Functional Analysis: Problem Set I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

Problem 1. Let Y be a closed subspace of a normed vector space E. Show that the dual of E/Y is isometrically isomorphic to $Y^{\perp}$.

Solution.

Question 2.

Problem 2. (Duality–Min-Max Principle) Let E be a normed vector space, and G a linear subspace of E. For any $x \in E$, define $m(x) = \inf_{y \in G} |x - y|$. Show that $m(x) = M(x)$, where $M(x) = \max_{\|f\|_{E^*} \leq 1, f=0 \text{ on } G} |\langle f, x \rangle|$.

Similarly, for any $g \in E^*$, we define $\|g\|_G = \sup\{|g(y)| : y \in G, \quad \|y\| \leq 1\}$. Then $\|g\|_G = \min\{\|g - h\|_{E^*} : h \in G^{\perp}\}$.

Solution.

Question 3.

Problem 3. Let E be a normed vector space.
(i) If Y is a closed proper subspace of E, then there is $x \in E$ such that $\|x\| = 1$ and $\|x - y\| > \frac{1}{2}$ for any $y \in Y$.
(ii) If E is of infinite dimension, then the unit ball $B_1 = \{x \in E : \|x\| \leq 1\}$ is never compact in strong topology.

Solution.
(i)

(ii) We proceed to construct a sequence $\{x_n\} \subset B_1$ such that there is no convergent subsequence, which shows that B_1 is not compact in strong topology through sequential characterization of compactness (strong topology is trivially metrizable).

Choose any $x \in E$ such that $||x|| = 1$ and set $x_1 = x$. Then, for any n, using (i), choose x_n such that

$$||x_n|| = 1 \text{ and } ||x_n - y|| > \frac{1}{2},$$

for any $y \in$ span$(x_1, ..., x_{n-1})$, where the validity comes from the fact that any finite dimensional subspace is a proper, closed subspace of an infinite dimensional space. Then, it is clear that $\{x_n\}$ has no convergent subsequence, because for any $n \geq 1$, there exists $k, l \geq n$ with $k \neq l$, such that $||x_k - x_l|| > \dfrac{1}{2}$. Since being cauchy is a necessary condition for being convergent, we are done.

Question 4.

Problem 4. Let $L^\infty[0,1]$ be the space of bounded, Lebesgue measurable functions on $[0,1]$. We define $l(f) = \int_0^1 f(t)\mathrm{d}t$. Then l is a positive, linear, continuous functional on $L^\infty[0,1]$. Here l is called positive if $l(f_2) \geq l(f_1)$ whenever $f_2 \geq f_1$. Define, for any bounded real-valued function g, $p(g) = \inf\{l(f) : g \leq f \in L^\infty[0,1]\}$. Show that
(i) p is a positive homogeneous, subadditive and $p(g) \leq 0$ whenever $g \leq 0$. Moreover $p(f) = l(f)$ if $f \in L^\infty[0,1]$.
(ii) l can be extended to a positive linear functional on the space of all bounded functions.

Solution.

Question 5.

Problem 5. Let E be a normed vector space. The norm of E is called uniformly convex if $\|\frac{x+y}{2}\| \leq 1 - \varepsilon(\|x-y\|)$ for all $x, y \in \{z \in E : \|z\| = 1\}$. Here $\varepsilon(r)$ is an increasing and positive function defined for $r > 0$ such that $\lim\limits_{r\to 0^+} \varepsilon(r) = 0$. Let K be a closed, convex subset in a Banach space E with uniformly convex norm. Prove that for any $x \in E$, there is a unique $y \in K$ such that $\|x - y\| = \inf\{\|x - z\| : z \in K\}$. (Hint: Assume $x = 0$ not in K, and let $\{y_n\} \subset K$ be a minimizing sequence, then consider $x_n = \frac{y_n}{\|y_n\|}$ and $\frac{x_m + x_n}{2}$.)

Solution.

Question 6.

Problem 6. Let E be a vector space with a metric, and O be a bounded open set in E such that it is convex and symmetric with respect to $\underline{0} \in O$ (i.e., $x \in O \Rightarrow -x \in O$). Then show that the Minkowski functional associated with O introduces a norm of E.

Solution.

Question 7.

Problem 7. Let E be the space of bounded Lebesque measurable functions on [a,b]. Find a sequence $\{f_n\} \subset E^*$ such that $f_n(x) \to 0$ for all $x \in E$ and $\|f_n\|_{E^*} = 1$. Is E separable, that is, does E has a countable dense set?

Solution.