DiffGeoI:
Problem Set I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

1. Show that a topological manifold M is connected iff M is path-connected.

Solution.
For any topological space, path-connected implies connected. We prove the converse. Suppose X is connected. Let $x \in X$, and set

$$U = \{y \in X \mid \text{ there is a path bewteen } x \text{ and } y\}.$$

Observe that $x \in U$ so U is non-empty. Now, as X is connected, if U and U^c are both open, then U^c is empty, and $U = X$, so X is path-connected. We show that U is open. Let $y \in U$. Then, there exists an open nbd of y, O, such that O is homeomorphic to an open ball in R^n(this is equivalent to the locally euclidean condition of topological manifold). Since path-connectedness is preserved through homeomorphism, we conclude that O is path connected and $O \subset U$. Therefore, U is open and similarly U^c is open, and we are done. $\square$

Question 2.

2. Let $\mathbb{R}P^n$ be the n-dimensional projective space, with the atlas given by the following functions

$$\phi_i : U_i \to \mathbb{R}^n, [x_1, \ldots x_{n+1}] \mapsto (\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_{n+1}}{x_i}),$$

where U_i, $i = 1, \ldots n+1$ are open subsets $\{[x_1, \ldots, x_{n+1}], x_i \neq 0\}$.

(a) Show that (U_i, ϕ_i) is a smooth atlas.

(b) Show that $\mathbb{R}P^1$ is diffeomorphic to S^1.

(c) Let $\pi : S^2 \to \mathbb{R}P^2$ be a map that sends a point (x, y, z) to a unique line through this point. Show that π is smooth and that π is local diffeomorphism: for any point $p \in S^2$ there exists an open neighborhood $U \subset M$ such that $\pi_U : U \to \pi(U)$ is a diffeomorphism on an open subset of $\mathbb{R}P^2$.

Solution.
(a) Let $1 \leq i < j \leq n$. By symmetry, it suffices to show that

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

is smooth. For any $a = (\frac{x_1}{x_i}, ..., \frac{x_{i+1}}{x_i}, ..., \frac{x_{n+1}}{x_i}) \in \phi_i(U_i \cap U_j)$ with $x_i, x_j \neq 0$,

$$a \mapsto_{\phi_i^{-1}} [x_1, ..., x_{n+1}] \mapsto_{\phi_j} (\frac{x_1}{x_j}, ..., \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, ..., \frac{x_{n+1}}{x_i}).$$

Since each coordinate map is smooth, we see that the transition map is smooth and since the indices were arbitrary, the atlas is smooth. $\square$

(b) Here, we choose to work with the stereographic projection chart on S^1. Consider the map $\pi : S^1 \to \mathbb{R}P1$ defined by

$$(x, y) \quad \mapsto \quad [1 - y : x]$$

for $(x, y) \in S^1$ such that $y \neq 1$, and

$$(x, y) \quad \mapsto \quad [0 : 1]$$

for $(x, y) \in S^1$ such that $y = 1$. Now, in chosen coordinates

$$\pi_{1,1}(p) = p$$
$$\pi_{2,1}(p) = \frac{1}{p}$$
$$\pi_{1,2}(p) = \frac{1}{p}$$
$$\pi_{2,2}(p) = p$$

.

(This result is well-known; we show a more detailed computation in part c.) Hence, all coordinate expressions are smooth, so π is smooth. π is one-to-one and in local charts the inverses are smooth as well. Hence, we see that π is a diffeomorphism. $\square$

(c) We choose the smooth structure of S^2 to be the one that contains the projection charts. Let $p = (x^*, y^*, z^*) \in S^2$ and assume without loss of generality that $z^* > 0$. Choose a chart (U_z^+, ψ_z) of p where

$$U_{z^+} \quad = \quad \{p = (x, y, z) \in S^2 \ : \ z > 0\}$$

with $\psi_z : U_{z^+} \to \mathbb{R}^2$ defined by

$$p = (x, y, z) \in U_{z^+} \mapsto (x, y).$$

Now, observe that with $z^* > 0$, $\pi(z^*) \in U_3$, so we can choose (U_3, ϕ_3) for a chart at $\pi(z^*)$. We now claim that

$$\phi_3 \circ \pi \circ \psi_z^{-1} : \psi_z(U_z^+) \to U_3$$

is smooth at $\psi(p)$. For each $(x, y) \in \psi(U_z^+)$, we have

$$\begin{aligned} (x,y) &\mapsto_{\psi_z^{-1}} & (x, y, (1-x^2-y^2)^{\frac{1}{2}}) \mapsto_\pi [x, y, (1-x^2-y^2)^{\frac{1}{2}}] \\ &\mapsto_{\phi_3} & (\frac{x}{(1-x^2-y^2)^{\frac{1}{2}}}, \frac{y}{(1-x^2-y^2)^{\frac{1}{2}}}) \end{aligned}$$

which simplifies to

$$(x,y) \mapsto_{\phi_3 \circ \pi \circ \psi_z^{-1}} (\frac{x}{(1-x^2-y^2)^{\frac{1}{2}}}, \frac{y}{(1-x^2-y^2)^{\frac{1}{2}}}).$$

Therefore, we see that each component is smooth, and $\phi^3 \circ \pi \circ \psi_z^{-1}$ is smooth, so π is smooth.

Now, it is easy to see that π in fact gives the diffeomorphism as claimed. By symmetry, on any chart, π is a bijection from U to $\pi(U)$, as we have eliminated the $2-1$ mapping by only considering positive or negative parts in all dimensions. It is also true that $\pi(U)$ is open. Restricted to $\pi(U)$ by the same computation as above, π^{-1} is smooth, so we see that π is a local diffeomorphism. Formally, fix $p \in S^2$. Choose a chart that contains p, U, then, π restircted to U gives a diffeomorphism as required. $\square$

Question 3.

4. Let M be a manifold of class C^k. Let $A, B \subset M$ be closed subsets such that $A \cap B = \emptyset$. Show that there is a function $f \in C^k(M)$ with values in $[0, 1]$ and such that f is identically 0 on A and identically 1 on B.

Solution.
Without loss of generality, we assume smooth structure on M and prove that a smooth map with the desired property exists. One should remark that this version of smooth urysohn's lemma is easier to prove than the version on the normal space that is merely continuous, as we have more structure from the smooth manifold assumption. It is sufficient to prove that for any open set O there exists a smooth $f : M \to [0, \infty)$ such that $f^{-1}(0) = (M \setminus O)$. First suppose that O is a product of open intervals in some $\mathbb{R}^n$. Then, there exists a bump function, positive on O and vanishes on the complement. Now, for O open in $\mathbb{R}^n$, write it as a union of open cubes such that for any point in O, there is finitely many open cubes that cover it. Sum of the bump functions from the previous result gives the desired construction. Now, consider an open set that has a chart defined on it. Then, by pull-back and using the previous result, we obtain the desired function. Again, for the general open set, use the locally finite cover of charts and sum then constructed functions.

Now let C_1, C_2 be closed subsets of M such that $C_1 \cap C_2 =$. Choose f_1 and f_2 smooth functions on M such that $f_1^{-1}(0) = C_1$ and $f_2^{-1}(0) = C_2$. Set

$$f = \frac{f_1}{f_1 + f_2}.$$

It follows that f is the desired function. $\square$