Functional Analysis:
Problem Set II

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the exercises of the problem set II.

Question 1.

Problem 1. (Exercise 2.2 in the textbook) Let E be a vector space and let $p : E \to \mathbb{R}$ be a function with the following three properties:
(i) $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in E$,
(ii) for each fixed $x \in E$ the function $\lambda \to p(\lambda x)$ is continuous from $\mathbb{R}$ into $\mathbb{R}$,
(iii) whenever a sequence (y_n) in E satisfies $p(y_n) \to 0$, then $p(\lambda y_n) \to 0$ for every $\lambda \in \mathbb{R}$.

Assume that (x_n) is a sequence in E such that $p(x_n) \to 0$ and (α_n) is a bounded sequence in $\mathbb{R}$. Prove that $p(0) = 0$ and that $p(\alpha_n x_n) \to 0$.

Solution.
Fix $\epsilon > 0$. Suppose for sake contradiction that there exists a subsequence $\{a_{n_k} x_{n_k}\}$ such that

$$|p(a_{n_k} x_{n_k})| \geq 2\epsilon \quad (*)$$

for all $k \geq 1$ Since $\{a_n\}$ is bounded, passing to a further subsequence, and relabeling, we may suppose that

$$|p(a_n x_n)| \geq 2\epsilon \quad \text{and} \quad \lim_{n\to\infty} a_n = a$$

for any $n \geq 1$ and for some $a \in \mathbb{R}$. Now, observe that $\phi_k : \mathbb{R} \to \mathbb{R}$ defined by

$$\lambda \quad \mapsto \quad |p(\lambda x_k)| \quad (\lambda \in \mathbb{R})$$

for each $k \geq 1$ is continuous by (ii). Therefore,

$$F_n \quad = \quad \bigcap_{k=n}^{\infty} {\phi_k}^{-1}([-\epsilon, \epsilon])$$

is closed for each $n \geq 1$ (F_n given in the hint). By assumption and (iii), it follows that

$$\bigcup_n F_n \quad = \quad \mathbb{R}$$

and by Baire-Category, we can choose $n_0 \in \mathbb{N}$ such that there exists $\lambda_0 \in \mathbb{R}$ and $\delta > 0$ such that

$$B(\lambda_0, \delta) \subset F_{n_0}.$$

Now, by (i), we obtain

$$p(a_k x_k) \quad \leq \quad p((\lambda_0 + a_k - a)x_k) + p((a - \lambda_0)x_k)$$

and

$$-p(a_k x_k) \quad \leq \quad -p((\lambda_0 + a_k - a)x_k) + p((\lambda_0 - a)x_k)$$

for each $k \geq 1$. Now for all k large enough, since $(a - \lambda_0), (\lambda_0 - a)$ are fixed constants, we have

$$(\lambda_0 + a_k - a) \in B(\lambda_0, \delta) \text{ and } |p((a - \lambda_0)|, |p(\lambda_0 - a)| < \epsilon$$

so

$$|p(a_k x_k)| \quad < \quad 2\epsilon,$$

which contradicts $(*)$. By (i), $p(0) \leq 2p(0)$, and $p(0) \leq p(x_n) + p(-x_n)$ for all $n \geq 1$, so $p(0) \leq 0$. Therefore, $p(0) = 0$. $\square$

Question 2.

Problem 2. (Exercise 2.4 in the textbook) Let E and F be two Banach spaces and let $a : E \times F \to \mathbb{R}$ be a bilinear form satisfying:
(i) for each fixed $x \in E$, the map $y \to a(x, y)$ is continuous;
(ii) for each fixed $y \in F$, the map $x \to a(x, y)$ is continuous.
Prove that there exists a constant $C \geq 0$ such that

$$|a(x, y)| \leq C\|x\|\|y\| \quad \forall x \in E, \quad \forall y \in F.$$

Solution.
Define a map T on E by

$$x \mapsto T_x \quad \text{with} \quad < T_x, y >= a(x, y).$$

By (i), T_x is bounded for all $x \in E$, so T maps E into F^*. We now show that T is bounded. Fix $y \in F$. Observe that

$$\{< Tx, y >: \ x \in B_E(0, 1)\} \quad = \quad \{a(x, y) \ : \ x \in B_E(0, 1)\}$$

is bounded by (ii). Therefore, by Corollary 2.5, which follows from Uniform boundedness principle, T is bounded. Now, by the boundedness of T, there exists $C > 0$ such that

$$\sup_{y \in F; ||y|| \neq 0} \frac{|a(x, y)|}{||y||} = ||T_x|| \leq C||x||$$

for any $x \in E$, and hence

$$a(x, y) \leq C||x||||y||$$

for any $x \in E$ and $y \in F$. $\square$

Question 3.

Problem 3. (Exercise 2.10 in the textbook) Let E and F be two Banach spaces and let $T \in L(E, F)$ be surjective.
1. Let M be any subset of E. Prove that $T(M)$ is closed in F iff $M + N(T)$ is closed in E.
2. Deduce that if M is a closed vector space in E and $\dim N(T) < \infty$, then $T(M)$ is closed.

Solution.
Suppose that $T(M)$ is closed. Then, by continuity of T,

$$M + N(T) = T^{-1}(T(M)) \quad \text{is closed.}$$

Conversely, suppose that $M + N(T)$ is closed. We contend that

$$T((M + N(T))^c) = T(M)^c. \quad (*)$$

Let $x \in T(M + N(T)^c)$. Then, there exists $y \in (M + N(T))^c$, such that $T(y) = x$. Suppose for sake of contradiction that $x \in T(M)$. Then, there exists $x_0 \in M$, such that $T(x_0) = y$, and by linearity of T, $x - x_0 \in N(T)$, so $x \in M + N(T)$, a contradiction. For the other inclusion, let $x \in T(M)^c$. By surjectivity of T, there exists $z \in E$ such that $T(z) = x$. Suppose for sake of contradiction that $z = m + n$ with some $m \in M$ and $n \in N(T)$. By linearity of T,

$$x = T(z) = T(m),$$

which contradicts that $x \in T(M)^c$. Therefore, (*) is true. To conclude, observe that, by open mapping theorem, $T(M)^c$ is open, so $T(M)$ is closed. Now, (ii) follows from the problem 5 and (i).
□

Question 4.

Problem 4. (Exercise 2.14 in the textbook) Let E and F be two Banach spaces.
1. Let $T \in \mathcal{L}(E, F)$. Prove that $R(T)$ is closed iff there exists a constant C such that $dist(x, N(T)) \leq C\|Tx\|, \quad \forall x \in E$.
2. Let $A : D(A) \subset E \to F$ be a closed unbounded operator. Prove that $R(A)$ is closed iff there exists a constant C such that $dist(u, N(A)) \leq C\|Au\| \quad \forall u \in D(A)$.

Solution.
(1) Let $\tilde{E} = E/N(T)$, π be the canonical quotient map, and $\tilde{T} : E/N(T) \to F$ such that $T = \tilde{T} \circ \pi$. From definition, we know that $\tilde{T}$ is linear, injective, and $R(T) = R(\tilde{T})$.

Suppose the given estimate holds. We claim that $R(\tilde{T})$ is closed, then by $R(T) = R(\tilde{T})$, we will be done. Observe that

$$||[x]|| = d(x, N(T)) \leq C||Tx|| = C||\tilde{T}[x]||(*)$$

for all $x \in E$. Suppose $\{y_n\} \subset R(\tilde{T})$ such that $y_n \to y$ for some y. Then, there exists $\{[x_n]\} \subset \tilde{E}$. By the above estimate, $\{[x_n]\}$ is cauchy, and by continuity $\tilde{T}[x_n] \to \tilde{T}[x]$ where $[x]$ is the limit of $\{[x_n]\}$. Therefore, $R(\tilde{T})$ is closed.

Suppose $R(T)$ is closed. Then, $R(\tilde{T})$ is closed, and hence Banach. Since $\tilde{T}$ is bijective, by Corollary 2.7, $\tilde{T}^{-1}$ is continuous so we again have (*).

(2) Consider $D(A)$ with graph norm. Then, as A is closed, $D(A)$ is Banach. Since graph norm only increases the norm from each norm, T is bounded and by (1) we are done.

Question 5.

Problem 5. Let G be a closed subspace of a Banach space E. Assume L is a finite dimensional subspace of E, then $G+L$ is a closed linear subspace. Moreover, $G+L$ admits a complement if and only if G does.

Solution.
Let π be the canonical projection of E onto E/G. As L is finite dimensional space, we see that $\pi(L)$ is finite dimensional, hence closed. By continuity of π, it follows that $\pi^{-1}(\pi(L)) = G + L$ is closed.

Suppose $G + L$ admits a complement A in E. Since $G \cap L$ is finite dimensional, it admits a complement B in L. $A + B$ is closed by (i). We claim that $A + B$ is the complement of G. If $g \in A + B \cap G$, then $g = a + b$ for some $b \in B$ and $a \in A$. By re-arranging, we see that $a = 0$ and $g = 0$. Therefore, it follows that for any $x \in A$, we can express it as an unique sum of an element in $A + B$ and G, so $A + B$ is a complement of G.

Suppose G admits a complement H in E. Let π_G and π_H be canonical projections of G and H respectively. Observe that $\pi_H(L)$ is finite dimensional, so it admits a complement $A \subset H$ in H. We claim that A is a complement of $G + L$. Note that A is closed trivially. Suppose $x \in A \cap G + L$. Then, $x = g + l$ for some $g \in G$ and $l \in L$. By projection properties, one can see that $x = 0$, and again see that x can be written as a unique sum of an element in $G + L$ and A which completes the proof. $\square$

Question 6.

Problem 6. Let $S_N(f,x)$ be the N^{th}-partial sum of the Fourier series of $f(x) \in L^1[-\pi,\pi]$, that is,

$$S_N(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{sin(N+\frac{1}{2})(x-\theta)}{sin\frac{1}{2}(x-\theta)} f(\theta)d\theta.$$

Show that there is a continuous 2π-periodic function $f(x)$ such that $|S_N(f,0)| \to +\infty$ as $N \to \infty$.

Solution.
For convenience, we identify $\mathbb{T}$ with $[0,2\pi]$ and consider 0 as the point, where we study the divergence. Let $\{\phi_n\}$ be a collection of continuous, linear functionals (standard property of Fourier series), defined on $C(\mathbb{T})$ given by

$$\phi_n(f) \quad = \quad S_n(f,0)$$

for all $f \in C(\mathbb{T})$ and $n \in \mathbb{N}$. Suppose for a moment that $\{\phi_n\}$ are not uniformly bounded. Then, by uniform boundedness principle, there exists $f \in C(\mathbb{T})$ such that $\{\phi_n(f)\}$ is not bounded.

We now show that $\{|\phi_n|\}$ is not bounded. Since $|\sin(t)| \le t$ for any $t \in [0,2\pi]$,

$$\begin{aligned} \int_0^{2\pi} |\frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})}|dx &\geq \int_0^{2\pi} |\sin(n+\frac{1}{2})x|\frac{2}{x}dx = \int_0^{2\pi(n+\frac{1}{2})} |\sin(x)|\frac{2}{x}dx \\ &\geq \sum_{k=1}^{n}\frac{1}{k}\int_{2\pi(k-1)}^{2\pi k} |\sin(x)|dx \geq \sum_{k=1}^{n}\frac{1}{k} \end{aligned}$$

for all $n \in \mathbb{N}$. The above estimate shows that the L^1 norms of the n-th Dirichlet kernels associated with ϕ_n diverges to ∞. Now, it is well-known that the functional norm of ϕ_n is exactly the L^1 norm of the n-th Dirichlet kernel for all $n \in \mathbb{N}$. This can be formally shown by considering the sign of the kernel as the continuous function, and using DCT to swap the order of limit and integration. $\square$

Question 7.

Problem 7. Let $L^1(S^1)$ be the space of Lebesgue integrable functions on the unit circle S^1. We define a product on $L^1(S^1)$(convolution):

$$\forall f, g \in L^1(S^1), \quad f * g(\theta) = \int_0^{2\pi} f(\theta - x)g(x)dx.$$

Show that $\|f * g\| \leq \|f\|\|g\|$, when $\|h\| = \int_0^{2\pi} |h(\theta)|d\theta$. (This makes $L^1(S^1)$ a Banach algebra).

Solution.
By Tonelli's theorem and the translation invariance property of Lebesgue measure,

$$\begin{aligned} ||f * g|| &= \int_0^{2\pi} |\int_0^{2\pi} f(t-x)g(x)dx|dt \leq \int_0^{2\pi}\int_0^{2\pi} |f(t-x)g(x)|dxdt \\ &= \int_0^{2\pi}\int_0^{2\pi} |f(t-x)g(x)|dtdx = \int_0^{2\pi} |g(x)| \int_0^{2\pi} |f(t-x)|dtdx \\ &= ||f|| \int_0^{2\pi} |g(x)| = ||f||||g|| \end{aligned}$$

for any $f, g \in L^1(S^1)$. □

Question 8.

Problem 8. Let $\mathcal{A} = \{f(\theta) = \sum_{n=-\infty}^{+\infty} c_n e^{in\theta}, \theta \in [0, 2\pi], c_n \in \mathbb{C}\}$ with the norm $\|f\| = \sum_{n=-\infty}^{+\infty} |c_n| < \infty$. Show that

(a) $(\mathcal{A}, \|\cdot\|)$ is a Banach space.
(b) Show that $\|fg\| \leq \|f\|\|g\|$ (In fact, $(\mathcal{A}, \|\cdot\|)$ is a Banach Algebra).
(c) $f_0 \equiv 1$ is the unit element of this Algebra.

(d) A homomorphism $h : \mathcal{A} \to \mathbb{C}$ means $h(f \cdot g) = h(f)h(g)$. For example, given any $\theta_0 \in [0, 2\pi]$, $h_{\theta_0} : \mathcal{A} \to \mathbb{C}$ defined by $h_{\theta_0}(f) = f(\theta_0)$ is a homomorphism. Show that every homomorphism $h : \mathcal{A} \to \mathbb{C}$ is of the form h_{θ_0} for some $\theta_0 \in [0, 2\pi]$. [Hint: $h(f_0) = 1$ and show first that $h(e^{i\theta}) = e^{i\theta_0}$ for some $\theta_0 \in [0, 2\pi]$].

Note that if $f \in \mathcal{A}$ with $|f| > 0$ on $[0, 2\pi]$, then $\frac{1}{f} \in \mathcal{A}$. The last conclusion is an interesting statement for Fourier series.

Solution.