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# Functional Analysis:

## Problem Set II

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### Abstract

This work contains solutions to the exercises of the problem set II.

#### Question 1.

**Problem 1.** (Exercise 2.2 in the textbook) Let  $E$  be a vector space and let  $p : E \rightarrow \mathbb{R}$  be a function with the following three properties:

- (i)  $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in E$ ,
- (ii) for each fixed  $x \in E$  the function  $\lambda \rightarrow p(\lambda x)$  is continuous from  $\mathbb{R}$  into  $\mathbb{R}$ ,
- (iii) whenever a sequence  $(y_n)$  in  $E$  satisfies  $p(y_n) \rightarrow 0$ , then  $p(\lambda y_n) \rightarrow 0$  for every  $\lambda \in \mathbb{R}$ .

Assume that  $(x_n)$  is a sequence in  $E$  such that  $p(x_n) \rightarrow 0$  and  $(\alpha_n)$  is a bounded sequence in  $\mathbb{R}$ . Prove that  $p(0) = 0$  and that  $p(\alpha_n x_n) \rightarrow 0$ .

#### Solution.

Fix  $\epsilon > 0$ . Suppose for sake contradiction that there exists a subsequence  $\{a_{n_k} x_{n_k}\}$  such that

$$|p(a_{n_k} x_{n_k})| \geq 2\epsilon \quad (*)$$

for all  $k \geq 1$ . Since  $\{a_n\}$  is bounded, passing to a further subsequence, and relabeling, we may suppose that

$$|p(a_n x_n)| \geq 2\epsilon \quad \text{and} \quad \lim_{n \rightarrow \infty} a_n = a$$

for any  $n \geq 1$  and for some  $a \in \mathbb{R}$ . Now, observe that  $\phi_k : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\lambda \mapsto |p(\lambda x_k)| \quad (\lambda \in \mathbb{R})$$

for each  $k \geq 1$  is continuous by (ii). Therefore,

$$F_n = \bigcap_{k=n}^{\infty} \phi_k^{-1}([- \epsilon, \epsilon])$$

is closed for each  $n \geq 1$  ( $F_n$  given in the hint). By assumption and (iii), it follows that

$$\bigcup_n F_n = \mathbb{R}$$

and by Baire-Category, we can choose  $n_0 \in \mathbb{N}$  such that there exists  $\lambda_0 \in \mathbb{R}$  and  $\delta > 0$  such that

$$B(\lambda_0, \delta) \subset F_{n_0}.$$

Now, by (i), we obtain

$$p(a_k x_k) \leq p((\lambda_0 + a_k - a)x_k) + p((a - \lambda_0)x_k)$$

and

$$-p(a_k x_k) \leq -p((\lambda_0 + a_k - a)x_k) + p((\lambda_0 - a)x_k)$$

for each  $k \geq 1$ . Now for all  $k$  large enough, since  $(a - \lambda_0), (\lambda_0 - a)$  are fixed constants, we have

$$(\lambda_0 + a_k - a) \in B(\lambda_0, \delta) \quad \text{and} \quad |p((a - \lambda_0)x_k)|, |p((\lambda_0 - a)x_k)| < \epsilon$$

so

$$|p(a_k x_k)| < 2\epsilon,$$

which contradicts (\*). By (i),  $p(0) \leq 2p(0)$ , and  $p(0) \leq p(x_n) + p(-x_n)$  for all  $n \geq 1$ , so  $p(0) \leq 0$ . Therefore,  $p(0) = 0$ .  $\square$

## Question 2.

**Problem 2.** (Exercise 2.4 in the textbook) Let  $E$  and  $F$  be two Banach spaces and let  $a : E \times F \rightarrow \mathbb{R}$  be a bilinear form satisfying:

- (i) for each fixed  $x \in E$ , the map  $y \rightarrow a(x, y)$  is continuous;
- (ii) for each fixed  $y \in F$ , the map  $x \rightarrow a(x, y)$  is continuous.

Prove that there exists a constant  $C \geq 0$  such that

$$|a(x, y)| \leq C \|x\| \|y\| \quad \forall x \in E, \quad \forall y \in F.$$

## Solution.

Define a map  $T$  on  $E$  by

$$x \mapsto T_x \quad \text{with} \quad \langle T_x, y \rangle = a(x, y).$$

By (i),  $T_x$  is bounded for all  $x \in E$ , so  $T$  maps  $E$  into  $F^*$ . We now show that  $T$  is bounded. Fix  $y \in F$ . Observe that

$$\{\langle T_x, y \rangle : x \in B_E(0, 1)\} = \{a(x, y) : x \in B_E(0, 1)\}$$

is bounded by (ii). Therefore, by Corollary 2.5, which follows from Uniform boundedness principle,  $T$  is bounded. Now, by the boundedness of  $T$ , there exists  $C > 0$  such that

$$\sup_{y \in F; \|y\| \neq 0} \frac{|a(x, y)|}{\|y\|} = \|T_x\| \leq C \|x\|$$

for any  $x \in E$ , and hence

$$a(x, y) \leq C \|x\| \|y\|$$

for any  $x \in E$  and  $y \in F$ . □

### Question 3.

**Problem 3.** (Exercise 2.10 in the textbook) Let  $E$  and  $F$  be two Banach spaces and let  $T \in L(E, F)$  be surjective.

1. Let  $M$  be any subset of  $E$ . Prove that  $T(M)$  is closed in  $F$  iff  $M + N(T)$  is closed in  $E$ .
2. Deduce that if  $M$  is a closed vector space in  $E$  and  $\dim N(T) < \infty$ , then  $T(M)$  is closed.

### Solution.

Suppose that  $T(M)$  is closed. Then, by continuity of  $T$ ,

$$M + N(T) = T^{-1}(T(M)) \text{ is closed.}$$

Conversely, suppose that  $M + N(T)$  is closed. We contend that

$$T((M + N(T))^c) = T(M)^c. \quad (*)$$

Let  $x \in T((M + N(T))^c)$ . Then, there exists  $y \in (M + N(T))^c$ , such that  $T(y) = x$ . Suppose for sake of contradiction that  $x \in T(M)$ . Then, there exists  $x_0 \in M$ , such that  $T(x_0) = y$ , and by linearity of  $T$ ,  $x - x_0 \in N(T)$ , so  $x \in M + N(T)$ , a contradiction. For the other inclusion, let  $x \in T(M)^c$ . By surjectivity of  $T$ , there exists  $z \in E$  such that  $T(z) = x$ . Suppose for sake of contradiction that  $z = m + n$  with some  $m \in M$  and  $n \in N(T)$ . By linearity of  $T$ ,

$$x = T(z) = T(m),$$

which contradicts that  $x \in T(M)^c$ . Therefore,  $(*)$  is true. To conclude, observe that, by open mapping theorem,  $T(M)^c$  is open, so  $T(M)$  is closed. Now, (ii) follows from the problem 5 and (i).  $\square$

#### Question 4.

**Problem 4.** (Exercise 2.14 in the textbook) Let  $E$  and  $F$  be two Banach spaces.

1. Let  $T \in \mathcal{L}(E, F)$ . Prove that  $R(T)$  is closed iff there exists a constant  $C$  such that  $\text{dist}(x, N(T)) \leq C\|Tx\|$ ,  $\forall x \in E$ .
2. Let  $A : D(A) \subset E \rightarrow F$  be a closed unbounded operator. Prove that  $R(A)$  is closed iff there exists a constant  $C$  such that  $\text{dist}(u, N(A)) \leq C\|Au\|$   $\forall u \in D(A)$ .

#### Solution.

(1) Let  $\tilde{E} = E/N(T)$ ,  $\pi$  be the canonical quotient map, and  $\tilde{T} : E/N(T) \rightarrow F$  such that  $T = \tilde{T} \circ \pi$ . From definition, we know that  $\tilde{T}$  is linear, injective, and  $R(T) = R(\tilde{T})$ .

Suppose the given estimate holds. We claim that  $R(\tilde{T})$  is closed, then by  $R(T) = R(\tilde{T})$ , we will be done. Observe that

$$\|[x]\| = d(x, N(T)) \leq C\|Tx\| = C\|\tilde{T}[x]\| (*)$$

for all  $x \in E$ . Suppose  $\{y_n\} \subset R(\tilde{T})$  such that  $y_n \rightarrow y$  for some  $y$ . Then, there exists  $\{[x_n]\} \subset \tilde{E}$ . By the above estimate,  $\{[x_n]\}$  is Cauchy, and by continuity  $\tilde{T}[x_n] \rightarrow \tilde{T}[x]$  where  $[x]$  is the limit of  $\{[x_n]\}$ . Therefore,  $R(\tilde{T})$  is closed.

Suppose  $R(T)$  is closed. Then,  $R(\tilde{T})$  is closed, and hence Banach. Since  $\tilde{T}$  is bijective, by Corollary 2.7,  $\tilde{T}^{-1}$  is continuous so we again have (\*).

(2) Consider  $D(A)$  with graph norm. Then, as  $A$  is closed,  $D(A)$  is Banach. Since graph norm only increases the norm from each norm,  $T$  is bounded and by (1) we are done.

**Question 5.**

**Problem 5.** Let  $G$  be a closed subspace of a Banach space  $E$ . Assume  $L$  is a finite dimensional subspace of  $E$ , then  $G + L$  is a closed linear subspace. Moreover,  $G + L$  admits a complement if and only if  $G$  does.

**Solution.**

Let  $\pi$  be the canonical projection of  $E$  onto  $E/G$ . As  $L$  is finite dimensional space, we see that  $\pi(L)$  is finite dimensional, hence closed. By continuity of  $\pi$ , it follows that  $\pi^{-1}(\pi(L)) = G + L$  is closed.

Suppose  $G$  admits a complement  $H$  in  $E$ . Let  $\pi_G$  and  $\pi_H$  be canonical projections of  $G$  and  $H$  respectively. Observe that  $\pi_H(L)$  is finite dimensional, so it admits a complement  $A \subset H$  in  $H$ . We claim that  $A$  is a complement of  $G + L$ . Note that  $A$  is closed trivially.

**Question 6.**

**Problem 6.** Let  $S_N(f, x)$  be the  $N^{\text{th}}$ -partial sum of the Fourier series of  $f(x) \in L^1[-\pi, \pi]$ , that is,

$$S_N(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N + \frac{1}{2})(x - \theta)}{\sin \frac{1}{2}(x - \theta)} f(\theta) d\theta.$$

Show that there is a continuous  $2\pi$ -periodic function  $f(x)$  such that  $|S_N(f, 0)| \rightarrow +\infty$  as  $N \rightarrow \infty$ .

**Solution.**

For convenience, we identify  $\mathbb{T}$  with  $[0, 2\pi]$  and consider 0 as the point, where we study the divergence. Let  $\{\phi_n\}$  be a collection of continuous, linear functionals (standard property of Fourier series), defined on  $C(\mathbb{T})$  given by

$$\phi_n(f) = S_n(f, 0)$$

for all  $f \in C(\mathbb{T})$  and  $n \in \mathbb{N}$ . Suppose for a moment that  $\{\phi_n\}$  are not uniformly bounded. Then, by uniform boundedness principle, there exists  $f \in C(\mathbb{T})$  such that  $\{\phi_n(f)\}$  is not bounded.

We now show that  $\{|\phi_n|\}$  is not bounded. Since  $|\sin(t)| \leq t$  for any  $t \in [0, 2\pi]$ ,

$$\begin{aligned} \int_0^{2\pi} \left| \frac{\sin(n + \frac{1}{2})x}{\sin(\frac{x}{2})} \right| dx &\geq \int_0^{2\pi} \left| \sin(n + \frac{1}{2})x \right| \frac{2}{x} dx = \int_0^{2\pi(n + \frac{1}{2})} |\sin(x)| \frac{2}{x} dx \\ &\geq \sum_{k=1}^n \frac{1}{k} \int_{2\pi(k-1)}^{2\pi k} |\sin(x)| dx \geq \sum_{k=1}^n \frac{1}{k} \end{aligned}$$

for all  $n \in \mathbb{N}$ . The above estimate shows that the  $L^1$  norms of the  $n$ -th Dirichlet kernels associated with  $\phi_n$  diverges to  $\infty$ . Now, it is well-known that the functional norm of  $\phi_n$  is exactly the  $L^1$  norm of the  $n$ -th Dirichlet kernel for all  $n \in \mathbb{N}$ . This can be formally shown by considering the sign of the kernel as the continuous function, and using DCT to swap the order of limit and integration.  $\square$

**Question 7.**

**Problem 7.** Let  $L^1(S^1)$  be the space of Lebesgue integrable functions on the unit circle  $S^1$ . We define a product on  $L^1(S^1)$  (convolution):

$$\forall f, g \in L^1(S^1), \quad f * g(\theta) = \int_0^{2\pi} f(\theta - x)g(x)dx.$$

Show that  $\|f * g\| \leq \|f\|\|g\|$ , when  $\|h\| = \int_0^{2\pi} |h(\theta)|d\theta$ . (This makes  $L^1(S^1)$  a Banach algebra).

**Solution.**

By Tonelli's theorem and the translation invariance property of Lebesgue measure,

$$\begin{aligned} \|f * g\| &= \int_0^{2\pi} \left| \int_0^{2\pi} f(t - x)g(x)dx \right| dt \leq \int_0^{2\pi} \int_0^{2\pi} |f(t - x)g(x)| dx dt \\ &= \int_0^{2\pi} \int_0^{2\pi} |f(t - x)g(x)| dt dx = \int_0^{2\pi} |g(x)| \int_0^{2\pi} |f(t - x)| dt dx \\ &= \|f\| \int_0^{2\pi} |g(x)| = \|f\|\|g\| \end{aligned}$$

for any  $f, g \in L^1(S^1)$ . □



### Question 8.

**Problem 8.** Let  $\mathcal{A} = \{f(\theta) = \sum_{n=-\infty}^{+\infty} c_n e^{in\theta}, \theta \in [0, 2\pi], c_n \in \mathbb{C}\}$  with the norm  $\|f\| =$

$\sum_{n=-\infty}^{+\infty} |c_n| < \infty$ . Show that

(a)  $(\mathcal{A}, \|\cdot\|)$  is a Banach space.

(b) Show that  $\|fg\| \leq \|f\|\|g\|$  (In fact,  $(\mathcal{A}, \|\cdot\|)$  is a Banach Algebra).

(c)  $f_0 \equiv 1$  is the unit element of this Algebra.

(d) A homomorphism  $h : \mathcal{A} \rightarrow \mathbb{C}$  means  $h(f \cdot g) = h(f)h(g)$ . For example, given any  $\theta_0 \in [0, 2\pi]$ ,  $h_{\theta_0} : \mathcal{A} \rightarrow \mathbb{C}$  defined by  $h_{\theta_0}(f) = f(\theta_0)$  is a homomorphism. Show that every homomorphism  $h : \mathcal{A} \rightarrow \mathbb{C}$  is of the form  $h_{\theta_0}$  for some  $\theta_0 \in [0, 2\pi]$ . [Hint:  $h(f_0) = 1$  and show first that  $h(e^{i\theta}) = e^{i\theta_0}$  for some  $\theta_0 \in [0, 2\pi]$ ].

Note that if  $f \in \mathcal{A}$  with  $|f| > 0$  on  $[0, 2\pi]$ , then  $\frac{1}{f} \in \mathcal{A}$ . The last conclusion is an interesting statement for Fourier series.

### Solution.