perp$, so

$$||g||_G \quad \le \quad \inf_{h\in G^\perp} ||g-h||_{E^*}$$

for any $g \in E^*$. Now, g restricted to G is linear and bounded, so by a corollary of Hahn-Banach, there exists f such that $||f||_{E^*} = ||g||_G$. Set $h = g - f$. Since $f = g$ on G, $h \in G^\perp$, and $||g-h||_{E^*} = ||f||_{E^*} = ||g||_G$, so we are done. $\square$

Question 3.

Problem 3. Let E be a normed vector space.
(i) If Y is a closed proper subspace of E, then there is $x \in E$ such that $\|x\| = 1$ and $\|x - y\| > \frac{1}{2}$ for any $y \in Y$.
(ii) If E is of infinite dimension, then the unit ball $B_1 = \{x \in E : \|x\| \leq 1\}$ is never compact in strong topology.

Solution.
(i) We prove the following generalization, known as the Riesz Lemma: for each $\epsilon > 0$, there exists $x \in E$ such that $||x - y|| \geq 1 - \epsilon$, for any $y \in Y$.

Let $0 < \epsilon < 1$. Let $x \in E \setminus Y$. As Y is closed,

$$d \quad := \quad \text{dist}(x, Y) > 0.$$

Choose y^* in Y such that

$$d \leq ||x - y^*|| \leq \frac{d}{1 - \epsilon}. \quad (1)$$

Set $x^* = \dfrac{x - y^*}{||x - y^*||}$. Clearly, $||x^*|| = 1$, and, for any $y \in Y$,

$$\begin{aligned} ||x^* - y|| &= ||\frac{x - y^*}{||x - y^*||} - y|| = \frac{1}{||x - y^*||}||x - (y^* + y||x - y||^*)|| \\ &\geq \frac{d}{||x - y^*||} \leq 1 - \epsilon, \end{aligned}$$

where the last inequality follows from (1), and we are done. □

(ii) We proceed to construct a sequence $\{x_n\} \subset B_1$ such that there is no convergent subsequence, which shows that B_1 is not compact in strong topology through sequential characterization of compactness (strong topology is trivially metrizable).

Choose any $x \in E$ such that $||x|| = 1$ and set $x_1 = x$. Then, for any n, using (i), choose x_n such that

$$||x_n|| = 1 \ \text{ and } \ ||x_n - y|| > \frac{1}{2},$$

for any $y \in \text{span}(x_1, ..., x_{n-1})$, where the validity comes from the fact that any finite dimensional subspace is a proper, closed subspace of an infinite dimensional space. Then, it is clear that $\{x_n\}$ has no convergent subsequence, because for any $n \geq 1$, there exists $k, l \geq n$ with $k \neq l$, such that $||x_k - x_l|| > \dfrac{1}{2}$. Since being cauchy is a necessary condition for being convergent, we are done. □

Question 4.

Problem 4. Let $L^\infty[0,1]$ be the space of bounded, Lebesgue measurable functions on $[0,1]$. We define $l(f) = \int_0^1 f(t)\mathrm{d}t$. Then l is a positive, linear, continuous functional on $L^\infty[0,1]$. Here l is called positive if $l(f_2) \geq l(f_1)$ whenever $f_2 \geq f_1$. Define, for any bounded real-valued function g, $p(g) = \inf\{l(f) : g \leq f \in L^\infty[0,1]\}$. Show that
(i) p is a positive homogeneous, subadditive and $p(g) \leq 0$ whenever $g \leq 0$. Moreover $p(f) = l(f)$ if $f \in L^\infty[0,1]$.
(ii) l can be extended to a positive linear functional on the space of all bounded functions.

Solution.
(i) Let g be a bounded real-valued function and $\lambda > 0$. Then, by linearity of Lebesgue integration,

$$\begin{aligned} p(\lambda g) &= \inf\{l(h) : \lambda g \leq h \in L^\infty\} \\ \lambda p(g) &= \inf\{l(\lambda h) : g \leq h \in L^\infty\}. \end{aligned}$$

We claim that

$$A := \{l(h) : \lambda g \leq h \in L^\infty\} \quad = \quad \{l(\lambda h) : g \leq h \in L^\infty\} =: B$$

If $\lambda g \leq h \in L^\infty$, then $g \leq \dfrac{h}{\lambda} \in L^\infty$, so $l(\lambda \dfrac{h}{\lambda}) = l(\lambda) \in B$. Conversely, if $g \leq h \in L^\infty$ then, $\lambda g \leq \lambda h \in L^\infty$, so $l(\lambda h) \in A$. Hence, $p(\lambda g) = \lambda p(g)$.

We now show that p is sub-additive. Let f, g be bounded real functions. Then, for any $h_1, h_2 \in L^\infty$ such that $f \leq h_1$ and $g \leq h_2$,

$$f + g \quad \leq \quad h_1 + h_2 \in L^\infty,$$

so, again by linearity of integration,

$$p(f+g) \quad \leq \quad l(h_1 + h_2) = l(h_1) + l(h_2).$$

Taking infs for h_1, then h_2, gives

$$p(f+g) \quad \leq \quad p(f) + p(g),$$

as required.

For any f, g bounded real-valued functions,

$$\begin{aligned} p(f+g) &= \inf\{l(f+g) : f + g \leq h, h \in L^\infty[0,1]\} \\ &= \inf\{l(f) + l(g) : f + g \leq h, h \in L^\infty[0,1]\}. \end{aligned}$$

Suppose $g \leq 0$. Then, as $0 \in L^\infty[0,1]$ and $l(0) = 0$, by definition, $p(g) \leq 0$.

We show that $p(f) = l(f)$ if $f \in L^\infty[0,1]$. For all $h \in L^\infty[0,1]$ such that $f \leq h$, then, by monotonicity of Lebesgue integration, $l(f) \leq l(h)$. Since $f \leq f$ trivially, it follows that $p(f) = l(f)$. $\square$

(ii) Now, as $l = p$ on $L^\infty[0,1]$, by Hahn-Banach, l can be extended to the entire space of bounded real-valued functions. This shows that we can make sense of integration for any bounded functions in a weaker sense, sacrificing some nice properties, such as countable additivity and so on(probably if such properties hold, then it will contradict existence of non-measurable sets by considering appropriate indicators). $\square$

Question 5.

Problem 5. Let E be a normed vector space. The norm of E is called uniformly convex if $\|\frac{x+y}{2}\| \leq 1-\varepsilon(\|x-y\|)$ for all $x, y \in \{z \in E : \|z\| = 1\}$. Here $\varepsilon(r)$ is an increasing and positive function defined for $r > 0$ such that $\lim_{r\to 0^+} \varepsilon(r) = 0$. Let K be a closed, convex subset in a Banach space E with uniformly convex norm. Prove that for any $x \in E$, there is a unique $y \in K$ such that $\|x-y\| = \inf\{\|x-z\| : z \in K\}$. (Hint: Assume $x = 0$ not in K, and let $\{y_n\} \subset K$ be a minimizing sequence, then consider $x_n = \frac{y_n}{\|y_n\|}$ and $\frac{x_m+x_n}{2}$.)

Solution.
Without loss of generality, assume $x = 0$ and $x \notin K$. Let $\{y_n\}$ be the minimizing sequence. As E is Banach, and K is closed, it suffices to show that $\{y_n\}$ is cauchy. Let $d = \lim_{n\to\infty} ||y_n||$. We claim that

$$||\frac{x_n + x_m}{2}|| \to 1 \quad \text{as} \quad n, m \to \infty.$$

For each $n, m \geq 1$,

$$\begin{aligned} ||\frac{x_n + x_m}{2}|| &= \frac{1}{2}\frac{||y_m|| + ||y_n||}{||y_n||||y_m||}||\frac{||y_m||y_n}{||y_n|| + ||y_m||} + \frac{||y_n||y_m}{||y_n|| + ||y_m||}|| \\ &\geq \frac{1}{2}\frac{||y_m|| + ||y_n||}{||y_n||||y_m||}d \end{aligned}$$

by convexity of K. The RHS goes to 1 as $n, m \to \infty$ via $d = \lim_{n\to\infty} ||y_n||$ condition, and $||\frac{x_n + x_m}{2}|| \geq 1$. We have shown the claimed limit. Now, by uniform convexity, $\{x_n\}$ is cauchy. Observe that

$$||y_n - y_m|| \quad \leq \quad ||y_n||||x_n - x_m|| + |||y_n|| - ||y_m||||x_m||$$

for each $n, m \geq 1$. Since the RHS goes to 0 as $n, m \to \infty$, it implies that $\{y_n\}$ is cauchy and we are done. $\square$

Question 6.

Problem 6. Let E be a normed vector space , and O be a bounded open set in E such that it is convex and symmetric with respect to $\underline{0} \in O$ (i.e., $x \in O \Rightarrow -x \in O$). Then show that the Minkowski functional associated with O introduces a new norm of E.
(ii) Is this new norm equivalent to the original norm $\|\cdot\|$.
(iii) If E is reflexive with respect to the original norm, does it implies then E is also reflexive with respect to the new norm?

Solution.
By ordinary properties of Minkowski functionals, it suffices to show that for any $x \in E$ and $\lambda \in \mathbb{R}$,

$$p(\lambda x) = |\lambda| p(x) \quad \text{and} \quad p(x) = 0 \implies x = 0.$$

We first prove the absolute homogeneity of p. Now, if $\lambda \geq 0$, then $p(\lambda x) = \lambda p(x)$ by positive homogeneity of Minkowski functionals. Now, if $\lambda \geq 0$, then, by symmetry, and positive homogeneity again, we obtain

$$p(\lambda x) = p(-\lambda x) = -\lambda p(x),$$

which completes the proof of absolute homogeneity.

Now, observe that, for any $0 < \alpha < \beta$, and $x \in E$,

$$\alpha^{-1} x \in C \quad \implies \beta^{-1} x \in C, \quad (**)$$

because by convexity

$$(1 - \frac{\beta^{-1}}{\alpha^{-1}})0 + \frac{\beta^{-1}}{\alpha^{-1}} \alpha^{-1} x = \beta^{-1} x \in C.$$

Let $x \in E$ such that $p(x) = 0$, Then, by the above discussion, it follows that

$$\alpha^{-1} x \quad \in \quad C \quad (*),$$

for any $\alpha \in (0, \infty)$. Suppose $x \neq 0$, and let $r > 0$ large enough that $C \subset B(0, r)$. Then, it follows that, from $(*)$, $\frac{r}{||x||} x \in C$, which contradicts the fact that $C \subset B(0, r)$. Therefore, $x = 0$ and we are done. ☐

(i) The new norm is equivalent with the original norm. Choose $r > 0$ such that $\overline{B(0, r)} \subset O$. Then,

$$\frac{r}{||x||} \quad \in \quad B(0, r) \subset O$$

and

$$rp(x) \quad \leq \quad ||x||$$

for any $x \in E$. For the other direction, choose $R > 0$ such that $O \subset B(0, R)$ with $\partial B(0, R) \cap O = \emptyset$. Then,

$$\frac{R}{||x||} x \quad \notin \quad O$$

for any $x \in E$. By the contrapositive of $(**)$, it follows that

$$||x|| \quad \leq \quad Rp(x)$$

for any $x \in E$. Hence, the claimed norm equivlaence holds.

(ii) Reflexive implies that the space is Banach (so there is no confusion with definition).

We prove the following claim: Let E be Banach space that is reflexive. For any normed linear space, defined by an equivalent norm on E, is reflexive.

Let Q be a norm on E such that E is reflexive, and P be a norm that is equivalent to Q. Since the topology induced by Q and P are the same, we have that $E_Q^* = E_P^*$, which we denote as E^*. Now, we claim that the dual norms Q^* and P^* on E^*, induced by Q and P are again equvialent. From the norm equivalence and defintion of dual norm,

$$|f(x)| \quad \leq \quad Q^*(f)Q(x)$$

and

$$|f(x)| \quad \leq \quad Q^*(f)CP(x)$$

for some $C > 0$, and for any $f \in E^*$ and $x \in E$. Therefore,

$$P^*(f) \quad \leq \quad CQ^*(f)$$

for any $f \in E^*$. The other direction can be shown similarly, so we have shown that the induced dual norms are again equvialent. Now, by the same argument as above, we see that $E_Q^{**} = E_P^{**}$, from which it follows that $J_Q = J_P$. Therefore, by reflexivity assumption, we have

$$E_P^{**} \quad = \quad E_Q^{**} = J_Q(E) = J_P(E)$$

which shows that (E, P) is reflexive as required. Therefore, we have shown that the new norm is reflexive, whenever the original norm is reflexive. $\square$

Question 7.

Problem 7. Let E be the space of bounded Lebesgue measurable functions on [a,b]. Find a sequence $\{f_n\} \subset E^*$ such that $f_n(x) \to 0$ for all $x \in E$ and $\|f_n\|_{E^*} = 1$. Is E separable, that is, does E has a countable dense set?

Solution.
Consider $\{f_n\}$ defined by

$$g \quad \mapsto \quad \int_{\mathbb{T}} g(t)e^{-2\pi int}dt = \hat{g}(n) \quad (g \in L^\infty(\mathbb{T}))$$

for each $n \geq 1$. As $L^\infty(\mathbb{T}) \subset L^1(\mathbb{T})$, by Riemann-Lebesgue lemma, for any $g \in L^\infty(\mathbb{T})$,

$$f_n(g) = \hat{g}(n) \to 0.$$

For any $n \geq 1$, $g \in L^\infty(\mathbb{T})$ with $||g|| = 1$,

$$|\hat{g}(n)| \quad \leq \quad \int_{\mathbb{T}} |g| \leq ||g||_\infty = 1,$$

Now, for any $n \geq 1$, take $g = e^{2\pi int}$ to see

$$|\hat{g}(n)| \quad = \quad |\int_{\mathbb{T}} 1dt| = 1,$$

so, combined with the above estimate, we have

$$||f_n|| = 1,$$

as required.

It is a well-known fact that L^∞ is not separable, given that the ambient space is not finite. Instead of appealing to this general result, we provide a construction of an uncountable family of functions in $L^\infty[a,b]$ with $a < b$ such that the distance is at least 1 apart between any two elements to contradict the separability assumption. Take $[a,b]$ such that $a < b$. Fix $x_0 \in (a,b)$ and consider $\mathscr{A} = \{1_{B(x_0,r)}\}$ where $0 < r \leq \min(x_0 - a, b - x_0)$. Then, the family is uncountable, but for any $f, g \in \mathscr{A}$,

$$||f - g||_\infty \quad = \quad 1.$$

Now, suppose the space is separable, hence there exists a countable dense subset C. Now, observe that

$$\bigcup_{x \in \mathscr{A}} B(x, \frac{1}{2}) \cap C \subset C,$$

but the left hand side is uncountable, because its an uncountable disjoint union of countable sets. Therefore, $L^\infty[a,b]$ is not separable. $\square$