DiffGeoI:
Problem Set I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

1. Show that the vector bundle constructed in 6e) of the Howework 1 is isomorphic to the tangent bundle of $\mathbb{R}P^n$.

Solution.
Consider $\mathbb{RP}^n$ as a quotient space of S^n, where you identify the antipodal points. Let p be the natural projection from S^n to $\mathbb{RP}^n$. For any $x \in S^n$, we have

$$T_x(S^n) \quad = \quad \{(x,v) \in S^n \times \mathbb{R}^{n+1} : x \cdot v = 0\}.$$

Further observe that (x,v) and $(-x,-v)$ both maps to $T_{[x]}\mathbb{RP}^n$ via the tangent map. Also, p is a local diffeomorphism, so $T_p(x)$ is an isomorphism for each $x \in S^n$. Hence, we can identify E' with $T\mathbb{RP}^n$ via f which sends $T_{[x]}\mathbb{RP}^n$ to $([x],y)$.

Question 2.

2. Let M be a compact manifold. Let $p : E \to M$ be a vector bundle. Show that one could embed E as a subbundle of a trivial vector bundle over M.

Solution.
Let $\{U_x\}_{x\in B}$ be a collection of trivializing open sets in B. By Urysohn's lemma, there exists $\psi_x : B \to [0,1]$ such that ψ_x vanishes outside of U_x and is nonzero at x. Consider $\{\psi_x^{-1}((0,1])\}$ as an open cover of B and via compactness choose the finite subcover. We index the corresponding trivializing open sets and the Urysohn functions as $\{U_i, \psi_i\}$. We now define $f_i : E \to \mathbb{R}^n$ by

$$v \quad \mapsto \quad \psi_i(p(v))(\pi_i\phi_i(v))$$

where ϕ_i is the local trivialization of U_i and π_i is the projection from $M \times \mathbb{R}^n$ to $\mathbb{R}^n$. It follows that f_i is a linear inection on each fiber over $\psi_i^{-1}((0,1])$, so $f = (f_1, ..., f_n)$ is a linear injection on each fiber which maps E to some $\mathbb{R}^k$. Finally, take $h : E \to B\mathbb{R}^k$ by first coordinate being p and the second coordiate g. Then, the image of h is a subbundle of $B \times \mathbb{R}^k$, since each $\mathbb{R}^n$ factor corresponds to the second coordinate of local trivialization over $\psi_i^{-1}((0,1])$. This is the type-1 vector bundle isomorphism from E to a subbundle of $B \times \mathbb{R}^k$, and since local trivializations are all diffeomorphism this does give a smooth embedding in a precise sense, so we are done. $\square$

Question 3.

3. Let M be an n-manifold of class C^k. Show that M is trivializable if and only if $\Gamma_k(TM)$ is a free $C^k(M)$-module of rank n.

Solution.
This statement essentially boils down to the equivalaence between a local trivalization and local frame of a vector bundle. Obviously, we view the tangent bundle TM as a vector bundle, and the C_k vector fields as C_k sections.

We elaborate on the equivalence more precisely. Let $\{e_i\}$ be a fixed basis for a typical fiber V. If ϕ is a local trivialization over U open, then by defining $\sigma_i(p) = \phi^{-1}(p, e_i)$, gives $\sigma_\phi = (\sigma_1, ..., \sigma_k)$ as a local frame over U. Conversely, if $\sigma_\phi = (\sigma_1, ..., \sigma_k)$ is a local frame over U, then for any $v \in \pi^{-1}(U)$ has the form $v = \sum v^i \sigma_i(p)$ for unique elements. Then, $f : U \times V \to \pi^{-1}(U)$ defined by $(p, v) \mapsto \sum v^i \sigma_i(p)$ is a diffeomorphism, so $\sigma = f^{-1}$ is the desired trivialization. Therefore, if we have a global frame field, then we obtain a gloal trivalization, so the vector bundle is trivial.

Suppose $\Gamma_k(TM)$ is a free $C^k(M)-$module of rank n. Then, there exists $\{e_1, ..., e_n\}$ such that, for any $X \in \Gamma_k(TM)$, there exists unique $f_1, ..., f_n \in C^k(M)$ such that

$$X_p = f_1(p)e_1(p) + ... + f_n(p)e_n(p) \in T_p(M) \quad (*)$$

for any $p \in M$. We claim that $\{e_1, ..., e_n\}$ are global frame field. Let $p \in M$, and $\pi^{-1}(p) = \{p\} \times T_p(M)$. Fix $(p, v) \in \{p\} \times T_p(M)$. By lemma 2.65 in Jeff M. Lee's book, there exists $X \in \gamma^k(TM)$ such that $X_p = v$. By (*), we see that v can be expressed as a linear combination of $\{e_1(p), ..., e_n(p)\}$. Since (p, v) were arbitrary, we see that $\{e_1, ..., e_n\}$ is a global frame field, and by the above discussion this implies that TM is trivializable.

Suppose M is trivializable, thus the tangent bundle TM is trivializable. This means that TM is vector bundle isomorphic to the product vector bundle $pr_1 : M \times V \to M$. Since this is equivalent to an existence of vector bundle trivialization over the entire manifold M (pg. 271; Jeff Lee), we can choose a global vector bundle trivizialization of TM, which we denote as ϕ Then, as motivated from above, set $\sigma_i(p) = \phi^{-1}(p, e_i)$, where e_is are the basis of the typical fiber. Vary p, to obtain $\{\sigma_i\}$ that are C^k sections on TM. Now, let $X \in \Gamma_k(TM)$. Then, from $\sigma_i(p) = \phi^{-1}(p, e_i)$, we see that X can be uniquely expressed as a linear combination of $\{\sigma_i\}$, so $\Gamma_k(TM)$ is free. $\square$