DiffGeoI:
Problem Set I

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

1. Show that a topological manifold M is connected iff M is path-connected.

Solution.
For any topological space, path-connected implies connected. We prove the converse. Suppose X is connected. Let $x \in X$, and set

$$U = \{y \in X \mid \text{ there is a path bewteen } x \text{ and } y\}.$$

Observe that $x \in U$ so U is non-empty. Now, as X is connected, if U and U^c are both open, then U^c is empty, and $U = X$, so X is path-connected. We show that U is open. Let $y \in U$. Then, there exists an open nbd of y, O, such that O is homeomorphic to an open ball in R^n(this is equivalent to the locally euclidean condition of topological manifold). Since path-connectedness is preserved through homeomorphism, we conclude that O is path connected and $O \subset U$. Therefore, U is open and similarly U^c is open, and we are done. $\square$

Question 2.

2. Let $\mathbb{R}P^n$ be the n-dimensional projective space, with the atlas given by the following functions

$$\phi_i : U_i \to \mathbb{R}^n, [x_1, \ldots x_{n+1}] \mapsto (\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_{n+1}}{x_i}),$$

where U_i, $i = 1, \ldots n+1$ are open subsets $\{[x_1, \ldots, x_{n+1}], x_i \neq 0\}$.

(a) Show that (U_i, ϕ_i) is a smooth atlas.

(b) Show that $\mathbb{R}P^1$ is diffeomorphic to S^1.

(c) Let $\pi : S^2 \to \mathbb{R}P^2$ be a map that sends a point (x, y, z) to a unique line through this point. Show that π is smooth and that π is local diffeomorphism: for any point $p \in S^2$ there exists an open neighborhood $U \subset M$ such that $\pi_U : U \to \pi(U)$ is a diffeomorphism on an open subset of $\mathbb{R}P^2$.

Solution.
(a) Let $1 \leq i < j \leq n$. By symmetry, it suffices to show that

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is smooth. For any $a = (\frac{x_1}{x_i}, ..., \frac{x_{i+1}}{x_i}, ..., \frac{x_{n+1}}{x_i}) \in \phi_i(U_i \cap U_j)$ with $x_i, x_j \neq 0$,

$$a \mapsto_{\phi_i^{-1}} [x_1, ..., x_{n+1}] \mapsto_{\phi_j} (\frac{x_1}{x_j}, ..., \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, ..., \frac{x_{n+1}}{x_i}).$$

Since each cordinate map is smooth, we see that the transition map is smooth and since the indices were arbitrary, the atlas is smooth. $\square$

(b)

(c) We choose the smooth structure of S^2 to be the one that contains the projection charts. Let $p = (x^*, y^*, z^*) \in S^2$ and assume without loss of generality that $z^* > 0$. Choose a chart (U_z^+, ψ_z) of p where

$$U_{z^+} = \{p = (x, y, z) \in S^2 : z > 0\}$$

with $\psi_z : U_{z^+} \to \mathbb{R}^2$ defined by

$$p = (x, y, z) \in U_{z^+} \mapsto (x, y).$$

Now, observe that with $z^* > 0$, $\pi(z^*) \in U_3$, so we can choose (U_3, ϕ_3) for a chart at $\pi(z^*)$. We now claim that

$$\phi_3 \circ \pi \circ \psi_z^{-1} : \psi_z(U_z^+) \to U_3$$

is smooth at $\psi(p)$. For each $(x, y) \in \psi(U_z^+)$, we have

$$\begin{aligned} (x, y) \quad &\mapsto_{\psi_z^{-1}} \quad (x, y, (1 - x^2 - y^2)^{\frac{1}{2}}) \mapsto_\pi [x, y, (1 - x^2 - y^2)^{\frac{1}{2}}] \\ &\mapsto_{\phi_3} \quad (\frac{x}{(1 - x^2 - y^2)^{\frac{1}{2}}}, \frac{y}{(1 - x^2 - y^2)^{\frac{1}{2}}}) \end{aligned}$$

which simplifies to

$$(x, y) \mapsto_{\phi_3 \circ \pi \circ \psi_z^{-1}} (\frac{x}{(1 - x^2 - y^2)^{\frac{1}{2}}}, \frac{y}{(1 - x^2 - y^2)^{\frac{1}{2}}}).$$

Therefore, we see that each component is smooth, and $\phi^3 \circ \pi \circ \psi_z^{-1}$ is smooth, so π is smooth.

Now, it is easy to see that π in fact gives the diffeomorphism as claimed. By symmetry, on any chart, π is a bijection from U to $\pi(U)$, as we have eliminated the $2-1$ mapping by only considering positive or negative parts in all dimensions. It is also true that $\pi(U)$ is open. Restricted to $\pi(U)$ by the same computation as above, π^{-1} is smooth, so we see that π is a local diffeomorphism. Formally, fix $p \in S^2$. Choose a chart that contains p, U, then, π restircted to U gives a diffeomorphism as required. $\square$

Question 3.

4. Let M be a manifold of class C^k. Let $A, B \subset M$ be closed subsets such that $A \cap B = \emptyset$. Show that there is a function $f \in C^k(M)$ with values in $[0,1]$ and such that f is identically 0 on A and identically 1 on B.

Solution.
This statement corresponds to