DiffGeoI:
Problem Set VI

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

1. Show that a topological group is generated (as a group) by any neighborhood of its neutral element. Deduce that if G is a Lie group, then $exp(\mathfrak{g})$ generates G.

Solution.
The first part of the problem should be corrected to a connected topological space. Suppose U is a neighborhood of e. Set

$$U_n \quad = \quad \{u_1 ... u_n \mid u_i \in U \ ; \ \forall 1 \leq i \leq n\}$$

for any $n \geq 1$, and $W = \cup_{n=1}^{\infty} U_n$. For topological groups, UV is open, if U and V are open, so U_n is open for all $n \geq 1$, which implies W is open. At this point, by connectivity of G, it suffices to show that W is closed. Let $g \in \overline{W}$. Then, as gU^{-1} is an open neighborhood of g, there exists $h \in W \cap gU^{-1}$, so

$$h = gu^{-1} \quad \text{and} \quad h = u_1 ... u_n$$

for some $u \in U$, and $u_1, ..., u_n \in U$, and hence

$$g = u_1 ... u_n u \in U^{n+1} \subset W.$$

Therefore, W is closed, and we are done.

By property of $\exp$,

$$T_e \exp(v) = v$$

for any $v \in \mathbf{g}$. Therefore, $\exp$ is a local diffeomorphism at e, so $\exp(\mathbf{g})$ contains a neighborhood of e. By (1), the result follows.

$\square$

Question 2.

2. Let G be a smooth manifold and assume that G has a group structure such that the map $(x, y) \mapsto xy$ is smooth. Show that G is a Lie group. (Hint: consider the map $(x, y) \mapsto (x, xy)$ in a neighborhood of (e, e).)

Solution.
Set

$$\triangle' = \{(g, g^{-1}) \in G \times G\}.$$

As the multiplication map, m, is constant rank, surjective and smooth, it is a submersion. Hence

$$\triangle' = m^{-1}(e)$$

is an embedded submanifold of $G \times G$ with dimension n. Let π_1, π_2 be standard projections of first, and second coordinates. Moreover, i be an inclusion map from $\triangle' \to G \times G$, and $d : G \to \triangle'$ be defined by $g \mapsto (g, g^{-1})$. Observe that

$$\text{inv} = \pi_2 \circ i \circ d.$$

Therefore, it suffices to show that d is smooth. Note that

$$d = (\pi_1 \circ i)^{-1}.$$

Now, as $\pi_1 \circ i$ is a smooth bijection, where the differential is nonsingular everywhere, by the inverse function theorem, d is smooth, and we are done. $\square$

Question 3.

3. Show that the vector space $\mathbb{R}^3$ with the vector product $\wedge$ is a Lie algebra. Consider the vector fields X, Y, Z on $\mathbb{R}^3$:

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad Y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Show that they generate a Lie subalgebra, in the Lie algebra $\mathfrak{X}(\mathbb{R}^3)$, isomorphic to $\mathbb{R}^3$ with the product structure $\wedge$.

Solution.
From ordinary calculus, we see that the cross product, when viewed as a binary operation has bilinearity, alternativity and jacobi identity. Hence, it is a Lie-algebra on $\mathbb{R}^3$.

We compute

$$\begin{aligned}
[X, Y] &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = Z \\
[X, Z] &= -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} = -Y \\
[Y, Z] &= Z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} = X
\end{aligned}$$

and hence

$$\begin{aligned}
[a_1X + b_1Y + c_1Z, a_2X + b_2Y + c_2Z] &= (a_1b_2 - b_1a_2)[X, Y] + (a_1c_2 - c_1a_2)[X, Z] \\
&+ (b_1c_2 - c_1b_2)[Y, Z] \\
&= (a_1b_2 - b_1a_2)Z - (-a_1c_2 - c_1a_2)Y \\
&+ (b_1c_2 - c_1b_2)X \qquad (1)
\end{aligned}$$

for any a_i, b_i, c_i with $i = 1, 2$. The closure implies that X, Y, Z generates a Lie subalgebra in $\mathscr{X}(\mathbb{R}^3)$ with $[,]$.

Let Φ be the map that sends an element in the subalgebra to $\mathbb{R}^3$ by the coefficients of X, Y, Z in order. If $X_1 = a_1 X + b_1 Y + c_1 Z$ and $X_2 = a_2 X + b_2 Y + c_2 Z$, then, from (1),

$$\begin{aligned} \Phi([X_1, X_2]) &= (b_1 c_2 - c_1 b_2, -a_1 c_2 + c_1 a_2, a_1 b_2 - b_1 a_2) \\ &= (a_1, b_1, c_1) \wedge (a_2, b_2, c_2) = \Phi(X_1) \wedge \Phi(X_2). \end{aligned}$$

Hence, Φ is a Lie Algebra isomorphism. $\square$