DiffGeoI:
Problem Set I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

3. Let $0 < r \leq m \leq n$. Let $V_r \subset M_{n\times m}(\mathbb{R})$ be the set of matrices of rank r. Show that V_r is a smooth submanifold of $M_{n\times m}(\mathbb{R})$ and compute its dimension.

Solution.
Let $M_k(m \times n, \mathbb{R})$ be the subset matrices of rank k. We claim that it is an embedded submanifold of dimension $mn - (m-k)(n-k)$. Let $E_0 \in M_k(m \times n, \mathbb{R})$. We write

$$E_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix},$$

where A_0 is non-singular $k \times k$, and D_0 is $(m-k) \times (m-k)$. Define U by

$$U = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(m \times n, \mathbb{R}) \ : \ \det A \neq 0\}.$$

Since $\det$ is continuous, U is a neighborhood of E_0. For any $E = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U$, set

$$P = \begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I_{n-k} \end{pmatrix}.$$

Observe that P is non-singular, so EP has the same rank as E. We compute

$$EP = \begin{pmatrix} I_k & 0 \\ CA^{-1} & D - CA^{-1}B \end{pmatrix},$$

so EP has rank r iff $D - CA^{-1}B$ is 0. Now, define $\Phi : U \to M((m-k) \times (n-k), \mathbb{R})$ by

$$\Phi \begin{pmatrix} A & B \\ C & D \end{pmatrix} = D - CA^{-1}B.$$

Consider $D\Phi(E)$. Define a curve $\gamma : \mathbb{R} \to U$ by

$$\lambda(t) = \begin{pmatrix} A & B \\ C & D + tX \end{pmatrix}.$$

We compute

$$\Phi_* \gamma^{'}(0) = dt_{t=0}(D + tX - CA^{-1}B) = X.$$

Therefore, Φ is a submersion, which shows that $M_k(m \times n, \mathbb{R}) \cap U$ is an embedded submanifold of U. Suppose E_0' is any matrix of rank k. Through row-column operations, R, which is linear and preserves rank, U' is a neighborhood E_0'. Then, for $\Phi \circ R$ is a submersion whose zero set is $M_k(m \times n, \mathbb{R}) \cap U'$. Therefore, $M_k(m \times n, \mathbb{R})$ is an embedded submanifold. $\square$

Question 2.

5. Is product of two smooth manifolds with boundary a smooth manifold with boundary?

Solution.
It is not true in general. Consider $[0,1]$ with a smooth maximal half-space atlas, induced by a smooth half-space atlas $\{([0,\frac{2}{3}),x),((\frac{1}{3},1],1-x)\}$. Now, consider the product manifold, and choosing the x charts, we see that $x\times x([0,\frac{2}{3})\times[0,\frac{2}{3}))=[0,\frac{2}{3})\times[0,\frac{2}{3})$, which is not open in $\mathbb{R}\times[0,\infty)$. This violates the fact that the chart needs to map into an open set, which shows that a product of manifold with boundaries is not a manifold with boundary. $\square$

Question 3.

6. Let $n > 0$ be an integer and let $\langle , \rangle$ be the euclidean scalar product on $\mathbb{R}^{n+1}$, and $S^n := \{x \in \mathbb{R}^{n+1}, ||x|| = 1\}$. If $x \in \mathbb{R}^{n+1} \setminus 0$, we denote by $[x]$ the corresponding point of $\mathbb{RP}^n$.

   (a) Show that the map $f : S^n \times S^n \to \mathbb{R}$ defined by $(x, y) \mapsto \langle x, y\rangle$ is smooth. Find all points were it is a submersion.

   (b) Let $M \subset S^n \times S^n$ consists of orthogonal couples. Show that M is a smooth submanifold of $S^n \times S^n$.

   (c) Let $M' \subset \mathbb{RP}^n \times \mathbb{RP}^n$ consists of couples of orthogonal lines (L, L') of $\mathbb{R}^{n+1}$. Show that M' is a smooth submanifold of $\mathbb{RP}^n \times \mathbb{RP}^n$.

   (d) Let E be the set of triples (x, x', y) of $S^n \times S^n \times \mathbb{R}^{n+1}$ such that $\langle x, x'\rangle = \langle x, y\rangle = \langle x', y\rangle = 0$ and $\pi : E \to M$ be the map $(x, x', y) \mapsto (x, x')$. Show that π is a smooth vector bundle over M.

   (e) Let E' be the set of couples $([x], y)$ of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ such that $\langle x, y\rangle = 0$ and $\pi' : E' \to \mathbb{RP}^n$ be the map $([x], y) \mapsto [x]$. Show that π' is a smooth vector bundle, isomorphic to the tangent bundle of $\mathbb{RP}^n$.

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   (f) Let E'' be the set of couples $(([x], y), ([x'], y'))$ of $E' \times E'$ such that $\langle x, x'\rangle = 0$ and $y = y'$ and let $\pi'' : E'' \to M$ be the map defined by $(([x], y), ([x'], y')) \mapsto ([x], [x'])$. Show that π'' is a is a smooth vector bundle.

Solution.
(a) Let $p = (x_1^*, ..., x_{n+1}^*, y_1^*, ..., y_{n+1}^*) \in S^n \times S^n$. Without loss of generality, suppose that $p \in U_N \times U_S$. Then, we have the associated projection being $\phi = \psi_N \times \psi_S$, and choose $(\mathbb{R}, id)$ chart for $f(p)$. We aim to show that

$$id \circ f \circ \phi^{-1} = f \circ \phi^{-1}$$

is smooth. For any $q = (x_1, ..., x_n, y_1, ...y_n) \in \phi^{-1}(U_N \times U_S)$,

$$\begin{aligned} q \quad &\mapsto_{\phi}^{-1} \quad \left(\frac{S^2-1}{S^2+1}, \frac{2x_1}{S^2+1}, ..., \frac{2x_n}{S^2+1}, \frac{S^2-1}{S^2+1}, \frac{P^2-1}{P^2+1}, \frac{2y_1}{P^2+1}, ..., \frac{2y_n}{P^2+1}\right) \\ &\mapsto_f \quad \frac{(S^2-1)(P^2-1)}{(S^2+1)(P^2+1)} + \sum_{j=1}^{n} \frac{4x_j y_j}{(S^2+1)(P^2+1)}, \end{aligned}$$

where $S^2 = \sum_{j=1}^n x_j^2$ and $P^2 = \sum_{j=1}^n y_n^2$. This shows that the map is well-defined rational function, so it is smooth. By definition, submersion points are points where the differential at the point is surjective. These can be seen to be points with $S^2 \neq 1$ and $P^2 \neq 1$, which can be written as $(S_{n-1} \times S_n) \cap (S_n \times S_{n-1}) = S_{n-1} \times S_{n-1}$.

(b) Let $p \in M$, and suppose without loss of generality, $p \in U_N \times U_S$ (up to re-ordering). Then, it suffices to show that

$$\begin{aligned} \psi_N \times \psi_S((U_N \times U_S) \cap M) &= \psi_N \times \psi_S(M \setminus ((p_S \times S_{n-1}) \cup (S_{n-1} \times p_N))) \\ &= \psi_N \times \psi_S(M) \setminus \psi_N \times \psi_S(p_S \times S_{n-1} \cup S_{n-1} \times p_N) \end{aligned}$$

where p_N and p_S denote north and south poles, is a submanifold of $\psi_N \times \psi_S(U_N \times U_S) = (\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$. This follows, because

$$(p_S \times S_{n-1}) \cup (S_{n-1} \times p_N) \subset S_{n-1} \times S_{n-1}$$

and from (a), we see that f provides the submersion on the manifold, which shows the submanifold relation.

(c) This can be shown through an analogous argument in (a) and (b).

(d) Suppose $n \geq 2$. We first show that π defined is surjective. Suppose $(x, x') \in M$. Then, since the dimension of $\mathbb{R}^{n+1}$ is larger than 2 we can choose a vector y orthogonal to both x and x'. It follows that $(x, x', y) \in E$ such that $\pi((x, x', y)) = (x, x')$. Hence, π is surjective. Now, π is by definition smooth, because it is a projection defined on a product manifold. One can show the smoothness in both direction by choosing the same chart around p and taking a product with an arbitrary chart on one side. So far, π is surjective and smooth. Now, for any $p = (x, x') \in M$, we have

$$\pi^{-1}(p) \quad = \quad \{(x, x', y) \mid (x, x') \in M \text{ and } < x, y >=< x', y >= 0\}.$$

This clearly has a structure of linear space of dimension $n - 1$. Take $\{x, x'\}$ as a two linearly independent vectors of $\mathbb{R}^{n+1}$ and complete it to obtain $A = \{y_1, ..., y_{n-1}\}$, where these are basis elements, excluding $\{x, x'\}$. Now, an identity map from $span(A)$ to $\pi^{-1}(p)$ is a linear isomorphism. Pick $p \in M$ and without loss of genearlity suppose $p \in U_s \times U_s \cap M$. Then, define

$$\phi : \pi^{-1}((U_s \times U_s) \cap M) \to ((U_s \times U_s) \cap M) \times \mathbb{R}^{n-1},$$

by

$$(x, x', y) \in \pi^{-1}((U_s \times U_s) \cap M) \quad \mapsto \quad (x, x', \alpha_1, ..., \alpha_{n-1}),$$

where the $\alpha_1, ..., \alpha_{n-1}$ are the coefficients of the basis representation of $\{x, x', y_1, ...y_{n-1}\}$, restricted to the $\{y_1, ...y_n\}$ part. The restriction to a point is clearly a vector space isomorphism by the above discussion. It remains to show that ϕ defined is fiber-preserving. We compute, for $p = (x, x', y) \in (U_s \times U_s) \cap M$,

$$\phi(\pi^{-1}(p)) = \phi(\pi^{-1}(x, x', y)) = \phi((x, x') \times \mathbb{R}^{n+1}) \subset (x, x') \times \mathbb{R}^{n-1} = \pi'^{-1}(p),$$

where π' is the induced differomorphism for the diagram. It also follows that ϕ is a differomorphism as above, and we are done.

(e) Analogous to (d), $\pi^{'}$ is surjective and smooth. Fix $p = [x] \in \mathbb{RP}$. Then,

$$\pi'^{-1}(p) \quad = \quad \{([x], y) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \ \mid \ < x, y >= 0\}.$$

By considering the same construction as (d), it shows that the considered fiber is isomorphic to $\mathbb{R}^n$. In fact, by considering the ϕ, but with completed basis $\{x, y_1, ..., y_n\}$ from $\{x\}$, shows that π' is a vector bundle.

(f) Analogous to (d), $\pi^{''}$ is surjective and smooth. Note that M should be M'. Fix $p = ([x], [x']) \in \mathbb{RP}^n \times \mathbb{RP}^n$. Then,

$$\pi''^{-1}([x], [x']) \quad = \quad \{(([x], y), ([x'], y') \mid \ < x, x' >=< x, y >=< x', y' >= 0 \ \ y = y'\}.$$

By considering the same construction as above, the fiber is isomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. Take x, x' as two linearly independent and then complete the basis. Since $y = y'$ condition holds, we will have the isomorphism with the product of $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. $\square$