Functional Analysis:
Problem Set I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

Problem 1. Let Y be a closed subspace of a normed vector space E. Show that the dual of E/Y is isometrically isomorphic to $Y^{\perp}$.

Solution.
First, define a map $\Phi : Y^{\perp} \to (E \setminus Y)^*$ naturally by

$$f \quad \mapsto \quad ([x] \mapsto f(x))$$

where $[x] \in E \setminus Y$. The map is well-defined, because for any $[x] = [x']$,

$$x' = x + y \text{ for some } y \in Y \text{ and } f(x') = f(x) + f(y) = f(x).$$

We now claim that Φ is a surjective isometry. By definition, for any $f \in Y^{\perp}$,

$$\begin{aligned} ||\Phi(f)|| &= \sup_{||[x]||=1} | < \Phi(f), [x] > | = \sup_{||[x]||=1} | < f, x > | \\ &= \sup_{\inf_{y \in Y} ||x-y||=1} | < f, x > | \end{aligned}$$

For sake of completeness, we note that if both pre-image and image are Banach, then a surjective isometry is a isometric isomorphism. This is a direct consequence of open mapping theorem. First, isometry implies injectivity: for $T \in \mathscr{L}(E, F)$ and $x, y \in E$, if $T(x) = T(y)$, then $0 = ||T(x - y)|| = ||x - y||$, so $x = y$. Therefore, T^{-1} is well-defined as a map. Now, by open mapping theorem, we see that for any O open in E, $(T^{-1})^{-1}(O) = T(O)$ is open in F. Therefore, T^{-1} is continuous as required.

It remains to be shown that $Y^{\perp}$ and $E \setminus Y$ are Banach. Suppose $f \in E^*$ such that there exists $\{f_n\} \subset Y^{\perp}$ with $f_n \to f$. It suffices to show that for any $yinY$, we have $< f, y >= 0$. But this is true, because norm convergence implies pointwise convergence and for all $n \geq 1$, $< f_n, x >= 0$. Now, $E \setminus Y$ is Banach with respect to the quotient norm, because

Question 2.

Problem 2. (Duality–Min-Max Principle) Let E be a normed vector space, and G a linear subspace of E. For any $x \in E$, define $m(x) = \inf_{y \in G} |x - y|$. Show that $m(x) = M(x)$, where $M(x) = \max_{\|f\|_{E^*} \leq 1, f=0 \text{ on } G} |\langle f, x \rangle|$.

Similarly, for any $g \in E^*$, we define $\|g\|_G = \sup\{|g(y)| : y \in G, \quad \|y\| \leq 1\}$. Then $\|g\|_G = \min\{\|g - h\|_{E^*} : h \in G^\perp\}$.

Solution.
If $x \in \bar{G}$, then $m(x) = 0$, and $M(x) = 0$, because by continuity for any $f \in E^*$ such that $f = 0$ on G, it follows that $f = 0$ on $\overline{G}$. Suppose $x \in E \setminus \overline{G}$.

Question 3.

Problem 3. Let E be a normed vector space.
(i) If Y is a closed proper subspace of E, then there is $x \in E$ such that $\|x\| = 1$ and $\|x - y\| > \frac{1}{2}$ for any $y \in Y$.
(ii) If E is of infinite dimension, then the unit ball $B_1 = \{x \in E : \|x\| \leq 1\}$ is never compact in strong topology.

Solution.
(i) We prove the following generalization, known as the Riesz Lemma: for each $\epsilon > 0$, there exists $x \in E$ such that $||x - y|| \geq 1 - \epsilon$, for any $y \in Y$.

Let $0 < \epsilon < 1$. Let $x \in E \setminus Y$. As Y is closed,

$$d \quad := \quad \text{dist}(x, Y) > 0.$$

Choose y^* in Y such that

$$d \leq ||x - y^*|| \leq \frac{d}{1-\epsilon}. \quad (1)$$

Set $x^* = \dfrac{x - y^*}{||x - y^*||}$. Clearly, $||x^*|| = 1$, and, for any $y \in Y$,

$$\begin{aligned} ||x^* - y|| \quad &= \quad ||\frac{x - y^*}{||x - y^*||} - y|| = \frac{1}{||x - y^*||}||x - (y^* + y||x - y||^*)|| \\ &\geq \quad \frac{d}{||x - y^*||} \leq 1 - \epsilon, \end{aligned}$$

where the last inequality follows from (1), and we are done. □

(ii) We proceed to construct a sequence $\{x_n\} \subset B_1$ such that there is no convergent subsequence, which shows that B_1 is not compact in strong topology through sequential characterization of compactness (strong topology is trivially metrizable).

Choose any $x \in E$ such that $||x|| = 1$ and set $x_1 = x$. Then, for any n, using (i), choose x_n such that

$$||x_n|| = 1 \text{ and } ||x_n - y|| > \frac{1}{2},$$

for any $y \in \text{span}(x_1, ..., x_{n-1})$, where the validity comes from the fact that any finite dimensional subspace is a proper, closed subspace of an infinite dimensional space. Then, it is clear that $\{x_n\}$ has no convergent subsequence, because for any $n \geq 1$, there exists $k, l \geq n$ with $k \neq l$, such that $||x_k - x_l|| > \dfrac{1}{2}$. Since being cauchy is a necessary condition for being convergent, we are done. □

Question 4.

Problem 4. Let $L^\infty[0,1]$ be the space of bounded, Lebesgue measurable functions on $[0,1]$. We define $l(f) = \int_0^1 f(t)\mathrm{d}t$. Then l is a positive, linear, continuous functional on $L^\infty[0,1]$. Here l is called positive if $l(f_2) \geq l(f_1)$ whenever $f_2 \geq f_1$. Define, for any bounded real-valued function g, $p(g) = \inf\{l(f) : g \leq f \in L^\infty[0,1]\}$. Show that
(i) p is a positive homogeneous, subadditive and $p(g) \leq 0$ whenever $g \leq 0$. Moreover $p(f) = l(f)$ if $f \in L^\infty[0,1]$.
(ii) l can be extended to a positive linear functional on the space of all bounded functions.

Solution.
(i) Let g be a bounded real-valued function and $\lambda > 0$. Then, by linearity of Lebesgue integration,

$$\begin{aligned} p(\lambda g) &= \inf\{l(h) : \lambda g \leq h \in L^\infty\} \\ \lambda p(g) &= \inf\{l(\lambda h) : g \leq h \in L^\infty\}. \end{aligned}$$

We claim that

$$A := \{l(h) : \lambda g \leq h \in L^\infty\} \quad = \quad \{l(\lambda h) : g \leq h \in L^\infty\} =: B$$

If $\lambda g \leq h \in L^\infty$, then $g \leq \dfrac{h}{\lambda} \in L^\infty$, so $l(\lambda \dfrac{h}{\lambda}) = l(\lambda) \in B$. Conversely, if $g \leq h \in L^\infty$ then, $\lambda g \leq \lambda h \in L^\infty$, so $l(\lambda h) \in A$. Hence, $p(\lambda g) = \lambda p(g)$.

We now show that p is sub-additive. Let f, g be bounded real functions. Then, for any $h_1, h_2 \in L^\infty$ such that $f \leq h_1$ and $g \leq h_2$,

$$f + g \quad \leq \quad h_1 + h_2 \in L^\infty,$$

so, again by linearity of integration,

$$p(f+g) \quad \leq \quad l(h_1 + h_2) = l(h_1) + l(h_2).$$

Taking infs for h_1, then h_2, gives

$$p(f+g) \quad \leq \quad p(f) + p(g),$$

as required.

For any f, g bounded real-valued functions,

$$\begin{aligned} p(f+g) &= \inf\{l(f+g) : f+g \leq h, h \in L^\infty[0,1]\} \\ &= \inf\{l(f) + l(g) : f+g \leq h, h \in L^\infty[0,1]\}. \end{aligned}$$

Suppose $g \leq 0$. Then, as $0 \in L^\infty[0,1]$ and $l(0) = 0$, by definition, $p(g) \leq 0$.

We show that $p(f) = l(f)$ if $f \in L^\infty[0,1]$. For all $h \in L^\infty[0,1]$ such that $f \leq h$, then, by monotonicity of Lebesgue integration, $l(f) \leq l(h)$. Since $f \leq f$ trivially, it follows that $p(f) = l(f)$. □

(ii) Now, as $l = p$ on $L^\infty[0,1]$, by Hahn-Banach, l can be extended to the entire space of bounded real-valued functions. This shows that we can make sense of integration for any bounded functions in a weaker sense, sacrificing some nice properties, such as countable additivity and so on(probably if such properties hold, then it will contradict existence of non-measurable sets by considering appropriate indicators). □

Question 5.

Problem 5. Let E be a normed vector space. The norm of E is called uniformly convex if $\|\frac{x+y}{2}\| \leq 1 - \varepsilon(\|x-y\|)$ for all $x, y \in \{z \in E : \|z\| = 1\}$. Here $\varepsilon(r)$ is an increasing and positive function defined for $r > 0$ such that $\lim_{r \to 0^+} \varepsilon(r) = 0$. Let K be a closed, convex subset in a Banach space E with uniformly convex norm. Prove that for any $x \in E$, there is a unique $y \in K$ such that $\|x - y\| = \inf\{\|x - z\| : z \in K\}$. (Hint: Assume $x = 0$ not in K, and let $\{y_n\} \subset K$ be a minimizing sequence, then consider $x_n = \frac{y_n}{\|y_n\|}$ and $\frac{x_m + x_n}{2}$.)

Solution.
Without loss of generality, assume $x = 0$ and $x \notin K$. Let $\{y_n\}$ be the minimizing sequence. As E is Banach, and K is closed, it suffices to show that $\{y_n\}$ is cauchy. We claim that, it suffices to show,

$$||\frac{x_n + x_m}{2}|| \to 1 \quad \text{as} \quad n, m \to \infty \quad (*)$$

By uniform convexity assumption, the above implies that $\{x_n\}$ is cauchy. Now, it follows that that $\{y_n\}$ is cauchy.

Now, we prove $(*)$. by convexity of K, for any $n, m \geq 1$,

$$d \leq ||\frac{y_n + y_m}{2}|| \leq ||\frac{y_n}{2}|| + ||\frac{y_m}{2}||$$

so

$$||\frac{y_n + y_m}{2}|| \to d \qquad \text{as} \qquad n, m \to \infty.$$

Then,

$$\begin{aligned}
\lim_{n,m\to\infty} ||\frac{x_n + x_m}{2}|| &= ||\lim_{n\to\infty} \frac{y_n}{2||y_n||} + \lim_{n\to\infty} \frac{y_m}{2||y_m||}|| = \frac{1}{d}||\lim_{n\to\infty} \frac{y_n}{2} + \lim_{m\to\infty} \frac{y_m}{2}|| \\
&= \frac{1}{d} \lim_{n,m\to\infty} ||\frac{y_n + y_m}{2}|| = 1
\end{aligned}$$

and we are done. $\square$

Question 6.

Problem 6. Let E be a normed vector space , and O be a bounded open set in E such that it is convex and symmetric with respect to $\underline{0} \in O$ (i.e., $x \in O \Rightarrow -x \in O$). Then show that the Minkowski functional associated with O introduces a new norm of E.
(ii) Is this new norm equivalent to the original norm $\|\cdot\|$.
(iii) If E is reflexive with respect to the original norm, does it implies then E is also reflexive with respect to the new norm?

Solution.
By ordinary properties of Minkowski functionals, it suffices to show that for any $x \in E$ and $\lambda \in \mathbb{R}$,

$$p(\lambda x) = |\lambda| p(x) \quad \text{and} \quad p(x) = 0 \implies x = 0.$$

We first prove the absolute homogeneity of p. Now, if $\lambda \geq 0$, then $p(\lambda x) = \lambda p(x)$ by positive homogeneity of Minkowski functionals. Now, if $\lambda \geq 0$, then, by symmetry, and positive homogeneity again, we obtain

$$p(\lambda x) = p(-\lambda x) = -\lambda p(x),$$

which completes the proof of absolute homogeneity.

Now, observe that, for any $0 < \alpha < \beta$, and $x \in E$,

$$\alpha^{-1} x \in C \quad \implies \beta^{-1} x \in C,$$

because by convexity

$$(1 - \frac{\beta^{-1}}{\alpha^{-1}})0 + \frac{\beta^{-1}}{\alpha^{-1}} \alpha^{-1} x = \beta^{-1} x \in C.$$

Let $x \in E$ such that $p(x) = 0$, Then, by the above discussion, it follows that

$$\alpha^{-1} x \quad \in \quad C \quad (*),$$

for any $\alpha \in (0, \infty)$. Suppose $x \neq 0$, and let $r > 0$ large enough that $C \subset B(0, r)$. Then, it follows that, from $(*)$, $\frac{r}{||x||} x \in C$, which contradicts the fact that $C \subset B(0, r)$. Therefore, $x = 0$ and we are done. $\square$

(i) The norm is equivalent

(ii) Reflexive implies that the space is Banach (so there is no confusion with definition).

We prove the following claim: Let E be Banach space that is reflexive. For any normed linear space, defined by an equivalent norm on E, is reflexive.

Consider an equivalent norm. Since the norm is equivalent with the original norm, the induced topology is equal to the original topology, hence the dual and induced weak topology coincide as well. We know that Banach space is reflexive iff the closed unit ball is weakly compact. Now, it suffices to show that, for any topology, compactness of the closed unit ball B_1 implies, for any $c > 0$, B_c is compact. Suppose $\{U_\lambda\}$ is a finite cover of B_1. Define $\{\tilde{U}_\lambda\}$ by

Question 7.

Problem 7. Let E be the space of bounded Lebesgue measurable functions on [a,b]. Find a sequence $\{f_n\} \subset E^*$ such that $f_n(x) \to 0$ for all $x \in E$ and $\|f_n\|_{E^*} = 1$. Is E separable, that is, does E has a countable dense set?

Solution.
Consider $\{f_n\}$ defined by

$$g \quad \mapsto \quad \int_{\mathbb{T}} g(t)e^{-2\pi i n t}dt = \hat{g}(n) \quad (g \in L^\infty(\mathbb{T}))$$

for each $n \geq 1$. As $L^\infty(\mathbb{T}) \subset L^1(\mathbb{T})$, by Riemann-Lebesgue lemma, for any $g \in L^\infty(\mathbb{T})$,

$$f_n(g) = \hat{g}(n) \to 0.$$

For any $n \geq 1$, $g \in L^\infty(\mathbb{T})$ with $||g|| = 1$,

$$|\hat{g}(n)| \quad \leq \quad \int_{\mathbb{T}} |g| \leq ||g||_\infty = 1,$$

Now, for any $n \geq 1$, take $g = e^{2\pi i n t}$ to see

$$|\hat{g}(n)| \quad = \quad |\int_{\mathbb{T}} 1dt| = 1,$$

so, combined with the above estimate, we have

$$||f_n|| = 1,$$

as required.

It is a well-known fact that L^∞ is not separable, given that the ambient space is not finite. Instead of appealing to this general result, we provide a construction of uncountable family of functions in $L^\infty[a,b]$ with $a < b$ such that the distance is at least 1 apart to contradict the separability assumption. Take $[a,b]$ such that $a < b$. Fix $x_0 \in (a,b)$ and consider $\mathscr{A} = \{1_{B(x_0,r)}\}$ where $0 < r \leq \min(x_0 - a, b - x_0)$. Then, the family is uncountable, but for any $f, g \in \mathscr{A}$,

$$||f - g||_\infty \quad = \quad 1.$$

Now, suppose the space is separable, hence there exists a countable dense subset C. Now, observe that

$$\bigcup_{x \in \mathscr{A}} B(x, \frac{1}{2}) \cap C \subset C,$$

but the left hand side is uncountable, because its an uncountable disjoint union of countable sets. Therefore, $L^\infty[a,b]$ is not separable. $\square$