ProbLimI:
Problem Set I

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Abstract

This work contains solutions to the exercises of the problem set I. The chosen problems are 1,2, and 4.

Question 1.

1. Show that a topological manifold M is connected iff M is path-connected.

Solution.

Question 2.

2. Let $\mathbb{R}P^n$ be the n-dimensional projective space, with the atlas given by the following functions

$$\phi_i : U_i \to \mathbb{R}^n, [x_1, \dots x_{n+1}] \mapsto (\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots \frac{x_{n+1}}{x_i}),$$

where U_i, $i = 1, \dots n+1$ are open subsets $\{[x_1, \dots, x_{n+1}], x_i \neq 0\}$.

(a) Show that (U_i, ϕ_i) is a smooth atlas.

(b) Show that $\mathbb{R}P^1$ is diffeomorphic to S^1.

(c) Let $\pi : S^2 \to \mathbb{R}P^2$ be a map that sends a point (x, y, z) to a unique line through this point. Show that π is smooth and that π is local diffeomorphism: for any point $p \in S^2$ there exists an open neighborhood $U \subset M$ such that $\pi_U : U \to \pi(U)$ is a diffeomorphism on an open subset of $\mathbb{R}P^2$.

Solution.

Question 3.

4. Let M be a manifold of class C^k. Let $A, B \subset M$ be closed subsets such that $A \cap B = \emptyset$. Show that there is a function $f \in C^k(M)$ with values in $[0,1]$ and such that f is identically 0 on A and identically 1 on B.

Solution.